MA1301 Introductory Mathematics

Chapter 2
Derivatives

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Overview

- Derivative - using the concept of limit
- Derivative - Gradient of tangent line
- Derivative - Instantaneous rate of change of function
- Rules of Differentiation
- Implicit Differentiation
- Higher Order Derivative
- Parametric Equations and Differentiation
- Tangent and Normal
- Linear Approximation
- Connected Rate of Change
- Increasing and Decreasing Functions
- Stationary Points
- First Derivative Test
- Concavity Test
- Points of Inflection
- Second Derivative Test
- Applications

Derivative

$$y = x^n \qquad\qquad f(x) = x^n$$

$$\frac{dy}{dx} = nx^{n-1} \qquad\qquad f'(x) = nx^{n-1}$$

The derivative of y with respect to x.
The derivative of $f(x)$ with respect to x.

with respect to —— w.r.t

$f'(x)$ or simply f' is called the derivative (or gradient function) of $f(x)$ or f.
When $y = f(x)$, $f'(x)$ is also commonly written as $\frac{dy}{dx}$ (or $\frac{df}{dx}$).
The process of finding $f'(x)$ of a function f is called differentiation.

Some results

$\frac{d}{dx}(c) = 0$, where c is any constant.
$\frac{d}{dx}(x^n) = nx^{n-1}$, where n is any constant.
$\frac{d}{dx}(\ln x) = \frac{1}{x}$ $\qquad$ $\frac{d}{dx}(e^x) = e^x$

Trigonometry Functions

$\frac{d}{dx}(\sin x) = \cos x$
$\frac{d}{dx}(\tan x) = \sec^2 x$
$\frac{d}{dx}(\sec x) = \sec x \tan x$

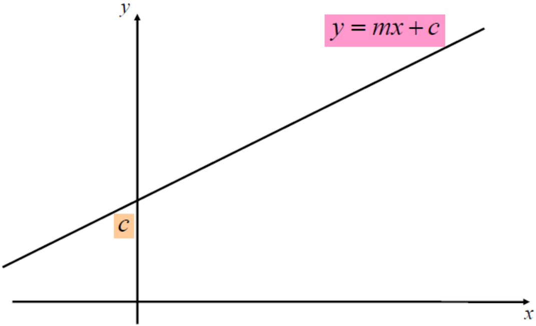
$\frac{d}{dx}(\cos x) = -\sin x$
$\frac{d}{dx}(\cot x) = -\csc^2 x$
$\frac{d}{dx}(\csc x) = -\csc x \cot x$

Question: How to derive these results? Using limits

1. Derivative — using the concept of limit

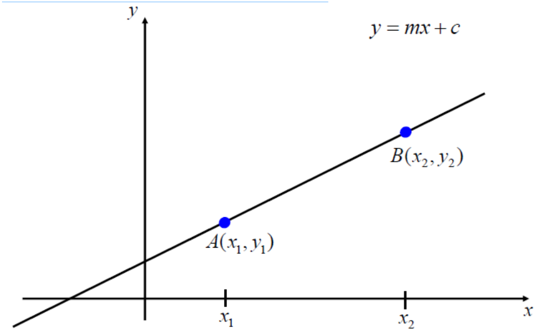
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

2. Derivative — (Geometrically) Gradient (Slope) of the tangent line
3. Derivative — instantaneous rate of change of the function



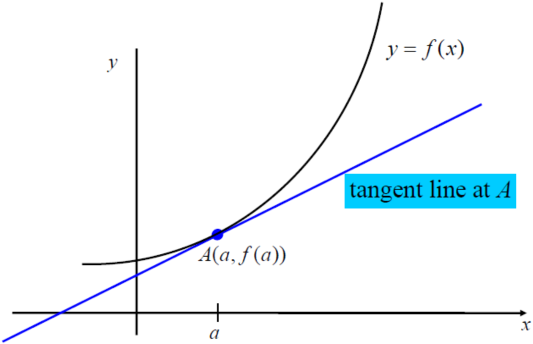
m —— gradient (slope) of the line
c —— y-intercept

Straight line — find gradient

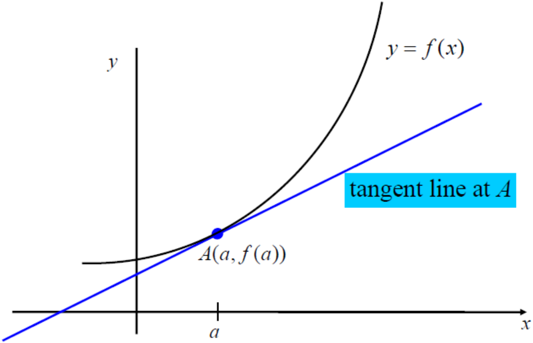


$$\text{gradient (slope)} = m = \frac{y_2 - y_1}{x_2 - x_1}$$

Curve — to find gradient

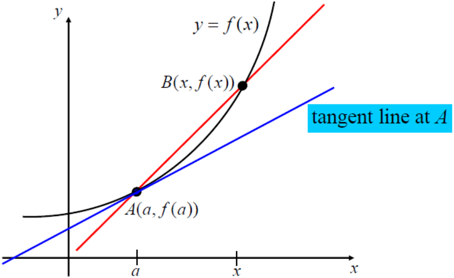


$$\text{gradient at } A = \text{ gradient of tangent line at } A$$
$$= f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$



Question: Why we define

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$



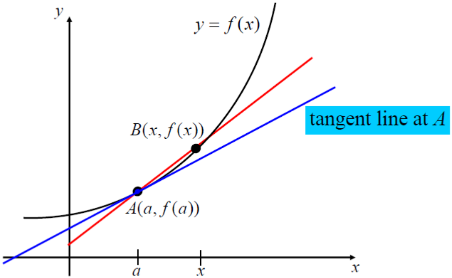
Why we define

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$\text{gradient of line } AB = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x) - f(a)}{x - a}$$

gradient at A = gradient of tangent line at A

gradient at $A \neq$ gradient of line AB



If we choose B close to A, then gradient of line $AB \approx$ gradient of tangent line at A

