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ANALYSIS OF THE SPECTRAL VANISHING VISCOSITY METHOD FOR PERIODIC CONSERVATION LAWS*

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Abstract. The convergence of the spectral vanishing method for both the spectral and pseudospectral discretizations of the inviscid Burgers' equation is analyzed. It is proved that this kind of vanishing viscosity is responsible for a spectral decay of those Fourier coefficients located toward the end of the computed spectrum; consequently, the discretization error is shown to be spectrally small, independently of whether or not the underlying solution is smooth. This in turn implies that the numerical solution remains uniformly bounded and convergence follows by compensated compactness arguments.

Key words. Burgers' equation, entropy solution, spectral viscosity method, compensated compactness, convergence

AMS(MOS) subject classifications. 35L65, 65M10, 65M15

Introduction. In this paper, we extend the analysis of the spectral vanishing viscosity method for stabilizing spectral approximations of nonlinear conservation laws. Spectral vanishing viscosity was introduced in [3], where L^∞-bounded spectral-Galerkin approximations are shown to converge strongly in $L^2_{\text{loc}}(x, t)$ to the exact entropy solutions of such conservation laws.

The analysis is performed on the 2π-periodic inviscid Burgers' equation

$$\frac{\partial}{\partial t} u(x, t)+\frac{\partial}{\partial x}\left(\frac{u^2(x, t)}{2}\right)=0, \tag{1.1}$$

submitted to the additional entropy condition

$$\frac{\partial}{\partial t}\left(\frac{u^2(x, t)}{2}\right)+\frac{\partial}{\partial x}\left(\frac{u^3(x, t)}{3}\right) \leqq 0, \tag{1.2}$$

which singles out the unique "physically relevant" weak solution of (1.1). Both the spectral-Galerkin and pseudospectral-collocation methods for (1.1), (1.2) are treated, and to this end we proceed as follows.

Denote by $S_N u(x, t)$ the spectral-Fourier projection of $u(x, t)$,

$$S_N u(x, t)=\sum_{|k| \leqq N} \hat{u}(k, t)\, e^{ikx}, \qquad \hat{u}(k, t)=\frac{1}{2\pi} \cdot \int_0^{2\pi} u(x, t)\, e^{-ikx}\, dx, \tag{1.3}$$

and let $I_N u(x, t)$ denote the pseudospectral-Fourier projection of $u(x, t)$, which interpolate $u(x, t)$ at the $2N+1$ equidistant collocation points $x_\nu = \nu h$, $h = 2\pi/(2N+1)$,

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$\nu=0,\cdots,2N$,

$$I_N u(x,t)=\sum_{|k|\leqq N}\tilde{u}(k,t)\,e^{ikx},\qquad \tilde{u}(k,t)=\frac{h}{2\pi}\cdot\sum_{\nu=0}^{2N}u(x_\nu,t)\,e^{-ikx_\nu}. \tag{1.4}$$

These two projection operators differ by aliasing error—that is, we have

$$I_N=S_N+A_N \tag{1.5}$$

where the aliasing projection A_N is given by [2]

$$A_N u(x,t)=\sum_{|k|\leqq N}\left[\sum_{j\neq 0}\hat{u}(k+j(2N+1),t)\right]e^{ikx}. \tag{1.6}$$

Throughout this paper, we use

$$P_N=S_N+a\cdot A_N \tag{1.7}$$

as a concise notation for the two kinds of Fourier projections: those having either $a=0$, corresponding to the spectral projection, or $a=1$, which corresponds to the pseudospectral interpolation.

We approximate the Fourier projection of the exact solution $P_N u(x,t)$, by an N-trigonometric polynomial $u_N(x,t)$,

$$u_N(x,t)=\sum_{|k|\leqq N}\hat{u}_k(t)\,e^{ikx}, \tag{1.8}$$

which is determined by the approximate evolution equation

$$\frac{\partial}{\partial t}u_N(x,t)+\frac{\partial}{\partial x}\left(P_N\frac{1}{2}u_N^2(x,t)\right)=\varepsilon\frac{\partial}{\partial x}\left(Q_N\frac{\partial}{\partial x}u_N(x,t)\right). \tag{1.9}$$

The expression on the right-hand side of (1.9) represents the spectral vanishing viscosity term. Here Q_N is the spectral viscosity operator defined as a convolution with a symmetric viscosity kernel $Q_N(x)$,

$$Q_N\frac{\partial}{\partial x}u_N(x,t)\equiv Q_N(x)*\frac{\partial}{\partial x}u_N(x,t),\qquad Q_N(x)=\sum_{|k|\leqq N}\hat{Q}(k)\,e^{ikx}. \tag{1.10}$$

In the spectral case where $a=0$, (1.9) amounts to

$$\frac{\partial}{\partial t}u_N(x,t)+\frac{\partial}{\partial x}\left[S_N\frac{1}{2}u_N^2(x,t)\right]=\varepsilon\frac{\partial}{\partial x}\left[Q_N(x)*\frac{\partial}{\partial x}u_N(x,t)\right],$$

consisting of a nonlinear system of ordinary differential equations for the Fourier coefficients $\hat{u}_k(t)$ that are coupled through the standard spectral convolution treatment of the nonlinear term. The interpretation of scheme (1.9) in the pseudospectral case, where $a=1$, leads us to

$$\frac{\partial}{\partial t}u_N(x_\nu,t)+\frac{\partial}{\partial x}\left[I_N\frac{1}{2}u_N^2(x,t)\right]_{|x=x_\nu}=\varepsilon\frac{\partial}{\partial x}\left[Q_N(x)*\frac{\partial}{\partial x}u_N(x,t)\right]_{|x=x_\nu},\qquad 0\leqq\nu\leqq 2N,$$

and consists in a complete statement of a standard collocation method with a pseudospectral treatment of the nonlinear term.

In both the spectral and pseudospectral cases, the spectral viscosity operator can be efficiently implemented in the Fourier rather than the physical space, i.e.,

$$\varepsilon\frac{\partial}{\partial x}\left(Q_N\frac{\partial}{\partial x}u_N(x,t)\right)\equiv\varepsilon\frac{\partial}{\partial x}\left[Q_N(x)*\frac{\partial}{\partial x}u_N(x,t)\right]=-\varepsilon\cdot\sum_{|k|\leqq N}k^2\hat{Q}(k)\hat{u}_k(t)\,e^{ikx}.$$

An essential feature of our spectral viscosity operator Q_N is that it should operate only on the high portion of the spectrum to retain the formal spectral accuracy of the method. Hence we make the following assumption.

ASSUMPTION I. There exists a constant $m \equiv m(N) < \frac{1}{4}N$, such that

$$\begin{aligned} &\hat{Q}(k) \equiv 0, && |k| < m, \\ 0 \leqq\ &\hat{Q}(k) \leqq 1, && m \leqq |k| \leqq 2m, \\ &\hat{Q}(k) \equiv 1, && 2m < |k| \leqq N. \end{aligned}$$

Then, with $Q_N = I - R_m$, we can rewrite (1.9) as

$$\frac{\partial}{\partial t} u_N(x, t) + \frac{\partial}{\partial x}\left(P_N \frac{1}{2} u_N^2(x, t)\right) = \varepsilon \frac{\partial}{\partial x}\left[(I - R_m)\frac{\partial}{\partial x} u_N(x, t)\right] \tag{1.11}$$

where the corresponding kernel $R_m(x)$,

$$R_m(x) = \sum_{|k| \leqq 2m} \hat{R}(k)\, e^{ikx}, \tag{1.12}$$

is a trigonometric polynomial of degree less than or equal to $2m$, with Fourier coefficients

$$\begin{aligned} &\hat{R}(k) \equiv 1, && |k| < m, \\ 0 \leqq\ &\hat{R}(k) \leqq 1, && m \leqq |k| \leqq 2m. \end{aligned} \tag{1.13}$$

To guarantee the uniform boundedness of our approximation $u_N(x, t)$, we need to control the size of this kernel; we therefore make the following assumption.

ASSUMPTION II. There exists a constant such that

$$\|R_m(\cdot)\|_{L^1(x)} \leqq \text{Const.}\log m. \tag{1.14}$$

We remark that the assumption of a logarithmic upper bound for the size of $R_m(x)$ is plausible, since typical applications involve $\hat{R}(k)$ that decrease *monotonically* to zero and (1.14) is automatically fulfilled in such cases (see Appendix A). To obtain, with the help of Assumption II, the promised uniform bound on $u_N(x, t)$, we need L^∞-bounded initial data, $u_N(x, 0)$. For technical reasons we shall need a slightly stronger assumption.

