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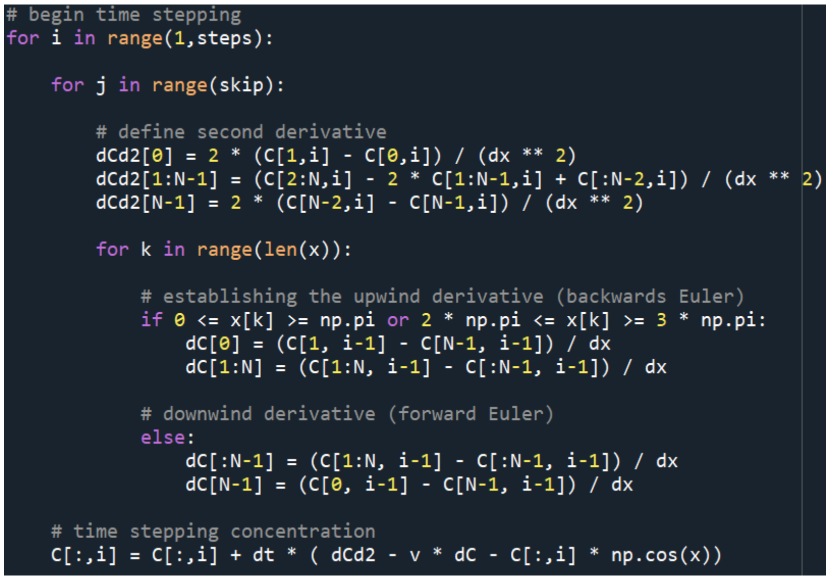
PHYS 305 homework 6 write-up

1. A steady-state solution is one that does not change over time, therefore the derivative of the function with respect to time is zero. Looking at the lecture notes from Part 15, we can see that the given equation is representative of an advection-diffusion problem with spatially dependent velocity where the velocity is sin(x) and the concentration is not changing with respect to time.

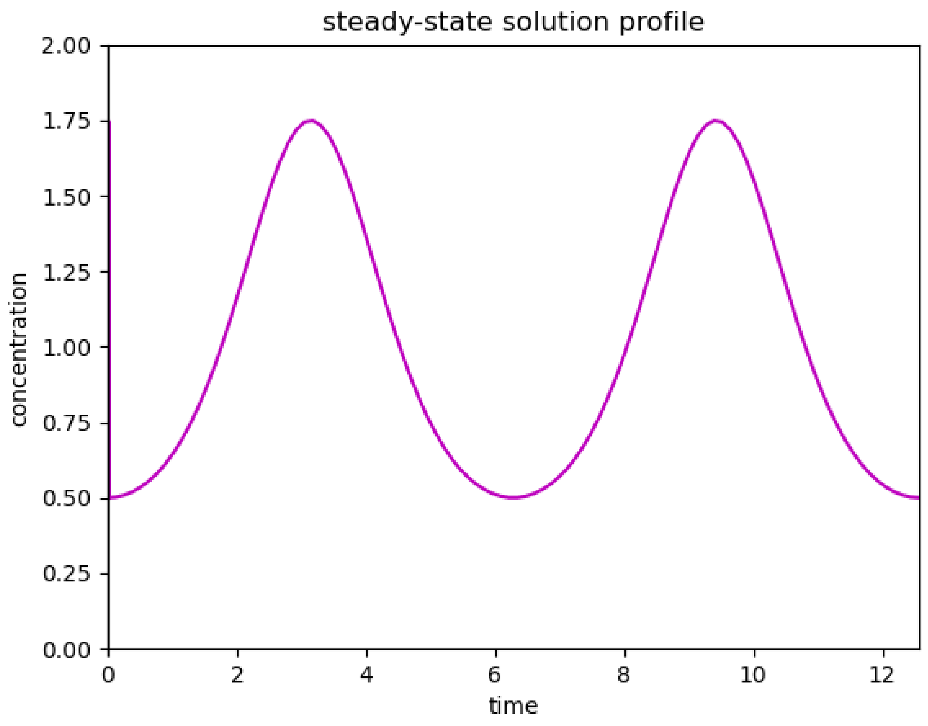
$$\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial x^2} - \frac{\partial}{\partial x}(sin(x) \ * \ C) = 0$$

For the code, I combined the code that was used for the diffusion and advection problems. The upwind derivative was a little difficult to code due to the fact that the velocity is not constant, but rather it is dependent on x (v = sin(x)). I had originally tried to do a for loop to index over the values of x to see if they would produce a positive or negative velocity and then try to update the concentration from there. The issue was with how the concentration is indexed with N and therefore the concentration was not updating with every iteration and stayed constant at 1. The code that did *not* work can be seen below:



