MA1301 Introductory Mathematics

Chapter 3
Integrals

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Overview

- Antiderivatives and Indefinite Integrals
- Standard Integrals
- Definite Integrals
- Integration by Substitution
- Integration by Parts
- Area
- Volume of Solids of Revolution
- First Order Ordinary Differential Equations

Suppose $\frac{dF}{dx} = f(x)$.
F is called the *antiderivative* of f.
Example.

$$\frac{d}{dx}(x^2 + 5) = 2x \qquad \frac{d}{dx}(x^2 - 9) = 2x$$

Both $x^2 + 5$ and $x^2 - 9$ are antiderivatives of $2x$.
In fact, $\frac{d}{dx}(x^2 + C) = 2x$ for any constant C.
Thus, $x^2 + C$ is an aniderivatives of $2x$ for any constant C.

In general, F is an anti-derivative of f,
then so is $F + C$ for any real constant C.
(since $\frac{d}{dx}(F + C) = \frac{dF}{dx} = f$).

We can also rewrite $\frac{dF}{dx} = f(x)$ as

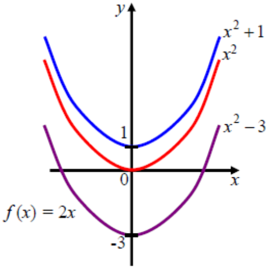
$$F(x) = \int f(x)\,dx$$

and call F the indefinite integral of f.

The geometrical interpretation of the process on ***integration*** is to find all curves

$$y = F(x) + C$$

which have their slopes $f(x)$ at x.



Integral Formulae

1. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$, n rational
   $\int 1 \, dx = \int dx = x + C$ (Special case, $n = 0$)

2. $\int \sin kx \, dx = -\frac{\cos kx}{k} + C$

3. $\int \cos kx \, dx = \frac{\sin kx}{k} + C$

4. $\int \sec^2 x \, dx = \tan x + C$

5. $\int \csc^2 x \, dx = -\cot x + C$

Integral Formulae

6. $\int \sec x \tan x \, dx = \sec x + C$
7. $\int \csc x \cot x \, dx = -\csc x + C$
8. $\int \frac{1}{x} \, dx = \ln |x| + C$
9. $\int a^x \, dx = \frac{a^x}{\ln a} + C$
10. $\int e^x \, dx = e^x + C$

Standard Integrals

1. $\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{(n+1)a} + C \quad (n \neq -1)$
2. $\int \frac{1}{ax+b} \, dx = \frac{1}{a} \ln |ax+b| + C$
3. $\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + C$
4. $\int \sin(ax+b) \, dx = -\frac{1}{a} \cos(ax+b) + C$
5. $\int \cos(ax+b) \, dx = \frac{1}{a} \sin(ax+b) + C$
6. $\int \tan(ax+b) \, dx = \frac{1}{a} \ln |\sec(ax+b)| + C$

Standard Integrals

7. $\int \sec(ax+b)\,dx = \frac{1}{a}\ln|\sec(ax+b)+\tan(ax+b)|+C$

8. $\int \csc(ax+b)\,dx = -\frac{1}{a}\ln|\csc(ax+b)+\cot(ax+b)|+C$

9. $\int \cot(ax+b)\,dx = -\frac{1}{a}\ln|\csc(ax+b)|+C$

10. $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b)+C$

11. $\int \csc^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b)+C$

12. $\int \sec(ax+b)\cdot\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b)+C$

Standard Integrals

13. $\int \csc(ax+b)\cdot\cot(ax+b)\,dx = -\frac{1}{a}\csc(ax+b)+C$

14. $\int \frac{1}{a^2+(x+b)^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x+b}{a}\right)+C$

15. $\int \frac{1}{\sqrt{a^2-(x+b)^2}}\,dx = \sin^{-1}\left(\frac{x+b}{a}\right)+C$

16. $\int \frac{-1}{\sqrt{a^2-(x+b)^2}}\,dx = \cos^{-1}\left(\frac{x+b}{a}\right)+C$

17. $\int \frac{1}{a^2-(x+b)^2}\,dx = \frac{1}{2a}\ln\left|\left(\frac{x+b+a}{x+b-a}\right)\right|+C$

18. $\int \frac{1}{(x+b)^2-a^2}\,dx = \frac{1}{2a}\ln\left|\left(\frac{x+b-a}{x+b+a}\right)\right|+C$

19. $\int \frac{1}{\sqrt{(x+b)^2+a^2}}\,dx = \ln\left|(x+b)+\sqrt{(x+b)^2+a^2}\right|+C$

20. $\int \frac{1}{\sqrt{(x+b)^2-a^2}}\,dx = \ln\left|(x+b)+\sqrt{(x+b)^2-a^2}\right|+C$

Trigonometric Identities Useful for Integration

- $\sec^2\theta - 1 = \tan^2\theta$
- $\csc^2\theta - 1 = \cot^2\theta$
- $\sin A \cos A = \frac{1}{2}\sin 2A$
- $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$
- $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$
- $\sin A \cdot \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$
- $\cos A \cdot \sin B = \frac{1}{2}[\sin(A + B) - \sin(A - B)]$
- $\cos A \cdot \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$
- $\sin A \cdot \sin B = -\frac{1}{2}[\cos(A + B) - \cos(A - B)]$

Example

(i) $\int \frac{6}{\sqrt{5-3x}}\,dx$ (ii) $\int \frac{6}{5+3x}\,dx$ (iii) $\int \frac{6}{(x-1)(x^2-1)}\,dx$