ASSUMPTION III. There exists a constant such that

$$\|u_N(x, t=0)\|_{L^\infty(x)} \leqq \sum_{|k| \leqq N} |\hat{u}_k(t=0)| \leqq \text{Const}_0.$$

The spectral viscosity term on the right of (1.11) depends on two free parameters: the viscosity amplitude $\varepsilon \equiv \varepsilon(N)$ and the effective size of the inviscid spectrum $m \equiv m(N)$. These two parameters should be chosen to ensure the convergence of the method. In [3] it is proved that in the absence of such a viscosity term $\varepsilon = 0$, strong as well as weak convergence to the exact entropy solution fails.

The main result of this paper asserts the following theorem.

THEOREM 1.1. *Consider the Fourier approximation* (1.11) *of either spectral or pseudospectral type. Let the spectral viscosity in* (1.12)–(1.14) *be parameterized with* (ε, m) *as follows*:

$$\varepsilon \equiv \varepsilon(m) \sim \alpha \frac{1}{m^2 \cdot \|R_m(\cdot)\|_{L^1(x)}}, \quad m \equiv m(N) \sim \text{Const. } N^\beta, \quad 0 < \beta < \frac{1}{4}. \tag{1.15}$$

Then $u_N(x, t)$ *converges boundedly almost everywhere to the unique entropy solution of the conservation law* (1.1).

Let us examine, for example, the viscosity operator $Q_N = I - S_m$. Here $R_m(x)$ coincides with Dirichlet kernel $D_m(x)$, where [5, Chap. II]

$$D_m(x) = \sum_{|k|\leqq m} e^{ikx} \equiv \frac{\sin(m+\frac{1}{2})x}{2\sin\frac{1}{2}x}, \qquad \|D_m(\cdot)\|_{L^1(x)} \sim \frac{4}{\pi}\log m,$$

so that Assumption II is fulfilled and Theorem 1.1 yields the following corollary.

COROLLARY 1.2. *Consider the Fourier approximation*

$$\frac{\partial}{\partial t} u_N(x,t) + \frac{\partial}{\partial x}\left(P_N \frac{1}{2} u_N^2(x,t)\right) = \varepsilon \frac{\partial}{\partial x}\left[(I - S_m)\frac{\partial}{\partial x} u_N(x,t)\right], \tag{1.16}$$

with

$$\varepsilon = \varepsilon(N) \sim \text{Const.}\frac{N^{-2\beta}}{\log N}, \quad m = m(N) \sim \text{Const.}\, N^{\beta}, \quad 0 < \beta < \frac{1}{4}. \tag{1.17}$$

Then $u_N(x,t)$ converges boundedly almost everywhere to the unique entropy solution of the conservation law (1.1).

The spectral portion of this result ($a = 0$) is derived in [3, Thm. 4.1] under the assumption that the numerical solution $u_N(x,t)$ remains uniformly bounded. The extension of Corollary 1.2 includes the pseudospectral approximation ($a = 1$), and in addition, because of the slightly more stringent parametrization than that of [3, Thm. 4.1], contains a proof of the previously assumed L^∞-bound.

In the last example the viscosity symbols $\hat{Q}(k)$ are discontinuous at $|k| = m$. It is suggested in [3] that the use of viscosity operators Q_N with smoothly varying symbols would be advantageous for the spectral viscosity method in (1.9). As our second and final example we consider the simplest viscosity operator of this type, namely

$$\begin{aligned} &\hat{Q}(k) \equiv 0, && |k| < m,\\ &\hat{Q}(k) = \frac{|k| - m}{m}, && m \leqq |k| \leqq 2m,\\ &\hat{Q}(k) \equiv 1, && 2m < |k| \leqq N. \end{aligned}$$

This kind of spectral viscosity is intimately related to the Fejér operator $F_m = (1/m)\cdot\sum_{k=0}^{m-1} S_k$: if we let $K_m(x)$ denote the corresponding Fejér kernel [3, Chap. III]

$$K_m(x) = \sum_{|k|\leqq m}\left(1 - \frac{|k|}{m}\right) e^{ikx} \equiv \frac{2}{m}\left(\frac{\sin\frac{1}{2}mx}{2\sin\frac{1}{2}x}\right)^2, \qquad \|K_m(\cdot)\|_{L^1(x)} = \pi,$$

then for $Q_N = I - R_m$ we have $R_m(x) = 2K_{2m}(x) - K_m(x)$. Hence the kernel associated with

$$R_m = 2F_{2m} - F_m \equiv \frac{1}{m}\cdot\sum_{k=m}^{2m-1} S_k$$

is L^1-uniformly bounded:

$$\|R_m(\cdot)\|_{L^1(x)} \leqq 2\|K_{2m}(\cdot)\|_{L^1(x)} + \|K_m(\cdot)\|_{L^1(x)} \leqq 3\pi,$$

so that Assumption II is fulfilled and Theorem 1.1 yields the following corollary.

COROLLARY 1.3. *Consider the Fourier approximation*

$$\frac{\partial}{\partial t} u_N(x,t) + \frac{\partial}{\partial x}\left(P_N \frac{1}{2} u_N^2(x,t)\right) = \varepsilon \frac{\partial}{\partial x}\left[\left(I - \frac{1}{m}\cdot\sum_{k=m}^{2m-1} S_k\right)\frac{\partial}{\partial x} u_n(x,t)\right], \tag{1.18}$$

with

$$\varepsilon = \varepsilon(N) \sim \text{Const. } N^{-2\beta}, \quad m = m(N) \sim \text{Const. } N^{\beta}, \quad 0 < \beta < \tfrac{1}{4}. \tag{1.19}$$

Then $u_N(x, t)$ *converges boundedly almost everywhere to the unique entropy solution of the conservation law* (1.1).

The paper is organized as follows. In § 2 we derive some basic L^2-type a priori energy estimates. In § 3 these estimates are used to study the spectral decay rate of the Fourier coefficients. This enables us to obtain an L^∞ a priori estimate on the numerical solution in § 4. Finally, on the basis of the a priori estimates prepared in §§ 2-4, Theorem 1.1 is proved in § 5 along the lines of [3], using compensated compactness arguments.

2. L^2-type a priori estimates. We consider the approximate Fourier method (1.9), which we rewrite as

$$\frac{\partial}{\partial t} u_N + \frac{\partial}{\partial x}\left(\frac{1}{2} u_N^2\right) = \frac{\partial}{\partial x}\left[(I - P_N)\frac{1}{2} u_N^2\right] + \varepsilon \frac{\partial}{\partial x}\left[Q_N \frac{\partial}{\partial x} u_N\right] \equiv \mathrm{I} + \mathrm{II}. \tag{2.1}$$

To prove the convergence of this method we need some a priori estimates on its solution. To this end, we multiply (2.1) by u_N:

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{1}{2} u_N^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{3} u_N^3\right) &= u_N \frac{\partial}{\partial x}\left[(I - P_N)\frac{1}{2} u_N^2\right] + \varepsilon u_N \frac{\partial}{\partial x}\left[Q_N \frac{\partial}{\partial x} u_N\right] \\ &\equiv \mathrm{III} + \mathrm{IV}, \end{aligned} \tag{2.2}$$

and integrate over a 2π-period: the integral of the second term on the left vanishes by periodicity, and after integration by parts for the second term on the right we are left with

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_N(\cdot, t)\|^2_{L^2(x)} &+ \varepsilon \int_0^{2\pi} \frac{\partial}{\partial x} u_N(x, t) Q_N \frac{\partial}{\partial x} u_N(x, t)\, dx \\ &= \int_0^{2\pi} u_N \frac{\partial}{\partial x}\left[(I - P_N)\frac{1}{2} u_N^2\right] dx. \end{aligned} \tag{2.3}$$

Using (1.7) and the fact that $I - S_N$ is orthogonal to our N-space, we find that the right-hand side of (2.3) equals

$$\begin{aligned} \int_0^{2\pi} u_N \frac{\partial}{\partial x}\left[(I - P_N)\frac{1}{2} u_N^2\right] dx &= -a \cdot \int_0^{2\pi} u_N \frac{\partial}{\partial x}\left[A_N \frac{1}{2} u_N^2\right] dx \\ &= -a \cdot \sum_{|p| \leqq N} \hat{u}_{-p}\, ip \left(\widehat{A_N \frac{1}{2} u_N^2}\right)_p, \end{aligned}$$

and by the aliasing relation (1.6), this does not exceed

$$\int_0^{2\pi} u_N \frac{\partial}{\partial x}\left[(I - P_N)\frac{1}{2} u_N^2\right] dx = \frac{a}{2} \sum_{|p+q+r| = 2N+1} ip \hat{u}_p \hat{u}_q \hat{u}_r \leqq \frac{|a| N}{2} \sum_{|p+q+r| = 2N+1} |\hat{u}_p||\hat{u}_q||\hat{u}_r|.$$