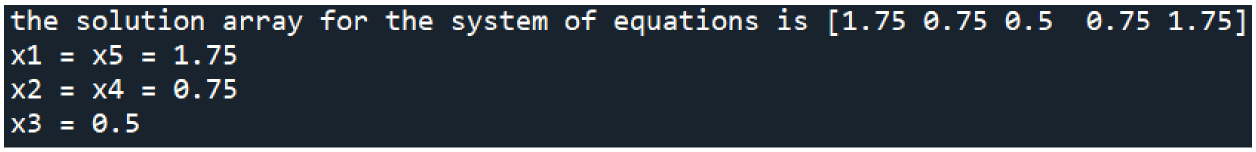
```
# begin time stepping
for i in range(1,steps):

    for j in range(skip):

        # define second derivative
        dCd2[0] = 2 * (C[1,i] - C[0,i]) / (dx ** 2)
        dCd2[1:N-1] = (C[2:N,i] - 2 * C[1:N-1,i] + C[:N-2,i]) / (dx ** 2)
        dCd2[N-1] = 2 * (C[N-2,i] - C[N-1,i]) / (dx ** 2)

        for k in range(len(x)):

            # establishing the upwind derivative (backwards Euler)
            if 0 <= x[k] >= np.pi or 2 * np.pi <= x[k] >= 3 * np.pi:
                dC[0] = (C[1, i-1] - C[N-1, i-1]) / dx
                dC[1:N] = (C[1:N, i-1] - C[:N-1, i-1]) / dx

            # downwind derivative (forward Euler)
            else:
                dC[:N-1] = (C[1:N, i-1] - C[:N-1, i-1]) / dx
                dC[N-1] = (C[0, i-1] - C[N-1, i-1]) / dx

    # time stepping concentration
    C[:,i] = C[:,i] + dt * ( dCd2 - v * dC - C[:,i] * np.cos(x))
```

Because of this, I figured that there needed to be other intervals used rather than just 0 to N or one to N and they needed to reflect the necessary conditions for upwind and downwind so that the backward Euler and forward Euler would be used properly. This work can be seen in the uploaded Python file. That method resulted in the following plot:



This is the steady state profile for the given equation involving diffusion and advection. We can see from the plot above that the concentration seems to stabilize at 0.50 for the minimums and 1.75 for the maximums.

2. Below is the written work for the solution for the given system of equations. The first boxed matrix is the upper triangular matrix using Gauss-Jordan elimination. From this matrix, I solved for each x value using the new equations. I also used Gauss-Jordan elimination on the upper triangular matrix to form the reduced row echelon form (RREF) of the system of equations. With only ones on the diagonal and zeros elsewhere, the values in the rightmost column represent the x values for the solved system of equations. The output from the written code is below (which matches the written work):



```
the solution array for the system of equations is [1.75 0.75 0.5  0.75 1.75]
x1 = x5 = 1.75
x2 = x4 = 0.75
x3 = 0.5
```

$$\begin{aligned} x_2 - x_1 &= -1 \\ x_3 - 3x_2 + x_1 &= 0 \\ x_4 - 3x_3 + x_2 &= 0 \\ x_5 - 3x_4 + x_3 &= 0 \\ x_5 - x_4 &= 1 \end{aligned} \longrightarrow \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \longrightarrow \left[\begin{array}{ccccc|c} -1 & 1 & 0 & 0 & 0 & -1 \\ 1 & -3 & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] R_2+R_1 \longrightarrow \left[\begin{array}{ccccc|c} -1 & 1 & 0 & 0 & 0 & -1 \\ 0 & -2 & 1 & 0 & 0 & -1 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] \updownarrow$$

$$\left[\begin{array}{ccccc|c} -1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] R_3+2R_2 \longrightarrow \left[\begin{array}{ccccc|c} -1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & -5 & 2 & 0 & -1 \\ 0 & 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] \updownarrow \longrightarrow \left[\begin{array}{ccccc|c} -1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & -5 & 2 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] R_4+5R_3$$

$$\left[\begin{array}{ccccc|c} -1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & -13 & 5 & -1 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] \updownarrow \longrightarrow \left[\begin{array}{ccccc|c} -1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -13 & 5 & -1 \end{array}\right] R_5-13R_4 \longrightarrow \begin{array}{c} \left[\begin{array}{ccccc|c} -1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -8 & -14 \end{array}\right] \end{array} \begin{array}{l} -R_1 \\ \\ \\ -R_4 \\ -\tfrac{1}{8}R_5 \end{array}$$

