Part-A

The following notation is used throughout the question paper.

- $\mathbb{C}$ denotes the set of complex numbers.
- $\mathbb{R}$ denotes the set of real numbers.
- $|S|$ denotes the cardinality of a set S.

1. Suppose there are three types of people in the world.

    - A person is “honest” if the person always speaks the truth.
    - A person is a “liar” if the person always lies.
    - A person is “normal” if the person sometimes speaks the truth and sometimes lies.

    In a city, a crime has been committed. `A`, `B` and `C` are the three suspects. It is known from reliable sources that among `A`, `B` and `C`, only one person is honest and that person has committed the crime. The following is a conversation between the three suspects.

    `A`: I did not commit the crime.

    `B`: Yes, `A` did not commit the crime.

    `C`: `B` is not normal.

    Based on the above, deduce with appropriate justifications who among `A`, `B` and `C` has committed the crime. [5]

2. Recall that $\mathbb{C}$ denotes the set of complex numbers. A set $S \subseteq \mathbb{C}$ is called *engulfing* if either $|S| = 1$ or S satisfies the following:

   - $|S^2| = |S|/2$
   - S^2 is also engulfing,

   where

   $$S^2 = \{x^2 \mid x \in S\}.$$

   (a) Find an engulfing set of cardinality 4.

   (b) Find an engulfing set of cardinality ≥ 6.

   [4 + 4]

3. A *minimum vertex cover* of a graph is the smallest set of vertices that includes at least one endpoint of every edge of the graph. An algorithm claims to compute a minimum vertex cover of a graph by repeatedly choosing a vertex of maximum degree and then removing it from the graph. The algorithm stops as soon as a vertex cover is obtained. Construct a graph for which this algorithm will fail to return a minimum vertex cover. [7]

Part-B

CS Group

1. Consider two binary search trees (BSTs) named T_1 and T_2, rooted at $Root_1$ and $Root_2$, respectively. Both T_1 and T_2 are AVL (Adelson-Velsky-Landis) trees with $M(> 0)$ and $N(> 0)$ nodes, respectively. Every node of T_1 and T_2 has three fields: value, left-link and right-link. T_1 and T_2 have no common value.

   Define *merging* of a BST T_2 with a BST T_1 as a process that combines the two BSTs to form a single BST by attaching the root of T_2 to a left- or right-link of T_1 that is free. Thus, the merge operation excludes one by one insertion of the nodes of one BST into the other.

   (a) Create two AVL trees T_A (with 6 integer values) and T_B (with 7 integer values), such that T_B cannot be merged with T_A as per the above definition. Note that T_A and T_B must not have any common value.

   (b) Design an algorithm that takes two AVL trees T_1 and T_2 as input and merges T_2 with T_1, if possible, otherwise reports an error. Explain your algorithm. Deduce the time complexity of your algorithm, and also find the maximum possible height of the merged tree in terms of the heights of T_1 and T_2.

   $[4 + (4 + 2)]$

2. Consider an array X of n distinct elements such that

   $$X[0] < X[1] < \cdots < X[i-1] < X[i] > X[i+1] > \cdots > X[n-1].$$

   Suggest a linear time algorithm to sort the elements in X.

   [10]

3. Let A be an $n \times n$ matrix with entries $a_{ij} = \pm 1$ for all $1 \leq i, j \leq n$. Prove that 2^{n-1} divides the determinant of A. [10]

4. For a graph, a *proper vertex colouring* assigns a colour to each vertex such that no two adjacent vertices have the same colour. A vertex colouring is *optimal* if it is proper and uses the minimum number of colours.