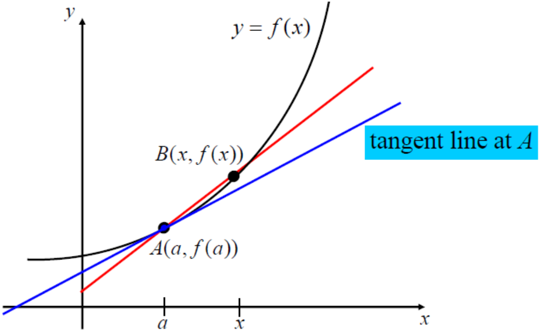
Why we define

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$\text{gradient of line } AB = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x) - f(a)}{x - a}$$

choosing B closer and closer to A is the same as letting x approaches a.

$$\text{Taking limit, we have, gradient at } A = \lim_{x \to a} (\text{gradient of } AB) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$



$$\text{gradient at } A = \text{gradient of tangent line at } A$$
$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Similarly, gradient at A gives instantaneous rate of change of the function $f(x)$.

Derivative

Let $f(x)$ be a function.
The derivative of f at a is defined to be

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided the limit exists.

If $f'(a)$ exists, we say that f is ***differentiable*** at $x = a$.
If f is ***differentiable*** at any point a in the domain of f, we say that f is ***differentiable***.

Function	Derivative
$(f(x))^n$	$nf'(x)(f(x))^{n-1}$
$\cos(f(x))$	$-f'(x)\cdot\sin(f(x))$
$\sin(f(x))$	$f'(x)\cdot\cos(f(x))$
$\tan(f(x))$	$f'(x)\cdot\sec^2(f(x))$
$\sec(f(x))$	$f'(x)\cdot\sec(f(x))\tan(f(x))$
$\csc(f(x))$	$-f'(x)\cdot\csc(f(x))\cot(f(x))$
$\cot(f(x))$	$-f'(x)\cdot\csc^2(f(x))$
$e^{f(x)}$	$f'(x)\cdot e^{f(x)}$
$\ln(f(x))$	$\frac{f'(x)}{f(x)}$
$\sin^{-1}(f(x))$	$\frac{f'(x)}{\sqrt{1-(f(x))^2}}$
$\cos^{-1}(f(x))$	$-\frac{f'(x)}{\sqrt{1-(f(x))^2}}$
$\tan^{-1}(f(x))$	$\frac{f'(x)}{1+(f(x))^2}$

Product Rule

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$

Derivative - Rules of Differentiation

Quotient Rule

$$\frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Show that $\frac{d}{dx}\tan x = \sec^2 x$.

$$u = \sin x$$
$$\frac{d}{dx}(\sin x) = \cos x$$

$$\begin{aligned}\frac{d}{dx}\tan x &= \frac{d}{dx}\frac{\sin x}{\cos x} \\ &= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x\end{aligned}$$

$$v = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\cos^2 x + \sin^2 x = 1$$

Example

Differentiate with respect to x:

(i) $x^2 \tan^{-1} x$ (ii) $\cos^{-1}\left(\frac{\ln x}{x}\right)$ (iii) $(\sin^{-1}(e^x))^4$

The Chain Rule - Example

Find $\frac{d}{dx}\sin(x^3)$.

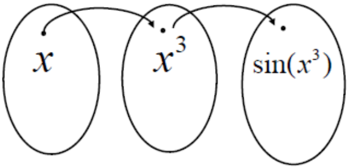
$$\begin{aligned}&\frac{d}{dx}\sin(x^3)\\=&\cos(x^3)\cdot\frac{d}{dx}x^3\\=&3x^2\cos(x^3)\end{aligned}$$

Fix a value for x

Let $x=\pi$

Step 1. π^3

Step 2. $\sin(\pi^3)$



The Chain Rule - Example

Find $\frac{d}{dx}\sin^5(e^x)$.

Note: $y = \sin^5(e^x) = (\sin(e^x))^5$.

To find derivative, go Reverse order

Fix a value for x

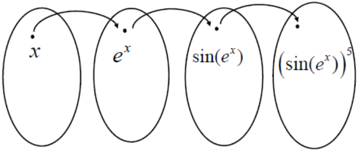
Let $x = \pi$

Step 1. e^π

Step 2. $\sin(e^\pi)$

Step 3. $(\sin(e^\pi))^5$

$$e^x \to \sin(\quad) \to (\quad)^5$$



Find $\frac{d}{dx}\sin^5(e^x)$. Note: $y = \sin^5(e^x) = (\sin(e^x))^5$.

$$\frac{dy}{dx} = 5(\sin(e^x))^4 \cos(e^x)\ e^x$$

Differentiate First

Differentiate Last

Differentiate Second

To find derivative, go Reverse order

$e^x \to \sin(\ ) \to (\quad)^5$

Fix a value for x.

