

LEGENDRE PSEUDOSPECTRAL VISCOSITY METHOD FOR NONLINEAR CONSERVATION LAWS*

YVON MADAY[†], SIDI M. OULD KABER[†], AND EITAN TADMOR[‡]

Abstract. In this paper, the Legendre spectral viscosity (SV) method for the approximate solution of initial boundary value problems associated with nonlinear conservation laws is studied. The authors prove that by adding a small amount of SV, bounded solutions of the Legendre SV method converge to the exact scalar entropy solution. The convergence proof is based on compensated compactness arguments, and therefore applies to certain 2×2 systems. Finally, numerical experiments for scalar as well as the one-dimensional system of gas dynamics equations are presented, which confirm the convergence of the Legendre SV method. Moreover, these numerical experiments indicate that by post-processing the SV approximation, one can recover the entropy solution within spectral accuracy.

Key words. conservation laws, Legendre polynomials, spectral viscosity, post-processing, compensated compactness, convergence, spectral accuracy

AMS subject classifications. 35L65, 65M10, 65M15

1. Introduction. We are concerned here with the extension of the spectral viscosity (SV) method [Ta1], [MT], [Ta2], [Ta3], [CDT] to initial boundary value problems associated with the nonlinear conservation law,

$$(1.1) \quad \frac{\partial}{\partial t} u(x, t) + \frac{\partial}{\partial x} f(u(x, t)) = 0, \quad (x, t) \in [-1, 1] \times [0, \infty),$$

which is augmented with appropriate initial values at $t = 0$, and the necessary boundary data prescribed at $x = \pm 1$.

We concentrate on the Legendre pseudospectral method, and we show that by adding a spectrally small amount of SV to the high modes, one can achieve stability (and hence convergence) without sacrificing the spectral accuracy of the underlying Legendre approximation.

The paper is organized as follows. In §2 we present the details of the Legendre SV method. In §3 we collect the necessary estimates concerning linear operators in the spectral (polynomial) space which are required in the sequel. In §§4 and 5 we deal with the scalar case: assuming the Legendre SV approximation is uniformly bounded, we prove in §4 an a priori estimate on its gradient, which in turn is used, together with compensated compactness arguments, to prove convergence in §5. The building block of our proof is a “weak” representation of the truncation error provided in Lemma 5.1, which shows that the entropy production plus entropy dissipation of the Legendre SV method belongs to a compact subset of $H_{\text{loc}}^{-1}(x, t)$; using the results of [D] (see also [C]), our proof applies therefore to certain 2×2 systems. Finally, we present numerical experiments, which demonstrate the application of the Legendre SV method for scalar equations in §6, as well as systems of conservation laws in §7.

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[†] Laboratoire d'Analyse Numérique de l'Université Pierre et Marie Curie, Paris, France. The research of these authors was supported in part by Direction des Recherches Etudes et Techniques contract N89.34.190.00.470.75.01.

[‡] School of Mathematical Sciences, Tel-Aviv University, Tel-Aviv, Israel. The research of this author was supported in part by the Office of Naval Research contract N00014-91-J-1076, and by the Basic Research Foundation, Israel Academy of Sciences and Humanities.

1.1. Notation. We let \mathcal{P}_N denote the space of algebraic polynomials of degree $\leq N$, and we let $(L_k)_{k \geq 0}$ denote the orthogonal family of Legendre polynomials in this space

$$(1.2) \quad (L_j, L_k) = \frac{2}{2k+1} \delta_{jk}.$$

Here, (\cdot, \cdot) and $\|\cdot\|$ represent the usual $L^2[-1, 1]$ -inner product and norm. Next we let $\{\xi_j\}_{j=0}^N$ denote the zeros of $(1-x^2)L'_N(x)$ with $\xi_0 = -1 < \xi_1 < \dots < \xi_N = 1$. In the sequel we shall use the Legendre Gauss–Lobatto quadrature rule, stating that there exist weights, ω_j , such that for all $\phi \in \mathcal{P}_{2N-1}[-1, 1]$ we have (see, e.g., [DR], [CHQZ])

$$(1.3) \quad \int_{-1}^1 \phi(x) dx = \sum_{j=0}^N \omega_j \phi(\xi_j).$$

This suggests to define a discrete inner product, $(\cdot, \cdot)_N$,

$$(\phi, \psi)_N = \sum_{j=0}^N \omega_j \phi(\xi_j) \psi(\xi_j),$$

and we let $\|\cdot\|_N$ denote the corresponding discrete norm. Indeed, this discrete norm is equivalent with the usual L^2 -norm over $\mathcal{P}_N[-1, 1]$ (consult [CQ]):

$$(1.4) \quad \|\phi\| \leq \|\phi\|_N \leq \sqrt{3} \|\phi\| \quad \forall \phi \in \mathcal{P}_N[-1, 1],$$

and of course, due to (1.3) we obtain

$$(1.5) \quad (\phi, \psi) = (\phi, \psi)_N \quad \text{if } \deg \phi + \deg \psi \leq 2N - 1.$$

Associated with the $N + 1$ points of the Legendre Gauss–Lobatto quadrature rule, $\{\xi_j\}_{j=0}^N$, is a unique \mathcal{P}_N -interpolant which we denote by \mathcal{I}_N :

$$\mathcal{I}_N(\phi)(x) \equiv \sum_{k=0}^N \frac{(\phi, L_k)_N}{\|L_k\|_N^2} L_k(x), \quad \mathcal{I}_N(\phi)(\xi_j) = \phi(\xi_j) \quad j = 0, 1, \dots, N.$$

The projection \mathcal{I}_N can be viewed as an “approximate identity” in the \mathcal{P}_N -space; in this context we recall the recent result of [M] which provides us with the estimate

$$(1.6) \quad \left\| \frac{\partial}{\partial x} \mathcal{I}_N \phi \right\| + N \cdot \|\phi - \mathcal{I}_N \phi\| \leq C \left\| \frac{\partial}{\partial x} \phi \right\|.$$

We note in passing that similar estimates hold for some other “approximate identities” in the \mathcal{P}_N -space. For instance, (1.6) remains valid if we replace $\mathcal{I}_N \phi$ with $\mathcal{J}_N \phi \equiv \int_{-1}^x \pi_{N-1} \frac{\partial}{\partial x} \phi dx$, where $\pi_{N-1} \phi$ denotes the usual L^2 -projection of ϕ in \mathcal{P}_{N-1} . Indeed, using standard estimates of the latter (consult [CHQZ]), we obtain

$$(1.7) \quad \left\| \frac{\partial}{\partial x} \mathcal{J}_N \phi \right\| + N \cdot \|\phi - \mathcal{J}_N \phi\| \leq C \left\| \frac{\partial}{\partial x} \phi \right\|.$$

Finally, using (1.5) with $\psi \equiv \mathcal{I}_{N-1} \psi + (\psi - \mathcal{I}_{N-1} \psi)$ followed by (1.6), imply that the error of Gauss quadrature for $\mathcal{P}_{2N}[-1, 1]$ -polynomials does not exceed

$$(1.8) \quad |(\phi, \psi) - (\phi, \psi)_N| \leq C \|\psi - \mathcal{I}_{N-1} \psi\| \|\phi\| \leq \frac{C}{N} \left\| \frac{\partial}{\partial x} \psi \right\| \cdot \|\phi\| \quad \forall \phi, \psi \in \mathcal{P}_N[-1, 1].$$

2. The Legendre SV approximation. In the spectral viscosity approximation of (1.1) we seek a \mathcal{P}_N -polynomial of the form $u_N(x, t) = \sum_{k=0}^N \hat{u}_k(t) L_k(x)$, such that for all $\phi \in \mathcal{P}_N[-1, 1]$, we have

$$(2.1) \quad \left(\frac{\partial}{\partial t} u_N + \frac{\partial}{\partial x} \mathcal{I}_N f(u_N), \phi \right)_N = -\varepsilon_N \left(Q \frac{\partial}{\partial x} u_N, \frac{\partial}{\partial x} \phi \right)_N + (B(u_N), \phi)_N.$$

The approximation (2.1) involves the boundary operator, $B(u_N)$, and the SV operator, Q . Here, $B(u_N)$ is a forcing polynomial in $\mathcal{P}_N[-1, 1]$ of the form

$$B(u_N) = [\lambda(t)(1-x) + \mu(t)(1+x)] L'_N(x),$$

involving (at most) two nonzero free parameters, $\lambda(t)$ and $\mu(t)$, which should enable $u_N(x, t)$ to match the *inflow* boundary data prescribed at $x = \pm 1$ whenever $\pm f'(u_N(\pm 1, t)) < 0$. And Q denotes the spectral viscosity operator

$$Q\phi \equiv \sum_{k=0}^N \hat{Q}_k \hat{\phi}_k L_k \quad \forall \phi = \sum_{k=0}^{\infty} \hat{\phi}_k L_k,$$

which is associated with bounded viscosity coefficients,

$$(2.2) \quad \begin{aligned} \hat{Q}_k &\equiv 0, & k &\leq m_N, \\ 1 &\geq \hat{Q}_k \geq 1 - \left(\frac{m_N}{k} \right)^4, & k &> m_N. \end{aligned}$$

The free pair of spectral viscosity parameters (ε_N, m_N) will be chosen later, such that $\varepsilon_N \downarrow 0$ and $m_N \uparrow \infty$, in order to retain the formal spectral accuracy of (2.1) with (1.1).

