Geometric Ray Theory

Huygen's Principle

Fermat's Principle: Minimum time path

Approximate to elastic wave solution to derive ray theory:

Equations of plane wave:

$$\Phi = A\,(x)e^{i\,(\pm \omega t \pm k.x)} \qquad \text{------(1)}$$

$$[x = (x_1, x_2, x_3)]$$

k→ point in the direction of wave propagation – represents a ray

For homogeneous material k is a straight line

Now, let ρ, λ, μ have small gradients, i.e. velocity V changes smoothly.

Original wave equations in homogeneous space

$$\left.\begin{aligned}&\nabla^2 \Phi = \frac{1}{V^2}\frac{\partial^2\Phi}{\partial t^2}\\ &\text{Or,}\quad \frac{\partial^2\Phi}{\partial x_1{}^2} + \frac{\partial^2\Phi}{\partial x_2{}^2} + \frac{\partial^2\Phi}{\partial x_3{}^2} = \frac{1}{v^2(x)}\frac{\partial^2\Phi}{\partial t^2}\end{aligned}\right] \quad \text{---(2)}$$

In the smoothly varying space approx. wave equations:

$$\nabla^2 \Phi = \frac{1}{V(x)^2}\frac{\partial^2\Phi}{\partial t^2} \quad \text{------(3)}$$

Let $\omega W(x)/V_0$ replace k.x (V_0 is a ref. velocity)

then, $\Phi = A(x)e^{i\omega\,(W(x)/V_0 - t)}$

Or,

$$\nabla^2\left[A(x)e^{i\omega\,(W(x)/V_0 - t)}\right] = \frac{1}{V(x)^2}\frac{\partial^2\left[A(x)e^{i\omega\,(W(x)/V_0 - t)}\right]}{\partial t^2} \quad \text{------(4)}$$

Compute,

$$\frac{\partial^2\Phi}{\partial x_1^2} = \frac{\partial}{\partial x}\left\{\frac{\partial A(x)}{\partial x_1}e^{i\omega\,(W(x)/V_0 - t)} + A(x)\frac{i\omega}{V_0}\frac{\partial W(x)}{\partial x_1}e^{i\omega\,(W(x)/V_0 - t)}\right\}$$

$$= \left[\frac{\partial^2 A(x)}{\partial x_1^2} - \frac{\omega^2 A(x)}{V_0^2}\left(\frac{\partial W(x)}{\partial x_1}\right)^2 + i\left(\frac{2\omega}{V_0}\frac{\partial A(x)}{\partial x_1}\frac{\partial W(x)}{\partial x_1} + A(x)\frac{\omega}{V_0}\frac{\partial^2 W(x)}{\partial x_1^2}\right)\right] e^{i\omega\,(W(x)/V_0 - t)}$$

Similarly calculate $\frac{\partial^2\Phi}{\partial t^2}$, $\frac{\partial^2\Phi}{\partial x_2^2}$, $\frac{\partial^2\Phi}{\partial x_3^2}$ and put in (4)

- **Equate real and imaginary part separately equal to zero. We get**

$$\nabla^2 A(x) - A(x)\ \frac{\omega^2}{V_0^2}\left[\left(\frac{\partial W(x)}{\partial x_1}\right)^2 + \left(\frac{\partial W(x)}{\partial x_2}\right)^2 + \left(\frac{\partial W(x)}{\partial x_3}\right)^2\right] = \frac{-\omega^2}{V^2(x)} A(x) \text{ -------- } (5)$$

$$2\left(\frac{\partial W(x)}{\partial x_1}\frac{\partial A(x)}{\partial x_1} + \frac{\partial W(x)}{\partial x_2}\frac{\partial A(x)}{\partial x_2} + \frac{\partial W(x)}{\partial x_3}\frac{\partial A(x)}{\partial x_3}\right) + A(x)\nabla^2 W(x) = 0 \text{ -------- } (6)$$

From (5)

$$\left(\frac{\partial W(x)}{\partial x_1}\right)^2 + \left(\frac{\partial W(x)}{\partial x_2}\right)^2 + \left(\frac{\partial W(x)}{\partial x_3}\right)^2 - \frac{V_0^2}{V^2(x)} = \frac{V_0^2}{A(x)\,\omega^2}\left(\nabla^2 A(x)\right) \text{ -------- } (7)$$

For high frequency, and small variation in A(x) caused by small variation in V(x) R.H.S. is negligibly small $\simeq$ 0

$$\left(\frac{\partial W(x)}{\partial x_1}\right)^2 + \left(\frac{\partial W(x)}{\partial x_2}\right)^2 + \left(\frac{\partial W(x)}{\partial x_3}\right)^2 = \frac{V_0^2}{V^2(x)} \text{ -------- } (8)$$

Eikonal Equations

Solution of this is not exact, but for many regions inside the earth the necessary restrictions on spatial variations of elastic parameters are satisfied- so solution are useful.

Eikonal equations are **partial differential equations that relates rays to the seismic velocity distribution**.

$$W(x)\ \omega/V_0 = k(x).x$$

Conditions for geometric ray theory to be useful approximation of wave equations, grad A(x) over one wavelength must be smaller than A(x). Let reference wavelength $\lambda_0 = 2\ \pi\ V_0/\omega$. The for equation 8 to hold we require that

For equations (8) to hold we require that (from eqs. 7 $V_0/\omega=\lambda_0$)

$$\lambda_0^2 \left(\nabla^2 A(x) / A(x)\right) << \nabla W(x) . \nabla W(x)$$

this gives $$\lambda_0^2 \left(\nabla^2 A(x) / A(x)\right) << V_0^2 / V^2(x) \qquad \text{---(9)}$$

for weak inhomogenity $$V_0^2 / V^2(x) \sim 1 \qquad \text{---(10)}$$

Therefore, $$\lambda_0^2 \left(\nabla^2 A(x) / A(x)\right) << 1 \qquad \text{---(11)}$$

To gain physical insight into this equation

$$\nabla W(x) . \nabla W(x) = V_0^2 / V^2(x) \qquad \textbf{implies}$$

$$\nabla W(x) \simeq V_0 / V(x) \qquad \text{---(12)}$$

From eq. (6) we can write

$$\nabla^2 W(x) \approx -2 \nabla W(x) . \nabla A(x) / A(x) \qquad \text{---(13)}$$

or,

$$\frac{\nabla A(x)}{A(x)} \approx -\tfrac{1}{2}\frac{\nabla^2 W(x)}{\nabla W(x)} \approx -\tfrac{1}{2}\frac{\nabla (V_0/V(x))}{V_0/V(x)} \quad \text{------(14)}$$

$$\approx \tfrac{1}{2}\nabla V(x)/V(x)$$

If we further compute gradient over a wavelength and multiply by $\boldsymbol{\lambda_0}^2$ → using (9), we can write

$$\lambda_0^2 \frac{\nabla^2 A(x)}{A(x)} \approx \tfrac{1}{2}\frac{\lambda_0\, \delta\, \Delta V(x)}{V(x)} \ll 1 \qquad \text{------(15)}$$

