MA1301 Introductory Mathematics

Chapter 1
Sequences and Series

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Overview

- Infinite Sequences
  - Definition of Sequence
  - Series and the Sigma Notation
  - Arithmetic Sequences/Arithmetic Progressions
  - Geometric Sequences/Geometric Progressions
  - Infinite Geometric Series
  - Binomial Theorem/Generalised Binomial Theorem
  - Telescoping Series/Telescoping Sum

Sequence

A **sequence** is an infinite list of numbers written in a definite order.

$$2, 4, 8, 16, 32, \cdots$$

The numbers in the list are called the terms of the sequence.

How can we figure out the 10th term in this sequence?

How to find a **formula** for the sequence,
i.e., a formula for how the nth term depends on n.

Different Ways of Writing a Sequence

It's often clearer when writing a sequence to provide a formula for the nth term immediately.
One method is to include the formula among the list of terms.

$$2, 4, 8, 16, 32, \cdots, 2^n, \cdots$$

Sometimes, it is convenient to write only the formula for a sequence.
The convention is that any formula surrounded by braces specifies a sequence

$$\{2^n\}_{n=1}^{\infty} \qquad \text{or} \qquad \text{simply } \{2^n\}$$

When talking about a sequence in general, we write the terms using variables.

$$a_1, a_2, a_3, a_4, \cdots, a_n, \cdots$$

Such a sequence may also be written using braces.

$$\{a_n\}_{n=1}^{\infty} \qquad \text{or} \qquad \text{simply } \{a_n\}$$

COMMON SEQUENCES

$$\{2^n\} : 2, 4, 8, 16, 32, 64, \cdots$$
$$\{3^n\} : 3, 9, 27, 81, 243, 729, \cdots$$
$$\{n^2\} : 1, 4, 9, 16, 25, 36, \cdots$$
$$\{n^3\} : 1, 8, 27, 64, 125, 216, \cdots$$
$$\{n!\} : 1, 2, 6, 24, 120, 720, \cdots$$

The last of these is the sequence of **factorials**, which you may not be familiar with.
The n^{th} term in this sequence (written $n!$, and pronounced "n factorial") is the product of all the whole numbers between 1 and n. For example:

$$5! = 1 \times 2 \times 3 \times 4 \times 5 = 120.$$

Example 1

Find the n^{th} term of the following sequences:
(i) $7^2, 9^2, 11^2, 13^2, \cdots$
(ii) $13, 17, 21, 25, \cdots$

Fibonacci Sequence

$$\{1, 1, 2, 3, 5, \cdots\}$$

Let f_n denote the n^{th} term, then

$$f_1 = 1, \quad f_2 = 1, \quad f_3 = f_2 + f_1 = 2, \quad f_4 = f_3 + f_2 = 3, \quad \cdots$$

In general, we have the following **recurrence equation** (or difference equation)

$$f_{n+2} = f_n + f_{n+1}$$

Fibonacci Sequence

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<u>For interest</u>

It can be shown that

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

The number $\frac{1+\sqrt{5}}{2}$ is known as the **golden ratio**

Example 2

The sequence of real numbers $\{x_1, x_2, \cdots\}$ is such that $x_1 = 1$ and $x_{n+1} = (n+1)x_n$ for all positive integers n.
Find the values of x_n for $n = 2, 3$ and 4.
Write down an expression for x_n in terms of n.

Series and the Sigma Notation

A series is built from a sequence, but differs from it in that the terms are *added* together.

$$2, 4, 8, 16, 32, \cdots \qquad \text{is a sequence,}$$

$$2 + 4 + 8 + 16 + 32 + \cdots \qquad \text{is a series.}$$

Series and the Sigma Notation

Series: sum of terms of a sequence
Let $u_1, u_2, \cdots, u_n$ be a sequence.
Then $u_1 + u_2 + \cdots + u_n$ is called the n^{th} partial sum of the sequence and is denoted by S_n.

$$S_n = u_1 + u_2 + \cdots + u_n$$

Result: $u_1 = S_1$ and $u_n = S_n - S_{n-1}$ for $n \geqslant 2$.

Example 3

Given that the sum of the first n terms of a sequence is given by $S_n = n(n+1)$, find the first term and the n^{th} term.

