MA1301 Introductory Mathematics

Chapter 4
Vectors

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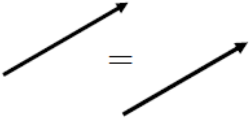
Vectors - Definitions

A directed line segment PQ

Direction: direction of the arrow
Magnitude: length of the line segment
Examples: velocity, gravitational force, magnetic field e.t.c.

Two vectors are equal if they have the same direction and length.

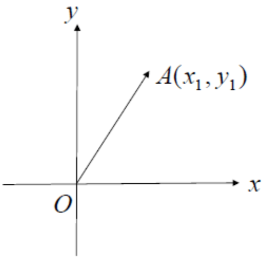


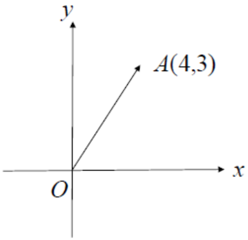
Magnitude of a vector $\mathbf{a}$, denoted by $|\mathbf{a}|$ (or $\|\mathbf{a}\|$) $=$ length of vector $\mathbf{a}$
On the $x - y$ plane, if A has coordinates (x_1, y_1) and O is the origin, then the position vector of A,

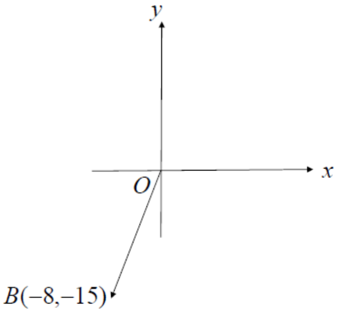
$$\overrightarrow{OA} = \mathbf{a}$$

has magnitude

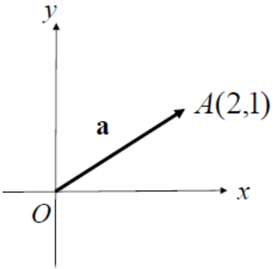
$$|\mathbf{a}| = \sqrt{x_1^2 + y_1^2}$$




y
A(4,3)
x
O

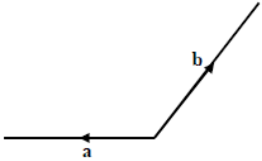

y
x
O
B(−8,−15)

Let λ be a scalar.
$\lambda\mathbf{a}$ is the vector that is parallel to $\mathbf{a}$ and has magnitude $|\lambda||\mathbf{a}|$.
$\lambda > 0 \Rightarrow$ $\mathbf{a}$ and $\lambda\mathbf{a}$ are are in the same direction
$\lambda < 0 \Rightarrow$ $\mathbf{a}$ and $\lambda\mathbf{a}$ are are in the opposite direction



Example

Draw $\mathbf{a} + 2\mathbf{b}$, $\frac{1}{2}\mathbf{a} - \mathbf{b}$, $\mathbf{b} - \mathbf{a}$



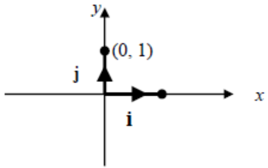
Given $A(x_1, y_1)$ and $B(x_2, y_2)$,

Then $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$

Let the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be denoted by **i** and **j** respectively.

Then, any vector $\begin{pmatrix} x \\ y \end{pmatrix}$ can be expressed as

$$x\mathbf{i} + y\mathbf{j}$$



Example

If $\mathbf{u} = \mathbf{i} + \mathbf{j}$ and $\mathbf{v} = \mathbf{i} - 6\mathbf{j}$, find $|2\mathbf{u} + \mathbf{v}|$.

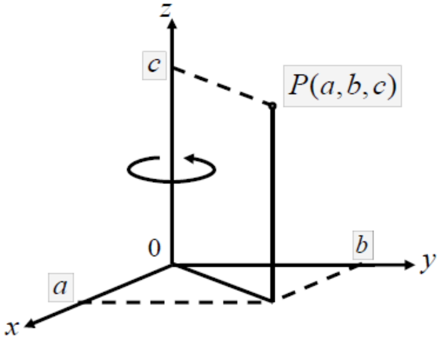
Example

Given that P is $(3, -2)$ and $\overrightarrow{QP} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, find the point Q.

