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Simulating the Motion of a Massive Object Orbiting a Black Hole

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Abstract

Due to the extreme nature of black holes, the astronomical objects that orbit them differ from "simple" orbits found throughout the universe, such as an exoplanet orbiting a star or a belt of asteroids existing as part of a solar system. For any ordinary orbit, we can approximate the nature of the orbits using Newtonian physics, essentially ignoring the energy-based nature of gravity that is described in Einstein's Theory of General Relativity. For black holes, general relativity is an essential component in understanding these complex orbits. In this paper, I discuss the process of using Python script to analyze the orbit of a massive object orbiting a black hole. The two radii that are most influential when determining the type of orbit of a massive object orbiting a black hole is the Innermost Stable Circular Orbit (ISCO) and the Innermost Bound Circular Orbit (IBCO). The ISCO is defined as having a larger radius value than the IBCO. Once at a smaller radius than the IBCO, the massive object will begin to spiral toward the center of the BH, eventually passing the event horizon, the inescapable boundary of the BH. Outside of the ISCO, the orbit of the massive object would likely be relatively stable, being a stable circular orbit or resembling the orbit of Mercury, an elliptical orbit with a slight precession over time. Using general relativity, these orbits were reproduced and analyzed at varying radii: outside of the ISCO, between the ISCO and the IBCO, and inside the IBCO.

Introduction

Black holes are an extreme astronomical object and much of the information about them is still highly theoretical. The governing knowledge about black holes relies on Einstein's Theory of General Relativity. General Relativity (GR) is essential to understanding our universe because it provides a generalized interpretation of our universe without a focus on our choice of coordinates by mathematical objects called tensors. At large distances where the curvature of spacetime is not greatly affecting either astronomical body, Newtonian mechanics can be used instead of GR [Chisari1 and Zaldarriaga, 2011] but for interacting massive bodies, GR is able to explain what Newtonian mechanics cannot, including relativistic speeds, geodesic geometry, and wobbling orbits [Andjelka B. Kovačević and Popović, 2024].

The boundaries that are essential to understanding the different types of orbits around a black hole are the innermost Stable Circular Orbit (ISCO) and the Innermost Bound Circular Orbit (IBCO). These two radii are defined using the Schwarzschild radius which defines the event horizon of the black hole. The Schwarzschild radius can be defined as follows:

$$r_s = 2GM/(c^2)$$

Furthermore, the expressions for the ISCO and IBCO are as follows:

$$r_{ISCO} = 3r_s = 6GM/(c^2)$$

$$r_{IBCO} = (3/2)r_s = 3GM/(c^2)$$

From these two radii, we can establish three different regions: outside of the ISCO, in between the ISCO and the IBCO, and inside of the IBCO.

Outisde of the ISCO

Three different types of orbits can occur outside of the ISCO. The first orbit is a stable circular orbit, resulting from the minimum of the relativistic potential shown in figure 3. The second orbit is an unbound orbit, meaning that the total energy of the orbit is positive, resulting in a parabolic or hyperbolic unbound orbit around the black hole. The third orbit is a processing elliptical orbit [A. M. Ghez and Duchêne, 2005]. This elliptical orbit typically reflects environments where a Newtonian potential and a Keplarian orbit can be assumed [Kareem El-Badry, 2022], such as within a relatively simple solar system. Once the radius is less than that of the ISCO, there is no stable circular orbit.

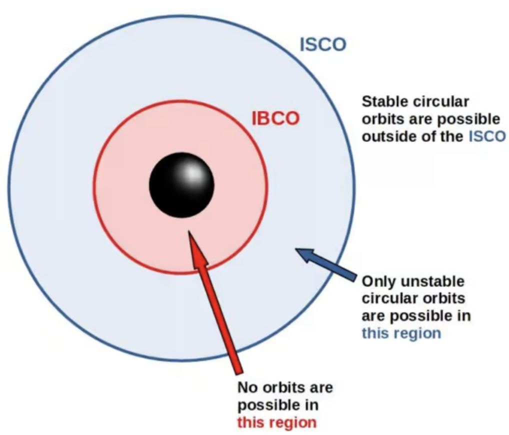

Fig. 1: This diagram shows the ISCO (solid blue line) and the IBCO (solid red line). The region between the ISCO and the IBCO is shaded in blue while the region inside of the IBCO is shaded in red [Hirvonen, 2023]