The Sigma Notation

$$\sum_{i=m}^{n} u_i = u_m + u_{m+1} + u_{m+2} + \cdots + u_n$$

Since $S_n = u_1 + u_2 + \cdots + u_n$

we can write $S_n = \sum_{i=1}^{n} u_i$

Example 4

(i) $\displaystyle\sum_{i=2}^{5} \frac{i}{i+1} =$

(ii) $\displaystyle\sum_{r=0}^{3} \sqrt{r^2+1} =$

(iii) $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots + \frac{1}{99} = \sum$

(iii) $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots + \frac{1}{99} = \sum$

(iv) $\frac{2}{1} + \frac{4}{2} + \frac{8}{3} + \frac{16}{4} + \cdots + \frac{1024}{10}$

(v) $\sqrt{1\times 2}+\sqrt{2\times 3}+\sqrt{3\times 4}+\cdots+$ to n^{th} term $=\sum$

Fact For any constants α, β

$$\sum_{r=m}^{n}(\alpha u_r + \beta v_r) = \alpha \sum_{r=m}^{n} u_r + \beta \sum_{r=m}^{n} v_r$$

Example: $\displaystyle\sum_{t=5}^{100}(3t^2 - 4\sqrt{t}) = 3\sum_{t=5}^{100}t^2 - 4\sum_{t=5}^{100}\sqrt{t}$

Fact For any constant C,

$$\sum_{r=m}^{n} C = C(n - m + 1)$$

Example 5

It is given that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$. Find the following

(i) $\sum_{r=1}^{n}(4r^2+3)$ (ii) $101^2 + 102^2 + 103^2 + \cdots + 200^2$

Arithmetic Sequence/Arithmetic Progression (A.P.)

$u_1, u_2, u_3 \cdots$ is an A.P.

$a, a+d, a+2d, \cdots$

Arithmetic Sequences

- $\{a_n\}$ is an Arithmetic Sequence with Common Difference d if
  - the difference of any two consecutive terms is d.
    - $d = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \cdots$.
    - $\boxed{a_{n+1} - a_n = d}$ for all $n \in \mathbb{Z}^+$.
- Let $\{a_n\}$ be an Arithmetic Sequence.
  - Let $a = a_1$ be the first term and d be the common difference.
    - $a_1 = a$;
    - $a_2 = a_1 + d = a + d$;
    - $a_3 = a_2 + d = (a+d) + d = a + 2d$;
    - $a_4 = a_3 + d = (a+2d) + d = a + 3d$;
    - $a_5 = a_4 + d = (a+3d) + d = a + 4d$;
    - $\cdots$
    - $\boxed{a_n = a + (n-1)d}$ for all $n \in \mathbb{Z}^+$

The Set of Positive Integers is denoted by $\mathbb{Z}^+$.

Arithmetic Sequence/Arithmetic Progression (A.P.)

$u_1, u_2, u_3 \cdots$ is an A.P.
$a, a+d, a+2d, \cdots$

$u_1 = a$ is the first term
$d = u_n - u_{n-1}$ is the common difference

Note that
x, y and z are three consecutive terms of an arithmetic progression
$\Leftrightarrow y - x = z - y$

Example 6

Find the value of x for which 3^{x-1}, 3^x and $3^x + 6$ are consecutive terms of an arithmetic sequence.

Sum of Arithmetic Sequences

- **Example.** Find the sum of the first 1000 terms of
  - arithmetic sequence: $12, 15, 18, 21, \cdots$
- **Solution.** First term: $a = 12$; Common difference: $d = 3$.
  - The 1000^{th} term: $a + 999d = 12 + 999 \times 3 = 3009$.

$$\begin{array}{ccccccc} 12 & +15 & +18 & +\cdots & +3003 & +3006 & +3009 \\ +3009 & +3006 & +3003 & +\cdots & +18 & +15 & +12 \\ \hline 3021 & +3021 & +3021 & +\cdots & +3021 & +3021 & +3021 \end{array}$$

  - Sum $\times 2 = 3021 \times 1000 = 3021000$
  - Sum $= 3021000 \div 2 = 1510500$
- Sum of an Arithmetic Sequence:
  - (First Term $+$ Last Term) $\times$ Number of Terms $\div 2$

Formulae

Let u_n and s_n denote the n^{th} term and sum of the first n terms of the arithmetic sequence $\{a, a+d, a+2d, \cdots\}$ respectively

- $u_n = a + (n-1)d$
- $s_n = \frac{n}{2}[2a + (n-1)d]$
- $s_n = \frac{n}{2}(u_1 + u_n)$
- $n = \frac{u_n - u_1}{d} + 1$

Example 7

Find the sum of all multiples of 7 between 100 and 300.

Example 8

Given an arithmetic sequence such that
(i) the fifth term is 41, and
(ii) the sum of the third and fourth term is 70,
find the eighth term of the arithmetic sequence.