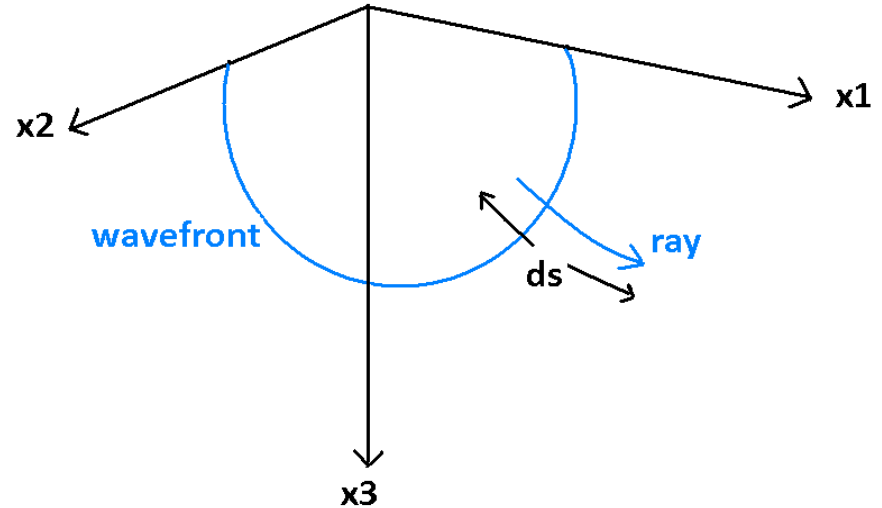
This means Eikonal equations will approximate the wave equations well if the fractional change in velocity gradient over one seismic wavelength is small compared to the velocity.

The Eikonal Equations and Ray Geometry

Ray W(x) is characterised by travelling an arc length ‘**s**’, in time ‘**t**’.

The direction cosines associated with ray is

dx_1/ds, dx_2/ds and dx_3/ds and satisfies



$$\left(\frac{dx_1}{ds}\right)^2 + \left(\frac{dx_2}{ds}\right)^2 + \left(\frac{dx_3}{ds}\right)^2 = 1 \qquad \text{---- (16)}$$

Physical connection between W(x) and 's'

W(x) is constant over the wave front , it's gradient represents ray direction, ray path length is ds

(i.e. gradient of a fn. which is constant over a surface (fn.) is oriented normal to that surface.

Therefore, $(\partial W(x)/\partial x_i) \ \alpha \ (dx_i/ds)$

Equations (16) can be written as (a= constant of proportionality)

$$\left(a \frac{\partial W(x)}{\partial x_1}\right)^2 + \left(a \frac{\partial W(x)}{\partial x_2}\right)^2 + \left(a \frac{\partial W(x)}{\partial x_3}\right)^2 = 1 \qquad \text{---- (17)}$$

Compare (17) with eq (8), ➔ (17) is Eikonal equations if $a = V(x)/V_0$ and $a^{-1} = n = V_0/V(x)$ = index of refraction

Equations (16) and (17) can be combined to get normal equations.

$$\left.\begin{aligned} n\frac{dx_1}{ds} &= \frac{\partial W(x)}{\partial x_1} \\ n\frac{dx_2}{ds} &= \frac{\partial W(x)}{\partial x_2} \\ n\frac{dx_3}{ds} &= \frac{\partial W(x)}{\partial x_3} \end{aligned}\right] \quad \text{----} \; (18)$$

How normal equations change along the ray path ?

$$\frac{d}{ds}\left(n\frac{dx_1}{ds}\right) = \frac{d}{ds}\left(\frac{\partial W(x)}{\partial x_1}\right)$$

$$= \frac{\partial}{\partial x_1}\left(\frac{\partial W(x)}{\partial x_1}\frac{dx_1}{ds} + \frac{\partial W(x)}{\partial x_2}\frac{dx_2}{ds} + \frac{\partial W(x)}{\partial x_3}\frac{dx_3}{ds}\right)$$

$$= \frac{\partial}{\partial x_1}\left[n\left\{\left(\frac{dx_1}{ds}\right)^2 + \left(\frac{dx_2}{ds}\right)^2 + \left(\frac{dx_3}{ds}\right)^2\right\}\right]$$

$$= \frac{\partial}{\partial x_1} n \qquad \text{------}(19)$$

The generalized form of this equation is called the **Raypath equations**

$$\frac{d}{ds}\left(\frac{n\, dx_i}{ds}\right) = \frac{\partial n}{\partial x_i}$$

$$\frac{d}{ds}\left(\frac{1}{V(x)}\frac{dx}{ds}\right) = \nabla\left(\frac{1}{V(x)}\right) \qquad \text{------(20)}$$

2^{nd} order differential equations for x- which is just the raypath. Raypath is α to spatial change in velocity distribution.

Two initial conditions control behavior of (20)

1. **Direction in which the ray leaves some arbitrary reference point $(\partial x/ \partial s)\,|_{s=0}$**
2. **Position of reference point S_0**

Example to gain physical insight in equation (20)

Ex: When $V = V(x_3)$, $n = n(x_3)$, $\partial n / \partial x_1 = \partial n / \partial x_2 = 0$

Then eqs (20) reduces to

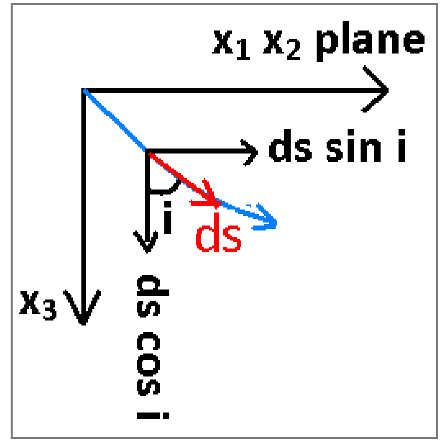
$$\frac{d}{ds}\left(n \frac{dx_3}{ds}\right) = \frac{dn}{dx_3} \qquad \text{----(21)}$$

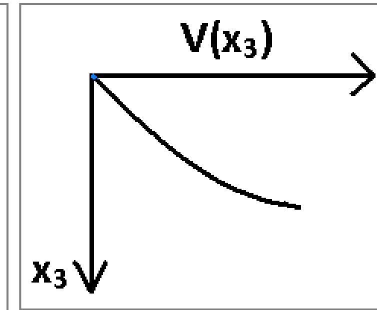
$$n \frac{dx_1}{ds} = V_1 = \text{constant}$$

$$n \frac{dx_2}{ds} = V_2 = \text{constant}$$

The ratio of V_1 / V_2 confines the raypath to a plane perpendicular to x_1 x_2 plane, i.e. its projection into the x_1 x_2 plane is a straight line.

