UNIVERSITI TEKNOLOGI MARA
FINAL EXAMINATION

COURSE	:	PARTIAL DIFFERENTIAL EQUATIONS
COURSE CODE	:	MAT612
EXAMINATION	:	OCTOBER 2010
TIME	:	3 HOURS

INSTRUCTIONS TO CANDIDATES

1. This question paper consists of five (5) questions.
2. Answer ALL questions in the Answer Booklet. Start each answer on a new page.
3. Do not bring any material into the examination room unless permission is given by the invigilator.
4. Please check to make sure that this examination pack consists of:

   i) the Question Paper
   ii) a four – page Appendix 1
   iii) an Answer Booklet – provided by the Faculty

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO

This examination paper consists of 4 printed pages

QUESTION 1

The end $x = 0$ of a rod of length 4 is kept at zero temperature and the other end $x = 4$ is insulated. The lateral surface of the rod is also insulated. The initial temperature distribution of the rod is $f(x)$. If $u(x,t)$ is the lateral temperature of the rod at any point x and time t, the related boundary value problem for the temperature of the rod is

$$u_t = \alpha^2 u_{xx}, \quad 0 < x < 4, t > 0$$
$$u(0,t) = 0, \quad t \geq 0$$
$$u_x(4,t) = 0, \quad t \geq 0$$
$$u(x,0) = f(x) = \begin{cases} x & 0 < x \leq 2 \\ 4 - x & 2 < x < 4 \end{cases}$$

If the thermal diffusivity is 1:

a) find the temperature distribution $u(x,t)$.

(14 marks)

b) sketch the temperature distribution $u(x,t)$ for various time t.

(3 marks)

c) determine the limiting temperature.

(3 marks)

QUESTION 2

Consider the following nonhomogeneous one-dimensional wave boundary value problem.

$$u_{tt}(x,t) = u_{xx}(x,t) + \pi^2 t \sin \pi x, \quad 0 < x < 1, \quad t > 0$$

subject to the boundary conditions

$$u(0,t) = u(1,t) = 0, t \geq 0$$

and initial conditions

$$u(x,0) = \pi, \quad u_t(x,0) = 2\pi \sin 2\pi x, \quad 0 < x < 1.$$

a) Identify the corresponding eigenvalues and eigenfunctions.

(1 mark)

b) Using the method of eigenfunction expansion, show that the general solution for $u(x,t)$ is

$$u(x,t)=\sum_{n=1}^{\infty}(A_n\cos n\pi t+B_n\sin n\pi t)\sin n\pi x+t\sin\pi x.$$

(15 marks)

c) Determine the coefficients A_n and B_n and hence write the particular solution for $u(x,t)$.

(4 marks)

QUESTION 3

Consider the following nonhomogeneous BVP,

$$\begin{aligned}
&u_t(x,t)=u_{xx}+2e^{-x} \quad 0\le x\le 1,\quad t>0\\
&u(0,t)=0,\quad t>0\\
&u_x(1,t)+u(1,t)=0,\quad t>0\\
&u(x,0)=-4e^{-x},\quad 0<x<1
\end{aligned}$$

a) Show that by letting $u(x,t)=v(x,t)+w(x)$, the system is transformed into the following homogeneous BVP

$$\begin{aligned}
&v_t(x,t)=v_{xx}(x,t),\quad 0<x<1,\ t>0\\
&v(0,t)=0, t\ge 0\\
&v_x(0,t)+v(1,t)=0,\ t\ge 0\\
&v(x,0)=u(x,0)-w(x)=-6e^{-x}-2
\end{aligned}$$

(9 marks)

b) Show that the solution for the above boundary value problem is

$$u(x,t)=\sum_{n=1}^{\infty}\frac{4}{(1+\cos^2 z_n)}\left(\frac{1}{z_n}-\frac{1}{z_n}\cos z_n-\frac{3z_n}{1+z_n^{\,2}}\right)\sin(z_n x)e^{-z_n^{\,2}t}+2(1+e^{-x})$$

(11 marks)

QUESTION 4

The vibrations of a square membrane are governed by the two-dimensional wave equation

$$u_{tt}(x,y,t)=\frac{1}{\pi^2}\left(u_{xx}(x,y,t)+u_{yy}(x,y,t)\right),\ 0<x<1,\ 0<y<1$$

where $u(x,y,t)$ is the deflection of the membrane at any point (x,y) and time t. The edges of the
membrane are held fixed at all time. Initially, the membrane is stretched into a shape modeled by the function $f(x,y)=xy(x-1)(y-1), 0<x<1, 0<y<1$. The membrane starts to vibrate according to $g(x,y)=2\sin\pi x\sin\pi y$.

a) Set up the BVP for the above problem.

(3 marks)

b) Determine the deflection $u(x,y,t)$ of the membrane.