Example 9

Sum all the **positive** terms in the sequence

$$2036, 2016, 1996, 1976, 1956, \cdots$$

Example 10

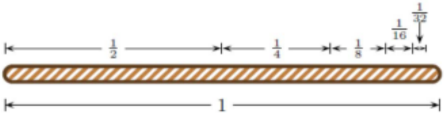
Find the value of n such that the sum of the first n terms of the sequence

$$\lg 3, \lg 27, \lg 243, \cdots$$

first exceeds 150.

(For this course, $\lg x$ denotes $\log_{10} x$.)

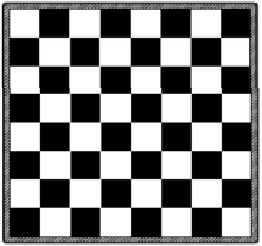
Examples



- Consider a segment of length 1.
  - Cut half on the first day.
  - Cut half of the remaining in the second day.
  - In general, cut half of the remaining everyday.
    How much have we cut by the 100^{th} day?
  - $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^{100}} = \sum_{i=1}^{100} \frac{1}{2^i}$

Geometric Sequences (Progressions)

Examples



- Consider an 8×8 chessboard.
  - Put 1 grain of rice in the first square of the chessboard.
  - Doubling the number in the next square.

    How much rice do we need to fill the chessboard?
  - $1 + 2 + 2^2 + 2^3 + 2^4 + \cdots + 2^{63} = \displaystyle\sum_{i=1}^{64} \frac{1}{2^{i-1}}$

The sequence

$$a, ar, ar^2, ar^3, \cdots$$

is called a geometric sequence (or geometric progression) and is denoted by G.P.

Geometric Sequences

- $\{a_n\}$ is a Geometric Sequence with Common Ratio r if
  - the ratio of any two Consecutive Terms is r.

$$r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \cdots = \frac{a_n}{a_{n-1}} = \frac{a_{n+1}}{a_n} = \cdots$$

  - Let $\{a_n\}$ be the First Term and r be the Common Ratio:
    - $a_1 = a$;
    - $a_2 = a_1 r = ar$;
    - $a_3 = a_2 r = (ar)r = ar^2$;
    - $a_4 = a_3 r = (ar^2)r = ar^3$;
    - $a_5 = a_4 r = (ar^3)r = ar^4$;
    - $\cdots$
    - $\boxed{a_n = ar^{n-1}}$

    We always assume that $a \neq 0$.

Example 11

A geometric sequence is given such that
(i) the first term exceeds the third term by 112, and
(ii) the second term exceeds the fourth term by 84.
Find the first term and the common ratio.
Find also the number of terms which exceed 50.

- A geometric sequence with first term a and common ratio r:
  - $a, ar, ar^2, ar^3, \cdots, ar^{n-1}, \cdots$
    What is the sum of the first n terms?
- **Solution.** Consider the sum, S_5, of the first 5 terms:
  - $S_5 = a + ar + ar^2 + ar^3 + ar^4$.
    $rS_5 = ar + ar^2 + ar^3 + ar^4 + ar^5$.

    $$\begin{aligned} S_5 - rS_5 &= (a + ar + ar^2 + ar^3 + ar^4) - (ar + ar^2 + ar^3 + ar^4 + ar^5) \\ &= a - ar^5. \end{aligned}$$
  - $(1-r)S_5 = a(1-r^5) \Rightarrow \boxed{S_5 = \frac{a}{1-r}(1-r^5)}$

Sum of Geometric Sequences

- A geometric sequence with first term a and common ratio r:
  - $a, ar, ar^2, ar^3, \cdots, ar^{n-1}, \cdots$
    What is the sum of the first n terms?
- Sum of geometric Sequence:
  - The sum of the first n terms is
    - $\boxed{S_n = \dfrac{a}{1-r}(1-r^n), \quad (r \neq 1)}$
  - If $r = 1$, then every term is a, and the sum is
    - $\boxed{S_n = na}$
  - Conclusion: $\boxed{S_n = \begin{cases} \frac{a}{1-r}(1-r^n), & \text{if } r \neq 1, \\ na, & \text{if } r = 1. \end{cases}}$

Remark

S_n, the sum of the first n terms of the sequence, is given by

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}.$$

Note that

x, y and z are three consecutive terms of an geometric progression
$\Leftrightarrow xz = y^2$

Example 12

Sum the series $\frac{1}{64} + \frac{1}{16} + \frac{1}{4} + 1 + \cdots + 4^{123}$

Example 13

Find the least n so that the sum of the first n terms of the geometric sequence

$$\frac{5}{6}, 1, \frac{6}{5}, \cdots$$

exceeds 30.