Example

(i) $\int (2x - \frac{5}{x^2})\, dx$ (ii) $\int (5\sec^2 x - 2\sin 3x)\, dx$

Example

(i) $\int \frac{1}{4x^2-8x+12}\,dx$ (ii) $\int \frac{1}{\sqrt{40x-4x^2}}\,dx$

Example

(i) $\int (2\cos 2x - \sin 2x)^2 \, dx$

(ii) $\int \frac{3\cos^2 x + 2}{\sin^2 x} \, dx$

Definite Integral

Suppose $\int f(x)\,dx = F(x) + C$.
Then the definite integral of $f(x)$ from a to b is

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

Example
We know that $\int x\,dx = \frac{1}{2}x^2 + C$

Hence $\int_3^5 x\,dx =$

Definite Integrals - Terminology

$$\int_a^b f(x)\,dx$$

$[a, b]$: the interval of integration
a: lower limit of integration
b: upper limit of integration
x: variable of integration
$f(x)$: the integrand

Note:
x is a dummy variable, i.e.,

$$\int_a^b f(x)\,dx = \int_a^b f(u)\,du = \int_a^b f(t)\,dt, \text{ etc.}$$

1. $\int_a^a f(x)\,dx = 0$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$, where k is a constant
   In particular, $\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$
   Take $k = -1$

4. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Example

Prove the identity

$$(2\cos 4\theta - 2\cos 2\theta + 1)\cos\theta = \cos 5\theta.$$

Hence, show that

$$\int_0^{\frac{\pi}{4}} \frac{\cos 5\theta}{\cos\theta}\, d\theta = \frac{\pi}{4} - 1$$

Example

If $\int_{-5}^{2} f(x)\,dx = 5$ and $\int_{5}^{2} f(x)\,dx = -2$, find
(i) $\int_{2}^{5} 2f(x)\,dx$ (ii) $\int_{-5}^{5}(-f(x))\,dx$

Integration by Substitution

To evaluate $\int f'(x)g(f(x))\,dx$
Method: Let $u = f(x)$

Example

Evaluate $\int (x^2 + 2x - 3)^2 (x+1)\, dx$.

Let $u = x^2 + 2x - 3$.

Then $\dfrac{du}{dx} = 2(x+1)$.

$$du = 2(x+1)\, dx$$

$$\frac{1}{2} du = (x+1)\, dx$$

$$\begin{aligned}\int (x^2 + 2x - 3)^2 (x+1)\, dx &= \int u^2 \frac{1}{2}\, du \\ &= \frac{1}{6}u^3 + C \\ &= \frac{1}{6}(x^2 + 2x - 3)^3 + C\end{aligned}$$

Example

Evaluate $\int \sin^4 x \cos x \, dx$.

Let $u = \sin x$.

Then $\dfrac{du}{dx} = \cos x$.

$$du = \cos x \, dx$$

$$\begin{aligned} \int \sin^4 x \cos x \, dx &= \int u^4 \, du \\ &= \frac{1}{5} u^5 + C \\ &= \frac{1}{5} \sin^5 x + C \end{aligned}$$

Example

Evaluate $\int \frac{(\ln x)^5}{x}\,dx$.

Let $u = \ln x$.

Then $\dfrac{du}{dx} = \dfrac{1}{x}$.

$$du = \frac{1}{x}\,dx$$

$$\begin{aligned} \int \frac{(\ln x)^5}{x}\,dx &= \int u^5\,du \\ &= \frac{1}{6}u^6 + C \\ &= \frac{1}{6}(\ln x)^6 + C \end{aligned}$$

Example

Find $\int 2x^2 \cdot (1+x^3)^{2021}\,dx$ and hence evaluate $\int_{-1}^{0} 2x^2 \cdot (1+x^3)^{2021}\,dx$.

Example

Find $\int e^{2x}\sqrt{1+e^{2x}}\,dx$

Example

Find $\int \frac{2}{x(1+\ln x)}\, dx$

Example

Find $\int \frac{e^{\tan^{-1} x}}{(1+x^2)} \, dx$

Example

Find $\int \frac{8}{x\sqrt{\ln x}}\, dx$

Integration by Parts

Recall the product rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

In differential form it becomes

$$d(uv) = u\,dv + v\,du$$

or equivalently,

$$u\,dv = d(uv) - v\,du$$

Integration by Parts

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Thus we have the Integration-by-parts Formula:

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.$$

$$\int u\,dv = uv - \int v\,du.$$

Integration by Parts

$$\underbrace{\int u\,dv}_{\text{tough}} = uv - \underbrace{\int v\,du}_{\text{easier}}$$

Must choose u and dv correctly
The part you choose as u, you differentiate to find du
The part you choose as dv, you integrate to find v

Evaluate $\int x \ln x \, dx$. $\int u \, dv = uv - \int v \, du$.

