UNIVERSITI TEKNOLOGI MARA
FINAL EXAMINATION

COURSE	:	**PARTIAL DIFFERENTIAL EQUATIONS**
COURSE CODE	:	**MAT612**
EXAMINATION	:	**APRIL 2010**
TIME	:	**3 HOURS**

INSTRUCTIONS TO CANDIDATES

1. This question paper consists of five (5) questions.
2. Answer ALL questions in the Answer Booklet. Start each answer on a new page.
3. Do not bring any material into the examination room unless permission is given by the invigilator.
4. Please check to make sure that this examination pack consists of:

   i) the Question Paper
   ii) a three-page Appendix 1
   iii) an Answer Booklet – provided by the Faculty

DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO

This examination paper consists of 4 printed pages

QUESTION 1

Consider the following BVP,

$$u_{tt}(x,t) = c^2 u_{xx}(x,t) \quad 0 < x < L, \quad t > 0$$

$$\left.\begin{array}{l} u(0,t) = 0 \\ u_x(L,t) = 0 \end{array}\right\} \; t > 0$$

$$u(x,0) = f(x) \quad 0 < x < L$$
$$u_t(x,0) = g(x) \quad 0 < x < L$$

a) By letting $u(x,t) = X(x)T(t)$, show that the eigenvalues and the eigenfunction for the problem are

$$\lambda_n = \frac{(2n-1)\pi}{2L}, \quad n = 1, 2, \ldots \text{ and}$$
$$X_n(x) = a_6 \sin\left(\frac{(2n-1)\pi}{2L} x\right), \quad n = 1, 2, \ldots$$

b) Use SOV to derive the solution $u(x,t)$ for the boundary value problem.

(20 marks)

QUESTION 2

Consider the following nonhomogeneous BVP,

$$u_t(x,t) = u_{xx} + 2x \qquad 0 \le x \le 1, \qquad t > 0$$
$$u(0,t) = 0, \qquad t > 0$$
$$u_x(1,t) + 3u(1,t) = 0, \quad t > 0$$
$$u(x,0) = 1, \qquad 0 < x < 1$$

a) Show that the system can be transformed into the following homogeneous BVP

$$v_t(x,t) = v_{xx}(x,t)$$
$$v(0,t) = 0$$
$$v_x(1,t) + 3v(1,t) = 0$$
$$v(x,0) = u(x,0) - \psi(x) = 1 + \frac{x^3}{3} - \frac{x}{2}$$

b) Show that the solution for the above boundary value problem is

$$v(x,t)=\sum_{n=1}^{\infty}\frac{\cos(z_n)\left(16-6(z_n)^2\right)+6(z_n)^2}{(z_n)^3\left(3+\cos^2(z_n)\right)}\sin(z_n x)e^{-(z_n)^2 t}.$$

c) Consequently determine the solution for $u(x,t)$.

(20 marks)

QUESTION 3

Consider the following homogeneous two-dimensional wave boundary value problem.

$$u_{tt}(x,y,t)=u_{xx}(x,y,t)+u_{yy}(x,y,t) \qquad 0<x<2\pi, \quad 0<y<2\pi, \quad t>0$$

subject to the boundary conditions

$$u(x,0,t)=u(x,2\pi,t)=0, \;\; 0<x<2\pi, \;\; t>0$$
$$u(0,y,t)=u(2\pi,y,t)=0, \;\; 0<y<2\pi, \;\; t>0$$

and initial conditions

$$u(x,y,0)=x^2\sin y, \;\; 0<x<2\pi, 0<y<2\pi$$
$$u_t(x,y,0)=0, \;\; 0<x<2\pi, 0<y<2\pi$$

a) Evaluate the double integral

$$\int_0^{2\pi}\int_0^{2\pi} x^2\sin y\,\sin\left(\frac{nx}{2}\right)\sin\left(\frac{my}{2}\right)dydx$$

b) Use the solution in part a) to determine the solution $u(x,y,t)$ for the boundary value problem.

(20 marks)

QUESTION 4

The steady state temperature $u(r,\theta)$ in a circular disc of radius 2 is modeled by the Dirichlet problem

$$u_{rr}(r,\theta)+\frac{1}{r}u_r(r,\theta)+\frac{1}{r^2}u_{\theta\theta}(r,\theta)=0, \quad 0<r<2, \quad 0<\theta<2\pi$$

and the temperature on the circumference is

$$u(2,\theta)=\begin{cases}\sin\theta & 0<\theta<\pi \\ 0 & \pi\le\theta<2\pi\end{cases}$$

Determine the solution $u(r,\theta)$.

(20 marks)

QUESTION 5

Consider the following nonhomogeneous wave equation

$$u_{tt}(x,t)=4u_{xx}(x,t)+xe^{-2t}, \quad 0<x<1, \quad t>0.$$
$$u(0,t)=0$$
$$u(1,t)=0$$
$$u(x,0)=0$$
$$u_t(x,0)=0$$

a) Using the method of eigenfunction expansion, show that $F_n(t)=2\dfrac{(-1)^{n+1}e^{-2t}}{n\pi}$.

b) Determine the general solution for $u_n(t)$ using the method of undetermined coefficients together with the condition $u(x,0)=0$.

c) Using the initial condition given, find the specific solution for the boundary value problem.

