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ISI Indian Statistical Institute Introduction Class

Special class

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Test Codes: UGA (Multiple-choice Type) and UGB (Short Answer Type) 2023

Questions will be based on the following and related topics.

Algebra: Sets, operations on sets. Prime numbers, factorization of integers and divisibility. Rational and irrational numbers. Permutations and combinations, basic probability. Binomial Theorem. Logarithms. Polynomials: Remainder Theorem, Theory of quadratic equations and expressions, relations between roots and coefficients. Arithmetic and geometric progressions. Inequalities involving arithmetic, geometric and harmonic means. Complex numbers. Matrices and determinants.

Geometry: Plane geometry. Geometry of 2 dimensions with Cartesian and polar coordinates. Equation of a line, angle between two lines, distance from a point to a line. Concept of a Locus. Area of a triangle. Equations of circle, parabola, ellipse and hyperbola and equations of their tangents and normals. Mensuration.

Trigonometry: Measures of angles. Trigonometric and inverse trigonometric functions. Trigonometric identities including addition formulae, solutions of trigonometric equations. Properties of triangles. Heights and distances.

Calculus: Sequences - bounded sequences, monotone sequences, limit of a sequence. Functions, one-one functions, onto functions. Limits and continuity. Derivatives and methods of differentiation. Slope of a curve. Tangents and normals. Maxima and minima. Using calculus to sketch graphs of functions. Methods of integration, definite and indefinite integrals, evaluation of area using integrals. Homogeneous differential equations of first order and first degree

1. Suppose, for some $\theta \in [0, \frac{\pi}{2}]$, $\frac{\cos 3\theta}{\cos \theta} = \frac{1}{3}$. Then $(\cot 3\theta) \tan \theta$ equals

   (A) $\frac{1}{2}$ (B) $\frac{1}{3}$
   (C) $\frac{1}{8}$ (D) $\frac{1}{7}$

2. Any positive real number x can be expanded as

   $$x = a_n \cdot 2^n + a_{n-1} \cdot 2^{n-1} + \cdots + a_1 \cdot 2^1 + a_0 \cdot 2^0 + a_{-1} \cdot 2^{-1} + a_{-2} \cdot 2^{-2} + \cdots,$$

   for some $n \geq 0$, where each $a_i \in \{0, 1\}$. In the above-described expansion of 21.1875, the smallest positive integer k such that $a_{-k} \neq 0$ is:

   (A) 3 (B) 2 (C) 1 (D) 4

3. Amongst all polynomials $p(x) = c_0 + c_1 x + \cdots + c_{10} x^{10}$ with real coefficients satisfying $|p(x)| \leq |x|$ for all $x \in [-1, 1]$, what is the maximum possible value of $(2c_0 + c_1)^{10}$?

   (A) 4^{10} (B) 3^{10} (C) 2^{10} (D) 1

4. The locus of points z in the complex plane satisfying $z^2 + |z|^2 = 0$ is

   (A) a straight line

   (B) a pair of straight lines

   (C) a circle

   (D) a parabola

5. Let A and B be two 3×3 matrices such that $(A + B)^2 = A^2 + B^2$. Which of the following must be true?

   (A) A and B are zero matrices.

   (B) AB is the zero matrix.

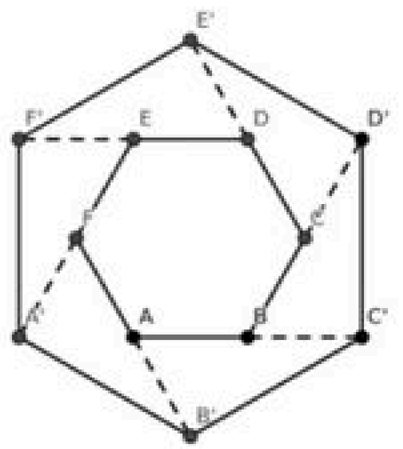
   (C) $(A - B)^2 = A^2 - B^2$

   (D) $(A - B)^2 = A^2 + B^2$

6. Let $\mathbb{Z}$ denote the set of integers. Let $f : \mathbb{Z} \to \mathbb{Z}$ be such that $f(x)f(y) = f(x+y) + f(x-y)$ for all $x, y \in \mathbb{Z}$. If $f(1) = 3$, then $f(7)$ equals

   (A) 840 (B) 844 (C) 843 (D) 842

7. The sides of a regular hexagon $ABCDEF$ is extended by doubling them to form a bigger hexagon $A'B'C'D'E'F'$ as in the figure below.