We close this section by explaining how the SV method (2.1) can be implemented as a collocation method. Let us “test” (2.1) against $\phi = \phi_i$, where ϕ_i is the standard characteristic polynomial of $\mathcal{P}_N[-1, 1]$ satisfying $\phi_i(\xi_j) = \delta_{ij}$, $0 \leq i, j \leq N$. At the interior points we obtain

$$(2.3a) \quad \frac{d}{dt} u_N(\xi_i, t) + \frac{\partial}{\partial x} \mathcal{I}_N f(u_N)(\xi_i, t) = \varepsilon_N \frac{\partial}{\partial x} Q \left(\frac{\partial}{\partial x} u_N \right) (\xi_i, t), \quad 1 \leq i \leq N-1.$$

These equations are augmented, at the outflow boundaries (say at $x = +1$) with

$$(2.3b) \quad \frac{d}{dt} u_N(+1, t) + \frac{\partial}{\partial x} \mathcal{I}_N f(u_N)(+1, t) = \varepsilon_N \frac{\partial}{\partial x} Q \left(\frac{\partial}{\partial x} u_N \right) (+1, t) - \frac{\varepsilon_N}{\omega_N} Q \left(\frac{\partial}{\partial x} u_N \right) (+1, t).$$

We note that the last term on the right of (2.3b) prevents the creation of a boundary layer. Equations (2.3a), (2.3b) together with the prescribed inflow data (say at $x = -1$) furnish a complete equivalent statement of the pseudospectral (collocation) viscosity approximation (2.1).

The SV approximation (2.3a), (2.3b) enjoys *formal* spectral accuracy, i.e., its truncation error decays as fast as the global smoothness of the underlying solution permits. However, it is essential to keep in mind that this superior accuracy cannot be realized in the presence of shock discontinuities, unless the final SV solution is post-processed. The rest of this section is devoted to clarify this point.

2.1. Epilogue. It is well known that spectral projections like $\pi_N u$, $\mathcal{I}_N u$, etc., provide highly accurate approximations of u , provided u itself is sufficiently smooth. Indeed, these projections enjoy spectral convergence rate. This superior accuracy is destroyed if u contains discontinuities; both $\pi_N u$ and $\mathcal{I}_N u$ produce spurious $\mathcal{O}(1)$ Gibbs' oscillations which are *localized* in the neighborhoods of the discontinuities, and moreover, their *global* accuracy is deteriorated to first order.

To accelerate the convergence rate in such cases, we follow a similar treatment in [GT1] for the Fourier projections of discontinuous data. We introduce a mollifier of the form

$$(2.4) \quad \Psi^{\alpha,p}(x; y) = \rho\left(\frac{x-y}{\alpha}\right) K_p(x; y),$$

which consists of the following two ingredients:

- $\rho(x)$ is a $C_0^\infty(-1, 1)$ -localizer satisfying $\rho(0) = 1$;
- $K_p(x; y)$ is the Christoffel–Darboux kernel

$$K_p(x; y) \equiv \sum_{j=0}^p \frac{L_j(x)L_j(y)}{\|L_j\|^2} = \frac{(p+1)}{2} \frac{L_{p+1}(x)L_p(y) - L_{p+1}(y)L_p(x)}{x-y}.$$

We let $F^{\alpha,\beta}$ denote the smoothing filter

$$(2.5) \quad F^{\alpha,\beta} w(x) \equiv \int_{x=-1}^1 \Psi^{\alpha,p=[N^\beta]}(x; y) w(y) dy,$$

depending on the two fixed parameters, $\alpha, \beta \in (0, 1)$. Then, the following spectral error estimate was derived in [O]: for all $s \geq 1$ there exists a constant $C_{s,\alpha}$ such that

$$(2.6) \quad |u(x) - F^{\alpha,\beta}(\pi_N u)(x)| \leq C_{s,\alpha} \left[N^{2\beta - (1-2\beta)s} \|u\|_{L^2[-1,1]} + N^{-(\frac{3}{4}-s)\beta} \max_{\substack{|x-y| < \alpha \\ 0 \leq j \leq s}} |D^j u(y)| \right].$$

A similar estimate holds for \mathcal{I}_N . These estimates show (at least for $\beta < \frac{1}{2}$) that except for a small neighborhood of the discontinuities (measured by the free parameter α), one can filter the Legendre projections, $\pi_N u$ and $\mathcal{I}_N u$, in order to recover *pointwise* values of u within spectral accuracy.

Next, let u be the desired exact solution of a given problem. The purpose of a spectral method is to compute an approximation to the *projection* of u rather than u itself. Consequently, if the underlying solution of our problem is discontinuous, then the approximation computed by a spectral method, u_N , exhibits the two difficulties of local Gibbs' oscillations, and global, low- (i.e., first-) order accuracy.

With this in mind, we now turn to discuss the present context of nonlinear conservation laws. The standard, viscous-free spectral method supports the spurious Gibbs' oscillations which render the overall approximation unstable; consult [Ta1]. The task of the SV is therefore twofold:

1. *Stability: To stabilize the standard spectral method, which is otherwise unstable.*
2. *Spectral accuracy: To retain the overall spectral accuracy of the underlying spectral method.*

The question of stability is addressed in the following sections. We prove that SV guarantees the H^{-1} -stability (and hence the convergence) of the Legendre SV approximation,

$$L^p_{\text{loc}} - \lim u_N(x, t) = u(x, t) \quad \forall p < \infty.$$

The question of spectral accuracy requires further clarification. As noted above, the Legendre SV solution, $u_N(\cdot, t)$, should be considered as an accurate approximation of $\mathcal{I}_N u(\cdot, t)$, rather than $u(\cdot, t)$ itself. Therefore, the convergence rate of the SV method is limited by the first order convergence rate of $\mathcal{I}_N u(\cdot, t)$. (Of course, this limitation arises once shock-discontinuities are formed.) We recall that according to (2.6), this first-order limitation can be avoided by filtering $\mathcal{I}_N u$: the *filtered* interpolant, $F^{\alpha, \beta}(\mathcal{I}_N u)$, retains a spectral convergence rate, at least in smooth regions of the discontinuous entropy solution $u(\cdot, t)$. This suggests to apply the same filtering procedure (2.5) to $u_N(\cdot, t)$, in order to accelerate the convergence rate of the SV method.

Let $\{\hat{u}_k(t)\}_{k=0}^N$ denote the computed coefficients of the Legendre SV method. The computation of the SV solution is based on adding spectral viscosity only to the “high” modes—those with wavenumbers $k > m_N$. Therefore, one *expects* the computation of the viscous-free coefficients, at least, $\hat{u}_k(t) \equiv (u_N, L_k)_N / \|L_k\|_N^2$, $k = 1, \dots, m_N$, to be spectrally accurate approximation of the exact pseudospectral Legendre coefficients, $(u, L_k)_N / \|L_k\|_N^2$. Assuming that indeed this is the case, then according to (2.6) one can post-process the SV solution, $u_N(\cdot, t)$, in order to recover spectral convergence rate in smooth regions of the entropy solutions. Thus, at the final stage of the SV method, (2.3a),(2.3b) should be augmented with the post-processing procedure

$$(2.7) \quad F^{\alpha, \beta} u_N(x, t) = \int_{x=-1}^1 \Psi^{\alpha, p=N^\beta}(x; y) u_N(y) dy.$$

We conclude by noting that the post-processing of the SV solution plays a necessary key role in realizing the spectral accuracy of the SV method in smooth regions of the underlying solution. The treatment of Gibbs’ oscillations in the neighborhood of discontinuities requires an alternative “one-sided” filtering procedure, which is currently under investigation, e.g., [GSV].

3. Preliminaries. We collect here a couple of a priori estimates associated with linear operators on $\mathcal{P}_N[-1, 1]$, which will be needed in the sequel. We begin with the following lemma.

LEMMA 3.1. *Let R denote the “smoothing” operator*

$$R\phi \equiv \sum_{k=0}^N \hat{R}_k \hat{\phi}_k L_k \quad \forall \phi = \sum_{k=0}^{\infty} \hat{\phi}_k L_k,$$

and let $\|\psi\|_R^2$ denote the weighted inner product $(R\psi, \psi)$. Then the following estimate holds:

$$(3.1) \quad \|\phi'\|_R^2 \leq CN^2 \sum_{k=1}^N k |\hat{R}_k| \cdot \|\phi\|^2 \quad \forall \phi \in \mathcal{P}_N[-1, 1].$$

Proof. If $\phi = \sum_{k=0}^N \hat{\phi}_k L_k$, then $\phi' = \sum_{k=0}^{N-1} \hat{\phi}'_k L_k$ with $\hat{\phi}'_k$ given by

$$\hat{\phi}'_k = (2k+1) \sum_{j \in J_{k,N}} \hat{\phi}_j, \quad J_{k,N} \equiv \{j \mid k+1 \leq j \leq N, j+k \text{ odd}\}.$$

Recalling (1.2), we find

$$\begin{aligned}
 (R\phi', \phi') &= \sum_{k=0}^{N-1} \hat{R}_k |\hat{\phi}'_k|^2 \|L_k\|^2 \\
 &= \sum_{k=0}^{N-1} (2k+1)^2 \hat{R}_k \cdot \left| \sum_{j \in J_{k,N}} \hat{\phi}_j \right|^2 \|L_k\|^2 \\
 &\leq 4 \sum_{k=0}^{N-1} (2k+1) |\hat{R}_k| \cdot \sum_{j \in J_{k,N}} |\hat{\phi}_j|^2 \|L_j\|^2 \cdot \sum_{j \in J_{k,N}} \frac{1}{\|L_j\|^2} \\
 &\leq C \sum_{j=0}^N |\hat{\phi}_j|^2 \|L_j\|^2 \cdot \sum_{k=0}^N k |\hat{R}_k| \cdot (N^2 - k^2) \leq CN^2 \sum_{k=0}^N k |\hat{R}_k| \cdot \|\phi\|^2,
 \end{aligned}$$

and (3.1) follows. \square

As an immediate corollary from (2.1) with $\hat{R}_k \equiv 1$ we obtain [CQ]

$$(3.2) \quad \|\phi'\| \leq CN^2 \|\phi\| \quad \forall \phi \in \mathcal{P}_N[-1, 1].$$

Another consequence of Lemma 3.1 is the following corollary.