Two choices:

---

(1) Let $u = x$

$\frac{du}{dx} = 1$

$du = dx$

$dv = \ln x \, dx$

$v = \int \ln x \, dx$

Difficult to find v

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(2) Let $u = \ln x$

$du = \frac{1}{x} \, dx$

$dv = x \, dx$

$v = \frac{1}{2} x^2$

Good Choice

Integration by Parts - Example

Evaluate $\int x \ln x \, dx$. $\qquad \int u \, dv = uv - \int v \, du.$

Let $u = \ln x$
$du = \frac{1}{x} \, dx$

$dv = x \, dx$
$v = \frac{1}{2} x^2$

$$\int x \ln x \, dx = \overbrace{\frac{1}{2} x^2}^{v} \overbrace{\ln x}^{u} - \int \overbrace{\frac{1}{2} x^2}^{v} \overbrace{\frac{1}{x} \, dx}^{du}$$

$$= \frac{1}{2} x^2 \ln x - \boxed{\frac{1}{2} \int x \, dx}$$

easy to solve

$$= \frac{1}{2} x^2 \ln x - \frac{x^2}{4} + C$$

Integration by Parts - Example

Evaluate $\int xe^x\,dx$. $\int u\,dv = uv - \int v\,du.$

Integration by Parts - Example

Evaluate $\int x^2 e^x \, dx$. $\int u \, dv = uv - \int v \, du.$

Let $u = x^2$

$du = 2x \, dx$

$dv = e^x \, dx$

$v = \int e^x \, dx = e^x$

$$\begin{aligned}
\int x^2 e^x \, dx &= x^2 e^x - \int e^x 2x \, dx \\
&= x^2 e^x - 2 \int x e^x \, dx \\
&= x^2 e^x - 2(x e^x - e^x) + C \\
&= x^2 e^x - 2x e^x + 2e^x + C
\end{aligned}$$

Integration by Parts - Example

Evaluate $\int \ln x \, dx$. $\qquad \int u \, dv = uv - \int v \, du.$

Let $u = \ln x$ $\qquad dv = dx$

$du = \dfrac{1}{x}\, dx$ $\qquad v = \int 1 \, dx = x$

$$\begin{aligned}
\int \ln x \, dx &= (\ln x)x - \int x \left(\frac{1}{x}\right) dx \\
&= x \ln x - \int 1 \, dx \\
&= x \ln x - x + C
\end{aligned}$$

Integration by Parts - Example

Evaluate $\int e^x \cos x \, dx$. $\int u \, dv = uv - \int v \, du$.

Let $u = e^x$

$du = e^x \, dx$

$dv = \cos x \, dx$

$v = \int \cos x \, dx = \sin x$

$$\int e^x \cos x \, dx = e^x \sin x - \int (\sin x) e^x \, dx$$
$$= e^x \sin x - \boxed{\int e^x \sin x \, dx}$$

Need integration by parts again

To find $\int e^x \sin x\, dx$. $\int u\, dv = uv - \int v\, du.$

Similarly to evaluate $\int e^x \cos x\, dx$,

Let $u = e^x$

$du = e^x\, dx$

$dv = \sin x\, dx$

$v = \int \sin x\, dx = -\cos x$

$$\int e^x \sin x\, dx = e^x(-\cos x) - \int (-\cos x)e^x\, dx$$

$$= -e^x \cos x - \boxed{\int e^x \cos x\, dx}$$

Get back the integral we started with

$$\int e^x \sin x\, dx = -e^x \cos x + \int e^x \cos x\, dx$$

Integration by Parts - Example

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

$$\begin{aligned}
\boxed{\int e^x \cos x \, dx} &= e^x \sin x - \boxed{\int e^x \sin x \, dx} \\
&= e^x \sin x - \boxed{\left(-e^x \cos x + \int e^x \cos x \, dx\right)} \\
&= e^x \sin x + e^x \cos x - \boxed{\int e^x \cos x \, dx} \\
\boxed{2 \int e^x \cos x \, dx} &= e^x \sin x + e^x \cos x \\
\int e^x \cos x \, dx &= \frac{1}{2}\left(e^x \sin x + e^x \cos x\right)
\end{aligned}$$

Integration by Parts - Remark

The method is suitable for other integrands such as $x^n e^x, x^n \ln x, x^n \cos x, x^n \sin x$, etc.

$$\underbrace{\int u\,dv}_{\text{tough}} = uv - \underbrace{\int v\,du}_{\text{easier}}$$

Must choose u and dv correctly
The part you choose as u, you differentiate to find du
The part you choose as dv, you integrate to find v

$$\underbrace{\int u\,dv}_{\text{tough}} = uv - \underbrace{\int v\,du}_{\text{easier}}$$

Must choose u and dv correctly
The part you choose as u, you differentiate to find du
The part you choose as dv, you integrate to find v

Question:
Which function to integrate and which to differentiate?

Question:
Which function to integrate and which to differentiate?

Types of Functions	Examples	Remark
Logarithmic Function	$\ln(ax+b)$ or its higher powers	Make the substitution $u = ax+b$ to simplify the integral
Inverse Trigonometric Functions	$\sin^{-1}(ax+b)$, $\cos^{-1}(ax+b)$, $\tan^{-1}(ax+b)$	
Algebraic Functions	Power functions x^{a}, polynomials	
Trigonometric Functions	$\sin(ax+b)$, $\cos(ax+b)$, $\tan(ax+b)$, $\csc(ax+b)$, $\sec(ax+b)$, $\cot(ax+b)$, combinations of these	
Exponential Functions	$\exp(ax+b)$	

