

# **Computer and Systems Sciences**

## **CLARIFICATIONS**

### **Page 13, Problem 2**

The first sentence should read as follows.

Let  $X_1$  and  $X_2$  be two independent and identically distributed real-valued random variables taking values in the unit interval  $[0, 1]$ .

### **Page 15, after Problem 6**

Please ignore the blank space on the page. Problem 7 appears on page 16.

## Part A: Common Part

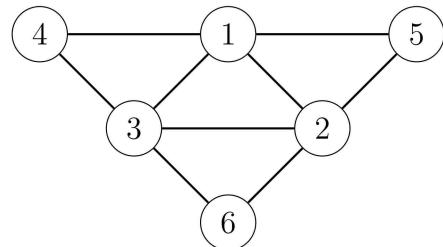
- 1.** Suppose the following statements about three persons in a room are true.

*Chandni, Sooraj and Tara are in a room. Nobody else is in the room. Chandni is looking at Sooraj. Sooraj is looking at Tara. Chandni is married. Tara is not married. A married person in the room is looking at an unmarried person.*

Then, which of the following is necessarily true?

- (a) Sooraj is married
- (b) Sooraj is unmarried
- (c) The situation described is impossible
- (d) There is insufficient information to conclude if Sooraj is married or unmarried ✓
- (e) None of the above

- 2.** Consider the graph shown below.



The following experiment is performed using this graph. First, an edge  $e = \{i, j\}$  of the graph is chosen uniformly at random from the set of nine possibilities. Next, a common neighbour  $k$  of  $i$  and  $j$  is chosen, again uniformly from the set of possibilities. (Note that the set of possibilities is always non-empty.) Thus,  $\{i, j, k\}$  is a triangle in the graph. What is the probability that the triangle finally picked is  $\{1, 2, 3\}$ ?

- (a)  $\frac{1}{6}$  ✓
- (b)  $\frac{1}{4}$
- (c)  $\frac{1}{3}$
- (d)  $\frac{2}{3}$
- (e)  $\frac{5}{6}$

- 3.** Consider the following set of  $3n$  linear equations in  $3n$  variables:

$$\begin{array}{llll} x_1 - x_2 = 0 & x_4 - x_5 = 0 & \cdots & x_{3n-2} - x_{3n-1} = 0 \\ x_2 - x_3 = 0 & x_5 - x_6 = 0 & & x_{3n-1} - x_{3n} = 0 \\ x_1 - x_3 = 0 & x_4 - x_6 = 0 & & x_{3n-2} - x_{3n} = 0 \end{array}$$

Let  $S \subseteq \mathbb{R}^{3n}$  be the set of solutions to this set of equations. Then,

- (a)  $S$  is empty
  - (b)  $S$  is a subspace of  $\mathbb{R}^{3n}$  of dimension 1
  - (c)  $S$  is a subspace of  $\mathbb{R}^{3n}$  of dimension  $n$**  ✓
  - (d)  $S$  is a subspace of  $\mathbb{R}^{3n}$  of dimension  $n - 1$
  - (e)  $S$  has exactly  $n$  elements
- 4.** There are  $n$  balls  $b_1, \dots, b_n$  and  $n$  boxes. Each ball is placed in a box chosen independently and uniformly at random. We say that  $(b_i, b_j)$  is a *colliding pair* if  $i < j$ , and  $b_i$  and  $b_j$  are placed in the same box. What is the expected number of colliding pairs?

- (a)  $\frac{n-1}{2}$**  ✓
- (b) 0
- (c) 1
- (d)  $n/4$
- (e)  $\binom{n}{2}$

- 5.** For a positive integer  $N \geq 2$ , let

$$\begin{aligned} A_N &:= \sum_{n=2}^N \frac{1}{n}; \\ B_N &:= \int_{x=1}^N \frac{1}{x} dx. \end{aligned}$$

Which of the following statements is true?