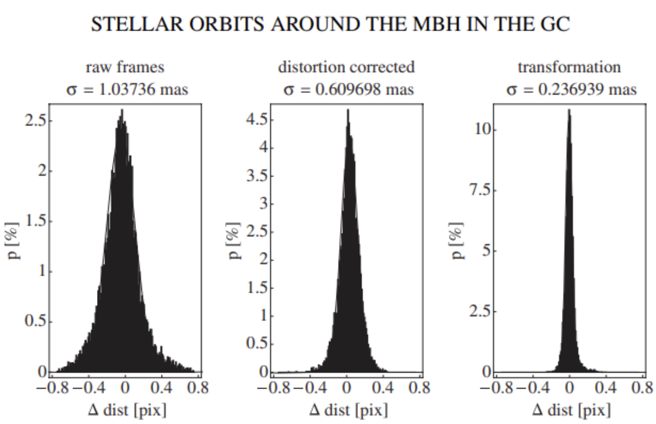

Fig. 2: This diagram shows the distortion in the orbit of S2, a star with a mass of roughly 14 to 15 $M_\odot$ that orbits Sagittarius A*. From this, it can be determined that the orbits represented by this star are not heavily distorted and can, in fact, be modeled as Keplarian. The perihelion of S2 is roughly 120 AU and the Schwarzschild radius of Sgr A* is roughly 0.08 AU so therefore, the orbit of S2 is far from the ISCO [S. Gillessen and Ott, 2009].

Between the ISCO and the IBCO

Two different types of orbits can occur within this region surrounding the black hole. The first type of orbit that can occur is an unstable circular orbit, resulting from the maximum of the relativistic potential shown in figure 3. The second type of orbit is known as the zoom-whirl orbit, which does not exist in Newtonian mechanics. These orbits essentially have two different paths: "whirls" which refers to a small orbit around the BH and "zooms" which refers to a larger orbit [Hirvonen, 2023].

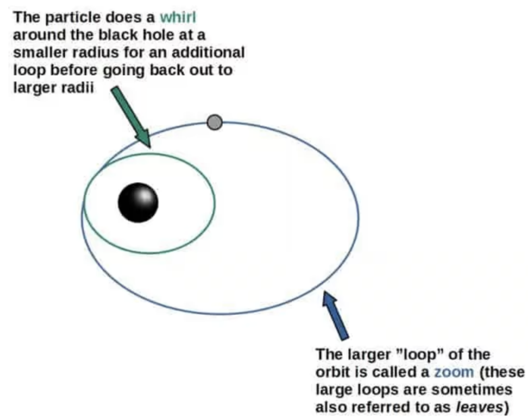

Fig. 3: This diagram demonstrates the zoom-whirl orbit that exists in the region between the ISCO and the IBCO. The smaller orbit is referred to as a "whirl" and the larger orbit is referred to as a "zoom" [Hirvonen, 2023].

Inside the IBCO

One type of orbit exists when the radius of the orbit is less than that of the IBCO and this orbit is extremely unstable and will begin to spiral towards the event horizon. Given this, the term for this orbit is the spiral fall orbit. The thing that can orbit within this radius is photons moving at the speed of light. This light orbiting the event horizon is commonly referred to as the photon sphere.

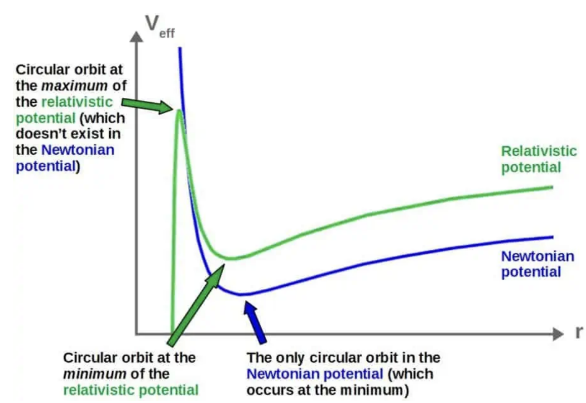

Fig. 4: This figure displays the difference in potentials for orbits based on Newtonian physics and orbits based on General Relativity. For those governed by General Relativity, there are two extrema, one minimum and one maximum, where circular orbits can occur [Hirvonen, 2023].