LIATE Rule

Example

Find $\int \sin^{-1} \frac{x}{2} \, dx$

Example

Find $\int_1^e (\ln x)^2 \, dx$

Example

(a) Use integration by parts to find

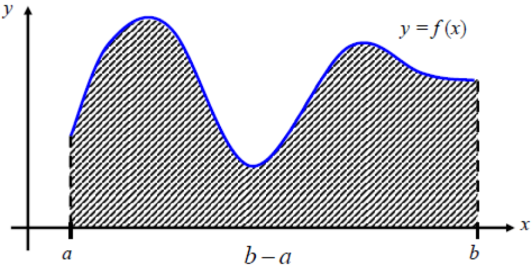
$$\int \ln\sqrt{2x}\,dx.$$

(b) Use an appropriate substitution to find

$$\int \frac{\ln\sqrt{2x}}{x}\,dx.$$

Integrals

Area under the curve of $f(x)$



Area under curve

$$A = \int_a^b f(x)\,dx$$

Question

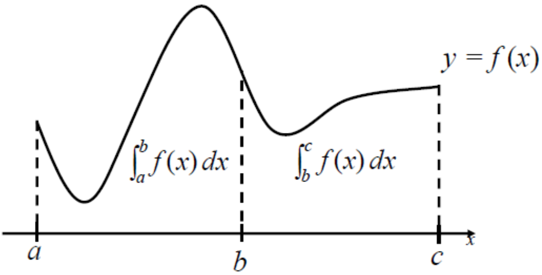
What is the difference between finding the value of an integral and finding area bounded?

To find value of integral:

$$\begin{aligned}\int_0^{\pi} \cos x \, dx &= [\sin x]_0^{\pi} \\ &= \sin \pi - \sin 0 \\ &= 0 - 0 \\ &= 0\end{aligned}$$

Rules of algebra for Definite Integrals

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

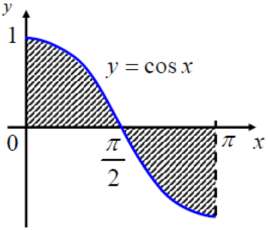


Question

What is the difference between finding the value of an integral and finding area bounded?

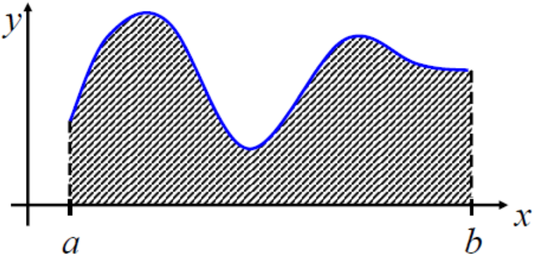
To find shaded area:

$$\int_0^{\frac{\pi}{2}} \cos x\, dx = [\sin x]_0^{\frac{\pi}{2}}$$
$$= \sin\frac{\pi}{2} - \sin 0 = 1$$
$$\int_{\frac{\pi}{2}}^{\pi} \cos x\, dx = [\sin x]_{\frac{\pi}{2}}^{\pi}$$
$$= \sin\pi - \sin\frac{\pi}{2} = -1$$



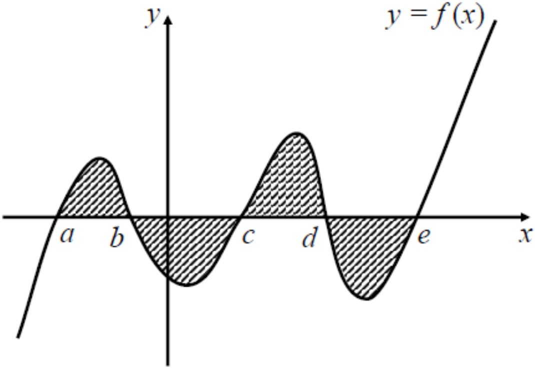
Shaded Area $= 1 + |-1| = 2$

If $f(x) \geqslant 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \geqslant 0$



Area under the curve of $f(x)$, $A = \int_a^b f(x)\,dx$

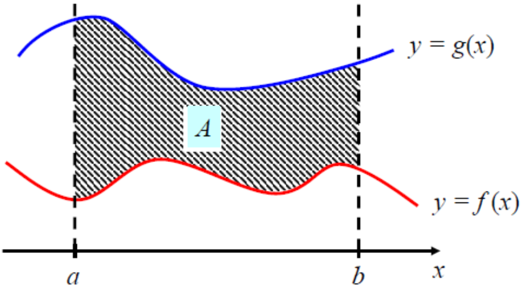
Note



$$\int_a^b f(x)\,dx = +\text{ve}, \quad \int_b^c f(x)\,dx = -\text{ve}$$

$$\int_c^d f(x)\,dx = +\text{ve}, \quad \int_d^e f(x)\,dx = -\text{ve}$$

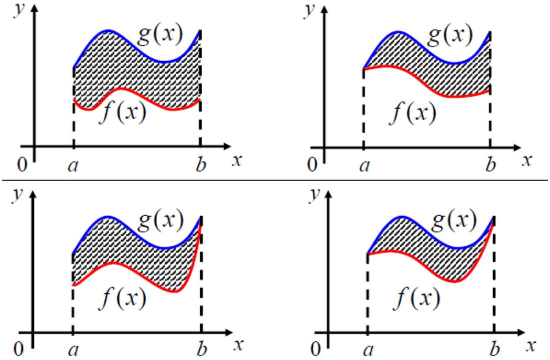
Area between two curves



$$A = \int_a^b \underbrace{(g(x) - f(x))}_{\text{Top curve } - \text{ bottom curve}} \, dx$$

