Vanishing Viscosity Limit of the Navier-Stokes Equations to the Euler Equations for Compressible Fluid Flow

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Abstract

We establish the vanishing viscosity limit of the Navier-Stokes equations to the isentropic Euler equations for one-dimensional compressible fluid flow. For the Navier-Stokes equations, there exist no natural invariant regions for the equations with the real physical viscosity term so that the uniform sup-norm of solutions with respect to the physical viscosity coefficient may not be directly controllable. Furthermore, convex entropy-entropy flux pairs may not produce signed entropy dissipation measures.

To overcome these difficulties, we first develop uniform energy-type estimates with respect to the viscosity coefficient for solutions of the Navier-Stokes equations and establish the existence of measure-valued solutions of the isentropic Euler equations generated by the Navier-Stokes equations. Based on the uniform energy-type estimates and the features of the isentropic Euler equations, we establish that the entropy dissipation measures of the solutions of the Navier-Stokes equations for weak entropy-entropy flux pairs, generated by compactly supported C^2 test functions, are confined in a compact set in H^{-1}, which leads to the existence of measure-valued solutions that are confined by the Tartar-Murat commutator relation.

A careful characterization of the unbounded support of the measure-valued solution confined by the commutator relation yields the reduction of the measure-valued solution to a Dirac mass, which leads to the convergence of solutions of the Navier-Stokes equations to a finite-energy entropy solution of the isentropic Euler equations with finite-energy initial data, relative to the different end-states at infinity. © 2010 Wiley Periodicals, Inc.

1 Introduction

We are concerned with the vanishing viscosity limit of the motion of a compressible viscous, barotropic fluid in Eulerian coordinates $\mathbb{R}^2_+ := [0,\infty)\times\mathbb{R}$, which is described by the system of Navier-Stokes equations

$$\begin{cases}\rho_t + (\rho u)_x = 0,\\ (\rho u)_t + (\rho u^2 + p)_x = \varepsilon u_{xx},\end{cases} \tag{1.1}$$

Communications on Pure and Applied Mathematics, Vol. LXIII, 1469–1504 (2010)

with the initial conditions

$$\rho(0,x) = \rho_0(x), \qquad u(0,x) = u_0(x), \tag{1.2}$$

such that $\lim_{x\to\pm\infty}(\rho_0(x), u_0(x)) = (\rho^\pm, u^\pm)$, where ρ denotes the density, u represents the velocity of the fluid when $\rho > 0$, p is the pressure, $m = \rho u$ is the momentum, and $(\rho^\pm, u^\pm)$ are constant states with $\rho^\pm > 0$. The physical viscosity coefficient ε is restricted to $\varepsilon \in (0, \varepsilon_0]$ for some fixed $\varepsilon_0 > 0$.

The pressure p is a function of the density through the internal energy $e(\rho)$:

$$p(\rho) = \rho^2 e'(\rho) \quad \text{for } \rho \geq 0.$$

In particular, for a polytropic perfect gas,

$$p(\rho) = \kappa\rho^\gamma, \qquad e(\rho) = \frac{\kappa}{\gamma - 1}\rho^{\gamma-1}, \tag{1.3}$$

where $\gamma > 1$ is the adiabatic exponent and, by scaling, the constant κ in the pressure-density relation may be chosen as $\kappa = (\gamma-1)^2/4\gamma$ without loss of generality. One of the fundamental features of this system is that strict hyperbolicity fails when $\rho \to 0$.

The vanishing artificial/numerical viscosity limit to the isentropic Euler equations with general L^∞ initial data has been studied by Chen [4, 5]; Ding [9]; Ding, Chen, and Luo [10]; DiPerna [12]; Lions, Perthame, and Souganidis [21]; and Lions, Perthame, and Tadmor [22] via the method of compensated compactness. See Chen and LeFloch [7] for the isentropic Euler equtions with general pressure laws. Also, see DiPerna [11]; Morawetz [23]; Perthame and Tzavaras [25]; and Serre [28] for the vanishing artificial/numerical viscosity limit to general 2×2 strictly hyperbolic systems of conservation laws. The vanishing artificial viscosity limit to general, strictly hyperbolic systems of conservation laws with general small BV initial data was first established by Bianchini and Bressan [3] via direct BV estimates with small oscillation. Also, see LeFloch and Westdickenberg [20] for the existence of finite-energy solutions to the isentropic Euler equations with finite-energy initial data for the case $1 < \gamma \leq \frac{5}{3}$.

The idea of regarding inviscid gases as viscous gases with vanishing real physical viscosity dates back to the seminal paper by Stokes [30] and the important contribution of Rankine [26], Hugoniot [17], Rayleigh [27], and Taylor [32] (cf. Dafermos [8]). However, the first rigorous convergence analysis of vanishing physical viscosity from the Navier-Stokes equations (1.1) to the isentropic Euler equations was made by Gilbarg [13] in 1951, when he established the mathematical existence and vanishing viscous limit of the Navier-Stokes shock layers. For the convergence analysis confined in the framework of piecewise smooth solutions, see Hoff and Liu [16]; Gùes, Métivier, Williams, and Zumbrun [14]; and the references cited therein. The convergence of vanishing physical viscosity with general initial data was first studied by Serre and Shearer [29] for a 2×2 system in nonlinear elasticity with severe growth conditions on the nonlinear function in the system.

In this paper, we first develop new uniform estimates with respect to the real physical viscosity coefficient for the solutions of the Navier-Stokes equations with finite-energy initial data, relative to the different end-states at infinity. Then we establish the H^{-1}-compactness of weak entropy dissipation measures of the solutions of the Navier-Stokes equations for any weak entropy-entropy flux pairs generated by compactly supported C^2 test functions. With these, the existence of measure-valued solutions with possibly unbounded support is established, which are confined by the Tartar-Murat commutator relation with respect to two pairs of weak entropy-entropy flux kernels. Then we establish the reduction of the measure-valued solution with unbounded support for the case $\gamma \geq 3$ and, as a corollary, we obtain the existence of global finite-energy entropy solutions of the Euler equations with general initial data for $\gamma \geq 3$.

We further simplify the reduction proof of the measure-valued solution with unbounded support for the case $1 < \gamma \leq \frac{5}{3}$ in LeFloch and Westdickenberg [20] and extend to the whole interval $1 < \gamma < 3$. Then we establish the first convergence result for the vanishing physical viscosity limit of solutions of the Navier-Stokes equations to a finite-energy entropy solution of the isentropic Euler equations with finite-energy initial data relative to the different end-states at infinity. We remark that, combining Propositions 6.2 and 7.2 in this paper with the uniform estimates in Lemmas 3.1 through 3.4, we obtain the existence of finite-energy solutions of the isentropic Euler equations with finite-energy initial data relative to the different end-states at infinity for the case $\gamma > \frac{5}{3}$, which is in addition to the existence result in [20] for $1 < \gamma \leq \frac{5}{3}$.

This paper is organized as follows. In Section 2, we analyze some basic properties of weak entropy-entropy flux pairs in the unbounded phase plane and introduce the notion of finite-energy entropy solutions. In Section 3, we make several uniform estimates for the solutions of the Navier-Stokes equations that are independent of the real physical viscosity coefficient $\varepsilon > 0$. These estimates are essential for establishing the convergence of the vanishing viscosity limit of the solutions of the Cauchy problem (1.1)–(1.2) for the Navier-Stokes equations. In Section 4, we establish the H^{-1}-compactness of entropy dissipation measures for solutions of (1.1)–(1.2) with initial data (1.2) for any weak entropy-entropy flux pairs generated by compactly supported C^2 test functions.

In Section 5, we employ the estimates in Sections 3 and 4 to construct the measure-valued solutions with possibly unbounded support determined by the solutions of the Navier-Stokes equations (1.1) with initial data (1.2) and show that they are confined by the Tartar-Murat commutator relation for any two pairs of weak entropy-entropy flux kernels. In Sections 6 and 7, we prove that the measure-valued solution must be a Dirac mass in the phase coordinates (ρ, m), $m = \rho u$, when $\gamma > 1$. Finally, in Section 8, we prove the strong convergence of the vanishing viscosity limit of solutions of the Navier-Stokes equations to a finite-energy entropy solution of the isentropic Euler equations with finite-energy initial data, relative to the different end-states at infinity.

2 Entropy for the Isentropic Euler Equations

In this section we analyze some basic properties of weak entropy pairs in the unbounded phase plane and introduce the notion of finite-energy entropy solutions of the isentropic Euler equations of the form

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = 0. \end{cases} \tag{2.1}$$

System (2.1) is an archetype of nonlinear hyperbolic systems of conservation laws,

$$U_t + F(U)_x = 0.$$

For our case, $U = (\rho, m)^\top$ and $F(U) = (m, \frac{m^2}{\rho} + p)^\top$ for $m = \rho u$.

For $\gamma > 1$, the eigenvalues of system (2.1) are

$$\lambda_j = u + (-1)^j \theta \rho^\theta, \quad j = 1, 2, \tag{2.2}$$

and the Riemann invariants are

$$w_j = u + (-1)^{j-1} \rho^\theta, \quad j = 1, 2, \tag{2.3}$$

where $\theta = \frac{\gamma - 1}{2}$. From (2.2),

$$\lambda_2 - \lambda_1 = 2\theta\rho^\theta \to 0 \quad \text{as } \rho \to 0.$$

Therefore, system (2.1) is strictly hyperbolic when $\rho > 0$. However, near the vacuum $\rho = 0$, the two characteristic speeds of (2.1) may coincide and the system may be nonstrictly hyperbolic.

A pair of mappings $(\eta, q) : \mathbb{R}^2_+ := \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^2$ is called an entropy-entropy flux pair (or entropy pair for short) of system (2.1) if the pair satisfies the 2×2 hyperbolic system

$$\nabla q(U) = \nabla \eta(U) \nabla F(U). \tag{2.4}$$

Furthermore, $\eta(\rho, m)$ is called a weak entropy if

$$\eta \Big|_{\substack{\rho = 0, \\ u = m/\rho \text{ fixed}}} = 0. \tag{2.5}$$

An entropy pair is said to be convex if the Hessian $\nabla^2 \eta(\rho, m)$ is nonnegative in the region under consideration. See Lax [19].

For example, the mechanical energy (a sum of the kinetic and internal energy) and the mechanical energy flux

$$\begin{aligned} \eta^*(\rho, m) &= \frac{1}{2}\frac{m^2}{\rho} + \rho e(\rho), \\ q^*(\rho, m) &= \frac{1}{2}\frac{m^3}{\rho^2} + m e(\rho) + \rho m e'(\rho), \end{aligned} \tag{2.6}$$

form a special entropy pair; $\eta^*(\rho,m)$ is convex for any $\gamma>1$ in the region $\rho\ge 0$.

Let $(\overline{\rho}(x),\overline{u}(x))$ be a pair of smooth monotone functions satisfying $(\overline{\rho}(x),\overline{u}(x)) = (\rho^\pm,u^\pm)$ when $\pm x\ge L_0$ for some large $L_0>0$. The relative total mechanical energy for (1.1) in $\mathbb{R}$ with respect to the end-states $(\rho^\pm,u^\pm)$ through $(\overline{\rho},\overline{u})$ is

$$\begin{aligned} E[\rho,u](t) &:= \int_{\mathbb{R}} \big(\eta^*(\rho,m)-\eta^*(\overline{\rho},\overline{m})-\nabla\eta^*(\overline{\rho},\overline{m})\cdot(\rho-\overline{\rho},m-\overline{m})\big)dx \\ &\ge 0, \end{aligned} \tag{2.7}$$

where $\overline{m}=\overline{\rho u}$.

In the coordinates (ρ,u), any weak entropy function $\eta(\rho,\rho u)$ is governed by the second-order linear wave equation

$$\begin{cases} \eta_{\rho\rho}-\frac{p'(\rho)}{\rho^2}\eta_{uu}=0, & \rho>0,\\ \eta|_{\rho=0}=0. \end{cases} \tag{2.8}$$

Therefore, any weak entropy pair (η,q) can be represented by

$$\begin{cases} \eta^\psi(\rho,\rho u)=\int_{\mathbb{R}}\chi(\rho;s-u)\psi(s)ds,\\ q^\psi(\rho,\rho u)=\int_{\mathbb{R}}(\theta s+(1-\theta)u)\chi(\rho;s-u)\psi(s)ds, \end{cases} \tag{2.9}$$

for any continuous function $\psi(s)$, where the weak entropy kernel $\chi(\rho,s-u)$ is determined by

$$\begin{cases} \chi_{\rho\rho}-\frac{p'(\rho)}{\rho^2}\chi_{uu}=0,\\ \chi(0,u;s)=0, \quad \chi_\rho(0,u;s)=\delta_{u=s}, \end{cases} \tag{2.10}$$

where $\delta_{u=s}$ is the Dirac mass concentrated at $u=s$.