In view of $|p + q + r| = 2N + 1$, at least two of the three indices $|p| \leqq N$, $|q| \leqq N$, and $|r| \leqq N$ are greater in absolute value than $N/2$, and hence

$$\int_0^{2\pi} u_N \frac{\partial}{\partial x}\left[(I-P_N)\frac{1}{2}u_N^2\right] dx \leqq \frac{4|a|}{N} \sum_{N/2\leqq|p|\leqq N} \sum_{N/2\leqq|q|\leqq N} |p||\hat{u}_p||q||\hat{u}_q||\hat{u}_{\pm(2N+1)-(p+q)}|$$

$$\leqq \frac{4|a|}{N}\left[\sum_{N/2\leqq|p|\leqq N} p^2|\hat{u}_p|^2 \sum_{N/2\leqq|q|\leqq N} q^2|\hat{u}_q|^2\right]^{1/2}\left[\sum_{N/2\leqq|p|\leqq N}\sum_{N/2\leqq|q|\leqq N}|\hat{u}_{\pm(2N+1)-(p+q)}|^2\right]^{1/2}$$

Consequently, since $\hat{Q}(p)=\hat{Q}(q)\equiv 1$ for $|p|, |q|\geqq N/2$, the expression on the right of (2.3) can be upper-bounded by

$$\text{(2.4a)} \qquad \int_0^{2\pi} u_N \frac{\partial}{\partial x}\left[(I-P_N)\frac{1}{2}u_N^2\right] dx \leqq \frac{4|a|}{N^{1/2}} \cdot \left\|Q_N \frac{\partial}{\partial x} u_N(\cdot, t)\right\|^2_{L^2(x)} \cdot \|u_N(\cdot, t)\|_{L^2(x)}.$$

Moreover, since $0\leqq \hat{Q}(k)\leqq 1$, for the second term on the left of (2.3) we have

$$\text{(2.4b)} \qquad \begin{aligned} \varepsilon \int_0^{2\pi} \frac{\partial}{\partial x} u_N(x,t) Q_N \frac{\partial}{\partial x} u_N(x,t)\, dx &= \varepsilon \sum_{|k|\leqq N} k^2 \hat{Q}(k)|\hat{u}_k(t)|^2 \\ &\geqq \varepsilon \left\|Q_N \frac{\partial}{\partial x} u_N(\cdot, t)\right\|^2_{L^2(x)}. \end{aligned}$$

Inserting this together with (2.4a) into (2.3), we end up with

$$\text{(2.5)} \qquad \frac{1}{2}\frac{d}{dt}\|u_N(\cdot,t)\|^2_{L^2(x)} + \left[\varepsilon - \frac{4|a|}{N^{1/2}}\|u_N(\cdot,t)\|_{L^2(x)}\right]\cdot \left\|Q_N\frac{\partial}{\partial x}u_N(\cdot,t)\right\|^2_{L^2(x)} \leqq 0.$$

Thus, as long as

$$\text{(2.6)} \qquad \varepsilon - \frac{4|a|}{N^{1/2}}\|u_N(\cdot,t)\|_{L^2(x)} > \frac{\varepsilon}{2},$$

we obtain

$$\text{(2.7)} \qquad \frac{d}{dt}\|u_N(\cdot,t)\|^2_{L^2(x)} + \varepsilon\left\|Q_N \frac{\partial}{\partial x}u_N(\cdot,t)\right\|^2_{L^2(x)} \leqq 0.$$

In particular, (2.7) implies that for $u_N(x,t)=\sum_{|k|\leqq N}\hat{u}_k(t)\, e^{ikx}$ we have our first L^2-type a priori estimate

$$\text{(2.8)} \qquad \|u_N(\cdot,t)\|^2_{L^2(x)} = \sum_{|k|\leqq N}|\hat{u}_k(t)|^2 \leqq E_0^2, \qquad E_0=\|u_N(\cdot, t=0)\|_{L^2(x)}\leqq \text{Const}_0.$$

Hence (2.6), (2.7), and consequently (2.8) prevail for all time provided (2.6) is valid at $t=0$, i.e., we require that in the pseudospectral case, where $a=1$, we have

$$\text{(2.9)} \qquad \varepsilon(N) > 8E_0 \cdot N^{-1/2}.$$

Indeed, Assumption II tells us that this requirement is fulfilled, at least for sufficiently large N, for

$$\text{(2.10)} \qquad \varepsilon > \text{Const.}\frac{N^{-2\beta}}{\log N} > 8\cdot E_0 \cdot N^{-1/2}, \qquad 2\beta < \frac{1}{2}.$$

Furthermore, temporal integration of (2.7) then gives us the second a priori estimate

$$\text{(2.11)} \qquad \varepsilon\left\|Q_N\frac{\partial}{\partial x}u_N\right\|^2_{L^2_{\text{loc}}(x,t)} = \varepsilon\int_t \sum_{m<|k|\leqq N} k^2|\hat{Q}(k)\hat{u}_k(t)|^2\, dt \leqq J_0^2, \qquad J_0\leqq \text{Const}_0.$$

3. The decay rate of the Fourier coefficients. Our Fourier approximation (2.1)

$$\frac{\partial}{\partial t}u_N+\frac{\partial}{\partial x}\left(\frac{1}{2}u_N^2\right)=\frac{\partial}{\partial x}\left[(I-P_N)\frac{1}{2}u_N^2\right]+\varepsilon\frac{\partial}{\partial x}\left[Q_N\frac{\partial}{\partial x}u_N\right], \tag{3.1}$$

consists of two kinds of errors. The first term, $\mathrm{I}\equiv(\partial/\partial x)[(I-P_N)\frac{1}{2}u_N^2]$, represents the discretization error, which includes spectral truncation errors $(\partial/\partial x)[(I-S_N)\frac{1}{2}u_N^2]$ as well as additional aliasing errors $-a\cdot(\partial/\partial x)[A_N\frac{1}{2}u_N^2]$ in the pseudospectral case. In this section, we borrow from Kreiss [1], to show that, due to the second error term of spectral vanishing viscosity, $\mathrm{II}\equiv\varepsilon(\partial/\partial x)[Q_N(\partial/\partial x)u_N]$, there is spectral decay of the Fourier coefficients $|\hat{u}_k(t)|$, $|k|>\frac{1}{2}N$, and therefore, the discretization error is spectrally small.

We begin by taking the $I-S_{2k}$ projection of (1.9). For $k>m$ we have, by Assumption I, $(I-S_{2k})Q_N=I-S_{2k}$, and hence

$$\begin{aligned}&\frac{\partial}{\partial t}[(I-S_{2k})u_N]+(I-S_{2k})\frac{\partial}{\partial x}\left[P_N\frac{1}{2}u_N^2\right]\\&\qquad=\varepsilon\frac{\partial}{\partial x}\left[(I-S_{2k})\frac{\partial}{\partial x}u_N\right],\qquad m<k\leqq N.\end{aligned} \tag{3.2}$$

Multiplying by $(I-S_{2k})u_N$ and integrating by parts over a 2π-period, we find that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(I-S_{2k})u_N(\cdot,t)\|_{L^2(x)}^2&=\frac{1}{2}\int_0^{2\pi}(I-S_{2k})\frac{\partial}{\partial x}u_N\cdot(I-S_{2k})P_Nu_N^2\,dx\\&\quad-\varepsilon\left\|(I-S_{2k})\frac{\partial}{\partial x}u_N(\cdot,t)\right\|_{L^2(x)}^2.\end{aligned} \tag{3.3}$$

The first integral on the right does not exceed

$$\begin{aligned}&\frac{1}{2}\int_0^{2\pi}(I-S_{2k})\frac{\partial}{\partial x}u_N\cdot(I-S_{2k})P_Nu_N^2\,dx\\&\quad\leqq\frac{1}{2}\left\|(I-S_{2k})\frac{\partial}{\partial x}u_N(\cdot,t)\right\|_{L^2(x)}\cdot\|(I-S_{2k})P_Nu_N^2(\cdot,t)\|_{L^2(x)}.\end{aligned} \tag{3.4}$$

To estimate the second term of the last product, we use the following lemma, whose proof is postponed to the end of this section.

LEMMA 3.1. *Let $f_N\equiv f_N(x)$ and $g_N\equiv g_N(x)$ be two N-trigonometric polynomials. Then for any $0<2k<N$ we have*

$$\begin{aligned}&\|(I-S_{2k})P_N(f_Ng_N)\|_{L^2(x)}\\&\quad\leqq\frac{2}{\sqrt{k}}\left[\|f_N\|_{L^2(x)}\cdot\left\|(I-S_k)\frac{\partial}{\partial x}g_N\right\|_{L^2(x)}+\|g_N\|_{L^2(x)}\cdot\left\|(I-S_k)\frac{\partial}{\partial x}f_N\right\|_{L^2(x)}\right].\end{aligned} \tag{3.5}$$

Lemma 3.1 with $f_N(\cdot)=g_N(\cdot)=u_N(\cdot,t)$ implies

$$\|(I-S_{2k})P_Nu_N^2(\cdot,t)\|_{L^2(x)}\leqq\frac{4}{\sqrt{k}}\|u_N(\cdot,t)\|_{L^2(x)}\cdot\left\|(I-S_k)\frac{\partial}{\partial x}u_N(\cdot,t)\right\|_{L^2(x)}. \tag{3.6}$$

Equipped with (3.4), (3.6), and (2.8), we return to (3.3) to find that

$$\frac{1}{2}\frac{d}{dt}\|(I-S_{2k})u_N(\cdot,t)\|^2_{L^2(x)} \leqq -\varepsilon\left\|(I-S_{2k})\frac{\partial}{\partial x}u_N(\cdot,t)\right\|^2_{L^2(x)} + \frac{2}{\sqrt{k}}E_0\cdot\left\|(I-S_{2k})\frac{\partial}{\partial x}u_N(\cdot,t)\right\|_{L^2(x)}\left\|(I-S_k)\frac{\partial}{\partial x}u_N(\cdot,t)\right\|_{L^2(x)}, \tag{3.7}$$

which brings us to the following theorem.