Example 14

The second, fifth and ninth terms of an arithmetic progression with non-zero common difference are consecutive terms of a geometric progression. Find
(i) the common ratio of the geometric progression
(ii) the ratio $S_{20} : S_{10}$, where S_n is the partial sum of the arithmetic progression.

Geometric Series

- A Geometric Series is the series associated to a Geometric Sequence. If the first term is a, and the common ratio is r.
  - $S_\infty = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + \cdots$.
- Recall that $S_n = \begin{cases} \frac{a}{1-r}(1-r^n), & \text{if } r \neq 1, \\ na, & \text{if } r = 1. \end{cases}$
  - Suppose $-1 < r < 1$. Then
    - $n \to \infty \Rightarrow r^n \to 0 \Rightarrow 1 - r^n \to 1 \Rightarrow S_n \to \frac{a}{1-r} \cdot 1 = \frac{a}{1-r}$.
- Conclusion:
  - If $-1 < r < 1$, $\boxed{S_\infty = \dfrac{a}{1-r}}$
  - If $r \leqslant -1$ or $r \geqslant 1$, S_∞ does NOT exist.

If the sum to infinity S_∞ exists, we say that the series converges. Geometric series converges for $-1 < r < 1$.

Examples

- Consider a segment of length 1.
  - Cut half on the first day.
  - Cut half of the the remaining everyday.
    How much have we cut by the 100^{th} day?
- Recall that first term $a = \frac{1}{2}$, common ratio $r = \frac{1}{2}$.
  - $S_n = \frac{a}{1-r}(1 - r^n)$.
  - $S_{100} = \frac{\frac{1}{2}}{1-\frac{1}{2}}\left[1 - \left(\frac{1}{2}\right)^{100}\right] = 1 - \frac{1}{2^{100}}$
- In general, $S_n = 1 - \frac{1}{2^n}$
  - As n gets larger, S_n gets close to 1.

Geometric Series

- Express $0.321321321321\cdots$ as a rational number.
  - $0.321321321321\cdots$ is a recurring decimal number.
- **Method 1**:

$$\begin{aligned} 0.321321321321\cdots = {} & 0.321 \\ & + 0.000321 \\ & + 0.000000321 \\ & + 0.000000000321 \\ & + \cdots \end{aligned}$$

Geometric series with first term 0.321 & common ratio 0.001.
  - $S_\infty = \frac{a}{1-r} = \frac{0.321}{1-0.001} = \frac{0.321}{0.999} = \frac{107}{333}$.

- Express $0.321321321321\cdots$ as a rational number.
  - $0.321321321321\cdots$ is a recurring decimal number.
- **Method 2**: Let $S = 0.32132132131\cdots$.

$$\begin{array}{rl} 1000S &= 321.321312321321321\cdots \\ -S &= 0.321321321321321\cdots \\ \hline 999S &= 321.000000000000000\cdots \end{array}$$

  - $999S = 312 \Rightarrow S = \frac{321}{999} = \frac{107}{333}$.
- **Example.** Express $1.123232323\cdots$ as a rational number.

$$\begin{aligned} S &= 1.1 + (0.023232323\cdots) \\ &= \frac{11}{10} + (0.023 + 0.00023 + 0.0000023 + \cdots) \\ &= \frac{11}{10} + \frac{0.023}{1-0.01} = \frac{11}{10} + \frac{23}{990} = \frac{556}{495}. \end{aligned}$$

Geometric Series

- $S = 1 + x + x^2 + x^3 + x^4 + \cdots \quad (-1 < x < 1)$.
  - **Method 1**: Geometric series: first term 1, common ratio x.
    - $S = \frac{a}{1-r} = \frac{1}{1-x}$.
  - **Method 2**:

$$\begin{array}{rl} S &= 1 + x + x^2 + x^3 + x^4 + \cdots \\ -xS &= x + x^2 + x^3 + x^4 + \cdots \\ \hline S - xS &= 1 \end{array}$$

    - $S(1-x) = 1 \Rightarrow \boxed{S = \dfrac{1}{1-x}}$

- **Examples.**
  - $1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$.
  - $1 + 2 + 4 + 8 + \cdots + 2^n + \cdots = \frac{1}{1-2} = -1$ WRONG!