Let $x = \pi$

Step 1. e^{π}

Step 2. $\sin(e^{\pi})$

Step 3. $(\sin(e^{\pi}))^5$

The Chain Rule - Example

Let $y = (x^5 + \cos(3x^2))^9$. Find $\frac{dy}{dx}$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(x^5 + \cos(3x^2))^9 \\
&= 9(x^5 + \cos(3x^2))^8 \cdot \frac{d}{dx}(x^5 + \cos(3x^2)) \\
&= 9(x^5 + \cos(3x^2))^8 (5x^4 - \sin(3x^2) \cdot 6x) \\
&= 9x(x^5 + \cos(3x^2))^8 (5x^3 - 6\sin(3x^2))
\end{aligned}$$

Other Types of Differentiation

Cartesian equation — An equation connecting x and y

$$y = x^3 + 4x$$

$$\frac{dy}{dx} = 3x^2 + 4$$

$$y = x^2 + \sqrt{x}$$

$$\frac{dy}{dx} = 2x + \frac{1}{2\sqrt{x}}$$

$$x^2 + y^2 = 9$$

Use Implicit Differentiation

Ordinary differentiation	Implicit differentiation
$\frac{d}{dx}(x^2) = 2x$	$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$
$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\sin y) = \cos y\frac{dy}{dx}$
$\frac{d}{dx}(e^x) = e^x$	$\frac{d}{dx}(e^y) = e^y\frac{dy}{dx}$
$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\frac{d}{dx}(\ln y) = \frac{1}{y}\frac{dy}{dx}$
$\frac{d}{dx}(x) = 1$	$\frac{d}{dx}(y) = 1\frac{dy}{dx}$

Example

$\frac{d}{dx}(x^5y^6) =$

Example

Find $\frac{dy}{dx}$ if $x \ln y + y \ln x = 2x + 3y$

Implicit Differentiation - Example

Find $\frac{dy}{dx}$ if $2y = x^2 + \sin y$.
Differentiate both sides with respect to x,

$$2\frac{dy}{dx} = 2x + \cos y \cdot \frac{dy}{dx}$$

So,

$$(2 - \cos y)\frac{dy}{dx} = 2x \Rightarrow \frac{dy}{dx} = \frac{2x}{2 - \cos y}$$

Find $\frac{dy}{dx}$

(a) $x^3 + y^3 - 9xy = 0$

(b) $x^3 e^y + \cos(xy) = 0$

Implicit Differentiation - Example

Find $\frac{dy}{dx}$ if $x^3e^y + \cos(xy) = 2021.$
Differentiate both sides with respect to x,

$$\overbrace{3x^2e^y + x^3e^y\frac{dy}{dx}}^{\text{Applying product rule to } x^3e^y} \quad - \sin(xy)\left(x\frac{dy}{dx} + y\right) = 0.$$

Solving for $\frac{dy}{dx}$, we get $\frac{dy}{dx} = \frac{3x^2e^y - y\sin(xy)}{x\sin(xy) - x^3e^y}$

Implicit Differentiation

What is $\frac{d}{dx}x^x$, where $x > 0$?

Let $y = x^x$.

$$\text{Then } \ln y = \ln x^x$$
$$= x \ln x. \qquad \text{Note: } \ln a^b = b \ln a$$

Differentiating both sides *w.r.t.* x yields

$$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$$

So,

$$\frac{dy}{dx} = y(1 + \ln x) = x^x(1 + \ln x) = x^x + x^x \ln x$$

.

Implicit Differentiation

To differentiate $\frac{d}{dx} f(x)^{g(x)}$
Let $y = f(x)^{g(x)}$.

$$\text{Consider } \ln y = \ln f(x)^{g(x)}$$
$$= g(x) \ln f(x)$$

Implicit differentiation and product rule

Example

Let $y = 5^{x \ln x}$. Find $\frac{dy}{dx}$.

$$\boxed{\frac{d}{dx} a^x = a^x \ln a}$$

$$\boxed{\frac{d}{dx}\left(a^{g(x)}\right) = a^{g(x)}(g'(x) \ln a)}$$

Example

Let $y = (x^2 - e^{3x})^{4\tan x}$. Find $\frac{dy}{dx}$.

Example

Differentiate $6^{x\cos x}$ with respect to x.

Other Types of Differentiation

Higher Order Derivatives

Higher order derivatives are obtained when we differentiate repeatedly. Let $y = f(x)$, then the following notation is used:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x) = D^2 f(x),$$

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = f'''(x) = D^3 f(x).$$

Other Types of Differentiation

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x) = D^2 f(x),$$

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = f'''(x) = D^3 f(x).$$

$$\frac{d^4y}{dx^4} = f''''(x) \qquad \frac{d^5y}{dx^5} = f'''''(x)$$

$$\frac{d^{2021}y}{dx^{2021}} = f''''' \cdots ''' (x)$$

Other Types of Differentiation

In general, the n-th derivative is denoted by

$$\frac{d^n y}{dx^n} \quad \text{or} \quad f^{(n)}(x) \quad \text{or} \quad D^n f(x)$$

Higher Order Derivatives - Example

Let $f(x) = \sqrt{x}$. Compute $f'''(x)$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}, \quad f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}, \quad f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}$$

Example

Let $y = xe^x$. Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^ny}{dx^n}$.

Example

If $y = x \ln x$, find $\frac{d^3y}{dx^3}$.

Cartesian equation — An equation connecting x and y

$$y = x^3 + 4x \qquad y = x^2 + \sqrt{x} \qquad x^2 + y^2 = 9$$

Parametric equations

1. $x = 2t$ and $y = t^2 + 1$
2. $x = \sin\theta + 2$ and $y = \cos\theta - 5$
3. $x = 1 + e^t$ and $y = e^{2t}$

Parametric Differentiation
Given $y = f(x)$, where

$$\begin{cases} y = u(t) \\ x = v(t), \end{cases}$$

we have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{u'(t)}{v'(t)}.$$

Parametric Differentiation - Example

Let $x = a(t - \sin t)$ and $y = a(1 - \cos t)$. Find $\frac{dy}{dx}$.

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{a \sin t}{a(1 - \cos t)} \\
&= \frac{2 \sin\left(\frac{t}{2}\right) \cos\left(\frac{t}{2}\right)}{2 \sin^2\left(\frac{t}{2}\right)} \\
&= \cot\left(\frac{t}{2}\right)
\end{aligned}
$$

$$x = v(t) \quad y = u(t) \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{u'(t)}{v'(t)}$$

Pause and Think!!!
True or false??

$$\frac{d^2y}{dx^2} = \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}} = \frac{u''(t)}{v''(t)}$$

Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}(y) = \frac{d}{du}(y) \cdot \frac{du}{dx}$$

Pause and Think!!!