- (a) As  $N \rightarrow \infty$ ,  $A_N$  increases to infinity but  $B_N$  converges to a finite number
- (b)  $A_N < B_N$  and the difference decreases as  $N \rightarrow \infty$
- (c)  $A_N < B_N < A_N + 1$**  ✓
- (d)  $B_N < A_N < B_N + 1$
- (e) As  $N \rightarrow \infty$ ,  $B_N$  increases to infinity but  $A_N$  converges to a finite number

6. Which of the following statements about the eigen values of  $I_n$ , the  $n \times n$  identity matrix (over complex numbers), is true?
- (a) The eigen values are  $1, \omega, \omega^2, \dots, \omega^{n-1}$ , where  $\omega$  is a primitive  $n$ -th root of unity
  - (b) The only eigen value is  $-1$
  - (c) Both 0 and 1 are eigen values, but there are no other eigen values
  - (d) The eigen values are  $1, 1/2, 1/3, \dots, 1/n$
  - (e) **The only eigen value is 1 ✓**
7. Let  $S$  be the  $4 \times 4$  square grid  $\{(x, y) : x, y \in \{0, 1, 2, 3\}\}$ . A *monotone path* in this grid starts at  $(0, 0)$  and at each step either moves one unit up or one unit right. For example, from the point  $(x, y)$  one can in one step either move to  $(x + 1, y) \in S$  or  $(x, y + 1) \in S$ , but never leave  $S$ . Let the number of distinct monotone paths to reach point  $(2, 2)$  starting from  $(0, 0)$  be  $z$ . How many distinct monotone paths are there to reach point  $(3, 3)$  starting from  $(0, 0)$ ?
- (a)  $2z + 6$
  - (b)  $3z + 6$
  - (c)  **$2z + 8$  ✓**
  - (d)  $3z + 8$
  - (e)  $3z + 4$
8. Let  $A$  and  $B$  be finite sets such that  $A \subseteq B$ . Then, what is the value of the expression:
- $$\sum_{C:A \subseteq C \subseteq B} (-1)^{|C \setminus A|},$$
- where  $C \setminus A = \{x \in C : x \notin A\}$ ?
- (a) Always 0
  - (b) Always 1
  - (c) 0 if  $A = B$  and 1 otherwise.
  - (d) **1 if  $A = B$  and 0 otherwise ✓**
  - (e) Depends on the size of the universe

9. Suppose a rectangular farm has area 100 square metres. The lengths of its sides are not known. It is known, however, that all the edges are at least 2 metres in length. Which of the following statements about the rectangle's perimeter  $p$  (in metres) is FALSE?
- (a)  $p$  can take all values between 45 and 50
  - (b)  $p$  can be 52 for some configuration
  - (c)  $p$  can take all values between 55 and 60
  - (d)  $p$  can be 70 for some configuration
  - (e)  **$p$  can be 39 for some configuration ✓**
10. Consider the sequence  $\langle s_n : n \geq 0 \rangle$  defined as follows:  $s_0 = 0$ ,  $s_1 = 1$ ,  $s_2 = 1$ , and  $s_n = s_{n-1} + s_{n-2} + s_{n-3}$ , for  $n \geq 3$ . Which of the following statements is FALSE?
- (a)  $s_{4k}$  is even, for any  $k \geq 0$
  - (b)  $s_{4k+1}$  is odd, for any  $k \geq 0$
  - (c)  $s_{4k+2}$  is odd, for any  $k \geq 0$
  - (d)  **$s_n$  is a multiple of 3, for only finitely many values of  $n$  ✓**
  - (e)  $s_{4k+3}$  is even, for any  $k \geq 0$
11. In one of the islands that his travels took him to, Gulliver noticed that the probability that a (uniformly) randomly chosen inhabitant has height at least 2 metres is 0.2. Also, 0.2 is the probability that a (uniformly) randomly chosen inhabitant has height at most 1.5 metres. What can we conclude about the average height  $h$  in metres of the inhabitants of the island?
- (i)  $1.5 \leq h \leq 2$
  - (ii)  $h \geq 1.3$
  - (iii)  $h \leq 2.2$
- Which of the above statements is necessarily true?
- (a) **(ii) only ✓**
  - (b) (iii) only
  - (c) (i), (ii), and (iii)
  - (d) (ii) and (iii) only
  - (e) None of the above

**12.** There are two rocks  $A$  and  $B$ , located close to each other, in a lily pond. There is a frog that jumps randomly between the two rocks at time  $t = 0, 1, 2, \dots$ . The location of the frog is determined as follows. Initially, at time  $t = 0$ , the frog is at  $A$ . From then on, the frog's location is determined as follows. If the frog is at  $A$  at time  $t$ , then at time  $t + 1$ , with probability  $2/3$  it jumps to  $B$  and with probability  $1/3$ , it jumps on the spot and stays at  $A$ . If the frog is at  $B$  at time  $t$ , then at time  $t + 1$ , with probability  $1/2$  it jumps to  $A$  and with probability  $1/2$  it jumps on the spot and stays at  $B$ . What is the probability that the frog is at  $B$  at time 3 (just after its third jump)?