THEOREM 3.2. *For any integer* $s\geqq 0$ *there exists a constant* $C_s=\text{Const.}\,(s, E_0)$, *such that for sufficiently large* N, $N>2^s\cdot 4m$, *we have*

$$\|(I-S_{N/2})u_N(\cdot,t)\|_{L^2(x)} \leqq C_s\cdot\left[\left(\frac{1}{\varepsilon\sqrt{N}}\right)^s+\left(1+\frac{1}{\varepsilon\sqrt{N}}\right)^s\cdot e^{-4\cdot 8^{-s}\cdot\varepsilon N^2 t}\right]. \tag{3.8}$$

Proof. Let $E_k(t)$ abbreviate the quantity

$$E_k(t)\equiv\|(I-S_k)u_N(\cdot,t)\|_{L^2(x)}. \tag{3.9}$$

In view of the inverse inequalities

$$2kE_{2k}(t)\leqq\left\|(I-S_{2k})\frac{\partial}{\partial x}u_N(\cdot,t)\right\|_{L^2(x)}\leqq NE_{2k}(t),$$

it follows from (3.7) that $E_k(t)$ satisfies

$$\frac{d}{dt}E_{2k}(t)\leqq -4\varepsilon k^2E_{2k}(t)+\frac{2E_0\cdot N^2}{\sqrt{k}}E_k(t), \qquad m<k\leqq N. \tag{3.10}$$

Temporal integration yields that for any $0<t_0<t$ we have

$$E_{2k}(t)\leqq\frac{2E_0\cdot N^2}{\sqrt{k}}\int_{\tau=t_0}^{t} e^{-4\varepsilon k^2(t-\tau)}E_k(\tau)\,d\tau+e^{-4\varepsilon k^2(t-t_0)}\cdot E_{2k}(t_0), \tag{3.11}$$

and therefore

$$E_{2k}(t)\leqq\frac{E_0\cdot N^2}{2\varepsilon\sqrt{k}\cdot k^2}\cdot\max_{t_0\leqq\tau\leqq t}E_k(\tau)+e^{-4\varepsilon k^2(t-t_0)}\cdot E_{2k}(t_0). \tag{3.12}$$

The a priori estimate (2.8) implies that

$$\max_{0\leqq\tau\leqq t}E_{2k}(\tau)<\max_{0\leqq\tau\leqq t}E_0(\tau)\leqq E_0,$$

and in view of (3.12) we have

$$E_{2k}(t)\leqq\frac{E_0\cdot N^2}{2\varepsilon\sqrt{k}\cdot k^2}\cdot E_0+e^{-4\varepsilon k^2 t}\cdot E_0, \qquad k>m. \tag{3.13$_{2k}$}$$

If we choose $t_0=t/2$ in (3.12) we find that

$$E_{2k}(t)\leqq\frac{E_0\cdot N^2}{2\varepsilon\sqrt{k}\cdot k^2}\cdot\max_{t/2\leqq\tau\leqq t}E_k(\tau)+e^{-2\varepsilon k^2 t}\cdot E_{2k}\left(\frac{t}{2}\right), \qquad k>m, \tag{3.14}$$

and following Kreiss [1], we can use this to improve our estimate (3.13). Namely, for $k>2m$ we can use $(3.13)_k$ to upper-bound the terms $\max_\tau E_k(\tau)$ and $E_{2k}(t/2)$ on the right of (3.14), and obtain the improved bound

$$E_{2k}(t)<\left(\frac{8E_0\cdot N^2}{\varepsilon\sqrt{k}\cdot k^2}\right)^2\cdot E_0+\left(1+\frac{8E_0\cdot N^2}{\varepsilon\sqrt{k}\cdot k^2}\right)e^{-2^{-1}\cdot\varepsilon k^2 t}\cdot E_0,\qquad k>2m.$$

Now we can repeat this process, and by induction we obtain that for $k>2^s\cdot m$ we have

$$E_{2k}(t)\leqq\left(\frac{8^sE_0\cdot N^2}{\varepsilon\sqrt{k}\cdot k^2}\right)^{s+1}\cdot E_0+\left(1+\frac{8^sE_0\cdot N^2}{\varepsilon\sqrt{k}\cdot k^2}\right)^s e^{-4\cdot 8^{-s}\cdot\varepsilon k^2 t}\cdot E_0. \tag{3.15}$$

Verification of the induction step is left to the Appendices. Finally, (3.15) implies that for sufficiently large $k=N/4>2^s\cdot m$ we have

$$\|(I-S_{N/2})u_N(\cdot,t)\|_{L^2(x)}\equiv E_{N/2}(t)$$
$$\leqq\left(\frac{32\cdot 8^s\cdot E_0}{\varepsilon\sqrt{N}}\right)^{s+1}\cdot E_0+\left(1+\frac{32\cdot 8^s\cdot E_0}{\varepsilon\sqrt{N}}\right)^s e^{-4\cdot 8^{-s}\cdot\varepsilon N^2 t}\cdot E_0,$$

and (3.8) follows.

Our parameterization in (1.14), (1.15) implies that for sufficiently large N we have

$$4\cdot 8^{-s}\cdot\varepsilon N^2 t\geqq\text{Const.}\frac{N^{2-2\beta}}{4\cdot 8^s\cdot\log N}\cdot t\geqq N^{3/2}\cdot t,\qquad 0<2\beta<\frac{1}{2},$$

as well as

$$\frac{1}{\varepsilon\sqrt{N}}\leqq\text{Const.}\frac{\log N}{N^{(1/2-2\beta)}}\leqq\frac{1}{N^\gamma},\qquad 0<\gamma<\frac{1}{2}-2\beta,$$

and Theorem 3.2 leads to the following corollary.

COROLLARY 3.3. *For any integer $s\geqq 0$ there exists a constant C_s such that*

$$\|(I-S_{N/2})u_N(\cdot,t)\|_{L^2(x)}\leqq C_s\cdot(N^{-s}+e^{-N^{3/2}\cdot t}). \tag{3.16}$$

Corollary 3.3 indicates the spectral decay of the Fourier coefficients $|\hat{u}(k)|$ with wavenumbers $|k|\geqq\frac{1}{2}N$, which in turn implies a similar decay for the discretization error I on the right of (3.1). For the latter we have

$$\|(I-P_N)\tfrac{1}{2}u_N^2(\cdot,t)\|^2_{L^2(x)}\equiv\|(I-S_N)\tfrac{1}{2}u_N^2(\cdot,t)\|^2_{L^2(x)}+a^2\|A_N\tfrac{1}{2}u_N^2(\cdot,t)\|^2_{L^2(x)}. \tag{3.17}$$

The Fourier coefficients of the two expressions on the right are given, respectively, by

$$\left(\widehat{(I-S_N)\frac{1}{2}u_N^2(\cdot,t)}\right)_k=\frac{1}{2}\sum_{p+q-k=0}\hat{u}_p(t)\hat{u}_q(t),\qquad |k|>N,$$

$$\left(\widehat{A_N\frac{1}{2}u_N^2(\cdot,t)}\right)_k=\frac{1}{2}\sum_{|p+q-k|=2N+1}\hat{u}_p(t)\hat{u}_q(t),\qquad |k|\leqq N.$$

In both cases, either $|p|>N/2$ or $|q|>N/2$; hence each one of these coefficients can be upper-bounded in a standard fashion to yield

$$\left\|(I-S_N)\frac{1}{2}u_N^2(\cdot,t)\right\|^2_{L^2(x)}+a^2\left\|A_N\frac{1}{2}u_N^2(\cdot,t)\right\|^2_{L^2(x)}$$
$$\leqq\sum_{N\leqq|k|\leqq 2N}(1+a^2)\cdot\left[\sum_{|q|\leqq N}|\hat{u}_q(t)|^2\right]\cdot\left[\sum_{|p|\geqq N/2}|\hat{u}_p(t)|^2\right]$$
$$\leqq 4E_0^2\cdot N\cdot\|(I-S_{N/2})u_N(\cdot,t)\|^2_{L^2(x)};$$

by (3.17) this is the same as

$$\|(I-P_N)\tfrac{1}{2}u_N^2(\cdot,t)\|_{L^2(x)} \leqq 2E_0\cdot\sqrt{N}\cdot\|(I-S_{N/2})u_N(\cdot,t)\|_{L^2(x)}. \tag{3.18}$$

Corollary 3.3, together with (3.18), shows that due to the presence of the spectral viscosity term II on the right of (2.1), the discretization error I decays to zero at a spectral rate independently whether or not the underlying solution is smooth. We state this as our next corollary.