Then the ratio of the areas of the bigger to the smaller hexagon is:

(A) $\sqrt{3}$ (B) 3 (C) $2\sqrt{3}$ (D) 4

8. Let $(n_1, n_2, \cdots, n_{12})$ be a permutation of the numbers $1, 2, \cdots, 12$. The number of arrangements with

$$n_1 > n_2 > n_3 > n_4 > n_5 > n_6$$

and

$$n_6 < n_7 < n_8 < n_9 < n_{10} < n_{11} < n_{12}$$

equals:

(A) $\binom{12}{5}$ (B) $\binom{12}{6}$ (C) $\binom{11}{6}$ (D) $\frac{11!}{2}$

9. Suppose the numbers $71, 104$ and 159 leave the same remainder r when divided by a certain number $N > 1$. Then, the value of $3N + 4r$ must equal:

(A) 53 (B) 48 (C) 37 (D) 23

10. In how many ways can we choose $a_1 < a_2 < a_3 < a_4$ from the set $\{1, 2, \ldots, 30\}$ such that a_1, a_2, a_3, a_4 are in arithmetic progression?

(A) 135 (B) 145 (C) 155 (D) 165

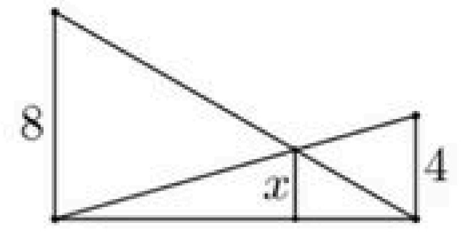
11. What is the minimum value of the function $|x-3|+|x+2|+|x+1|+|x|$ for real x?

(A) 3 (B) 5 (C) 6 (D) 8

12. If x, y are positive real numbers such that $3x + 4y < 72$, then the maximum possible value of $12xy(72 - 3x - 4y)$ is:

(A) 12240 (B) 13824 (C) 10656 (D) 8640

13. A straight road has walls on both sides of height 8 feet and 4 feet respectively. Two ladders are placed from the top of one wall to the foot of the other as in the figure below. What is the height (in feet) of the maximum clearance x below the ladders?



(A) 3 (B) $2\sqrt{2}$ (C) $\frac{8}{3}$ (D) $2\sqrt{3}$

14. Consider a differentiable function $u : [0,1] \to \mathbb{R}$. Assume the function u satisfies

$$u(a) = \frac{1}{2r}\int_{a-r}^{a+r} u(x)\,dx, \quad \text{for all } a \in (0,1) \text{ and all } r < \min(a, 1-a).$$

Which of the following four statements must be true?

(A) u attains its maximum but not its minimum on the set $\{0, 1\}$.

(B) u attains its minimum but not maximum on the set $\{0, 1\}$.

(C) If u attains either its maximum or its minimum on the set $\{0, 1\}$, then it must be constant.

(D) u attains both its maximum and its minimum on the set $\{0, 1\}$.

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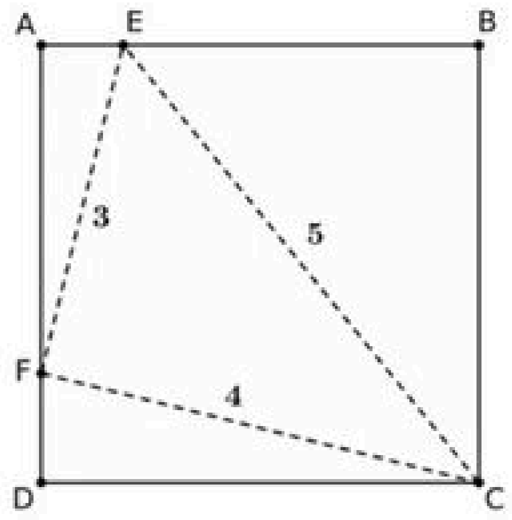
15. In the figure below, $ABCD$ is a square and ΔCEF is a triangle with given sides inscribed as in the figure. Find the length BE.



(A) $\frac{13}{\sqrt{17}}$ (B) $\frac{14}{\sqrt{17}}$

(C) $\frac{15}{\sqrt{17}}$ (D) $\frac{16}{\sqrt{17}}$

16. Let $y = x + c_1$, $y = x + c_2$ be the two tangents to the ellipse $x^2 + 4y^2 = 1$. What is the value of $|c_1 - c_2|$?