$$\boxed{\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 7/4 \end{array}\right]} R_4+R_5 \longrightarrow \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 3/4 \\ 0 & 0 & 0 & 0 & 1 & 7/4 \end{array}\right] R_3+3R_4 \longrightarrow \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 9/4 \\ 0 & 0 & 0 & 1 & 0 & 3/4 \\ 0 & 0 & 0 & 0 & 1 & 7/4 \end{array}\right] R_3-R_5$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2/4 \\ 0 & 0 & 0 & 1 & 0 & 3/4 \\ 0 & 0 & 0 & 0 & 1 & 7/4 \end{array}\right] R_2+3R_3 \longrightarrow \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 6/4 \\ 0 & 0 & 1 & 0 & 0 & 2/4 \\ 0 & 0 & 0 & 1 & 0 & 3/4 \\ 0 & 0 & 0 & 0 & 1 & 7/4 \end{array}\right] R_2-R_4 \longrightarrow \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 3/4 \\ 0 & 0 & 1 & 0 & 0 & 2/4 \\ 0 & 0 & 0 & 1 & 0 & 3/4 \\ 0 & 0 & 0 & 0 & 1 & 7/4 \end{array}\right] R_1+R_2$$

$$\text{RREF: } \boxed{\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 7/4 \\ 0 & 1 & 0 & 0 & 0 & 3/4 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 3/4 \\ 0 & 0 & 0 & 0 & 1 & 7/4 \end{array}\right]} \longrightarrow \boxed{\begin{aligned} x_1 &= x_5 = 7/4 \\ x_2 &= x_4 = 3/4 \\ x_3 &= 1/2 \end{aligned}}$$

from row 5: $x_5 = 7/4$

from row 4: $x_4 - x_5 = -1 \rightarrow x_4 = \frac{7}{4} - 1 = 3/4$

from row 3: $x_3 - 3x_4 + x_5 = 0 \rightarrow x_3 = 3x_4 - x_5 = \frac{9}{4} - \frac{7}{4} = \frac{1}{2}$

from row 2: $x_2 - 3x_3 + x_4 = 0 \rightarrow x_2 = 3x_3 - x_4 = \frac{3}{2} - \frac{3}{4} = \frac{3}{4}$

from row 1: $x_1 - x_2 = 1 \rightarrow x_1 = 1 + x_2 = 1 + \frac{3}{4} = 7/4$

3. For this problem involving Fisher's equation, I followed the steps shown in lecture part 15 to write the backward Euler discretization of the equation for the time and spatial components (using the central difference derivative which yields the least error). From this discretization, we can then use the domain of five nodes to express the system as a system of five equations that can finally be expressed in matrix form and could be solved using Gauss-Jordan elimination, as completed in problem 2. The jth time node is non-linear; therefore, we want to group these terms together (in the solution vector) and use the forward Euler method. The (j+1)th node is linear and therefore we can use the backward Euler method.

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backward euler on diffusion eqtn: $\frac{\partial u}{\partial t} = D \cdot \frac{\partial^2 u}{\partial t^2} + u(1-u)$ w/ BC: $u(0,t) = 1$, $u(L,t) = 0$ $\rightarrow$ following the slides from lecture part 15

$$\frac{u_i^{j+1} - u_i^j}{\Delta t} = \frac{D}{(\Delta x)^2}\left(u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}\right) + u_i^j(1-u_i^j) \quad \longrightarrow \quad u_i^{j+1} - \frac{D(\Delta t)}{(\Delta x)^2}\left(u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}\right) + \Delta t \cdot u_i^j(1-u_i^j) = u_i^j$$

with 5 nodes:

$$\begin{aligned}
u_0^{j+1} - \frac{D(\Delta t)}{(\Delta x)^2}\left(u_1^{j+1} - 2u_0^{j+1} + u_{-1}^{j+1}\right) + u_0^j \Delta t(1-u_0^j) &= u_0^j \\
u_1^{j+1} - \frac{D(\Delta t)}{(\Delta x)^2}\left(u_2^{j+1} - 2u_1^{j+1} + u_0^{j+1}\right) + u_1^j \Delta t(1-u_1^j) &= u_1^j \\
u_2^{j+1} - \frac{D(\Delta t)}{(\Delta x)^2}\left(u_3^{j+1} - 2u_2^{j+1} + u_1^{j+1}\right) + u_2^j \Delta t(1-u_2^j) &= u_2^j \\
u_3^{j+1} - \frac{D(\Delta t)}{(\Delta x)^2}\left(u_4^{j+1} - 2u_3^{j+1} + u_2^{j+1}\right) + u_3^j \Delta t(1-u_3^j) &= u_3^j \\
u_4^{j+1} - \frac{D(\Delta t)}{(\Delta x)^2}\left(u_5^{j+1} - 2u_4^{j+1} + u_3^{j+1}\right) + u_4^j \Delta t(1-u_4^j) &= u_4^j
\end{aligned}$$