- (a)  $\frac{1}{2}$     (b)  $\frac{31}{54}$  ✓    (c)  $\frac{14}{27}$     (d)  $\frac{61}{108}$     (e)  $\frac{2}{3}$ .

**13.** Let  $n \geq 2$  be any integer. Which of the following statements is not necessarily true?

- (a)  $\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1}$ , where  $1 \leq i \leq n - 1$   
 (b)  $n!$  divides the product of any  $n$  consecutive integers  
 (c)  $\sum_{i=0}^n \binom{n}{i} = 2^n$   
**(d)**  $n$  divides  $\binom{n}{i}$ , for all  $i \in \{1, 2, \dots, n - 1\}$  ✓  
 (e) If  $n$  is an odd prime, then  $n$  divides  $2^{n-1} - 1$

**14.** A *diagonal* in a polygon is a straight line segment that connects two non-adjacent vertices, and is contained in the interior of the polygon (except for its end points). Two such diagonals are said to cross if they have a point in common in the interior of the polygon. In one such polygon with  $n$  vertices, a certain number (say  $k$ ) of non-crossing diagonals were drawn to cut up the inside of the polygon into regions, each of which was a quadrilateral. How many diagonals were drawn, that is, what is  $k$ ?

- (a) Cannot be determined from the information given  
**(b)**  $\frac{n}{2} - 2$  ✓  
 (c)  $\frac{n}{4} - 1$   
 (d)  $n - 4$   
 (e)  $n^2 - 9.5n + 22$

15. In a tournament with 7 teams, each team plays one match with every other team. For each match, the team earns two points if it wins, one point if it ties, and no points if it loses. At the end of all matches, the teams are ordered in the descending order of their total points (the order among the teams with the same total are determined by a whimsical tournament referee). The first three teams in this ordering are then chosen to play in the next round. What is the minimum total number of points a team must earn in order to be guaranteed a place in the next round?
- (a) 13    (b) 12    (c) 11    **(d) 10 ✓**    (e) 9

## Part B: Computer Science

1. A Boolean formula is said to be a *tautology* if it evaluates to TRUE for all assignments to its variables. Which one of the following is NOT a tautology?

- (a)  $((p \vee q) \wedge (r \vee s)) \Rightarrow ((p \wedge r) \vee q \vee s)$
- (b)**  $((p \vee q) \wedge (r \vee s)) \Rightarrow (q \vee s)$  ✓
- (c)  $((p \vee q) \wedge (r \vee s)) \Rightarrow (r \vee q \vee s)$
- (d)  $((p \vee q) \wedge (r \vee s)) \Rightarrow (p \vee q \vee s)$
- (e)  $((p \vee q) \wedge (r \vee s)) \Rightarrow (p \vee q)$

2. Which language class has the following properties?

It is closed under union and intersection but not complement.

- (a) Regular language
- (b) Context-free languages
- (c) Recursive languages
- (d) Recursively enumerable languages** ✓
- (e) Languages that are not recursively enumerable

3. Assume  $P \neq NP$ . Which of the following is not TRUE?

- (a) 2-SAT is in NP
- (b) 2-SAT is in coNP
- (c) 3-SAT is polynomial-time reducible to 2-SAT** ✓
- (d) 4-SAT is polynomial-time reducible to 3-SAT
- (e) 2-SAT is in P

4. In the following,  $A$  stands for a set of apples, and  $S(x, y)$  stands for “ $x$  is sweeter than  $y$ ”. Let

$$\begin{aligned}\Psi &\equiv \exists x : x \in A \\ \Phi &\equiv \forall x \in A : \exists y \in A : S(x, y).\end{aligned}$$

Which of the following statements implies that there are infinitely many apples (i.e.,  $A$  is an infinite set)?