Methods

For the analysis of these orbits of massive objects around black holes, I started with a relatively simple Python script that described the motion of two rotating bodies affected by a Yukawa

Potential. The Yukawa potential is characterized by an exponential decay rather than the inverse-square law which characterizes Newton's gravitational potential. The Yukawa model is ideal for models with short distances, such as inside an atom. Depending on the parameters used, Yukawa orbits are typically unstable and therefore seemed like a reasonable starting point for the strange orbits observed around black holes.

Outisde of the ISCO

These types of orbits can be seen in situations where black holes are not involved. For these orbits, I continued to use the Yukawa orbit as a template. The Yukawa Potential is defined as follows:

$$U(r) = -g^2 e^{-\lambda r}/r$$

where g is the coupling constant, lambda is the decay length, and r is the distance between the two objects. This potential results in a force expression as follows:

$$\mathbf{F} = -\nabla U = g^2 e^{-\lambda r}/r^3(\lambda r + 1)(\boldsymbol{r_2} - \boldsymbol{r_1})$$

where $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ represent the distance vectors of each object from the center of mass.

Between the ISCO and the IBCO

For orbits that involve the ISCO, we must account for Einstein's Theory of General Relativity (GR). We are able to do so by replacing the effective potential used in the Yukawa Orbit simulation with a modified version of the Newtonian potential with a term accounting for GR. The new potential expression for the Schwarzschild metric is as follows [Rubin H. Landau, 2024]:

$$V_{eff}(r) = -GMm/r + l^2/2mr^2 - GMl^2/mc^2r^3$$

where G is the gravitational constant, M is the mass of the BH, m is the mass of the object, r is the distance from the BH to the massive object, and l is the angular momentum per unit rest mass. The first term of the righthand side represents the general attraction term seen in classical mechanics. The second term represents the angular momentum barrier. The third term is the term that accounts for GR which accounts for the additional strong attraction at very short distances. This potential results in a force expression as follows:

$$F(r) = -GMm/r^2 + l^2/mr^3 - 3GMl^2/mc^2r^4$$

Inside the IBCO

As previously mentioned, the only orbit that is possible inside the IBCO is extremely unstable. If a massive object crosses the radius for the IBCO, it will begin to spiral toward the event horizon of the BH and will be unable to escape its intense gravity. Therefore, the code must reflect this spiral nature.

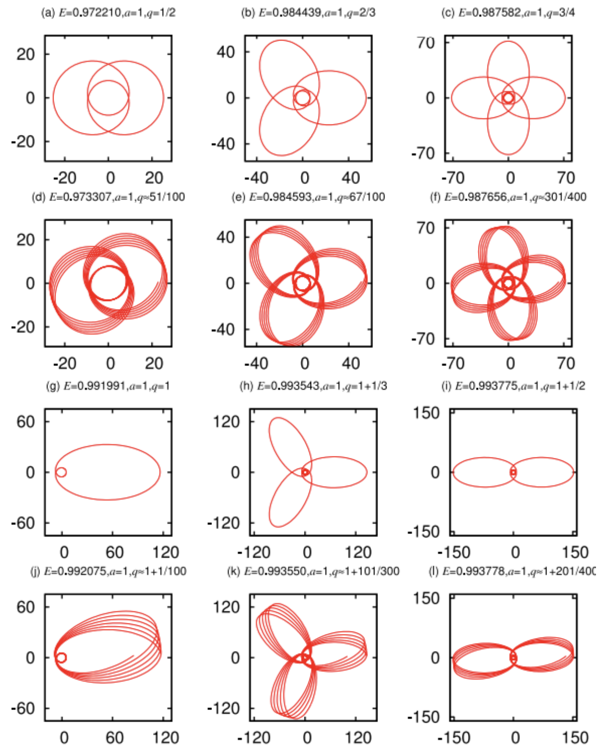

Fig. 5: This image shows the many different kinds of orbits that can occur around a black hole [Deng, 2020].

