Chapter 12

Solving Nonhomogeneous PDEs (Eigenfunction Expansions)

12.1 Goal

We know how to solve diffusion problems for which both the PDE and the BCs are homogeneous using the separation of variables method. Unfortunately, this method requires that both the PDE and the BCs be homogeneous. We also learned how to apply certain transformations so that nonhomogeneous BCs are transformed into homogeneous ones. Unfortunately, these transformations may in some cases, transform the PDE into a nonhomogeneous one. To complete the set of tools we have to solve diffusion problems, we must learn how to handle nonhomogeneous PDEs. More specifically, we will show how to solve the IBVP

$$\left\{\begin{array}{cccc} \text{PDE} & u_t = \alpha^2 u_{xx} + f(x,t) & 0 < x < 1 & 0 < t < \infty \\ \text{BC} & \begin{array}{c} \alpha_1 u_x(0,t) + \beta_1 u(0,t) = 0 \\ \alpha_2 u_x(1,t) + \beta_2 u(1,t) = 0 \end{array} & & 0 < t < \infty \\ \text{IC} & u(x,0) = \phi(x) & 0 \le x \le 1 & \end{array}\right. \tag{12.1}$$

by finding a series solution of the form

$$u(x,y) = \sum_{n=1}^{\infty} T_n(t) X_n(t)$$

where $X_n(t)$ are the eigenfunctions we find when solving the associated homogeneous problem

$$\left\{\begin{array}{cccc} \text{PDE} & u_t = \alpha^2 u_{xx} & 0 < x < 1 & 0 < t < \infty \\ \text{BC} & \begin{array}{c} \alpha_1 u_x(0,t) + \beta_1 u(0,t) = 0 \\ \alpha_2 u_x(1,t) + \beta_2 u(1,t) = 0 \end{array} & & 0 < t < \infty \\ \text{IC} & u(x,0) = \phi(x) & 0 \le x \le 1 & \end{array}\right. \tag{12.2}$$

and $T_n(t)$ are functions which can be found by solving a sequence of ODEs.

12.2 Outline

Recall that the solution of 12.2 is of the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{(-\lambda_n \alpha)^2 t} X_n(x)$$

Where λ_n and $X_n(x)$ are the eigenvalues and eigenfunctions of the problem

$$\begin{cases} X'' + \lambda^2 X = 0 \\ \alpha_1 X'(0) + \beta_1 X(0) = 0 \\ \alpha_2 X'(1) + \beta_2 X(1) = 0 \end{cases}$$

For the problem in 12.1, we will look for a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$$

The physical reason for this is that without $f(x,t)$, there is no heat source, so it is normal to expect the temperature to decrease with time, hence the damping term $e^{(-\lambda_n \alpha)^2 t}$. With a heat source, temperature will no longer decrease, hence we might expect the part which depends on t to be different.

12.3 General Idea

We illustrate this method with the following nonhomogeneous IBVP:

$$\begin{cases} \text{PDE} & u_t = \alpha^2 u_{xx} + f(x,t) & 0 < x < 1 & 0 < t < \infty \\ \text{BC} & \begin{array}{c} u(0,t) = 0 \\ u(1,t) = 0 \end{array} & & 0 < t < \infty \\ \text{IC} & u(x,0) = \phi(x) & 0 \le x \le 1 & \end{cases} \tag{12.3}$$

The general idea is to decompose $f(x,t)$ into simple components

$$f(x,t) = f_1(t) X_1(x) + f_2(t) X_2(x) + ... + f_n(t) X_n(x) + ...$$

and find the response $u_n(x,t) = T_n(t) X_n(x)$ to each of these individual components. The solution to our problem will then be

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

We will break the procedure of solving this problem into several steps.