- (a)  $\Psi \wedge \Phi \wedge [\forall x \in A : \neg S(x, x)]$
  - (b)  $\Psi \wedge \Phi \wedge [\forall x \in A : S(x, x)]$
  - (c)  $\Psi \wedge \Phi \wedge [\forall x, y \in A : S(x, x) \wedge S(x, y) \rightarrow S(y, y)]$
  - (d)  $\Psi \wedge \Phi \wedge [\forall x \in A : \neg S(x, x)] \wedge [\forall x, y, z \in A : S(x, y) \wedge S(y, z) \rightarrow S(y, x)]$
  - (e)  $\Psi \wedge \Phi \wedge [\forall x \in A : \neg S(x, x)] \wedge [\forall x, y, z \in A : S(x, y) \wedge S(y, z) \rightarrow S(x, z)] \checkmark$
5. Consider the following recursive function `mc91`.

```
int mc91(int n)
{
    print n
    if (n > 100) {
        return n - 10;
    }
    else {
        return mc91(mc91(n+11));
    }
}
```

Let

$\text{Out} = \{n : \text{there is an } x \in \{0, 1, \dots, 100\} \text{ such that } n \text{ is one of the integers printed by } \text{mc91}(x)\}.$

Then, which of the following is  $\text{Out}$ ?

- (a)  $\{n : -\infty < n \leq 100\}$
- (b)  $\{n : 0 \leq n \leq 101\}$
- (c)  $\{n : 0 \leq n \leq 110\}$
- (d)  $\{n : 0 \leq n \leq 111\} \checkmark$
- (e)  $\{n : 0 \leq n < +\infty\}$

**6.** A subset  $X$  of  $\mathbb{R}^n$  is convex if for all  $x, y \in X$  and all  $\lambda \in (0, 1)$ , we have  $\lambda x + (1 - \lambda)y \in X$ . If  $X$  is a convex set, which of the following statements is necessarily TRUE?

- (a) For every  $x \in X$ , there exist  $y, z \in X - \{x\}$  and  $\lambda \in (0, 1)$  so that  $x = \lambda y + (1 - \lambda)z$
- (b) If  $x, y \in X$  and  $\lambda \geq 0$ , then  $\lambda x + (1 - \lambda)y \in X$
- (c) **If**  $x_1, \dots, x_n \in X$  ( $n \geq 1$ ), **then**  $(x_1 + \dots + x_n)/n \in X$  ✓
- (d) If  $x \in X$ , then  $\lambda x \in X$  for all scalars  $\lambda$
- (e) If  $x, y \in X$ , then  $x - y \in X$

**7.** Let  $n = m!$ . Which of the following is TRUE?

- (a)  $m = \Theta(\log n / \log \log n)$  ✓
- (b)  $m = \Omega(\log n / \log \log n)$  but not  $m = O(\log n / \log \log n)$
- (c)  $m = \Theta(\log^2 n)$
- (d)  $m = \Omega(\log^2 n)$  but not  $m = O(\log^2 n)$
- (e)  $m = \Theta(\log^{1.5} n)$

**8.** Consider the following language

$$\text{PRIMES} = \left\{ \underbrace{111 \cdots 11}_{p \text{ times}} : p \text{ is prime} \right\}.$$

Then, which of the following is TRUE?

- (a) PRIMES is regular
- (b) PRIMES is undecidable
- (c) PRIMES is decidable in polynomial time ✓
- (d) PRIMES context free but not regular
- (e) PRIMES is NP-complete and  $P \neq NP$

9. Which of the following graphs DOES NOT have an Eulerian circuit? (Recall that an Eulerian circuit in an undirected graph is a walk in the graph that starts at a vertex and returns to the vertex after travelling on each edge exactly once.)

- (a)  $K_{9,9}$  ✓
- (b)  $K_{8,8}$
- (c)  $K_{12,12}$
- (d)  $K_9$
- (e) The graph  $G$  on vertex set  $\{1, 2, \dots, 9\}$  with edge set

$$E(G) = \{\{i, j\} : 1 \leq i < j \leq 5 \text{ or } 5 \leq i < j \leq 9\}.$$

10. A *vertex cover* in an undirected graph  $G$  is a subset  $C \subseteq V(G)$  such that every edge of  $G$  has an endpoint in  $C$ . An independent set in  $G$  is a subset  $I \subseteq V(G)$  such that no edge has both its endpoints in  $I$ . Which of the following is TRUE of every graph  $G$  and every vertex cover  $C$  of  $G$ ?

- (a) There exists an independent set of size  $|C|$
- (b)  $V(G) - C$  is an independent set ✓**
- (c)  $|C| \geq |E(G)|/2$
- (d)  $|C| \geq |V(G)|/2$
- (e)  $C$  intersects every independent set

11. Let  $n \geq 4$  be an integer. Regard the set  $\mathbb{R}^n$  as a vector space over  $\mathbb{R}$ . Consider the following undirected graph  $H$ .

$$\begin{aligned} V(H) &= \{S \subseteq \mathbb{R}^n : S \text{ is a basis for } \mathbb{R}^n\}; \\ E(H) &= \{\{S, T\} : |S \setminus T| = 1 \text{ and } |T \setminus S| = 1\}, \end{aligned}$$

where  $S \setminus T = \{x \in S : x \notin T\}$ . Which of the following statements is FALSE?

