Chapter 6

DECIDING THE CLASS K BY AN ORDERING REFINEMENT

6.1 THE CLASS K

Class K is a decidable class of formulas in first order locic described in [Mas 68]. It contains a lot of known decidable classe (e.g. all classes described in the chapter 2 of [DG 79] can be reduced to the class K).
Let us present the description of this class first in quantificational and then in its Skolemized form (as in [Zam 89]).

DEFINITION 6.1.: The variables in the domain of a substitution λ are called proper variables of λ.

DEFINITION 6.2.: Let F be a formula and G a subformula of F. The F-prefix of the formula G is a sequence of quantifiers of the formula F which bind the free variables of G.

The F-prefix is the F-prefix of any atomic subformula of the formula F.

DEFINITION 6.3.: (The original definition of the class K was given by S. Maslov in a dual form). The formula F belongs to the class K in there exist variables $x_1, \dots, x_k$, $k \geq 0$, which are not in the scope of any existential quanifier, such that each nonempty F-prefix:

- either is of length 1 (singular F-prefix),
- or ends with an existential quantifier,
- or is of the form $\forall x_1, \forall x_2, \dots, \forall x_k$.

W.l.o.g. we can assume that every formula of the class K has the form

$$(\forall x_1)(\forall x_2) \dots (\forall x_k)(Qy_1) \dots (Qy_l) \mathop{\&}_{i=1}^{n} \bigvee_{j=1}^{m_i} A_{i,j} \qquad (1)$$

where $k \geq 0$, $l \geq 0$, $n > 0$, $m_i > 0$, $A_{i,j}$ are literals.

After Skolemisation the formula (I) will be in the for

$$\forall(\mathop{\&}_{i=1}^{n} \bigvee_{j=1}^{m_i} B_{i,j}) \qquad (2)$$

where each literal $B_{i,j}$ is obtained from $A_{i,j}$ by replacing $\exists$ - variables and free variables by corresponding Skolem terms.
The class of formulas of the kind (2) obtained from formulas of the class K will be will denoted by KS.

<u>DEFINITION</u> 6.4.: A nonconstant argument of a literal is called <u>essential</u> and <u>inessential</u> otherwise.

Let us define some properties of formulas from the class KS referring to corresponding properties of formulas from the class K.

Table 1.

$F \in K$	$G \in KS$
F is a formula in prenex normal form	For any two functional G - terms t and s the argument list of t is the beginning of the arguments list of s or vice versa.
The quantifier $\exists x$ of a formula F is in the scope of a quantifier $\forall y$	The Skolem G - term replacing the variable x contains the variable y as an argument
F - prefix	The set of essential arguments of the corresponding atom contains a functional term t such that every variable of this set is an argument of t, and the list of arguments of any other functional term from this set is a beginning of the list of arguments of the term t

The F-prefix has the form $\forall x_1 \dots \forall x_k$	The set of essential arguments consists of variables $x_1, \dots, x_k$
The F-prefix has a length 1	The set of essential arguments is a singleton set.

DEFINITION 6.5.: The term t dominates the term s if at least one of the following conditions holds:

(1) $t = s$,

(2) t is a functional term and s is a variable argument of t,

(3) $t = f(t_1, \dots, t_n)$, $s = g(t_1, \dots, t_m)$, $n \geq m \geq 0$.

DEFINITION 6.6.: The set T_1 of terms dominates the set T_2 of terms if for every term t_2 from T_2 there exists a term t_1 from T_1 such that t_1 dominates t_2.

DEFINITION 6.7.: The literal L_1 dominates the literal L_2 if the set of arguments of L_1 dominates the set of arguments of L_2

DEFINITION 6.8.: Terms t and s are similar if t dominates s and s dominates t. The similarity relation for literals and term sets is defined analogously.

EXAMPLE 6.1.: The term $f(x, y, g(x))$ dominates the terms $x, y, g(x)$, $h(x,y)$, $h(x, y, g(x))$. The literal $A(x, y, f(x,y))$ dominates the literals $B(y)$, $C(g(x), y)$, $D(y, h(x,y), x)$. The terms $f(x, y, g(x, y))$ and $h(x, y, g(x, y))$ are similar but $f(x, y, g(x, y))$ and $h(y, x, g(x, y))$ are not. The literals $A(x, y, g(x))$, $A(y, x, g(x))$, $B(x,y,x,y, h(x))$ are similar.