Step 1 Find the functions $X_n(x)$. It turns out that the functions $X_n(x)$ are the eigenfunctions of the associated homogeneous problem when we solve it by separation of variables. We derive this problem one more time. The associated homogeneous PDE is $u_t = \alpha^2 u_{xx}$. If we look for a solution of the form $u(x,t) = T(t)X(x)$. Replacing in the PDE gives $T'(t)X(x) = \alpha^2 T(t)X''(x)$. Dividing each side by $\alpha^2 T(t)X(x)$ gives $\dfrac{T'(t)}{\alpha^2 T(t)} = \dfrac{X''(x)}{X(x)}$. We concluded these had to be equal to a negative constant we called $-\lambda^2$. Thus, to find X, we solve the second order ODE $X'' + \lambda^2 X = 0$. For X, the boundary condition meant that $X(0) = 0$ and $X(1) = 0$. Thus, we see that finding X amounts to solving the initial value problem

$$\begin{cases} X'' + \lambda^2 X = 0 \\ X(0) = 0 \\ X(1) = 0 \end{cases}$$

When we do so, we say that we are finding the eigenfunctions of this problem. You will recall that the solutions are $X(x) = A\sin\lambda x + B\cos\lambda x$. Using the boundary conditions gives $0 = X(0) = B$, so that $X(x) = A\sin\lambda x$. Also, we have $0 = X(1) = A\sin\lambda$. It follows that we must have $\sin\lambda = 0$ which means that $\lambda = n\pi$ for $n = 1, 2, 3, 4, ...$ If we call $\lambda_n = n\pi$ for each $n = 1, 2, 3, 4, ...$ then we have $X_n(x) = \sin n\pi x$. Note that we omitted the constant, it will be part of the other components $f_n(t)$.

Step 2 Find the functions $f_n(t)$. So far, we have

$$f(x,t) = f_1(t)\sin\pi x + f_2(t)\sin 2\pi x + ... + f_n(t)\sin n\pi x + ...$$

To find $f_n(t)$ we simply multiply each side by $\sin m\pi x$ and integrate from 0 to 1 with respect to x. We have already used this method. We will have

$$\begin{aligned} \int_0^1 f(x,t)\sin m\pi x dx &= \sum_{n=1}^{\infty} f_n(t)\int_0^1 \sin m\pi x dx \sin n\pi x dx \\ &= \frac{1}{2} f_m(t) \end{aligned}$$

Thus

$$f_n(t) = 2\int_0^1 f(x,t)\sin n\pi x dx \tag{12.4}$$

Step 3 Find the response $u_n(x,t) = T_n(t)X_n(x)$. We can replace the nonhomogeneous term $f(x,t)$ by its decomposition

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\sin n\pi x$$

and we try to find the individual responses

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin n\pi x$$

So, we have to find the functions $T_n(t)$ which solve the IBVP 12.3. If we replace u in that problem with the expression we have, we obtain

$$\begin{cases} \text{PDE} & \sum_{n=1}^{\infty} T_n'(t)\sin n\pi x = -\alpha^2 \sum_{n=1}^{\infty} (n\pi)^2 T_n(t)\sin n\pi x + \sum_{n=1}^{\infty} f_n(t)\sin n\pi x & 0<x<1 & 0<t \\ \text{BC} & \begin{matrix} \sum_{n=1}^{\infty} T_n(t)\sin 0 = 0 \\ \sum_{n=1}^{\infty} T_n(t)\sin n\pi = 0 \end{matrix} & & 0<t \\ \text{IC} & \sum_{n=1}^{\infty} T_n(0)\sin n\pi x = \phi(x) & 0\le x\le 1 & \end{cases}$$

The BCs do not give us any information, they simply say $0=0$. We are left with

$$\begin{cases} \sum_{n=1}^{\infty}\left[T_n'(t) + (n\pi\alpha)^2 T_n(t) - f_n(t)\right]\sin n\pi x = 0 \\ \sum_{n=1}^{\infty} T_n(0)\sin n\pi x = \phi(x) \end{cases}$$