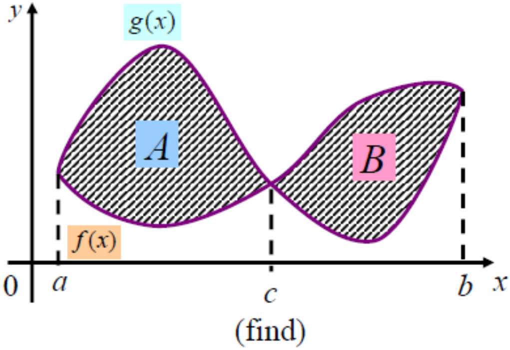
Area between two curves

Consider $g(x) \geqslant f(x)$



Area between the two curves $= \int_a^b g(x) - f(x)\,dx$

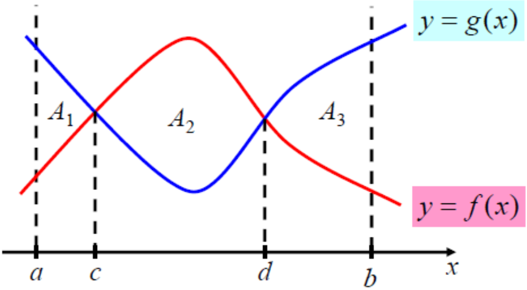
Area between two curves



$$A = \int_a^c g(x) - f(x)\, dx$$

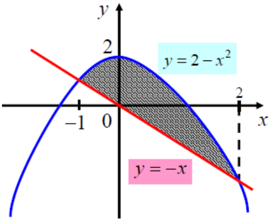
$$B = \int_c^b f(x) - g(x)\, dx$$

Area between two curves



$$A_1 + A_2 + A_3 = \int_a^c (g(x) - f(x))\,dx + \int_c^d (f(x) - g(x))\,dx + \int_d^b (g(x) - f(x))\,dx$$

Find the area enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.



Consider

$$\begin{aligned} -x &= 2 - x^2 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \\ x = 2 &\text{ or } x = -1 \end{aligned}$$

$$\begin{aligned} \text{Area } &= \int_{-1}^{2} 2 - x^2 - (-x)\,dx \\ &= \int_{-1}^{2} 2 - x^2 + x\,dx \\ &= \left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{2} = 4\frac{1}{4} \text{ units}^2 \end{aligned}$$

Find the area of the region in the first quadrant bounded by the curves $y = \sqrt{x}$ and $y = x - 2$.

Consider $x - 2 = \sqrt{x}$ $\qquad$ $y = \sqrt{x}$ Domain : $x \geqslant 0$

$$x - \sqrt{x} - 2 = 0$$

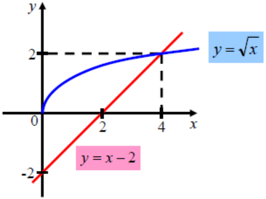
$$(\sqrt{x})^2 - \sqrt{x} - 2 = 0$$

$$(\sqrt{x} - 2)(\sqrt{x} + 1) = 0$$

$$\sqrt{x} = 2 \text{ or } \boxed{\sqrt{x} = -1 \quad \text{(Not possible)}}$$

$$x = 4 \qquad \sqrt{x} = \text{take positive root}$$

$$\begin{aligned} \text{Area } &= \int_0^4 \sqrt{x} - (x - 2)\, dx \\ &= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2 + 2x\right]_0^4 \\ &= \frac{16}{3} \text{ units}^2 \end{aligned}$$



Pause and Think !!!

1. Find the area of the region in the first quadrant bounded by the curves $y = \sqrt{x}$ and $y = x - 2$.

2. Find the area bounded by the curves $y = \sqrt{x}$, $y = x - 1$ and the y-axis.

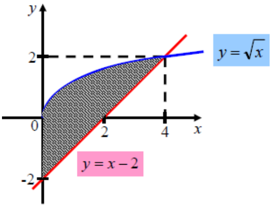
What is the difference between the two questions???

2. Find the area bounded by the curves $y = \sqrt{x}$, $y = x - 1$ and the y-axis.

Consider $\quad x - 2 = \sqrt{x} \qquad\qquad y = \sqrt{x}$ Domain : $x \geqslant 0$

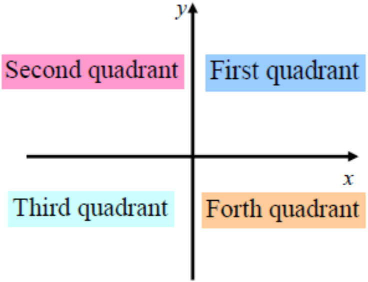
$$\begin{aligned} x - 2 &= \sqrt{x} \\ x - \sqrt{x} - 2 &= 0 \\ (\sqrt{x})^2 - \sqrt{x} - 2 &= 0 \\ (\sqrt{x} - 2)(\sqrt{x} + 1) &= 0 \\ \sqrt{x} &= 2 \text{ or } \boxed{\sqrt{x} = -1 \quad \text{(Not possible)}} \\ x &= 4 \qquad \sqrt{x} = \text{take positive root} \end{aligned}$$

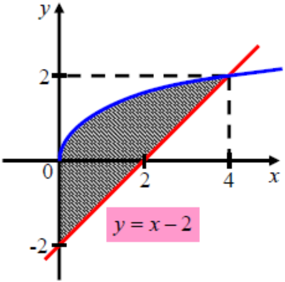
$$\begin{aligned} \text{Area } &= \int_0^4 \sqrt{x} - (x - 2)\, dx \\ &= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2 + 2x\right]_0^4 \\ &= \frac{16}{3} \text{ units}^2 \end{aligned}$$



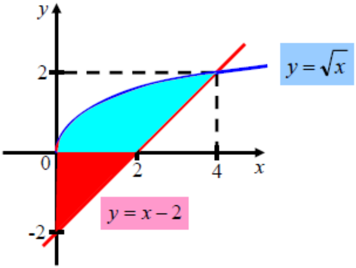
1. Find the area of the region in the $\boxed{\text{first quadrant}}$ bounded by the curves $y = \sqrt{x}$ and $y = x - 2$.