COROLLARY 3.2. *As before, we let $\|w\|_Q^2$ denote the weighted inner product (Qw, w) . Then, the following estimate holds:*

$$(3.3) \quad \left\| \frac{\partial}{\partial x} u_N \right\|_Q^2 \leq \left\| \frac{\partial}{\partial x} u_N \right\|_Q^2 + Cm_N^4 \ln N \|u_N\|^2.$$

Proof. We first note that $\|(\partial/\partial x)u_N\|^2 \equiv \|(\partial/\partial x)u_N\|_Q^2 + \|(\partial/\partial x)u_N\|_R^2$ where, according to (2.2), $\hat{R}_k \equiv 1 - \hat{Q}_k$ satisfy

$$\hat{R}_k \equiv 1 - \hat{Q}_k \begin{cases} = 1, & k \leq m_N, \\ \leq \frac{m_N^4}{k^4}, & k > m_N. \end{cases}$$

It remains to upper bound $\|(\partial/\partial x)u_N\|_R^2$, and to this end, we decompose $u_N(x, t)$ as a dyadic sum

$$u_N(x, t) = \sum_{k=0}^{m_N} \hat{u}_k L_k + \sum_{j=1}^J u_N^j(x, t), \quad u_N^j(x, t) \equiv \sum_{k > 2^{j-1}m_N}^{2^j m_N} \hat{u}_k L_k, \quad J = \log_2 \frac{N}{m_N}.$$

By Lemma 3.1 we have

$$\left\| \frac{\partial}{\partial x} u_N^j \right\|_R^2 \leq C(2^j m_N)^2 \sum_{k > 2^{j-1}m_N}^{2^j m_N} k \frac{m_N^4}{k^4} \|u_N^j\|^2 \leq 2Cm_N^4 \|u_N^j\|^2,$$

and the result (3.3) follows, from

$$\begin{aligned}
 (3.4) \quad \left\| \frac{\partial}{\partial x} u_N \right\|_R^2 &\leq 2 \left\| \frac{\partial}{\partial x} \sum_{k=0}^{m_N} \hat{u}_k L_k \right\|_R^2 + 2J \sum_{j=0}^J \left\| \frac{\partial}{\partial x} u_N^j \right\|_R^2 \\
 &\leq 2Cm_N^4 \left\| \sum_{k=0}^{m_N} \hat{u}_k L_k \right\|^2 + 4CJm_N^4 \sum_{j=1}^J \|u_N^j\|^2 \leq Cm_N^4 \ln N \|u_N\|^2. \quad \square
 \end{aligned}$$

4. A priori estimates. The SV approximation (2.3a)–(2.3b) amounts to a nonlinear system of ordinary differential equations (ODEs). It can be shown that under appropriate assumptions on ε_N and m_N , this nonlinear system admits a unique solution, $u_N(\cdot, t)$, $t \in [0, T]$; consult [MOT]. In order to focus on its essential features, we skip the existence/uniqueness proof of such a solution, and furthermore, we make the following assumption regarding its uniform stability.

ASSUMPTION (L^∞ -boundedness). There holds

$$(4.1) \quad \|u_N(\cdot, t)\|_{L^\infty_{\text{loc}}(x, [0, T])} \leq M < \infty.$$

The assumption of L^∞ -boundedness is indeed confirmed by the numerical experiments reported later in this paper. (For a proof of L^∞ -boundedness in a similar situation of the periodic Fourier SV method, we refer to [MT] and [CDT] for a treatment of the one- and, respectively, multi-dimensional scalar cases.)

To simplify the presentation, we shall deal here with the prototype case where one boundary, say $x = -1$, is an inflow boundary, while $x = +1$ is an outflow one. In this case, the boundary operator $B(u_N)$ takes the form

$$(4.2a) \quad B(u_N) = \lambda(t)(1 - x)L'_N(x),$$

where $\lambda(t)$ is a free “Lagrange multiplier” parameter which enables $u_N(x, t)$ to match the prescribed inflow boundary data

$$(4.2b) \quad u_N(-1, t) = g(t), \quad g \in H^1_{\text{loc}}(t).$$

We begin by noting that

$$(1 - x)L'_N(x) = \begin{cases} 0 & \text{for } x = \xi_i > -1, \\ -\frac{2(-1)^N}{\omega_0} & \text{for } x = \xi_0 = -1. \end{cases}$$

Consequently, the contribution of the boundary operator $B(u_N)$ to (2.1) amounts to

$$(B(u_N), \phi)_N \equiv -2(-1)^N \lambda(t) \phi(-1, t).$$

To gain a better insight on the role of $\lambda(t)$, we set $\phi \equiv 1$ in (2.1), and obtain

$$\frac{d}{dt}(u_N, 1) + f(u_N)|_{x=-1}^{x=+1} = -2(-1)^N \lambda(t).$$

Thus, $\lambda(t)$ measures the rate of change of total mass over the whole $[-1, +1]$ interval. The last equality implies that

$$(4.3a) \quad |\lambda(t)| \leq \frac{1}{\sqrt{2}} \left\| \frac{\partial}{\partial t} u_N \right\| + |f|_\infty, \quad |f|_\infty \equiv \max_{|u| \leq M} |f(u)|,$$

and the following “pessimistic” upper bound on the total mass holds

$$(4.3b) \quad \left| \Lambda(t) \equiv \int_{s=0}^t \lambda(s) ds \right| \leq \text{Const}, \quad \text{Const} = \frac{1}{\sqrt{2}} \|u_N(\cdot, t)\| + t|f|_\infty.$$

Equipped with (4.3a), (4.3b), we now turn to derive an a priori estimate on the gradient of the Legendre viscosity approximation $u_N(x, t)$. To this end we proceed as follows.

Let $F(u) \equiv \int^u w f'(w) dw$ be the entropy flux corresponding to the quadratic entropy $\frac{1}{2}u^2$. We set $\phi = u_N$ in (2.1) and obtain, in view of (1.5), (1.6), and Corollary 3.2,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|u_N\|_N^2 + F(u_N)_{-1}^{+1} + \varepsilon_N \left(Q \frac{\partial}{\partial x} u_N, \frac{\partial}{\partial x} u_N \right)_N + 2(-1)^N \lambda(t) u_N(-1, t) \\ &= \left(\frac{\partial}{\partial x} f(u_N), u_N \right) - \left(\frac{\partial}{\partial x} \mathcal{I}_N f(u_N), u_N \right)_N \\ &\equiv - \left((I - \mathcal{I}_N) f(u_N), \frac{\partial}{\partial x} u_N \right) \\ &\leq \frac{C}{N} \left\| \frac{\partial}{\partial x} f(u_N) \right\| \left\| \frac{\partial}{\partial x} u_N \right\| \leq \frac{C}{N} \left[\left\| \frac{\partial}{\partial x} u_N \right\|_Q^2 + m_N^4 \ln N \|u_N\|^2 \right]. \end{aligned}$$

By (1.4) and (1.5), $\|u_N\| \sim \|u_N\|_N$ and $(Q \frac{\partial}{\partial x} u_N, \frac{\partial}{\partial x} u_N)_N = \|\frac{\partial}{\partial x} u_N\|_Q^2$, and hence

$$\begin{aligned} (4.4) \quad & \frac{1}{2} \frac{d}{dt} \|u_N\|_N^2 + \left(\varepsilon_N - \frac{C}{N} \right) \left\| \frac{\partial}{\partial x} u_N \right\|_Q^2 \\ & \leq \frac{C m_N^4 \ln N}{N} \|u_N\|^2 + 2|F|_\infty - 2(-1)^N \lambda(t) g(t). \end{aligned}$$

Thus, abbreviating $L_{\text{loc}}^2(x, t) = L_{\text{loc}}^2(x, [0, T])$, we arrive at the desired estimate on the gradient of the Legendre viscosity approximation.

LEMMA 4.1. *Assume that the spectral viscosity parameters (ε_N, m_N) satisfy*

$$(4.5) \quad 0 \downarrow \varepsilon_N \sim \frac{1}{N^\theta}, \quad m_N < \text{Const} \cdot N^{\frac{q}{4}} \quad \text{with} \quad 0 < q < \theta \leq 1.$$

Then the following estimate holds:

$$(4.6) \quad \left\| \frac{\partial}{\partial x} u_N \right\|_{L_{\text{loc}}^2(x, t)}^2 + \left\| \frac{\partial}{\partial t} u_N \right\|_{L_{\text{loc}}^2(x, t)}^2 \leq \text{Const} \left[1 + \|g\|_{H_{\text{loc}}^1(t)}^2 \right] \frac{1}{\varepsilon_N} \leq \text{Const} \frac{1}{\varepsilon_N}.$$

Proof. According to (4.3b) we have

$$\left| \int_{t=0}^T \lambda(t) g(t) dt \right| \equiv \left| [g(t) \Lambda(t)]_{t=0}^{t=T} - \int_{t=0}^T \frac{d}{dt} g(t) \Lambda(t) dt \right| \leq \text{Const} + \|g\|_{H_{\text{loc}}^1(t)}^2,$$

and hence, temporal integration of (4.4) yields

$$\begin{aligned} \|u_N(\cdot, T)\|_{L^2(x)}^2 + \varepsilon_N \left\| \frac{\partial}{\partial x} u_N \right\|_{L_{\text{loc}}^2(Q(x), t)}^2 &\leq \text{Const} \left[1 + \left| \int_{t=0}^T \lambda(t) g(t) dt \right| \right] \\ &\leq \text{Const} \left[1 + \|g\|_{H_{\text{loc}}^1(t)}^2 \right]. \end{aligned}$$

Thanks to the last estimate together with Corollary 3.2 we have

$$\begin{aligned} (4.7a) \quad & \varepsilon_N \left\| \frac{\partial}{\partial x} u_N \right\|_{L_{\text{loc}}^2(x, t)}^2 \leq \varepsilon_N \left\| \frac{\partial}{\partial x} u_N \right\|_{L_{\text{loc}}^2(Q(x), t)}^2 + C \varepsilon_N m_N^4 \ln N \|u_N\|_{L_{\text{loc}}^2(x, t)}^2 \\ & \leq \text{Const} \left[1 + \|g\|_{H_{\text{loc}}^1(t)}^2 \right], \end{aligned}$$

and the first half of (4.6) follows.