This implies that, for the γ-law case, the weak entropy kernel as the unique solution of (2.10) is

$$\chi(\rho;s-u)=[\rho^{2\theta}-(s-u)^2]_+^\lambda, \tag{2.11}$$

where $\lambda=\frac{3-\gamma}{2(\gamma-1)}>-\frac12$. Then the weak entropy pairs have the form

$$\begin{aligned} \eta^\psi(\rho,m)=\eta^\psi(\rho,\rho u) &= \int_{\mathbb{R}}[\rho^{2\theta}-(u-s)^2]_+^\lambda\psi(s)ds \\ &= \rho\int_{-1}^1\psi(u+\rho^\theta s)[1-s^2]_+^\lambda\,ds, \end{aligned} \tag{2.12}$$

$$\begin{aligned} q^\psi(\rho,m)=q^\psi(\rho,\rho u) &= \int_{\mathbb{R}}(\theta s+(1-\theta)u)[\rho^{2\theta}-(u-s)^2]_+^\lambda\psi(s)ds \\ &= \rho\int_{-1}^1(u+\theta\rho^\theta s)\psi(u+\rho^\theta s)[1-s^2]_+^\lambda\,ds. \end{aligned} \tag{2.13}$$

In particular, when $\psi_\sharp(w) = \frac{1}{2}w|w|$, the corresponding entropy pair $(\eta^\sharp, q^\sharp) := (\eta^{\psi_\sharp}, q^{\psi_\sharp})$ satisfies that there exists $C > 0$, depending only on $\gamma > 1$, such that

$$|\eta^\sharp(\rho,m)| \le C(\rho|u|^2 + \rho^\gamma), \quad q^\sharp(\rho,m) \ge C^{-1}(\rho|u|^3 + \rho^{\gamma+\theta}), \tag{2.14}$$

$$|\eta^\sharp_m(\rho,m)| \le C(|u| + \rho^\theta), \quad |\eta^\sharp_{mm}(\rho,m)| \le C\rho^{-1}, \tag{2.15}$$

and, regarding $\eta^\sharp_m$ in the coordinates (ρ,u),

$$|\eta^\sharp_{mu}(\rho,\rho u)| \le C, \quad |\eta^\sharp_{m\rho}(\rho,\rho u)| \le C\rho^{\theta-1}, \tag{2.16}$$

for all $\rho \ge 0$ and $u \in \mathbb{R}$ (also see, e.g., [22]).

Furthermore, we have the following:

LEMMA 2.1 *For a C^2 function $\psi : \mathbb{R} \to \mathbb{R}$, compactly supported on the interval $[a,b]$, we have*

$$\operatorname{supp}\eta^\psi, \operatorname{supp} q^\psi \subset \{(\rho,m) = (\rho,\rho u) : \rho^\theta + u \ge a,\ u - \rho^\theta \le b\}.$$

Furthermore, there exists a constant $C_\psi > 0$ such that, for any $\rho \ge 0$ and $u \in \mathbb{R}$, we have the following:

(i) *For $\gamma \in (1,3]$,*

$$|\eta^\psi(\rho,m)| + |q^\psi(\rho,m)| \le C_\psi \rho.$$

(ii) *For $\gamma > 3$,*

$$|\eta^\psi(\rho,m)| \le C_\psi\rho, \quad |q^\psi(\rho,m)| \le C_\psi \rho \max\{1,\rho^\theta\}.$$

(iii) *If η^ψ is considered as a function of $(\rho,m), m = \rho u$, then*

$$|\eta^\psi_m(\rho,m)| + |\rho\eta^\psi_{mm}(\rho,m)| \le C_\psi;$$

and, if η^ψ_m is considered as a function of (ρ,u), then

$$|\eta^\psi_{mu}(\rho,\rho u)| + |\rho^{1-\theta}\eta^\psi_{m\rho}(\rho,\rho u)| \le C_\psi.$$

PROOF: We first notice that, if (ρ,u) is such that $\rho^\theta + u < a$, then $u + \rho^\theta s < a$ for any $s \in [-1,1]$. Similarly, if $u - \rho^\theta > b$, then $u + s\rho^\theta > b$ for any $s \in [-1,1]$.

For (i), since ψ has compact support, it is clear from (2.12) that

$$|\eta^\psi(\rho,m)| \le C_\psi\rho.$$

When $\gamma = 3$,

$$q^\psi(\rho,m) = \rho\int_{-1}^{1}(u+\rho s)\psi(u+\rho s)\,ds,$$

which implies that $|q^\psi(\rho,m)| \le C_\psi\rho$ since ψ has compact support.

When $\gamma < 3$, we use the first formula in (2.13) to obtain

$$|q^\psi(\rho,m)| \le C_\psi\rho^{2\theta\lambda+\theta} \le C_\psi\rho.$$

For (ii), since ψ has compact support, it is clear from the formulas in (2.12)–(2.13) that

$$|\eta^{\psi}(\rho,m)| \leq C_{\psi}\rho, \quad |q^{\psi}(\rho,m)| \leq C_{\psi}\rho\max\{1,\rho^{\theta}\}.$$

To prove (iii), we first notice that

$$\eta^{\psi}_{m}(\rho,m) = \int \psi'(\tfrac{m}{\rho} + \rho^{\theta}s)[1-s^2]_{+}^{\lambda}\,ds,$$

which implies that $|\eta^{\psi}_{m}| \leq C_{\psi}$. Furthermore, we have

$$\eta^{\psi}_{mm}(\rho,m) = -\frac{1}{\rho}\int \psi''(\tfrac{m}{\rho} + \rho^{\theta}s)[1-s^2]_{+}^{\lambda}\,ds,$$

which yields that $|\rho\eta^{\psi}_{mm}(\rho,m)| \leq C_{\psi}$.

When η^{ψ}_{m} is regarded as a function of (ρ,u),

$$\eta^{\psi}_{m}(\rho,\rho u) = \int \psi'(u + \rho^{\theta}s)[1-s^2]_{+}^{\lambda}\,ds.$$

Then

$$\eta^{\psi}_{mu}(\rho,\rho u) = \int \psi''(u + \rho^{\theta}s)[1-s^2]_{+}^{\lambda}\,ds, \tag{2.17}$$

which leads to $|\eta^{\psi}_{mu}(\rho,\rho u)| \leq C_{\psi}$, while

$$\eta^{\psi}_{m\rho}(\rho,\rho u) = \theta\rho^{\theta-1}\int \psi''(u + \rho^{\theta}s)s[1-s^2]_{+}^{\lambda}\,ds, \tag{2.18}$$

which implies that $|\eta^{\psi}_{m\rho}(\rho,\rho u)| \leq C_{\psi}\rho^{\theta-1}$. This completes the proof. □

DEFINITION 2.2 Let (ρ_0,u_0) be given initial data with relative finite energy with respect to the end-states $(\rho^{\pm},u^{\pm})$ at infinity, i.e., $E[\rho_0,u_0] \leq E_0 < \infty$. A pair of measurable functions $(\rho,u) : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ is called a *finite-energy entropy solution of the Cauchy problem* (2.1) and (1.2) if the following holds:

(i) The relative total energy with respect to the end-states $(\rho^{\pm},u^{\pm})$ is bounded in time: There is a bounded function $C(E,t)$, defined on $\mathbb{R}^+ \times \mathbb{R}^+$ and continuous in t for each $E \in \mathbb{R}^+$, such that, for a.e. $t > 0$,

$$E[\rho,u](t) \leq C(E_0,t).$$

(ii) The entropy inequality

$$\eta^{\psi}(\rho,m)_t + q^{\psi}(\rho,m)_x \leq 0$$

is satisfied for the test function $\psi(s) \in \{\pm 1, \pm s, s^2\}$ in the sense of distributions.

(iii) The initial data (ρ_0,u_0) are attained in the sense of distributions.

The existence of entropy solutions in L^∞ was established by DiPerna [12] for the case $\gamma = (N+2)/N$, $N \geq 5$ odd, by Chen [4] and Ding, Chen, and Luo [10] for the general case $1 < \gamma \leq \frac{5}{3}$ for usual gases, by Lions, Perthame, and Tadmor [22] for the cases $\gamma \geq 3$, and by Lions, Perthame, and Souganidis [21] for closing the gap $\frac{5}{3} < \gamma < 3$. The existence of finite-energy solutions was recently established by LeFloch and Westdickenberg [20] for the case $1 < \gamma \leq \frac{5}{3}$. As a corollary of Theorem 8.1 in this paper, the existence of finite-energy entropy solutions for the isentropic Euler equations with finite-energy initial data with respect to the different end-states at infinity is established for all $\gamma > 1$.

3 Uniform Estimates for the Solutions of the Navier-Stokes Equations

Consider the Cauchy problem (1.1)–(1.2) for the Navier-Stokes equations in $\mathbb{R}^2_+ := [0,\infty)\times\mathbb{R}$. Assume that $(\rho^\varepsilon(t,x), u^\varepsilon(t,x))$ are smooth solutions of (1.1)–(1.2), globally in time, with $\rho^\varepsilon(t,x) \geq c_\varepsilon(t)$ for some $c_\varepsilon(t) > 0$ for $t \geq 0$ and

$$\lim_{x\to\pm\infty} (\rho^\varepsilon(t,x), u^\varepsilon(t,x)) = (\rho^\pm, u^\pm).$$

We now make several uniform estimates for the solutions $(\rho^\varepsilon(t,x), u^\varepsilon(t,x))$ of (1.1)–(1.2), which are independent of the physical viscosity coefficient $\varepsilon > 0$. These estimates are essential for establishing the convergence of the vanishing viscosity limit of solutions of the Cauchy problem (1.1)–(1.2) for the Navier-Stokes equations to a finite-energy entropy solution of the isentropic Euler equations (2.1) with finite-energy initial data (1.2) relative to the end-states $(\rho^\pm, u^\pm)$ at infinity.

To simplify the notation, throughout this section we denote $\int = \int_{\mathbb{R}}$, $(\rho, u) = (\rho^\varepsilon, u^\varepsilon)$, and $C > 0$ is a universal constant independent of ε.

3.1 Estimate I: Energy Estimate

The relative total mechanical energy for (1.1) in $\mathbb{R}$ with respect to the end-states $(\rho^\pm, u^\pm)$ through $(\overline{\rho}, \overline{u})$ introduced in (2.7) is equal to

$$E[\rho,u](t) = \int \left(\frac{1}{2}\rho(t,x)|u(t,x)-\overline{u}(x)|^2 + e^*(\rho(t,x),\overline{\rho}(x))\right)dx,$$

where $e^*(\rho,\overline{\rho}) = \rho e(\rho) - \overline{\rho}e(\overline{\rho}) - (\overline{\rho}e'(\overline{\rho}) + e(\overline{\rho}))(\rho-\overline{\rho}) \geq 0$ satisfies

$$e^*(\overline{\rho},\overline{\rho}) = e^*_\rho(\overline{\rho},\overline{\rho}) = 0, \quad e^*_{\rho\rho}(\rho,\overline{\rho}) = \frac{(\gamma-1)^2}{4}\rho^{\gamma-2} \geq 0 \ \text{ for } \gamma > 1.$$

This implies that $e^*(\rho,\overline{\rho})$ is a convex function in $\rho \geq 0$ that behaves like ρ^γ for large ρ and like $(\rho-\overline{\rho})^2$ for ρ close to $\overline{\rho}$. In particular, for later use, we notice that there exists $C_0 > 0$ such that

$$\rho(\rho^\theta - \overline{\rho}^\theta)^2 \leq C_0 e^*(\rho,\overline{\rho}) \quad \text{for } \rho \in [0,\infty), \tag{3.1}$$

where C_0 is a continuous function of $\overline{\rho}$ and γ.

We start with the standard energy estimate.

LEMMA 3.1 (Energy Estimate) *Let $E[\rho_0, u_0] \leq E_0 < \infty$, where $E_0 > 0$ is independent of ε. Then there exists $C = C(E_0, t, \bar{\rho}, \bar{u}) > 0$, independent of ε, such that*

$$\sup_{\tau \in [0,t]} E[\rho, u](\tau) + \int_0^t \int \varepsilon |u_x|^2 \, dx \, d\tau \leq C. \tag{3.2}$$

This can be seen through the following direct calculation:

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt} \int \eta^*(\rho, m) dx - \frac{d}{dt} \int \eta^*(\bar{\rho}, \bar{m}) dx \\ &\quad - \int \nabla \eta^*(\bar{\rho}, \bar{m}) \cdot (\rho_t, m_t) dx. \end{aligned} \tag{3.3}$$

Since (η^*, q^*) is an entropy pair, we have

$$\eta^*(\rho, m)_t + q^*(\rho, m)_x - \varepsilon \eta^*_m(\rho, m) \, u_{xx} = 0,$$

from which we conclude that

$$\frac{d}{dt} \int \eta^*(\rho, m) dx + \varepsilon \int |u_x|^2 \, dx = q^*(\rho^-, m^-) - q^*(\rho^+, m^+). \tag{3.4}$$

The second integral in (3.3) is independent of t, which implies that the second term on the right-hand side of (3.3) vanishes. For the last integral, we employ (1.1) to obtain

$$\begin{aligned} \left| \int \nabla \eta^*(\bar{\rho}, \bar{m}) \cdot (\rho_t, m_t) dx \right| &= \left| - \int \nabla \eta^*(\bar{\rho}, \bar{m}) \cdot (m_x, (\rho u^2 + p)_x - \varepsilon u_{xx}) dx \right| \\ &= \left| \int (\nabla \eta^*(\bar{\rho}, \bar{m}))_x \cdot (m, \rho u^2 + p - \varepsilon u_x) dx \right| \\ &\leq \frac{\varepsilon}{2} \int |u_x|^2 \, dx + C \int \rho |u - \bar{u}|^2 \, dx \\ &\quad + C \left(1 + \int_{-L_0}^{L_0} (\rho + p) dx \right), \end{aligned}$$

where we used that the compact support of $(\bar{\rho}_x, \bar{u}_x)$ lies in the interval $[-L_0, L_0]$ for some $L_0 > 0$. Since

$$\int_{-L_0}^{L_0} (\rho + p) dx \leq C \left(1 + \int_{-L_0}^{L_0} e^*(\rho, \bar{\rho}) dx \right),$$

we obtain

$$\left| \int \nabla \eta^*(\bar{\rho}, \bar{m}) \cdot (\rho_t, m_t) dx \right| \leq \frac{\varepsilon}{2} \int |u_x|^2 \, dx + C(E + 1)$$

for some C depending only on $(\gamma, \bar{\rho}, \bar{u})$. Combining this with (3.4), we have

$$\frac{dE}{dt} + \frac{\varepsilon}{2} \int |u_x|^2 \, dx \leq CE + C.$$

Then the lemma follows by Gronwall's inequality.