   Consider a graph G with an optimal vertex colouring. Show that in G, for each colour, there exists a vertex of that colour which has vertices of every other colour adjacent to it. [10]

5. Prime numbers p and q are said to be twin primes if $q = p + 2$. Consider the following language

   $$L := \{1^n \mid \text{there exist twin primes } p \text{ and } q \text{ with } q \geq p \geq n\}.$$

   Prove that L is regular irrespective of whether the number of twin primes is finite or infinite. [10]

6. For the gate T shown in Fig. A, the truth-table of the function $\mathsf{f(A, B, C)}$ is given in Fig. B.

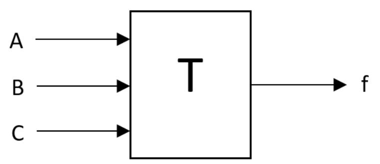


Fig. A

C \ AB	00	01	11	10
0	0	0	1	0
1	0	1	0	1

Fig. B

   If you are allowed to set any of the inputs to 1, show that the gate T is functionally complete (i.e., any 2-input Boolean function can be implemented with it). [10]

7. You are given a processor having a single-level cache hierarchy with only an L1 cache that has a 95% hit rate. You have to re-design the cache hierarchy without increasing the Effective Memory Access Time (EMAT), using a smaller and cheaper L1 cache with a lower hit rate of 0.9, and an L2 cache.

   (a) Show that the access time of the L2 cache should be no greater than 1/2 of the main memory.

   (b) Determine the minimum hit rate of the L2 cache to achieve the required EMAT?

   (c) If the actual access time of the L2 cache is 1/10 of the main memory access time, then determine the minimum hit rate of the L2 cache to achieve the required EMAT?

[10]

8. Consider a bit stuffing framing method where both the start and the end of a frame are indicated by the flag 01^k0 where 1^k denotes k consecutive ones.

   Recall that for a flag 01^k0, stuffing a 0 bit is required whenever 01^{k-1} appears in the original data stream. On the other hand, if 0 is preceded by 01^{k-1} in the transmitted bit stream, destuffing the 0 bit is required at the receiver.

   (a) Is it necessary to stuff a 0 bit in $01^{k-1}0$? Justify your answer.

   (b) In the worst case, how many bits need to be stuffed for a data packet of length L?

[$5 + 5$]

9. Consider the following information about a magnetic hard disk, with cylinders numbered 0 (innermost) to 511 (outermost):

- at time $t = 0$ ms, the disk head is at cylinder **120**, moving outwards towards cylinder 511;
- the disk arm takes **1** ms to seek from cylinder i to cylinder $i + 1$ or cylinder $i - 1$;
- the time taken to read sectors may be neglected.

The following I/O requests arrive at the disk controller.

Request no.	**Arrival time (ms)**	**Cylinder no.**
r_1	1	480
r_2	10	240
r_3	15	360
r_4	20	100
r_5	25	140
r_6	30	185

(a) In what order will the six requests be serviced if the controller uses the following scheduling algorithms?

i. LOOK

ii. **preemptive** SHORTEST SEEK TIME FIRST (SSTF)

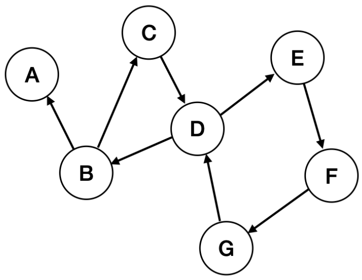
The standard definitions of these algorithms are given towards the end of this question.

(b) Let $T_{LOOK}(S)$ and $T_{SSTF}(S)$ denote the total time taken to service a given sequence S of requests using LOOK and SSTF, respectively. Two requests arrive at time $t = 0$ for cylinder nos. x_1 and x_2. Show that, for any $x_1, x_2 \in [0, 511]$, $T_{LOOK}(S) \geq T_{SSTF}(S)$ for $S = x_1, x_2$.

In LOOK, the disk head alternatively moves in the outward and inward directions servicing any currently pending requests as it reaches each cylinder. While travelling in a particular direction, the arm goes only as far as the last request currently pending *in that direction*, before reversing direction.

