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NOTE

A Simple Approach for the Estimation of Circular Arc Center and Its Radius

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A simple exact solution for estimating the location of the center of a circular arc and its radius is suggested. Its features are demonstrated with the help of a computer program.

1. INTRODUCTION

A common problem of quality control and inspection of mechanical parts is the estimation of an arc center and its radius. Given a set of coordinates in two dimensions $(x_1, y_1), \ldots, (x_N, y_N)$ which represents an outline of a mechanical part and assumed to belong to a circular arc, the problem is to find the best estimation of the location of the center of the arc and its radius. An appropriate vision system could be used to extract the coordinates of the outline of the 2-dimensional image of the object system. By applying the formulas given here to the set of coordinate points the center and radius could be easily found.

A parametric approach to this problem was treated in [1–3]. By transforming the curves in two dimensions to a parameter space and by minimizing the LMS errors between the given set of data points and the curve, it is possible to find explicitly the parameters.

U. M. Landau [4] has suggested an iterative algorithm for estimating the location of the center of a circular arc and its radius. This algorithm is based on the minimization of the error between a set of given points and the estimated arc. Using vector notation, he was able to express the minimization condition. But since he was unable to solve simultaneously the two equations, he has suggested an iterative algorithm to estimate the circular arc center and its radius. In this paper, we wish to show that it is possible to find an exact solution if we carefully redefine our estimated error; the resulting simultaneous equation can be solved exactly to find the center of the arc and the radius of the estimated circle. The idea behind the method is based on the notion of what the "measure" is for best fitting in a 2-dimensional problem. We propose that the "measure" is an "area" rather than a "length." With this idea, we wish to redefine the error. However, because the minimization is now performed on the errors in area, an estimation bias results. This

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bias is small and approaches zero as the number of data points approaches infinity, i.e., the estimation is consistent.

2. DERIVATION OF THE EQUATION

Given a set of coordinates $(x_1, y_1)\ldots(x_i, y_i)\ldots(x_N, y_N)$, define a circle with center $(\bar{x}, \bar{y})$ and radius R. We define the error as the difference between the constant area πR^2 and the area of the circle centered at $\bar{x}, \bar{y}$ and has a radius

$$\left[(x_i - \bar{x})^2 + (y_i - \bar{y})^2\right]^{1/2}.$$

Summing up the squares of the errors we have

$$e(R, \bar{x}, \bar{y}) = \sum_{i=1}^{N} \left[\pi R^2 - \pi\left\{(x_i - \bar{x})^2 + (y_i - \bar{y})^2\right\}\right]^2$$

or

$$J = \frac{e}{\pi^2} = \sum_{i=1}^{N} \left[R^2 - \left\{(x_i - \bar{x})^2 + (y_i - \bar{y})^2\right\}\right]^2. \tag{2.1}$$

The function $J(R, \bar{x}, \bar{y})$ should be minimized with respect to R, $\bar{x}$, and $\bar{y}$. Differentiating (2.1) with respect to R yields

$$\frac{\partial J}{\partial R} = 2\sum_{i=1}^{N} \left[R^2 - \left\{(x_i - \bar{x})^2 + (y_i - \bar{y})^2\right\}\right](2R) = 0$$

or

$$NR^2 = \sum_{i=1}^{N} \left\{(x_i - \bar{x})^2 + (y_i - \bar{y})^2\right\}. \tag{2.2}$$

Differentiating (2.1) with respect to $\bar{x}$ yields

$$\frac{\partial J}{\partial \bar{x}} = 2\sum_{i=1}^{N} \left[R^2 - \left\{(x_i - \bar{x})^2 + (y_i - \bar{y})^2\right\}\right]2(x_i - \bar{x})(-1) = 0$$

or

$$\sum_{i=1}^{N} \left[R^2 - \left\{(x_i - \bar{x})^2 + (y_i - \bar{y})^2\right\}\right]x_i$$
$$= \sum_{i=1}^{N} \left[R^2 - \left\{(x_i - \bar{x})^2 + (y_i - \bar{y})^2\right\}\right]\bar{x} = 0$$