$$\frac{d}{ds}\left(n\frac{dx_3}{ds}\right) = \frac{dn}{dx_3} \qquad \text{----(21)}$$





$$l_1 = \frac{dx_1}{ds} = \sin i \qquad \text{---(22)}$$

$$l_3 = \frac{dx_3}{ds} = \cos i \qquad \text{---(23)}$$

Therefore, $n\frac{dx_1}{ds} = \frac{V_0}{V(x)}\sin i$ =constant = n sin i

sin i / V(x) =constant = p = ray parameter ---(24) snell's law

= horizontal slowness

p=0 → perpendicular path, p =1/V → horizontal path,
i= angle of incidence

From eqs (21) $\Rightarrow \dfrac{dn}{dx_3} = \dfrac{d}{ds}\left(n \dfrac{dx_3}{ds}\right) = \dfrac{d}{ds}(n \cos i)$

$$= \frac{dn}{ds}\cos i + n\frac{d \cos i}{ds} = \cos i \frac{dn}{dx_3}\frac{dx_3}{ds} + n\frac{d \cos i}{di}\frac{di}{ds}$$

$$= -n \sin i \frac{di}{ds} + \cos^2 i \frac{dn}{dx_3}$$

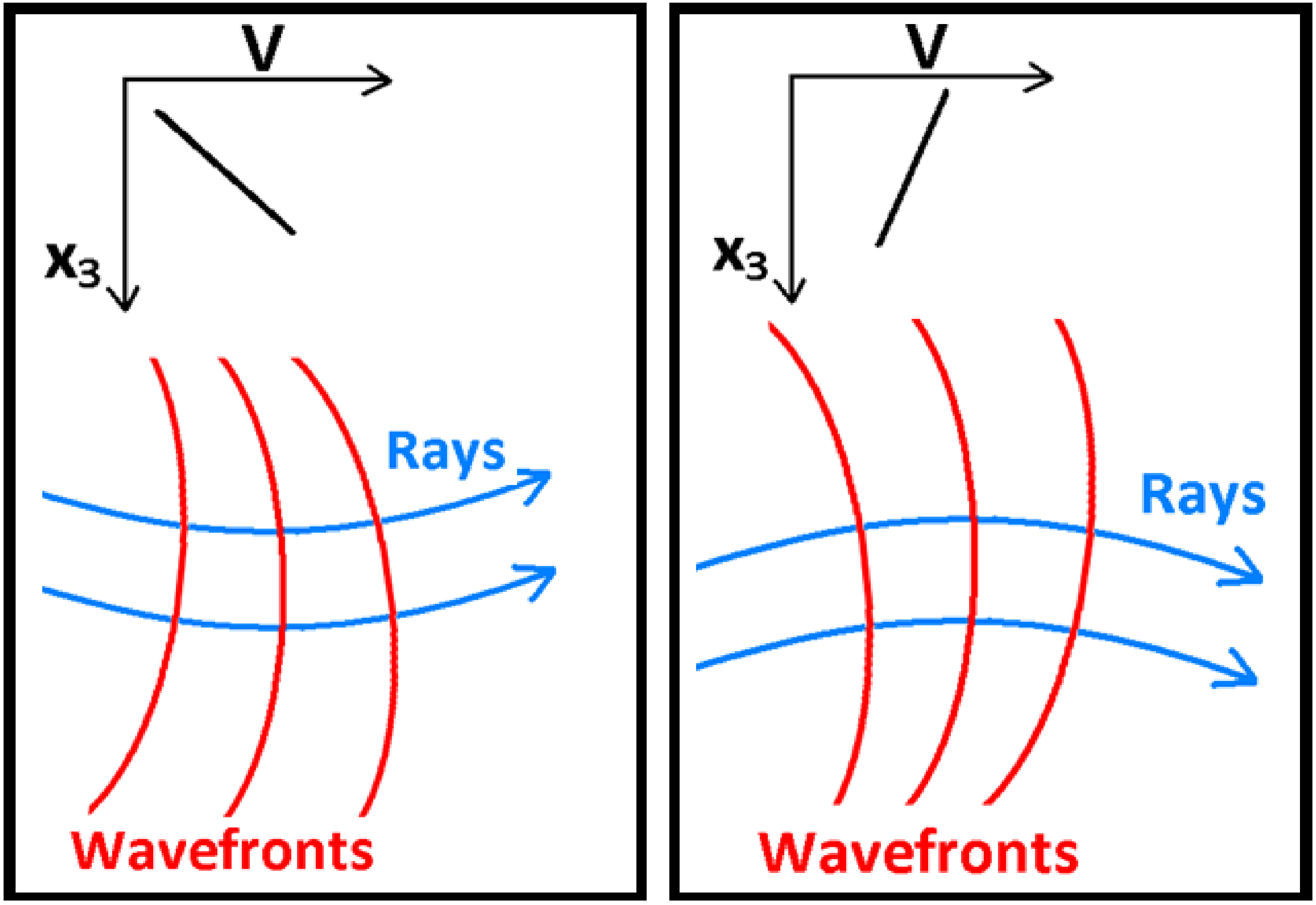
or, $(1-\cos^2 i)\dfrac{dn}{dx_3} = -n \sin i \dfrac{di}{ds}$ $\quad$ [here, $n = V_0 / V(x_3)$]

or, $\sin^2 i \dfrac{dn}{dx_3} = -n \sin i \dfrac{di}{ds}$

Or, $\dfrac{di}{ds} = -\dfrac{\sin i}{n}\dfrac{dn}{dx_3} = -\dfrac{\sin i}{V_0 / V(x_3)}\dfrac{d}{dx_3}\left(\dfrac{V_0}{V(x_3)}\right)$

$$= -\sin i \, . V(x_3) \, . \frac{d}{dx_3}\left(\frac{1}{V(x_3)}\right) = +\frac{\sin i}{V(x_3)}\frac{dV(x_3)}{dx_3}$$

$$\text{or,}\quad \frac{di}{ds} = p\,\frac{dV(x_3)}{dx_3} \qquad \text{----(25)}$$



i.e. curvature of ray α velocity gradient

Equation (21) has several interesting aspects:

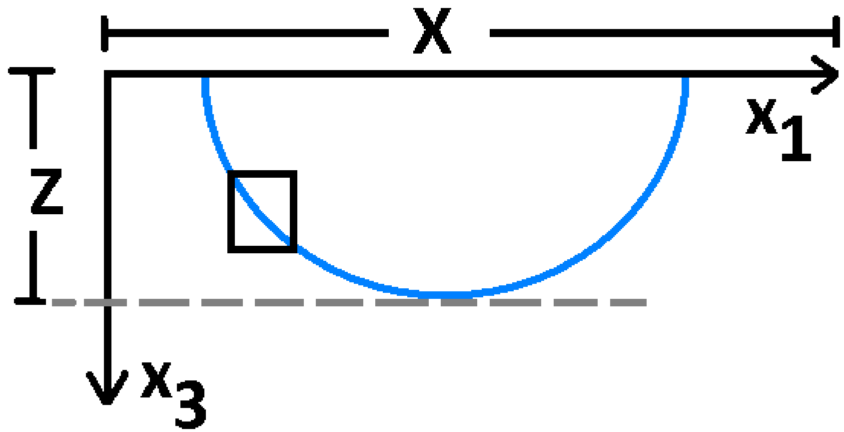
$$\frac{d}{ds}\left(n\frac{dx_3}{ds}\right) = \frac{dn}{dx_3} \quad \text{----(21)}$$

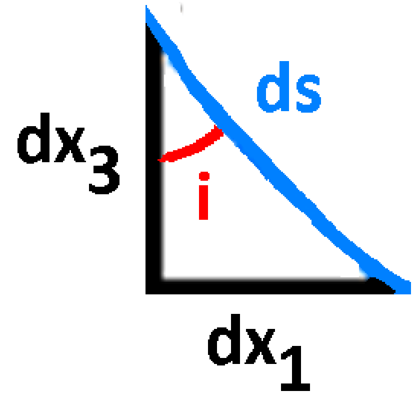
$$n\frac{dx_1}{ds} = V_1 = \text{constant} \qquad n\frac{dx_2}{ds} = V_2 = \text{constant}$$

i) For each angle i, a specific ray leaves the source and follows a specific raypath.

ii) The initial angle and the velocity structure determine the distance at which the ray will emerge at the surface

iii) For a given source- receiver geometry several possible connecting raypaths may exist, which means that a multiplicity of arrival will occur, all with different initial angles and travel times- this is the basis for seismic interpretation of Earth's structure.