Program codes

As discussed in the previous section, Python language was used for this analysis. For each function written for each kind of orbit, the parameters were the same: the mass of the black hole, the mass of the massive object that orbits the black hole, the total time allowed for the program to simulate the orbit, the change in time noted as dt, and the skip parameter which allows the function to skip some iterations.

Outside of the ISCO

For the region outside of the ISCO, I used the code for the Yukawa Orbit as a rough approximation for the elliptical orbits observed. Below is the script used to produce figure 6 which shows an elliptical orbit and figure 7 which shows the massive object being ejected away from the BH. The following code was also used to produce figure 8 which although does not use GR, shows a similar nature of precession of the perihelion of the massive object with respect to the BH.

```
def YukawaOrbit(mBH, mObj, T, dt, skip):
    g = 1
    lam = 0.5
    mu = mBH * mObj / (mBH + mObj)
    steps = round(T/dt/skip)
    rBH = np.zeros((steps,2), 'float')
    rObj = np.zeros((steps,2), 'float')
    vBH = np.zeros((steps,2), 'float')
    vObj = np.zeros((steps,2), 'float')
    time = np.zeros((steps,1), 'float')
    rBH[0,0] = -mObj / (mBH + mObj)
    rObj[0,0] = mBH / (mBH + mObj)
    vBH[0,1] = -np.sqrt(3*mu*np.exp(-0.5)/2)/mBH
```

```
    vObj[0,1] = np.sqrt(3*mu*np.exp(-0.5)/2)/mObj
    rBHmid = np.zeros((1,2), 'float')
    rObjmid = np.zeros((1,2), 'float')
    vBHmid = np.zeros((1,2), 'float')
    vObjmid = np.zeros((1,2), 'float')
    metadata = dict(title = "YukawaOrbit", \
                    artist = "Matplotlib")
    writer = FFMpegWriter(fps = 15, metadata = metadata)
    fig1 = plt.figure()
    lBH, = plt.plot([], [], 'ko', ms=25)
    lObj, = plt.plot([], [], 'yo', ms=5)
    plt.xlim(-5, 5)
    plt.ylim(-5, 5)
    with writer.saving(fig1, "YukawaOrbit.mp4", 100):
        for i in range(1, steps):
            rBH[i,:] = rBH[i - 1,:]
            vBH[i,:] = vBH[i - 1,:]
            rObj[i,:] = rObj[i - 1,:]
            vObj[i,:] = vObj[i - 1,:]
            time[i] = time[i - 1]
            for j in range(skip):
                rBHmid[0,:] = rBH[i,:] + dt * vBH[i,:] / 2
                rObjmid[0,:] = rObj[i,:] + dt * vObj[i,:] / 2
                r = np.sqrt((rBHmid[0,0] - rObjmid[0,0]) \
                    ** 2 + (rBHmid[0,1] - rObjmid[0,1]) ** 2)
                Fmag = g ** 2 * np.exp(-lam*r) * \
                    (lam*r + 1) / r ** 3
                vBHmid[0,:] = 0.5 * vBH[i,:]
                vObjmid[0,:] = 0.5 * vObj[i,:]
                vBH[i,:] = vBH[i,:] + dt * Fmag * \
                    (rObj[i,:] - rBH[i,:]) / mBH
                vObj[i,:] = vObj[i,:] + dt * Fmag * \
                    (rBH[i,:] - rObj[i,:]) / mObj
                vBHmid[0,:] += 0.5 * vBH[i,:]
                vObjmid[0,:] += 0.5 * vObj[i,:]
                rBH[i,:] = rBH[i,:] + dt * vBHmid[0,:]
                rObj[i,:] = rObj[i,:] + dt * vObjmid[0,:]
            lBH.set_data(rBH[i, 0], rBH[i, 1])
            lObj.set_data(rObj[i, 0], rObj[i, 1])
            writer.grab_frame()
            plt.pause(0.02)
    plt.figure()
    plt.title("Yukawa orbit of massive object around BH")
    plt.xlabel("x position")
    plt.ylabel("y position")
    plt.plot(rBH[0,0], rBH[0,1], 'go', label = "start")
    plt.plot(rBH[-1,0], rBH[-1,1], 'co', label = "end")
    plt.plot(rObj[0,0], rObj[0,1], 'go')
    plt.plot(rObj[-1,0], rObj[-1,1], 'co')
    plt.plot(rBH[:,0], rBH[:,1], 'k', ms=50)
    plt.plot(rObj[:,0], rObj[:,1], 'y', ms=5)
    plt.legend()
    return rBH, rObj, vBH, vObj
```