True or false??

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\
&= \frac{d}{\boxed{dt}}\left(\frac{dy}{dx}\right) \cdot \left(\frac{\boxed{dt}}{dx}\right) \\
&= \left(\frac{d}{dt}\left(\frac{dy}{dx}\right)\right) \cdot \left(\frac{dt}{dx}\right)
\end{aligned}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{d}{dx}(y) = \frac{d}{\boxed{dt}}(y) \cdot \frac{\boxed{dt}}{dx}$$

Parametric Differentiation - Example

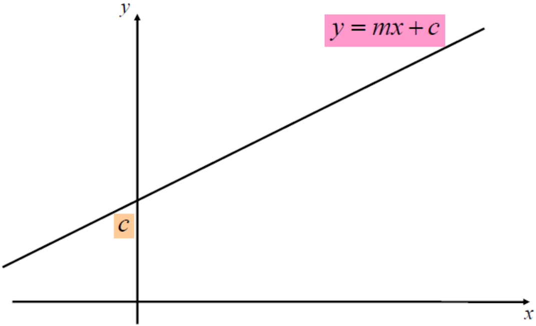
Let $x = a(t - \sin t)$ and $y = a(1 - \cos t)$. Find $\frac{dy}{dx}$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{a\sin t}{a(1-\cos t)} \\ &= \frac{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}{2\sin^2\left(\frac{t}{2}\right)} \\ &= \cot\left(\frac{t}{2}\right)\end{aligned}$$

Find also $\dfrac{d^2y}{dx^2}$.

Example

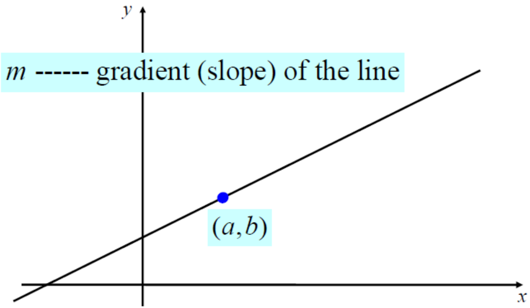
Let $x = 1 - \cos 2t$, $y = 2t - \sin 2t$.
Show that $\frac{dy}{dx} = \tan t$ and find $\frac{d^2y}{dx^2}$.



m —— gradient (slope) of the line

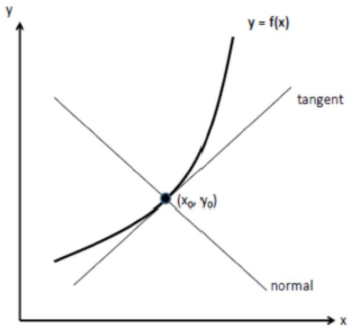
c —— y-intercept

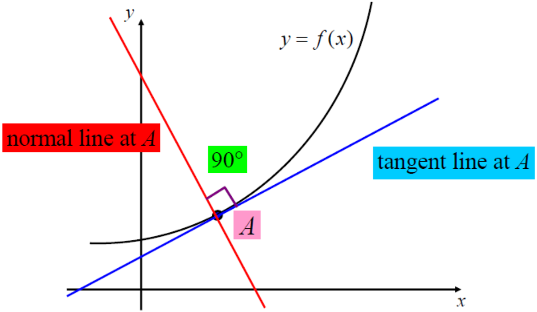
Point - Slope form



$$y - b = m(x - a)$$

Tangent and Normal





Result:
(gradient of tangent line) $\times$ (gradient of normal line) $= -1$

Tangent and Normal

Result:
(gradient of tangent line) $\times$ (gradient of normal line) $= -1$

Let $y = f(x)$ and let $m = f'(x_0)$. At (x_0, y_0)

- Tangent: $y - y_0 = m(x - x_0)$
- Normal: $y - y_0 = -\frac{1}{m}(x - x_0)$

Tangent and Normal

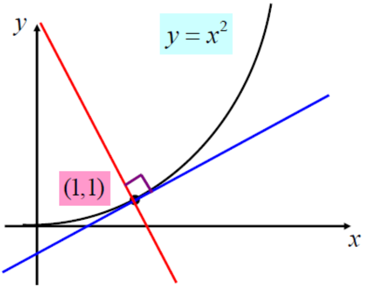
Result:
(gradient of tangent line) $\times$ (gradient of normal line) $= -1$
Recall that the straight line passing through point (a, b) with slope m is:

- $\frac{y-b}{x-a} = m$; or equivalently, $\boxed{y = m(x-a) + b}$

Let $y = f(x)$ be a function.

- Tangent Line of $y = f(x)$ at $x = a$:
  - It is the line passing through point $(a, f(a))$ with slope $f'(a)$.
    - $\boxed{y = f'(a)(x-a) + f(a)}$
- Normal Line of $y = f(x)$ at $x = a$:
  - It is the line passing through point $(a, f(a))$ and Perpendicular to the tangent line at $x = a$, i.e., of slope $-\frac{1}{f'(a)}$.
    - $\boxed{y = -\dfrac{1}{f'(a)}(x-a) + f(a)}$

Find equation of lines which are tangent and normal to the curve $y = x^2$ at $x = 1$ respectively.

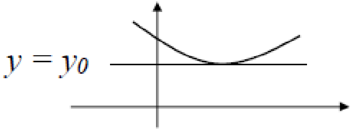


$\frac{dy}{dx} = 2x \qquad x = 1, \frac{dy}{dx} = 2$
gradient of tangent $= 2$
gradient of normal $= -\frac{1}{2}$

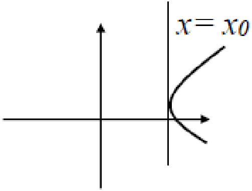
Equation of tangent: $y - 1 = 2(x - 1)$
Equation of normal: $y - 1 = -\frac{1}{2}(x - 1)$

Example

Find the coordinates of the points of the curve $x^2 + 3xy + y^2 + 4 = 0$ at which the tangents are parallel to the line $y = x + 7$.