- (a)  $H$  has an infinite number of vertices
- (b) The diameter of  $H$  is infinite ✓**
- (c)  $H$  is connected
- (d)  $H$  contains an infinite clique
- (e)  $H$  contains an infinite independent set

12. A computer program computes a function  $f : \{0, 1\}^* \times \{0, 1\}^* \rightarrow \{0, 1\}^*$ . Suppose  $f(a, b)$  has length  $|b|^2$ , where  $|a|$  and  $|b|$  are the lengths of  $a$  and  $b$ . Suppose, using this program, the following computation is performed.

```
x = "01"
for i = 1, . . . , n do
    x = f("01", x)
```

Suppose at the end, the length of the string  $x$  is  $t$ . Which of the following is TRUE (assume  $n \geq 10$ )?

- (a)  $t \leq 2n$
- (b)  $n < t \leq n^2$
- (c)  $n^2 < t \leq n^{\log_2 n}$
- (d)  $n^{\log_2 n} < t \leq 2^{(2^n)}$  ✓**
- (e)  $2^{(2^n)} < t$

13. An undirected graph  $G = (V, E)$  is said to be  $k$ -colourable if there exists a mapping  $c : V \rightarrow \{1, 2, \dots, k\}$  such that for every edge  $\{u, v\} \in E$  we have  $c(u) \neq c(v)$ . Which of the following statements is FALSE?

- (a)  $G$  is  $|V|$ -colourable
- (b)  $G$  is 2-colourable iff there are no odd cycles in  $G$
- (c)  $G$  is  $(\Delta + 1)$ -colourable where  $\Delta$  is the maximum degree in  $G$
- (d) There is a polynomial time algorithm to check if  $G$  is 2-colourable
- (e) If  $G$  has no triangle then it is 3-colourable ✓**

14. Consider a family  $\mathcal{F}$  of subsets of  $\{1, 2, \dots, n\}$  such that for any two distinct sets  $A$  and  $B$  in  $\mathcal{F}$  we have:  $A \subset B$  or  $B \subset A$  or  $A \cap B = \emptyset$ . Which of the following statements is TRUE? (Hint: what does the Venn diagram of this family look like?)

- (a)  $|\mathcal{F}| \leq 2n$  and there exists a family  $\mathcal{F}$  such that  $|\mathcal{F}| = 2n$  ✓**
- (b)  $|\mathcal{F}| \leq n^2$  and there exists a family  $\mathcal{F}$  such that  $|\mathcal{F}| = n^2$
- (c)  $|\mathcal{F}| \leq 2n^2$  and there exists a family  $\mathcal{F}$  such that  $|\mathcal{F}| = 2n^2$
- (d)  $|\mathcal{F}| \leq 2^{n-1}$  and there exists a family  $\mathcal{F}$  such that  $|\mathcal{F}| = 2^{n-1}$
- (e) None of the above

15. Let  $G$  be an undirected graph. For a pair  $(x, y)$  of distinct vertices of  $G$ , let  $\text{mincut}(x, y)$  be the least number of edges that should be deleted from  $G$  so that the resulting graph has no  $x-y$  path. Let  $a, b, c$  be three vertices in  $G$  such that  $\text{mincut}(a, b) \leq \text{mincut}(b, c) \leq \text{mincut}(c, a)$ . Consider the following possibilities:

- (i)  $\text{mincut}(a, b) < \text{mincut}(b, c) < \text{mincut}(c, a)$
- (ii)  $\text{mincut}(a, b) = \text{mincut}(b, c) < \text{mincut}(c, a)$
- (iii)  $\text{mincut}(a, b) < \text{mincut}(b, c) = \text{mincut}(c, a)$
- (iv)  $\text{mincut}(a, b) = \text{mincut}(b, c) = \text{mincut}(c, a)$

Which of the following is TRUE?