Z is maximum depth of penetration

$$\sin i = \frac{dx_1}{ds} = pV \quad [V = V(x_3)] \qquad \cos i = \frac{dx_3}{ds} = \sqrt{1 - \sin^2 i} = \sqrt{1 - p^2 V^2}$$

$$dx_1 = ds \sin i = \frac{dx_3}{\cos i}\, p\, V = \frac{p\, V}{\sqrt{1 - p^2 V^2}}\, dx_3$$

$$\boxed{X(p) = 2 \int_0^z \frac{p\, V}{\sqrt{1 - p^2 V^2}}\, dx_3}$$

$$\text{Now, } dT = \frac{ds}{V} \longrightarrow T = 2 \int_{\text{path}} \frac{ds}{V} = 2 \int_0^2 \frac{dx_3}{V \cos i} = \text{Travel time}$$

$$X = 2p\int_0^2 \frac{dx_3}{\sqrt{\gamma^2 - p^2}} \qquad \text{-----}(26)$$

where $\gamma = 1/V$

$$T = 2\int_0^2 \frac{\gamma^2 dx_3}{\sqrt{\gamma^2 - p^2}} \qquad \text{-----}(27)$$

$$T = 2\int_0^2 \frac{\gamma^2 dx_3}{\sqrt{\gamma^2 - p^2}} = 2\int_0^2 \left(\frac{p^2}{\sqrt{\gamma^2 - p^2}} + \sqrt{\gamma^2 - p^2} \right) dx_3$$

$$= p\,X + 2\int_0^2 \sqrt{\gamma^2 - p^2}\; dx_3 \qquad \text{-----}(28)$$

Travel time equations: Two separable terms

$$T = pX + 2\int_0^2 \eta \, dx_3$$

one depends on X

other depends on Z

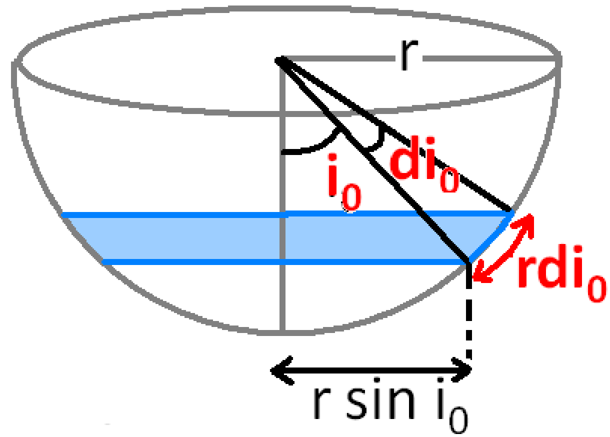
Horizontal Vertical

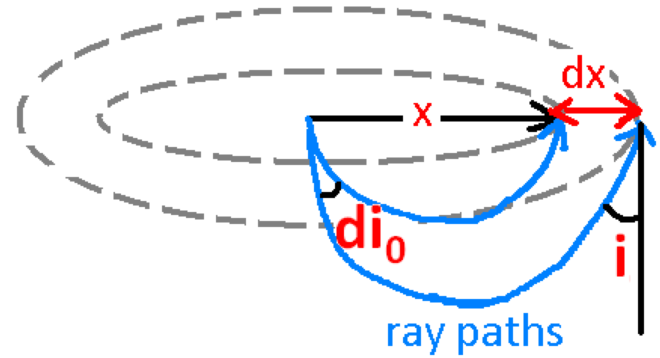
Slowness

$$\frac{dT}{dX} = p$$

Change in travel time with distance = ray parameter

(used extensively in interpreting the structure of the Earth)





Amplitude:

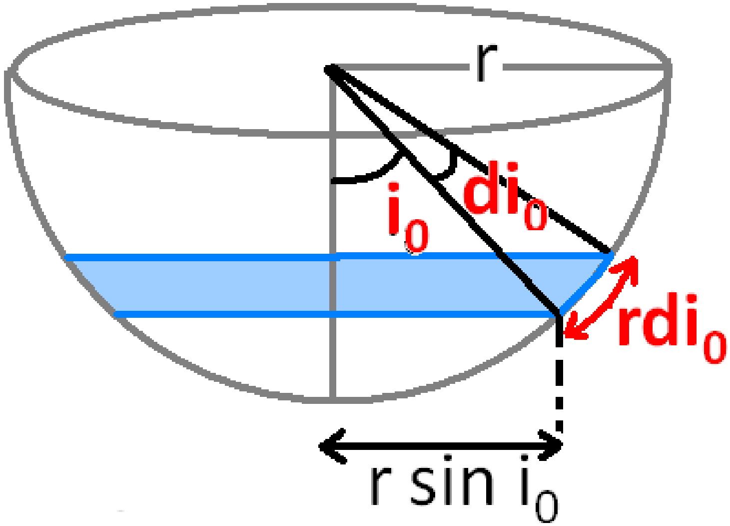
Start from equations (21)- get insight about amplitude variation.

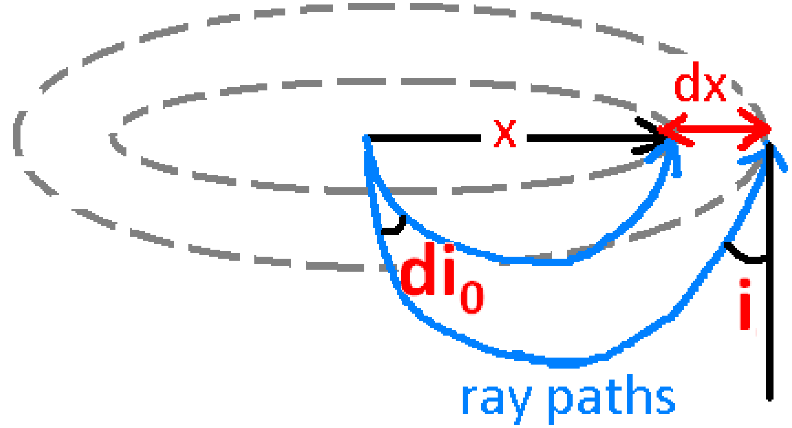
Consider a spherical wave at small distance from the seismic source in a region of uniform velocity.