Next we set $\phi = (\frac{\partial}{\partial t})u_N$ in (2.1), obtaining

$$\begin{aligned} & \left\| \frac{\partial}{\partial t} u_N \right\|_N^2 + \left(\frac{\partial}{\partial x} \mathcal{I}_N f(u_N), \frac{\partial}{\partial t} u_N \right)_N \\ &= -\varepsilon_N \left(Q \frac{\partial}{\partial x} u_N, \frac{\partial^2}{\partial x \partial t} u_N \right) + \left(B(u_N), \frac{\partial}{\partial t} u_N \right)_N \\ &= -\frac{\varepsilon_N}{2} \frac{d}{dt} \left\| \frac{\partial}{\partial x} u_N \right\|_Q^2 - 2(-1)^N \lambda(t) \frac{\partial}{\partial t} u_N(-1, t). \end{aligned}$$

Hence, using (1.6) and (4.3a) we find

$$\begin{aligned} & \left\| \frac{\partial}{\partial t} u_N \right\|^2 + \frac{\varepsilon_N}{2} \frac{d}{dt} \left\| \frac{\partial}{\partial x} u_N(\cdot, t) \right\|_Q^2 \\ (4.7b) \quad & \leq \left\| \frac{\partial}{\partial x} \mathcal{I}_N f(u_N) \right\| \cdot \left\| \frac{\partial}{\partial t} u_N \right\| + 2|\lambda(t)| \cdot \left| \frac{d}{dt} g(t) \right| \\ & \leq \text{Const} \left\| \frac{\partial}{\partial x} u_N \right\|^2 + \frac{1}{4} \left\| \frac{\partial}{\partial t} u_N \right\|^2 + \frac{1}{4} \left\| \frac{\partial}{\partial t} u_N \right\|^2 + \text{Const} \left| \frac{d}{dt} g(t) \right|^2. \end{aligned}$$

Temporal integration of (4.7b) followed by (4.7a) imply the second half of (4.6), for

$$\begin{aligned} (4.7c) \quad & \frac{1}{2} \left\| \frac{\partial}{\partial t} u_N \right\|_{L^2_{\text{loc}}(x, t)}^2 \leq \text{Const} \left[\left\| \frac{\partial}{\partial x} u_N \right\|_{L^2_{\text{loc}}(x, t)}^2 + \|g(t)\|_{H^1_{\text{loc}}(t)}^2 \right] \\ & \leq \text{Const} \left[1 + \|g\|_{H^1_{\text{loc}}(t)}^2 \right] \frac{1}{\varepsilon_N}. \quad \square \end{aligned}$$

5. Convergence of the Legendre viscosity approximation. Equipped with the a priori estimates of §4 we now turn to prove the convergence of (2.1) by compensated compactness arguments. To this end we want to show that $\frac{\partial}{\partial t} U(u_N) + \frac{\partial}{\partial x} F(u_N)$ belongs to a compact subset of $H^{-1}_{\text{loc}}(x, t)$ for all convex entropy pairs $(U(u_N), F(u_N))$. Our main tool in this direction is the following lemma.

LEMMA 5.1. *A weak representation of the truncation error of the Legendre viscosity approximation (2.1) is given by*

$$(5.1) \quad \left(\frac{\partial}{\partial t} u_N + \frac{\partial}{\partial x} f(u_N), \phi \right) = \sum_{j=1}^6 I_j(\phi), \quad \phi(x, t) \in \mathcal{D}([-1, 1]),$$

where the following estimates hold:

$$(5.2a) \quad \sum_{j=1}^3 |I_j(\phi)| \leq \mathcal{O} \left(\frac{1}{\sqrt{\varepsilon_N}} \right) \left[\|\phi - \phi_N\| + \frac{1}{N} \left\| \frac{\partial}{\partial x} \phi_N \right\| \right],$$

$$(5.2b) \quad |I_4(\phi)| \leq \mathcal{O}(\varepsilon_N m_N^2 \sqrt{\ln N}) \left\| \frac{\partial}{\partial x} \phi_N \right\|,$$

$$(5.2c) \quad |I_5(\phi) \equiv -\varepsilon_N \left(\frac{\partial}{\partial x} u_N, \frac{\partial}{\partial x} \phi_N \right)| \leq \mathcal{O}(\sqrt{\varepsilon_N}) \left\| \frac{\partial}{\partial x} \phi_N \right\|,$$

$$(5.2d) \quad I_6(\phi) \equiv 2(-1)^{N+1} \int_{t=0}^T \lambda(t) \phi_N(-1, t) dt.$$

Here, $\phi_N(\cdot, t)$ is an arbitrary \mathcal{P}_N -polynomial at our disposal.

Proof. We proceed in three steps.

Step 1. For arbitrary $\phi \in \mathcal{D}([-1, 1])$ and $\phi_N \in \mathcal{P}_N[-1, 1]$ we have the identity

$$\begin{aligned} & \left(\frac{\partial}{\partial t} u_N + \frac{\partial}{\partial x} f(u_N), \phi \right) \\ & \equiv \left(\frac{\partial}{\partial t} u_N + \frac{\partial}{\partial x} f(u_N), \phi - \phi_N \right) + \left(\frac{\partial}{\partial t} u_N + \frac{\partial}{\partial x} f(u_N), \phi_N \right) \\ & \equiv \left(\frac{\partial}{\partial t} u_N + \frac{\partial}{\partial x} f(u_N), \phi - \phi_N \right) + \left(\frac{\partial}{\partial x} f(u_N) - \frac{\partial}{\partial x} \mathcal{I}_N f(u_N), \phi_N \right) \\ & \quad + \left(\frac{\partial}{\partial t} u_N + \frac{\partial}{\partial x} \mathcal{I}_N f(u_N), \phi_N \right) \\ & \equiv \left(\frac{\partial}{\partial t} u_N + \frac{\partial}{\partial x} f(u_N), \phi - \phi_N \right) \\ & \quad + \left(\frac{\partial}{\partial x} (f - \mathcal{I}_N f)(u_N), \phi_N \right) \\ & \quad + \left[\left(\frac{\partial}{\partial t} u_N + \frac{\partial}{\partial x} \mathcal{I}_N f(u_N), \phi_N \right) - \left(\frac{\partial}{\partial t} u_N + \frac{\partial}{\partial x} \mathcal{I}_N f(u_N), \phi_N \right)_N \right] \\ & \quad + \left(\frac{\partial}{\partial t} u_N + \frac{\partial}{\partial x} \mathcal{I}_N f(u_N), \phi_N \right)_N \equiv I_1(\phi) + I_2(\phi) + I_3(\phi) + J_4(\phi). \end{aligned}$$

Thus, we conclude the first step noting that the truncation error of the Legendre viscosity approximation (in its weak form) can be represented as

$$(5.3) \quad \left(\frac{\partial}{\partial t} u_N + \frac{\partial}{\partial x} f(u_N), \phi \right) = \sum_{j=1}^3 I_j(\phi) + J_4(\phi).$$

In the second step we shall estimate $\{I_j(\phi)\}_{j=1}^3$, and in the third step we conclude with the desired representation of $J_4(\phi)$ as $\sum_{k=4}^6 I_k(\phi)$.

Step 2. We begin with the first expression,

$$(5.4a) \quad I_1(\phi) \equiv \left(\frac{\partial}{\partial t} u_N + \frac{\partial}{\partial x} f(u_N), \phi - \phi_N \right).$$

Lemma 4.1 implies that this expression does not exceed

$$(5.4b) \quad |I_1(\phi)| \leq \left[\left\| \frac{\partial}{\partial t} u_N \right\| + a_\infty \left\| \frac{\partial}{\partial x} u_N \right\| \right] \cdot \|\phi - \phi_N\| \leq \frac{\text{Const}}{\sqrt{\varepsilon_N}} \|\phi - \phi_N\|.$$

Integration by parts shows that the second term equals

$$(5.5a) \quad I_2(\phi) \equiv \left(\frac{\partial}{\partial x} (f - \mathcal{I}_N f)(u_N), \phi_N \right) = - \left((I - \mathcal{I}_N) f(u_N), \frac{\partial}{\partial x} \phi_N \right),$$

and the inequality (1.6) followed by Lemma 4.1 implies

$$(5.5b) \quad \begin{aligned} |I_2(\phi)| &\leq \left| \left((f - \mathcal{I}_N f)(u_N), \frac{\partial}{\partial x} \phi_N \right) \right| \leq \frac{\text{Const}}{N} \left\| \frac{\partial}{\partial x} f(u_N) \right\| \left\| \frac{\partial}{\partial x} \phi_N \right\| \\ &\leq \frac{\text{Const}}{N\sqrt{\varepsilon_N}} \left\| \frac{\partial}{\partial x} \phi_N \right\|. \end{aligned}$$

The exactness of Gauss quadrature rule for \mathbb{P}_{2N-1} -polynomials implies that the third expression equals

$$(5.6a) \quad \begin{aligned} I_3(\phi) &\equiv \left[\left(\frac{\partial}{\partial t} u_N + \frac{\partial}{\partial x} \mathcal{I}_N f(u_N), \phi_N \right) - \left(\frac{\partial}{\partial t} u_N + \frac{\partial}{\partial x} \mathcal{I}_N f(u_N), \phi_N \right)_N \right] \\ &= \left(\frac{\partial}{\partial t} u_N, \phi_N \right) - \left(\frac{\partial}{\partial t} u_N, \phi_N \right)_N. \end{aligned}$$

To upper bound this expression, we use the error estimate of the Gauss quadrature rule for \mathbb{P}_{2N} -polynomials (consult (1.8)), and together with Lemma 4.1 we find

$$(5.6b) \quad \begin{aligned} |I_3(\phi)| &= \left| \left(\frac{\partial}{\partial t} u_N, \phi_N \right) - \left(\frac{\partial}{\partial t} u_N, \phi_N \right)_N \right| \leq \text{Const} \frac{1}{N} \left\| \frac{\partial}{\partial x} \phi_N \right\| \left\| \frac{\partial}{\partial t} u_N \right\| \\ &\leq \frac{\text{Const}}{N\sqrt{\varepsilon_N}} \left\| \frac{\partial}{\partial x} \phi_N \right\|. \end{aligned}$$

The inequalities (5.4b), (5.5b), (5.6b) complete the proof of (5.2a).