$$\longrightarrow$$

$$\begin{aligned}
u_0^{j+1} - \frac{D(\Delta t)}{(\Delta x)^2}\left(u_1^{j+1} - 2u_0^{j+1} + u_{-1}^{j+1}\right) + \Delta t\, u_0^{j+1} - \Delta t\, u_0^{j^2} &= u_0^j \\
u_1^{j+1} - \frac{D(\Delta t)}{(\Delta x)^2}\left(u_2^{j+1} - 2u_1^{j+1} + u_0^{j+1}\right) + \Delta t\, u_1^{j+1} - \Delta t\, u_1^{j^2} &= u_1^j \\
u_2^{j+1} - \frac{D(\Delta t)}{(\Delta x)^2}\left(u_3^{j+1} - 2u_2^{j+1} + u_1^{j+1}\right) + \Delta t\, u_2^{j+1} - \Delta t\, u_2^{j^2} &= u_2^j \\
u_3^{j+1} - \frac{D(\Delta t)}{(\Delta x)^2}\left(u_4^{j+1} - 2u_3^{j+1} + u_2^{j+1}\right) + \Delta t\, u_3^{j+1} - \Delta t\, u_3^{j^2} &= u_3^j \\
u_4^{j+1} - \frac{D(\Delta t)}{(\Delta x)^2}\left(u_5^{j+1} - 2u_4^{j+1} + u_3^{j+1}\right) + \Delta t\, u_4^{j+1} - \Delta t\, u_4^{j^2} &= u_4^j
\end{aligned}$$

$$\begin{aligned}
u_0^{j+1} - \frac{D(\Delta t)}{(\Delta x)^2}\left(u_1^{j+1} - 2u_0^{j+1} + u_{-1}^{j+1}\right) + \Delta t\, u_0^{j+1} &= u_0^j + \Delta t\, u_0^{j^2} \\
u_1^{j+1} - \frac{D(\Delta t)}{(\Delta x)^2}\left(u_2^{j+1} - 2u_1^{j+1} + u_0^{j+1}\right) + \Delta t\, u_1^{j+1} &= u_1^j + \Delta t\, u_1^{j^2} \\
u_2^{j+1} - \frac{D(\Delta t)}{(\Delta x)^2}\left(u_3^{j+1} - 2u_2^{j+1} + u_1^{j+1}\right) + \Delta t\, u_2^{j+1} &= u_2^j + \Delta t\, u_2^{j^2} \\
u_3^{j+1} - \frac{D(\Delta t)}{(\Delta x)^2}\left(u_4^{j+1} - 2u_3^{j+1} + u_2^{j+1}\right) + \Delta t\, u_3^{j+1} &= u_3^j + \Delta t\, u_3^{j^2} \\
u_4^{j+1} - \frac{D(\Delta t)}{(\Delta x)^2}\left(u_5^{j+1} - 2u_4^{j+1} + u_3^{j+1}\right) + \Delta t\, u_4^{j+1} &= u_4^j + \Delta t\, u_4^{j^2}
\end{aligned}$$

$$\longrightarrow \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
-\frac{D\Delta t}{(\Delta x)^2} & 1 + \frac{2D\Delta t}{(\Delta x)^2} - \Delta t & -\frac{D\Delta t}{(\Delta x)^2} & 0 & 0 \\
0 & -\frac{D\Delta t}{(\Delta x)^2} & 1 + \frac{2D\Delta t}{(\Delta x)^2} - \Delta t & -\frac{D\Delta t}{(\Delta x)^2} & 0 \\
0 & 0 & -\frac{D\Delta t}{(\Delta x)^2} & 1 + \frac{2D\Delta t}{(\Delta x)^2} - \Delta t & -\frac{D\Delta t}{(\Delta x)^2} \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} u_0^{j+1} \\ u_1^{j+1} \\ u_2^{j+1} \\ u_3^{j+1} \\ u_4^{j+1} \end{bmatrix}
=
\begin{bmatrix} 1 \\ u_1^j + \Delta t(u_1^j)^2 \\ u_2^j + \Delta t(u_2^j)^2 \\ u_3^j + \Delta t(u_3^j)^2 \\ 0 \end{bmatrix}$$

1 and 0 come from the given boundary conditions