Inside the ISCO and the IBCO

Once the radii of the ISCO is reached, Newtonian mechanics is no longer the driving ideology behind the orbits of massive objects around a BH. The following code is modeled after the work of Dr. Xue-Mei Deng [Deng, 2020] as well as Dr. Rubin H. Landau, Dr. Manuel J. Páez, and Dr. Cristian C. Bordeianu [Rubin H. Landau, 2024] and accounts for the effects of General Relativity:

```
# this function calculates the orbit of a massive object orbiting
# a black hole with parameters E (energy), u (initial value of
# the potential), l (angular momentum), and dt (time step)
def GROrbit(E, u, l, dt):
    # scaling constants for easier math
    G = 1   # gravitational constant
    M = 1   # mass of BH
    c = 1   # speed of light
    # derivative of u WRT phi (angle of precession)
    du = np.sqrt(2*E/(l**2) + 2*G*u/(l**2) - G*u**2 + 2*G*(u**3))
    def dudphi(du):
        return du
    def du2dphi2(u):
        V = -u + G*M/(l**2) + 3*G*M*u**2/(c**2)
        return V
    # create an array or phi values
    phi = np.arange(0, 12*np.pi, dt)
    # initializing position vectors to be the shape of phi
    r_vals = np.zeros(phi.shape)
    x = np.zeros(phi.shape)
    y = np.zeros(phi.shape)
    # integrating using fourth order Runge-Kutta
    for i, ph in enumerate(phi):
        k1 = dt*dudphi(du)
        k2 = dt*dudphi(du + k1/2)
        k3 = dt*dudphi(du + k2/2)
        k4 = dt*dudphi(du + k3)
        l1 = dt*du2dphi2(u)
        l2 = dt*du2dphi2(u + l1/2)
        l3 = dt*du2dphi2(u + l2/2)
        l4 = dt*du2dphi2(u + l3)
        u += (k1 + 2*k2 + 2*k3 + k4)/6
        du += (l1 + 2*l2 + 2*l3 + l4)/6
        r = 1 / u
        r_vals[i] = r
        x[i] = r * np.cos(ph)
        y[i] = r * np.sin(ph)
    plt.subplots(figsize=(10,10))
    plt.title("Massive Object Orbiting a BH")
    plt.xlabel("x position")
    plt.ylabel("y position")
    plt.plot(x, y, 'm', label = "GR orbit")
    plt.plot(x[0], y[0], 'co', label = "start", ms = 5)
    plt.plot(x[-1], y[-1], 'ro', label = "end", ms = 5)
    plt.plot(0, 0, 'ko', label = "BH", ms = 15)
    plt.legend()
    plt.show()
```

Results

For all Yukawa Orbits, I used a dt value of 0.01 and a skip value of 10 for times less than 100, and a skip value of 100 for times greater than 100.

Outside of the ISCO

These orbits are typically elliptical in nature and are relatively stable, considering they orbit an extremely chaotic environment. For these orbits, I used the Yukawa Orbit Python function.

