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On the Hough Technique for Curve Detection

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Abstract—**The usual Hough procedure for detecting curves embedded in digitized images involves a transformation from image space $\mathscr{I}$ to parameter space $\mathscr{P}$. In this note we suggest that insight into this transformation may be enhanced by mapping $\mathscr{P}$ back into $\mathscr{I}$.**

We generalize the Hough procedure to multiple gray-level images, and show that under certain conditions the Hough technique is a form of matched spatial filtering.

Index Terms—**Curve detection, image processing, line detection, matched filter, pattern recognition.**

I. INTRODUCTION

THE HOUGH technique is a procedure for detecting and finding—within a noisy image—straight lines, circles, parabolas, and other curves that can be specified by a small number of parameters. The basic concept is to compute the locus, in parameter space, of the set of curves passing through each candidate edge element in the given image. The parameter space is quantized into cells, and an accumulator is assigned to each cell [2]. The contents of the accumulator of every cell subtended by each locus is incremented by 1. The accumulators with the largest resulting contents determine the existence and location of the most likely curves, from the specified class, in the given image. The Hough technique is described in more detail in [1]–[5].

One of the problems associated with the Hough technique is that a uniform quantization of parameter space results in a nonuniform precision of the computed curves in image space. Toward dealing with this problem, we suggest a means to an improved insight into the relation between parameter space and image space. We also describe a relationship between the Hough technique and matched spatial filtering in those cases where all the curves or images to be detected are translations of each other.

II. PARAMETER SPACE AND IMAGE SPACE

It is usually possible to find geometric objects in image space that fully represent the parameters of each curve passing through a candidate edge element. For example, the straight line passing through $\boldsymbol{x} = (x_1, x_2)$ is specified by (ρ, θ) in the equation

$$\rho = x_1 \cos \theta + x_2 \sin \theta.$$

The quantities ρ and θ are the length and angle, with respect to the x_1-axis, of the position vector at the foot of the perpendicular to the line [2]. This is illustrated in Fig. 1.

The set of all straight lines passing through $\boldsymbol{x}$ is thus specified by the locus of the feet of the perpendiculars to these straight lines. This locus is a circle circumscribing the segment $(\boldsymbol{0}, \boldsymbol{x})$, and centered at $\boldsymbol{x}/2$. This circle, shown as the dashed curve in Fig. 1, is the locus *in image space* of the sinusoidal locus of (ρ, θ) in parameter space.

The geometry of the Hough detection of the straight line determined by three colinear points $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ is illustrated in Fig. 2. In this figure the point of intersection P of the three circumscribing circles is the foot of the perpendicular of the straight line through $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

Another example of this approach to plotting parameter loci in image space is the case where the curves to be detected are restricted to circles. Here the parameters of the curves are the circles' centers and radii. In an earlier paper [3], we showed how the locus of the centers of the circles, for a fixed radius, are conveniently plotted in image space.

A third example is the case where the curves are parabolas

Manuscript received July 5, 1977; revised December 10, 1977. This work was supported by the National Institute of General Medical Sciences under U.S. Public Health Service Grant GM-17632.

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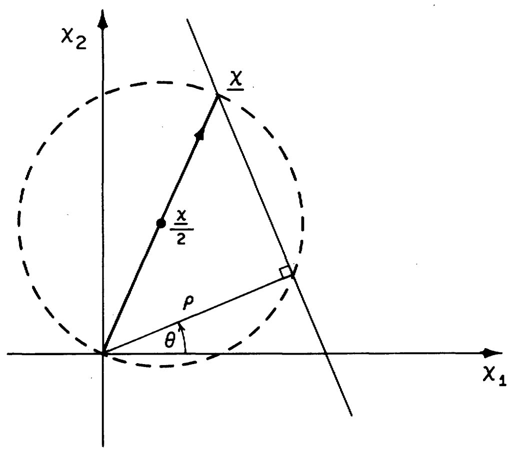

Fig. 1. Relation between x and (ρ, θ). (Variables with underbars in figures appear boldface italic in text.)

pointing in a common direction. Here the locus of the foci, for a fixed distance of the focus from the directrix, is conveniently plotted in image space. (We applied this technique successfully to the detection of rib contours in chest radiographs [7].) Each such trajectory in image space is a parabola pointing in a direction opposite to that of the set of parabolas to be detected. This observation (of oppositeness of direction) leads us to the generalization of the Hough technique described in the next section.

The advantage of this approach to plotting the loci of parameters is that it provides a geometric insight into the Hough technique—and, in particular, an insight into the relation between the accuracy and precision of the estimated parameters and the accuracy and precision of the corresponding curves in image space.

III. The Hough Technique and the Matched Filter

In this section, we present a generalized Hough algorithm for digitized binary-valued pictures and their translations. We also show how this technique may be extended to digitized m-valued pictures, in which case the technique is equivalent to a "matched filter."