(20 marks)

END OF QUESTION PAPER

SOLUTION GUIDELINES FOR BVPs

	BVP	GENERAL SOLUTION
1.	$u_t(x,t)=\alpha^2 u_{xx}(x,t) \quad 0<x<L,\ t>0$ $\left.\begin{matrix} u(0,t)=0 \\ u(L,t)=0 \end{matrix}\right\},\quad t>0$ $u(x,0)=f(x), \quad 0<x<L$	$u(x,t)=\sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} sin\left(\frac{n\pi x}{L}\right)$ $A_n=\frac{2}{L}\int_0^L f(x) sin\left(\frac{n\pi x}{L}\right)dx, \quad n=1, 2, 3,\ldots$
2.	$u_t(x,t)=\alpha^2 u_{xx}(x,t), \quad 0<x<L,\ t>0$ $\left.\begin{matrix} u(0,t)=0 \\ u_x(L,t)=0 \end{matrix}\right\},\quad t>0$ $u(x,0)=f(x), \quad 0<x<L$	$u(x,t)=\sum_{n=1}^{\infty} A_n e^{-\left(\frac{(2n-1)\pi\alpha}{2L}\right)^2 t} sin\left(\frac{(2n-1)\pi x}{2L}\right)$ where $A_n=\frac{2}{L}\int_0^L f(x) sin\left(\frac{(2n-1)\pi x}{2L}\right)dx \quad n=1, 2, 3,\ldots$
3.	$u_{tt}(x,t)=\alpha^2 u_{xx}(x,t), \quad 0<x<L, \quad t>0$ $\left.\begin{matrix} u(0,t)=0 \\ u(L,t)=0 \end{matrix}\right\},\quad t>0$ $\left.\begin{matrix} u(x,0)=f(x) \\ u_t(x,0)=g(x) \end{matrix}\right\},\quad 0<x<L$	$u(x,t)=\sum_{n=1}^{\infty}\left(A_n cos\frac{n\pi\alpha t}{L}+B_n sin\frac{n\pi\alpha t}{L}\right) sin\frac{n\pi x}{L}$ where $A_n=\frac{2}{L}\int_0^L f(x) sin\left(\frac{n\pi x}{L}\right)dx, \quad n=1, 2, 3,\ldots$ $B_n=\frac{2}{n\pi\alpha}\int_0^L g(x) sin\left(\frac{n\pi x}{L}\right)dx \quad n=1, 2, 3,\ldots$

4.	$u_{tt} = k^2\left(u_{xx} + u_{yy}\right) \quad 0 < x < b,\ 0 < y < c,\ t > 0$ $u(x,0,t) = u(x,c,t) = 0 \quad 0 < x < b,\ t > 0$ $u(0,y,t) = u(b,y,t) = 0 \quad 0 < y < c,\ t > 0$ $u(x,y,0) = f(x,y),\ 0 < x < b, 0 < y < c$ $u_t(x,y,0) = g(x,y),\ 0 < x < b, 0 < y < c$	$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left\{A_{mn}\cos(a\omega t) + B_{mn}\sin(a\omega t)\right\} \sin\left(\frac{n\pi x}{b}\right)\sin\left(\frac{m\pi y}{c}\right)$ where $\omega = \sqrt{\frac{n^2\pi^2}{b^2} + \frac{m^2\pi^2}{c^2}}$ $A_{nm} = \frac{4}{bc}\int_0^c\int_0^b f(x,y)\sin(\frac{n\pi x}{b})\sin(\frac{m\pi y}{c})dxdy$ $B_{nm} = \frac{4}{abc\omega}\int_0^c\int_0^b g(x,y)\sin(\frac{n\pi x}{b})\sin(\frac{m\pi y}{c})dxdy$
5.	$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0,\ 0 < \theta < 2\pi,\ 0 < r < c$ $u(c,\theta) = f(\theta), \quad 0 < \theta < 2\pi$	$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n\left(A_n\cos n\theta + B_n\sin n\theta\right)$ where $A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$ $A_n = \frac{1}{c^n\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta$ $B_n = \frac{1}{c^n\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta$

Double Fourier sine formula

$$f(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}[A_{nm}]\sin\left(\frac{n\pi x}{b}\right)\sin\left(\frac{m\pi y}{c}\right)$$

$$A_{n_0m_0}=\frac{4}{bc}\int_0^b\int_0^c f(x,y)\sin\left(\frac{n_0\pi x}{b}\right)\sin\left(\frac{m_0\pi y}{c}\right)dydx$$

Identities

$$sin^2 a=\frac{1-cos\,2a}{2}$$

$$cos^2 a=\frac{1+cos\,2a}{2}$$

$$sin\,a\,sin\,b=\frac{cos(a-b)-cos(a+b)}{2}$$

$$cos\,a\,cos\,b=\frac{cos(a+b)+cos(a-b)}{2}$$

$$sin\,a\,cos\,b=\frac{sin(a+b)+sin(a-b)}{2}$$

$$cos\,a\,sin\,b=\frac{sin(a+b)-sin(a-b)}{2}$$