- (a) All of (i), (ii), (iii), (iv) are possible
- (b) (i), (ii), (iii) are possible but not (iv)
- (c) (i) and (iv) are possible but neither (ii) nor (iii)
- (d) (ii) and (iv) are possible but neither (i) nor (iii) ✓**
- (e) (iii) and (iv) are possible but neither (i) nor (ii)

## Part C: Systems Science

1. Suppose  $f(x) = cx^{-\alpha}$  for some  $c > 0$  and  $\alpha > 0$  such that  $\int_1^\infty f(x)dx = 1$ . Then, which of the following is possible?
  - (a)  $\int_1^\infty xf(x)dx = \infty$  ✓
  - (b)  $\int_1^\infty \frac{f(x)}{1+x}dx = \infty$
  - (c)  $\int_1^\infty (\ln x)f(x)dx = \infty$
  - (d)  $\int_1^\infty \frac{f(x)}{1+\ln x}dx = \infty$
  - (e) None of the above
  
2. Let  $X_1$  and  $X_2$  be two *independent* continuous real-valued random variables taking values in the unit interval  $[0, 1]$ . Let  $Y = \max\{X_1, X_2\}$  and  $Z = \begin{cases} 1 & \text{if } X_1 = Y \\ 2 & \text{otherwise} \end{cases}$ . Which of the following is true?
  - (a)  $\Pr[Z = 1] = \Pr[Z = 1 \mid Y \geq 0.3]$  ✓
  - (b)  $\Pr[Z = 1] = \Pr[Z = 1 \mid Y = 0.3, X_1 = 0.2]$
  - (c)  $\Pr[Z = 1] = \Pr[Z = 1 \mid Y = 0.3, X_1 = 0.3, X_2 = 0.2]$
  - (d)  $\Pr[Z = 1] > \Pr[Z = 2] = \frac{1}{2}$
  - (e)  $\Pr[Z = 1] < \Pr[Z = 2]$
  
3. Let  $(X, Y)$  be a pair of independent random variables. Suppose  $X$  takes values in  $\{1, \dots, 6\}$  with equal probability, and  $Y$  takes values in  $\{2, 3\}$  with  $\Pr[Y = 2] = p$ . Let  $Z = (X \bmod Y) + 1$ .
 

Which of the statements is true?

  - (a)  $\Pr[Z = 1] = \frac{2}{5}$  **for some value of  $p$**  ✓
  - (b)  $\Pr[Z = 1] = \frac{1}{2}$  for no value of  $p$
  - (c)  $\Pr[Z = 1] = \frac{1}{2}$  for  $p = \frac{1}{2}$
  - (d)  $\Pr[Z = 1] = p(1 - p)$
  - (e) None of the above

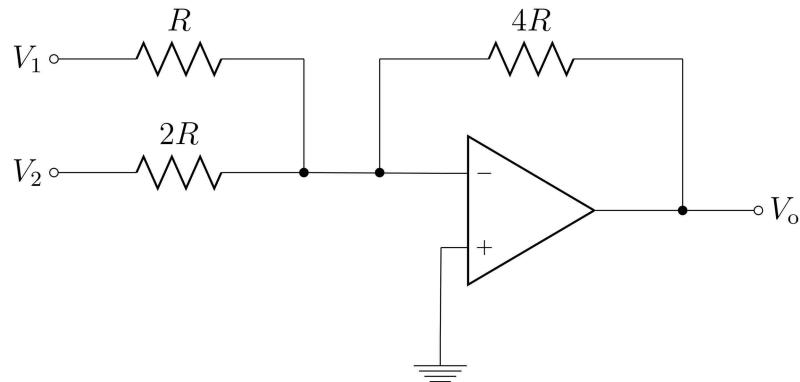
4. Consider a system which in response to input  $x(t)$  outputs

$$y(t) = x(t^2).$$

Which of the following describes the system?

- (a) linear, time-invariant, causal
- (b) linear, time-invariant, non-causal
- (c) linear, time-variant ✓**
- (d) non-linear, time-invariant
- (e) non-linear, time-variant

5. Consider the opamp circuit in the figure. Approximately what is  $V_o$ ?



- (a)  $-(\frac{V_1}{2} + V_2)$
- (b)  $-(\frac{V_1}{4} + \frac{V_2}{2})$
- (c)  $-(V_1 + 2V_2)$
- (d)  $-(4V_1 + 2V_2)$  ✓**
- (e) None of the above

6. What is the Laplace transform  $F(s)$  of the signal  $f(t), t \geq 0$  defined below? In  $t \in [0, 1]$ ,

$$f(t) = \begin{cases} 1, & t \in [0, \frac{1}{2}) \\ 0, & t \in [\frac{1}{2}, 1] \end{cases}$$

and in  $t \geq 1$ ,  $f(t) = f(t - n)$ ,  $n \leq t < n + 1$ ,  $n = 1, 2, 3, \dots$