(17 marks)

QUESTION 5

Consider the Dirichlet problem for steady-state temperature of an annulus,

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0,\ 2<r<4,\ 0<\theta<2\pi$$

a) By using SOV with $u(r,\theta)=R(r)H(\theta)$, show that the solution for $u(r,\theta)$ has the form

$$u(r,\theta)=A_0+B_0\ln r+\sum_{n=1}^{\infty}\left(A_nr^n+B_nr^{-n}\right)\cos n\theta+\left(C_nr^n+D_nr^{-n}\right)\sin n\theta.$$

(13 marks)

b) Hence, determine $u(r,\theta)$ if the temperature along the boundaries of the annulus are given by

$$u(2,\theta)=6\cos\theta+10\sin\theta$$
$$u(4,\theta)=15\cos\theta+17\sin\theta$$

(7 marks)

END OF QUESTION PAPER

SOLUTION GUIDELINES FOR BVPs

	BVP	GENERAL SOLUTION
1.	$u_t(x,t)=\alpha^2 u_{xx}(x,t) \quad 0<x<L,\ t>0$ $\left.\begin{matrix} u(0,t)=0 \\ u(L,t)=0 \end{matrix}\right\},\quad t>0$ $u(x,0)=f(x), \quad 0<x<L$	$u(x,t)=\sum_{n=1}^{\infty} A_n\, e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$ $A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n=1,2,3,\ldots$
2.	$u_t(x,t)=\alpha^2 u_{xx}(x,t), \quad 0<x<L,\ t>0$ $\left.\begin{matrix} u(0,t)=0 \\ u_x(L,t)=0 \end{matrix}\right\},\quad t>0$ $u(x,0)=f(x), \quad 0<x<L$	$u(x,t)=\sum_{n=1}^{\infty} A_n e^{-\left(\frac{(2n-1)\pi\alpha}{2L}\right)^2 t} \sin\left(\frac{(2n-1)\pi x}{2L}\right)$ where $A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{(2n-1)\pi x}{2L}\right)dx \quad n=1,2,3,\ldots$
3.	$u_t(x,t)=\alpha^2 u_{xx}(x,t),\ 0<x<L, t>0$ $u(0,t)=0$ $u_x(L,t)+hu(L,t)=0, \quad t>0$ $u(x,0)=f(x), \quad 0<x<L$	$u(x,t)=\sum_{n=1}^{\infty} A_n \sin\left(\frac{z_n x}{L}\right)e^{-\left(\frac{z_n\alpha}{L}\right)^2 t}$ where $A_n=\dfrac{\int_0^L f(x)\sin\left(\frac{z_n x}{L}\right)dx}{\int_0^L \sin^2\left(\frac{z_n x}{L}\right)dx} \quad n=1,2,3\ldots$

4.	$u_{tt}(x,t)=\alpha^2 u_{xx}(x,t), \quad 0<x<L, \quad t>0$ $\left.\begin{array}{l} u(0,t)=0 \\ u(L,t)=0 \end{array}\right\}, \quad t>0$ $\left.\begin{array}{l} u(x,0)=f(x) \\ u_t(x,0)=g(x) \end{array}\right\}, \quad 0<x<L$	$u(x,t)=\sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi\alpha t}{L}+B_n \sin\frac{n\pi\alpha t}{L}\right)\sin\frac{n\pi x}{L}$ where $A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \qquad n=1,2,3,\ldots$ $B_n=\frac{2}{n\pi\alpha}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad n=1,2,3,\ldots$
5.	$u_{tt}=a^2\left(u_{xx}+u_{yy}\right), \; 0<x<b, \; 0<y<c,$ $t>0$ $u(x,0,t)=u(x,c,t)=0 \quad 0<x<b, \; t>0$ $u(0,y,t)=u(b,y,t)=0 \quad 0<y<c, \; t>0$ $u(x,y,0)=f(x,y), \; 0<x<b, 0<y<c$ $u_t(x,y,0)=g(x,y), \; 0<x<b, 0<y<c$	$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{A_{mn}\cos(a\omega t)+B_{mn}\sin(a\omega t)\right\}\sin\left(\frac{n\pi x}{b}\right)\sin\left(\frac{m\pi y}{c}\right)$ where $\omega=\sqrt{\frac{n^2\pi^2}{b^2}+\frac{m^2\pi^2}{c^2}}$ $A_{nm}=\frac{4}{bc}\int_0^c\int_0^b f(x,y)\sin\left(\frac{n\pi x}{b}\right)\sin\left(\frac{m\pi y}{c}\right)dxdy$ $B_{mn}=\frac{4}{bc\omega}\int_0^c\int_0^b g(x,y)\sin\left(\frac{n\pi x}{b}\right)\sin\left(\frac{m\pi y}{c}\right)dxdy$

$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0,\ 0 < \theta < 2\pi,\ 0 < r < c$ $u(c,\theta) = f(\theta), \quad 0 < \theta < 2\pi$	$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$ where $A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) d\theta$ $A_n = \frac{1}{c^n \pi}\int_0^{2\pi} f(\theta) \cos n\theta \, d\theta$ $B_n = \frac{1}{c^n \pi}\int_0^{2\pi} f(\theta) \sin n\theta \, d\theta$