Tangent // x-axis $\qquad \frac{dy}{dx} = 0$



Tangent // y-axis $\qquad \frac{dy}{dx} = \pm\infty$

Example

A curve C has equation

$$y^2 - 4xy + 8x^2 = 100.$$

(i) Find $\frac{dy}{dx}$ in terms of x and y.
(ii) Find the equations of the two tangents which are parallel to the y-axis.
(iii) Find the equation of the normal to the curve C at the point $(0, 10)$.

Example

Given that $4x^2 + 8x + 9y^2 - 36y + 4 = 0$,
(i) find $\frac{dy}{dx}$.
(ii) Write down the equation(s) of the tangent(s) to the curve that are parallel to

(a) the x-axis (b) the y-axis.

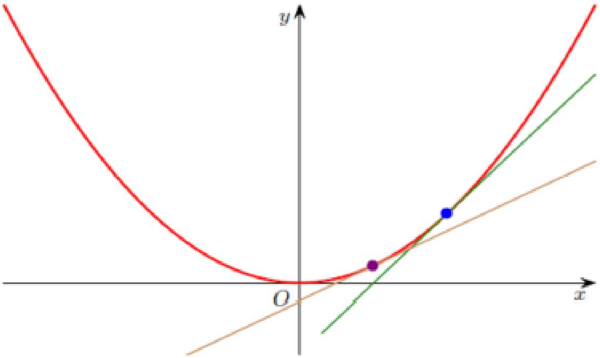
Linear Approximation

- How to Approximate the value of a function at given point?
  - If $x \approx a$, then $f(x) \approx f(a)$. Is this a good approximation?
    - $101 \approx 100 \Rightarrow \sqrt{101} \approx \sqrt{100} = 10$.
      $\sqrt{101} = 10.04988\cdots$. Error $\approx 0.5\%$.
    - $101 \approx 100 \Rightarrow 101^2 \approx 100^2 = 10000$.
      $101^2 = 10201$. Error $\approx 2\%$.
  - Recall: $f'(a)$ is the rate of change of $y = f(x)$ at $x = a$.
    - $x \approx a \Rightarrow \frac{f(x)-f(a)}{x-a} \approx f'(a) = \left.\frac{dy}{dx}\right|_{x=a}$.
    - $f(x) \approx f'(a)(x-a) + f(a)$.
- Note that $y = f'(a)(x-a) + f(a)$ is the Tangent Line of $y = f(x)$ at $x = a$.

Let $y = f(x)$ be a function.

- If $x \approx a$, then $f(x) \approx f'(a)(x-a) + f(a)$.

In other words, for x near a, the value of a function at x can be approximated by the Tangent Line of the function at a. (The nearer, the better!)



Example

- Approximate 101^2.
  - $y = f(x) = x^2$ at $x = 100$.
    1. $\frac{dy}{dx} = 2x \Rightarrow \left.\frac{dy}{dx}\right|_{x=100} = f'(100) = 200.$
    2. Tangent line at $x = 100$ is given by $y = 200 \cdot (x - 100) + 100^2$.
    3. $101^2 \approx 200 \cdot (101 - 100) + 10000 = 10200.$
- Approximate $\sqrt{101}$.
  - $y = f(x) = \sqrt{x}$ at $x = 100$.
    1. $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \Rightarrow \left.\frac{dy}{dx}\right|_{x=100} = f'(100) = \frac{1}{20} = 0.05.$
    2. Tangent line at $x = 100$ is given by $y = 0.05 \cdot (x - 100) + \sqrt{100}$.
    3. $\sqrt{101} \approx 0.05 \cdot (101 - 100) + 10 = 10.05.$

Example

- Let $y = f(x) = \sqrt{x}$.
  1. $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow \left.\frac{dy}{dx}\right|_{x=a} = \frac{1}{2\sqrt{a}}$.
  2. Tangent line at $x = a$: $y = \frac{1}{2\sqrt{a}}(x-a) + \sqrt{a}$.
  3. $x \approx a \Rightarrow \sqrt{x} \approx \frac{1}{2\sqrt{a}}(x-a) + \sqrt{a}$.
- In order to approximate $\sqrt{x}$, we shall find a number a near x, such that $\sqrt{a}$ can be easily evaluated.
  - $\sqrt{2} : 2 \approx 1.96 = 1.4^2$. Use $a = 1.4$.
    - $\sqrt{2} \approx \frac{1}{2 \cdot 1.4}(2 - 1.96) + 1.4 \approx 1.414 \cdots$

Example

- Let $y = f(x) = \sqrt{x}$.
  1. $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow \frac{dy}{dx}\Big|_{x=a} = \frac{1}{2\sqrt{a}}$.
  2. Tangent line at $x = a$: $y = \frac{1}{2\sqrt{a}}(x-a) + \sqrt{a}$.
  3. $x \approx a \Rightarrow \sqrt{x} \approx \frac{1}{2\sqrt{a}}(x-a) + \sqrt{a}$.
- In order to approximate $\sqrt{x}$, we shall find a number a near x, such that $\sqrt{a}$ can be easily evaluated.
  - $\sqrt{3} : 3 \approx 2.89 = 1.7^2$. Use $a = 1.7$.
    - $\sqrt{3} \approx \frac{1}{2 \cdot 1.7}(3 - 2.89) + 1.7 \approx 1.732 \cdots$