Thus, T_n must satisfy the initial value problem

$$\begin{cases} T_n'(t) + (n\pi\alpha)^2 T_n(t) - f_n(t) = 0 \\ T_n(0) = 2\int_0^1 \phi(x)\sin n\pi x dx \end{cases}$$

Let $a_n = 2\int_0^1 \phi(x)\sin n\pi x dx$. This is a first order linear ODE which can be solved using the integration factor technique. Recall, if we multiply each side of the ODE by $e^{(n\pi\alpha)^2 t}$, we obtain

$$T_n'(t)e^{(n\pi\alpha)^2 t} + (n\pi\alpha)^2 T_n(t)e^{(n\pi\alpha)^2 t} = f_n(t)e^{(n\pi\alpha)^2 t}$$

which is

$$\left(T_n(t)e^{(n\pi\alpha)^2 t}\right)' = f_n(t)e^{(n\pi\alpha)^2 t}$$

Integrating from 0 to t on each side, we get

$$\int_0^t \left(T_n(\tau)e^{(n\pi\alpha)^2\tau}\right)' d\tau = \int_0^t f_n(t)e^{(n\pi\alpha)^2 t} d\tau$$

$$T_n(t)e^{(n\pi\alpha)^2 t} - T_n(0) = \int_0^t f_n(\tau)e^{(n\pi\alpha)^2\tau} d\tau$$

Recall we set $a_n = T_n(0)$, so we have

$$\begin{aligned} T_n(t) &= a_n e^{-(n\pi\alpha)^2 t} + e^{-(n\pi\alpha)^2 t}\int_0^t f_n(\tau)e^{(n\pi\alpha)^2\tau} d\tau \\ &= a_n e^{-(n\pi\alpha)^2 t} + \int_0^t f_n(\tau)e^{-(n\pi\alpha)^2(\tau-t)} d\tau \end{aligned}$$

Thus, the solution to the IBVP 12.3 is

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} T_n(t)\sin n\pi x \\
&= \sum_{n=1}^{\infty} a_n e^{-(n\pi\alpha)^2 t}\sin n\pi x + \sum_{n=1}^{\infty} \sin n\pi x \int_0^t f_n(\tau)\, e^{-(n\pi\alpha)^2(\tau - t)} d\tau
\end{aligned}
$$

This shows in particular that the temperature in the rod is due to two parts. One comes from the initial condition. The other one from the heat source.

12.4 A specific Problem

We now apply the above procedure to a specific example. Consider the following IBVP:

$$
\begin{cases}
\text{PDE} & u_t = \alpha^2 u_{xx} + \sin 3\pi x & 0 < x < 1 & 0 < t < \infty \\
\text{BC} & \begin{matrix} u(0,t) = 0 \\ u(1,t) = 0 \end{matrix} & & 0 < t < \infty \\
\text{IC} & u(x,0) = \sin \pi x & 0 \le x \le 1 &
\end{cases}
$$

The eigenfunctions $X_n(x)$ depend on the corresponding homogeneous PDE and the BCs. Since they are the same in this problem as in the previous one, the $X_n(x)$ will be the same. Thus, we have to compute the coefficients $T_n(t)$ in the expansion $u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin n\pi x$. Using our work from the previous example, we see that $T_n(t)$ must satisfy

$$
\begin{aligned}
T_n' + (n\pi\alpha)^2 T_n &= f_n \\
&= 2\int_0^1 \sin 3\pi x \sin n\pi x dx \\
&= \begin{cases} 0 & \text{if } n \neq 3 \\ 1 & \text{if } n = 3 \end{cases}
\end{aligned}
$$

And

$$
\begin{aligned}
T_n(0) &= 2\int_0^1 \sin \pi x \sin n\pi x dx \\
&= \begin{cases} 0 & \text{if } n \neq 1 \\ 1 & \text{if } n = 1 \end{cases}
\end{aligned}
$$