(A) $\sqrt{2}$ (B) $\sqrt{5}$ (C) $\frac{\sqrt{5}}{2}$ (D) 1

17. For $n \in \mathbb{N}$, let a_n be defined as

$$a_n = \int_0^n \frac{1}{1 + nx^2}\, dx.$$

Then $\lim_{n\to\infty} a_n$

(A) equals 0 (B) equals $\frac{\pi}{4}$

(C) equals $\frac{\pi}{2}$ (D) does not exist

18. Let p and q be two non-zero polynomials such that the degree of p is less than or equal to the degree of q, and $p(a)q(a) = 0$ for $a = 0, 1, 2, \ldots, 10$. Which of the following must be true?

(A) degree of $q \neq 10$

(B) degree of $p \neq 10$

(C) degree of $q \neq 5$

(D) degree of $p \neq 5$

19. The number of positive integers n less than or equal to 22 such that 7 divides $n^5 + 4n^4 + 3n^3 + 2022$ is

(A) 7 (B) 8 (C) 9 (D) 10

20. A 3×3 magic square is a 3×3 rectangular array of positive integers such that the sum of the three numbers in any row, any column or any of the two major diagonals, is the same. For the following incomplete magic square

27	36	
31		

the column sum is

(A) 90 (B) 96 (C) 94 (D) 99

21. Let $1, \omega, \omega^2$ be the cube roots of unity. Then the product

$$(1-\omega+\omega^2)(1-\omega^2+\omega^{2^2})(1-\omega^{2^2}+\omega^{2^3})\cdots(1-\omega^{2^9}+\omega^{2^{10}})$$

is equal to:

(A) 2^{10} (B) 3^{10} (C) $2^{10}\omega$ (D) $3^{10}\omega^2$

22. In a class of 45 students, three students can write well using either hand. The number of students who can write well only with the right hand is 24 more than the number of those who write well only with the left hand. Then, the number of students who can write well with the right hand is:

(A) 33 (B) 36 (C) 39 (D) 41

23. The number of triples (a, b, c) of positive integers satisfying the equation

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1+\frac{2}{abc}$$

and such that $a < b < c$, equals:

(A) 3 (B) 2 (C) 1 (D) 0

24. The function $x^2 \log_e x$ in the interval $(0, 2)$ has:

(A) exactly one point of local maximum and no points of local minimum.

(B) exactly one point of local minimum and no points of local maximum.

(C) points of local maximum as well as local minimum.

(D) neither a point of local maximum nor a point of local minimum.

25. A triangle has sides of lengths $\sqrt{5}, 2\sqrt{2}, \sqrt{3}$ units. Then, the radius of its inscribed circle is :

(A) $\frac{\sqrt{5}+\sqrt{3}+2\sqrt{2}}{2}$ (B) $\frac{\sqrt{5}+\sqrt{3}+2\sqrt{2}}{3}$

(C) $\sqrt{5}+\sqrt{3}+2\sqrt{2}$ (D) $\frac{\sqrt{5}+\sqrt{3}-2\sqrt{2}}{2}$

26. An urn contains 30 balls out of which one is special. If 6 of these balls are taken out at random, what is the probability that the special ball is chosen?

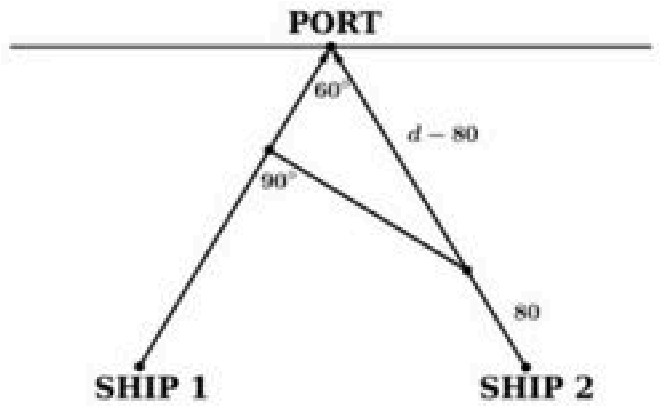
(A) $\frac{1}{30}$ (B) $\frac{1}{6}$ (C) $\frac{1}{5}$ (D) $\frac{1}{15}$

27. If $x_1 > x_2 > \cdots > x_{10}$ are real numbers, what is the least possible value of

$$\left(\frac{x_1 - x_{10}}{x_1 - x_2}\right)\left(\frac{x_1 - x_{10}}{x_2 - x_3}\right)\cdots\left(\frac{x_1 - x_{10}}{x_9 - x_{10}}\right)?$$