The Cartesian Coordinate System

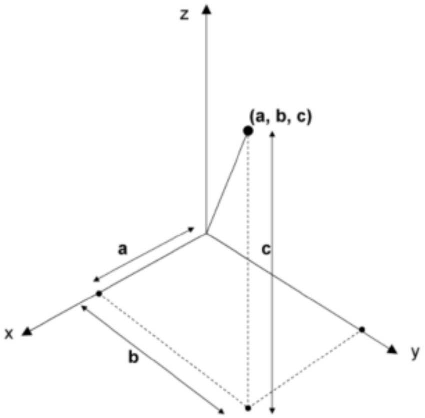
Rectangular Coordinates

Right-handed coordinates system



If we rotate the x-axis counter-clockwise toward the y-axis, then a right-handed screw will move in the positive z direction.

3-D Coordinate System

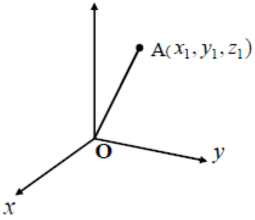


Position vectors in 3-D Cartesian system

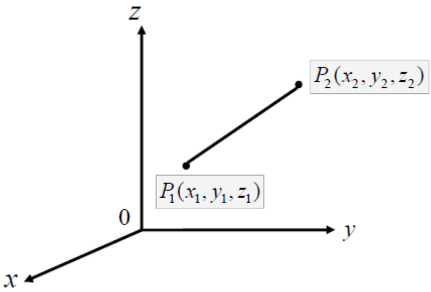
For any point $A(x_1, y_1, z_1)$ the vector $\overrightarrow{OA}$ = position vector of A with respect to O.

$$\overrightarrow{OA} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k} \text{ where } \mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$|\overrightarrow{OA}| = \sqrt{x_1^2 + y_1^2 + z_1^2}$$



Distance between two points



For two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, the length of P_1P_2 is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Two vectors **a** and **b** are $//$ $\Leftrightarrow$ $\mathbf{a} = \lambda \mathbf{b}$ for some scalar $\lambda \neq 0$.

A, B and C are collinear (that is, they lie on the same straight line) if and only if

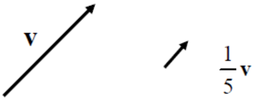
$$\overrightarrow{AB} // \overrightarrow{AC} \quad (// \overrightarrow{BC})$$

Example

Given $A(1, p, 3)$ and $B(1, 5, -1)$, find the possible values of p if $|\overrightarrow{AB}| = 5$.

Unit Vectors

Vectors of length 1



Suppose $\|\mathbf{v}\| = 5$, then $\frac{1}{5}\mathbf{v}$ will have length 1.

To find unit vector: $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$

Example

Suppose $\|\mathbf{v}\| = 5$.
Find a vector with length 7 and in the direction $\mathbf{v}$.

Unit Vectors

$\hat{a}$ = unit vector in the direction of **a**

Thus

$$\hat{a} = \frac{1}{|\mathbf{a}|}\mathbf{a},$$

in other words,

$$\hat{a} \text{ is the vector } // \ \mathbf{a}$$

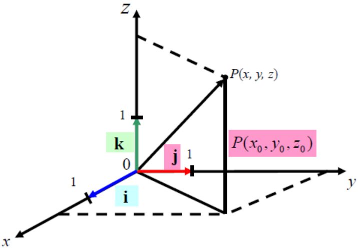
and **has magnitude** 1

$\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors in the direction of x-, y- and z-axis respectively.

Note that:

every vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ can be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$



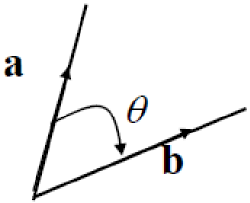
Example

Given $A(2,-2,1)$, $B(-2,1,1)$, $C(h,3,k)$
find
(i) $\hat{a}$
(ii) the two vectors // **a** and having magnitude 18
(iii) the values of h and k, given that A, B and C are collinear.

Scalar Product

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

where θ is the angle between **a** and **b**.



Results

$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

$\lambda(\mathbf{a} \cdot \mathbf{b}) = (\lambda\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\lambda\mathbf{b})$ for any $\lambda \in \mathbb{R}$.