Example

- Let $y = f(x) = \sqrt{x}$.
  1. $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow \left.\frac{dy}{dx}\right|_{x=a} = \frac{1}{2\sqrt{a}}$.
  2. Tangent line at $x = a$: $y = \frac{1}{2\sqrt{a}}(x-a) + \sqrt{a}$.
  3. $x \approx a \Rightarrow \sqrt{x} \approx \frac{1}{2\sqrt{a}}(x-a) + \sqrt{a}$.
- In order to approximate $\sqrt{x}$, we shall find a number a near x, such that $\sqrt{a}$ can be easily evaluated.
  - $\sqrt{23} : 23 \approx 23.04 = 4.8^2$. Use $a = 4.8$.
    - $\sqrt{23} \approx \frac{1}{2 \cdot 4.8}(23 - 23.04) + 4.8 \approx 4.796\cdots$

Connected Rate of Change

Connected Rate of Change

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

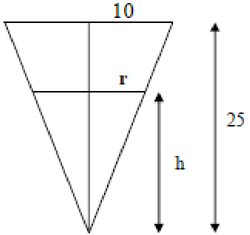
relates the rate of change of y to rate of change of x.

Example

A conical flask, initially empty and of base radius 10 cm and vertical height 25 cm, is being filled with water at a rate of 4 cm^3 s^{-1}. Let h be the height of the water t seconds later. Show that the volume V cm^3 of water at time t is given by

$$V = \frac{4\pi h^3}{75}$$

Hence, calculate the rate at which h is changing when $V = 180\pi$ cm^3



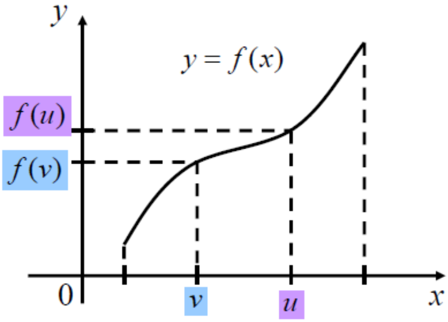
Example

The surface area of a sphere is decreasing at 2 cm^2 s^{-1}. Find the rate at which the volume sphere is changing at the instant when the surface area is 16π cm^2.

Increasing and Decreasing Functions

Increasing functions

Let f be a function defined on an interval I.



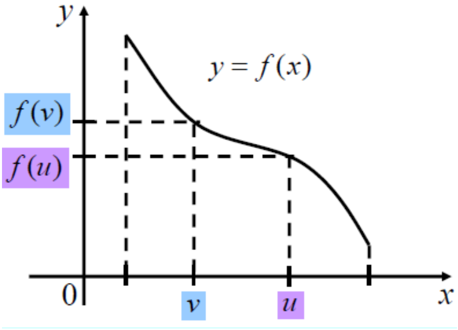
f is ***increasing*** on I if $u > v \Rightarrow f(u) > f(v)$

Bigger x value, bigger $f(x)$ value

y increases as x increases

Decreasing functions

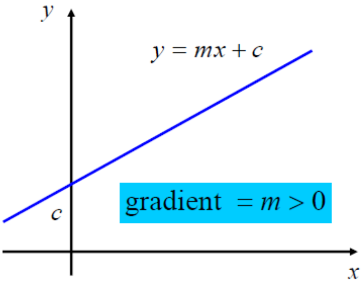
Let f be a function defined on an interval I.



f is ***decreasing*** on I if $u > v \Rightarrow f(u) < f(v)$
Bigger x value, smaller $f(x)$ value
y decreases as x increases

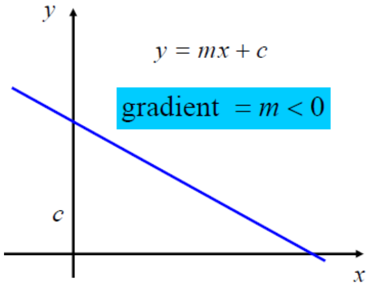
Question:
How to check a function $f(x)$ is increasing/decreasing??

What does the sign of m tell you?



$m > 0$

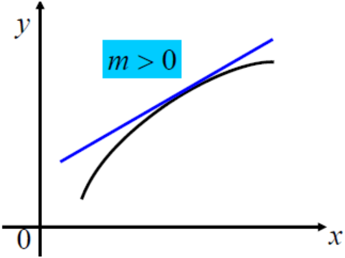
y increases as x increases

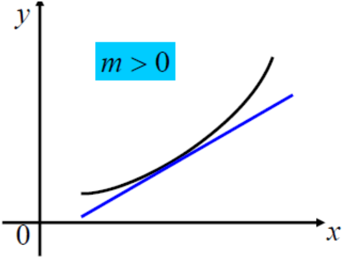


$m < 0$

y decreases as x increases

What does the sign of $\frac{dy}{dx}$ tell you?

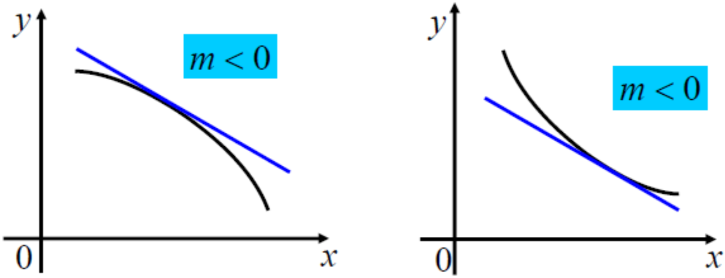




For both graphs, y is increasing.
For both graphs, $\frac{dy}{dx} > 0$.

$\therefore y$ increases if $\frac{dy}{dx} > 0$.
$\therefore f(x)$ increases if $f'(x) > 0$.

What does the sign of $\frac{dy}{dx}$ tell you?



For both graphs, y is decreasing.
For both graphs, $\frac{dy}{dx} < 0$.

$\therefore y$ decreases if $\frac{dy}{dx} < 0$.
$\therefore f(x)$ decreases if $f'(x) < 0$.