COROLLARY 3.4. *For any integer* $s \geqq 0$ *there exists a constant* C_s *such that for sufficiently large* N *we have*

$$\|I \equiv [(I-P_N)\tfrac{1}{2}u_N^2(\cdot,t)]\|_{L^2(x)} \leqq C_s\cdot\sqrt{N}\cdot(N^{-s}+e^{-N^{3/2}\cdot t}), \qquad s\geqq 0. \tag{3.19}$$

We close this section with the promised proof.

Proof of Lemma 3.1. Starting with the identity

$$f_Ng_N \equiv f_N(I-S_k)g_N+(I-S_k)f_N\cdot S_kg_N+S_kf_N\cdot S_kg_N$$

and subtracting from this $(I-S_{2k})P_N[S_kf_N\cdot S_kg_N]\equiv(I-S_{2k})[S_kf_N\cdot S_kg_N]\equiv 0$, we can write

$$(I-S_{2k})P_N(f_Ng_N)\equiv(I-S_{2k})P_N[f_N(I-S_k)g_N+(I-S_k)f_N\cdot S_kg_N].$$

The quantity inside the right brackets is a trigonometric polynomial of degree less than or equal to $2N$, and hence, by the Parseval relation, its $L^2(x)$ norm dominates the $L^2(x)$ norm of its P_N projection, i.e.,

$$\begin{aligned}\|(I-S_{2k})P_N(f_Ng_N)\|_{L^2(x)} &\leqq \|P_N[\cdots]\|_{L^2(x)}\\ &\leqq \|f_N(I-S_k)g_N+(I-S_k)f_N\cdot S_kg_N\|_{L^2(x)}.\end{aligned} \tag{3.20}$$

The norm on the right of (3.20) is upper-bounded by

$$\begin{aligned}&\|f_N(I-S_k)g_N+(I-S_k)f_N\cdot S_kg_N\|_{L^2(x)}\\ &\quad\leqq \|f_N\|_{L^2(x)}\cdot\|(I-S_k)g_N\|_{L^\infty(x)}+\|g_N\|_{L^2(x)}\cdot\|(I-S_k)f_N\|_{L^\infty(x)}.\end{aligned} \tag{3.21}$$

Finally, for h_N equal to either f_N or g_N, we have

$$\begin{aligned}\|(I-S_k)h_N\|_{L^\infty(x)} &\leqq \sum_{|p|>k}|\hat{h}_p|\\ &\leqq \left[\sum_{|p|>k}\frac{1}{p^2}\right]^{1/2}\cdot\left[\sum_{|p|>k}p^2|\hat{h}_p|^2\right]^{1/2} < \frac{2}{\sqrt{k}}\left\|(I-S_k)\frac{\partial}{\partial x}h_N\right\|_{L^2(x)}\end{aligned} \tag{3.22}$$

and (3.5) follows from (3.20)-(3.22).

4. L^∞ a priori estimate. The classical energy method can be used to show that the solution of (2.1) remains uniformly bounded during a small finite time. The method reflects the fact that for sufficiently smooth initial data, say with $(\partial^2/\partial x^2)u_N(x,t=0)$ that are L^2-bounded, the process of shock formation takes a finite time, during which $(\partial/\partial x)u_N(x,t)$ remains uniformly bounded and a few Sobolev norms could be a priori estimated during that time.

For brief initial time intervals, we can do better with regard to the smoothness of the initial data, as Lemma 4.1 shows.

LEMMA 4.1. *Consider the Fourier approximation* (1.11)-(1.14) *with initial data* $u_N(x,t=0)$ *such that Assumption* III *holds, i.e.,*

$$\sum_{|k|\leqq N}|\hat{u}_k(t=0)|\leqq \text{Const}_0. \tag{4.1}$$

Then for $t \lesssim 1/N$ we have

$$\|u_N(\cdot,t)\|_{L^\infty(x)} \leqq 2 \cdot \sum_{|k|\leqq N} |\hat{u}_k(t=0)|, \qquad t \leqq \frac{1}{8\,\mathrm{Const}_0 \cdot N}. \tag{4.2}$$

Proof. The Fourier transform of (1.9) reads:

$$\frac{d}{dt}\hat{u}_k(t) + ik\left[\sum_{p+q=k} \hat{u}_p(t)\hat{u}_q(t) + a \cdot \sum_{|p+q-k|=2N+1} \hat{u}_p(t)\hat{u}_q(t)\right] = -\varepsilon k^2 \hat{Q}(k)\hat{u}_k(t). \tag{4.3}$$

Multiply the real (respectively, imaginary) part of this by sgn (Re $\hat{u}_k(t)$) (respectively, sgn (Im $\hat{u}_k(t)$)) and sum over all k's: since the right-hand side is negative, after summing both parts we obtain

$$\begin{aligned} \frac{d}{dt}\sum_{|k|\leqq N} |\hat{u}_k(t)| &\leqq (1+|a|)\cdot 2N \cdot \sum_{|p|\leqq N} |\hat{u}_p(t)| \cdot \sum_k |\hat{u}_{k-p}(t)| \\ &\leqq 4N \cdot \left(\sum_{|k|\leqq N} |\hat{u}_k(t)|\right)^2. \end{aligned} \tag{4.4}$$

Integration in time yields

$$\|u_N(\cdot,t)\|_{L^\infty(x)} \leqq \sum_{|k|\leqq N} |\hat{u}_k(t)| \leqq \frac{1}{1-4Nt\cdot \sum_{|k|\leqq N} |\hat{u}_k(t=0)|} \cdot \sum_{|k|\leqq N} |\hat{u}_k(t=0)|,$$

and (4.2) follows.

To obtain an L^∞ bound for later time, we shall carefully iterate on the $L^p(x)$ norms of $u_N(x,t)$. To this end, we multiply (2.1) by pu_N^{p-1} and integrate over the 2π-period, obtaining

$$\begin{aligned} &\frac{d}{dt}\|u_N(\cdot,t)\|^p_{L^p(x)} + \frac{p}{p+1} u_N^{p+1}(x,t)\Big|_{x=0}^{x=2\pi} \\ &\quad = p \cdot \int_0^{2\pi} u_N^{p-1} \frac{\partial}{\partial x}\left[(I-P_N)\frac{1}{2}u_N^2\right] dx + p \cdot \int_0^{2\pi} u_N^{p-1} \varepsilon \frac{\partial}{\partial x}\left[Q_N \frac{\partial}{\partial x} u_N\right] dx. \end{aligned} \tag{4.5}$$

By Corollary 3.4, the discretization error is negligibly small. Using (3.19) and the fact that $(I-P_N)\frac{1}{2}u_N^2$ is a trigonometric polynomial of degree less than or equal to $2N$, we have for any $s \geqq 0$

$$\begin{aligned} \left\|\frac{\partial}{\partial x}\left[(I-P_N)\frac{1}{2}u_N^2(\cdot,t)\right]\right\|_{L^p(x)} &\leqq \sqrt{2N}\cdot \left\|\frac{\partial}{\partial x}\left[(I-P_N)\frac{1}{2}u_N^2(\cdot,t)\right]\right\|_{L^2(x)} \\ &\leqq C_s N^2 \cdot (N^{-s} + e^{-N^{3/2}\cdot t}). \end{aligned} \tag{4.6}$$

Therefore, by the Hölder inequality, the first integral on the right of (4.5) does not exceed

$$\begin{aligned} &p \cdot \int_0^{2\pi} u_N^{p-1} \frac{\partial}{\partial x}\left[(I-P_N)\frac{1}{2}u_N^2\right] dx \\ &\quad \leqq p \cdot \|u_N^{p-1}(\cdot,t)\|_{L^q(x)} \left\|\frac{\partial}{\partial x}\left[(I-P_N)\frac{1}{2}u_N^2(\cdot,t)\right]\right\|_{L^p(x)} \\ &\quad \leqq p \cdot \|u_N(\cdot,t)\|^{p-1}_{L^p(x)} \cdot C_s \cdot (N^{2-s} + N^2 e^{-N^{3/2}\cdot t}), \qquad \frac{1}{p}+\frac{1}{q} = 1. \end{aligned} \tag{4.7}$$