$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

$\mathbf{a} \cdot \mathbf{b} = 0$ if and only if $\mathbf{b} \perp \mathbf{a}$ for non-zero vectors $\mathbf{a}$ and $\mathbf{b}$

Example

The vectors $\mathbf{p}$ and $\mathbf{q}$ are such that $|\mathbf{p}| = 5$, $|\mathbf{q}| = 6$ and the angle between $\mathbf{p}$ and $\mathbf{q}$ is 120°.
Calculate the exact value of
(i) $\mathbf{p} \cdot \mathbf{q}$ (ii) $(\mathbf{p} + 2\mathbf{q}) \cdot (2\mathbf{p} - \mathbf{q})$.

Example

$$\text{Let } \mathbf{u} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$$
$$\mathbf{v} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$$
$$\text{Then } \mathbf{u} \cdot \mathbf{v} = x_1x_2 + y_1y_2 + z_1z_2$$

Example

Given $A(3,-4,0)$ and $B(2,2,-1)$, find

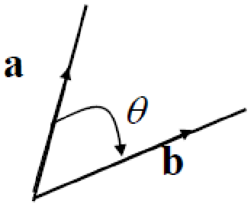
(i) $\overrightarrow{OA} \cdot \overrightarrow{OB}$ (ii) $\overrightarrow{OA} \cdot \overrightarrow{BA}$.

Angle between vectors **a** and **b**

Formula:

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2+y_1^2+z_1^2}\cdot\sqrt{x_2^2+y_2^2+z_2^2}}$$

where $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$



Example

Given $A(2,-2,1)$, $B(-2,1,2)$ and $C(1,0,-1)$, calculate angle CBA. Hence, find the shortest distance between C and the line segment AB.

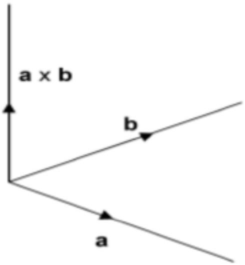
Definition and properties

Vector Product of $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a} \times \mathbf{b}$ is defined as follows:

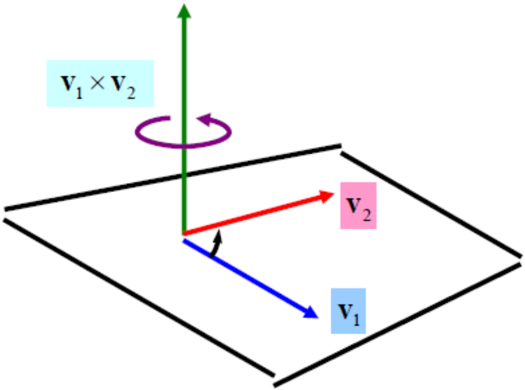
(i) $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$

(ii) direction of $\mathbf{a} \times \mathbf{b}$ is given by the right-hand rule

(iii) $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$, where $\theta =$ angle between $\mathbf{a}$ and $\mathbf{b}$



$\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$



Right hand

Results

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

$$\mathbf{a} \times \mathbf{a} = \mathbf{0}$$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\lambda(\mathbf{a} \times \mathbf{b}) = (\lambda\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\lambda\mathbf{b}) \text{ for any scalar } \lambda$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

Vector Product

Method 1

Let $\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$.

Then their ***vector product*** or ***cross product*** is the vector

$$\begin{aligned} \mathbf{v}_1 \times \mathbf{v}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \\ &= (y_1 z_2 - y_2 z_1)\mathbf{i} - (x_1 z_2 - x_2 z_1)\mathbf{j} + (x_1 y_2 - x_2 y_1)\mathbf{k} \end{aligned}$$

Recall that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ $\qquad \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} = y_1 z_2 - y_2 z_1$

$$\begin{array}{ccc} + & - & + \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{array}$$

Let

$$\mathbf{u} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$$
$$\mathbf{v} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$$

Find

$$\mathbf{u} \times \mathbf{v}$$

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \times \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - x_2 z_1) \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$$

$$\begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} = y_1 z_2 - y_2 z_1$$

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_1 y_2 - x_2 y_1$$

$$\begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix} = x_1 z_2 - x_2 z_1$$

These are the so-called determinants of 2×2 square matrices

Let $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 3 \\ -1 \\ -3 \end{pmatrix}$.

