NATIONAL UNIVERSITY OF SINGAPORE

MA1301 Introductory Mathematics **Tutorial 8**

1. Solve the following differential equations.

(a) $(1-x)\dfrac{dy}{dx} = 6 \ (x < 1), \quad y = 5$ when $x = 0$.

(b) $x\dfrac{dy}{dx} + (x^2+1) = 0 \ (x > 0), \quad y = \dfrac{1}{2}$ when $x = 1$.

(c) $\dfrac{dy}{dx} = \dfrac{3}{ye^{y-1}} \ (y > 1), \quad y = 1$ when $x = 0$.

(d) $e^{y+1}\dfrac{dy}{dx} - e^{1-2y} = 0, \quad y = 0$ when $x = 1$.

2. Solve the following differential equation

$$\frac{dy}{dx} + \frac{8x+4y+1}{4x+2y+1} = 0, \quad y = 1 \text{ when } x = 1.$$

3. The angle between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is 120°. If $|\boldsymbol{a}| = 3$ and $|\boldsymbol{b}-\boldsymbol{a}| = 7$, find $|\boldsymbol{b}|$ and $|\boldsymbol{a}+\boldsymbol{b}|$.

4. Let $A(0,3,4)$, $B(-2,p,3)$, $C(q,1,3)$ and $D(4,7,r)$ be points in $\mathbb{R}^3$.

(a) Find the value of p for which the length $|\overrightarrow{AB}| = 3$.

(b) Find the values of p and r for which A, B and D are collinear.

(c) Find the value of q for which $\overrightarrow{AC} \perp \overrightarrow{OC}$, where O is the origin.

(d) Find the angle $\angle ABC$ if $p = 1$ and $q = 2$.

5. (a) Find the unit vector in the direction of $-4\boldsymbol{i} + 3\boldsymbol{j}$.

(b) Find two vectors which have magnitude 34 units and are parallel $4\boldsymbol{i} - \dfrac{15}{2}\boldsymbol{j}$.

6. Relative to the origin O, the position vectors of A, B and C are $3\boldsymbol{i} - \boldsymbol{j}$, $-\boldsymbol{i} + 2\boldsymbol{j}$ and $3\boldsymbol{j}$ respectively.

(a) Show that $\triangle ABC$ is an isosceles triangle.

(b) Find $\angle BAC$, and hence find the area of $\triangle ABC$.

7. In $\triangle OAB$, $\angle AOB = 90°$. Let C be the point on the segment AB such that $\overrightarrow{OC} \perp \overrightarrow{AB}$. Show that

$$\frac{|\overrightarrow{CA}|}{|\overrightarrow{CB}|} = \frac{|\overrightarrow{OA}|^2}{|\overrightarrow{OB}|^2}.$$

Solutions and Hints

1. (a) $y = -6\ln(1-x) + 5$; (b) $y = -\frac{1}{2}x^2 - \ln x + C$; (c) $x = \frac{1}{3}e^{y-1}(y-1)$;
(d) $x = \frac{1}{3}e^{3y} + \frac{2}{3}$.

2. $(2x+y)^2 + (x+y) = 11$. *Hint*: Set $y = v - 2x$. Then convert the equation in x and v.

3. 5, $\sqrt{19}$. *Hint*: Use law of cosine: $c^2 = a^2 + b^2 - 2ab\cos\theta$.

4. (a) 1 or 5. (b) $p = 1$, $r = 6$. *Hint*: $\overrightarrow{AB} \parallel \overrightarrow{AD}$, so $\overrightarrow{AB} = \lambda\overrightarrow{AD}$ for some $\lambda \in \mathbb{R}$.
(c) $\sqrt{5}$ or $-\sqrt{5}$. *Hint*: $\boldsymbol{u} \perp \boldsymbol{v} \Leftrightarrow \boldsymbol{u} \bullet \boldsymbol{v} = 0$.
(d) $\cos^{-1}\left(-\frac{2}{3}\right)$ (or $\pi - \cos^{-1}\left(\frac{2}{3}\right)$). *Hint*: $\cos\theta = \dfrac{\boldsymbol{u} \bullet \boldsymbol{v}}{|\boldsymbol{u}|\,|\boldsymbol{v}|}$.

5. (a) $-\frac{4}{5}\boldsymbol{i} + \frac{3}{5}\boldsymbol{j}$. (b) $16\boldsymbol{i} - 30\boldsymbol{j}$ and $-16\boldsymbol{i} + 30\boldsymbol{j}$.

6. (a) *Hint*: Evaluate $|\overrightarrow{AB}|$, $|\overrightarrow{AC}|$ and $|\overrightarrow{BC}|$.
(b) $\cos^{-1}\frac{24}{25}$, $\frac{7}{2}$. *Hint*: Use a formula for the area of a triangle: $\frac{1}{2}ab\sin\theta$.

7. *Hint*: Suppose $\overrightarrow{AC} = \lambda\overrightarrow{AB}$. Then express $\overrightarrow{OC}$ in terms of $\overrightarrow{OA}$, $\overrightarrow{OB}$ and λ. Then use $\overrightarrow{OC} \bullet \overrightarrow{AB} = 0$ to determine the value of λ. A diagram will be very helpful.