The second integral on the right of (4.5), with $Q_N = I - R_m$, equals

$$p \cdot \int_0^{2\pi} u_N^{p-1} \varepsilon \frac{\partial}{\partial x}\left[(I-R_m)\frac{\partial}{\partial x} u_N\right] dx$$
$$= -\varepsilon p(p-1) \cdot \int_0^{2\pi} u_N^{p-2}\left(\frac{\partial u_N}{\partial x}\right)^2 dx - \varepsilon p \cdot \int_0^{2\pi} u_N^{p-1}\frac{\partial^2}{\partial x^2}(R_m u_N)\, dx.$$

The first term on the right-hand side is negative for any even integer $p \geqq 2$; for the second term we use the Hölder inequality as before, obtaining

$$\text{(4.8a)}\qquad \varepsilon p \cdot \int_0^{2\pi} u_N^{p-1}\frac{\partial^2}{\partial x^2}(R_m u_N)\, dx \leqq \varepsilon p \|u_N(\cdot, t)\|_{L^p(x)}^{p-1} \cdot \left\|\frac{\partial^2}{\partial x^2}(R_m u_N)\right\|_{L^p(x)}$$

Now, since $R_m u_N \equiv R_m(x) * u_N(x, t)$ is a trigonometric polynomial of degree less than or equal to $2m$ (see (1.12)), we can estimate the $L^p(x)$ norm of its derivatives as follows [5, Chap. X]:

$$\text{(4.8b)}\qquad \begin{aligned}\left\|\frac{\partial^2}{\partial x^2}(R_m u_N)\right\|_{L^p(x)} &\leqq (2m)^2 \cdot \|R_m(\cdot) * u_N(\cdot, t)\|_{L^p(x)} \\ &\leqq 4m^2 \|R_m(\cdot)\|_{L^1(x)} \cdot \|u_N(\cdot, t)\|_{L^p(x)}.\end{aligned}$$

Using (4.8a) and (4.8b) we conclude that the second integral on the right of (4.5) is upper-bounded by

$$\text{(4.9)}\quad p \cdot \int_0^{2\pi} u_N^{p-1} \varepsilon \frac{\partial}{\partial x}\left[Q_N \frac{\partial}{\partial x} u_N\right] dx \leqq 4p \cdot \varepsilon m^2 \cdot \|R_m(\cdot)\|_{L^1(x)} \cdot \|u_N(\cdot, t)\|_{L^p(x)}^p.$$

We recall that, according to our parameterization (1.15), $\varepsilon m^2 \cdot \|R_m(\cdot)\|_{L^1(x)} \leqq \alpha$. Hence, equipped with (4.7) and (4.9), we return to (4.5) to find that

$$\begin{aligned}&p\|u_N(\cdot, t)\|_{L^p(x)}^{p-1} \cdot \frac{d}{dt}\|u_N(\cdot, t)\|_{L^p(x)} \\ &\quad \leqq p\|u_N(\cdot, t)\|_{L^p(x)}^{p-1}[4\alpha \cdot \|u_N(\cdot, t)\|_{L^p(x)} + C_s \cdot (N^{2-s} + N^2 e^{-N^{3/2}\cdot t})],\end{aligned}$$

or, after division by the common factor of $p\|u_N(\cdot, t)\|_{L^p(x)}^{p-1}$,

$$\frac{d}{dt}\|u_N(\cdot, t)\|_{L^p(x)} \leqq 4\alpha \cdot \|u_N(\cdot, t)\|_{L^p(x)} + C_s \cdot (N^{2-s} + N^2 e^{-N^{3/2}\cdot t}).$$

Finally, we integrate in time, obtaining by Gronwall's inequality that for any $0 \leqq t_0 \leqq t$,

$$\text{(4.10)}\qquad \begin{aligned}\|u_N(\cdot, t)\|_{L^p(x)} &\leqq e^{4\alpha(t-t_0)} \\ &\quad \cdot [\|u_N(\cdot, t_0)\|_{L^p(x)} + C_s \cdot (N^{2-s} \cdot (t-t_0) + \sqrt{N} \cdot e^{-N^{3/2}\cdot t_0})].\end{aligned}$$

If we let p even tend to infinity, then (4.10) with $t_0 \equiv t_0(N) = 1/(8\,\mathrm{Const}_0 \cdot N)$ gives us

$$\text{(4.11)}\quad \|u_N(\cdot, t)\|_{L^\infty(x)} \leqq e^{4\alpha t} \cdot [\|u_N(\cdot, t_0)\|_{L^\infty(x)} + C_s(N^{2-s} \cdot t + \sqrt{N}\, e^{-\mathrm{Const}\cdot\sqrt{N}})]$$

and together with Lemma 4.1 we conclude with the desired L^∞ bound.

THEOREM 4.2. *Consider the Fourier approximation* (1.11)–(1.14). *Then for any* $s \geqq 0$ *there exist constants* $\alpha > 0$ *and* C_s *such that*

$$\text{(4.12)}\qquad \|u_N(\cdot, t)\|_{L^\infty(x)} \leqq e^{4\alpha t} \cdot \left[4 \cdot \sum_{|k| \leqq N} |\hat{u}_k(t=0)| + C_{s+2} \cdot (N^{-s} \cdot t + \sqrt{N}\, e^{-\mathrm{Const.}\cdot\sqrt{N}})\right].$$

Remarks. (1) We observe that the exponential time growth in (4.12) does not exceed 4α, where $\alpha \sim \varepsilon m^2 \cdot \|R_m(\cdot)\|_{L^1(x)} \leqq \text{Const.}$

(2) The a priori $L^p(x)$ estimate derived in (4.10) is valid for arbitrary L^2-bounded initial data. We note, however, that the resulting L^∞ bound in such case is not uniform with respect to the initial time t_0. That is, with arbitrary L^2-bounded initial data, the solution $\|u_N(\cdot, t)\|_{L^\infty(x)}$ may still grow by a factor of $O(\sqrt{N})$. The point made in Lemma 4.1 is that, with slightly strengthened regularity assumption on the initial data,

$$\sum_{|k| \leqq N} |\hat{u}_k(t=0)| \leqq \text{Const}_0,$$

this growth is bounded for a brief time interval of length $\sim 1/N$, after which the spectral viscosity becomes effective and guarantees the L^∞ bound later.

5. Convergence to the entropy solution. We follow [3], using compensated compactness arguments to conclude that $u_N(x, t)$ converges to the entropy solution of (1.1), (1.2).

Proof of Theorem 1.1. Consider the four terms on the right-hand side of (2.1) and (2.2). Using (3.18) together with (2.11) along the lines of [3, Lemma 3.1], we find that term I satisfies

$$\begin{aligned} &\left\| \mathrm{I} \equiv \frac{\partial}{\partial x}\left[(I-P_N)\frac{1}{2}u_N^2(\cdot, t)\right]\right\|_{H^{-1}_{\text{loc}}(x,t)} \\ &\leqq \left\|(I-P_N)\frac{1}{2}u_N^2(\cdot, t)\right\|_{L^2_{\text{loc}}(x,t)} \leqq 4E_0 \cdot J_0 \cdot \frac{1}{\sqrt{\varepsilon(N)\cdot N}} \xrightarrow[N\to\infty]{} 0. \end{aligned} \tag{5.1}$$

Also, by the a priori estimate (2.11) we have

$$\begin{aligned} &\left\| \mathrm{II} \equiv \varepsilon \frac{\partial}{\partial x}\left[Q_N \frac{\partial}{\partial x} u_N\right]\right\|_{H^{-1}_{\text{loc}}(x,t)} \\ &\leqq \varepsilon \left\|Q_N \frac{\partial}{\partial x} u_N\right\|_{L^2_{\text{loc}}(x,t)} \leqq \sqrt{\varepsilon(N)} \cdot J_0 \xrightarrow[N\to\infty]{} 0. \end{aligned} \tag{5.2}$$

Thus, in view of (5.1) and (5.2), terms I and II on the right of (2.1) belong to the compact of $H^{-1}_{\text{loc}}(x, t)$.

