Hebbian Learning

```
import numpy as np
import matplotlib.pyplot as plt

# Display settings
%matplotlib inline
np.set_printoptions(precision=3, suppress=True)

seed = 42
np.random.seed(seed)
# For reproducibility
rng = np.random.default_rng(seed)

# hints: rng.multivariate_normal gives you 2-D gaussians.
# .       quiver let's you plot actual vector arrows.
```

Exercise 1 • Sampling inputs & visualisation

For each of the following zero-mean 2-D Gaussian input distributions

$$\Sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} 1 & 0.4 \\ 0.4 & 1 \end{pmatrix}, \quad \Sigma_3 = \begin{pmatrix} 1 & 0.9 \\ 0.9 & 1 \end{pmatrix}$$

1. Draw **500 samples** and show them in a single scatter plot (use different colours/markers).
2. Draw one random **weight vector** $\mathbf{w} \in \mathbb{R}^2$ from a standard normal distribution and add it to the scatter plot.
3. In a *second* panel visualise the linear activation

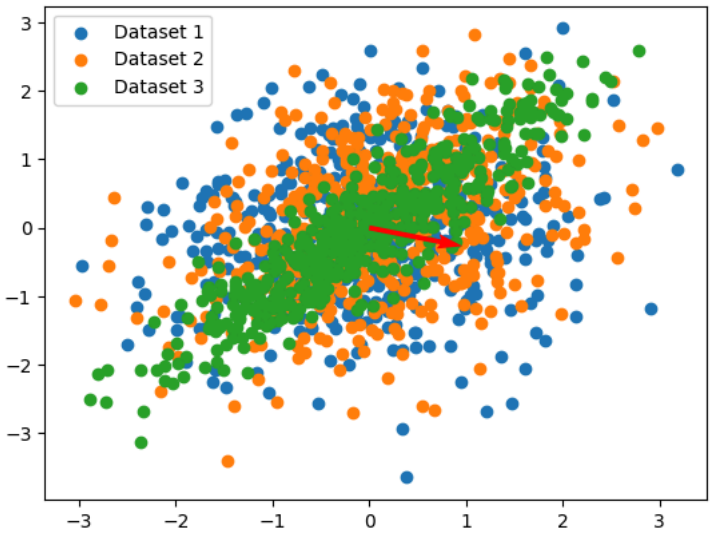
   $$v = \mathbf{w} \cdot \mathbf{u}$$

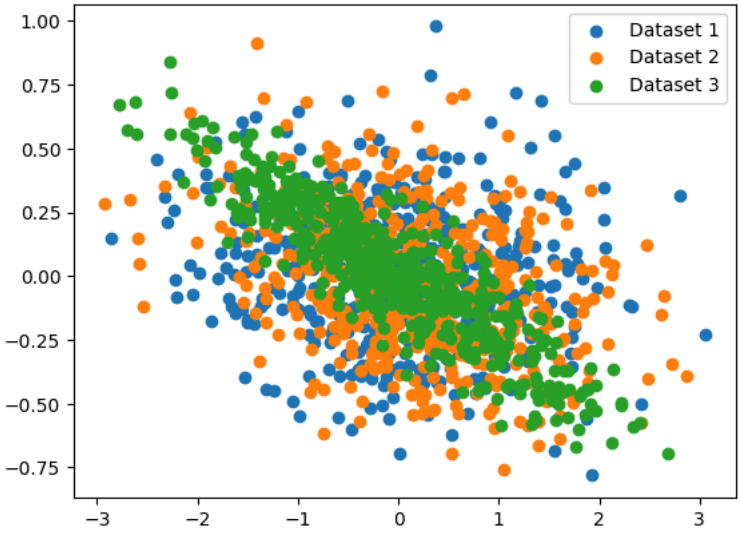
   on a 2-D grid covering $[-3, 3] \times [-3, 3]$ using a heatmap, together with the weight vector.