3.2 Estimate II: Space Derivative Estimate for the Density

We now develop an essential estimate for $\rho_x(t,x)$ involving the x-derivative of the density, motivated by an argument in [18].

LEMMA 3.2 *Let (ρ_0, u_0) be such that*

$$\varepsilon^2 \int \frac{|\rho_{0,x}(x)|^2}{\rho_0(x)^3}\,dx \le E_1 < \infty,$$

where E_1 is independent of ε. Then there exists $C = C(E_0, E_1, \bar\rho, \bar u, t) > 0$ independent of ε such that, for any $t > 0$,

$$\varepsilon^2 \int \frac{|\rho_x(t,x)|^2}{\rho(t,x)^3}\,dx + \varepsilon \int_0^t \int \rho^{\gamma-3}|\rho_x|^2\,dx\,d\tau \le C. \tag{3.5}$$

PROOF: Set $v = \frac{1}{\rho}$. Then the first equation in (1.1) can be written as

$$v_t + u v_x = v u_x.$$

Differentiating the above equation in x, we have

$$v_{xt} + (u v_x)_x = (v u_x)_x. \tag{3.6}$$

Then we multiply (3.6) by $2v_x$ to obtain

$$(|v_x|^2)_t + u(|v_x|^2)_x + 2u_x|v_x|^2 = 2v_x(v u_x)_x.$$

Multiplying this by ρ and using the equation of conservation of mass yields

$$(\rho|v_x|^2)_t + (\rho u|v_x|^2)_x + 2\rho u_x|v_x|^2 = 2\rho v_x(v u_x)_x$$

or

$$(\rho|v_x|^2)_t + (\rho u|v_x|^2)_x = 2v_x u_{xx}. \tag{3.7}$$

Using the second equation in (1.1) and (3.6), we obtain

$$\begin{aligned} 2v_x u_{xx} &= \frac{2}{\varepsilon} v_x(p_x + (\rho u)_t + (\rho u^2)_x) \\ &= \frac{2}{\varepsilon} v_x p_x + \frac{2}{\varepsilon}\Big((\rho(u-\bar u)v_x)_t - (\bar u(\ln\rho)_x)_t \\ &\qquad \underbrace{-\rho u(v u_x)_x + \rho u(u v_x)_x + v_x(\rho u^2)_x}_{J}\Big). \end{aligned} \tag{3.8}$$

By integration by parts, we have

$$\begin{aligned} \int J\,dx &= \int \big(v u_x(\rho u)_x - u v_x(\rho u)_x + v_x(u(\rho u)_x + \rho u u_x)\big)dx \\ &= \int \big(v u_x(\rho u)_x + \rho u v_x u_x\big)dx = \int |u_x|^2\,dx. \end{aligned} \tag{3.9}$$

Furthermore,

$$v_x p_x = -\frac{(\gamma-1)^2}{4}\rho^{\gamma-3}|\rho_x|^2. \tag{3.10}$$

Integrating (3.7) over $[0,t)\times\mathbb{R}$ and using the calculations in (3.8)–(3.10), we conclude

$$
\begin{aligned}
&\varepsilon^2\int\frac{|\rho_x(t,x)|^2}{\rho(t,x)^3}\,dx+\frac{(\gamma-1)^2}{2}\varepsilon\int_0^t\int\rho^{\gamma-3}|\rho_x|^2\,dx\,d\tau\\
&=-2\varepsilon\int\frac{\rho_x(t,x)(u(t,x)-\overline{u}(x))}{\rho(t,x)}\,dx+2\varepsilon\int\overline{u}(x)(\ln\rho)_x(t,x)dx\\
&\quad+2\varepsilon\int_0^t\int|u_x|^2\,dx\,d\tau+2\varepsilon\int\frac{\rho_{0,x}(x)(u_0(x)-\overline{u}(x))}{\rho_0(x)}\,dx\\
&\quad+2\varepsilon\int\overline{u}(x)(\ln\rho_0)_x(x)dx.
\end{aligned}
\tag{3.11}
$$

The first integral on the right-hand side is estimated by

$$
\begin{aligned}
\frac{\varepsilon^2}{4}\int\frac{|\rho_x(t,x)|^2}{\rho(t,x)^3}\,dx+8\int\rho(t,x)|u(t,x)-\overline{u}(x)|^2\,dx\le&\\
\frac{\varepsilon^2}{4}\int\frac{|\rho_x(t,x)|^2}{\rho(t,x)^3}dx+16E[\rho,u](t).&
\end{aligned}
\tag{3.12}
$$

Similarly, the fourth integral on the right-hand side is controlled by

$$
\frac{\varepsilon^2}{4}\int\frac{|\rho_{0,x}(x)|^2}{\rho_0(x)^3}\,dx+16E_0.
\tag{3.13}
$$

To estimate the second integral, we write

$$
\begin{aligned}
2\varepsilon\int\overline{u}(\ln\rho)_x\,dx=-2\varepsilon\int_{A_1}\overline{u}_x\ln\rho\,dx-2\varepsilon\int_{A_2}\overline{u}_x\ln\rho\,dx&\\
+2\varepsilon(u^+\ln\rho^+-u^-\ln\rho^-),&
\end{aligned}
$$

where

$$
A_1=\left\{x:\rho(t,x)\le\frac{\check{\rho}}{2}\right\},\quad A_2=A_1^{\mathrm{c}},
$$

for $\check{\rho}=\min\{\rho^-,\rho^+\}$. Since, on A_2, $|\ln\rho(t,x)|\le C\rho(t,x)$ and $\overline{u}_x$ is compactly supported, we can obtain

$$
\left|2\varepsilon\int_{A_2}\overline{u}_x\ln\rho\,dx\right|\le C\left(1+\int e^*(\rho(t,x),\overline{\rho}(x))dx\right).
\tag{3.14}
$$

If the set A_1 is not empty, then

$$
\left|2\varepsilon\int_{A_1}\overline{u}_x\ln\rho\,dx\right|\le C\varepsilon\sup_{x\in A_1}|\ln\rho(t,x)|\le C\varepsilon\sup_{x\in A_1}\frac{1}{\sqrt{\rho(t,x)}},
$$

and A_1 has finite measure, which can be estimated from (3.2) by

$$
|A_1|\le\frac{C}{e^*(\check{\rho}/2,\check{\rho})}=:d(t).
$$

In particular, for any (t,x), there is a point $x_0(t,x)$ such that $|x-x_0| \le d(t)$ and $\rho(t,x_0) = \frac{\check{\rho}}{2}$. Then we have

$$\begin{aligned}\varepsilon \sup_{x\in A_1} \frac{1}{\sqrt{\rho(t,x)}} &\le \varepsilon \sup_{x\in A_1} \left| \frac{1}{\sqrt{\rho(t,x)}} - \frac{1}{\sqrt{\rho(t,x_0)}} \right| + \frac{\varepsilon}{\sqrt{\check{\rho}/2}} \\ &\le \varepsilon \int_{x_0-d(t)}^{x_0+d(t)} \left| \left(\frac{1}{\sqrt{\rho(t,x)}} \right)_x \right| dx + \frac{\varepsilon}{\sqrt{\check{\rho}/2}} \\ &\le \left(\frac{\varepsilon^2}{2} \int \frac{|\rho_x|^2}{\rho^3}\, dx \right)^{1/2} \sqrt{d(t)} + \frac{\varepsilon}{\sqrt{\check{\rho}/2}} \\ &\le \frac{\varepsilon^2}{4} \int \frac{|\rho_x|^2}{\rho^3}\, dx + C(t).\end{aligned}$$

Thus, we obtain

$$2\varepsilon \left| \int \overline{u} (\ln \rho)_x \, dx \right| \le \frac{\varepsilon^2}{4} \int \frac{|\rho_x|^2}{\rho^3}\, dx + C.$$

Combining this with (3.12) in (3.11), we obtain

$$\begin{aligned}\varepsilon^2 \int \frac{|\rho_x(t,x)|^2}{\rho(t,x)^3}\, dx + \frac{(\gamma-1)^2}{2} \varepsilon \int_0^t \int \rho^{\gamma-3} |\rho_x|^2 \, dx\, d\tau \le \\ \frac{\varepsilon^2}{2} \int \frac{|\rho_x(t,x)|^2}{\rho(t,x)^3}\, dx + \varepsilon^2 \int \frac{|\rho_{0,x}(x)|^2}{\rho_0(x)^3}\, dx + C.\end{aligned}$$

The estimate of the lemma then follows. □

3.3 Estimate III: Higher Integrability

We now make uniform estimates for higher integrability of the solutions.

LEMMA 3.3 (Higher Integrability I) *Let $E[\rho_0, u_0] \le E_0 < \infty$ for E_0 independent of ε. Then, for any compact set $K \subset \mathbb{R}$ and all $t > 0$, there exists $C = C(K, E_0, \gamma, \overline{\rho}, \overline{u}, t)$, independent of $\varepsilon > 0$, such that*

$$\int_0^t \int_K \rho(t,x)^{\gamma+1}\, dx\, d\tau \le C.$$

PROOF: Let $\omega(x)$ be an arbitrary smooth, compactly supported function such that $0 \le \omega(x) \le 1$. Multiplying the second equation in (1.1) by $\omega(x)$ and then integrating with respect to the space variable over $(-\infty, x)$, we have

$$\rho u^2 \omega + p\omega = \varepsilon u_x \omega - \left(\int_{-\infty}^x \rho u \omega \, dy \right)_t + \int_{-\infty}^x ((\rho u^2 + p)\omega_x - \varepsilon u_x \omega_x) dy.$$

Multiply this by $\rho\omega$ and use the first equation in (1.1) to obtain

$$\begin{aligned}
\rho p\omega^2 &= -\rho^2 u^2\omega^2 + \varepsilon\rho u_x\omega^2 - \Big(\rho\omega\int_{-\infty}^{x}\rho u\omega\,dy\Big)_t \\
&\quad - (\rho u)_x\omega\int_{-\infty}^{x}\rho u\omega\,dy + \rho\omega\int_{-\infty}^{x}((\rho u^2+p)\omega_x - \varepsilon u_x\omega_x)dy \\
&= \varepsilon\rho u_x\omega^2 - \Big(\rho\omega\int_{-\infty}^{x}\rho u\omega\,dy\Big)_t - \Big(\rho u\omega\int_{-\infty}^{x}\rho u\omega\,dy\Big)_x \\
&\quad + \rho u\omega_x\int_{-\infty}^{x}\rho u\omega\,dy + \rho\omega\int_{-\infty}^{x}((\rho u^2+p)\omega_x - \varepsilon u_x\omega_x)dy.
\end{aligned}$$

Integrating the above equation over $(0,t)\times\mathbb{R}$, we have

$$\begin{aligned}
\int_0^t\int \rho p\,\omega^2\,dy\,d\tau &= \varepsilon\int_0^t\int \rho u_x\omega^2\,dy\,d\tau \\
&\quad - \int\rho\omega\Big(\int_{-\infty}^{x}\rho u\omega\,dy\Big)dx \\
&\quad + \int\rho_0\omega\Big(\int_{-\infty}^{x}\rho_0 u_0\omega\,dy\Big)dx + r_1(t),
\end{aligned}\tag{3.15}$$

where

$$\begin{aligned}
r_1(t) &= \int_0^t\int\rho u\omega_x\Big(\int_{-\infty}^{x}\rho u\omega\,dy\Big)dx\,d\tau \\
&\quad + \int_0^t\int\rho\omega\Big(\int_{-\infty}^{x}((\rho u^2+p)\omega_x - \varepsilon u_x\omega_x)dy\Big)dx\,d\tau.
\end{aligned}$$

Note that, by the Hölder inequality, for any $\delta > 0$,

$$\begin{aligned}
&\varepsilon\int_0^t\int\rho u_x\omega^2\,dx\,d\tau \\
&\quad \le \frac{\varepsilon^2}{\delta}\int_0^t\int|u_x|^2\,dx\,d\tau + \delta\int_0^t\int\rho^2\omega^4\,dx\,d\tau \\
&\quad \le \frac{\varepsilon_0}{\delta}\varepsilon\int_0^t\int|u_x|^2\,dx\,d\tau + C\delta\int_0^t\int(1+\rho^{\gamma+1})\omega^2\,dx\,d\tau \\
&\quad \le C + C\delta\int_0^t\int\rho^{\gamma+1}\omega^2\,dx\,d\tau,
\end{aligned}\tag{3.16}$$

since $\varepsilon \in (0, \varepsilon_0]$. By Lemma 3.1 and the Hölder inequality, we have

$$\begin{aligned}\left|\int_{-\infty}^{x} \rho u \omega \, dy\right| &\leq \int_{\operatorname{supp}\omega} |\rho u| dy \\ &\leq \left(\int_{\operatorname{supp}\omega} \rho \, dy\right)^{1/2} \left(\int_{\operatorname{supp}\omega} \rho u^2 \, dy\right)^{1/2} \\ &\leq C\left(\int_{\operatorname{supp}\omega} (1 + e^*(\rho, \bar{\rho}))dy\right)^{1/2} \left(\int_{\operatorname{supp}\omega} \rho u^2 \, dy\right)^{1/2} \leq C.\end{aligned}$$