A. A Generalized Hough Algorithm

The original Hough algorithm [1] was designed to detect colinear subsets of points within larger sets of points. Later this algorithm was generalized to circles and other curves [2]–[5], [7]. Here we generalize this to the detection of any specified subset of points or an arbitrary translation of this subset.

Let $\mathscr{C}$ denote any finite set of points in Euclidean space. Thus,

$$\mathscr{C} = \{x(t) \mid t = 0, 1, \cdots, m\}. \tag{1}$$

Note: $\mathscr{C}$ may be a digitized line drawing, a digitized blob, etc.

Let

$$y(t) = x(t) + r$$
$$= \text{a translation of } x(t). \tag{2}$$

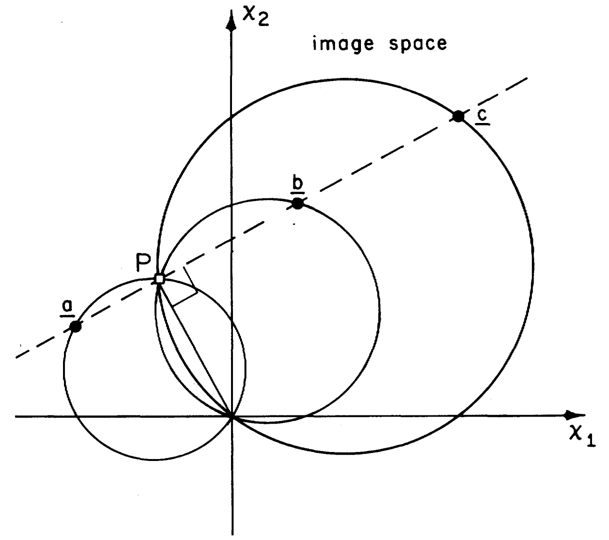

Fig. 2. Determination of straight line through $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

Let $\boldsymbol{b}$ denote any point in the image plane. The set of all $y(t)$'s passing through $\boldsymbol{b}$ for some $t = t'$ is given by $\boldsymbol{r}$ as a function of t' and $\boldsymbol{b}$. This function is found from the equation

$$y(t') = \boldsymbol{b} = x(t') + \boldsymbol{r}. \tag{3}$$

Thus,

$$\boldsymbol{r}(t', \boldsymbol{b}) = \boldsymbol{b} - x(t'). \tag{4}$$

Replacing t' by t, and plotting $\boldsymbol{r}$ in the image plane as a function of t, with $\boldsymbol{b}$ fixed, yields

$$\boldsymbol{r}(t) = \boldsymbol{b} - x(t), \tag{5}$$

which is a translated reflection of $x(t)$ about the origin. The origin may be any point in the plane of $\mathscr{C}$.

This result is illustrated in Fig. 3, where $x(t)$ is a parabola with its focus at the origin, the solid curves are translations of $x(t)$, and the dashed curve is the locus of the foci of translations of $x(t)$ passing through point $\boldsymbol{b}$. The four $\boldsymbol{r}_i$ in this figure are the foci of the four $y(t)$'s passing through $\boldsymbol{b}$. Note that the dashed curve is a translated reflection of $x(t)$ about the origin.

The function $\boldsymbol{r}(t)$, given in (5), is the equivalent of the point spread function of a *matched filter* for binary-valued pictures. A special case of this concept was described by Merlin and Farber [4].

In the Hough technique of curve detection, a function $\boldsymbol{r}(t, \boldsymbol{b})$ is computed for every candidate point $\boldsymbol{b}$ lying on one of the sought curves, and each of the accumulators met by $\boldsymbol{r}(t, \boldsymbol{b})$ is incremented by 1. In a template matching technique, on the other hand, several of the $y(t)$'s are passed through every candidate point $\boldsymbol{b}$, and the number of candidate points contained by each $y(t)$ is measured. The template with the largest count indicates the most likely position of one of the specified class of curves.

It is interesting to analyze the contents of the accumulator

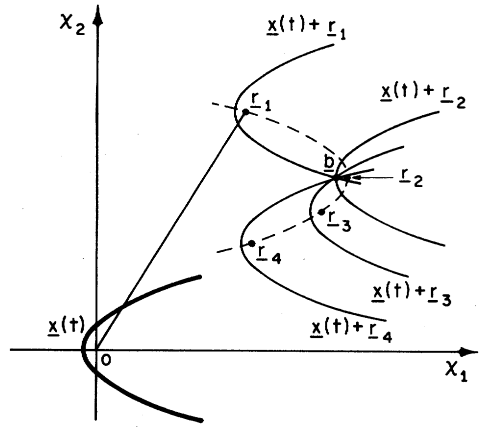

Fig. 3. The matched filter geometry for detecting parabolas.

at point x. Let $q(x)$ denote the contents of this accumulator. Let $h(x)$ be the characteristic function of $\mathscr{C}$, i.e., $h(x) = 1$ for $x \in \mathscr{C}$, 0 for $x \notin \mathscr{C}$. Let $p(x)$ denote a binary-valued picture function. Then

$$q(x) = \sum_y h(y - x)p(y). \tag{6}$$

Here we see that $h(y - x)$ is equivalent to $\boldsymbol{r}(t, \boldsymbol{b})$ and that $h(x)$ is the "template" or, equivalently, the point spread function of a matched filter.