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 3 & -1 & -3 \end{vmatrix}$$

$$= -\mathbf{i} + 12\mathbf{j} - 5\mathbf{k}$$

Recall that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{matrix} + & - & + \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{matrix}$$

Vector Product - Example

Let $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 3 \\ -1 \\ -3 \end{pmatrix}$.

$$\begin{aligned} \mathbf{v}_1 \times \mathbf{v}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 3 & -1 & -3 \end{vmatrix} \\ &= -\mathbf{i} + 12\mathbf{j} - 5\mathbf{k} \end{aligned}$$

Recall that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{matrix} + & - & + \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{matrix}$$

Example

$(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \times (2\mathbf{i} - \mathbf{j} - \mathbf{k}) =$

Homework: $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$ **Answer**: $\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$

$$
\begin{array}{ccccc}
 & \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} \\
\mathbf{a} & x_1 & y_1 & z_1 & x_1 \\
\mathbf{b} & x_2 & y_2 & z_2 & x_2
\end{array}
$$

$$
\begin{array}{ccc}
(x_1y_2 - x_2y_1) & (y_1z_2 - y_2z_1) & (z_1x_2 - z_2x_1) \\
\mathbf{k} & \mathbf{i} & \mathbf{j}
\end{array}
$$

Example

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

Applications of vector products

Area of triangle $ABC = \frac{1}{2}|\overrightarrow{CA} \times \overrightarrow{CB}|$

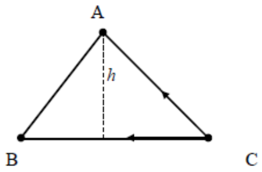
Area of triangle $ABC = \frac{1}{2}|\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}|$

Example

Find the area of triangle ABC given $A = (1, 2, 3)$, $B = (2, 4, 6)$ and $C = (-2, 3, -4)$

Shortest distance h from A to BC is

$$h = \frac{|\overrightarrow{CA} \times \overrightarrow{CB}|}{|\overrightarrow{CB}|} = \frac{|\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}|}{|\mathbf{b} - \mathbf{c}|}$$

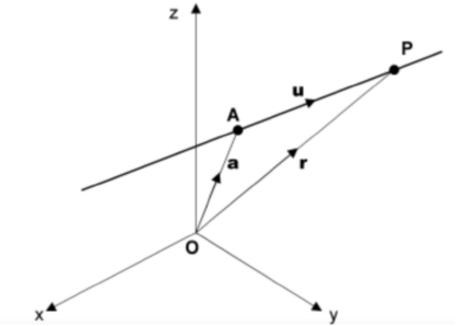


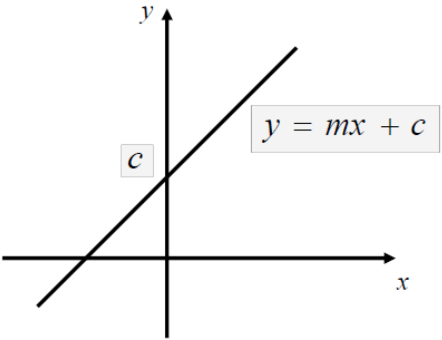
Lines in 3-D Space

The line L passes through a point A and is parallel to a vector $\mathbf{u}$ has vector equation

$$\mathbf{r} = \overbrace{\mathbf{a}}^{\text{point}} + \lambda \overbrace{\mathbf{u}}^{\substack{\text{direction}\\ \text{vector}}}$$

where $\lambda \in \mathbb{R}$

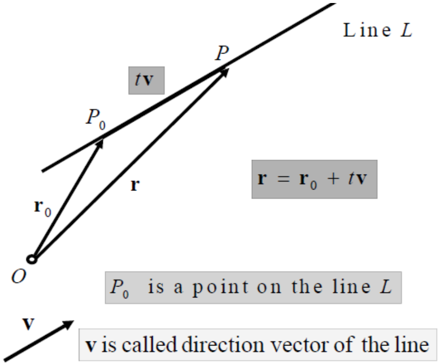




To determine a line, we need

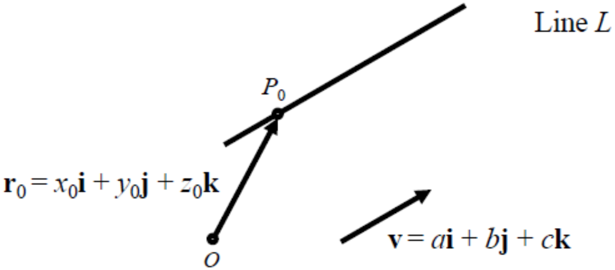
gradient m $\qquad$ y-intercept c

Find the equation of the line L



Different values of t gives different points on the line L.