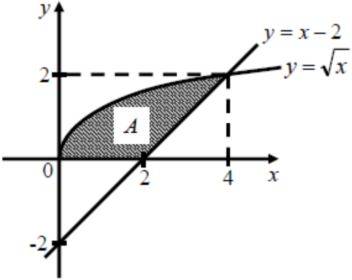
In Question 1, we consider first quadrant





1. Find the area of the region in the first quadrant bounded by the curves $y = \sqrt{x}$ and $y = x - 2$.





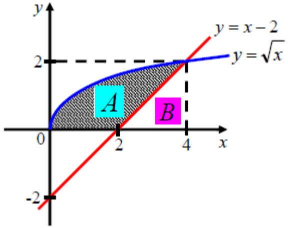
$$\begin{aligned}\text{Area} &= \int_0^4 \sqrt{x} - (x-2)\,dx \\ &= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2 + 2x\right]_0^4 \\ &= \frac{16}{3} \text{ units}^2\end{aligned}$$

$$\begin{aligned}\text{Area } A &= \frac{16}{3} - \text{ Area of red triangle} \\ &= \frac{16}{3} - \frac{1}{2} \times 2 \times 2 \\ &= \frac{10}{3}\end{aligned}$$

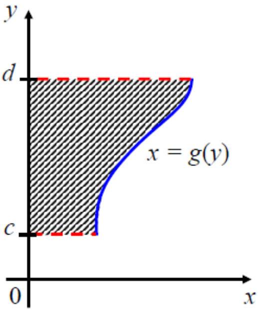
Find the area of the region in the first quadrant bounded by the curves $y = \sqrt{x}$ and $y = x - 2$.

Method 2

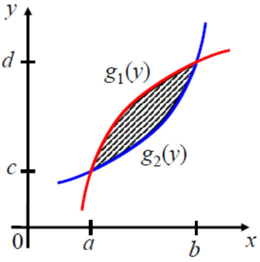
$$\text{Area } A = \int_0^4 \sqrt{x}\, dx - \text{ Area of triangle } B$$
$$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^2 - \frac{1}{2} \times 2 \times 2$$
$$= \frac{10}{3} \text{ units}^2$$



Area between two curves



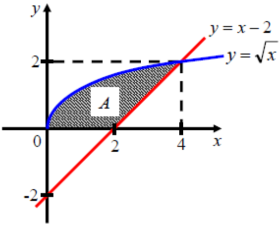
Area $= \int_c^d g(y)\, dy$



Area $= \int_c^d g_2(y) - g_1(y)\, dy$

Find $y = c$ and $y = d$

Find the area of the region in the first quadrant bounded by the curves $y = \sqrt{x}$ and $y = x - 2$.



Consider
$$
\begin{aligned}
x - 2 &= \sqrt{x} \\
x - \sqrt{x} - 2 &= 0 \\
(\sqrt{x})^2 - \sqrt{x} - 2 &= 0 \\
(\sqrt{x} - 2)(\sqrt{x} + 1) &= 0 \\
\sqrt{x} = 2 \text{ or } \sqrt{x} &= -1 \quad \text{(Not possible)} \\
x &= 4
\end{aligned}
$$

When $x = 4, y = \sqrt{4} = 2$

$$
\begin{aligned}
y &= \sqrt{x} \quad \to x = y^2 \\
y &= x - 2 \to x = y + 2
\end{aligned}
$$

$$
\begin{aligned}
A &= \int_0^2 ((y+2) - y^2)\, dy \\
&= \frac{10}{3} \text{ units}^2
\end{aligned}
$$

Example (Curve lies above the x-axis)

Find area of the region bounded by $y = x(6-x)$ and the x-axis.

Example (Curve lies below the x-axis)

Find area of the region bounded by $y = -\frac{6}{x^2}$ and the lines $x = 1$ and $x = 3$.

Example ($f(x)$ changes sign on the interval $[a, b]$)

Find area of the region bounded by $y = x(x^2 - x - 2)$ and the x-axis for $-1 \leqslant x \leqslant 2$.

Example (Area bounded between two curves $y = f(x)$ and $y = g(x)$)

Find area of the region in the first quadrant bounded by the curve $y = 1 - \cos x$ and the line $y = \frac{2x}{\pi}$ for $0 \leqslant x \leqslant \frac{\pi}{2}$.

Example (Curve lies to the right of the y-axis)

Find area of the region bounded by $y = \frac{3}{x}$, the y-axis and the lines $y = 1$ and $y = 2$.

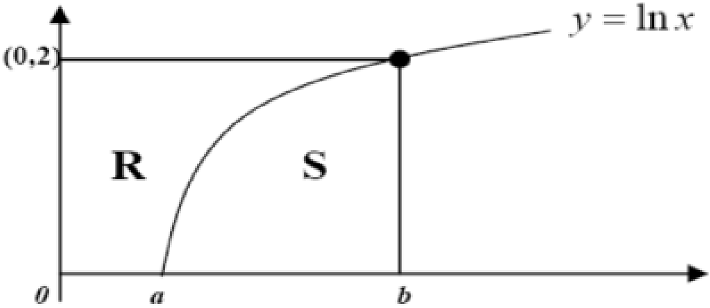
Example (Curve lies to the left of the y-axis)

Find area of the region bounded by $x = y^2 - 4y$, the y-axis and the lines $y = 1$ and $y = 2$.