(since RHS is zero from (2.2).) Therefore,

$$R^2 \sum_{i=1}^{N} x_i = \sum_{i=1}^{N} \left\{(x_i - \bar{x})^2 + (y_i - \bar{y})^2\right\}x_i. \tag{2.3}$$

Similarly differentiating (2.1) with respect to $\bar{y}$ yields

$$R^2 \sum_{i=1}^{N} y_i = \sum_{i=1}^{N} \left\{ (x_i - \bar{x})^2 + (y_i - \bar{y})^2 \right\} y_i. \tag{2.4}$$

These equations may look very similar to the equation in [4]. Equations (2.2), (2.3), and (2.4) can be solved, even though quadratic, using the following tricks. First, let us simplify the notations using the following conventions. We use summation over N indices as $\sum_i$ and let Σ_x stand for $\sum_i x_i = x_1 + x_2 + \cdots + x_i + \cdots + x_N$ and

$$\Sigma_y = \sum_i y_i \qquad \Sigma_{x^2} = \sum_i x_i^2 \qquad \Sigma_{y^2} = \sum_i y_i^2$$

$$\Sigma_{xy} = \sum_i x_i y_i \qquad \Sigma_{x^3} = \sum_i x_i^3 \qquad \Sigma_{y^3} = \sum_i y_i^3$$

$$\Sigma_{x^2y} = \sum_i x_i^2 y_i \qquad \Sigma_{xy^2} = \sum_i x_i y_i^2.$$

With these conventions, (2.2) becomes

$$NR^2 = \Sigma_{x^2} - 2\Sigma_x \bar{x} + N\bar{x}^2 + \Sigma_{y^2} - 2\Sigma_y \bar{y} + N\bar{y}^2 \tag{2.5}$$

and (2.3) becomes

$$R^2\Sigma_x = \Sigma_{x^3} - 2\Sigma_{x^2}\bar{x} + \Sigma_x\bar{x}^2 + \Sigma_{xy^2} - 2\Sigma_{xy}\bar{y} + \Sigma_x\bar{y}^2 \tag{2.6}$$

and (2.4) becomes

$$R^2\Sigma_y = \Sigma_{x^2y} - 2\Sigma_{xy}\bar{x} + \Sigma_y\bar{x}^2 + \Sigma_{y^3} - 2\Sigma_{y^2}\bar{y} + \Sigma_y\bar{y}^2. \tag{2.7}$$

Equations (2.5), (2.6), and (2.7) can be solved.

Multiply (2.5) by Σ_x and subtract N multiplied by (2.6) to give

$$\begin{aligned} &\Sigma_{x^2}\Sigma_x - N\Sigma_{x^3} - 2\bar{x}\left[\Sigma_x^2 - N\Sigma_{x^2}\right] + \Sigma_x\Sigma_{y^2} - N\Sigma_{xy^2} \\ &\quad -2\bar{y}\left[\Sigma_x\Sigma_y - N\Sigma_{xy}\right] = 0. \end{aligned} \tag{2.8}$$

Multiply (2.5) by Σ_y and subtract N by (2.7) to give

$$\begin{aligned} &\Sigma_{x^2}\Sigma_y - N\Sigma_{x^2y} - 2\bar{x}\left[\Sigma_x\Sigma_y - N\Sigma_{xy}\right] + \Sigma_y\Sigma_{y^2} - N\Sigma_{y^3} \\ &\quad -2\bar{y}\left[\Sigma_y^2 - N\Sigma_{y^2}\right] = 0. \end{aligned} \tag{2.9}$$

We can now solve (2.8) and (2.9) for $\bar{x}$ and $\bar{y}$; the result can be written in matrix form as

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \tag{2.10}$$

where

$$a_1 = 2\left(\Sigma_x^2 - N\Sigma_{x^2}\right), \qquad b_1 = 2\left(\Sigma_x \Sigma_y - N\Sigma_{xy}\right)$$
$$a_2 = 2\left(\Sigma_x \Sigma_y - N\Sigma_{xy}\right) = b_1, \qquad b_2 = 2\left(\Sigma_y^2 - N\Sigma_{y^2}\right)$$
$$c_1 = \left(\Sigma_{x^2}\Sigma_x - N\Sigma_{x^3} + \Sigma_x \Sigma_{y^2} - N\Sigma_{xy^2}\right),$$
$$c_2 = \left(\Sigma_{x^2}\Sigma_y - N\Sigma_{y^3} + \Sigma_y \Sigma_{y^2} - N\Sigma_{x^2 y}\right)$$

from which

$$\bar{x} = \frac{c_1 b_2 - c_2 b_1}{a_1 b_2 - a_2 b_1} \tag{2.11}$$

and

$$\bar{y} = \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}. \tag{2.12}$$