Example 15

Determine whether the following geometric series converge or diverge (= does not converge). Find the sum to infinity of any series that converges.
(i) $999 + 333 + 111 + \cdots$

(ii) $\frac{1}{32} + \frac{1}{16} + \frac{1}{8} + \cdots$

(iii) $e^{2011} + e^{2010} + e^{2009} + \cdots$

Example 16

The geometric series

$$\frac{1}{x^2-4}+\frac{1}{x+2}+\cdots$$

has a sum to infinity. Find the range of values of x.

More on Geometric Series

For a geometric series with first term $= 1$ and common ratio $= x$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{r=0}^{\infty} x^r$$

provided $-1 < x < 1$

We can use the above result to express certain functions as an infinite series.

Example 17

Express the following as an infinite series of the form

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

giving the **first three non-zero terms**. In each case, find the range of values of x for which the series is valid.

(i) $\frac{3}{4-5x}$

(ii) $\frac{1-x}{2+x}$

- It is not difficult to verify the following identities:
  - $(a+b)^2 = a^2 + 2ab + b^2$.
  - $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

  In general, what is $\boxed{(a+b)^n}$ for every positive integer n?
- Binomial Theorem:

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n.$$

  - $\boxed{\binom{n}{r} = \dfrac{n!}{r!(n-r)!}}$, where $n! = 1 \times 2 \times \cdots \times n$.

- $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ is the Binomial Coefficient, $0 \leqslant r \leqslant n$.
  - It represents the number of ways to form a group of r people from n people, read as "n choose r".
  - For example, choose 2 letters from $\{A, B, C, D, E\}$:
    - $\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}$,
    - $\{B, C\}, \{B, D\}, \{B, E\}$,
    - $\{C, D\}, \{C, E\}$,
    - $\{D, E\}$.

    $$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2! \times 3!} = \frac{120}{2 \times 6} = 10.$$
  - In a Scientific Calculator,
    - $\binom{n}{r}$ can be evaluated using $\boxed{\text{nCr}}$

Binomial Theorem

- Pascal's Triangle: Each number is the sum of the two above.

$$
\begin{array}{lccccccccccccc}
\binom{0}{r} & & & & & & & 1 & & & & & & \\
\binom{1}{r} & & & & & & 1 & & 1 & & & & & \\
\binom{2}{r} & & & & & 1 & & 2 & & 1 & & & & \\
\binom{3}{r} & & & & 1 & & 3 & & 3 & & 1 & & & \\
\binom{4}{r} & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
\binom{5}{r} & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
\binom{6}{r} & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

- For example,
  - $\binom{4}{0} = 1,\ \binom{4}{1} = 4,\ \binom{4}{2} = 6,\ \binom{4}{3} = 4,\ \binom{4}{4} = 1$
  - $\binom{5}{0} = 1,\ \binom{5}{1} = 5,\ \binom{5}{2} = 10,\ \binom{5}{3} = 10,\ \binom{5}{4} = 5,\ \binom{5}{5} = 1$

Binomial Theorem

- The expansion of $(a+b)^n$ is the sum of the terms:
  - $\boxed{\binom{n}{r} a^{n-r} b^r}$ $\quad r = 0, 1, 2, \cdots, n.$
- **Example.** $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
  - $r = 0 : \binom{4}{0} a^{4-0} b^0 = a^4$
  - $r = 1 : \binom{4}{1} a^{4-1} b^1 = 4a^3b$
  - $r = 2 : \binom{4}{2} a^{4-2} b^2 = 6a^2b^2$
  - $r = 3 : \binom{4}{3} a^{4-3} b^3 = 4ab^3$
  - $r = 4 : \binom{4}{4} a^{4-4} b^4 = b^4$

- **Example.** $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$
  - $r = 0 : \binom{5}{0} a^{5-0}b^0 = a^5$
  - $r = 1 : \binom{5}{1} a^{5-1}b^1 = 5a^4b$
  - $r = 2 : \binom{5}{2} a^{5-2}b^2 = 10a^3b^2$
  - $r = 3 : \binom{5}{3} a^{5-3}b^3 = 10a^2b^3$
  - $r = 4 : \binom{5}{4} a^{5-4}b^4 = 5ab^4$
  - $r = 5 : \binom{5}{5} a^{5-5}b^5 = b^5$
- Exercises: Find $(a+b)^6$ and $(a+b)^7$.