Total energy over hemispherical wavefront = K

Energy/unit area = $K/\ 2\pi r^2$

Let a bundle of rays leave the source between angle i_0 and $i_0 + di_0$





Fraction of energy in a circular ring on the wavefront defined by two takeoff angles:

$$E = (K/\ 2\pi r^2)\ (2\pi r \sin i_0)\ (di_0 r)$$

$$= K \sin i_0\, di_0$$

The corresponding energy spreads out on area

$$2\pi\ X\ dX \cos i_0$$

Therefore, Energy density = $$E(X) = \frac{K}{2\pi} \frac{\tan i_0}{X} \frac{di_0}{dx}$$

Now, $p = \sin i_0 / V_0 = dT/dX$; $\quad i_0 = \sin^{-1}(V_0\, dT/dX)$

Therefore, $$\frac{di_0}{dX} = \frac{V_0}{\sqrt{1 - V_0^2 (dT/dX)^2}} \frac{d^2T}{dX^2}$$

$$= \frac{V_0}{\cos i_0} \frac{d^2T}{dX^2}$$

$$\boxed{E(X) = \frac{K}{2\pi} V_0 \left(\frac{\tan i_0}{X \cos i_0} \right) \frac{d^2T}{dX^2} = \frac{K}{2\pi} \frac{V_0^2\, p}{(1 - p^2 V_0^2)} \frac{dp}{dX}}$$

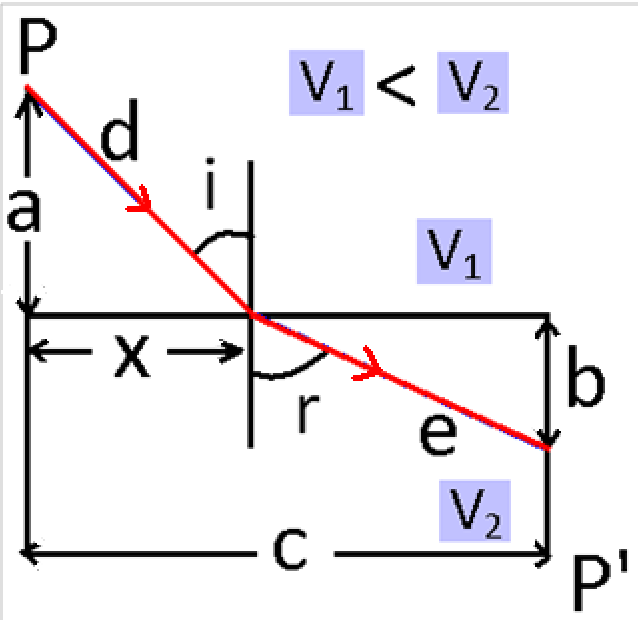
Amplitude $A(x) \propto \sqrt{E(x)}$

$A(x) \propto$ change in ray parameter with distance.

For velocity structure for which p changes rapidly yield large amplitude variation. When p is constant $\rightarrow$ very small amplitude variation

Geometric interpretation of Snell's law

(using simple ray geometry and Fermat's principle of least time)



$$T_{PP'} = \frac{d}{V_1} + \frac{e}{V_2} = \frac{\sqrt{a^2 + x^2}}{V_1} + \frac{\sqrt{b^2 + (c-x)^2}}{V_2}$$

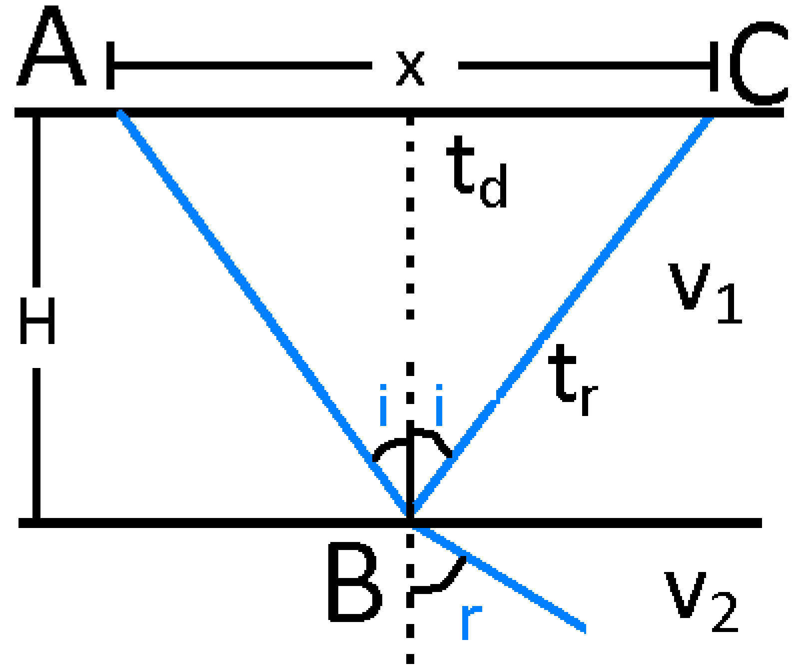
For minimum time path $dT/dx = 0$

$$\text{or, } \frac{x}{V_1\sqrt{a^2 + x^2}} - \frac{c-x}{V_2\sqrt{b^2 + (c-x)^2}} = 0$$

$$\text{or, } \frac{\sin i}{V_1} = \frac{\sin r}{V_2} = p$$

(ray parameter) [Snell's law]

Travel Time in a layered Earth

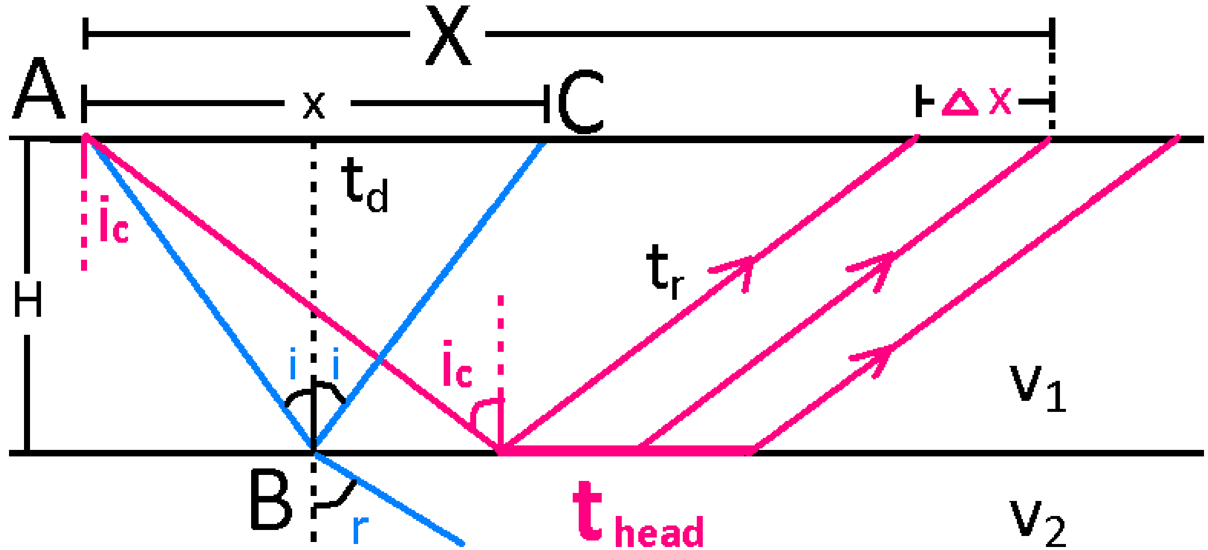


$$t_d = \frac{x}{V_1}$$

$$t_r = \frac{AB}{V_1} + \frac{BC}{V_1}$$

$$= \frac{2\left[\left(\frac{x}{2}\right)^2 + H^2\right]^{1/2}}{V_1} = \frac{\sqrt{x^2 + 4H^2}}{V_1}$$