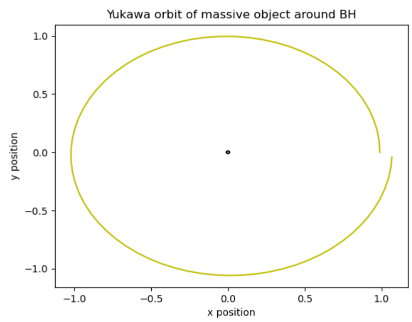

Fig. 6: This orbit reflects a BH mass to orbiting mass ratio of 100 over a total time of 20 and a coupling constant of 1.

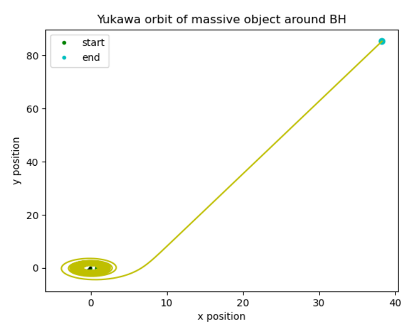

Fig. 7: This orbit reflects a BH mass to orbiting mass ratio of 100 over a total time of 1000 and a coupling constant of 1. The orbit is no longer stable and the massive object is ejected.

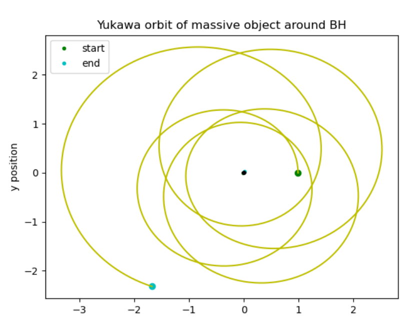

Fig. 8: This orbit reflects a BH mass to orbiting mass ratio of 100 over a total time of 1000 and a coupling constant of 0.9. Although the coupling constant has only been slightly altered from the other plots, the precession of the perihelion can be seen.

Inside the ISCO and the IBCO

For these orbits, I used the code that accounts for general relativity. The parameters that I changed for each orbit were the initial energy, initial potential, and angular momentum. For every plot, I used a dt value of 0.001. Specific parameters that were used for each plot are specified in the captions. It is important to note that with this code, it was necessary to use an energy with an absolute value less than 0.04 and an initial potential value greater than 0.04.

Changing the Initial Energy: E

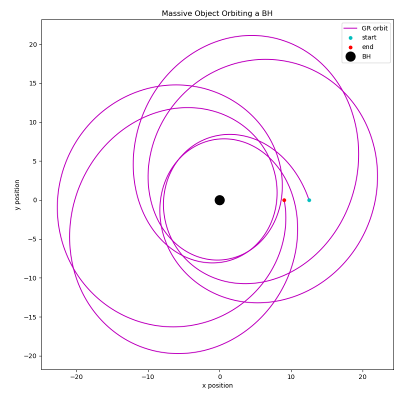

Fig. 9: The above plot shows the relativistic orbit of a massive object orbiting a black hole with the following parameters: E = -0.03, u = 0.08, and l = 4.

The plot in figure 9 is similar to the plot in figure 8, showing an orbit with intense precession although figure 9 accounts for GR.

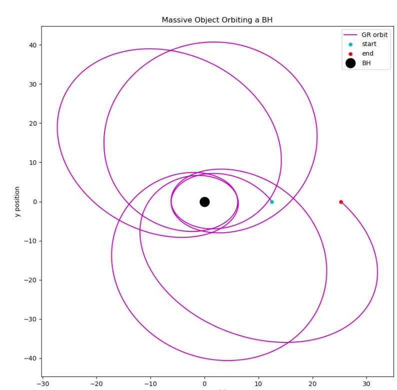

Fig. 10: The above plot shows the relativistic orbit of a massive object orbiting a black hole with the following parameters: E = -0.02, u = 0.08, and l = 4.

In Figure 10, it becomes easier to see the ”zoom-whirl” orbit that was previously described. The massive object has an orbit with a smaller radius and another orbit with a larger radius.