In **preemptive** SHORTEST SEEK TIME FIRST (SSTF), the disc controller runs the scheduling algorithm whenever a new I/O request arrives. $[(1 + 4) + 5]$

10. A directed graph can be represented by its edge-list and stored in a database table with attributes (`source`, `target`). For example, the following table stores the edge-list of the graph shown in the picture.



source	target
B	A
B	C
C	D
D	B
D	E
E	F
F	G
G	D

We define a triangle to be a directed cycle of length 3. Note that in the graph G, (B, C, D) is a triangle, whereas (D, E, F, G) is a cycle but not a triangle.

Write a relational algebra expression or an SQL query to output all the triangles in a directed graph G that is represented by its edge-list and stored in a table `G_edges(source, target)`. Briefly justify that your answer is correct. [10]

Non-CS Group

1. Let A be an $n \times n$ matrix with entries $a_{ij} = \pm 1$ for all $1 \leq i, j \leq n$. Prove that 2^{n-1} divides the determinant of A. [10]

2. For a graph, a *proper vertex colouring* assigns a colour to each vertex such that no two adjacent vertices have the same colour. A vertex colouring is *optimal* if it is proper and uses the minimum number of colours.

   Consider a graph G with an optimal vertex colouring. Show that in G, for each colour, there exists a vertex of that colour which has vertices of every other colour adjacent to it. [10]

3. A pair of prime numbers p and q are said to be twins if $q = p+2$. Consider the following language

   $$L := \{1^n \mid \text{there exists twin primes } p \text{ and } q \text{ with } q \geq p \geq n\}.$$

   Prove that L is regular irrespective of whether there are infinite or finite number of twin primes. [10]

4. W_1 and W_2 are non-zero, non-trivial, distinct subspaces of a vector space V (of dimension n) over $\mathbb{C}$.

   (a) Prove that there exists a non-zero element $\alpha \in V$ which is in neither of the subspaces W_1 and W_2.

   (b) Prove that there is a basis $\{b_1, \ldots, b_n\}$ of V, such that $b_i \notin W_1 \cup W_2$ for $i = 1, \ldots, n$. [4 + 6]

5. A person **A** rolls a fair die until the outcome is 6. After that, **B** rolls the die as many times as **A** had rolled it. Compute the probability that **B** will get exactly one 6. [10]

6. Find all values of $a \in \mathbb{C}$ for which the equation

$$x^4 + ax^2 + a^2x - 1 = 0$$

has all roots of the same absolute value. [10]

7. An oil-exploration company is considering 10 possible drilling sites $S_1, S_2, \ldots, S_{10}$. The drilling costs associated with these sites are $C_1, C_2, \ldots, C_{10}$ respectively. Due to land topography, there are some additional constraints as follows.

   - Drilling at sites S_1 and S_7 would prevent the company from drilling at site S_8.
   - Drilling at site S_3 or S_4 prevents the company from drilling at site S_5.
   - Out of the drilling sites S_5, S_6, S_7, S_8, only two sites may be drilled.

   Your job is to help the company select 5 out of 10 sites for drilling such that the above constraints are satisfied and the total drilling cost is minimised. Formulate the problem as an Integer Linear Program (ILP). [10]

8. Consider the sequence a_n, defined as

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{4}{a_n}\right).$$

   Prove that $\lim_{n\to\infty} a_n$ exists. Compute the value of the limit. [6 + 4]

9. Consider the set of points $\{(x, y, z) \mid x, y, z \in \{0, 1, \ldots, n\}\}$. Determine how many of these points lie on the Euclidean plane that contains the points $(n, 0, 0)$, $(0, n, 0)$ and $(0, 0, n)$. [10]

10. Given a permutation σ of $\{1, 2, \dots, n\}$, we define the symmetric binary matrix A^σ as follows:

$$A^\sigma_{ij} = \begin{cases} 1 & \text{if } (i-j)(\sigma(i) - \sigma(j)) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Is every binary symmetric $n \times n$ matrix of the form A^σ for some permutation σ of $\{1, 2, \dots, n\}$? Justify your answer. [10]