Step 3. We are left with the fourth expression, $J_4(\phi)$, which, using the Legendre viscosity approximation (2.1), can be rewritten as

$$(5.7) \quad \begin{aligned} J_4(\phi) &\equiv \left(\frac{\partial}{\partial t} u_N + \frac{\partial}{\partial x} \mathcal{I}_N f(u_N), \phi_N \right)_N \\ &= -\varepsilon_N \left(Q \frac{\partial}{\partial x} u_N, \frac{\partial}{\partial x} \phi_N \right)_N - 2(-1)^N \int_{t=0}^T \lambda(t) \phi_N(-1, t) dt \\ &= \varepsilon_N \left(R \frac{\partial}{\partial x} u_N, \frac{\partial}{\partial x} \phi_N \right) - \varepsilon_N \left(\frac{\partial}{\partial x} u_N, \frac{\partial}{\partial x} \phi_N \right) + 2(-1)^{N+1} \int_{t=0}^T \lambda(t) \phi_N(-1, t) dt \\ &\equiv I_4(\phi) + I_5(\phi) + I_6(\phi). \end{aligned}$$

Using (3.4) we find that (5.2b) holds for

$$\begin{aligned} |I_4(\phi)| &\equiv \left| \varepsilon_N \left(\frac{\partial}{\partial x} u_N, \frac{\partial}{\partial x} \phi_N \right)_R \right| \leq \varepsilon_N \left\| \frac{\partial}{\partial x} u_N \right\|_R \left\| \frac{\partial}{\partial x} \phi_N \right\| \\ &\leq \text{Const} \cdot \varepsilon_N m_N^2 \sqrt{\ln N} \|u_N\| \left\| \frac{\partial}{\partial x} \phi_N \right\|, \end{aligned}$$

which completes the proof. \square

For each $\phi \in H_0^1([-1, 1] \times [0, T])$ we assign an approximant in $\mathbb{P}_N[-1, 1]$, denoted by ϕ_N , such that

$$(5.8) \quad \phi_N(-1, t) = 0, \quad \left\| \frac{\partial}{\partial x} \phi_N \right\| + N \cdot \|\phi - \phi_N\| \leq \text{Const} \cdot \left\| \frac{\partial}{\partial x} \phi \right\|.$$

Clearly, there are several possibilities for such assignment; for example, $\phi_N = \mathcal{I}_N \phi$ or $\phi_N = \mathcal{J}_N \phi$ will serve our purpose; consult (1.6), (1.7).

We now return to Lemma 5.1; we find that *any* assignment of such ϕ_N (satisfying (5.8)) gives us

$$(5.9a) \quad \sum_{k=0}^3 |I_k(\phi)| \leq \mathcal{O}\left(\frac{1}{N\sqrt{\varepsilon_N}}\right) \left\| \frac{\partial}{\partial x} \phi \right\|,$$

$$(5.9b) \quad |I_4(\phi)| \leq \mathcal{O}(\varepsilon_N m_N^2 \sqrt{\ln N}) \left\| \frac{\partial}{\partial x} \phi \right\|,$$

$$(5.9c) \quad \left| I_5(\phi) \equiv -\varepsilon_N \left(\frac{\partial}{\partial x} u_N, \frac{\partial}{\partial x} \phi_N \right) \right| \leq \mathcal{O}(\sqrt{\varepsilon_N}) \left\| \frac{\partial}{\partial x} \phi \right\|.$$

$$(5.9d) \quad I_6(\phi) \equiv -2(-1)^N \int_{t=0}^T \lambda(t) \phi_N(-1, t) dt = 0.$$

We conclude that the SV approximation (2.1), (2.2) parameterized according to (4.5) satisfies

$$(5.10) \quad \left\| \frac{\partial}{\partial t} u_N + \frac{\partial}{\partial x} f(u_N) \right\|_{H_{\text{loc}}^{-1}(x,t)} \leq \mathcal{O}\left(\frac{1}{N\sqrt{\varepsilon_N}} + \varepsilon_N m_N^2 \sqrt{\ln N} + \sqrt{\varepsilon_N}\right) \rightarrow 0,$$

and hence, $(\frac{\partial}{\partial t})u_N + (\frac{\partial}{\partial x})f(u_N)$ belongs to a compact subset of $H_{\text{loc}}^{-1}(x, t)$. In fact, more is true, namely, Lemma 5.1 together with Lemma 4.1 imply that for all $\phi \in H_0^1([-1, 1] \times [0, T])$ we have

$$(5.11) \quad \begin{aligned} & \left| \left(\frac{\partial}{\partial t} U(u_N) + \frac{\partial}{\partial x} F(u_N), \phi \right) \right| = \left| \sum_{j=1}^5 I_j(U'(u_N)\phi) \right| \\ & \leq \text{Const} \cdot \left(\frac{1}{N\sqrt{\varepsilon_N}} + \varepsilon_N m_N^2 \sqrt{\ln N} + \sqrt{\varepsilon_N} \right) \left\| \frac{\partial}{\partial x} (U'(u_N)\phi) \right\| \\ & \leq \text{Const} \cdot \left(\frac{1}{N\sqrt{\varepsilon_N}} + \varepsilon_N m_N^2 \sqrt{\ln N} + \sqrt{\varepsilon_N} \right) \cdot \left[\left\| \frac{\partial}{\partial x} u_N \right\| \cdot \|\phi\|_{L^\infty} + \|\phi_x\| \cdot \|U'\|_\infty \right] \\ & \leq \text{Const} \cdot \left[\|\phi\|_{L^\infty} + \left(\frac{1}{N\sqrt{\varepsilon_N}} + \varepsilon_N m_N^2 \sqrt{\ln N} + \sqrt{\varepsilon_N} \right) \|\phi_x\| \right]. \end{aligned}$$

Thus, $(\frac{\partial}{\partial x})U(u_N) + (\frac{\partial}{\partial x})F(u_N)$ can be written as a sum of two terms which belong, respectively, to a compact subset of $H_{\text{loc}}^{-1}(x, t)$ and a bounded set of $L_{\text{loc}}^1(x, t)$, and hence by Murat's lemma, to a compact subset of $H_{\text{loc}}^{-1}(x, t)$. Using the div-curl lemma [Tr], it follows that the Legendre viscosity approximation, $u_N(x, t)$, converges strongly to a weak solution, $u(x, t)$, of (1.1). We arrive at the following theorem.

THEOREM 5.2. *Let $u_N(x, t)$ be the Legendre viscosity approximation of (2.1), (2.2), with spectral viscosity parameters (ε_N, m_N) which satisfy*

$$(5.12) \quad 0 \downarrow \varepsilon_N \sim \frac{1}{N^\theta}, \quad m_N < \text{Const} \cdot N^{q/4} \quad \text{with } 0 < q < \theta \leq 1.$$

Then (a subsequence of) $u_N(x, t)$ converges strongly (in L^p_{loc} , $p < \infty$) to a weak solution of the conservation law (1.1). Moreover, if $\theta < 1$, then (the whole sequence of) $u_N(x, t)$ converges strongly to the unique entropy solution of (1.1).

Proof. We have shown that (a subsequence of) $u_N(x, t)$ converges strongly to a weak solution, $\bar{u}(x, t)$, of (1.1). To show that $\bar{u}(x, t)$ is the entropy solution, we choose to implement Lemma 5.1 with a particular assignment of an approximant in \mathcal{P}_N , namely, we shall use (5.1) with $\phi_N = \mathcal{J}_N \phi$,
(5.13)

$$\phi_N = \int_{-1}^x \pi_{N-1} \left(\frac{\partial}{\partial x} \phi \right) dx, \quad \text{where } \pi_M = L^2[-1, 1] - \text{projection on } \mathcal{P}_M[-1, 1].$$

At this point we note that since this choice of ϕ_N satisfies (5.8) (consult (1.7)), it can be used in conjunction with Lemma 5.1 in order to show, as was done before, that $\frac{\partial}{\partial x} U(u_N) + \frac{\partial}{\partial x} F(u_N)$ belongs to a compact subset of $H^{-1}_{\text{loc}}(x, t)$. The advantage of using the special choice of $\phi_N = \mathcal{J}_N \phi$ will enable us to deduce more, namely, that $\frac{\partial}{\partial t} U(u_N) + \frac{\partial}{\partial t} F(u_N)$ tends to a negative measure. To this end we proceed as follows.