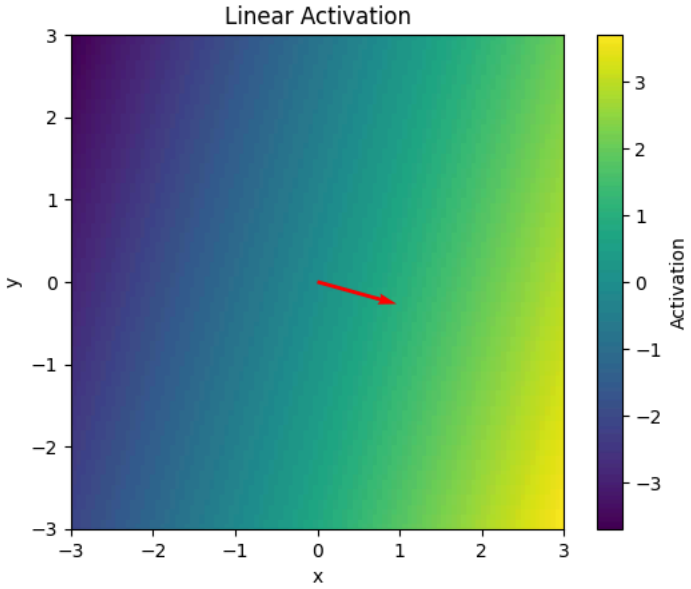
```
# hints: rng.multivariate_normal gives you 2-D gaussians.
#        quiver lets you plot actual vector arrows.

num_samples = 500
mean = [0, 0]
covs = [[[1, 0], [0, 1]], [[1, 0.4], [0.4, 1]], [[1, 0.9], [0.9, 1]]]

# create the 3 datasets
datasets = []
for distribution in covs:
    datasets.append(rng.multivariate_normal(mean, distribution, num_samples))

# plot all 3 datasets
for i, data in enumerate(datasets):
    plt.scatter(data[:, 0], data[:, 1], label=f"Dataset {i + 1}")


# draw and plot weight vector
weight = np.random.normal(size=2)
weight = weight / np.linalg.norm(weight)

print(weight)
plt.quiver(0, 0, *weight, color="red", angles="xy", scale_units="xy", scale=1)
plt.legend()
plt.show()


# create activations
activations = []
for dataset in datasets:
    activations.append(weight * dataset)

# plot all 3 activations
for i, data in enumerate(activations):
    plt.scatter(data[:, 0], data[:, 1], label=f"Dataset {i + 1}")

plt.legend()
# create grid for heatmap
x = np.linspace(-3, 3, 100)
y = np.linspace(-3, 3, 100)
```

```
y = np.linspace(-3, 3, 100)
X, Y = np.meshgrid(x, y)
grid_points = np.column_stack((X.ravel(), Y.ravel()))

# calculate activation for each point
activations = np.dot(grid_points, weight)
activations = activations.reshape(X.shape)

# plot heatmap
plt.figure()
plt.imshow(activations, extent=[-3, 3, -3, 3], origin="lower", aspect="equal")
plt.colorbar(label="Activation")

# plot weight vector on heatmap
plt.quiver(0, 0, *weight, color="red", angles="xy", scale_units="xy", scale=1)
plt.title("Linear Activation")
plt.xlabel("x")
plt.ylabel("y")
plt.show()

```

[0.963 -0.268]







Exercise 2 • Dynamics of Hebbian plasticity

2 a — Simple Hebb rule

Implement Hebbian learning

$$\Delta \mathbf{w}_n = v\,\mathbf{u}$$
$$v_n = \mathbf{w}_n^\top \mathbf{u}$$
$$\mathbf{w}_{n+1} = w + \eta \Delta \mathbf{w}_n$$

- Use a learning rate $\eta = 10^{-3}$.
- Start from a different random initial vector $\mathbf{w}_0$.
- Update once per randomly drawn input sample (2000 steps should suffice).
- **Plot** the trajectory of the weight vector on top of the corresponding input scatter for **each** of the three input distributions (use a marker for every 10 updates to keep the plot readable).

Describe qualitatively what you observe.

```
def hebb_update(w, u, lr):
    v = w * u
    grad = v * u
    return w + lr * grad


# you can use this function for other learning rules by changing update_fn. cov determines inputs.
def run_learning(update_fn, cov, lr=1e-3, steps=2000):
    # initialize weights, draw a data point, update weights. save weights to a vector.
    weights = np.random.normal(size=2)
    weights = weights / np.linalg.norm(weights)
    traj = []
    for i in range(steps):
        data = rng.multivariate_normal([0, 0], cov)
        weights = update_fn(weights, data, lr)
        if (i + 1) % 10 == 0:
            traj.append(weights.copy())
    return traj
    return traj


# it's helpful to standardize your plotting for the rest of the experiments.
# (ax=None lets you pass in subfigure axis)
def plot_trajectory(traj, data, title, ax=None):
    if ax is None:
        fig, ax = plt.subplots()
    ax.scatter(data[:, 0], data[:, 1], alpha=0.2, c="gray", label="Data")

    # Plot trajectory as connected line segments with larger dots
    traj = np.array(traj)
    ax.plot(traj[:, 0], traj[:, 1], "b-", alpha=0.5, label="Trajectory", linewidth=2)

    # Plot from the origin to the last trajectory point
    last_traj = traj[-1]
    ax.quiver(0, 0, last_traj[0], last_traj[1], color="red", scale=10, label="Last Trajectory")
    ax.set_title(title)
    ax.set_xlim(-5, 5)
    ax.set_ylim(-5, 5)
    ax.legend()
    plt.show()


# run the experiment for each cov. matrix and produce the plot for each condition.
for i in range(3):
    traj = run_learning(hebb_update, covs[i])
    plot_trajectory(traj, datasets[i], f"Weight trajectory for distribution {i + 1}")
    print(len(traj))