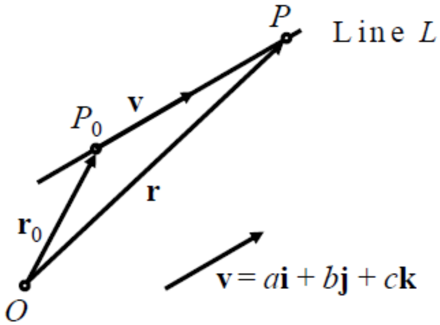
Find the equation of the line L



The point P_0 bring you to the line L.

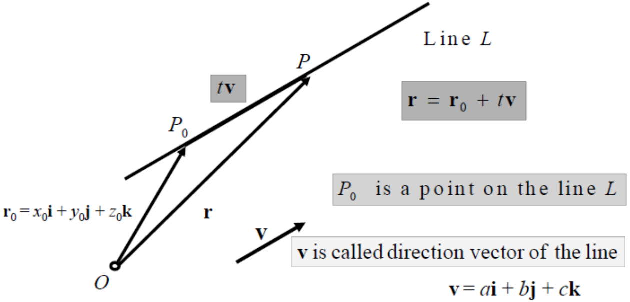
Vector Equation of a Line

Let $P(x, y, z)$ be any point on L with position vector **r**.



If you walk in the direction parallel to **v**, then you will be always on the line L.
In this way, you can reach any point on the line L.

Find the equation of the line L



Then a ***vector equation*** of L is

$$\mathbf{r} = (x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}) + t(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$$

or

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$$

$$
\begin{aligned}
\text{Write } \mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\
&= (x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}) + t(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}).
\end{aligned}
$$

Equating, we have

$$
\begin{cases}
x = x_0 + at \\
y = y_0 + bt \\
z = z_0 + ct
\end{cases}
$$

which are the parametric equations of the line L.

Vector equation of line L (denoted by $\mathbf{r}$) passing through

$$A(x_0, y_0, z_0)$$

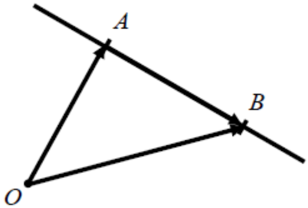
and parallel to

$$\mathbf{u} = d\mathbf{i} + e\mathbf{j} + f\mathbf{k}$$
$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{u} \qquad (\lambda \in \mathbb{R})$$

$$\begin{aligned}
\mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\
&= (x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}) + \lambda(d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) \\
&= (x_0 + \lambda d)\mathbf{i} + (y_0 + \lambda e)\mathbf{j} + (z_0 + \lambda f)\mathbf{k}
\end{aligned}$$
$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} d \\ e \\ f \end{pmatrix}$$

Example

The points A and B have position vectors $-3\mathbf{i}+2\mathbf{j}-3\mathbf{k}$ and $\mathbf{i}-\mathbf{j}+4\mathbf{k}$ respectively. Write down the parametric equations of the line passing through A and B.



$$\begin{aligned}\overrightarrow{AB} &= (\mathbf{i}-\mathbf{j}+4\mathbf{k})-(-3\mathbf{i}+2\mathbf{j}-3\mathbf{k})\\ &= 4\mathbf{i}-3\mathbf{j}+7\mathbf{k}\end{aligned}$$

We may take $\mathbf{v}=\overrightarrow{AB}$ as the direction vector of line AB.
The vector equation is

$$\mathbf{r}=(-3\mathbf{i}+2\mathbf{j}-3\mathbf{k})+t(4\mathbf{i}-3\mathbf{j}+7\mathbf{k}).$$

Example

The vector equation is

$$\mathbf{r} = (-3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + t(4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}).$$

Write $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Hence the parametric equations of line the passing through A and B are

$$\begin{cases} x = -3 + 4t \\ y = 2 - 3t \\ z = -3 + 7t \end{cases}$$

Example

Find the vector equation of the line that passes through $A(1,2,3)$ and // to the vector $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$. Does the point $C(-8,-3,0)$ lie on this line?

Example

Find the vector equation of the line that passes through $A(1,2,3)$ and $B(3,4,6)$. Show that the point $D(9,10,15)$ lies on this line.