Having solved $\bar{x}$, $\bar{y}$ we can substitute into (2.5) to get R^2, where

$$R^2 = \frac{1}{N}\left\{\Sigma_{x^2} - 2\Sigma_x \bar{x} + N\bar{x}^2 + \Sigma_{y^2} - 2\Sigma_y \bar{y} + N\bar{y}^2\right\}. \tag{2.13}$$

3. RESULTS AND CONCLUSION

Equations (2.11), (2.12), and (2.13) can be used to calculate the coordinates of the center and hence the radius of the best fitting circle. We have computed these using a simple Fortran program. Note that equations (2.10) may not always have a solution. For example, if there are only two data points, we cannot fit a unique circle through them and the matrix in (2.10) is singular. To test the accuracy of the calculation, we input 12 data points from a circle of radius 10 and centered at 10, 10. The program computed the x coordinate of the best fitting circle as 10.000, the y coordinate as 10.000, and radius of the best fitting circle as 10.000.

We modified the program such that the data points (400 points) are corrupted by a uniform random noise of (noise factor) $\times$ (± 0.5) along both x and y axes. Figures 1 and 2 show the plot and the best fitting circle. The noise factor is shown in each figure as well as the coordinates of the best fitting circle and the radius.

We also conducted a statistical summary of the result. In order to test the goodness of the fit we ran the program for 25 distinct runs each with a random noise ranging from $\pm$ (noise factor multiplied by 0.1), where the noise factor ranges from 1 to 25. The averages of these test runs were compared for the number of points N = 400, 200, 100, 50, 25, 12, 6, 3. It is found that the radius remains within statistical error. This experiment was conducted for a complete arc of 2π radians, and the results are in Table 1.

The root mean square error of the radius estimate is given by

$$\mathrm{RMSE} = \left[\frac{\sum_{i=1}^{25}\left(\hat{R}_i - R\right)^2}{25}\right]^{1/2},$$

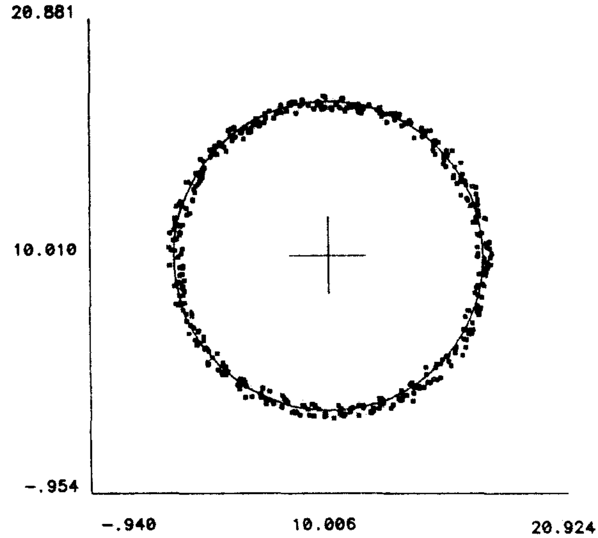

FIG. 1. The noise factor of this plot is 1: X coordinate of best fitting circle is 10.006; Y coordinate of best fitting circle is 10.010; radius of the best fitting circle is 9.994; minimum X coordinate is -0.440; minimum Y coordinate is -0.454; maximum X coordinate is 20.424; maximum Y coordinate is 20.381.