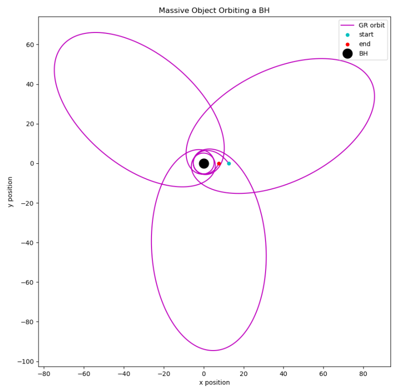

Fig. 11: The above plot shows the relativistic orbit of a massive object orbiting a black hole with the following parameters: E = -0.01, u = 0.08, and l = 4.

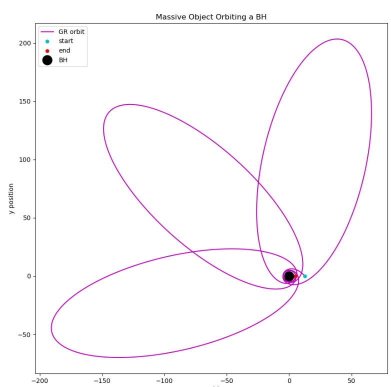

Fig. 12: The above plot shows the relativistic orbit of a massive object orbiting a black hole with the following parameters: E = -0.005, u = 0.08, and l = 4.

Figures 11 and 12 display more of the precession of the perihelion. These ”zooms” (the larger orbits) are also known as leaves. These orbits would exist in between the ISCO and ICBO and be unstable in nature. Figure 11 shows a three-leaf orbit that would likely continue to be offset slightly (as can be seen by the start and stop points marked in cyan and red, respectively). Figure 12 displays an orbit that has more zooms/leaves than figure 11 and if the total time were increased, this nature would continue around a complete orbit of the BH.

Changing the Initial Potential: u

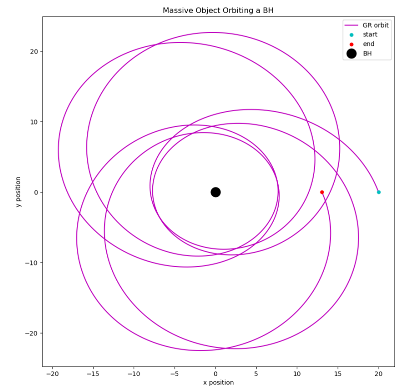

Fig. 13: The above plot shows the relativistic orbit of a massive object orbiting a black hole with the following parameters: E = -0.03, u = 0.05, and l = 4.

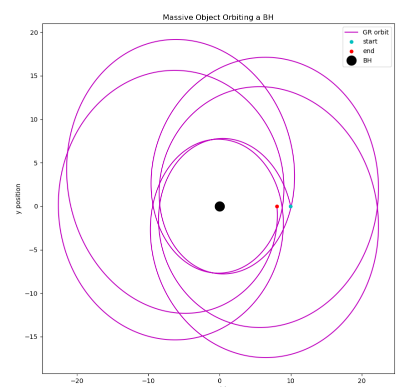

Fig. 14: The above plot shows the relativistic orbit of a massive object orbiting a black hole with the following parameters: E = -0.03, u = 0.1, and l = 4.

Both figures 13 and 14 continue to represent the ”zoom-whirl” orbits but one can see that as the potential increases, the orbit becomes slightly tighter and more elliptical. Still, the massive object continues to have a precession of the perihelion as it continues to orbit around the black hole. These orbits also exist within the region between the ISCO and the IBCO but continue to display precession as if the orbits were outside of the ISCO.

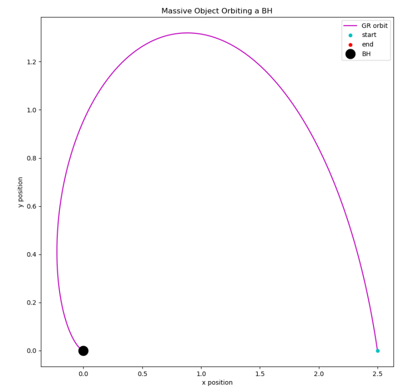

Fig. 15: The above plot shows the relativistic orbit of a massive object orbiting a black hole with the following parameters: E = -0.03, u = 0.4, and l = 4.