Example (Area bounded between two curves $x = g(y)$ and $x = h(y)$)

Find area of the region bounded by $x = y^2 - 2y - 3$ and $y = x - 1$.

Example

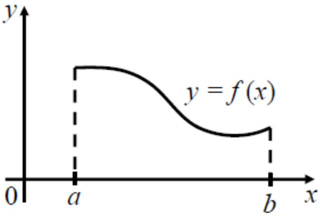


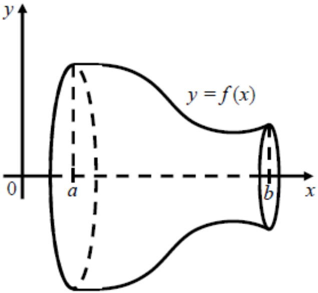
(i) Find a and b

(ii) Calculate the area of R

(iii) Use (ii) to find the area of S

(I) About x-axis



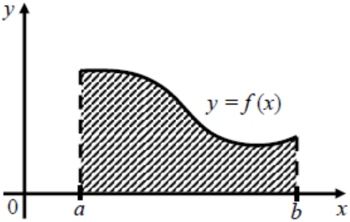


$$V = \int_a^b \pi y^2 \, dx \qquad \text{or} \qquad V = \int_a^b \pi [f(x)]^2 \, dx$$

Volume of Solids of Revolution

Volume of solid generated by revolving about the $\boxed{x\text{-axis}}$ from $x = a$ to $x = b$ is:

$$V = \int_a^b \pi[f(x)]^2\,dx$$
$$= \int_a^b \pi y^2\,dx$$



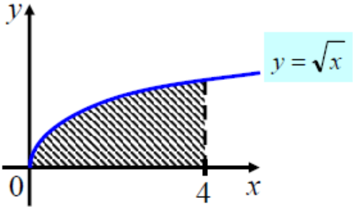
Can only use this formula if you revolve about x-axis

Note: revolving about x-axis is the same as revolving about the line $y = 0$

Example

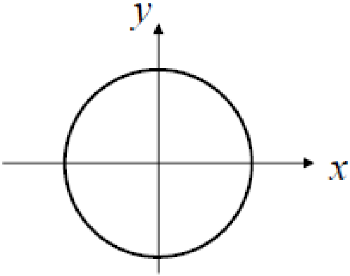
The region between $y = \sqrt{x}$, $0 \leqslant x \leqslant 4$, and the x-axis is revolved about the x-axis. Find the volume generated.

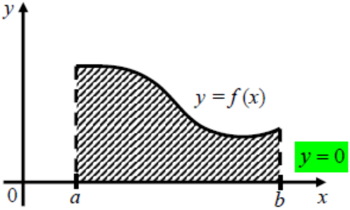
$$
\begin{aligned}
V &= \pi \int_a^b y^2 \, dx \\
&= \pi \int_0^4 (\sqrt{x})^2 \, dx \\
&= \pi \int_0^4 x \, dx \\
&= 8\pi \text{ units}^3
\end{aligned}
$$



Example

Derive formula for volume of a sphere: $V = \frac{4}{3}\pi r^3$.

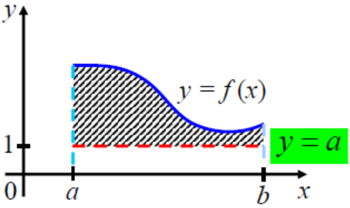




$$V = \int_a^b \pi y^2 \, dx$$

or

$$V = \int_a^b \pi [f(x)]^2 \, dx$$



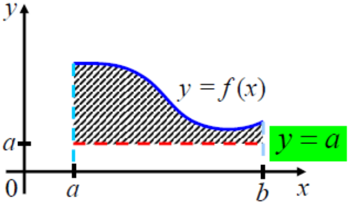
Question:
How to modify the formula to find volume???

About the line $y = a$.

$$V = \pi \int_a^b (y - a)^2 \, dx$$

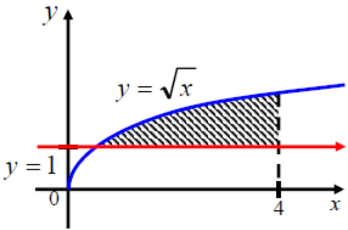
or

$$V = \pi \int_a^b (f(x) - a)^2 \, dx$$

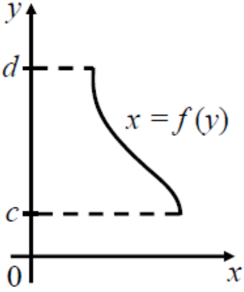


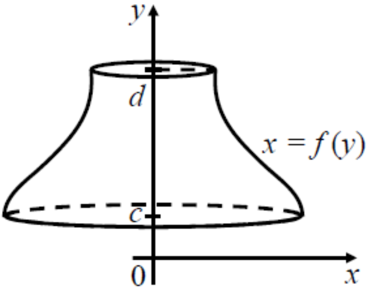
Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$ and $x = 4$ about the line $y = 1$.