Examples

- Expand $(3+2x)^5$.
  - Recall that $(a+b)^5$ equals

$$\binom{5}{0}a^5b^0+\binom{5}{1}a^4b^1+\binom{5}{2}a^3b^2+\binom{5}{3}a^2b^3+\binom{5}{4}a^1b^4+\binom{5}{5}a^0b^5$$
$$=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$$

  - Let $a=3$ and $b=2x$. Then

$$\begin{aligned}(3+2x)^5 &= (3)^5+5(3)^4(2x)+10(3)^3(2x)^2+10(3)^2(2x)^3\\ &\quad+5(3)(2x)^4+(2x)^5\\ &= 243+5\cdot 81\cdot 2x+10\cdot 27\cdot 4x^2+10\cdot 9\cdot 8x^3\\ &\quad+5\cdot 3\cdot 16x^4+32x^5\\ &= 243+810x+1080x^2+720x^3\\ &\quad+240x^4+32x^5.\end{aligned}$$

Examples

- Find the coefficient of x^3 in $(3+2x)^6$.
  - By Binomial Theorem, $(3+2x)^6$ is the sum of
    - $\binom{6}{r}(3)^{6-r}(2x)^r, r = 0, 1, 2, 3, 4, 5, 6.$

      Separate the coefficients: $\left[\binom{6}{r}3^{6-r}2^r\right] \cdot x^r.$
    - Coefficient of x^3: Let $r = 3$.

      $$\binom{6}{3}3^{6-3}2^3 = 20 \cdot 27 \cdot 8 = 4320.$$

Examples

- Find the coefficient of x^{-9} in $\left(3x^2 - \frac{2}{x}\right)^{12}$.
  - $\left(3x^2 - \frac{2}{x}\right)^{12} = \left[(3x^2) + \left(-\frac{2}{x}\right)\right]^{12}$.
  - By Binomial Theorem, it is the sum of
    - $\binom{12}{r}(3x^2)^{12-r}\left(-\frac{2}{x}\right)^r, r = 0, 1, 2, \cdots, 12.$
    - $\binom{12}{r}(3)^{12-r}(-2)^r \cdot (x^2)^{12-r}(x^{-1})^r$
    - $\binom{12}{r}(3)^{12-r}(-2)^r \cdot x^{24-3r}$.
  - Let $24 - 3r = -9$. Then $3r = 24 + 9 = 33 \Rightarrow r = 11$.
    - $\binom{12}{11}(3)^{12-11}(-2)^{11} = 12 \times 3 \times (-2048) = -73728.$

Examples

- Find the constant term in $\left(3x^2 - \frac{2}{x}\right)^{12}$.
  - $\left(3x^2 - \frac{2}{x}\right)^{12} = \left[(3x^2) + (-\frac{2}{x})\right]^{12}$.
  - By Binomial Theorem, it is the sum of
    - $\begin{pmatrix} 12 \\ r \end{pmatrix} (3x^2)^{12-r} \left(-\frac{2}{x}\right)^r, r = 0, 1, 2, \cdots, 12.$
    - $\begin{pmatrix} 12 \\ r \end{pmatrix} (3)^{12-r}(-2)^r \cdot x^{24-3r}, r = 0, 1, 2, \cdots, 12.$
  - Constant term is the coefficient of $1 = x^0$.
  - Let $24 - 3r = 0$. Then $r = 8$.
    - $\begin{pmatrix} 12 \\ 8 \end{pmatrix} (3)^{12-8}(-2)^8 = 495 \times 81 \times 256 = 10264320.$

Generalized Binomial Coefficients

- Recall that for all integers $0 \leqslant r \leqslant n$,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{1}{r!} \times \frac{1 \times 2 \times \cdots \times n}{1 \times 2 \times \cdots \times (n-r)}$$
$$= \frac{1}{r!} \times \frac{1 \times 2 \times \cdots \times (n-r) \times (n-r+1) \times \cdots \times n}{1 \times 2 \times \cdots \times (n-r)}$$
$$= \frac{(n-r+1) \times (n-r+2) \times \cdots \times (n-1) \times n}{r!}.$$

- We allow n to be any number, and define
    - $\binom{n}{r} = \frac{(n-r+1)\times(n-r+2)\times\cdots\times(n-1)\times n}{r!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{1\times2\times3\times\cdots\times r}$.
        - The number of factors in the numerator is r.
    - For example,
        - $\binom{\frac{1}{2}}{3} = \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!} = \frac{1}{16}$.

Find the value of:

$$\binom{\frac{5}{3}}{4}$$

$$\binom{-6}{5}$$

Generalized Binomial Theorem

- $(1+a)^n = 1 + na + \frac{n(n-1)}{2!}a^2 + \frac{n(n-1)(n-2)}{3!}a^3 + \cdots,$
  - n is not a nonnegative integer, and $-1 < a < 1$.
    We may use the sum of the first few terms to obtain an approximation.
- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, -1 < x < 1.$
  - $\frac{1}{1-x} \approx 1 + x + x^2 + x^3 + x^4.$
    For example, let $x = \frac{1}{3}$.

$$\text{L.H.S.} = \frac{1}{1-\frac{1}{3}} = 1.5$$

$$\text{R.H.S.} = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} = \frac{121}{81} \approx 1.494.$$

- The approximation is more accurate by taking more terms.