It then follows that

$$\left|\int \rho\omega\left(\int_{-\infty}^{x} \rho u \omega \, dy\right) dx\right| \leq C. \tag{3.17}$$

Similarly, we have

$$\begin{aligned}&\left|\int_0^t \int \rho u \omega_x \left(\int_{-\infty}^{x} \rho u \omega \, dy\right) dx \, d\tau\right| \\ &\quad + \left|\int_0^t \int \rho\omega\left(\int_{-\infty}^{x} (\rho u^2 + p)\omega_x \, dy\right) dx \, d\tau\right| \\ &\quad + \left|\int_0^t \int \rho\omega\left(\int_{-\infty}^{x} \varepsilon u_x \omega_x \, dy\right) dx \, d\tau\right| \leq C.\end{aligned} \tag{3.18}$$

Combining estimates (3.16), (3.17), and (3.18) for the terms on the right-hand side of (3.15), we obtain

$$\int_0^t \int \rho^{\gamma+1} \omega^2 \, dx \, d\tau \leq C\delta \int_0^t \int \rho^{\gamma+1} \omega^2 \, dx \, dt + C.$$

Choosing suitably small $\delta > 0$, we conclude that

$$\int_0^t \int \rho^{\gamma+1} \omega^2 \, dx \, d\tau \leq C.$$

□

LEMMA 3.4 (Higher Integrability II) *Let $(\rho_0(x), u_0(x))$ satisfy that, in addition to the conditions in Lemmas* 3.1 *and* 3.2*,*

$$\int_{-\infty}^{\infty} \rho_0(x) |u_0(x) - \bar{u}(x)| dx \leq M_0 < \infty, \tag{3.19}$$

where $M_0 > 0$ is a constant independent of ε. Then, for any compact set $K \subset \mathbb{R}$ and $t > 0$, there exists $C > 0$ independent of ε such that

$$\int_0^t \int_K (\rho |u|^3 + \rho^{\gamma+\theta}) dx \, d\tau \leq C. \tag{3.20}$$

PROOF: Choose $\psi_\sharp(w) = \frac{1}{2}w|w|$ in (2.12)–(2.13). Then the corresponding weak entropy pair $(\eta^\sharp, q^\sharp) = (\eta^{\psi_\sharp}, q^{\psi_\sharp})$ satisfies estimates (2.14)–(2.16).

Note also that

$$\eta^\sharp(\rho,0) = \eta^\sharp_\rho(\rho,0) = 0,$$
$$q^\sharp(\rho,0) = \frac{\theta}{2}\rho^{3\theta+1}\int |s|^3[1-s^2]_+^\lambda\,ds > 0,$$

and

$$\eta^\sharp_m(\rho,0) = \alpha\rho^\theta \quad \text{with } \alpha := \int |s|[1-s^2]_+^\lambda\,ds.$$

We also need the Taylor expansion of $\eta^\sharp(\rho,m)$ at $m=0$ for fixed ρ:

$$\eta^\sharp(\rho,m) = \alpha\rho^\theta m + r_2(\rho,m) \tag{3.21}$$

with

$$|r_2(\rho,m)| \le C\frac{m^2}{\rho} = C\rho|u|^2 \tag{3.22}$$

for some positive $C > 0$. Finally, we introduce an entropy pair $(\check\eta, \check q)$ by choosing the density function $\psi(s) = \psi_\sharp(s-u^-)$, where u^- is the left end-state of $u(t,x)$. Then

$$\check\eta(\rho,m) = \eta^\sharp(\rho, m-\rho u^-),$$
$$\check q(\rho,m) = q^\sharp(\rho, m-\rho u^-) - u^-\eta^\sharp(\rho, m-\rho u^-).$$

Moreover, from (3.21) and (3.22), we conclude

$$\check\eta(\rho,m) = \alpha\rho^{\theta+1}(u-u^-) + r_2(\rho, \rho(u-u^-)) \tag{3.23}$$

with

$$|r_2(\rho,\rho(u-u^-))| \le C\rho|u-u^-|^2. \tag{3.24}$$

Multiplying the first equation in (1.1) by $\check\eta_\rho$ and the second equation by $\check\eta_m$, adding them together, and integrating the result over $(0,t)\times(-\infty,x)$, we obtain

$$\begin{aligned}
&\int_{-\infty}^x \big(\check\eta(\rho,m) - \check\eta(\rho_0,m_0)\big)dy - t\tilde q \\
&\quad + \int_0^t q^\sharp(\rho,\rho(u-u^-)) - u^-\eta^\sharp(\rho,\rho(u-u^-))\,d\tau \\
&\quad - \varepsilon\int_0^t \check\eta_m u_x\,d\tau + \varepsilon\int_0^t\int_{-\infty}^x \check\eta_{mu}|u_x|^2\,dy\,d\tau \\
&\quad + \varepsilon\int_0^t\int_{-\infty}^x \check\eta_{m\rho}\rho_x u_x\,dy\,d\tau = 0,
\end{aligned} \tag{3.25}$$

where $\tilde{q} = q^\sharp(\rho^-, 0)$. From the pointwise estimate (2.16) on $(\eta^\sharp_{m\rho}, \eta^\sharp_{mu})$, which also holds for $(\check{\eta}_{m\rho}, \check{\eta}_{mu})$, and Lemmas 3.1 and 3.2, we have

$$\left|\varepsilon \int_0^t \int_{-\infty}^x \check{\eta}_{mu} |u_x|^2 \, dy \, d\tau \right| \leq C, \tag{3.26}$$

$$\left|\varepsilon \int_0^t \int_{-\infty}^x \check{\eta}_{m\rho} \rho_x u_x \, dy \, d\tau \right| \leq C. \tag{3.27}$$

Using estimates (2.14), (3.26), and (3.27) in (3.25), we obtain

$$\begin{aligned} \int_0^t \int_K (\rho|u-u^-|^3 + \rho^{\gamma+\theta}) dx\, dt & \\ \leq C(E_0, E_1, |K|, \overline{q}, t) & \\ + 2 \sup_{\tau\in[0,t]} \left| \int_K \left(\int_{-\infty}^x \check{\eta}(\rho(y,\tau), (\rho u)(y,\tau)) dy \right) dx \right| & \\ + \sup |\overline{u}| \int_0^t \int_K |\eta^\sharp(\rho, \rho(u-u^-))| d\tau \, dx & \\ + \varepsilon C \int_0^t \int_K |u||u_x| dx\, d\tau + \varepsilon C \int_0^t \int_K \rho^\theta |u_x| dx\, d\tau. & \end{aligned} \tag{3.28}$$

Clearly, by the Hölder inequality,

$$\begin{aligned} \varepsilon \int_0^t \int_K \rho^\theta |u_x| dx\, d\tau \leq \varepsilon \int_0^t \int |u_x|^2 \, dx\, d\tau & \\ + \varepsilon \int_0^t \int_K \rho^{\gamma-1} \, dx\, d\tau \leq C. & \end{aligned} \tag{3.29}$$

Similarly,

$$\begin{aligned} \varepsilon \int_0^t \int_K |u|\,|u_x| dx\, d\tau &\leq \varepsilon \int_0^t \int_K |u_x|^2 \, dx\, d\tau + \varepsilon \int_0^t \int_K |u|^2 \, dx\, d\tau \\ &\leq C + \varepsilon \int_0^t \int_K |u|^2 \, dx\, d\tau. \end{aligned} \tag{3.30}$$

Note from Lemma 3.1 that there exists a nondecreasing function $C(t) > 0$ such that, for any $t > 0$,

$$\int_{\{\rho(t,x)\leq\overline{\rho}/2\}} e^*(\rho(t,\cdot), \overline{\rho}) dx \leq C(t),$$

which implies that

$$\left|\left\{x : \rho(t,x) \le \frac{\check{\rho}}{2}\right\}\right| \le \frac{C(t)}{e^*(\frac{\check{\rho}}{2}, \check{\rho})}, \quad \check{\rho} = \min\{\rho^-, \rho^+\}.$$

Without loss of generality, we assume that K contains the interval $[a,b]$ of length $2C(t)/e^*(\frac{\check{\rho}}{2}, \check{\rho})$. It follows then that, for any $t \ge 0$, there is a (measurable) subset $A = A(t) \subset (a,b)$ of measure not less than $C(t)/e^*(\frac{\check{\rho}}{2}, \check{\rho})$ on which $\rho(t,x) \ge \frac{\check{\rho}}{2}$.

Denote

$$u_A(t) := \frac{1}{|A|} \int_A u(t,x)dx.$$

Then

$$|u(t,x)| \le |u_A(t)| + \int_K |u_x| dx \quad \text{for } x \in [a,b].$$

We estimate

$$\begin{aligned} |u_A(t)| \le \frac{1}{|A|} \int_A |u(t,x)| dx &\le \frac{1}{|A|} \sqrt{\frac{2}{\check{\rho}}} \int_A \sqrt{\rho(t,x)} |u(t,x)| dx \\ &\le \frac{1}{\sqrt{|A|}} \sqrt{\frac{2}{\check{\rho}}} \left(\int_K \rho(t,x) |u(t,x)|^2 \, dx \right)^{1/2} \\ &\le \sqrt{\frac{2e^*(\check{\rho}/2, \check{\rho})}{C(t)\check{\rho}}} \, C(E_0, K). \end{aligned}$$

Then

$$\varepsilon \int_0^t \int_K |u|^2 \, dx \, d\tau \le C\left(\varepsilon \int_0^t \int |u_x|^2 \, dx \, d\tau + \int_0^t |u_A(\tau)|^2 \, d\tau \right) \le C,$$

and, from (3.30),

$$\varepsilon \int_0^t \int_K |u||u_x| dx \, d\tau \le C. \tag{3.31}$$

Also, for the compact set K,

$$\int_0^t \int_K |\eta^\sharp(\rho, \rho(u - u^-))| d\tau \, dx \le C\left(1 + \int_0^t E[\rho, u](\tau) d\tau\right). \tag{3.32}$$

Finally, we estimate the term $\int_K (\int_{-\infty}^x \check{\eta}(\rho,\rho u)dy)dx$. Consider

$$
\begin{aligned}
&\left|\int_{-\infty}^x \check{\eta}(\rho,\rho u)dy\right| \\
&\quad = \left|\int_{-\infty}^x (\check{\eta}(\rho,\rho u) - \alpha\rho^{\theta+1}(u-u^-))dy\right| + \left|\int_{-\infty}^x \alpha\rho^{\theta+1}(u-u^-)dy\right| \\
&\quad = \left|\int_{-\infty}^x r_2(\rho,\rho(u-u^-))dy\right| + \left|\int_{-\infty}^x \alpha(\rho^\theta-(\rho^-)^\theta)\rho(u-u^-)dy\right| \\
&\qquad + \left|\alpha(\rho^-)^\theta \int_{-\infty}^x \rho(u-u^-)dy\right| \\
&\quad \le C\left(1+\int(\rho|u-u^-|^2 + e^*(\rho,\overline{\rho}))dx\right) + \alpha(\rho^-)^\theta\left|\int_{-\infty}^x \rho(u-u^-)dy\right| \\
&\quad \le C + \alpha(\rho^-)^\theta\left|\int_{-\infty}^x \rho(u-u^-)dy\right|,
\end{aligned} \tag{3.33}
$$

where we used (3.1)–(3.2) and (3.23)–(3.24) for $r_2(\rho,\rho(u-u^-))$, and the following inequality by using (3.1): For $x\in K$,

$$
\int_{-\infty}^x \rho(\rho^\theta-(\rho^-)^\theta)dx \le C\int_{-\infty}^x e^*(\rho,\rho^-)dx \le C\left(1+\int e^*(\rho,\overline{\rho})dx\right).
$$

It remains to estimate $|\int_{-\infty}^x \rho(u-u^-)dy|$. For this, we integrate the equations in (1.1) with respect to the space variable from $-\infty$ to x and the time variable from 0 to t:

$$
\begin{aligned}
&\int_{-\infty}^x \rho(t,y)(u(t,y)-u^-)dy \\
&\quad = \int_{-\infty}^x \rho_0(u_0-\overline{u})dy + \int_{-\infty}^x \rho_0(\overline{u}-u^-)dy \\
&\qquad - \int_0^t (\rho u^2 + p - p(\rho^-) + u^-(\rho u - \rho^- u^-))d\tau + \varepsilon\int_0^t u_x\, d\tau.
\end{aligned}
$$

Then, by a straightforward application of Lemma 3.1, we obtain

$$
\int_K \left|\int_{-\infty}^x \rho(t,y)(u(t,y)-u^-)dy\right| dx \le C.
$$

Combining this with (3.33), we have

$$
\int_K \left|\int_{-\infty}^x \check{\eta}(\rho,\rho u)(t,y)dy\right| dx \le C.
$$

Using this, (3.29), (3.31), and (3.32) in (3.28), we conclude the proof. □

Remark 3.5. In the uniform estimate above, we require that the initial functions $(\rho_0(x), u_0(x))$ satisfy the following:

(i) $\rho_0(x) > 0$, $\int \rho_0(x)|u_0(x)-\overline{u}(x)|dx < \infty$.