Next we note that since $Q_N + R_m = I$ we have

$$\sqrt{\varepsilon}\left\|\frac{\partial}{\partial x} u_N\right\|_{L^2_{\text{loc}}(x,t)} \leqq \sqrt{\varepsilon}\left\|Q_N \frac{\partial}{\partial x} u_N\right\|_{L^2_{\text{loc}}(x,t)} + \sqrt{\varepsilon}\left\|\frac{\partial}{\partial x}(R_m u_N)\right\|_{L^2_{\text{loc}}(x,t)}.$$

The first term on the right is bounded by J_0; the second one—being the derivative of a trigonometric polynomial of degree less than or equal to $2m$—does not exceed $\sqrt{\varepsilon} \cdot 2m \cdot \|u_N\|_{L^2_{\text{loc}}(x,t)} \leqq \text{Const.}$ Consequently,

$$\sqrt{\varepsilon}\left\|\frac{\partial}{\partial x} u_N\right\|_{L^2_{\text{loc}}(x,t)} \leqq \text{Const.} \tag{5.3}$$

Equipped with (5.3) we now turn to consider the right-hand side of (2.2). For the third term in (2.2),

$$\begin{aligned} \mathrm{III} &= u_N \frac{\partial}{\partial x}\left[(I-P_N)\frac{1}{2}u_N^2\right] \\ &= \frac{\partial}{\partial x}\left[u_N(I-P_N)\frac{1}{2}u_N^2\right] - \frac{\partial}{\partial x}u_N \cdot (I-P_N)\frac{1}{2}u_N^2 \equiv \mathrm{III}_1 + \mathrm{III}_2, \end{aligned} \tag{5.4a}$$

by Theorem 4.2 and estimate (5.1) we have

$$\begin{aligned}
&\left\|\mathrm{III}_1 \equiv \frac{\partial}{\partial x}\left[u_N(I-P_N)\frac{1}{2}u_N^2\right]\right\|_{H^{-1}_{\mathrm{loc}}(x,t)} \\
\text{(5.4b)}\qquad &\leqq \left\|u_N(I-P_N)\frac{1}{2}u_N^2\right\|_{L^2_{\mathrm{loc}}(x,t)} \\
&\leqq \|u_N(\cdot,t)\|_{L^\infty(x)}\cdot\left\|(I-P_N)\frac{1}{2}u_N\right\|^2_{L^2_{\mathrm{loc}}(x,t)} \xrightarrow[N\to\infty]{} 0,
\end{aligned}$$

and together with (5.3) we also have

$$\begin{aligned}
&\left\|\mathrm{III}_2 \equiv \frac{\partial}{\partial x}u_N\cdot(I-P_N)\frac{1}{2}u_N^2\right\|_{L^1_{\mathrm{loc}}(x,t)} \\
\text{(5.4c)}\qquad &\leqq \sqrt{\varepsilon}\left\|\frac{\partial}{\partial x}u_N\right\|_{L^2_{\mathrm{loc}}(x,t)}\cdot\frac{1}{\sqrt{\varepsilon}}\left\|(I-P_N)\frac{1}{2}u_N^2\right\|_{L^2_{\mathrm{loc}}(x,t)} \\
&\leqq \mathrm{Const}\cdot 4E_0J_0\cdot\frac{1}{\varepsilon(N)\cdot\sqrt{N}}\xrightarrow[N\to\infty]{} 0.
\end{aligned}$$

Finally, for the fourth term in (2.2),

$$\begin{aligned}
&\mathrm{IV}\equiv \varepsilon u_N\frac{\partial}{\partial x}\left(Q_N\frac{\partial}{\partial x}u_N\right) \\
\text{(5.5a)}\qquad &= \varepsilon\frac{\partial}{\partial x}\left[u_NQ_N\frac{\partial}{\partial x}u_N\right]-\varepsilon\frac{\partial}{\partial x}u_N\cdot Q_N\frac{\partial}{\partial x}u_N\equiv \mathrm{IV}_1+\mathrm{IV}_2,
\end{aligned}$$

we have by (2.8), (2.11), and the uniform bound in Theorem 4.2,

$$\text{(5.5b)}\qquad \left\|\mathrm{IV}_1\equiv\varepsilon\frac{\partial}{\partial x}\left[u_NQ_N\frac{\partial}{\partial x}u_N\right]\right\|_{H^{-1}_{\mathrm{loc}}(x,t)}\leqq \|u_N(\cdot,t)\|_{L^\infty(x)}\cdot\sqrt{\varepsilon(N)}\cdot J_0\xrightarrow[N\to\infty]{} 0,$$

while $\mathrm{IV}_2\equiv -\varepsilon[Q_N\,\partial/\partial x\,u_N]^2+\varepsilon(I-Q_N)\,\partial/\partial x\,u_N\cdot Q_N\,\partial/\partial x\,u_N$ satisfies

$$\begin{aligned}
\|\mathrm{IV}_2\|_{L^1_{\mathrm{loc}}(x,t)} &\leqq \varepsilon\left\|Q_N\frac{\partial}{\partial x}u_N\right\|^2_{L^2_{\mathrm{loc}}(x,t)} \\
&\quad +\sqrt{\varepsilon}\left\|R_m\frac{\partial}{\partial x}u_N\right\|_{L^2_{\mathrm{loc}}(x,t)}\cdot\sqrt{\varepsilon}\left\|Q_N\frac{\partial}{\partial x}u_N\right\|_{L^2_{\mathrm{loc}}(x,t)} \\
&\leqq J_0^2+\sqrt{\varepsilon}\cdot 2m\cdot E_0\cdot J_0\leqq \mathrm{Const}.
\end{aligned}$$

Therefore, (5.4), (5.5) together with Murat's Lemma [4] imply that the terms III and IV are also in the compact of $H^{-1}_{\mathrm{loc}}(x,t)$. In summary, we have shown that the right-hand sides of (2.1), (2.2) lie in the compact of $H^{-1}_{\mathrm{loc}}(x,t)$, and, according to Theorem 4.2, that $\|u_N(\cdot,t)\|_{L^\infty(x)}$ is bounded (in fact, $\|u_N\|_{L^p(x,t)}$ with $p>6$ will do for our purpose). Hence we can apply the div-curl lemma [4] to the left-hand sides of (2.1), (2.2) and obtain that (a subsequence of) $u_N(x,t)$ converges strongly in $L^2_{\mathrm{loc}}(x,t)$ to a weak limit solution $\bar{u}(x,t)$.

Moreover, we claim that this limit is the entropy solution of (1.1). To verify this claim we show that the right-hand side of (2.2), III+IV, tends weakly to a negative measure. Indeed, by (5.4) and (5.5b) the terms III and IV_1 tend weakly to zero, and hence it remains to show that the term IV_2,

$$\mathrm{IV}_2\equiv-\varepsilon\frac{\partial}{\partial x}u_N\cdot Q_N\frac{\partial}{\partial x}u_N=-\varepsilon(Q_N+R_m)\frac{\partial}{\partial x}u_N\cdot Q_N\frac{\partial}{\partial x}u_N,\qquad Q_N+R_m=I,$$

tends weakly to a negative measure. To this end we proceed as in [3, § 4] and rewrite IV_2 in the form

$$\text{(5.6)}\quad IV_2=-\varepsilon\left[Q_N\frac{\partial}{\partial x}u_N\right]^2-\varepsilon\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(R_mu_N)\cdot Q_Nu_N\right]+\varepsilon\frac{\partial^2}{\partial x^2}(R_mu_N)\cdot Q_Nu_N.$$

Denote the three terms on the right of (5.6) by IV_{21}, IV_{22}, and IV_{23}, respectively. By (2.11), IV_{21} tends weakly to a negative measure

$$\text{(5.7a)}\qquad w\lim_{N\to\infty}\left[IV_{21}\equiv-\varepsilon\left[Q_N\frac{\partial}{\partial x}u_N\right]^2\right]\leqq 0.$$

If we integrate the second term, IV_{22}, against any C_0^∞ test function $\psi(x,t)$, we find

$$\begin{aligned}\int_x\int_t\psi\cdot IV_{22}\,dx\,dt&\leqq\varepsilon\int_x\int_t\left|\frac{\partial\psi}{\partial x}\cdot\frac{\partial}{\partial x}(R_mu_N)\cdot Q_Nu_N\right|dx\,dt\\&\leqq\varepsilon\left\|\frac{\partial\psi}{\partial x}\right\|_{L^\infty_{\text{loc}}(x,t)}\cdot\left\|\frac{\partial}{\partial x}(R_mu_N)\right\|_{L^2_{\text{loc}}(x,t)}\cdot\|Q_Nu_N\|_{L^2_{\text{loc}}(x,t)},\end{aligned}$$

and since R_mu_N is a trigonometric polynomial of degree less than or equal to $2m$, this is less than

$$\text{(5.7b)}\qquad\begin{aligned}\int_x\int_t\psi\cdot IV_{22}\,dx\,dt&\leqq\varepsilon\cdot\left\|\frac{\partial\psi}{\partial x}\right\|_{L^\infty_{\text{loc}}(x,t)}\cdot 2m\cdot\|u_N\|^2_{L^2_{\text{loc}}(x,t)}\\&\leqq\text{Const.}\frac{1}{m(N)}\xrightarrow[N\to\infty]{}0.\end{aligned}$$

Finally, for the third term

$$IV_{23}=\varepsilon\frac{\partial^2}{\partial x^2}(R_mu_N)\cdot Q_N(u_N-\bar u)+\varepsilon\frac{\partial^2}{\partial x^2}(R_mu_N)\cdot Q_N\bar u\equiv IV_{231}+IV_{232}$$

we have

$$\text{(5.7c)}\qquad\begin{aligned}&\left\|IV_{231}\equiv\varepsilon\frac{\partial^2}{\partial x^2}(R_mu_N)\cdot Q_N(u_N-\bar u)\right\|_{L^1_{\text{loc}}(x,t)}\\&\quad\leqq\varepsilon\cdot 4m^2\cdot\|R_mu_N\|_{L^2_{\text{loc}}(x,t)}\cdot\|u_N-\bar u\|_{L^2_{\text{loc}}(x,t)}\\&\quad\leqq\text{Const.}\|u_N-\bar u\|_{L^2_{\text{loc}}(x,t)}\to 0,\end{aligned}$$

and since $\varepsilon R_mu_N\cdot Q_N\bar u$ tends weakly to zero, so does the term IV_{232},

$$\text{(5.7d)}\qquad w\lim_{N\to\infty}\left[IV_{232}\equiv\varepsilon\frac{\partial^2}{\partial x^2}(R_mu_N)\cdot Q_N\bar u\right]=0.$$

From (5.7a)-(5.7d) we conclude that term IV_2 in (5.6)—and therefore the right-hand side of (2.2)—tends weakly to a negative measure. Thus by taking the weak limit of (2.2) we recover (1.2) for our limit solution $\bar u(x,t)$. Consequently, the strong L^2_{loc} limit of $u_N(x,t)=\bar u(x,t)$ is the unique entropy solution of (1.1) as asserted.

