11 Wedges, annuli, exterior of a circle

Similar to the circle considered last time, we can apply separation of variables in polar coordinates for any polar rectangle. Such examples are:

$$\begin{aligned} \text{A wedge:} \quad & \{0 < r < a, 0 < \beta\}, \\ \text{An annulus:} \quad & \{0 < a < r < b\}, \\ \text{Exterior of a circle:} \quad & \{a < r < \infty\}. \end{aligned}$$

We next treat each case separately, applying separation of variables to arrive at a series solution.

11.1 The wedge

Let us consider the following boundary problem in a wedge $D = \{0 < r < a, 0 < \beta\}$.

$$\begin{cases} u_{xx} + u_{yy} = 0 \quad \text{in } D, \\ u(r, 0) = u(r, \beta) = 0, \\ u(a, \theta) = h(\theta). \end{cases} \tag{1}$$

Proceeding as in the circle case, we look for separated solutions in terms of the polar coordinates, $u(r, \theta) = R(r)\Theta(\theta)$. Laplace's equation in polar coordinates will reduce to the following ODE's for $R(r)$ and $\Theta(\theta)$

$$r^2 R'' + rR' - \lambda R = 0, \qquad \Theta'' = -\lambda\Theta. \tag{2}$$

Noticing that the homogeneous Dirichlet boundary conditions on the lateral sides of the wedge imply $\Theta(0) = \Theta(\beta) = 0$, we will have the eigenvalue problem

$$\begin{cases} \Theta'' = -\lambda\Theta, \\ \Theta(0) = \Theta(\beta). \end{cases}$$

The eigenvalues and eigenfunctions of this problem are

$$\lambda_n = \left(\frac{n\pi}{\beta}\right)^2, \quad \Theta_n(\theta) = \sin\frac{n\pi\theta}{\beta}, \quad n = 1, 2, \ldots.$$

Using these values of λ, we can find the solution to the R equation. As we saw last time, the solution to this Euler type equation has the form $R(r) = r^\alpha$, where $\alpha = \pm\sqrt{\lambda}$. So for $\lambda = (n\pi/\beta)^2$, we have

$$R_n(r) = A_n r^{n\pi/\beta} + B_n r^{-n\pi/\beta}.$$

The $B_n r^{-n\pi\beta}$ term is not defined at the origin, which is a boundary point of the wedge. Thus, this term is discarded, since we are looking for a harmonic function which is also continuous on the boundary (this can be thought of as a boundary condition at $r = 0$ that the solution is finite). Now using the separated solutions $R_n(r)\Theta_n(\theta)$, we can write the series solution as

$$u(r, \theta) = \sum_{n=1}^{\infty} R_n(r)\Theta_n(\theta) = \sum_{n=1}^{\infty} A_n r^{n\pi/\beta} \sin\frac{n\pi\theta}{\beta}. \tag{3}$$

The coefficients A_n are determined by the boundary condition on the boundary $r = a$. Indeed, checking this condition gives

$$u(a, \theta) = h(\theta) = \sum_{n=1}^{\infty} A_n a^{n\pi/\beta} \sin\frac{n\pi\theta}{\beta},$$

which implies

$$A_n a^{n\pi/\beta} = h_n = \frac{2}{\beta}\int_0^\beta h(\theta)\sin\frac{n\pi\theta}{\beta}\,d\theta \quad \Rightarrow \quad A_n = \frac{h_n}{a^{n\pi/\beta}}.$$

So the solution to problem (1) is

$$u(r,\theta) = \sum_{n=1}^{\infty} h_n \left(\frac{r}{a}\right)^{n\pi/\beta} \sin\frac{n\pi\theta}{\beta},$$

with h_n being the Fourier sine coefficients of the Dirichlet data $h(\theta)$.

Similarly, one can solve boundary value problems in a wedge with Neumann or Robin boundary conditions on the boundary $r = a$. The series solution (3) will still be the same, but the coefficients A_n will be determined by the new condition.

11.2 The annulus

We consider the following Dirichlet problem in the annulus,

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } 0 < a^2 < x^2 + y^2 < b^2, \\ u = g(\theta), & \text{for } x^2 + y^2 = a^2, \\ u = h(\theta), & \text{for } x^2 + y^2 = a^2. \end{cases} \tag{4}$$

In this case the boundary condition of the eigenvalue problem for Θ is the same as in the circle case, $\Theta(\theta + 2\pi) = \Theta(\theta)$, which leads to both $\cos n\theta$ and $\sin n\theta$ being eigenfunctions corresponding to the eigenvalue $\lambda_n = n^2$. Also, all the components $\log r$, r^n and r^{-n} of $R_n(r)$ are allowed, since they are defined everywhere in the annulus. Hence, the series solution in polar coordinates can be written as

$$u(r,\theta) = \frac{1}{2}(C_0 + D_0\log r) + \sum_{n=1}^{\infty}(C_n r^n + D_n r^{-n})\cos n\theta + (A_n r^n + B_n r^{-n})\sin n\theta.$$