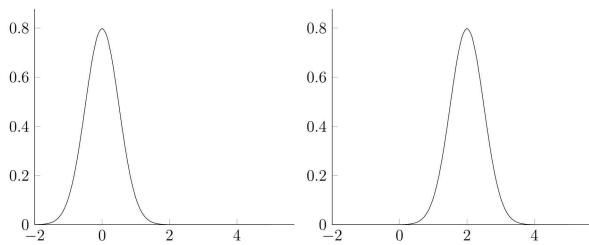
- (a)  $\frac{1-e^{-s/2}}{s}$
- (b)  $\frac{1+e^{-s/2}}{s}$
- (c)  $\frac{1}{s(1-e^{-s})}$

(d)  $\frac{1}{s(1-e^{-s/2})}$

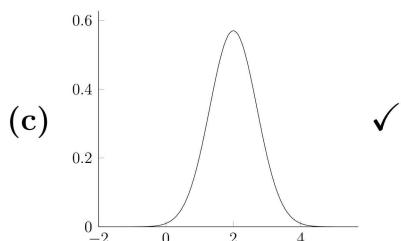
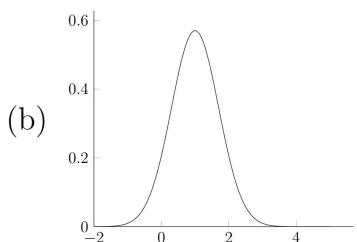
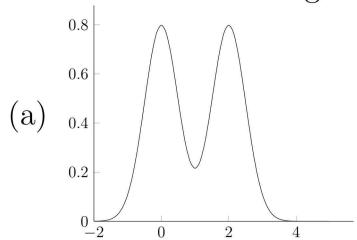
(e)  $\frac{1}{s(1+e^{-s/2})}$  ✓

Please ignore the blank space below.  
Problem 7 appears on page 16.

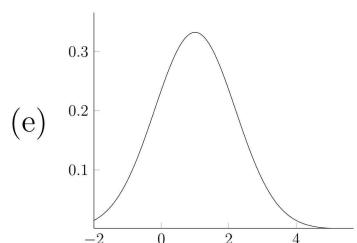
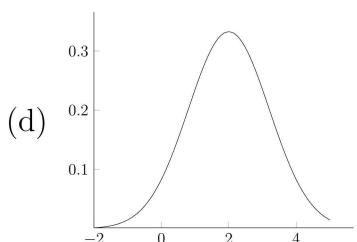
7. Suppose  $X$  and  $Y$  are independent Gaussian random variables, whose pdfs are represented below.



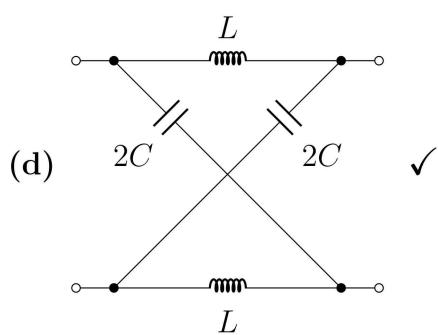
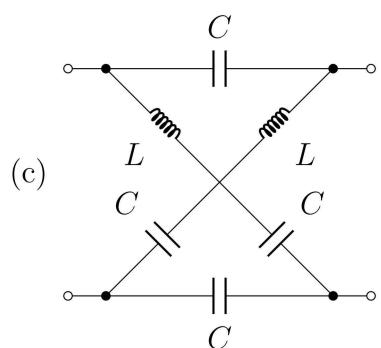
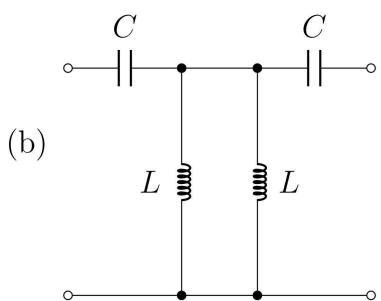
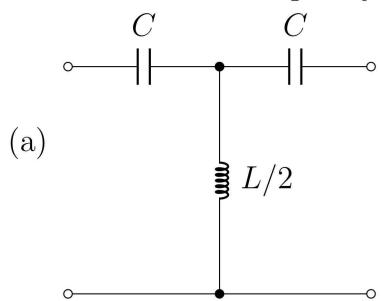
Which of the following describes the pdf of the  $X + Y$ ?