Travel Time in a layered Earth

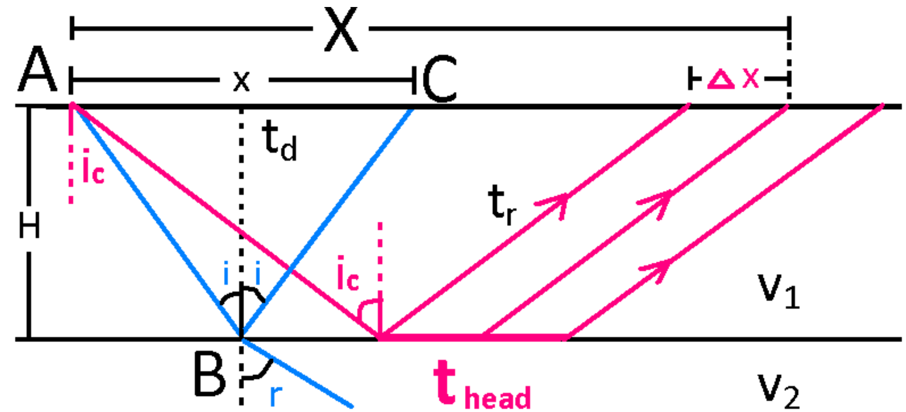


$\frac{\sin i}{V_1} = \frac{\sin r}{V_2}$ | If $V_1 < V_2$ then at $i = i_c$, $r = 90°$

Therefore, $ic = \sin^{-1}\left(\frac{V_1}{V_2}\right)$ → critically refracted ray

$\frac{x/2}{H} = \tan i_c$ | $X_{critical} = 2H \tan i_c$

$$t_{critical} = \frac{\sqrt{(2H \tan i_c)^2 + 4H^2}}{V_1}$$



$$t_{critical} = \frac{\dfrac{2H\sqrt{\sin^2 i_c + \cos^2 i_c}}{\cos^2 i_c}}{V_1} = \frac{2H}{V_1 \cos i_c}$$

If $X = x_{critical} + \Delta x$ --> where the ray travels the path Δx with velocity V_2.

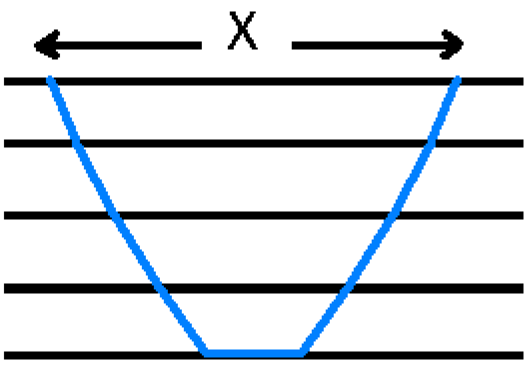
$X_{head} = 2H \tan i_c + \Delta x$

$$\text{and}\quad T_{head} = \frac{2H}{V_1 \cos i_c} + \frac{\Delta x}{V_2} = \frac{2H}{V_1 \cos i_c} + \frac{X_{head} - 2H \tan i_c}{V_2}$$

$$= \frac{X_{head}}{V_2} + \frac{2H}{V_1 \cos i_c}\left(1 - \frac{V_1}{V_2} \sin i_c\right)$$

$$\boxed{= \frac{X_{head}}{V_2} + \frac{2H}{V_1 \cos i_c}\left(1 - \sin^2 i_c\right) \qquad = \frac{X_{head}}{V_2} + \frac{2H \cos i_c}{V_1}}$$

Travel – time for a multi-layered Earth



$$\boxed{T = pX + 2\sum_{i=1}^{n} H_i \eta_i}$$

$$p = \frac{\sin i_i}{V_i} = \frac{1}{V_r}$$

$$\eta_i = (1 - p^2 V_i^2)^{1/2} / V_i$$

For continuous velocity variation

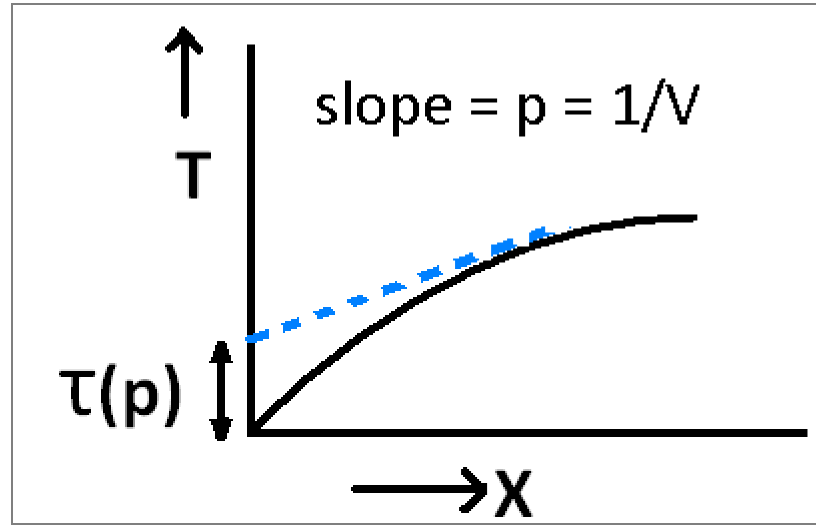
$$T = pX + 2\int_0^{z\,max} \eta\, dz \qquad = pX + 2\int_0^{z\,max} \sqrt{\gamma^{2.} - p^2}\, dx_3$$

Intercept time $\tau(p) = T - pX$

$$= 2\int_0^{z} \sqrt{\gamma^{2} - p^2}\, dx_3$$

$$\frac{d\tau}{dp} = \frac{d}{dp}\left(2\int_0^z \sqrt{\gamma^2 - p^2}\, dx_3\right)$$

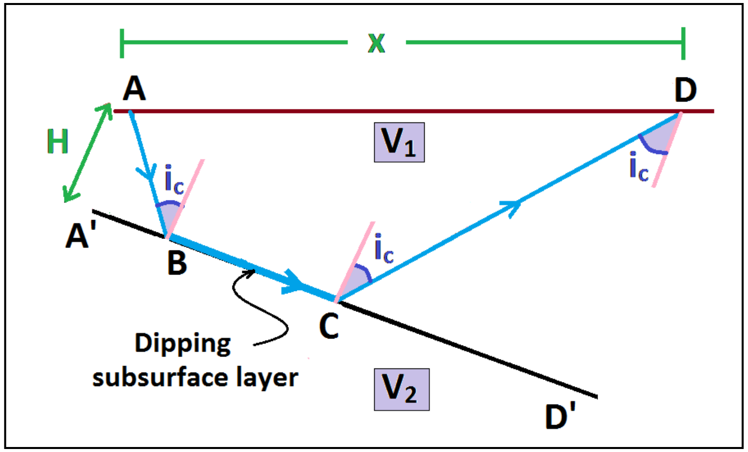
$$= 2\int_0^z \frac{-p}{\sqrt{\gamma^2 - p^2}}\, dx_3 = -X$$

