NATIONAL UNIVERSITY OF SINGAPORE

MA1301 Introductory Mathematics **Tutorial 7**

1. Using integration by parts, evaluate the following definite integrals.

(a) $\displaystyle\int_0^{\pi} x\cos\frac{x}{2}\,dx;$ (b) $\displaystyle\int_0^1 \frac{2x-1}{e^{2x}}\,dx;$

(c) $\displaystyle\int_1^e x^3\ln x\,dx;$ (d) $\displaystyle\int_0^{\pi/4} x\sec^2 x\,dx;$

(e) $\displaystyle\int_0^{\pi/2} e^{3x}\cos 2x\,dx;$ (f) $\displaystyle\int_e^{e^2} x\ln(x^4)\,dx;$

(g) $\displaystyle\int_0^{1/4} \sin^{-1} 2x\,dx;$ (h) $\displaystyle\int_0^{\pi/8} x\tan^2 2x\,dx;$

(i) $\displaystyle\int_0^{\pi} x\sin x\cos x\,dx;$ (j) $\displaystyle\int_0^{\pi/3} x\sin^2 3x\,dx.$

2. Calculate the area of the region bounded by the following curves/lines.

(a) $y = 4x(1-x)$ and $y = 0$.

(b) $y = 1 - x$ and $y^2 = 1 + x$.

(c) $y = 2\sin x + 1$ $(0 \le x \le \pi)$, the y-axis and the line $y = \dfrac{x}{\pi}$.

(d) $y = \sqrt{x}$, the x-axis and the line $y = 6 - x$.

3. A curve C has equation $y = 5 - e^x$.

i) Find the coordinates of the points at which C meets the axes.

ii) Find the equation of the asymptote of C.

iii) Sketch the curve C.

iv) Find the equation of the tangent line to C at the point where C meets the y-axis.

v) The region R is bounded by the curve C, the tangent line in (iv) and the x-axis. Find the volume of the solid generated by rotating R completely about the x-axis.

4. i) On a single xy-coordinate system, sketch the graphs of $y = x^2$ and $y = 2 - x^2$.

ii) Find the volume of the solid formed by rotating the region bounded between the two curves in (i) completely about the x-axis.

5. i) Sketch the graph of $y = x + \dfrac{4}{x}$ for $x > 0$.

ii) Find the area of the region S bounded by the curve in (i) and the line $y = 5$.

iii) Find the volume of the solid when S is rotated completely about the line $y = 5$.

6. The region R is bounded by $y = \tan^2 x$ $(0 \le x < \frac{\pi}{2})$, the y-axis and $y = 3$.

i) Find the area of the region R.

ii) Using the result in (i) or otherwise, evaluate $\displaystyle\int_0^3 \tan^{-1}\sqrt{y}\,dy$.

iii) Show that $\dfrac{d}{dx}(\tan^3 x - 3\tan x + 3x) = 3\tan^4 x$.

iv) Find the volume of the solid formed by rotating R completely about the x-axis.

Solutions and Hints

1. (a) $2\pi - 4$. *Hint*: $\displaystyle\int x\cos\frac{x}{2}\,dx = 2x\sin\frac{x}{2} + 4\cos\frac{x}{2} + C$.

(b) $-e^{-2}$. *Hint*: $\displaystyle\int \frac{2x-1}{e^{2x}}\,dx = -e^{-2x} - xe^{-2x} + C$.

(c) $\dfrac{3}{16}e^4 + \dfrac{1}{16}$. *Hint*: $\displaystyle\int x^3\ln x\,dx = \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C$.

(d) $\dfrac{1}{4}\pi - \dfrac{1}{2}\ln 2$. *Hint*: $\displaystyle\int x\sec^2 x\,dx = x\tan x + \ln|\cos x| + C$.

(e) $-\dfrac{3}{13}e^{3\pi/2} - \dfrac{3}{13}$. *Hint*: $\displaystyle\int e^{3x}\cos 2x\,dx = \frac{3}{13}e^{3x}\cos 2x + \frac{2}{13}e^{3x}\sin 2x + C$.

(f) $3e^4 - e^2$. *Hint*: $\displaystyle\int x\ln(x^4)\,dx = 2x^2\ln x - x^2 + C$.

(g) $\dfrac{\pi}{24} + \dfrac{\sqrt{3}}{4} - \dfrac{1}{2}$. *Hint*: $\displaystyle\int \sin^{-1}2x\,dx = x\sin^{-1}2x + \frac{1}{2}\sqrt{1-4x^2} + C$.

(h) $\dfrac{1}{16}\pi - \dfrac{1}{8}\ln 2 - \dfrac{1}{128}\pi^2$. *Hint*: $\displaystyle\int x\tan^2 2x\,dx = \frac{1}{2}x\tan 2x + \frac{1}{4}\ln|\cos 2x| - \frac{1}{2}x^2 + C$.

(i) $-\dfrac{1}{4}\pi$. *Hint*: $\displaystyle\int x\sin x\cos x\,dx = \frac{1}{4}x\cos 2x - \frac{1}{8}\sin 2x + C$.

(j) $\dfrac{1}{36}\pi^2$. *Hint*: $x\sin^2 3x\,dx = \dfrac{1}{4}x^2 - \dfrac{1}{12}x\sin 6x - \dfrac{1}{72}\cos 6x + C$.

2. (a) $\displaystyle\int_0^1 4x(1-x)\,dx = \frac{2}{3}$; (b) $\displaystyle\int_{-1}^2 [(1-y) - (y^2-1)]\,dy = \frac{9}{2}$;

(c) $\displaystyle\int_0^\pi \left[(2\sin x + 1) - \frac{x}{\pi}\right]dx = 4 + \frac{\pi}{2}$; (d) $\displaystyle\int_0^2 [(6-y) - y^2]\,dy = \frac{22}{3}$.

3. i) $(\ln 5, 0)$, $(0, 4)$; ii) $y = 5$; $y = -x + 4$;

iv) $\left(\dfrac{148}{3} - 25\ln 5\right)\pi$. *Hint*: $\displaystyle\frac{1}{3}\pi\cdot 4^2\cdot 4 - \int_0^{\ln 5}\pi(5-e^x)^2\,dx$.

4. ii) $\dfrac{16}{3}\pi$. *Hint*: $\displaystyle\int_{-1}^1 \pi(2-x^2)^2\,dx - \int_{-1}^1 \pi(x^2)^2\,dx$.

5. ii) $\dfrac{15}{2} - 8\ln 2$. *Hint*: $3 \times 5 - \displaystyle\int_1^4 \left(x + \frac{4}{x}\right) dx$.

iii) $57 - 80\ln 2$. *Hint*: $\displaystyle\int_1^4 \pi \left(5 - x - \frac{4}{x}\right)^2 dx$.

6. i) $\dfrac{4}{3}\pi - \sqrt{3}$. *Hint*: $3 \times \dfrac{\pi}{3} - \displaystyle\int_0^{\pi/3} \tan^2 x \, dx$.

ii) *Hint*: Express the area of R as an integral in y.

iv) $\dfrac{8}{3}\pi^2$. *Hint*: $\pi \cdot 3^2 \cdot 3 - \displaystyle\int_0^{\pi/3} \pi(\tan^2 x)^2 \, dx$.