The coefficients are determined by the boundary conditions of (4). Indeed,

$$u(a,\theta) = g(\theta) = \frac{1}{2}(C_0 + D_0\log a) + \sum_{n=1}^{\infty}(C_n a^n + D_n a^{-n})\cos n\theta + (A_n a^n + B_n a^{-n})\sin n\theta,$$

and similarly for the boundary condition on $r = b$. But then we must have

$$\begin{cases} C_0 + D_0\log a = A_0^g \\ C_0 + D_0\log b = A_0^h, \end{cases} \qquad \begin{cases} C_n a^n + D_n a^{-n} = A_n^g \\ C_n b^n + D_n b^{-n} = A_n^h, \end{cases} \quad \text{and} \quad \begin{cases} A_n a^n + B_n a^{-n} = B_n^g \\ A_n b^n + B_n b^{-n} = B_n^h, \end{cases}$$

where A_n^g, B_n^g are the Fourier coefficients of $g(\theta)$, and A_n^h, B_n^h are the Fourier coefficients of $h(\theta)$. It is not hard to see that the determinants of the coefficients matrices of the above linear systems of equations are nonzero, provided $a \neq b$, guaranteeing unique solutions for the coefficients C_n, D_n for $n = 0, 1, \ldots$ and the coefficients A_n, B_n for $n = 1, 2, \ldots$. Again, other types of boundary conditions can be handled in a similar way.

11.3 Exterior of a circle

We finally turn to the Dirichlet problem for the exterior of a circle.

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } x^2 + y^2 > a^2, \\ u = h(\theta), & \text{for } x^2 + y^2 = a^2, \\ u \text{ bounded as } x^2 + y^2 \to \infty. \end{cases} \tag{5}$$

We again look for a series solution in terms of the separated solutions. The eigenvalue problem for $\Theta(\theta)$ will be exactly the same as in the case of the interior of a circle and the annulus. Notice, however, that in this case the terms $\log r$ and r^n of $R_n(r)$ must be discarded, since both of them are unbounded as $r \to \infty$, which can be thought of as a boundary point for the exterior of a circle. Thus, the series solution of (5) will be

$$u(r, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^{-n}(A_n \cos n\theta + B_n \sin n\theta). \tag{6}$$

The boundary condition then gives

$$u(a, \theta) = h(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^{-n}(A_n \cos n\theta + B_n \sin n\theta),$$

from which we get

$$A_n = a^n A_n^h = \frac{a^n}{\pi} \int_0^{2\pi} h(\phi) \cos n\phi \, d\phi, \qquad B_n = a^n B_n^h = \frac{a^n}{\pi} \int_0^{2\pi} h(\phi) \sin n\phi \, d\phi.$$

We can then substitute these expressions for A_n, B_n into the series solution (6). Comparing this to the circle case, we can see that the only difference in the solutions is that r is replaced by r^{-a}, and a by a^{-1}. So we can follow the same procedure, and sum the series explicitly, since in this case the magnitude of the (complex) ratios of the geometric series will be a/r, which is less than one in the exterior of the circle, thus leading to summable series. Then Poisson's formula in the exterior of the circle will be

$$\begin{aligned} u(r, \theta) &= (a^{-2} - r^{-2}) \int_0^{2\pi} \frac{h(\theta)}{a^{-2} - 2a^{-1}r^{-1}\cos(\theta - \phi) + r^{-2}} \, d\phi \\ &= (r^2 - a^2) \int_0^{2\pi} \frac{h(\theta)}{a^2 - 2ar\cos(\theta - \phi) + r^2} \, d\phi. \end{aligned}$$

11.4 Conclusion

Similar to the Dirichlet problem on the circle, we separated variables in polar coordinates to solve boundary value problems for Laplace's equation in several examples of polar rectangles. In general, Laplace's equation in any polar rectangle $\{a < r < b, \alpha < \theta < \beta\}$ can be solved by separating variables in polar coordinates, just as Cartesian rectangles were handled by separation of variables in Cartesian coordinates.

We also found that in the case of the exterior of a circle the series solution of the Dirichlet problem can be explicitly summed, similar to the case of the interior of a circle discussed last time. This gave a representation formula for the solution, using which one can find the values of the harmonic function in the domain D from its values on the boundary of the domain ∂D. In the coming lectures we will use Green's functions to derive similar representation formulas for more general domains.