$$
\begin{aligned}
V &= \pi \int_a^b (y-a)^2 \, dx \\
&= \boxed{\pi \int_1^4 (\sqrt{x}-1)^2 \, dx} \\
&= \pi \int_1^4 (x - 2\sqrt{x} + 1) \, dx \\
&= \pi \left[\frac{x^2}{2} - \frac{4}{3} x^{\frac{3}{2}} + x \right]_1^4 \\
&= \frac{7\pi}{6} \text{ units}^3
\end{aligned}
$$



(II) About y-axis





$$V = \int_c^d \pi x^2 \, dy$$

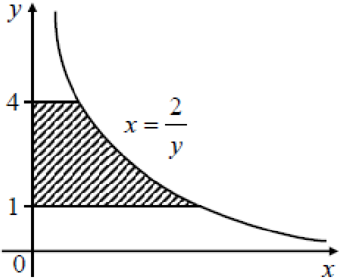
or

$$V = \int_c^d \pi [g(y)]^2 \, dy$$

Example

Find the volume of the solid generated by revolving the region bounded by $x = \frac{2}{y}$, $y = 1$ and $y = 4$ about the y-axis.

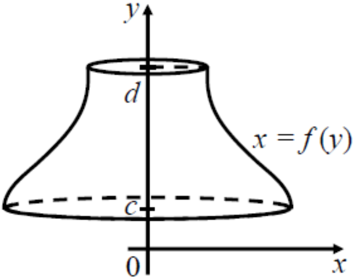
$$
\begin{aligned}
V &= \pi \int_1^4 x^2 \, dy \\
&= \pi \int_1^4 \left(\frac{2}{y}\right)^2 \, dy \\
&= 4\pi \int_1^4 y^{-2} \, dy \\
&= 4\pi \left[\frac{y^{-1}}{-1}\right]_1^4 = 3\pi \text{ units}^3
\end{aligned}
$$



Example

The region R is formed by $y = \ln x$, the axes and the line $y = 1$ is rotated completely about the y-axis. Calculate the volume of the solid formed.

Revolve about y-axis
Same as the line $x = 0$



$$V = \int_c^d \pi [g(y)]^2 \, dy$$

$$V = \int_c^d \pi x^2 \, dy$$

Revolve about the line $x = b$

$$V = \pi \int_c^d (g(y) - b)^2 \, dy$$

Example

Find the volume of the solid formed by rotating the region bounded by $y = x^2 + 1$ and $y = 9 - x^2$ about

(i) the y-axis (ii) x-axis.

Example

The diagram below shows the graph of $y = 3 - e^{-x}$ which cuts the y-axis at the point A. The line $y = c$ is a horizontal asymptote of the curve, and the tangent to the curve $y = 3 - e^{-x}$ at A is drawn. The region R is bounded by the curve, the tangent at A and the x-axis.

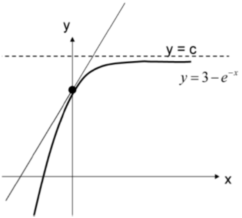
Find

(i) the coordinates of A,

(ii) the value of c,

(iii) the **exact** area of the region R,

(iv) the **exact** volume of the solid formed by rotating the region R completely about the x-axis.



Introduction to First Order Ordinary Differential Equations (ODE)

First Order Ordinary Differential Equations

A first order ordinary differential equation in x and y is an equation that contains terms involving $\frac{dy}{dx}$ and and one or both of x and y.

Example of first order ODE:

$$\frac{dy}{dx} + y = y^2$$

$$x\frac{dy}{dx} + x + y = 2021$$

Solutions of ODE

General Solution

For any constant c, $y = \cos 2x + c$ is the **general solution** of the ODE:

$$\frac{dy}{dx} + 2\sin 2x = 0$$

Verify:

Particular Solution

$$\frac{dy}{dx} + 2\sin 2x = 0$$

$y = \cos 2x - 1$ and $y = \cos 2x + 7$ are two **particular solutions** of the above ODE

Example

Find y in terms of x

$$x\frac{dy}{dx} = x + 2$$

$y = 3$ when $x = 1$.

Example

Find the particular solution of the DE

$$(2x-1)\frac{dx}{dy} - 2e^x = 0$$

for which $y = 2$ when $x = 0$.

ODE of the form $\frac{dy}{dx} = g(y)$

Example

Solve the ODE $6e^{2y+1}\frac{dy}{dx} = e^{1-y}$

Example

A curve C which passes through the point $(2, 1)$ is such that at any point (x, y) on C,

$$\frac{dy}{dx} = \frac{y}{1 + 2y^2}.$$

(i) Find the equation of the normal to the curve at the point $(2, 1)$.
(ii) Find the equation of the curve C, giving your answer in the form $x = f(y)$.

Example

A curve C passes through $(2, 1)$ and is such that at any point (x, y) on the curve,

$$x^2 \frac{dy}{dx} = y(x^3 + 4)$$

Find the equation of the curve.

DE by Substitution

To convert given ODE to the form

$$\frac{dy}{dx} = f(x) \qquad \text{or} \qquad \frac{dy}{dx} = g(y)$$

Example

Use the substitution $y = w + x$ to solve the ODE $\frac{dy}{dx} = 1 + \frac{1}{(y-x)^2}$ given that the solution curve passes through the origin.

Example

Show that the differential equation

$$x^3\frac{dy}{dx} + 4 = 2x^2y$$

can be reduced to

$$\frac{dz}{dx} = -\frac{4}{x^5}$$

by means of the substitution $y = zx^2$.
Hence, find y in terms of x, given that $y = 6$ when $x = 1$.