Intersecting Lines and Skew Lines

Given 2 lines

$$\mathbf{r} = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + \lambda \begin{pmatrix} r_1 \\ s_1 \\ t_1 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} + \lambda \begin{pmatrix} r_2 \\ s_2 \\ t_2 \end{pmatrix}$$

there are four possibilities:

- They are coincident/identical
- They are parallel, not coincident
- They are non-parallel and intersecting
- They are non-parallel and non-intersecting (skew lines)

Example

Find the position vector of the point of intersection of L_1 and L_2.

$$L_1 : \mathbf{r} = \mathbf{i} + t_1(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}),$$

$$L_2 : \mathbf{r} = (2\mathbf{i} + \mathbf{j}) + t_2(3\mathbf{i} + \frac{9}{2}\mathbf{j} + \frac{9}{2}\mathbf{k}).$$

Eliminating $\mathbf{r}$ from the vector equations of L_1 and L_2, we get

$$\boxed{\mathbf{i}} + \boxed{t_1(\mathbf{i}} + 2\mathbf{j} + 3\mathbf{k}) = (\boxed{2\mathbf{i}} + \mathbf{j}) + \boxed{t_2(3\mathbf{i}} + \frac{9}{2}\mathbf{j} + \frac{9}{2}\mathbf{k}).$$

Hence it follows that

$$\boxed{1 + t_1 = 2 + 3t_2}, \quad 2t_1 = 1 + \frac{9}{2}t_2, \quad 3t_1 = \frac{9}{2}t_2$$

from which we obtain $t_1 = -1$ and $t_2 = -\frac{2}{3}$.

Find the position vector of the point of intersection of L_1 and L_2.

$$L_1 : \mathbf{r} = \mathbf{i} + t_1(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}),$$

$$L_2 : \mathbf{r} = (2\mathbf{i} + \mathbf{j}) + t_2(3\mathbf{i} + \frac{9}{2}\mathbf{j} + \frac{9}{2}\mathbf{k}).$$

Putting $t_1 = -1$ into the vector equation of L_1, we obtain

$$\mathbf{r} = \mathbf{i} + (-1)(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = -2\mathbf{j} - 3\mathbf{k}.$$

So the position vector of the point of intersection P of the two lines is

$$\overrightarrow{OP} = -2\mathbf{j} - 3\mathbf{k}.$$

Example

Determine whether the following pair of lines intersect. If they do, find the position vector of their point of intersection.

(i) $\mathbf{r} = 2\mathbf{i} + 5\mathbf{k} + s(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$
$\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + t(-3\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$

(ii) $\mathbf{r} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} + s(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$
$\mathbf{r} = \mathbf{i} + 3\mathbf{j} + 10\mathbf{k} + t(-3\mathbf{i} + 4\mathbf{j} + \mathbf{k})$

(iii) $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} + s(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$
$\mathbf{r} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + t(-3\mathbf{i} + \mathbf{k})$

Example

(i) $\mathbf{r} = 2\mathbf{i} + 5\mathbf{k} + s(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$
$\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + t(-3\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$

Example

(ii) $\mathbf{r} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} + s(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

$\mathbf{r} = \mathbf{i} + 3\mathbf{j} + 10\mathbf{k} + t(-3\mathbf{i} + 4\mathbf{j} + \mathbf{k})$

Example

(iii) $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} + s(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$
$\mathbf{r} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + t(-3\mathbf{i} + \mathbf{k})$

Skew lines are lines on different parallel planes

Example

Show that L_1 and L_3 are skew, i.e., do not intersect each other.

$$L_1 : \mathbf{r} = \mathbf{i} + t_1(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}),$$

$$L_3 : \mathbf{r} = (2\mathbf{i} + \mathbf{j}) + t_3(3\mathbf{i} + \mathbf{j}).$$

Eliminating $\mathbf{r}$ from the vector equations of L_1 and L_3, we get

$$\mathbf{i} + t_1(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = (2\mathbf{i} + \mathbf{j}) + t_3(3\mathbf{i} + \mathbf{j}).$$

Hence it follows that

$$1 + t_1 = 2 + 3t_3, \quad 2t_1 = 1 + t_3, \quad 3t_1 = 0.$$

Solving the first two equations above gives $t_1 = \frac{2}{5}$ but the last equation says $t_1 = 0$, thus there is a contradiction.
So there is no solution to the equations and we conclude that L_1 and L_3 do not intersect.