Lemma 5.1 tells us that for all $\phi \in H^1_0([-1, 1] \times [0, T])$ we have

$$\begin{aligned} \left(\frac{\partial}{\partial t} U(u_N) + \frac{\partial}{\partial x} F(u_N), \phi \right) &= \sum_{j=1}^3 I_j(U'(u_N)\phi) + \sum_{j=4}^5 I_j(U'(u_N)\phi), \\ \phi &\in H^1_0([-1, 1] \times [0, T]). \end{aligned}$$

The first three terms on the right tend to zero, for by (5.8), (5.9a) we have for all $\phi \in H^1_0([-1, 1] \times [0, T])$,

$$\begin{aligned} \sum_{j=1}^3 |I_j(U'(u_N)\phi)| &\leq \text{Const} \frac{1}{N\sqrt{\varepsilon_N}} \left\| \frac{\partial}{\partial x} (U'(u_N)\phi)_N \right\| \\ &\leq \text{Const} \frac{1}{N\sqrt{\varepsilon_N}} \left[\left\| \frac{\partial}{\partial x} u_N \right\| \cdot \|\phi\|_{L^\infty} + \left\| \frac{\partial}{\partial x} \phi \right\| \cdot \|u_N\|_{L^\infty} \right] \\ &\leq \text{Const} \left[N^{\theta-1} \|\phi\|_{L^\infty} + N^{\frac{\theta}{2}-1} \left\| \frac{\partial}{\partial x} \phi \right\| \right] \rightarrow 0. \end{aligned}$$

The fourth term tends to zero, for by (5.8), (5.9b) we have

$$|I_4(U'(u_N)\phi)| \leq \mathcal{O}(\varepsilon_N m_N^2 \sqrt{\ln N}) \left\| \frac{\partial}{\partial x} (U'(u_N)\phi) \right\| \leq \text{Const} \cdot \sqrt{\ln N} \cdot N^{q-\theta/2} \left\| \frac{\partial}{\partial x} \phi \right\| \rightarrow 0.$$

Finally, we are left with the fifth term, $I_5(U'(u_N)\phi)$, and it is here that we take advantage of our special assignment for the ϕ_N approximant in (5.13). Using the orthogonality of $I - \pi_{N-1}$ to the $\mathcal{P}_{N-1}[-1, 1]$, we find that the fifth term tends to a negative measure, for by the convexity of U , we have for all $\phi \geq 0$,

$$\begin{aligned} I_5(U'(u_N)\phi) &\equiv -\varepsilon_N \left(\frac{\partial}{\partial x} u_N, \pi_{N-1} \frac{\partial}{\partial x} (U'(u_N)\phi) \right)_N \\ &\equiv -\varepsilon_N \left(\frac{\partial}{\partial x} u_N, U''(u_N)\phi \frac{\partial}{\partial x} u_N \right) - \varepsilon_N \left(\frac{\partial}{\partial x} u_N, U'(u_N) u_N \frac{\partial}{\partial x} \phi \right) \\ &\leq -\varepsilon_N \left(\frac{\partial}{\partial x} u_N, U'(u_N) u_N \frac{\partial}{\partial x} \phi \right) \leq \mathcal{O}(\sqrt{\varepsilon_N}) \left\| \frac{\partial}{\partial x} \phi \right\| \rightarrow 0. \end{aligned}$$

We conclude that $\bar{u}(x, t)$ satisfies the entropy inequality $\frac{\partial}{\partial t}U(\bar{u}) + \frac{\partial}{\partial x}F(\bar{u}) \leq 0$, and the convergence of (the whole sequence of) $u_N(x, t)$ follows. \square

6. Numerical results—the scalar case. In the next two sections we present numerical simulations of the Legendre SV method. We begin with the inviscid Burgers' equation

$$(6.1a) \quad \frac{\partial}{\partial t}u(x, t) + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2(x, t)\right) = 0, \quad (x, t) \in [-1, 1] \times [0, \infty),$$

which serves as a prototype for nonlinear scalar conservation laws. The equation is augmented with initial values at $t = 0$,

$$(6.1b) \quad u(x, t = 0) = 1 + \frac{1}{2}\sin \pi x, \quad x \in [-1, 1],$$

and with prescribed boundary conditions at the inflow boundary $x = -1$,

$$(6.1c) \quad u(-1, t) = g(t).$$

The inflow boundary data are taken from the outflow boundary, $g(t) = u(1, t)$, i.e., we solve the periodic problem associated with (6.1a), (6.1b), as an initial boundary value problem. This initial boundary value problem serves as an appropriate test case, for it admits a *nonstationary* shock discontinuity which originates at $x_c = (2 - \pi)/\pi$ at time $t_c = 2/\pi$, and continues its course dictated by the Rankine–Hugoniot relation.

The Legendre SV approximation of this problem reads (consult (2.3a), (2.3b))

$$(6.2a) \quad \frac{d}{dt}u_N(\xi_i, t) + \frac{\partial}{\partial x}\mathcal{I}_N\left(\frac{1}{2}u_N^2\right)(\xi_i, t) = \varepsilon_N \frac{\partial}{\partial x}Q\left(\frac{\partial}{\partial x}u_N\right)(\xi_i, t), \quad 1 \leq i \leq N-1,$$

together with boundary conditions

$$(6.2b) \quad \begin{aligned} \frac{d}{dt}u_N(1, t) + \frac{\partial}{\partial x}\mathcal{I}_N\left(\frac{1}{2}u_N^2\right)(1, t) &= \varepsilon_N \frac{\partial}{\partial x}Q\left(\frac{\partial}{\partial x}u_N\right)(1, t) - \frac{\varepsilon_N}{\omega_N}Q\left(\frac{\partial}{\partial x}u_N\right)(1, t), \\ u_N(-1, t) &= g(t), \quad g(t) = u_N(1, t). \end{aligned}$$

The resulting nonlinear system of $N + 1$ ODEs for $\{u_N(\xi_i, t)\}_{i=0}^N$ was integrated in time using the second-order Adams–Bashforth ODE solver with time step $\Delta t = 10^{-5}$ (consult [GT2] regarding the time step dictated by a CFL-like condition for such methods). All the numerical results presented in this section were recorded at time $t = 1$.

The first result we present shows the necessity of including a certain amount of SV term in the discretization of (6.1a), (6.1b) by spectral methods. Indeed, the standard Legendre method is inconsistent with the entropy condition, and hence it fails to converge to the entropy solution once shock discontinuities are formed; consult [Ta1], [Ta3] for a proof concerning the similar periodic case. We illustrate this fact in Fig. 6.1(a), showing the results obtained with a plain collocation scheme, that is, the standard pseudospectral Legendre method corresponding to (6.2a) with $\varepsilon_N = 0$. It is clear that the numerical solution does not converge to the exact one. Moreover, Fig. 6.1(b) shows that convergence fails even after the plain spectral solution was post-processed according to (2.7).

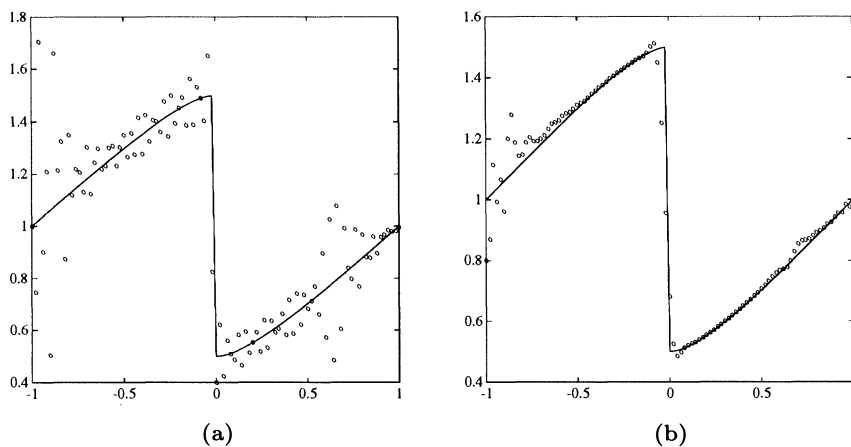


FIG. 6.1. Solution of the standard viscous-free pseudospectral Legendre method ($N = 128$) (a) before post-processing, (b) after post-processing.

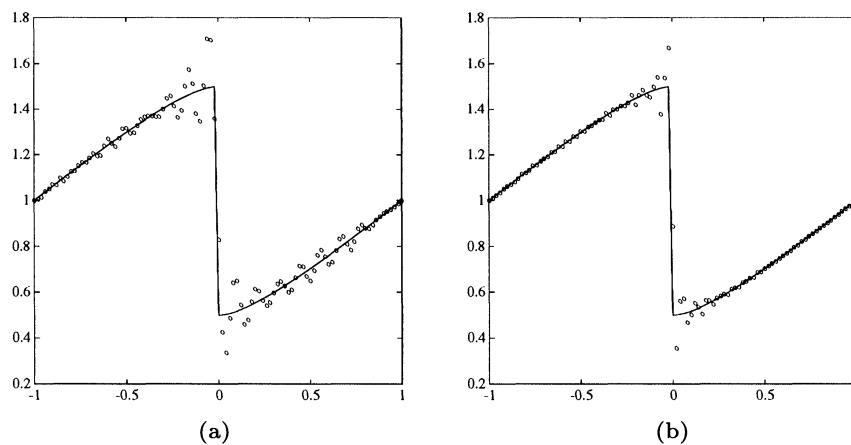


FIG. 6.2. Solution of the Legendre SV method based on (a) $N = 64$ modes, (b) $N = 128$ modes.

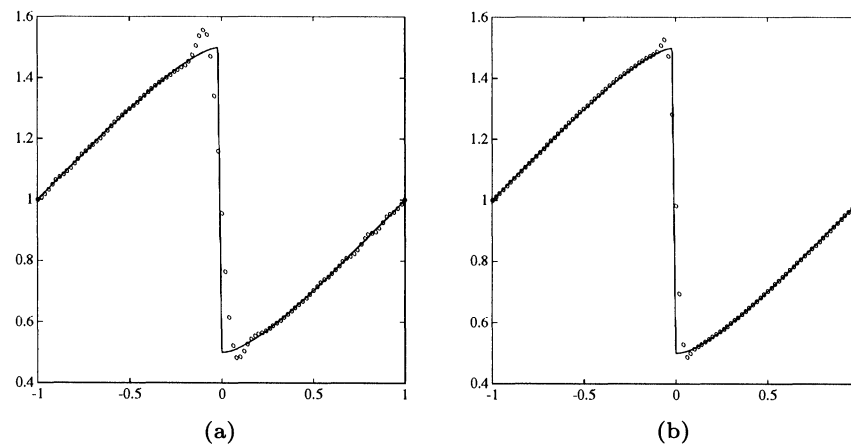


FIG. 6.3. The SV solution in Fig. 6.2 after post-processing.