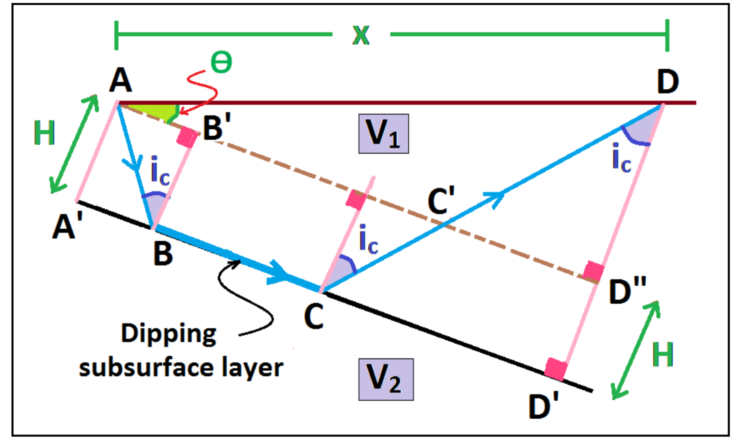


τ(p) increases as p decreases and X increases.

τ(p) is a single valued function of p and can simplify analysis of travel- time curves.

Travel time equation for down dip refraction





$AD'' = A'D'$ $\quad$ $AB = H / \cos i_C$ $\quad$ $DD' = x \sin \Theta + H$

$AD'' = AD \cos \Theta = x \cos \Theta$ $\quad$ $DD'' = AD \sin \Theta = x \sin \Theta$

$BC = A'D' - (A'B + CD') = x \cos \Theta - \tan i_C [2H + x \sin \Theta]$

$$\text{Travel time} = \frac{AB}{V_1} + \frac{BC}{V_2} + \frac{CD}{V_1} \qquad \left[\; CD = CC' + C'D \right.$$

$$= \frac{H}{V_1 \cos i_c} + \frac{1}{V_2}\left[x \cos \Theta - \left\{ \frac{H \sin i_c}{\cos i_c} + (x \sin \Theta + H) \tan i_c \right\} + \frac{H}{V_1 \cos i_c} + \frac{x \sin \Theta}{V_1 \cos i_c} \right]$$

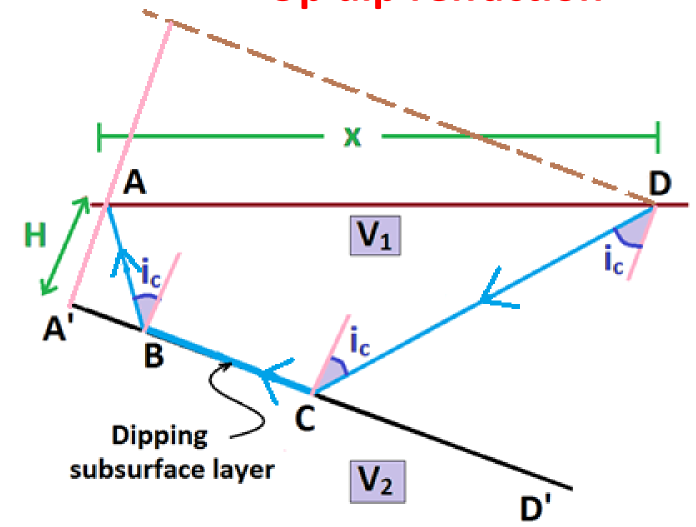
$$= \frac{H}{V_1 \cos i_c} + \frac{\sin i_c}{V_1}\left[x \cos \theta - \{H \tan i_c + x \sin \theta \tan i_c + H \tan i_c\} \right] + \frac{H}{V_1 \cos i_c} + \frac{x \sin \theta}{V_1 \cos i_c}$$

$$= \frac{2H}{V_1 \cos i_c} + \frac{x \cos \theta \sin i_c}{V_1} - \frac{2H \sin^2 i_c}{V_1 \cos i_c} - \frac{x \sin \theta \sin^2 i_c}{V_1 \cos i_c} + \frac{x \sin \theta}{V_1 \cos i_c}$$

$$= \frac{2H}{V_1 \cos i_c}\left[1 - \sin^2 i_c\right] + \frac{x \cos \theta \sin i_c}{V_1} + \frac{x \sin \theta}{V_1 \cos i_c}\left[1 - \sin^2 i_c\right]$$