✓



8. In terms of their frequency responses, which of the following is the odd one out?



(e) All four circuits are equivalent

9. Suppose  $Y = X + Z$ , where  $X$  and  $Z$  are independent zero-mean random variables each with variance 1. Let  $\hat{X}(Y) = aY$  be the optimal linear least-squares estimate of  $X$  from  $Y$ , i.e.,  $a$  is chosen such that  $E[(X - aY)^2]$  is minimized. What is the resulting minimum  $E[(X - \hat{X}(Y))^2]$ ?

(a) 1    (b)  $\frac{2}{3}$     (c)  $\frac{1}{2}$  ✓    (d)  $\frac{1}{3}$     (e)  $\frac{1}{4}$

10. Let  $U_1, U_2, U_3$  be independent random variables that are each uniformly distributed between zero and one. What is the probability that the second highest value amongst the three lies between  $1/3$  and  $2/3$ ?

(a)  $\frac{2}{9}$     (b)  $\frac{1}{27}$     (c)  $\frac{13}{27}$  ✓    (d)  $\frac{1}{3}$     (e)  $\frac{7}{18}$

11. Suppose that a random variable  $X$  has a probability density function (pdf) given by

$$f(x) = c \exp(-2x)$$

for  $x \geq 1$ , and  $f(x) = 0$ , for  $x < 1$ , where  $c$  is an appropriate constant so that  $f(x)$  is a valid pdf.

What is the expected value of  $X$  given that  $X \geq 5$ ?

(a)  $5\frac{1}{2}$  ✓    (b) 7    (c) 10    (d)  $8\frac{1}{2}$     (e) 6

12. Recall that the Shannon entropy of a random variables  $X$  taking values in a finite set  $S$  is given by

$$H[X] = - \sum_{x \in S} \Pr[X = x] \log_2 \Pr[X = x].$$

(We set  $0 \log_2 0 = 0$ .) For a pair of random variables  $(X, Y)$  taking values in the finite set  $S \times T$ , we write

$$H[X \mid Y = y] = - \sum_{x \in S} \Pr[X = x \mid Y = y] \log_2 \Pr[X = x \mid Y = y]$$

$$\text{and } H[X \mid Y] = - \sum_{y \in T} \Pr[Y = y] H[X \mid Y = y].$$

Now, consider an  $1024 \times 1024$  chess board. Suppose 1024 rooks are placed one after another randomly at distinct locations on a  $1024 \times 1024$  chess board so that no rook attacks another: that is, the  $i$ -th rook ( $i = 1, 2, \dots, 1024$ ) is placed at a location chosen uniformly from among the available possibilities so that it does not attack any of the previously placed rooks. Let  $R_i$  be the row number of the  $i$ -th rook and  $C_i$  its column number. What is  $H[R_{513}, C_{513} \mid R_1, R_2, \dots, R_{512}]$ ?

(a)  $\log_2 513$     (b) 9    (c) 10    (d) 19 ✓    (e) 81

**13.** Suppose  $m$  and  $n$  are positive integers,  $m \neq n$ , and  $A$  is an  $m \times n$  matrix with real entries. Consider the following statements.

- (i)  $\text{rank}(AA^T) = \text{rank}(A^TA)$
- (ii)  $\det(A^TA) = \det(AA^T)$
- (iii)  $\text{Trace}(AA^T) = \text{Trace}(A^TA)$

Which of the above statements is true for all such  $A$ ?

- (a) Only (i)
- (b) Only (ii)
- (c) Only (iii)
- (d) (i) and (iii) ✓**
- (e) None of them

**14.** Consider matrices  $A \in \mathbb{R}^{n \times m}$ ,  $B \in \mathbb{R}^{m \times m}$ , and  $C \in \mathbb{R}^{m \times n}$ . Let  $r = \text{rank}(ABC)$ . Which of the following must be true?

- (a)  $\min(m, n) \leq r \leq \max(m, n)$
- (b)  $r \leq \min(m, n)$  ✓**
- (c)  $\min(\text{rank}(A), \text{rank}(B), \text{rank}(C)) \leq r \leq \max(m, n)$
- (d)  $\min(m, n) \leq r \leq \max(\text{rank}(A), \text{rank}(B), \text{rank}(C))$
- (e) None of the above

**15.** What is

$$\max_{x,y} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

subject to

$$x^2 + y^2 = 1?$$

- (a) 1    (b)  $\sqrt{2}$     (c) 2    (d) 3    **(e) 4 ✓**