To overcome the failure of convergence, SV was added to the standard Legendre spectral method. It was implemented with SV amplitude $\varepsilon_N \simeq N^{-1}$, which was activated for modes $k \geq m_N \simeq 5\sqrt{N}$. A smooth viscosity kernel of the form

$$(6.2c) \quad \hat{Q}_k = \exp\left\{-\frac{(k-N)^2}{(k-m_N)^2}\right\}, \quad k > m_N,$$

was used in the numerical experiments reported in this section. As advocated in [Tal], the C^∞ -smoothness of \hat{Q}_k (as a function of k/N), improves the resolution of the SV method. The numerical results in Figs. 6.2(a) and 6.2(b) indicate the strong L^p -convergence ($p < \infty$) of the SV method, in contrast to the plain Legendre method.

These results confirm the stability of the Legendre SV method. However, the convergence rate in this case is rather low. According to the arguments presented in §2, this reflects the low- (i.e., first-) order convergence rate of $\mathcal{I}_N u$, rather than the lack of spectral accuracy of the SV solution itself. To amplify this point, we post-process the SV solution at the final time $t = 1$. The post-processed solution, $F^{\alpha,\beta} u_N(\cdot, t = 1)$ with $(\alpha, \beta) = (0.25, 0.8)$ was recorded in Fig. 6.3. The dramatic improvement in the convergence rate is evident. Indeed, Fig. 6.4 shows that the post-processed SV solution recovers the smooth parts of the exact entropy solution within spectral accuracy.

In fact, based on these figures, we can draw the two conclusions:

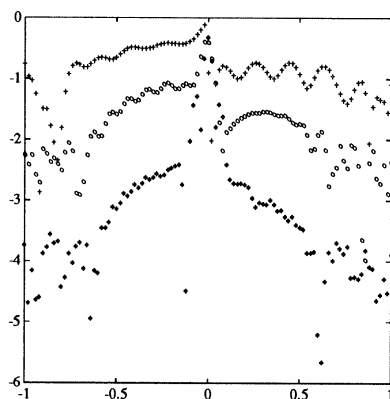
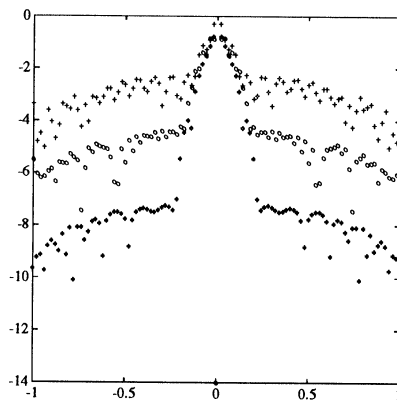
- the convergence rate of the post-processed SV solution is faster than any finite value, in agreement with the piecewise C^∞ -regularity of the entropy solution;
- spectral convergence rate can be observed already for quite low values of the parameter N .

This numerical evidence supports the assumption that the SV solution, u_N , serves as an accurate approximation to the *projected* solution, $\pi_m u$, rather than u itself. This assumption was proved for the linear periodic problem in [AGT], and is under current investigation in the nonlinear case. In this context it is important to check the optimality of the SV method, to provide further support to the above assumption. The optimal polynomial approximation of discontinuous functions like $u(\cdot, t)$, is achieved by the filtering of the “best” L^2 -fit projection of the *exact* solution, $\pi_N u(\cdot, t)$. In Fig. 6.5 we present the error between the exact solution and its filtered projection. Comparing Figs. 6.4 and 6.5, we see that the quality of results for the computed SV solution, u_N with $N = 128$ modes, is between that of the “optimal” results achieved for $\pi_N u(\cdot, t)$ with $N = 64$ and $N = 128$! One final remark is in order. One could have suspected that the results for the SV method would scale with m_N , the number of viscous-free modes; in our case, $N = 128$, $m_N \sim 50$, and thus the SV computation should have resulted in less resolution than the results obtained by the “best” L^2 -fit with $N = 64$ modes. This is clearly not the case.

7. Numerical results—the system of gas dynamics. In this section we will present numerical experiments which demonstrate the performance of the Legendre SV method for systems of conservation laws. We consider the approximate solution of the Euler equations of gas dynamics,

$$(7.1) \quad \frac{\partial}{\partial t} u(x, t) + \frac{\partial}{\partial x} f(u(x, t)) = 0, \quad u = \begin{bmatrix} \rho \\ \rho v \\ E \end{bmatrix}, \quad f(u) = \begin{bmatrix} \rho v \\ \rho v^2 + p \\ v(E + p) \end{bmatrix},$$

where ρ denotes the density of the gas, v its velocity, $m \equiv \rho v$ its momentum, E its energy per unit volume, and $p = (\gamma - 1) \cdot (E - \frac{1}{2} \rho v^2)$ its (polytropic) pressure, $\gamma = 1.4$.

FIG. 6.4. Error plot (in a logarithmic scale) of the post-processed SV solution for $N = 32, 64, 128$.FIG. 6.5. Error plot (in a logarithmic scale) of the post-processed best L^2 -fit for $N = 64, 128, 256$.

The Legendre SV approximation of this system reads

$$(7.2a) \quad \frac{d}{dt} u_N(\xi_i, t) + \frac{\partial}{\partial x} \mathcal{I}_N f(u_N)(\xi_i, t) = \varepsilon_N \frac{\partial}{\partial x} Q \left(\frac{\partial}{\partial x} u_N \right) (\xi_i, t), \quad 1 \leq i \leq N-1.$$

Here, $u_N \equiv {}^t(\rho_N, \rho_N v_N, E_N) \in \mathbb{P}_N^3[-1, 1]$ denotes the polynomial approximation of the 3-vector of (density, momentum, energy), and Q abbreviates a general 3×3 spectral viscosity matrix, $\{\hat{Q}_k^{\ell, j}\}_{k=m_N}^N$, $1 \leq \ell, j \leq 3$, which is activated only on “high” Legendre modes, i.e., $\hat{Q}_k^{\ell, j} = 0$ for all $k > m_N(\ell, j)$. The numerical results reported in this section were obtained using a simple scalar viscosity matrix,

$$(7.2b) \quad Q \left(\frac{\partial}{\partial x} u_N \right) = {}^t \left(Q \frac{\partial}{\partial x} \rho_N, Q \frac{\partial}{\partial x} \rho_N v_N, Q \frac{\partial}{\partial x} E_N \right),$$

with the same viscosity coefficients, \hat{Q}_k , as in (6.2c).

The Legendre SV method (7.2a), (7.2b) amounts to a nonlinear system of $(N+1)^3$ ODEs which was integrated in time using the second-order Adams–Bashforth ODE solver.

We implemented the SV method for two test problems.

(1) *The Riemann shock tube problem* [S]. Our first example is the Riemann problem (7.1), subject to initial conditions

$$(7.3) \quad u(x, 0) = \begin{cases} u_\ell = {}^t(1., 0, 2.5), & x < 0, \\ u_r = {}^t(0.125, 0, 0.25), & x > 0. \end{cases}$$

The exact solution of the Riemann problem (7.1), (7.3), develops four constant states separated by a rarefaction wave, a contact discontinuity and a shock wave [GR], [Sm], [La]. The exact solution at $t = 0.287$ was recorded by the solid lines in Figs. 7.1–7.3. The Legendre SV method, (7.2a), (7.2b), was implemented in this case with SV parameters $(\varepsilon_N, m_N) = (1/N, 6\sqrt{N})$. The circles in Figs. 7.1–7.3 display the numerical results of the corresponding SV approximation, which was integrated in time by the second-order Adams–Bashforth method with time step $\Delta t = 10^{-5}$. (Consult [GT2].)

Figures 7.1(a), 7.2(a) and 7.3(a) display the computed density ρ_N , velocity v_N , and pressure p_N , with $N = 128$ Legendre modes. The numerical results in these figures show that the presence of SV guarantees the convergence of the pseudospectral Legendre method that is otherwise unstable. However, Gibbs' oscillations which are inherited from the *projected* solution, $\mathcal{I}_N u(\cdot, t)$, are still present.

To remove these oscillations without sacrificing spectral accuracy, the SV solution on the left side of Figs. 7.1–7.3 was post-processed using the filtering procedure (2.5), $F^{\alpha, \beta}$ with $(\alpha, \beta) = (0.2, 0.85)$. Again, as in the scalar case, the post-processing leads to a dramatic improvement in the quality of the computed results, revealing the high resolution content of the SV computation. In particular, comparing the results obtained by the post-processed SV method in Figs. 7.1(b), 7.2(b), and 7.3(b), we find the representation of the rarefaction wave and the capturing of the contact discontinuity to be better than the results obtained by the finite difference methods in [S] or the high resolution schemes in [SO]. (It is worthwhile noting that these high resolutions results of the SV computations were obtained *without* the costly characteristic decompositions which are employed in the modern high resolution finite difference approximations.)

The resolution of the shock discontinuity, however, still suffers from a smearing of spurious Gibbs' oscillations. As told by the error estimate (2.6), the oscillations in the neighborhood of the discontinuities cannot be removed by the filtering procedure (2.5). Instead, these oscillations can be avoided by using an alternative “one-sided” filter which is currently under investigation [GSV].