$$\boxed{= \frac{2H}{V_1} + \frac{x}{V_1} \sin(i_c + \theta)}$$

Up dip refraction



For up dip refraction the travel time equation changes to

$$\boxed{= \frac{2H}{V_1} + \frac{x}{V_1} \sin(i_c - \theta)}$$

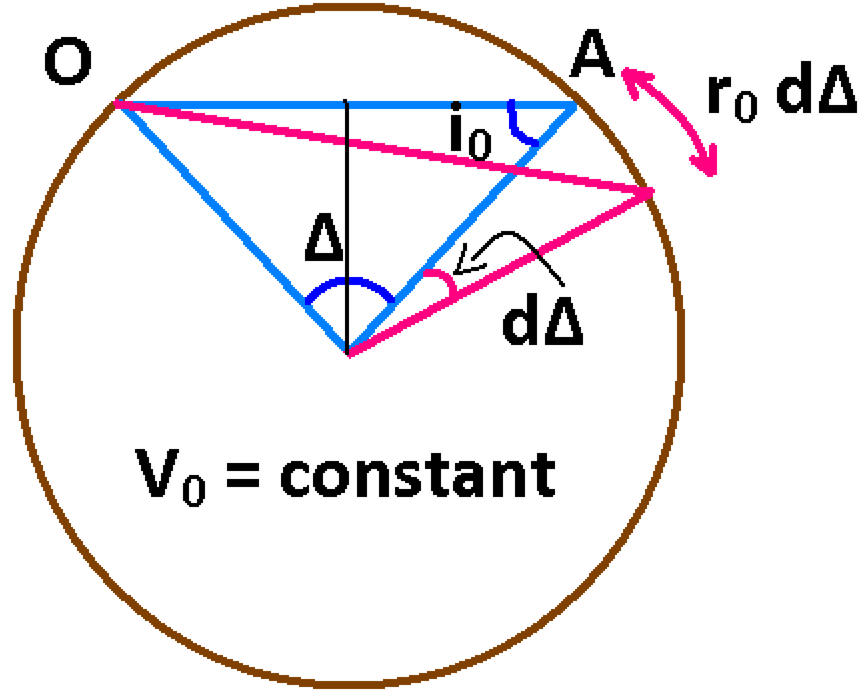
Travel Time Equations in a sphere

For a homogeneous sphere

$$T(\Delta) = \frac{OA}{V} = \frac{2\, r_0 \sin(\Delta/2)}{V_0}$$

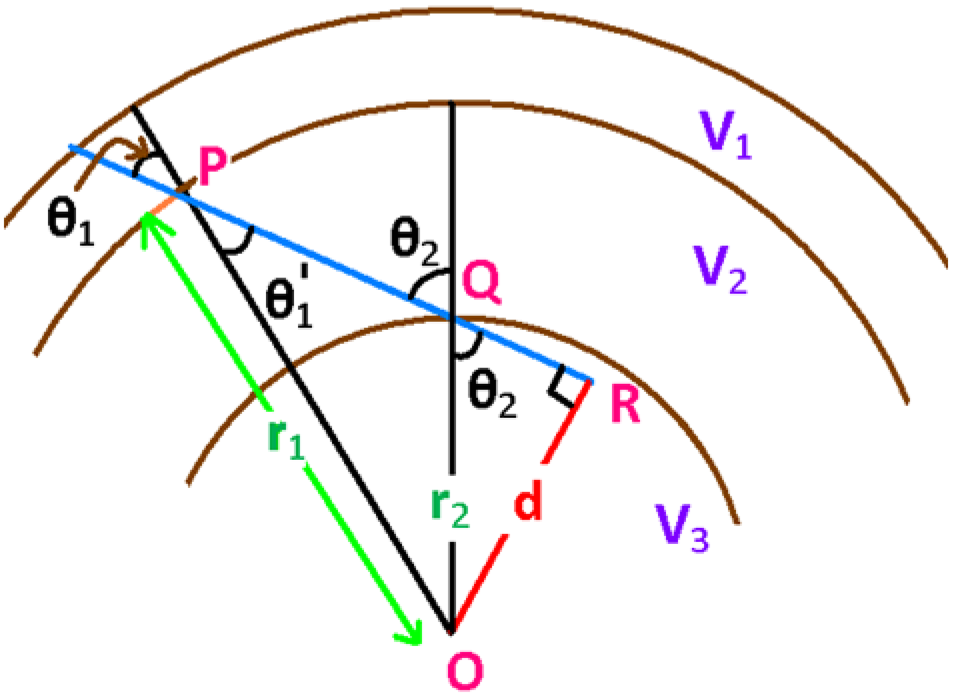
$$p = \frac{r_0 \sin i_0}{V_0} = \frac{r_0 \cos(\Delta/2)}{V_0}$$

[since $i_0 = 90° - \Delta/2$]



Important: **Even though velocity is constant, the travel time curve is not a straight line but has decreasing ray parameter with distance.**

Travel times in a layered spherical Earth



On local scale surface curvature is negligible. So at position P Snell's law must be satisfied:

$$\frac{\sin\theta_1}{V_1} = \frac{\sin\theta_1'}{V_2} \qquad \text{-----(1)}$$

If two right- angled triangles share length 'd' (Δ POR and Δ QOR, then $d = r_1 \sin\theta_1' = r_2 \sin\theta_2$ -----(2)

Then from (1) $$\frac{r_1 \sin\theta_1}{V_1} = \frac{r_1 \sin\theta_1'}{V_2} = \frac{r_2 \sin\theta_2}{V_2}$$

Therefore, $$\frac{r_1 \sin \theta_1}{V_1} = \frac{r_2 \sin \theta_2}{V_2}$$

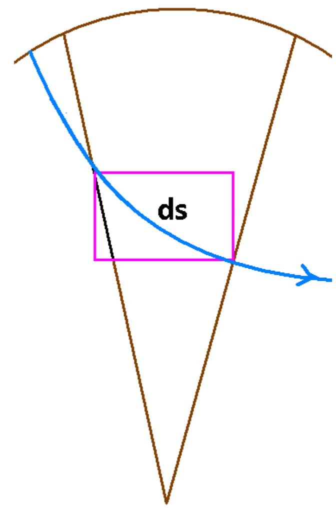
General equation along the entire raypath. Since r_1 and r_2 can be any value along the raypath.

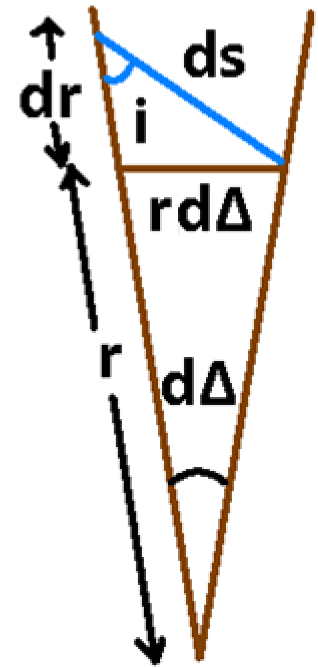
Therefore, for a spherical earth $\frac{r \sin i}{V} = p$ ---(3)

Note: Unit of p for spherical Earth differ from that in flat Earth, but the meaning is same.

For a sphere where velocity is not constant

$$(ds)^2 = (dr)^2 + r^2\,(d\Delta)^2 \qquad \text{---- (1)}$$





$$\sin i = \frac{r\,d\Delta}{ds}$$

$$p = \frac{r \sin i}{V} = \frac{r^2\,d\Delta}{V\ ds} \qquad \text{---- (2)}$$

$$ds = (1/p)\,r^2\,\frac{d\Delta}{V}$$

Use (2) in (1) to eliminate ds

Therefore, $\frac{1}{p^2}\frac{r^4}{V^2}(d\Delta)^2 - r^2 (d\Delta)^2 = (dr)^2$

or, $(d\Delta)^2 = \frac{(dr)^2\, p^2 V^2}{r^4 - r^2 p^2 V^2}$

or, $d\Delta = \frac{pV}{r}\frac{dr}{\sqrt{r^2 - p^2 V^2}} = \frac{p\, dr}{r\sqrt{\xi^2 - p^2}}$ (where $\xi = r/V$)

Therefore,

$$\Delta = 2p \int_{r_i}^{r_0} \frac{dr}{r\sqrt{\xi^2 - p^2}}$$

[r_0 is the radius of the Earth
r_i is the deepest point of penetration]

Use (2) in (1) to eliminate $d\Delta$ then

$$(ds)^2 = (dr)^2 + r^2 \left(\frac{pV\, ds}{r^2} \right)^2 \qquad \text{---- (1)}$$

$$= (dr)^2 + \frac{p^2V^2}{r^2} (ds)^2$$

$$\text{or, } ds = \frac{dr}{\sqrt{1-(p^2V^2 / r^2)}} = \frac{\xi\, dr}{\sqrt{\xi^2 - p^2}}$$

Then travel time along the path

$$T = \int ds / V = 2 \int_{r_t}^{r_0} \frac{\xi \; dr}{V \sqrt{\xi^2 - p^2}}$$

$$= 2 \int_{r_t}^{r_0} \frac{\xi^2 \; dr}{r \sqrt{\xi^2 - p^2}} \qquad \text{[where } r/V = \xi \text{]}$$

$$= 2 \int_{r_t}^{r_0} \left(\frac{p^2}{r \sqrt{\xi^2 - p^2}} + \frac{\xi^2 - p^2}{r \sqrt{\xi^2 - p^2}} \right) dr$$

$$= p \, \Delta + 2 \int_{r_t}^{r_0} \frac{\sqrt{\xi^2 - p^2}}{r} \; dr$$

$$= p\,\Delta + 2\int_{r_t}^{r_0} \frac{\sqrt{\xi^2 - p^2}}{r}\,dr$$

depends on Δ or surface horizontal distance

depends only on r, the ⊥ dimension. This is analogous to the flat earth case τ(p) function.

Three different velocity structures and corresponding ray geometries, travel time curves, ray parameter Vs distance and Tau Vs ρ