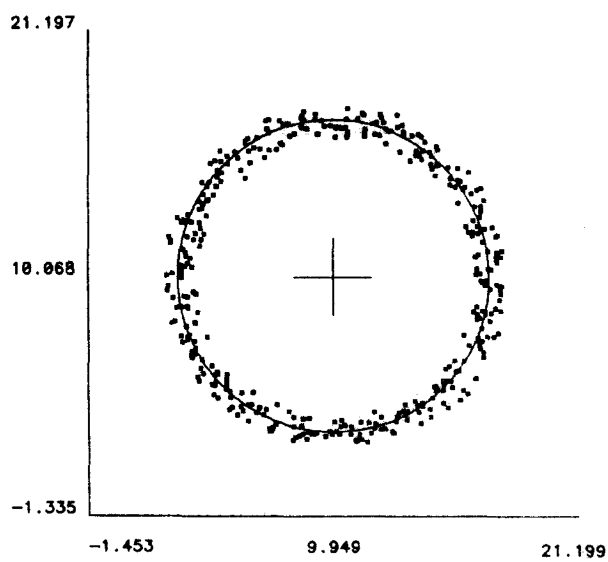

FIG. 2. The noise factor of this plot is 2: X coordinate of best fitting circle is 9.949; Y coordinate of best fitting circle is 10.068; radius of the best fitting circle is 10.058; minimum X coordinate is -0.953; minimum Y coordinate is -0.835; maximum X coordinate is 20.699; maximum Y coordinate is 20.697.

TABLE 1

Number of Points (N)	Noise Factor (M)	Average Center X-coordinate Actual = 10	Average Center y-coordinate Actual = 10	Average Radius using EQ (2.13) Actual = 10
400	1	9.9985739	10.0004789	10.0010154
400	6	9.9997513	9.9974331	10.0115466
400	11	10.0114473	10.0044434	10.0432726
400	16	10.0100685	9.9741293	10.0897615
400	21	9.9697464	9.9961919	10.1278938
200	1	10.0001748	10.0002807	9.9999973
200	6	10.0084820	10.0017519	10.0110382
200	11	9.9927242	9.9915394	10.0345454
200	16	10.0081068	9.9644038	10.0892568
200	21	9.9904823	9.9593583	10.1567724
100	1	9.9985943	10.0012976	10.0007876
100	6	10.0021564	9.9889549	9.9994034
100	11	10.0278977	9.9944288	10.0437132
100	16	9.9944577	10.0323156	10.0844023
100	21	9.9772840	9.9868947	10.1651771
50	1	9.9963887	10.0019022	9.9984908
50	6	9.9926281	9.9678267	10.0264716
50	11	10.0249448	9.9913807	10.0601056
50	16	9.9777196	9.9858302	10.1079065
50	21	9.9214509	9.9958168	10.1603779
25	1	10.0041456	9.9940041	9.9992253
25	6	10.0274718	10.0045303	10.0456138
25	11	9.9743052	10.0627367	10.0807912
25	16	9.9264993	9.9762541	10.0435113
25	21	9.9041944	9.9687544	10.1743551
12	1	9.9903169	9.9992849	10.0029320
12	6	9.9582489	10.0272809	9.9779538
12	11	9.9891634	10.0131387	10.0102818
12	16	10.0531139	10.0073709	10.1300286
12	21	9.9257072	9.9396633	10.1360055
6	1	9.9942958	10.0032020	9.9979513
6	6	9.9730454	10.0099192	10.0462949
6	11	9.9411750	10.1189518	10.0637641
6	16	9.9817839	10.0257210	10.0653427
6	21	10.3116101	10.1018914	10.1451083
3	1	10.0091631	10.0013427	9.9996959
3	6	9.9959756	10.0428319	10.0883417
3	11	10.1375569	10.0433213	10.1063313
3	16	10.0456089	9.8455176	10.1389265
3	21	10.2084082	9.9846273	10.0028788

where $\hat{R}_i$ is the estimated radius of each computer run i and $R = 10$ is the true radius. The quantity $10 \log_{10}$ RMSE is plotted against $10 \log_{10}$ SNR in Fig. 3 for $N = 200$, where SNR $= R^2/\sigma^2$ and σ^2 is the variance of the noise term. It is seen that the RMSE increases linearly with decreasing SNR. It does not exhibit a nonlinear phenomenon where the RMSE will increase precipitously where the SNR is below a certain threshold value.