Generalized Binomial Theorem

- Find the first 3 terms of the expansion of $\frac{1}{\sqrt{4+3x}}$.
  - $\frac{1}{\sqrt{4+3x}} = (4+3x)^{-\frac{1}{2}} = 4^{-\frac{1}{2}}\left(1+\frac{3x}{4}\right)^{-\frac{1}{2}}$.

$$
\begin{aligned}
(1+a)^{-\frac{1}{2}} &= 1 + (-\frac{1}{2})a + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!}a^2 + \cdots \\
&= 1 - \frac{1}{2}a + \frac{3}{8}a^2 + \cdots . \\
\left(1+\frac{3x}{4}\right)^{-\frac{1}{2}} &= 1 - \frac{1}{2}\left(\frac{3x}{4}\right) + \frac{3}{8}\left(\frac{3x}{4}\right)^2 + \cdots \\
&= 1 - \frac{3}{8}x + \frac{27}{128}x^2 + \cdots . \\
\frac{1}{\sqrt{4+3x}} &= \frac{1}{2}\left(1 - \frac{3}{8}x + \frac{27}{128}x^2 + \cdots\right) \\
&= \frac{1}{2} - \frac{3}{16}x + \frac{27}{256}x^2 + \cdots .
\end{aligned}
$$

Generalized Binomial Theorem

- Find the first 3 terms of the expansion of $\frac{1}{\sqrt{4+3x}}$.
  - $\frac{1}{\sqrt{4+3x}} = \frac{1}{2}\left(1+\frac{3x}{4}\right)^{-\frac{1}{2}} = \frac{1}{2} - \frac{3}{16}x + \frac{27}{256}x^2 + \cdots$.
    - It is valid $\Leftrightarrow -1 < \frac{3x}{4} < 1 \Leftrightarrow -\frac{4}{3} < x < \frac{4}{3}$.
  - Let $x = 1$.

$$\begin{aligned} \text{L.H.S.} &= \frac{1}{\sqrt{7}}, \\ \text{R.H.S.} &\approx \frac{1}{2} - \frac{3}{16} + \frac{27}{256} = \frac{107}{256}. \end{aligned}$$

  - $\sqrt{7} \approx \frac{256}{107}$. This approximation seems not accurate.
  - $\sqrt{7} \approx 2.646$, $\frac{256}{107} \approx 2.393$.

Generalized Binomial Theorem

- Find the first 3 terms of the expansion of $\frac{1}{\sqrt{4+3x}}$.
  - $\frac{1}{\sqrt{4+3x}} = \frac{1}{2}\left(1+\frac{3x}{4}\right)^{-\frac{1}{2}} = \frac{1}{2} - \frac{3}{16}x + \frac{27}{256}x^2 + \cdots$.
    - It is valid $\Leftrightarrow -1 < \frac{3x}{4} < 1 \Leftrightarrow -\frac{4}{3} < x < \frac{4}{3}$.
  - Let $x = \frac{1}{3}$.

$$\text{L.H.S.} = \frac{1}{\sqrt{5}},$$
$$\text{R.H.S.} \approx \frac{1}{2} - \frac{1}{16} + \frac{3}{256} = \frac{15}{256}.$$

    - $\sqrt{5} \approx \frac{256}{115}$. This is a better approximation.
    - $\sqrt{5} \approx 2.236$, $\frac{256}{115} \approx 2.226$.

- Find the first 3 terms of the expansion of $\frac{1}{\sqrt{4+3x}}$.
  - $\frac{1}{\sqrt{4+3x}} = \frac{1}{2}\left(1+\frac{3x}{4}\right)^{-\frac{1}{2}} = \frac{1}{2} - \frac{3}{16}x + \frac{27}{256}x^2 + \cdots.$
    - It is valid $\Leftrightarrow -1 < \frac{3x}{4} < 1 \Leftrightarrow -\frac{4}{3} < x < \frac{4}{3}$.
  - Let $x = \frac{1}{4}$.

$$\text{L.H.S.} = \frac{1}{\sqrt{4+\frac{3}{4}}} = \frac{2}{\sqrt{19}},$$

$$\text{R.H.S.} \approx \frac{1}{2} - \frac{3}{64} + \frac{27}{4096} = \frac{1883}{4096}.$$

    - $\sqrt{19} \approx \frac{8192}{1883}$. This approximation is more accurate.
    - $\sqrt{19} \approx 4.359$, $\frac{8192}{1883} \approx 4.351$.