Appendix A. The L^1-logarithmic bound of monotone viscosity kernels. We consider symmetric viscosity kernels of the form

$$Q_N(x)=\sum_{|k|\leqq 2m}\hat Q(k)\,e^{ikx}+\sum_{2m<|k|\leqq N}e^{ikx},$$

with monotonically increasing Fourier coefficients. Then the kernels that correspond to $R_m = I - Q_N$ are symmetric polynomials of degree less than or equal to $2m$,

$$R_m(x) = 2 \cdot \sum_{|k| \leqq 2m} \hat{R}(k) \cos kx \tag{A1}$$

whose Fourier coefficients are monotonically decreasing (compare (1.13)),

$$1 \geqq \hat{R}(k)\downarrow \geqq 0. \tag{A2}$$

Such kernels satisfy Assumption II, as shown by Lemma A1.

LEMMA A1. *There exists a constant such that*

$$\|R_m(\cdot)\|_{L^1(x)} \leqq \text{Const.} \log m. \tag{A3}$$

Proof. The result follows if we can show that $R_m(x)$ is majorized by Const. m and Const. $1/|x|$. If so, we have

$$\begin{aligned} \|R_m(\cdot)\|_{L^1(x)} &\leqq \int_{|x|\leqq 1/m} \text{Const.}\, m \cdot dx + \int_{1/m \leqq |x| \leqq \pi} \text{Const.} \frac{1}{|x|}\, dx \\ &\leqq \frac{2}{m} \text{Const.}\, m + 2 \cdot \text{Const.} \log |x| \Big|_{x=1/m}^{x=\pi} \qquad \text{(A4)} \\ &\leqq \text{Const.} \log m. \end{aligned}$$

Since $0 \leqq \hat{R}(k) \leqq 1$ we have

$$|R_m(x)| \leqq 2 \cdot \sum_{|k| \leqq 2m} |\hat{R}(k)| \leqq 4m;$$

furthermore, summation by parts yields

$$\begin{aligned} \left|\sin \frac{x}{2} \cdot R_m(x)\right| &= \left| \sum_{|k| \leqq 2m} \hat{R}(k) \cdot \left[\sin\left(k + \frac{1}{2}\right) x - \sin\left(k - \frac{1}{2}\right) x \right] \right| \\ &\leqq 4 + \sum_{1 \leqq |k| \leqq 2m-1} |\hat{R}(k+1) - \hat{R}(k)| \cdot \left| \sin\left(k + \frac{1}{2}\right) x \right|, \end{aligned}$$

and since $\hat{R}(k)$ are assumed to decrease monotonically, we have

$$|R_m(x)| \leqq \frac{6}{|\sin x/2|} \leqq \text{Const.} \frac{1}{|x|},$$

which completes the proof.

Appendix B. The decay rate of the Fourier coefficients revisited. In § 3, we concluded that the quantities $E_k(t) \equiv \|(I - S_k) u_N(\cdot, t)\|_{L^2(x)}$ satisfy, for $k > m$, the recursive inequality (3.14):

$$E_{2k}(t) \leqq \frac{E_0 \cdot N^2}{2\varepsilon\sqrt{k} \cdot k^2} \max_{t/2 \leqq \tau \leqq t} E_k(\tau) + e^{-2\varepsilon k^2 t} \cdot E_{2k}\left(\frac{t}{2}\right). \tag{B1}$$

Here we complete the details for the solution of these recurrence relations, and obtain that for $k > 2^s \cdot m$ we have

$$E_{2k}(t) \leqq \left(\frac{8^s E_0 \cdot N^2}{\varepsilon\sqrt{k} \cdot k^2}\right)^{s+1} \cdot E_0 + \left(1 + \frac{8^s E_0 \cdot N^2}{\varepsilon\sqrt{k} \cdot k^2}\right)^s e^{-4 \cdot 8^{-s} \cdot \varepsilon k^2 t} \cdot E_0, \tag{B2}$$

i.e., (3.15) holds. For $s=0$, (B2) is reduced to (3.13); now assume that (B2) is valid for any $k>2^s\cdot m$. In particular, for $k>2^{s+1}\cdot m$ we can use (B2) with k replaced by $k/2>2^s\cdot m$, and obtain that

$$\text{(B3)}\quad \max_{t/2\leqq\tau\leqq t} E_k(\tau)\leqq\left(\frac{8^{s+1}E_0\cdot N^2}{\varepsilon\sqrt{k}\cdot k^2}\right)^{s+1}\cdot E_0+\left(1+\frac{8^{s+1}E_0N^2}{\varepsilon\sqrt{k}\cdot k^2}\right)^{s}\cdot e^{-4\cdot 8^{-(s+1)}\cdot\varepsilon k^2 t}\cdot E_0.$$

Furthermore, we have

$$\text{(B4)}\qquad E_{2k}\left(\frac{t}{2}\right)\leqq\left(\frac{8^{s}E_0N^2}{\varepsilon\sqrt{k}\cdot k^2}\right)^{s+1}\cdot E_0+\left(1+\frac{8^{s}E_0\cdot N^2}{\varepsilon\sqrt{k}\cdot k^2}\right)^{s}\cdot e^{-4\cdot 8^{-(s+1)}\cdot\varepsilon k^2 t}\cdot E_0.$$

Using (B3) and (B4) to upper-bound the right-hand side of (B1), we find

$$\begin{aligned}E_{2k}(t)\leqq&\frac{E_0\cdot N^2}{2\varepsilon\sqrt{k}\cdot k^2}\cdot\left(\frac{8^{s+1}E_0\cdot N^2}{\varepsilon\sqrt{k}\cdot k^2}\right)^{s+1}\cdot E_0\\&+\frac{E_0N^2}{2\varepsilon\sqrt{k}\cdot k^2}\cdot\left(1+\frac{8^{s+1}E_0N^2}{\varepsilon\sqrt{k}\cdot k^2}\right)^{s}\cdot e^{-4\cdot 8^{-(s+1)}\cdot\varepsilon k^2 t}\cdot E_0\\&+e^{-2\varepsilon k^2 t}\cdot\left(\frac{8^{s}E_0N^2}{\varepsilon\sqrt{k}\cdot k^2}\right)^{s+1}\cdot E_0\\&+e^{-2\varepsilon k^2 t}\cdot\left(1+\frac{8^{s}E_0N^2}{\varepsilon\sqrt{k}\cdot k^2}\right)^{s}\cdot e^{-4\cdot 8^{-(s+1)}\cdot\varepsilon k^2 t}\cdot E_0.\end{aligned}$$

The first of the four terms on the right is less than $(8^{s+1}E_0\cdot N^2/\varepsilon\sqrt{k}\cdot k^2)^{s+2}\cdot E_0$; the sum of the remaining three terms does not exceed

$$\left(1+\frac{8^{s+1}E_0\cdot N^2}{\varepsilon\sqrt{k}\cdot k^2}\right)^{s+1}\cdot e^{-4\cdot 8^{-(s+1)}\cdot\varepsilon k^2 t}\cdot E_0,$$

and hence for $k>2^{s+1}\cdot m$ we have

$$E_{2k}(t)\leqq\left(\frac{8^{s+1}E_0\cdot N^2}{\varepsilon\sqrt{k}\cdot k^2}\right)^{s+2}\cdot E_0+\left(1+\frac{8^{s+1}E_0\cdot N^2}{\varepsilon\sqrt{k}\cdot k^2}\right)^{s+1} e^{-4\cdot 8^{-(s+1)}\cdot\varepsilon k^2 t}\cdot E_0,$$

which completes the induction proof of (B2).

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