Foot of Perpendicular and Shortest Distance From a Point to a Line

Exmaple

Find the position vector of the foot of perpendicular from $P(1, -4, 13)$ to the line

$$\mathbf{r} = \begin{pmatrix} 1-2s \\ 2-3s \\ 3+4s \end{pmatrix}, s \in \mathbb{R}$$

Planes In Three-dimensional Space

Planes In Three-dimensional Space

Let a vector perpendicular to a given plane be denoted by **n**. We call this a normal vector to the plane.
Fix a point A on the plane and let P be any point on the plane. Let the position vectors of A and P be **a** and **r**.
Then, the vector $\overrightarrow{AP}$ is perpendicular to the normal vector **n**. Hence,

$$(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$$

or

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$$

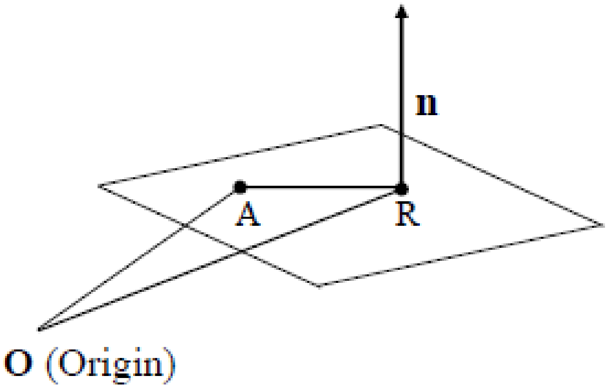
This is the vector equation of the plane.

The normal vector **n** can be found from

$$\mathbf{n} = \mathbf{u} \times \mathbf{v}$$

where **u** and **v** are any vectors that are parallel to the plane and **u** is **NOT** parallel to **v**.

A is a given point on plane with normal **n**
R is any point on plane



Example

Find the equation, in the form $\mathbf{r} \cdot \mathbf{n} = d$, of the plane which
(i) is perpendicular to the vector $4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ and which contains the point $(2, -2, 0)$
(ii) passes through $A(1, 2, 3)$, $B(2, 2, -1)$ and $C(0, 0, 1)$
(iii) contains $A(3, 4, 5)$ and line $L : \mathbf{r} = 4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k} + \mu(-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$

Cartesian Equation of Plane

A Cartesian equation of the plane $\mathbf{r} \cdot \mathbf{n} = d$ is

$$ax + by + cz = d$$

where $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

For example:
$\mathbf{r} \cdot (2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}) = 6$
Cartesian equation:

$\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$ meets $\mathbf{r} \cdot \mathbf{n} = d$ provided they are not parallel, i.e. $\mathbf{u} \cdot \mathbf{n} \neq 0$

Example

Find the position vector of the point of intersection of the line $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} + \mathbf{k})$ and the plane $\Pi : \mathbf{r} \cdot \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = 3.$

Example

Given that the line $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} + \mathbf{k})$ lies entirely on the plane $x + 2y + pz = q$, find the values of p and q.

Acute Angle Between Planes

Let $\mathbf{n}_1$, $\mathbf{n}_2$ be normal vectors to the planes

$$\cos\theta = \left|\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right|.$$

Example

Calculate the acute angle between the planes:

$$\Pi_1 : 3x - 4y + 5z = 6 \qquad \text{and} \qquad \Pi_2 : \mathbf{r} \cdot \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = 3.$$

Acute Angle Between Line and Plane

$\mathbf{u}$: direction vector of line

$\mathbf{n}$: normal vector of plane

$$\sin\theta = \left|\frac{\mathbf{u}\cdot\mathbf{n}}{|\mathbf{u}||\mathbf{n}|}\right|.$$

Example

Calculate the angle between the line $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} + \mathbf{k})$ and the plane $3x - 4y + 5z = 6$.

Intersection of Two Planes

Suppose two non-parallel planes

$$\Pi_1 : \mathbf{r} \cdot \mathbf{n}_1 = d_1 \qquad \text{and} \qquad \Pi_2 : \mathbf{r} \cdot \mathbf{n}_2 = d_2$$

intersect along the line L.

Vector equation of L:

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{n}_1 \times \mathbf{n}_2), \quad (\lambda \in \mathbb{R})$$

where **a** is the position vector of a point on both planes

Example

Find the line of intersection between:

$$\mathbf{r} \cdot (\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}) = 1 \qquad \text{and} \qquad 3x + 2y - 7z = 11.$$