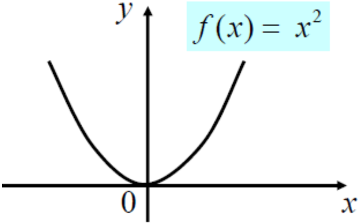
Test for Increasing/Decreasing function

$f'(x) > 0$ for all values of x in I,
then f is ***increasing*** on I.

$f'(x) < 0$ for all values of x in I,
then f is ***decreasing*** on I.

$f'(x) > 0$ for all values of x in I, then f is ***increasing*** on I.

$f'(x) < 0$ for all values of x in I, then f is ***decreasing*** on I.



$f'(x) = 2x$

For $x > 0$, $f'(x) = 2x > 0$, $f(x)$ is increasing
For $x < 0$, $f'(x) = 2x < 0$, $f(x)$ is decreasing

Example

$f(x) = \frac{2}{3}x^3 + x^2 + 2x + 1$
Prove that $f(x)$ is an increasing function.

$$\begin{aligned} f'(x) &= 2x^2 + 2x + 2 \\ &= 2\left(\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right) \qquad \text{(Completing square)} \end{aligned}$$

For all x, $f'(x) > 0$, $f(x)$ is increasing

Example

$f(x) = x^3(x-1)^2$.
Determine the intervals on which $f(x)$ is increasing/decreasing.

$$\begin{aligned} \text{Then } f'(x) &= x^3(2)(x-1) + 3x^2(x-1)^2 \\ &= x^2(x-1)(5x-3) \end{aligned}$$

Set $f'(x) = 0$, we have $x = 0, 1$ or $\frac{3}{5}$.

$f'(x)$	(+)(-)(-)	(+)(-)(-)	(+)(-)(+)	(+)(+)(+)
$f(x)$	↗	↗	↘	↗

$0 \qquad \frac{3}{5} \qquad 1$

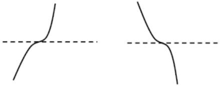
f has a *stationary point* at $x = c$ if $\frac{dy}{dx} = 0$ at $x = c$.

3 types of stationary points:

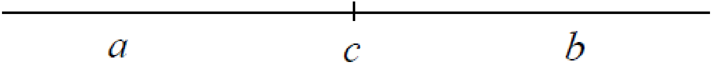
(i) *local maximum* point

(i) *local minimum* point

(i) *saddle* point



First Derivative Test

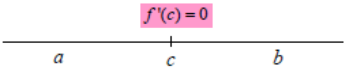


Suppose that f has a stationary point at $x = c$, i.e., $f'(c) = 0$

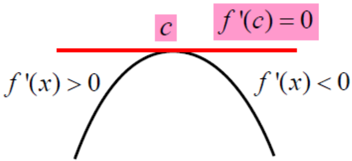
(i) $f'(x) > 0$ for $x \in (a, c)$ and $f'(x) < 0$ for $x \in (c, b)$,
then $f(c)$ is a ***local maximum***.

(ii) $f'(x) < 0$ for $x \in (a, c)$ and $f'(x) > 0$ for $x \in (c, b)$,
then $f(c)$ is a ***local minimum***.

Note: $f'(x)$ changes sign

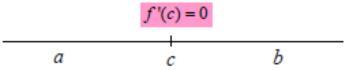


(i) $f'(x) > 0$ for $x \in (a, c)$ and $f'(x) < 0$ for $x \in (c, b)$, then $f(c)$ is a ***local maximum***.

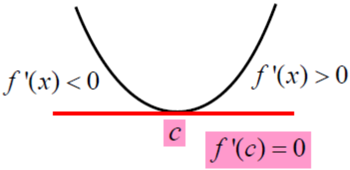


f' changes sign from $+$ to $-$ at c
(that is, f changes from increasing to decreasing at c)

First Derivative Test

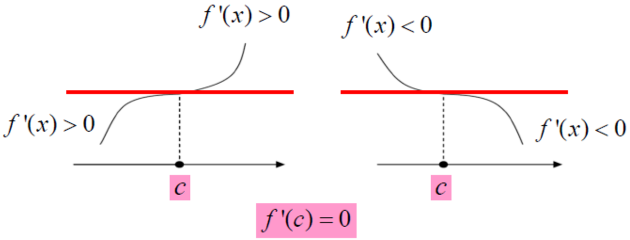


(ii) $f'(x) < 0$ for $x \in (a, c)$ and $f'(x) > 0$ for $x \in (c, b)$,
then $f(c)$ is a ***local minimum***.



f' changes sign from $-$ to $+$ at c
(that is, f changes from decreasing to increasing at c)

First Derivative Test



f' does not change sign at c, we say that
f has a saddle point at c.

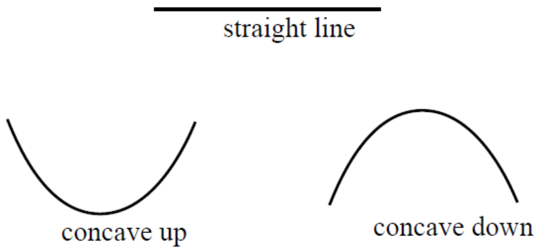
Example

(i) Find the interval(s) on which the given function is
  (a) increasing
  (b) decreasing
(ii) Find the stationary points and state their nature.