(2) *The shock-disturbance interaction* (e.g., [SO]). Our second example models the interaction of a sinusoidal disturbance and a shock wave due to initial conditions

$$(7.4) \quad (\rho(x, 0), v(x, 0), p(x, 0)) = \begin{cases} (3.857143, 10.333333, 2.629369), & x < -0.8, \\ (1. + 0.2 \sin(5\pi x), 0., 1.), & x > -0.8. \end{cases}$$

The exact solution of this problem, (7.1), (7.4), consists of a density wave that will emerge behind the shock discontinuity, and the fine structure of this density wave

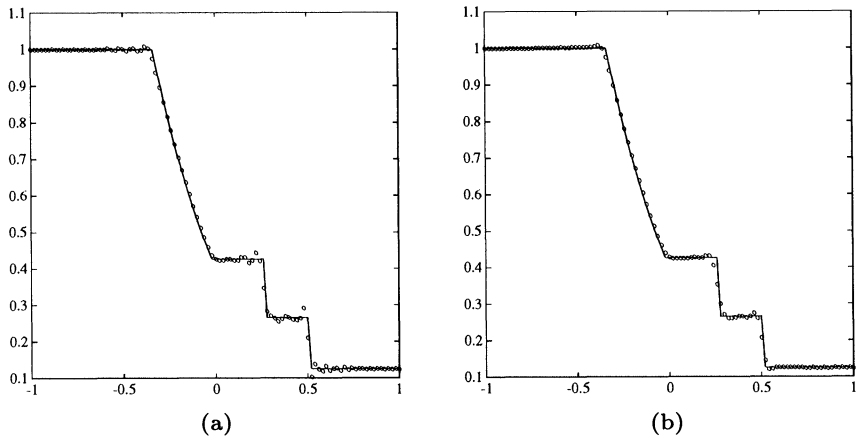


FIG. 7.1. Density ρ_N with $N = 128$ Legendre modes (a) before post-processing, (b) after post-processing.

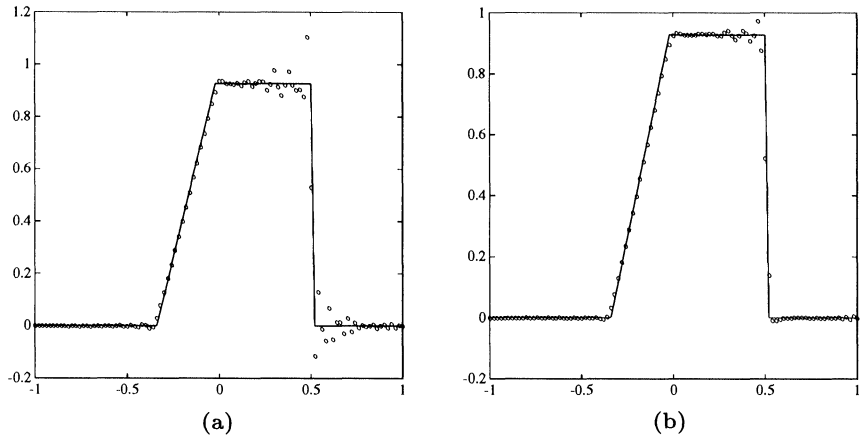


FIG. 7.2. Velocity v_N with $N = 128$ Legendre modes (a) before post-processing, (b) after post-processing.

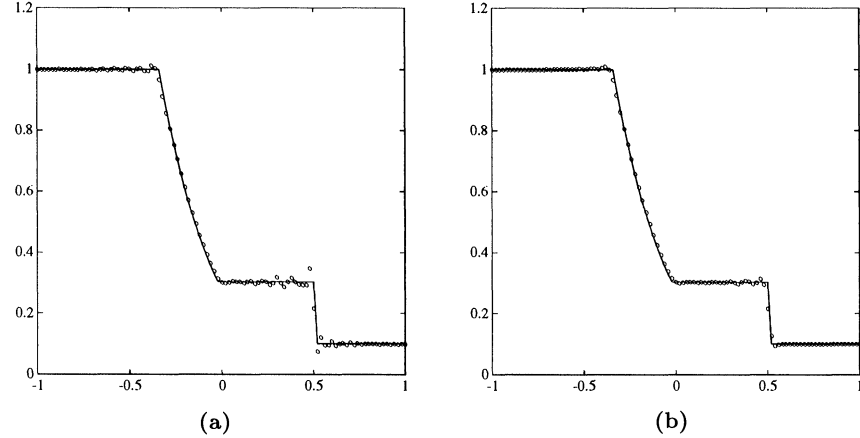


FIG. 7.3. Pressure p_N with $N = 128$ Legendre modes (a) before post-processing, (b) after post-processing.

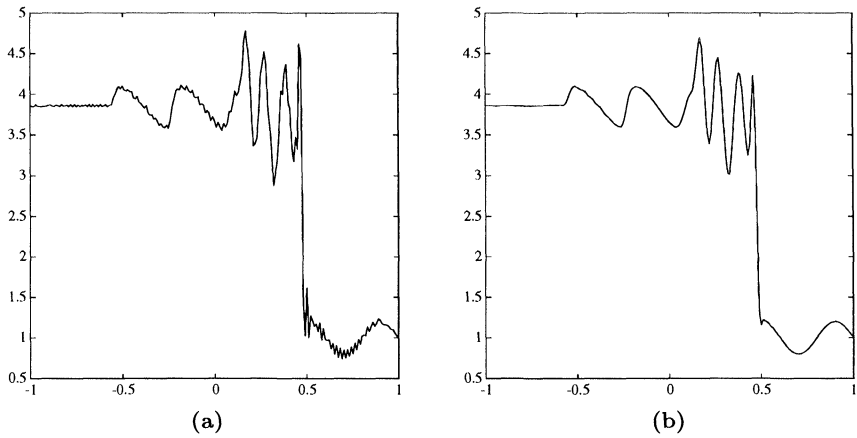


FIG. 7.4. Density ρ_N with $N = 220$ Legendre modes (a) before post-processing, (b) after post-processing.

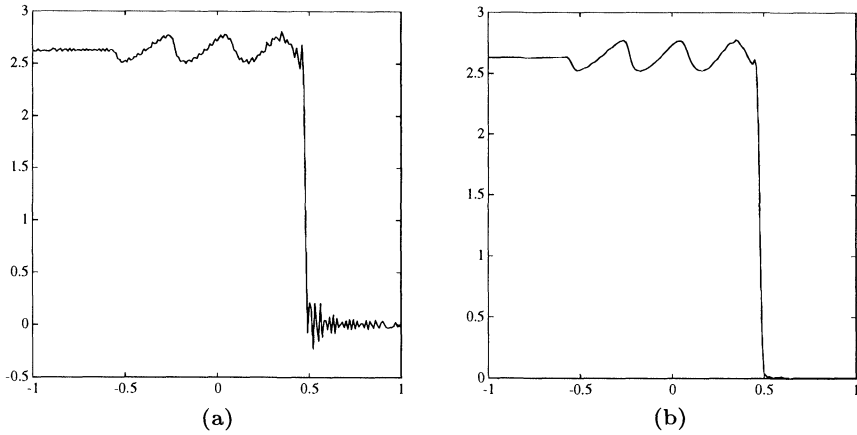


FIG. 7.5. Velocity v_N with $N = 220$ Legendre modes (a) before post-processing, (b) after post-processing.

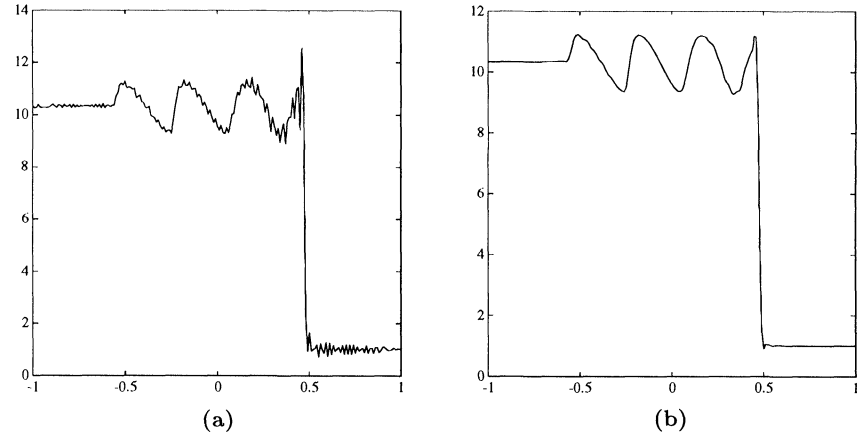


FIG. 7.6. Pressure p_N with $N = 220$ Legendre modes (a) before post-processing, (b) after post-processing.

makes the current problem a suitable test case for high-order methods. For example, second-order MUSCL-type schemes [Le] are unable to resolve the fine structure of the density wave unless the number of grid points is substantially increased

The Legendre SV method was implemented in this case with SV parameters $(\varepsilon_N, m_N) = (\frac{1}{N}, 8\sqrt{N})$. Figures 7.4–7.6 display the numerical results of the SV approximation which was integrated in time by the second-order Adams–Bashforth method with time step $\Delta t = 2.5 \cdot 10^{-6}$.

Figures 7.4(a), 7.5(a), and 7.6(a) show the approximated density ρ_N , velocity v_N , and pressure p_N at $t = 0.36$, computed with $N = 220$ Legendre modes. These results were post-processed by the filtering procedure (2.5), $F^{\alpha,\beta}$, with $(\alpha, \beta) = (0.1, 0.89)$. Figures 7.4(b), 7.5(b), and 7.6(b) present the post-processed results, which show that the velocity and pressure waves are well resolved. The density wave still contains Gibbs' oscillations in the neighborhood of the shock discontinuity, and its first extremum behind the shock is smeared by our smoothing filter. Here, a "one-sided" filter would be recommended instead. A better resolution of the density profile near the shock was obtained by a different spectral method presented in [CGH]. However, the latter is a shock fitting like method which might not be easy to extend to higher dimensions.

8. Concluding remarks. The numerical experiments reported in the last two sections show that SV may serve as a robust and *easily implemented* "fix" to the otherwise unstable Legendre method. The versatility and simplicity of the SV method is particularly relevant for multidimensional problems which could be easily handled *without* dimensional splitting. At the same time, the SV method retains spectral resolution, which could be realized by post-processing the SV solution with appropriate smoothing filter. With this in mind, it would be desirable to carry out further analytical and numerical work in order to explore optimal SV parameterizations and appropriate one-sided filtering procedures.

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