An apparent advantage of the Hough method over the template-matching method is that the local properties of the edge element at $\boldsymbol{b}$ can guide a restriction of the computation of $\boldsymbol{r}(t, \boldsymbol{b})$—i.e., only a small subset of the points on $\boldsymbol{r}(t, \boldsymbol{b})$ need be computed, depending on the properties of the edge in the neighborhood of $\boldsymbol{b}$—while a corresponding restriction in the construction of templates seems to be more difficult to implement.

B. Generalization to Multiple-Valued Pictures

In this section we extend the Hough technique to multiple-valued picture functions. This generalization is equivalent to North's "matched filter" [6]. Here, too, it is assumed that the picture function to be detected is either a given picture function or a translation of that function.

Suppose a picture function $p(x)$ is a sum of a "signal" picture $p_S(x)$ and a "noise" picture $p_N(x)$, where $p_N(x)$ is generated by a statistically independent random process. Communication theory tells us that the signal-to-noise ratio in the detection of $p_S(x)$ at $x = \boldsymbol{a}$ is minimized by a convolution with a matched filter [6]:

$$h^*(x) = p_S(\boldsymbol{a} - x). \tag{7}$$

That is, the signal-to-noise ratio at $x = \boldsymbol{a}$ in estimating the size of $p_S(x)$ in $p(x)$ from the function

$$q(x) = \sum_y p(y)h(x - y) \tag{8}$$

is minimized when

$$h(x) = h^*(x) = p_S(\boldsymbol{a} - x). \tag{9}$$

[The function $q(x)$ represents the contents of the accumulator at point x, in analogy to (6).] The function $p_S(\boldsymbol{a} - x)$ is the "point spread function" of the matched filter for digital pictures. Note that if $p(x)$ is a two-valued (black and white with no intermediate gray level) picture, then the point spread function $h^*(x)$ is a reflection of $p_S(x)$ about the point $x - \boldsymbol{a}$. Thus, the Hough technique is optimum in the mean-square-error sense (equivalent in this case to minimizing the signal-to-noise ratio) when the given picture function is a sum of the signal $p_S(x)$ and the noise $p_N(x)$.

For the reader's convenience we present a proof of the above result in the next section.

C. Proof of the Matched Filter Technique for Digital Pictures

A shift invariant two-dimensional digital filter $h(x)$ yields the output picture $q(x)$ in response to an input $p(x)$ by the discrete convolution

$$q(x) = \sum_y p(y)h(x - y). \tag{10}$$

Recall that

$$p(x) = p_S(x) + p_N(x) \tag{11}$$

and that the values of $p_N(x)$ are chosen from a statistically independent random process. Let σ^2 denote the mean of $p_N^2(x)$ over all x.

Since the filter is linear, the signal and noise components in $q(x)$ are

$$q_s(x) = \sum_y p_S(y)h(x - y)$$

$$q_N(x) = \sum_y p_N(y)h(x - y),$$

respectively. Since the $p_N(x)$'s are statistically independent, the mean-square value of $q_N(x)$ is

$$\langle q_N^2(x) \rangle = \sigma^2 \sum_x h^2(x),$$

where $\langle\ \rangle$ denotes a spatial averaging operation and σ^2 denotes the mean of $p_N^2(x)$ over all x.

At $x = \boldsymbol{a}$, the signal-to-noise ratio is

$$\frac{S}{N}(\boldsymbol{a}) = \frac{\left| \sum_y p_s(y)h(\boldsymbol{a} - y) \right|^2}{\sigma^2 \sum_x h(y)}. \tag{12}$$

By Schwartz's Inequality, it follows that $\left(\frac{S}{N}\right)(\boldsymbol{a})$ is a maximum when

$$h(\boldsymbol{a} - y) = p_S(y).$$

Hence, the filter that maximizes the signal-to-noise ratio at $x = \boldsymbol{a}$ is

$$h^*(x) = p_S(a - x),$$

which proves the validity of (9).

ACKNOWLEDGMENT

We are grateful to C. W. Barnes and to the reviewers of this paper for their comments and criticisms.

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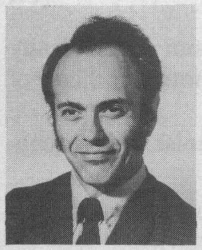

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