```

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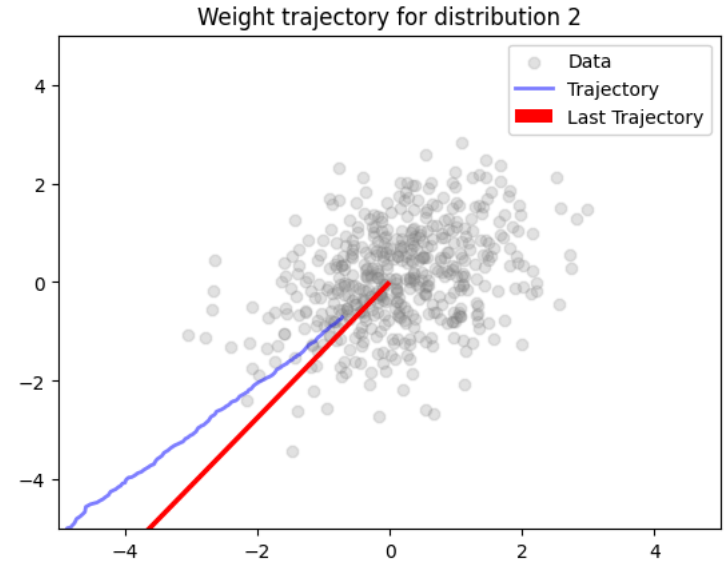

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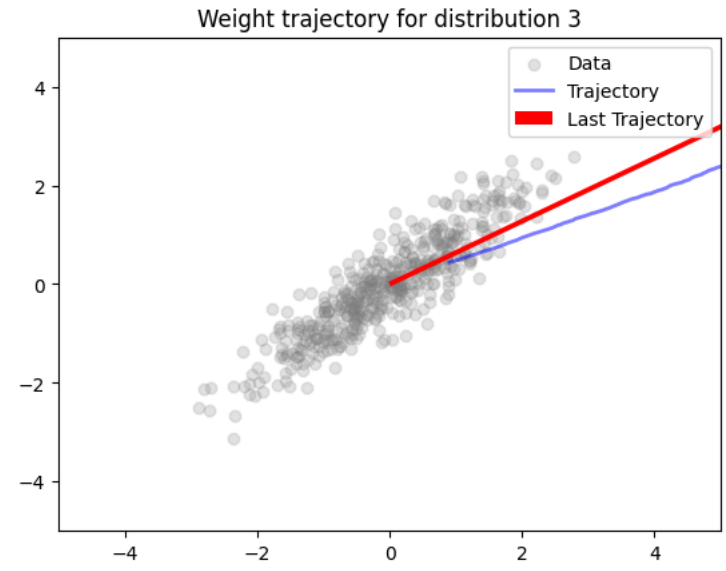

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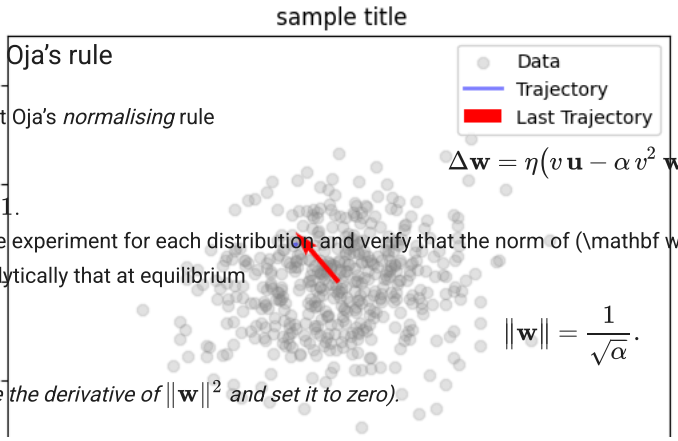
2 b — Catch your weight vectors!

Repeat part (a) but clip the weight vector to a fixed maximum length
$\|\mathbf{w}\| \leq w_{\max} 2$.

```
# implement new plasticity update
def hebb_clip_update(w, u, lr, w_max=1):
    v = np.dot(np.transpose(w), u)
    grad = np.dot(v, u)
    updated_w = w + lr * grad
    return updated_w / np.maximum(1, np.linalg.norm(updated_w) / w_max)


# repeat experiment (use run_learning and plot_trajectory)

# run the experiment for each cov. matrix and produce the plot for each condition.
for i in range(3):
    weight_history = run_learning(hebb_clip_update, covs[i])
    plot_trajectory(weight_history, datasets[i], " sample title")
```



2c — Oja's rule

Implement Oja's *normalising* rule

$$\Delta \mathbf{w} = \eta\big(v\,\mathbf{u} - \alpha\, v^2\,\mathbf{w}\big)$$

with $\alpha = 1$.

Repeat the experiment for each distribution and verify that the norm of (\mathbf w) converges.

Show analytically that at equilibrium

$$\|\mathbf{w}\| = \frac{1}{\sqrt{\alpha}}.$$

(Hint: take the derivative of $\|\mathbf{w}\|^2$ and set it to zero).

```
# implement new plasticity update
def oja_update(w, u, lr, alpha=1.0):
    v = w @ u
    return w + lr * (v * u - alpha * v**2 * w)


final_norms = []

for i in range(3):
    traj = run_learning(oja_update, covs[i])
    plot_trajectory(traj, datasets[i], " sample title")
    final_norms.append(np.linalg.norm(traj[-1]))

# repeat experiment (use run_learning and plot_trajectory)
print("Final weight nroms()", np.round(final_norms, 3))
```