(ii) The relative total mechanical energy with respect to the end-states $(\rho^{\pm}, u^{\pm})$ at infinity through $(\overline{\rho}, \overline{u})$ is finite:

$$\int \left(\frac{1}{2}\rho_0(x)|u_0(x) - \overline{u}(x)|^2 + e^*(\rho_0(x), \overline{\rho}(x))\right) dx =: E_0 < \infty.$$

(iii) $\varepsilon^2 \int \frac{|\rho_{0,x}(x)|^2}{\rho_0(x)^3}\, dx \le E_1 < \infty$.

Since our approach in dealing with the vanishing viscosity limit below allows the vacuum, i.e., $\rho(t,x) \ge 0$, the initial conditions (iii) and $\rho_0(x) > 0$ can be removed by the standard cutoff, $\max\{\rho_0(x), \varepsilon^{1/2}\}$, first and then by a mollification $(\rho_0^\varepsilon(x), u_0^\varepsilon(x)) \in C^\infty(\mathbb{R})$ so that $\rho_0^\varepsilon(x) \ge \varepsilon^{1/2}$ and

$$\varepsilon^2 \int \frac{|\rho_{0,x}^\varepsilon(x)|^2}{\rho_0^\varepsilon(x)^3}\, dx \le E_1 < \infty$$

for $E_1 > 0$ independent of ε.

4 H^{-1}-Compactness of the Weak Entropy Dissipation Measures

In this section we establish the H^{-1}-compactness of entropy dissipation measures for solutions to the Navier-Stokes equations (1.1) with initial data (1.2) for the weak entropy pairs generated by compactly supported C^2 test functions ψ.

PROPOSITION 4.1 *Let $\psi : \mathbb{R} \to \mathbb{R}$ be any compactly supported C^2 function. Let (η^ψ, q^ψ) be a weak entropy pair generated by ψ. Then, for the solutions $(\rho^\varepsilon, u^\varepsilon)$ with $m^\varepsilon = \rho^\varepsilon u^\varepsilon$ of the Navier-Stokes equations* (1.1)–(1.2)*, the entropy dissipation measures*

$$\eta^\psi(\rho^\varepsilon, m^\varepsilon)_t + q^\psi(\rho^\varepsilon, m^\varepsilon)_x \tag{4.1}$$

are confined in a compact subset of $H^{-1}_{\mathrm{loc}}(\mathbb{R}^2_+)$.

PROOF: Multiplying the first equation in (1.1) by $\eta^\psi_\rho(\rho^\varepsilon, m^\varepsilon)$ and the second by $\eta^\psi_m(\rho^\varepsilon, m^\varepsilon)$ and adding them up, we obtain

$$\begin{aligned} \eta^\psi(\rho^\varepsilon, m^\varepsilon)_t + q^\psi(\rho^\varepsilon, m^\varepsilon)_x &= \\ \varepsilon(\eta^\psi_m(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)u^\varepsilon_x)_x - \varepsilon\eta^\psi_{mu}&(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)|u^\varepsilon_x|^2 - \varepsilon\eta^\psi_{m\rho}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)\rho^\varepsilon_x u^\varepsilon_x, \end{aligned} \tag{4.2}$$

where $\eta^\psi_{m\rho}(\rho, \rho u) = \partial_\rho(\eta^\psi_m(\rho, \rho u))$ and $\eta^\psi_{mu}(\rho, \rho u) = \partial_u(\eta^\psi_m(\rho, \rho u))$.

Lemma 2.1 indicates that

$$|\eta^\psi_{mu}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)| + |(\rho^\varepsilon)^{1-\theta}\eta^\psi_{m\rho}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)| \le C,$$

where $C > 0$ is independent of ε. Using this and the Hölder inequality, we obtain that, for any $T \in (0, \infty)$,

$$\begin{aligned} \|\varepsilon\eta^\psi_{mu}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)|u^\varepsilon_x|^2 + \varepsilon\eta^\psi_{m\rho}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)\rho^\varepsilon_x u^\varepsilon_x\|_{L^1([0,T]\times\mathbb{R})} &\le \\ C_\psi\, \|(\sqrt{\varepsilon}u^\varepsilon_x, \sqrt{\varepsilon}\rho^{\frac{\gamma-3}{2}}\rho^\varepsilon_x)\|_{L^2([0,T]\times\mathbb{R})} &\le C. \end{aligned}$$

This yields that

$$(4.3)\qquad \begin{aligned}&-\varepsilon\eta^{\psi}_{mu}(\rho^\varepsilon,\rho^\varepsilon u^\varepsilon)|u^\varepsilon_x|^2-\varepsilon\eta^{\psi}_{m\rho}(\rho^\varepsilon,\rho^\varepsilon u^\varepsilon)\rho^\varepsilon_x u^\varepsilon_x\\ &\text{are uniformly bounded in } L^1([0,T]\times\mathbb{R}),\end{aligned}$$

which implies that they are confined in a compact subset of $W^{-1,q_1}_{\mathrm{loc}}(\mathbb{R}^2_+)$, $1<q_1<2$.

Furthermore, since $|\eta^{\psi}_m(\rho^\varepsilon,\rho^\varepsilon u^\varepsilon)|\le C$, we obtain

$$(4.4)\quad \|\varepsilon\eta^{\psi}_m(\rho^\varepsilon,\rho^\varepsilon u^\varepsilon)u^\varepsilon_x\|_{L^2([0,T]\times\mathbb{R})}\le C\sqrt{\varepsilon}\|\sqrt{\varepsilon}u^\varepsilon_x\|_{L^2([0,T]\times\mathbb{R})}\le C\sqrt{\varepsilon}\to 0$$

as $\varepsilon\to 0$. Combining (4.3) with (4.4) yields that

$$(4.5)\quad \eta^{\psi}(\rho^\varepsilon,m^\varepsilon)_t+q^{\psi}(\rho^\varepsilon,m^\varepsilon)_x\quad \text{are confined in a compact subset of } W^{-1,q_1}_{\mathrm{loc}}$$

for $1<q_1<2$.

On the other hand, using the estimates in Lemma 2.1(i)–(ii) and in Lemmas 3.3 and 3.4, we obtain that

$$\eta^{\psi}(\rho^\varepsilon,m^\varepsilon),\ q^{\psi}(\rho^\varepsilon,m^\varepsilon)\quad \text{are uniformly bounded in } L^{q_2}_{\mathrm{loc}}(\mathbb{R}^2_+)$$

for $q_2=\gamma+1>2$ when $\gamma\in(1,3]$ and $q_2=\frac{\gamma+\theta}{1+\theta}>2$ when $\gamma>3$. This implies that, for some $q_2>2$,

$$(4.6)\qquad \eta^{\psi}(\rho^\varepsilon,m^\varepsilon)_t+q^{\psi}(\rho^\varepsilon,m^\varepsilon)_x\quad \text{are uniformly bounded in } W^{-1,q_2}_{\mathrm{loc}}.$$

The interpolation compactness theorem (cf. [6, 10]) indicates that, for $q_1>1$, $q_2\in(q_1,\infty]$, and $p\in[q_1,q_2)$,

$$\begin{aligned}&(\text{compact set of } W^{-1,q_1}_{\mathrm{loc}}(\mathbb{R}^2_+))\cap(\text{bounded set of } W^{-1,q_2}_{\mathrm{loc}}(\mathbb{R}^2_+))\\ &\quad\subset(\text{compact set of } W^{-1,p}_{\mathrm{loc}}(\mathbb{R}^2_+)),\end{aligned}$$

which is a generalization of Murat's lemma in [24, 31]. Combining this interpolation compactness theorem for $1<q_1<2$, $q_2>2$, and $p=2$ with the facts in (4.5)–(4.6), we conclude the result. $\square$

5 Compensated Compactness and Measure-Valued Solutions

In this section, we use the estimates in Sections 3 and 4 to construct the measure-valued solutions of the Cauchy problem (1.1)–(1.2) for the Navier-Stokes equations and show that the measure-valued solutions are confined by the Tartar-Murat commutator relation for any two pairs of weak entropy-entropy flux kernels via the method of compensated compactness.

For convenience, we will work with measures defined on the phase space

$$\mathbb{H}=\{(\rho,u):\rho>0\}.$$

As in LeFloch and Westdickenberg [20], let $\overline{\mathcal{H}}$ be a compactification of $\mathbb{H}$ such that the space $C(\overline{\mathcal{H}})$ is equivalent (isometrically isomorphic) to the space

$$\bar{C}(\mathbb{H}) = \left\{ \phi \in C(\overline{\mathbb{H}}) : \begin{array}{l} \phi(\rho, u) \text{ is constant on } \{\rho = 0\} \text{ and} \\ \text{the map } (\rho, u) \to \lim_{s\to\infty} \phi(s\rho, su) \\ \text{belongs to } C(\mathbb{S}^1 \cap \overline{\mathbb{H}}) \end{array} \right\}$$

where $\mathbb{S}^1 \subset \mathbb{R}^2$ is the unit circle. These spaces allow us to deal with the two difficulties of the problem when $\rho = 0$ (vacuum) and when $\rho \gg 1$ in the large. As usual, we will not distinguish between the functions in $\bar{C}(\mathbb{H})$ and in $C(\overline{\mathcal{H}})$. The topology of $\overline{\mathcal{H}}$ is the weak-star topology induced by $C(\overline{\mathcal{H}})$, which is separable and metrizable. Note that the topology above does not distinguish points in the compactification of the set $\{\rho = 0\}$; that is, all points in the vacuum are equivalent. Denote by V the weak-star closure of $\{\rho = 0\}$ and define $\mathcal{H} = \mathbb{H} \cup V$.

Following Alberti and Müller [1] (also see Ball [2] and Tartar [31]), we find that, given any sequence of measurable functions $(\rho^\varepsilon, u^\varepsilon) : \mathbb{R}^2_+ \to \overline{\mathcal{H}}$, there exists a subsequence (still labeled $(\rho^\varepsilon, u^\varepsilon)$) and a function

$$\nu_{t,x} \in L^\infty_w(\mathbb{R}^2_+; \mathrm{Prob}(\overline{\mathcal{H}}))$$

such that, for all $\phi \in C(\overline{\mathcal{H}})$,

$$\phi(\rho^\varepsilon(t,x), u^\varepsilon(t,x)) \overset{*}{\rightharpoonup} \int_{\overline{\mathcal{H}}} \phi(\rho, u)\, d\nu_{t,x}(\rho, u) \quad \text{in } L^\infty(\mathbb{R}^2_+). \tag{5.1}$$

The sequence of functions $(\rho^\varepsilon, u^\varepsilon)$ converges to $(\rho, u) : \mathbb{R}^2_+ \to \overline{\mathcal{H}}$ if and only if

$$\nu_{t,x} = \delta_{(\rho(t,x), m(t,x))} \quad \text{a.e. } (t,x)$$

in the phase coordinates (ρ, m), $m = \rho u$.

In what follows we will often abbreviate $\nu_{t,x}$ as ν, implicitly assuming the dependence on (t,x) when no confusion arises.

Let B_R be a closed ball of radius R centered at the origin. The restriction of ν to $C(B_R \cap \overline{\mathbb{H}})$ can be identified with a Radon (regular, Borel) measure $\nu_R \in C(B_R \cap \overline{\mathbb{H}})^*$. By taking a sequence of radii, $R_n \to \infty$, we obtain a probability measure ν on $\mathbb{H}$ such that, for any

$$\phi \in C_0(\mathbb{H}) = \{\text{continuous functions, compactly supported on } \mathbb{H}\},$$

we have

$$\int_{\mathbb{H}} \phi\, d\nu = \langle \nu, \phi \rangle_{(\bar{C}(\mathbb{H}))^* \times \bar{C}(\mathbb{H})} \tag{5.2}$$

and

$$\phi(\rho^\varepsilon, u^\varepsilon) \overset{*}{\rightharpoonup} \int_{\mathbb{H}} \phi(\rho, u)\, d\nu \quad \text{in } L^\infty(\mathbb{R}^2_+). \tag{5.3}$$

We will often use the same letter ν for an element of $(\overline{C}(\mathbb{H}))^*$ or $(C(\overline{\mathcal{H}}))^*$ and for its restriction (a Radon measure on $\mathbb{H}$) to $(C_0(\mathbb{H}))^*$, but it will be clear from the context which one is being used.

Let $(\rho^\varepsilon, u^\varepsilon)$ be the sequence of solutions of the Navier-Stokes equations (1.1) with initial data (1.2). Let $\nu = \nu_{t,x}$ be a Young measure corresponding to this sequence of functions $(\rho^\varepsilon, u^\varepsilon)$.

In the following proposition (analogous to proposition 2.3 in [20]), we can extend the Young measure $\nu_{t,x}$ to a class of test functions larger than $\overline{C}(\mathbb{H})$.

PROPOSITION 5.1 *The following statements hold*:

(i) *For the Young measure $\nu_{t,x}$ introduced above,*

$$\int_{\mathbb{H}} (\rho^{\gamma+1} + \rho|u|^3) d\nu_{t,x} \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+). \tag{5.4}$$

(ii) *Let $\phi(\rho, u)$ be a function such that*

(a) *ϕ is continuous on $\overline{\mathbb{H}}$ and zero on $\partial\mathbb{H}$;*

(b) $\operatorname{supp}\phi \subset \{(\rho, u) : \rho^\theta + u \geq -a, u - \rho^\theta \leq a\}$ *for some constant* $a > 0$;

(c) $|\phi(\rho, u)| \leq \rho^{\beta(\gamma+1)}$ *for all (ρ, u) with large ρ and some $\beta \in (0, 1)$.*

Then ϕ is $\nu_{t,x}$-integrable and

$$\phi(\rho^\varepsilon, u^\varepsilon) \rightharpoonup \int_{\mathbb{H}} \phi \, d\nu_{t,x} \quad \textit{in } L^1_{\mathrm{loc}}(\mathbb{R}^2_+). \tag{5.5}$$

(iii) *For $\nu_{t,x}$ viewed as an element of $(C(\overline{\mathcal{H}}))^*$,*

$$\nu_{t,x}[\overline{\mathcal{H}} \setminus (\mathbb{H} \cup V)] = 0, \tag{5.6}$$

which means that $\nu_{t,x}$ is concentrated in $\mathbb{H}$ and/or on the vacuum $V = \{\rho = 0\}$.