We also conducted another experiment to study the effects of arc length. Figures 4 and 5 show the plots for arcs of $\pi/2$ and $3\pi/4$. The noise factor and the coordinates of the center and the radius of the best fitting circle are also shown. It is observed, as expected, that the result is better with lower noise and larger arc and more points.

It can be shown that the estimation of R and $\bar{x}$ and $\bar{y}$ is biased. This bias is a function of the number of points N and the noise variance and a derivation of the bias expression is included in the Appendix.

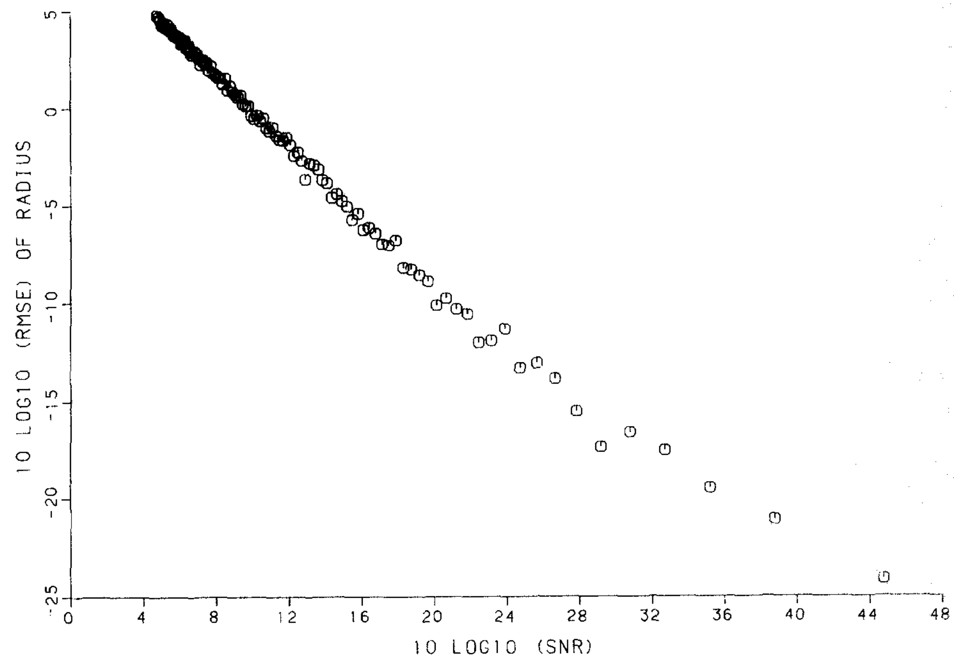

FIG. 3. Signal-to-noise ratio vs RMSE of radius.

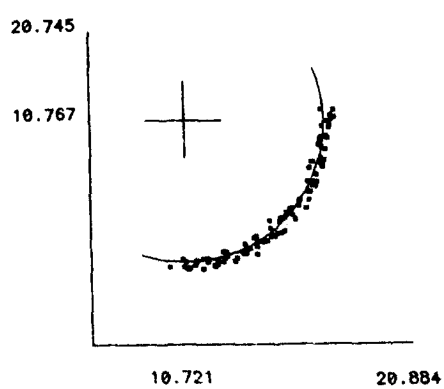

FIG. 4. The noise factor of this plot is 1: X coordinate of best fitting circle is 10.721; Y coordinate of best fitting circle is 10.767; radius of the best fitting circle is 9.091.

APPENDIX: BIAS FORMULAE FOR THE ESTIMATES

Let

$$a_1 = 2\left(\Sigma_x^2 - N\Sigma_{x^2}\right) \qquad b_1 = 2\left(\Sigma_x \Sigma_y - N\Sigma_{xy}\right)$$

$$a_2 = 2\left(\Sigma_x \Sigma_y - N\Sigma_{xy}\right) \qquad b_2 = 2\left(\Sigma_y^2 - N\Sigma_{y^2}\right)$$

$$c_1 = \Sigma_{x^2}\Sigma_x - N\Sigma_{x^3} + \Sigma_x \Sigma_{y^2} - N\Sigma_{xy^2}$$

$$c_2 = \Sigma_{x^2}\Sigma_y - N\Sigma_{y^3} + \Sigma_y \Sigma_{y^2} - N\Sigma_{x^2 y}.$$