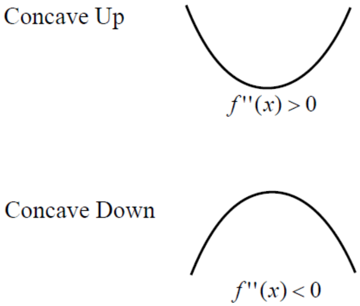
$$f(x) = 2x^3 + 3x^2 - 36x + 11.$$

Concave Upward and Concave Downward

Concavity



Concavity Test



Concavity Test

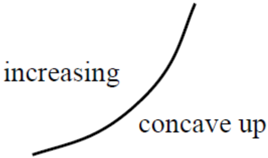
$f''(x) > 0$ concave up

$f''(x) < 0$ concave down

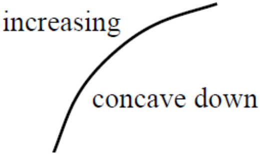
Concavity

$f'(x)$ determines if f is increasing or decreasing.

$f''(x)$ determines if f is concave up or down.

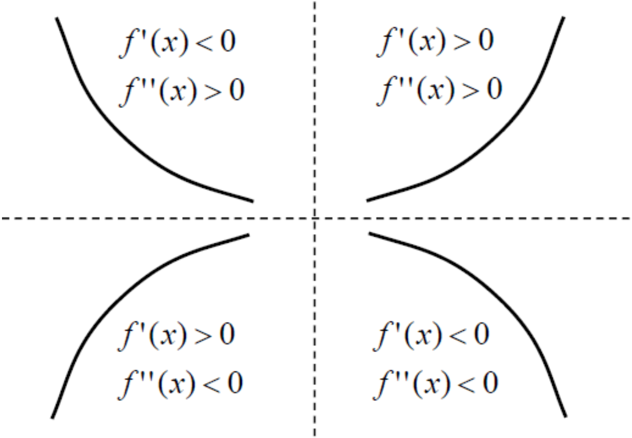


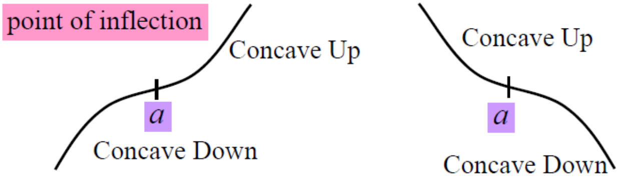
$$f'(x) > 0$$
$$f''(x) > 0$$



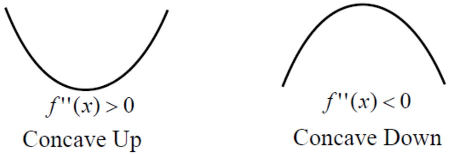
$$f'(x) > 0$$
$$f''(x) < 0$$

Concavity





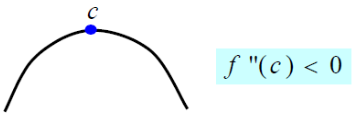
A point c is a point of inflection of the function f if f is continuous at c and there is an open interval containing c such that the graph of f changes from concave up (or down) before c to concave down (or up) after c.



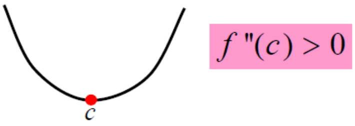
At point of inflection, $\frac{d^2y}{dx^2}$ changes sign

Second Derivative Test for Local Extreme Values

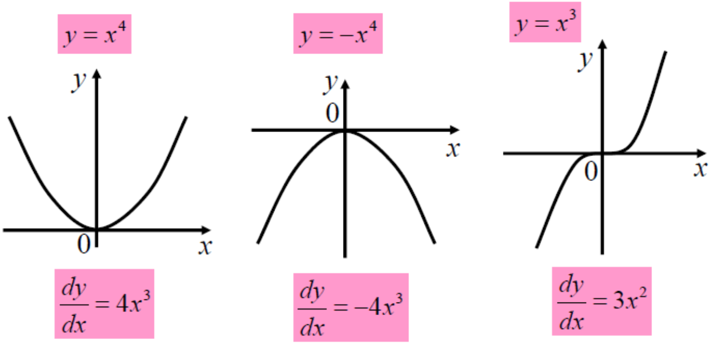
If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.



If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.



If $f'(c) = 0$ and $f''(c) = 0$, then the test fails.



Note: In all 3 cases, $y'(0) = y''(0) = 0$

Example

Apply the Second Derivative Test to the function

(i) $f(x) = x^2(x-2)$

(ii) $g(x) = x^3(x-2)$

Example

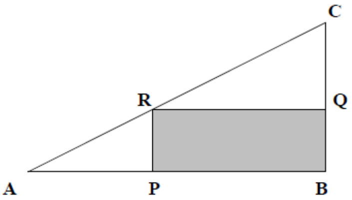
Determine the intervals on which the function
$f(x) = -2x^3 + 15x^2 - 24x + 7$ is
(i) increasing
(ii) decreasing
(iii) concave upward
(iv) concave downward.

Applications of Maxima and Minima

Problems typically require you to

- Identify and construct the function to be maximized/minimized
- Find the derivative of the function and set it to ZERO
- Calculate the required quantities
- Perform derivative test (If the Q asks for it)

Example



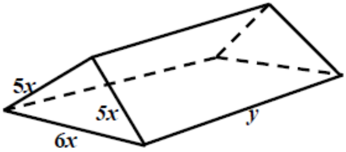
ABC is a right-angled triangle with $\angle B = 90^\circ$, $AC = 34$ cm and $BC = 16$ cm.
Find the largest rectangle $PBQR$ that can be inscribed in triangle ABC.

Example

The solid prism shown below has a total surface area of 450 cm^3. Show that its volume V cm^3 is given by

$$V = \frac{675}{2}x - 18x^3$$

Deduce the stationary value of V.
Determine whether this is a maximum or a minimum value.



Example

A rectangle is inscribed in a circle of fixed radius r as shown in the diagram. One vertex of the rectangle has coordinates (x, y), as shown below.

The Cartesian equation of the circle is $x^2 + y^2 = r^2$.

Find the area of the rectangle in terms of x only. Hence, show that the maximum area of the rectangle occurs when $x = y$ (that is, the rectangle is a square) and find the maximum area in terms of r.