PROOF: To prove (i), we define a cutoff function $\omega_k(\rho, u)$ that is nonnegative and continuous, equals 1 on the box

$$\{(\rho, u) : \rho^\theta \in [\tfrac{1}{k}, k], |u| \leq k\},$$

and equals 0 outside the box

$$\{(\rho, u) : \rho^\theta \in [\tfrac{1}{2k}, 2k], |u| \leq 2k\}.$$

Then the functions $((\rho^\varepsilon)^{\gamma+1} + \rho^\varepsilon|u^\varepsilon|^3)\omega_k(\rho^\varepsilon, u^\varepsilon)$ are in $\overline{C}(\mathbb{H})$ so that

$$\lim_{\varepsilon\to 0} \int_{[0,T]\times K} ((\rho^\varepsilon)^{\gamma+1} + \rho^\varepsilon|u^\varepsilon|^3)\omega_k(\rho^\varepsilon, u^\varepsilon) dx\, dt =$$
$$\int_{[0,T]\times K} \left(\int_{\mathbb{H}} (\rho^{\gamma+1} + \rho|u|^3)\omega_k(\rho, u) d\nu_{t,x} \right) dx\, dt,$$

where K is a compact subset of $\mathbb{R}$. Note that, by Lemmas 3.3 and 3.4,

$$\int_{[0,T]\times K} ((\rho^{\varepsilon})^{\gamma+1} + \rho^{\varepsilon}|u^{\varepsilon}|^3)\omega_k(\rho^{\varepsilon}, u^{\varepsilon})dx\,dt \le C,$$

where $C > 0$ is independent of $\varepsilon > 0$. By the monotone convergence theorem,

$$\lim_{k\to\infty}\int_{\mathbb{H}}(\rho^{\gamma+1} + \rho|u|^3)\omega_k(\rho,u)d\nu = \int_{\mathbb{H}}(\rho^{\gamma+1} + \rho|u|^3)d\nu$$

is a (t,x)-integrable function, which is finite a.e. $(t,x) \in [0,T]\times K$:

$$\int_{[0,T]\times K}\left(\int_{\mathbb{H}}(\rho^{\gamma+1} + \rho|u|^3)d\nu_{t,x}\right)dx\,dt < \infty.$$

To prove (ii), let $\omega_k(\rho,u)$ be the same cutoff function as in (i). Note that, with $\phi(\rho,u)$ satisfying (ii)(a)–(c), $\omega_k(\rho,u)\phi(\rho,u) \in \overline{C}(\mathbb{H})$, and thus $\langle \nu_{t,x}, \omega_k\,\phi\rangle$ is well-defined for a.e. (t,x).

By the Lebesgue dominated convergence theorem and (i), it follows that

$$\lim_{k\to\infty}\int_{\mathbb{H}}\phi\,\omega_k\,d\nu_{t,x} = \int_{\mathbb{H}}\phi\,d\nu_{t,x} \quad \text{a.e. } (t,x)\in[0,T]\times K$$

and

$$\lim_{k\to\infty}\int_{[0,T]\times K}\int_{\mathbb{H}}\phi\omega_k\,d\nu_{t,x}\,dx\,dt = \int_{[0,T]\times K}\int_{\mathbb{H}}\phi\,d\nu_{t,x}\,dx\,dt.$$

On the other hand, by the definition of Young measure, it implies that

$$\lim_{k\to\infty}\lim_{\varepsilon\to 0}\int_{[0,T]\times K}\langle \nu^{\varepsilon}_{t,x}, \phi\omega_k\rangle dx\,dt = \int_{[0,T]\times K}\int_{\mathbb{H}}\phi\,d\nu_{t,x}\,dx\,dt. \tag{5.7}$$

Claim. $\int_{[0,T]\times K}\langle \nu^{\varepsilon}_{t,x}, \phi\omega_k\rangle dx\,dt \to \int_{[0,T]\times K}\langle \nu^{\varepsilon}_{t,x}, \phi\rangle dx\,dt$ as $k \to \infty$ uniformly for $\varepsilon \in [0,\varepsilon_0)$.

If this is true, then we can interchange the limits in (5.7) to obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{[0,T]\times K}\phi(\rho^{\varepsilon}(t,x), u^{\varepsilon}(t,x))dx\,dt = \lim_{\varepsilon\to 0}\int_{[0,T]\times K}\langle \nu^{\varepsilon}_{t,x}, \phi\rangle dx\,dt\\
&= \lim_{\varepsilon\to 0}\lim_{k\to\infty}\int_{[0,T]\times K}\langle \nu^{\varepsilon}_{t,x}, \phi\omega_k\rangle dx\,dt = \lim_{k\to\infty}\lim_{\varepsilon\to 0}\int_{[0,T]\times K}\langle \nu^{\varepsilon}_{t,x}, \phi\omega_k\rangle dx\,dt\\
&= \lim_{k\to\infty}\int_{[0,T]\times K}\int_{\mathbb{H}}\phi\omega_k\,d\nu_{t,x}\,dx\,dt = \int_{[0,T]\times K}\int_{\mathbb{H}}\phi\,d\nu_{t,x}\,dx\,dt,
\end{aligned}$$

which is what we want.

PROOF OF THE CLAIM: With $k_1 < k_2$, consider

$$\int_{[0,T]\times K} \langle \nu^{\varepsilon}_{t,x}, (\omega_{k_1} - \omega_{k_2})\phi\rangle \,dx\,dt.$$

Notice that

$$\operatorname{supp}(\omega_{k_1} - \omega_{k_2}) \subset \left(\left\{\frac{1}{k_1} \le \rho^{\theta} \le k_1,\ |u| \le k_1\right\}\right)^{\mathsf{c}},$$
$$\sup_{\{0\le\rho^{\theta}\le 1/k_1\}} |\phi(\rho,u)| \le c_{k_1} \to 0 \quad \text{as } k_1 \to \infty,$$

and, if $(\rho,u) \in \operatorname{supp}\phi \cap (\{\frac{1}{k_1} \le \rho^{\theta} \le k_1,\ |u| \le k_1\})^{\mathsf{c}}$, then

$$\rho^{\theta} \ge k_1$$

for large k_1. Furthermore, by Young's inequality, for any $\alpha > 0$, there exists $C(\beta,\alpha) > 0$ such that

$$|\phi(\rho,u)| \le C(\beta,\alpha) + \alpha\rho^{\gamma+1}.$$

Thus we can estimate

$$\tag{5.8} \begin{aligned} &\left|\int_{[0,T]\times K} \langle \nu^{\varepsilon}_{t,x}, (\omega_{k_1} - \omega_{k_2})\phi\rangle \,dx\,dt\right| \\ &\quad \le T|K|\,c_{k_1} + C(\beta,\alpha)|([0,T]\times K) \cap \{(t,x) : (\rho^{\varepsilon})^{\theta} > k_1\}| \\ &\qquad + \alpha \int_{[0,T]\times K} |\rho^{\varepsilon}(t,x)|^{\gamma+1}\,dx\,dt. \end{aligned}$$

By the Chebyshev inequality,

$$|([0,T]\times K) \cap \{(t,x) : (\rho^{\varepsilon})^{\theta} > k_1\}| \le (k_1)^{-\frac{\gamma+1}{\theta}} \int_{[0,T]\times K} |\rho^{\varepsilon}(t,x)|^{\gamma+1}\,dx\,dt.$$

Using the uniform estimate in Lemma 3.3, we deduce from (5.8) that

$$\left|\int_{[0,T]\times K} \langle \nu^{\varepsilon}_{t,x}, (\omega_{k_1} - \omega_{k_2})\phi\rangle \,dx\,dt\right| \le T|K|\,c_{k_1} + C(\beta,\alpha)(k_1)^{-\frac{\gamma+1}{\theta}} + C\alpha,$$

where $C > 0$ and c_{k_1} are independent of ε, and $\alpha > 0$ is an arbitrary constant. The claim then follows. □

The result in (iii) follows directly from the uniform estimates for $(\rho^{\varepsilon}, u^{\varepsilon})$ in Lemmas 3.3 and 3.4 and Proposition 5.1. □

To simplify the notation, we denote the entropy kernel

$$\chi(\xi) := [\rho^{2\theta} - (u-\xi)^2]^{\lambda}_{+}$$

with $\lambda = (3-\gamma)/(2(\gamma-1))$ and, for any function $f(\rho,u)$ with growth slower than $\rho|u|^3 + \rho^{\gamma+\max\{1,\theta\}}$,

$$f(\rho^\varepsilon, u^\varepsilon) \rightharpoonup \overline{f(\rho,u)}(t,x) := \langle \nu_{t,x}, f(\rho,u)\rangle.$$

PROPOSITION 5.2 *Let $\nu_{t,x}$ be the Young measure determined by the solutions of the Navier-Stokes equations* (1.1) *with initial data* (1.2). *Then the Young measure $\nu_{t,x}$ is a measure-valued solution of* (1.1)–(1.2): *For the test functions $\psi \in \{\pm 1, \pm s, s^2\}$,*

$$\langle \nu_{t,x}, \eta^\psi\rangle_t + \langle \nu_{t,x}, q^\psi\rangle_x \le 0, \quad \langle \nu_{t,x}, \eta^\psi\rangle(0,\cdot) = \eta^\psi(\rho_0, \rho_0 u_0), \tag{5.9}$$

in the sense of distributions in $\mathbb{R}^2_+$. Furthermore, the measure-valued solution $\nu_{t,x}$ is confined by the following commutator relation: *For a.e. $s_1, s_2 \in \mathbb{R}$,*

$$\begin{aligned}\theta(s_2 - s_1)\big(\overline{\chi(s_1)\chi(s_2)} - \overline{\chi(s_1)}\,\overline{\chi(s_2)}\big) &= \\ &(1-\theta)\big(\overline{u\chi(s_2)}\,\overline{\chi(s_1)} - \overline{u\chi(s_1)\chi(s_2)}\big),\end{aligned} \tag{5.10}$$

where $\theta = (\gamma-1)/2$.

PROOF: We first employ (4.2) and (2.17) to obtain that the solutions $(\rho^\varepsilon, u^\varepsilon)$ of (1.1)–(1.2) satisfy

$$\begin{aligned}&\eta^\psi(\rho^\varepsilon, m^\varepsilon)_t + q^\psi(\rho^\varepsilon, m^\varepsilon)_x \\ &\quad = \varepsilon(\eta^\psi_m(\rho^\varepsilon, m^\varepsilon)u^\varepsilon_x)_x \\ &\qquad - \int \psi''\big(\tfrac{m^\varepsilon}{\rho^\varepsilon} + (\rho^\varepsilon)^\theta s\big)[1-s^2]^\lambda_+\big(\varepsilon|u^\varepsilon_x|^2 + \varepsilon\theta(\rho^\varepsilon)^\theta \rho^\varepsilon_x u^\varepsilon_x s\big)ds\end{aligned} \tag{5.11}$$

for any C^2 function $\psi : \mathbb{R} \to \mathbb{R}$.

When $\psi(s) \in \{\pm 1, \pm s, s^2\}$, $\psi''(s) \ge 0$, which implies

$$\eta^\psi(\rho^\varepsilon, m^\varepsilon)_t + q^\psi(\rho^\varepsilon, m^\varepsilon)_x \le \varepsilon(\eta^\psi_m(\rho^\varepsilon, m^\varepsilon)u^\varepsilon_x)_x, \tag{5.12}$$

where we have used the fact that $\int s[1-s^2]^\lambda_+\,ds = 0$.

Taking $\varepsilon \to 0$ in (5.12), we conclude (5.9).

Furthermore, combining Proposition 4.1 and the uniform estimates from Lemmas 3.3 and 3.4 with the div-curl lemma (cf. Murat [24] and Tartar [31]), we deduce that, for any C^2 compactly supported functions ϕ and ψ, the quadratic functions $\eta^\psi q^\phi - \eta^\phi q^\psi$ are weakly continuous with respect to the weakly convergent physical viscosity sequence $(\rho^\varepsilon, m^\varepsilon) \rightharpoonup (\rho, m)$:

$$\eta^\psi(\rho^\varepsilon, m^\varepsilon)q^\phi(\rho^\varepsilon, m^\varepsilon) - \eta^\phi(\rho^\varepsilon, m^\varepsilon)q^\psi(\rho^\varepsilon, m^\varepsilon) \rightharpoonup \overline{\eta^\psi q^\phi} - \overline{\eta^\phi q^\psi} \tag{5.13}$$

in the sense of distributions in $[0,\infty) \times \mathbb{R}$.