Telescoping Series

A series of the form

$$\sum_{r=m}^{n} (a_r - a_{r-1}) \text{ or } \sum_{r=m}^{n} (a_r - a_{r+1})$$

can be summed easily as terms other than the first term and the last term will cancel off.

Telescoping Sum

- Find $S = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{100\times 101}$.

$$
\begin{aligned}
\frac{1}{1\times 2} &= \frac{2-1}{1\times 2} = \frac{2}{1\times 2} - \frac{1}{1\times 2} = \frac{1}{1} - \frac{1}{2},\\
\frac{1}{2\times 3} &= \frac{3-2}{2\times 3} = \frac{3}{2\times 3} - \frac{2}{2\times 3} = \frac{1}{2} - \frac{1}{3},\\
\frac{1}{3\times 4} &= \frac{2-1}{3\times 4} = \frac{4}{3\times 4} - \frac{3}{3\times 4} = \frac{1}{3} - \frac{1}{4},\\
\cdots &= \cdots\\
\frac{1}{99\times 100} &= \frac{100-99}{99\times 100} = \frac{100}{99\times 100} - \frac{99}{99\times 100} = \frac{1}{99} - \frac{1}{100},\\
\frac{1}{100\times 101} &= \frac{101-100}{100\times 101} = \frac{101}{100\times 101} - \frac{100}{100\times 101} = \frac{1}{100} - \frac{1}{101},
\end{aligned}
$$

- $S = \displaystyle\sum_{n=1}^{100} \frac{1}{n(n+1)} = \frac{1}{1} - \frac{1}{101} = \frac{100}{101}$.

Telescoping Sum

- A Telescoping Sum is a sum which has a fixed number of terms after cancellation.
  - Consider a sequence $a_1, a_2, a_3, \cdots, a_n, a_{n+1}, \cdots$
    - Set

$$
\begin{aligned}
u_1 &= a_1 - a_2\\
u_2 &= a_2 - a_3\\
u_3 &= a_3 - a_4\\
\cdots &= \cdots\\
u_{n-1} &= a_{n-1} - a_n\\
u_n &= a_n - a_{n+1}
\end{aligned}
$$

- $\boxed{\sum_{i=1}^{n} u_i = u_1 + u_2 + \cdots + u_n = a_1 - a_{n+1}}$

Telescoping Sum

- A Telescoping Sum is a sum which has a fixed number of terms after cancellation.
  - Consider a sequence $a_1, a_2, a_3, \cdots, a_n, a_{n+1}, \cdots$
    - Set

$$\begin{aligned} u_m &= a_m - a_{m+1} \\ u_{m+1} &= a_{m+1} - a_{m+2} \\ u_{m+2} &= a_{m+2} - a_{m+3} \\ \cdots &= \cdots \\ u_{n-1} &= a_{n-1} - a_n \\ u_n &= a_n - a_{n+1} \end{aligned}$$

    - $\boxed{\sum_{i=m}^{n} u_i = u_m + u_{m+1} + \cdots + u_n = a_m - a_{n+1}}$

Example

Sum the following series

(i) $\displaystyle\sum_{r=2}^{400}(\sqrt{r}-\sqrt{r-1})$ (ii) $\displaystyle\sum_{r=1}^{n}\frac{1}{(r+1)(r+2)}$

Example

Show that $r(r+1)! - (r-1)r! = (r^2+1)r!$
Hence, sum the series

$$(11^2+1)11! + (12^2+1)!12! + \cdots + (100^2+1)100!$$

Telescoping Sum

- Recall that if $u_i = a_i - a_{i+1}$ for all $i = 1, 2, 3, \cdots$,
  - $\displaystyle\sum_{i=m}^{n} u_i = a_m - a_{n+1}$.
    If $n \to \infty \Rightarrow a_{n+1} \to 0$, then the Telescoping Series
  - $\boxed{\displaystyle\sum_{i=m}^{\infty} u_i = u_m + u_{m+1} + u_{m+2} + \cdots = a_m}$
- **Examples.**
  - $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) = \frac{1}{1} - \frac{1}{n+1}$.
    Note that $n \to \infty \Rightarrow \frac{1}{n+1} \to 0$.
  - $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i(i+1)} = \sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1}\right) = \frac{1}{1} = 1$.