Lecture 6 The Dirichlet Problem for the Disk

The Dirichlet problem in a disk of radius r_0 and center at $(0,0)$ can be expressed as

$$\begin{aligned} \text{PDE:} \quad & U_{rr} + \frac{U_r}{r} + \frac{U_{\theta\theta}}{r^2} = 0, \quad 0 < r < r_0, \ -\pi \le \theta \le \pi, \\ \text{BC:} \quad & U(r_0, \theta) = f(\theta), \ -\pi \le \theta \le \pi, \end{aligned} \tag{1}$$

where $f(\theta)$ is a given periodic, continuous function of period 2π ($f(\theta + 2\pi) = f(\theta)$). To solve the above problem, we use the method of separation of variables.

Step 1.(Writing the ODEs): Seek solutions of the form

$$U(r, \theta) = R(r)T(\theta),$$

where $0 \le r \le r_0$ and $-\pi \le \theta \le \pi$. Substituting into (1) and separating variables yield

$$\begin{aligned} & R''(r)T(\theta) + r^{-1}R'(r)T(\theta) + r^{-2}R(r)T''(\theta) = 0. \\ \Longrightarrow \quad & \frac{r^2 R''(r) + rR'(r)}{R(r)} = -\frac{T''(\theta)}{T(\theta)} = k. \end{aligned}$$

Which leads to the following two ODEs:

$$T''(\theta) + kT(\theta) = 0, \tag{2}$$

$$r^2 R''(r) + rR'(r) - kR(r) = 0. \tag{3}$$

Step 2.(Solving the ODEs):

Case (a): When $k < 0$, the general solution to (2) is the sum of two exponentials. Hence we have only trivial 2π-periodic solutions (see, Lecture 5).

Case (b): When $k = 0$, we find that $T(\theta) = A\theta + B$ is the solution to (2). This linear function is periodic only when $A = 0$, that is, $T_0(\theta) = B$ is the only 2π-periodic solution corresponding to $k = 0$.

Case (c): When $k > 0$, the general solution to (2) is

$$T(\theta) = A\cos(\sqrt{k}\theta) + B\sin(\sqrt{k}\theta).$$

In this case we get a nontrivial 2π-periodic solution only when $\sqrt{k} = n$, $n = 1, 2, \ldots$. Hence, we obtain the nontrivial 2π-periodic solutions

$$T_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta) \tag{4}$$

corresponding to $\sqrt{k} = n,\ n = 1, 2, \ldots$.

Now for $k = n^2$, $n = 0, 1, 2, \ldots$, equation (3) is the Cauchy-Euler equation

$$r^2 R''(r) + rR'(r) - n^2 R(r) = 0. \tag{5}$$

When $n = 0$, the general solution is

$$R_0(r) = C + D \ln r.$$

Since $\ln r \to \infty$ as $r \to 0^+$, this solution is unbounded near $r = 0$ when $D \neq 0$. Therefore, we must choose $D = 0$ if $U(r, \theta)$ is to be continuous at $r = 0$. We now have $R_0(r) = C$ and so $U_0(r, \theta) = R_0(r)T_0(\theta) = CB$. For convenience, we write $U_0(r, \theta)$ in the form

$$U_0(r, \theta) = \frac{A_0}{2}, \tag{6}$$

where A_0 is an arbitrary constant.

When $k = n^2$, $n = 1, 2, \ldots$, the general solution of (3) is given by

$$R_n(r) = C_n r^n + D_n r^{-n}.$$

Since $r^{-n} \to \infty$ as $r \to 0^+$, we must set $D_n = 0$ in order for $u(r, \theta)$ to be bounded at $r = 0$. Thus

$$R_n(r) = C_n r^n$$

Now for each $n = 1, 2, \ldots$, we have the solutions

$$U(r, \theta) = R_n(r)T_n(\theta) = C_n r^n [A_n \cos(n\theta) + B_n \sin(n\theta)].$$

By superposition principle, we write

$$U(r, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} C_n r^n [A_n \cos(n\theta) + B_n \sin(n\theta)].$$

This series may be written in the equivalent form

$$U(r, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^n [A_n \cos(n\theta) + B_n \sin(n\theta)], \tag{7}$$

where the A_n's and b_n's are constants. These constants can be determined from the boundary condition. With $r = r_0$ in (7), we have

$$f(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} [A_n \cos(n\theta) + B_n \sin(n\theta)].$$

Since $f(\theta)$ is 2π-periodic, we recognize that A_n, B_n are Fourier coefficients. Thus