In terms of the Young measure, (5.13) yields the Tartar-Murat commutator relation:

$$\overline{\eta_\psi q_\phi - \eta_\phi q_\psi} = \overline{\eta_\psi q_\phi} - \overline{\eta_\phi q_\psi}. \tag{5.14}$$

Thus, we have

$$\begin{aligned}
&\int \psi(s_1)\overline{\chi(s_1)}\,ds_1 \int \phi(s_2)\overline{(\theta s_2+(1-\theta)u)\chi(s_2)}\,ds_2\\
&\quad - \int \psi(s_2)\overline{\chi(s_2)}\,ds_2 \int \phi(s_1)\overline{(\theta s_1+(1-\theta)u)\chi(s_1)}\,ds_1\\
&\quad = \int \psi(s_1)\phi(s_2)\overline{\chi(s_1)(\theta s_2+(1-\theta)u)\chi(s_2)}\,ds_1\,ds_2\\
&\quad\quad - \int \psi(s_1)\phi(s_2)\overline{\chi(s_1)(\theta s_1+(1-\theta)u)\chi(s_1)\chi(s_2)}\,ds_1\,ds_2,
\end{aligned}$$

which holds for arbitrary functions ψ and ϕ. This yields

$$\begin{aligned}
\overline{\chi(s_1)}\;\overline{(\theta s_2+(1-\theta)u)\chi(s_2)} - \overline{\chi(s_2)}\;\overline{(\theta s_1+(1-\theta)u)\chi(s_1)} ={}&\\
\theta(s_2-s_1)\overline{\chi(s_1)\chi(s_2)},&
\end{aligned}$$

which implies (5.10). $\square$

6 Reduction of the Measure-Valued Solutions for $\gamma \in (3,\infty)$

In this section, we prove that the measure-valued solution $\nu = \nu_{t,x}$ is a Dirac mass in the phase coordinates (ρ, m) for a.e. $(t,x) \in \mathbb{R}^2_+$.

LEMMA 6.1 *Let $\gamma > 3$. Then*

$$\overline{\chi(s)} \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+; L^p(\mathbb{R})) \quad \textit{for } 1 \le p < \frac{\gamma-1}{\gamma-3}.$$

This can be seen by the following direct calculation: For any $K \Subset \mathbb{R}$ and $T \in (0,\infty)$,

$$\begin{aligned}
&\int_{[0,T]\times K} \|\overline{\chi(s)}\|_{L^p}\,dx\,dt\\
&\quad\le \int_{[0,T]\times K}\int_{\mathbb{H}} \left(\int [\rho^{2\theta}-(u-s)^2]_+^{p\lambda}\,ds\right)^{1/p} d\nu_{t,x}\,dx\,dt\\
&\quad= \int_{[0,T]\times K}\int_{\mathbb{H}} \rho^{\frac{\theta}{p}(2\lambda p+1)}\left(\int_{-1}^{1}(1-\tau^2)^{p\lambda}d\tau\right)^{1/p} d\nu_{t,x}\,dx\,dt\\
&\quad\le C\int_{[0,T]\times K}\int_{\mathbb{H}} \max\{1,\rho\}d\nu_{t,x}\,dx\,dt < \infty,
\end{aligned}$$

if $\frac{\theta}{p}(2\lambda p+1) > 0$ and $p\lambda > -1$, which hold if $1 \le p < \frac{\gamma-1}{\gamma-3}$.

Let A be the open set defined as

$$A := \bigcup\{(u-\rho^\theta, u+\rho^\theta) : (\rho,u) \in \operatorname{supp}\nu\},$$

and let J be any connected component of A.

PROPOSITION 6.2 *When $\gamma > 3$, J is bounded.*

PROOF: Note that

$$\operatorname{supp}\chi(s) = \{(\rho,u) : u - \rho^\theta \le s \le u + \rho^\theta\}.$$

By definition of J, $\overline{\chi(s)} > 0$ for a.e. $s \in J$.

From (5.10), we obtain that, if $\overline{\chi(s_1)}\,\overline{\chi(s_2)} \neq 0$, then

$$\frac{1-\theta}{\theta}\frac{1}{s_2-s_1}\left(\frac{\overline{u\chi(s_2)}}{\overline{\chi(s_2)}} - \frac{\overline{u\chi(s_1)}}{\overline{\chi(s_1)}}\right) = \frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}\,\overline{\chi(s_2)}} - 1. \tag{6.1}$$

Taking the limits $s_1, s_2 \to s$ in (6.1) (cf. [22, p. 426]), we conclude that

$$\frac{1-\theta}{\theta}\frac{\partial}{\partial s}\left(\frac{\overline{u\chi(s)}}{\overline{\chi(s)}}\right) = \frac{\overline{\chi^2(s)}}{\left(\overline{\chi(s)}\right)^2} - 1 \ge 0. \tag{6.2}$$

This implies that the function

$$\frac{1-\theta}{\theta}\frac{\overline{u\chi(s)}}{\overline{\chi(s)}} \quad \text{is nondecreasing on } J. \tag{6.3}$$

Consequently, from (6.1), we obtain

$$\frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} \ge \overline{\chi(s_2)} \quad \text{a.e. } s_1, s_2 \in J,\quad s_1 < s_2. \tag{6.4}$$

On the contrary, suppose now that J is unbounded from below, that is, $\inf\{s : s \in J\} = -\infty$.

We fix $M_0 > 0$ such that $M_0 + 1 \in J$ and restrict $s_2 \in (M_0, M_0+1)$. We will take $s_1 \le -2|M_0|$. For such s_1,

$$|M_0 - s_1| > \frac{|s_1|}{2}. \tag{6.5}$$

If $(\rho,u) \in \operatorname{supp}\chi(s_2) \cap \operatorname{supp}\chi(s_1)$, then, by the above assumptions on s_1 and M_0, we have

$$\rho^\theta - u + s_2 = \rho^\theta - u + s_1 + (s_2 - s_1) \ge s_2 - s_1 \ge M_0 - s_1 > \frac{|s_1|}{2}.$$

Since $\gamma > 3$, i.e., $\lambda < 0$, it follows that

$$\begin{aligned}\int \chi(s_1)\chi(s_2)\,d\nu &= \int \chi(s_1)[\rho^\theta - u + s_2]_+^\lambda[\rho^\theta + u - s_2]_+^\lambda\,d\nu \\ &\le 2^{-\lambda}|s_1|^\lambda \int_{\operatorname{supp}\chi(s_2)} \chi(s_1)[\rho^\theta + u - s_2]_+^\lambda\,d\nu.\end{aligned} \tag{6.6}$$

We integrate (6.6) in s_2 over the interval $(M_0,\ M_0+1)$ to obtain

$$\begin{aligned}\int_{M_0}^{M_0+1}&\int \chi(s_1)\chi(s_2)d\nu\, ds_2\\ &\le 2^{-\lambda}|s_1|^{\lambda}\int_{M_0}^{M_0+1}\int_{\operatorname{supp}\chi(s_2)} \chi(s_1)[\rho^\theta+u-s_2]_+^\lambda\, d\nu\, ds_2\\ &= 2^{-\lambda}|s_1|^{\lambda}\int \chi(s_1)\biggl(\int_{(M_0,M_0+1)\cap(u-\rho^\theta,u+\rho^\theta)} [\rho^\theta+u-s_2]_+^\lambda\, ds_2\biggr)d\nu.\end{aligned}\tag{6.7}$$

We now consider the integral in the parentheses in (6.7). When $\rho^\theta+u\ge M_0+2$, then $\rho^\theta+u-s_2\ge M_0+2-(M_0+1)=1$ and

$$\int_{(M_0,M_0+1)\cap(u-\rho^\theta,u+\rho^\theta)} [\rho^\theta+u-s_2]_+^\lambda\, ds_2\le 1,$$

since $\lambda<0$.

When $\rho^\theta+u<M_0+2$, then

$$\begin{aligned}\int_{(M_0,M_0+1)\cap(u-\rho^\theta,u+\rho^\theta)} [\rho^\theta+u-s_2]_+^\lambda\, ds_2 &\le \int_{M_0}^{M_0+1}[\rho^\theta+u-s_2]_+^\lambda\, ds_2\\ &\le \frac{1}{1+\lambda}[\rho^\theta+u-M_0]_+^{1+\lambda}\\ &\le \frac{1}{1+\lambda}2^{1+\lambda},\end{aligned}$$

since $1+\lambda>0$.

Combining the two observations above into (6.7), we find that there exists $C=C(\lambda)>0$ such that

$$\int_{M_0}^{M_0+1}\int \chi(s_1)\chi(s_2)d\nu\, ds_2\le C(\lambda)|s_1|^\lambda\overline{\chi(s_1)}.\tag{6.8}$$

Combining this with (6.4), we obtain

$$C(\lambda)|s_1|^\lambda\ge\int_{M_0}^{M_0+1}\overline{\chi(s_2)}ds_2\equiv C(M_0,\lambda)>0.$$

Since $\lambda<0$ and $|s_1|$ can be chosen arbitrarily large, we arrive at a contradiction.

The case when J is unbounded from above can be treated similarly. □

With this proposition, a simple argument (cf. [22, lemma 6]) implies that ν is reduced to a Dirac mass on the set $\{\rho>0\}$ or is supported completely in the vacuum $V=\{\rho=0\}$ for the case $\gamma>3$. This can be seen as follows: Let $J=(s_-,s_+)$ be the open connected component. Then the values (ρ,u) such that $\chi(s)>0$ in an interval $(s_+-\varepsilon,s_+)$ satisfy

$$u+\rho^\theta\ge s_+-\varepsilon.$$

Since $s_- \le u - \rho^\theta$ for these (ρ, u) values, we have

$$\lim_{s\to s_+} \frac{\overline{u\chi(s)}}{\overline{\chi(s)}} \ge \min\{u : (\rho,u) \in \operatorname{supp}\nu, u+\rho^\theta = s_+\} \ge \frac{s_+ + s_-}{2}. \tag{6.9}$$

Similarly, we have

$$\lim_{s\to s_-} \frac{\overline{u\chi(s)}}{\overline{\chi(s)}} \le \frac{s_+ + s_-}{2}. \tag{6.10}$$

Combining (6.9)–(6.10) with (6.3), we conclude that $\overline{u\chi(s)}/\overline{\chi(s)}$ is constant, which implies from (6.2) that

$$\overline{\chi(s)^2} = (\overline{\chi(s)})^2.$$

Since $\nu_{t,x}$ is a probability measure,

$$\langle \nu_{t,x}, (\chi(s) - \langle \nu_{t,x}, \chi(s)\rangle)^2\rangle = 0 \quad \text{for any } s \in \mathbb{R},$$

which yields

$$\operatorname{supp}\nu_{t,x} \subset \{\chi(s) = \langle \nu_{t,x}, \chi(s)\rangle\} \quad \text{for any } s \in \mathbb{R}.$$

This leads to the conclusion. That is, in the phase coordinates (ρ, m), $m = \rho u$,

$$\nu_{t,x} = \delta_{(\rho(t,x), m(t,x))}$$

for some $(\rho(t,x), m(t,x))$.

When $\gamma = 3$, then $\theta = 1$ and the commutator relation (5.10) reads

$$\overline{\chi(s_1)\chi(s_2)} = \overline{\chi(s_1)}\,\overline{\chi(s_2)},$$

which implies $\overline{\chi(s)^2} = (\overline{\chi(s)})^2$ by taking $s_1 = s_2 = s$. This again implies that $\nu_{t,x} = \delta_{(\rho(t,x), m(t,x))}$ for some $(\rho(t,x), m(t,x))$.

PROPOSITION 6.3 *When $\gamma \ge 3$, the measure-valued solution $\nu_{t,x}$ is a Dirac mass in the phase coordinates (ρ, m),*

$$\nu_{t,x} = \delta_{(\rho(t,x), m(t,x))}.$$

7 Reduction of the Measure-Valued Solutions for $\gamma \in (1,3)$

In this section, we directly prove that the measure-valued solution $\nu = \nu_{t,x}$ is a Dirac mass in the phase coordinates (ρ, m).

LEMMA 7.1 *When $\gamma \in (1,3)$, $\overline{\chi(s)}$ and $\overline{\chi(s_1)\chi(s_2)}$ are continuous and weakly differentiable functions of their arguments and*

$$\frac{\partial}{\partial s}\overline{\chi(s)} = \overline{\chi'(s)} \in L^1_{\text{loc}}(\mathbb{R}^2_+; L^1(\mathbb{R})),$$
$$\frac{\partial}{\partial s_1}\overline{\chi(s_1)\chi(s_2)} = \overline{\chi'(s_1)\chi(s_2)} \in L^1_{\text{loc}}(\mathbb{R}^2_+; L^1(\mathbb{R}^2)).$$

This can be seen as follows: We compute

$$\chi'(s) = 2\lambda(u-s)[\rho^{2\theta} - (u-s)^2]_+^{\lambda-1}$$

and

$$\begin{aligned}\int |\overline{\chi'(s)}|ds &\le 2\lambda \int \left(\int_{u-\rho^\theta}^{u} (u-s)[\rho^{2\theta} - (u-s)^2]_+^{\lambda-1}\, ds \right) d\nu_{t,x} \\ &\quad + 2\lambda \int \left(\int_{u}^{u+\rho^\theta} (s-u)[\rho^{2\theta} - (u-s)^2]_+^{\lambda-1}\, ds \right) d\nu_{t,x} \\ &\le C(\lambda) \int \rho^{2\theta\lambda}\, d\nu_{t,x} \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+),\end{aligned}$$

from the fact that $0 < 2\theta\lambda \le \gamma + 1$ and Proposition 5.1(i). The function $\overline{\chi(s_1)\chi(s_2)}$ is treated similarly.

Let A be the open set defined as

$$A := \bigcup \{(u-\rho^\theta, \rho^\theta + u) : (\rho, u) \in \operatorname{supp} \nu\},$$

and let J be any connected component of A.

PROPOSITION 7.2 *When $\gamma \in (1,3)$, J is bounded.*

PROOF: We divide the proof into three steps.