If we write these equations for each n, we have

$$\begin{cases} (n=1) & \begin{array}{c} T_1' + (\pi\alpha)^2 T_1 = 0 \\ T_1(0) = 1 \end{array} \\ (n=2) & \begin{array}{c} T_2' + (2\pi\alpha)^2 T_2 = 0 \\ T_2(0) = 0 \end{array} \\ (n=3) & \begin{array}{c} T_3' + (3\pi\alpha)^2 T_3 = 1 \\ T_3(0) = 0 \end{array} \\ (n \geq 4) & \begin{array}{c} T_n' + (n\pi\alpha)^2 T_n = 0 \\ T_n(0) = 0 \end{array} \end{cases}$$

Solution for $n=1$ $T_1(t) = Ae^{-(\pi\alpha)^2 t}$. Since $T_1(0) = 1$, it follows that $A = 1$. Thus, $T_1(t) = e^{-(\pi\alpha)^2 t}$.

Solution for $n=2$ $T_2(t) = Ae^{-(2\pi\alpha)^2 t}$. Since $T_2(0) = 0$, it follows that $A = 0$. Thus, $T_2(t) = 0$.

Solution for $n=3$ We use the integrating factor technique. We get $T_3(t) = \dfrac{1}{(3\pi\alpha)^2}\left(1 - e^{-(3\pi\alpha)^2 t}\right)$.

Solution for $n \geq 4$ $T_2(t) = Ae^{-(n\pi\alpha)^2 t}$. Since $T_n(0) = 0$, it follows that $A = 0$. Thus, $T_n(t) = 0$.

Thus, we see that the solution is

$$u(x,t) = e^{-(\pi\alpha)^2 t}\sin \pi x + \frac{1}{(3\pi\alpha)^2}\left(1 - e^{-(3\pi\alpha)^2 t}\right)\sin 3\pi x$$

Remark 63 *It is important to realize that the eigenfunctions $X_n(x)$ and the eigenvalues λ_n which appear in the solution of nonhomogeneous problems vary for each problem. They depend on the PDE used and the BCs.*

12.5 Problems

1. In the last example, find the solution in the case $n = 3$.
2. Solve the problem

$$\begin{cases} \text{PDE} & u_t = u_{xx} + \sin \pi x + \sin 2\pi x & 0 < x < 1 & 0 < t < \infty \\ \text{BC} & \begin{array}{c} u(0,t) = 0 \\ u(1,t) = 0 \end{array} & & 0 < t < \infty \\ \text{IC} & u(x,0) = 0 & 0 \leq x \leq 1 & \end{cases}$$

3. Solve the problem

$$\begin{cases} \text{PDE} & u_t = u_{xx} + \sin \pi x & 0 < x < 1 & 0 < t < \infty \\ \text{BC} & \begin{array}{c} u(0,t) = 0 \\ u(1,t) = 0 \end{array} & & 0 < t < \infty \\ \text{IC} & u(x,0) = 1 & 0 \leq x \leq 1 & \end{cases}$$

4. Solve the problem

$$\begin{cases} \text{PDE} & u_t = u_{xx} + \sin \lambda_1 x & 0 < x < 1 & 0 < t < \infty \\ \text{BC} & \begin{matrix} u(0,t) = 0 \\ u_x(1,t) + u(1,t) = 0 \end{matrix} & & 0 < t < \infty \\ \text{IC} & u(x,0) = 0 & 0 \le x \le 1 & \end{cases}$$

where λ_1 is the first root of the equation $\tan \lambda = -\lambda$. What are the eigenfunctions X_n in this problem?

5. Solve the problem

$$\begin{cases} \text{PDE} & u_t = u_{xx} & 0 < x < 1 & 0 < t < \infty \\ \text{BC} & \begin{matrix} u(0,t) = 0 \\ u(1,t) = \cos t \end{matrix} & & 0 < t < \infty \\ \text{IC} & u(x,0) = x & 0 \le x \le 1 & \end{cases}$$

by:

(a) Transforming it to one with homogeneous BCs.

(b) Solving the resulting problems using the techniques of this chapter.