(A) 10^{10} (B) 10^9 (C) 9^9 (D) 9^{10}

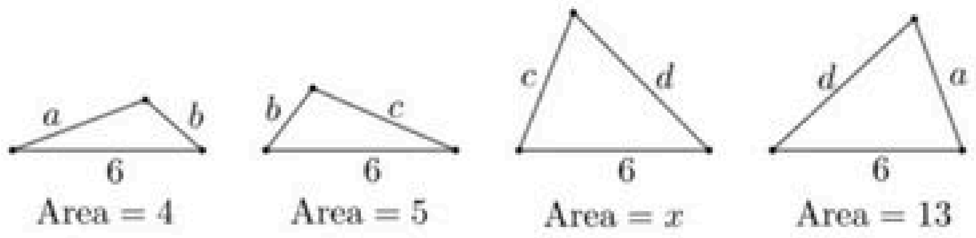
28. Two ships are approaching a port along straight routes at constant velocities. Initially, the two ships and the port formed an equilateral triangle. After the second ship travelled 80 km, the triangle became right-angled.



When the first ship reaches the port, the second ship was still 120 km from the port. Find the initial distance of the ships from the port.

(A) 240 km (B) 300 km (C) 360 km (D) 180 km

29. In the following diagram, four triangles and their sides are given. Areas of three of them are also given. Find the area x of the remaining triangle.



(A) 12 (B) 13 (C) 14 (D) 15

30. The range of values that the function

$$f(x) = \frac{x^2 + 2x + 4}{2x^2 + 4x + 9}$$

takes as x varies over all real numbers in the domain of f is:

(A) $\frac{3}{7} < f(x) \leq \frac{1}{2}$ (B) $\frac{3}{7} \leq f(x) < \frac{1}{2}$

(C) $\frac{3}{7} < f(x) \leq \frac{4}{9}$ (D) $\frac{3}{7} \leq f(x) \leq \frac{1}{2}$

Note. In this question-paper, $\mathbb{R}$ denotes the set of real numbers.

1. Consider a board having 2 rows and n columns. Thus there are $2n$ cells in the board. Each cell is to be filled in by 0 or 1.

   (a) In how many ways can this be done such that each row sum and each column sum is even?

   (b) In how many ways can this be done such that each row sum and each column sum is odd?

2. Consider the function

$$f(x) = \sum_{k=1}^{m} (x-k)^4, \quad x \in \mathbb{R},$$

   where $m > 1$ is an integer. Show that f has a unique minimum and find the point where the minimum is attained.

3. Consider the parabola $C : y^2 = 4x$ and the straight line $L : y = x + 2$. Let P be a variable point on L. Draw the two tangents from P to C and let Q_1 and Q_2 denote the two points of contact on C. Let Q be the mid-point of the line segment joining Q_1 and Q_2. Find the locus of Q as P moves along L.

4. Let $P(x)$ be an odd degree polynomial in x with real coefficients. Show that the equation $P(P(x)) = 0$ has at least as many *distinct* real roots as the equation $P(x) = 0$.

5. For any positive integer n, and $i = 1, 2$, let $f_i(n)$ denote the number of divisors of n of the form $3k + i$ (including 1 and n). Define, for any positive integer n,

$$f(n) = f_1(n) - f_2(n).$$

   Find the values of $f(5^{2022})$ and $f(21^{2022})$.

6. Consider a sequence $P_1, P_2, \ldots$ of points in the plane such that P_1, P_2, P_3 are non-collinear and for every $n \geq 4$, P_n is the mid-point of the line segment joining P_{n-2} and P_{n-3}. Let L denote the line segment joining P_1 and P_5. Prove the following:

   (a) The area of the triangle formed by the points P_n, P_{n-1}, P_{n-2} converges to zero as n goes to infinity.

   (b) The point P_9 lies on L.

7. Let
$$P(x) = 1 + 2x + 7x^2 + 13x^3, \; x \in \mathbb{R}.$$
Calculate for all $x \in \mathbb{R}$,
$$\lim_{n\to\infty} \left(P\left(\frac{x}{n}\right)\right)^n.$$

8. Find the minimum value of
$$|\sin x + \cos x + \tan x + \cot x + \sec x + \operatorname{cosec} x|$$
for real numbers x not multiple of $\pi/2$.

9. Find the smallest positive real number k such that the following inequality holds
$$|z_1 + \ldots + z_n| \geq \frac{1}{k}\left(|z_1| + \ldots + |z_n|\right).$$
for every positive integer $n \geq 2$ and every choice $z_1, \ldots, z_n$ of complex numbers with non-negative real and imaginary parts.

[Hint: First find k that works for $n = 2$. Then show that the same k works for any $n \geq 2$.]