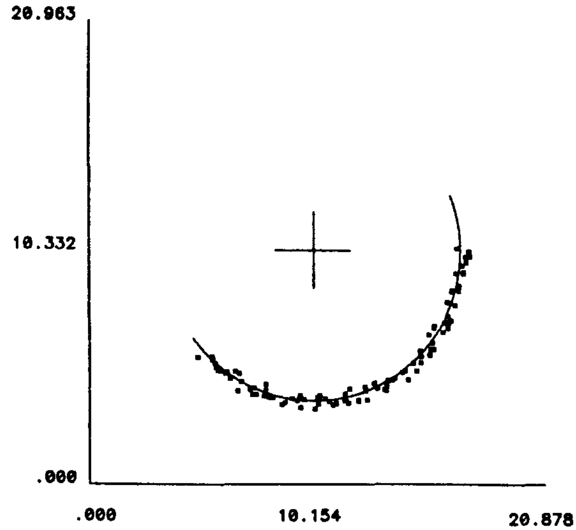

FIG. 5. The noise factor of this plot is 1: X coordinate of best fitting circle is 10.154; Y coordinate of best fitting circle is 10.332; radius of the best fitting circle is 9.720.

Then

$$\bar{x} = \frac{c_1 b_2 - c_2 b_1}{a_1 b_2 - a_2 b_1} \tag{A.1}$$

$$\bar{y} = \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}. \tag{A.2}$$

Suppose the data points x_i, y_i are corrupted by noise so that

$$x_i = \tilde{x}_i + \varepsilon_{x_i}, \qquad y_i = \tilde{y}_i + \varepsilon_{y_i}, \tag{A.3}$$

where ε_{x_i} and ε_{y_i} are independent white noise sequences of equal variance σ^2 and $\tilde{x}_i$ and $\tilde{y}_i$ are the true center coordinates. Then by substituting (A.3) into (A.1) and (A.2) and taking probability limits (plim) in the expression yields

$$\text{plim } \bar{x} = \bar{x}_0 + b_{\bar{x}} \tag{A.4}$$

$$\text{plim } \bar{y} = \bar{y}_0 + b_{\bar{y}}, \tag{A.5}$$

where $\bar{x}_0$, $\bar{y}_0$ are the true center coordinates and the biases

$$b_{\bar{x}} = \frac{4\sigma^2(2 - 2N)N\{\Sigma_{xy}\Sigma_y - \Sigma_{y^2}\Sigma_x\}}{a_1b_2 - a_2b_1} \tag{A.6}$$

$$b_{\bar{y}} = \frac{4\sigma^2(2 - 2N)N\{\Sigma_{xy}\Sigma_x - \Sigma_{x^2}\Sigma_y\}}{a_1b_2 - a_2b_1}. \tag{A.7}$$

The bias for R^2 is similarly found to be

$$b_{R^2} = \left(N\sigma^2 - 2\Sigma_x b_{\bar{x}} + 2N\bar{x}b_{\bar{x}} + Nb_{\bar{x}}^2 + N\sigma^2 - 2\Sigma_y b_{\bar{y}} + 2N\bar{y}b_{\bar{y}} + Nb_{\bar{y}}^2\right)N^{-1}$$

and, since $R = (R^2)^{1/2}$, the bias for R is approximately

$$b_R \sim b_{R^2}/2R. \tag{A.8}$$

For our problem, $b_{R^2} \sim 2\sigma^2$ and $R = 10$. The table below shows the theoretical and experimental average values of R.

M	Maximum deviation = δ	$\sigma^2 = \delta^2/3$	$b_R = \sigma^2/10$	Theoretical average R	Experimental average R
1	± 0.1	3.3×10^{-3}	3.3×10^{-4}	10.00033	10.00116
10	± 1	0.33	0.03	10.03	10.031
20	± 2	1.33	0.13	10.13	10.127
30	± 3	3	0.3	10.3	10.29122
40	± 4	5.3	0.53	10.53	10.49804
50	± 5	8.3	0.83	10.83	10.77041
60	± 6	12.0	1.2	11.2	11.11502

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