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)d\theta, \quad n = 0, 1, \ldots, \tag{8}$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)d\theta, \quad n = 1, \ldots, \tag{9}$$

We now summarize the Dirichlet problem for a disk as follows.

$$\boxed{\begin{array}{l} \text{In the Dirichlet problem(1), if} \\ f(\theta) = \dfrac{A_0}{2} + \displaystyle\sum_{n=1}^{\infty}[A_n\cos(n\theta) + B_n\sin(n\theta)], \\ \text{then the solution is given by} \\ U(r,\theta) = \dfrac{A_0}{2} + \displaystyle\sum_{n=1}^{\infty}\left(\frac{r}{r_0}\right)^n [A_n\cos(n\theta) + B_n\sin(n\theta)], \end{array}}$$

where A_n and B_n are given by (8) and (9), respectively.

Example 1. *Solve the following BVP*

$$\begin{aligned} \textit{PDE:} \qquad & U_{rr} + \frac{U_r}{r} + \frac{U_{\theta\theta}}{r^2} = 0, \quad 0 \le r < 1, \\ \textit{BC:} \qquad & U(1,\theta) = f(\theta), \end{aligned}$$

where $f(\theta) = 1 + r\sin\theta + \frac{r^3}{2}\sin(3\theta) + r^4\cos(4\theta)$.

Solution. Here $r_0 = 1$. Note that $f(\theta)$ is already in the form of Fourier series, with

$$A_n = \begin{cases} 2 & \text{for } n = 0 \text{ and } 1 \text{ for } n = 4 \\ 0 & \text{for other } n \end{cases} \qquad B_n = \begin{cases} 1 & n = 1 \\ \frac{1}{2} & n = 3 \\ 0 & \text{for other } n \end{cases}$$

The solution of the BVP is

$$\begin{aligned} U(r,\theta) &= \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{r_0}\right)^n [A_n\cos(n\theta) + B_n\sin(n\theta)] \\ &= 1 + r\sin\theta + \frac{r^3}{2}\sin(3\theta) + r^4\cos(4\theta). \end{aligned}$$

Exterior Dirichlet Problem: We shall discuss the exterior Dirichlet problem i.e., the Dirichlet problem outside the circle. The exterior Dirichlet problem is given by

$$\begin{aligned} \text{PDE:} \qquad & U_{rr} + \frac{U_r}{r} + \frac{U_{\theta\theta}}{r^2} = 0, \quad 1 \le r < \infty, \\ \text{BC:} \qquad & U(1,\theta) = f(\theta), \quad 0 \le \theta \le 2\pi. \end{aligned}$$

This problem is solved exactly in a manner similar to the interior Dirichlet problem. We assume that the solutions are bounded as $r \to \infty$. Basically, we throw out the solutions

$$r^n \cos(n\theta), \quad r^n \sin(n\theta), \quad \ln r$$

that are unbounded as $r \to \infty$.

The solution is given by

$$U(r,\theta) = \sum_{n=0}^{\infty} r^{-n}[A_n \cos(n\theta) + B_n \sin(n\theta)], \tag{10}$$

where A_n and B_n are given by

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta, \\ A_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta, \\ B_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta. \end{aligned}$$

The detail procedure is thus left as an exercise.

Practice Problems

1. Solve the Dirichlet problem

$$\begin{aligned} &U_{xx} + U_{yy} = 0, \quad (x^2 + y^2 < 1), \\ &u(1,\theta) = \sin^2\theta, \quad -\pi \le \theta \le \pi, \end{aligned} \tag{11}$$

for the disk $r \le 1$.

2. Solve the BVP

$$\begin{aligned} &U_{rr} + \frac{U_r}{r} + \frac{U_{\theta\theta}}{r^2} = 0 \quad 0 \le r < 2, \quad -\pi < \theta < \pi, \\ &U(2,\theta) = 1 + 8\sin\theta - 32\cos(4\theta) \quad -\pi < \theta < \pi. \end{aligned}$$

3. Show that the exterior Dirichlet problem

$$\begin{aligned} &U_{rr} + \frac{U_r}{r} + \frac{U_{\theta\theta}}{r^2} = 0 \quad 1 \le r < \infty, \\ &U(1,\theta) = 1 + \sin\theta + \cos(3\theta) \;\; 0 < \theta < 2\pi, \end{aligned}$$

has the solution

$$U(r,\theta) = 1 + \frac{1}{r}\sin\theta + \frac{1}{r^3}\sin(3\theta).$$