In figure 15, the orbit resembles that which is inside the ICBO because there is no "orbit" but rather a "falling" towards the event horizon. Once the potential gets large enough (roughly 0.4), the orbit no longer follows the pattern of a "zoom-whirl" orbit and therefore does not display any precession either.

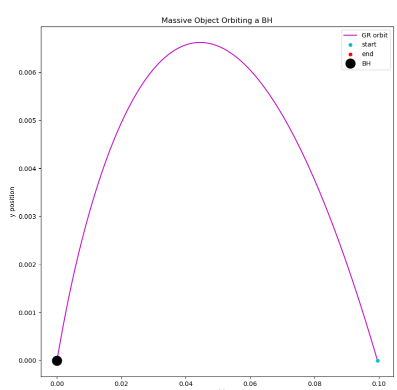

Fig. 16: The above plot shows the relativistic orbit of a massive object orbiting a black hole with the following parameters: E = -0.03, u = 10, and l = 4.

In figure 16, the potential has been increased to 10 and one can see that the shape of the orbit resembles an upside-down parabola. Rather than spiraling toward the event horizon of the BH, the massive object begins to orbit but is not able to travel in an orbit with any circular shape. Instead, the object falls straight into the BH. This orbit also would be found at a smaller radius than the ICBO due to its extreme instability.

Changing the Angular Momentum: l

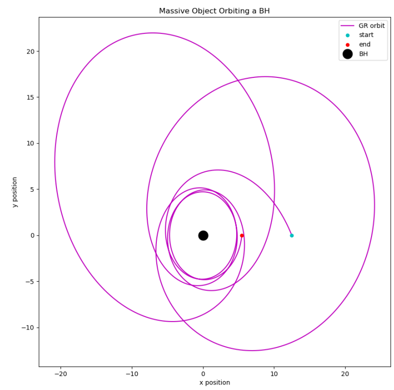

Fig. 17: The above plot shows the relativistic orbit of a massive object orbiting a black hole with the following parameters: E = -0.03, u = 0.08, and l = 3.75.

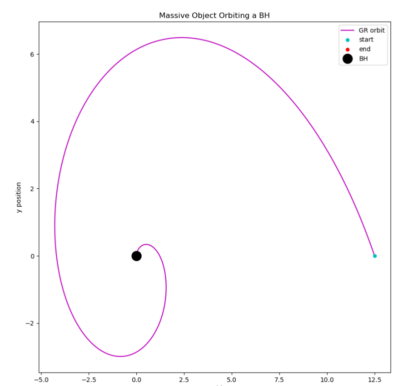

Fig. 18: The above plot shows the relativistic orbit of a massive object orbiting a black hole with the following parameters: E = -0.03, u = 0.08, and l = 3.6.

Figure 18 is similar to figure 16 in the sense that this orbit shows the object in an extremely unstable orbit. In figure 18, the object's orbit has more of a spiral shape than that of figure 16. Still starting at a radius smaller than the ICBO, the object orbits almost a full rotation before falling toward the event horizon of the BH.

Discussion

One difficulty with this project was properly adjusting the code to reflect the behavior of the orbits. In reality, the "fabric" of spacetime around a black hole is an extreme environment that is only recently being studied by observational astronomers in addition to theoretical astronomers. Einstein's Theory of General Relativity answers many questions that were left unanswered by Newtonian physics and GR has paved the way for the assumption of two important radii, the Innermost Stable Circular Orbit (ISCO) and the Innermost Bound Circular Orbit (IBCO), and the different orbits that exist as a result of these radii. Using code representing the Yukawa Orbit, I was able to reproduce all three types of orbits that exist outside of the ISCO. Using the code for GR Orbits, I was able to change the parameters of the function to produce orbits that represent those within the ISCO, the radius where orbits around the BH are unstable and exhibit behavior not seen in other parts of the universe. Now that we are able to produce simulations of these orbits using Einstein's Theory of General Relativity, the next goal will be to physically observe these orbits as we continue to discuss more black holes in our observable universe.

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