Step 1. On the contrary, suppose as before that J is unbounded from below and let $M_0 = \sup\{s : s \in J\} \in (-\infty, \infty]$.

Let $s_1, s_2, s_3 \in (-\infty, M_0)$ with $s_1 < s_2 < s_3$. From equation (5.10), it can be derived that

$$(7.1)\quad (s_2 - s_1)\frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} + (s_3 - s_2)\frac{\overline{\chi(s_3)\chi(s_2)}}{\overline{\chi(s_3)}} = (s_3 - s_1)\overline{\chi(s_2)}\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\,\overline{\chi(s_3)}}.$$

Differentiating this equation in s_2 and dividing by $s_3 - s_1$, we obtain

$$(7.2)\quad \begin{aligned}&\frac{s_2 - s_1}{s_3 - s_1}\frac{\overline{\chi(s_1)\chi'(s_2)}}{\overline{\chi(s_1)}} + \frac{s_3 - s_2}{s_3 - s_1}\frac{\overline{\chi(s_3)\chi'(s_2)}}{\overline{\chi(s_3)}} \\ &\quad + \frac{1}{s_3 - s_1}\frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} - \frac{1}{s_3 - s_1}\frac{\overline{\chi(s_3)\chi(s_2)}}{\overline{\chi(s_3)}} = \overline{\chi'(s_2)}\,\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\,\overline{\chi(s_3)}}.\end{aligned}$$

Our strategy is to take $s_1 \to -\infty$ and show that the left-hand side of (7.2) has a smaller order than the right-hand side, which leads to a contradiction.

Step 2. We start with a claim.

Claim. $\overline{\chi(s)} \to 0$ as $s \to -\infty$ and $s \to M_0$.

If $M_0 < \infty$, then the result follows by the definition of J and the fact that $\overline{\chi(s)}$ is continuous (which follows from Lemma 7.1).

We now show that $\overline{\chi(s)} \to 0$ as $|s| \to \infty$ for $M_0 = \infty$. Using Lemma 3.3 and Young's inequality, we have

$$\begin{aligned}
\overline{\chi(s)} &= \int_{\mathbb{H}} [\rho^{2\theta} - (u-s)^2]_+^{\lambda} \, d\nu \le \int_{\mathbb{H} \cap \operatorname{supp} \chi(s)} \rho^{2\theta\lambda} \, d\nu \\
&\le \varepsilon^{2\lambda} + \int_{\mathbb{H} \cap \{\rho^\theta \ge \varepsilon\} \cap \operatorname{supp} \chi(s)} \big(C(\delta) + \delta \rho^{\gamma+1}\big) d\nu \\
&\le \varepsilon^{2\lambda} + \delta C + C(\delta) \nu(\{\rho^\theta \ge R\} \cup \{\rho^\theta \ge \varepsilon, |u| \ge R\}),
\end{aligned}$$

where ε and δ are positive constants (to be taken small) and $C(\delta)$ is some constant depending on the negative powers of δ and $R := \frac{|s|}{4}$. Then, by Chebyshev's inequality and Proposition 5.1(i), we conclude that

$$\begin{aligned}
\nu(\{\rho^\theta \ge R\}) &\le \frac{\int \rho^{\gamma+1} d\nu}{R^{\frac{\gamma+1}{\theta}}} \le \frac{M}{R^{\frac{\gamma+1}{\theta}}}, \\
\nu(\{\rho^\theta \ge \varepsilon, |u| \ge R\}) &\le \frac{\int \rho |u|^3 d\nu}{\varepsilon^{1/\theta} R^3} \le \frac{N}{\varepsilon^{1/\theta} R^3},
\end{aligned}$$

where M and N are the constants depending only on (t, x). Thus, choosing first δ small, then ε small, and finally R (i.e., $|s|$) large, we can make $\overline{\chi(s)}$ as small as we want.

Step 3. Now we prove Proposition 7.2. Since $\overline{\chi(s)} \ge 0$ is not identically zero and

$$\overline{\chi(s)} \to 0 \quad \text{as } s \to \inf J, \ \sup J,$$

there exists s_2 such that

$$\overline{\chi'(s_2)} > 0, \quad \overline{\chi(s_2)} > 0. \tag{7.3}$$

Moreover, following the same argument for (6.4) from (5.10), we still have

$$\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\ \overline{\chi(s_3)}} \ge 1 \quad \text{for any } s_1, s_3 \in J. \tag{7.4}$$

Let $s_3 > s_2$ be the points such that $\overline{\chi(s_3)} > 0$ and let $s_1 \to -\infty$. Then, from (7.1), we conclude

$$\frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} = \frac{s_3 - s_1}{s_2 - s_1} \overline{\chi(s_2)} \frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\ \overline{\chi(s_3)}} + o(1) \quad \text{as } s_1 \to -\infty. \tag{7.5}$$

From (7.2), by throwing away the negative terms, we obtain

$$\overline{\chi'(s_2)}\,\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\,\overline{\chi(s_3)}} \le \frac{s_2-s_1}{s_3-s_1}\,\frac{\overline{\chi(s_1)[\chi'(s_2)]_+}}{\overline{\chi(s_1)}} + \frac{1}{s_3-s_1}\,\frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} + o(1), \tag{7.6}$$

where $[w]_+$ stands for the nonnegative part of w. For $(\rho,u)\in \operatorname{supp}[\chi'(s)]_+$, consider

$$\begin{aligned}
[\chi'(s)]_+ &= 2\lambda[\rho^\theta - s + u]_+^{\lambda-1}[\rho^\theta + s - u]_+^{\lambda-1}[u-s]_+\\
&= 2\lambda[\rho^\theta - s + u]_+^{\lambda}[\rho^\theta + s - u]_+^{\lambda}\frac{1}{[\rho^\theta + s - u]_+}\,\frac{[u-s]_+}{[u-s+\rho^\theta]_+}\\
&\le 2\lambda[\rho^\theta - s + u]_+^{\lambda}[\rho^\theta + s - u]_+^{\lambda}\frac{1}{[\rho^\theta + s - u]_+}.
\end{aligned}$$

Note that, if $(\rho,u)\in \operatorname{supp}\chi(s_1)$, then $\rho^\theta \ge u - s_1$. If, in addition, $(\rho,u)\in \operatorname{supp}\chi(s)$ with $s>s_1$, then

$$\rho^\theta + s - u \ge s - s_1.$$

Thus, we have

$$[\chi'(s)]_+ \le \frac{2\lambda}{s-s_1}[\rho^\theta - s + u]_+^{\lambda}[\rho^\theta + s - u]_+^{\lambda} = \frac{2\lambda}{s-s_1}\chi(s), \tag{7.7}$$

when $(\rho,u)\in \operatorname{supp}\chi(s_1)\cap\operatorname{supp}\chi(s)$ for $s_1<s$. Setting $s=s_2$ and using (7.7) in (7.6), we obtain

$$\overline{\chi'(s_2)}\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\,\overline{\chi(s_3)}} \le \frac{2\lambda+1}{s_3-s_1}\,\frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} + o(1). \tag{7.8}$$

From this, recalling (7.5), we obtain

$$\left(\overline{\chi'(s_2)} - \frac{2\lambda+1}{s_2-s_1}\overline{\chi(s_2)}\right)\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\,\overline{\chi(s_3)}} \le o(1). \tag{7.9}$$

Because of (7.3) and (7.4), the last inequality is a contradiction when $s_1\to-\infty$. This completes the proof. $\square$

Then, by the well-known result (see [4, 10, 12, 21]), the measure-valued solution ν reduces to a Dirac mass in the phase coordinates (ρ,m).

PROPOSITION 7.3 *When $\gamma\in(1,3)$, the measure-valued solution $\nu_{t,x}$ is a Dirac mass in the phase coordinates (ρ,m)*:

$$\nu_{t,x} = \delta_{(\rho(t,x),m(t,x))}.$$

Remark 7.4. The above proof provides another way to establish the reduction of measure-valued solutions, which simplifies the proof by LeFloch and Westdickenberg [20] for the case $1<\gamma\le\frac{5}{3}$.

8 Vanishing Viscosity Limit of the Navier-Stokes Equations to the Euler Equations with Finite-Energy Initial Data

Consider the Cauchy problem (1.1)–(1.2) for the Navier-Stokes equations in $\mathbb{R}^2_+ := \mathbb{R} \times [0, \infty)$. Hoff's theorem in [15] (also see Kanel′ [18] for the case of the same end-states) indicates that, when the initial functions $(\rho_0(x), u_0(x))$ are smooth with the lower bounded density $\rho_0(x) \geq c_0^\varepsilon > 0$ for $x \in \mathbb{R}$ and

$$\lim_{x \to \pm\infty} (\rho_0(x), u_0(x)) = (\rho^\pm, u^\pm),$$

then there exists a unique smooth solution $(\rho^\varepsilon(t,x), u^\varepsilon(t,x))$, globally in time, with $\rho^\varepsilon(t,x) \geq c_\varepsilon(t)$ for some $c_\varepsilon(t) > 0$ for $t \geq 0$ and

$$\lim_{x \to \pm\infty} (\rho^\varepsilon(t,x), u^\varepsilon(t,x)) = (\rho^\pm, u^\pm).$$

Combining the uniform estimates and Remark 3.5 in Section 3 and the compactness of weak entropy dissipation measures in H^{-1}_{loc} in Section 4 with the compensated compactness argument in Section 5 and the reduction of the measure-valued solution $\nu_{t,x}$ in Sections 6 and 7, we prove the main theorem of this paper.

THEOREM 8.1 *Let the initial functions $(\rho_0^\varepsilon, u_0^\varepsilon)$ be smooth and satisfy the following conditions: There exist $E_0, E_1, M_0 > 0$, independent of ε, and $c_0^\varepsilon > 0$ such that*

(i) $\rho_0^\varepsilon(x) \geq c_0^\varepsilon > 0$, $\int \rho_0^\varepsilon(x)|u_0^\varepsilon(x) - \overline{u}(x)|dx \leq M_0 < \infty$.

(ii) *The relative total mechanical energy with respect to the end-states $(\rho^\pm, u^\pm)$ at infinity through $(\overline{\rho}, \overline{u})$ is finite:*

$$\int \left(\frac{1}{2} \rho_0^\varepsilon(x)|u_0^\varepsilon(x) - \overline{u}(x)|^2 + e^*(\rho_0^\varepsilon(x), \overline{\rho}(x)) \right) dx \leq E_0 < \infty.$$

(iii) $\varepsilon^2 \int \frac{|\rho^\varepsilon_{0,x}(x)|^2}{\rho_0^\varepsilon(x)^3}\, dx \leq E_1 < \infty$.

(iv) $(\rho_0^\varepsilon(x), \rho_0^\varepsilon(x)u_0^\varepsilon(x)) \to (\rho_0(x), \rho_0(x)u_0(x))$ *in the sense of distributions as $\varepsilon \to 0$, with $\rho_0(x) \geq 0$ a.e.*

Here $(\overline{\rho}(x), \overline{u}(x))$ is a pair of smooth monotone functions satisfying $(\overline{\rho}(x), \overline{u}(x)) = (\rho^\pm, u^\pm)$ when $\pm x \geq L_0$ for some large $L_0 > 0$.

Let $(\rho^\varepsilon, m^\varepsilon)$, $m^\varepsilon = \rho^\varepsilon u^\varepsilon$, be the solution of the Cauchy problem (1.1)–(1.2) *for the Navier-Stokes equations with initial data $(\rho_0^\varepsilon(x), u_0^\varepsilon(x))$ for each fixed $\varepsilon > 0$.*

Then, when $\varepsilon \to 0$, there exists a subsequence of $(\rho^\varepsilon, m^\varepsilon)$ that converges almost everywhere to a relative finite-energy entropy solution (ρ, m) to the Cauchy problem (2.1) *and* (1.2) *with initial data $(\rho_0(x), \rho_0(x)u_0(x))$ for the isentropic Euler equations with $\gamma > 1$ in the sense of Definition* 2.2. *Moreover, there exists a bounded Radon measure $\mu(t,x,s)$ on $\mathbb{R}^2_+ \times \mathbb{R}$ such that*

$$\mu(U \times \mathbb{R}) \geq 0 \tag{8.1}$$

for any open set $U \subset \mathbb{R}^2_+$, *and the corresponding entropy kernel* $\chi(\rho, s-u)$ *defined by* (2.11) *satisfies*

$$\partial_t \chi(\rho, s-u) + \partial_x((\theta s + (1-\theta)u)\chi(\rho, s-u)) = \partial_s^2 \mu \tag{8.2}$$

in the sense of distributions on $\mathbb{R}^2_+ \times \mathbb{R}$.

Results (8.1)–(8.2) in Theorem 8.1 are direct corollaries of (5.11), the representation formulas (2.12)–(2.13) of the entropy-entropy flux pairs via the entropy kernel $\chi(\rho, s-u)$, the uniform estimates in Lemmas 3.1 and 3.2, and the strong convergence of $(\rho^\varepsilon, m^\varepsilon)$ from the first part of Theorem 8.1.

Acknowledgment. Gui-Qiang Chen's research was supported in part by the National Science Foundation under grants DMS-0935967, DMS-0807551, and DMS-0505473, the Natural Science Foundation of China under grant NSFC-10728101, and the Royal Society–Wolfson Research Merit Award (UK). This paper was written as part of the International Research Program on Nonlinear Partial Differential Equations at the Centre for Advanced Study at the Norwegian Academy of Science and Letters in Oslo during the academic year 2008–09.

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Received October 2009.