DEFINITION 6.9.: A term is called regular if it dominates all its arguments.

In particular, variables and constants are trivially regular, all terms of depth 1 are regular.

EXAMPLE 6.2.: The terms x, $f(x,y,z)$, $f(x,g(x),z)$, $f(x,g(x),\ h(x,g(x)))$ are regular. The term $f(x,h(x,g(x)))$ is not regular since it does not dominate its second argument.

We can see from the definition of a regular term, that it is possible to use a short form of any regular term:
we may omit the arguments of leading functional subterms, since the position of any function symbol completely determines its arguments (it is assumed that the arities of all functional symbols in a short from are known).

EXAMPLE 6.3.: The short forms of the last two regular terms from the example above are $f(x,g,z)$, $f(x,g,h)$.

DEFINITION 6.10.: A set of terms is called regular if it contains no functional terms or it contains some regular functional term which dominates all terms of this set. (We will call such a term a dominating term for this set).

DEFINITION 6.11.: A literal is called regular if the set of its arguments is regular.

EXAMPLE 6.4.: The literals $A(x,y,f(x,y),\ g(x))$, $B(x,f(x,g(x)))$, $C(x,a,y)$ are regular.

DEFINITION 6.12.: A literal of depth 0 is called unary if the set of its essential arguments consists of only one variable.

Some obvious properties of regular terms and of the dominating and similarity relations are given in the form of the following remarks.

Remark 1. If the term t dominates the terms t_1 and t_2 and the arity of the term t_1 is not less than the arity of the term t_2 then t_1 dominates t_2.

Remark 2. The dominating relation is transitive and, for functional terms, it is preserved under substitution.

Remark 3. The similarity relation is transitive.

Remark 4. Sets of essential arguments of any two similar literals of depth 0 are equal.

DEFINITION 6.13.: A set M of literals is called k-regular if the following conditions hold:

1) M contains regular literals only,
2) the nonnegative integer k is not greater than the minimum arity of function symbols occurring in literals of the set M,
3) M contains some literal which dominates every literal from the set M,
4) All nonunary literals of depth 0 are similar and the set of essential arguments of any such literal is similar to the set of the first k arguments of any functional term occuring in any literal from M.

A set M of literals is called regular if it is k-regular for some k, $k \geq 0$.

DEFINITION 6.14.: A regular literal of positive depth called a literal of type 1, a nonunary literal of depth zero is called a literal of type 2, an unary literal of depth zero is called a literal of type 3.

EXAMPLE 6.5.: The set of literals $\{A(x,z,f(x,y,z)),\ B(x,g(x,y)),\ C(x,y),\ D(z)\}$ is 2-regular.

DEFINITION 6.15.: A subset M_1 of the set M of literals is called a component of M if M_1 and $H-M_1$ are variable disjoint. A set M is called decomposable if it has at least two nonempty components and indecomposable otherwise.

DEFINITION 6.16.: A set of literals is called quasiregular if all of its indecomposable components are regular.

THEOREM 6.1.: Every clause C of a formula of the form (2) is quasiregular.

Proof:

We first show that C consists of regular literals only. Let A be a literal from C and let B be the corresponding literal in a formula of type (1). We will consider the following cases:

1) The F-prefix of the literal B ends with $\exists y$. Then the literal A contains the term t (replacing the variable y after Skolemization) which dominates all arguments of the literal A.
2) The F-prefix of B is of the form $\forall x_1 \ldots \forall x_k$. In this case B does not contain functional terms and therefore is regular.
3) The F-prefix of B consists of one $\forall$-quantifier. In this case B is regular too, since it does not contain any functional terms.
4) The F-prefix of B is empty. Similar to cases 1) and 2).

Now let us show that C is quasiregular. Let C_1 be an indecomposable component of C. Since C_1 is indecomposable it either consists of one literal - in this case C_1 is trivially regular - or it does not contain any literals. Let us consider the following cases:

1) Let A be a literal of positive depth from C_1 such that its dominating term t_1 has a maximum arity of all terms occurring in C_1. Let B be another literal from C_1. If B contains a functional term t_2 then the arity of t_1 is not less then the arity of t_2. Since F is a prenex formula and t_1, t_2 are Skolem terms, the argument list of t_2 is a beginning of the argument list of the term t_1, i.e. t_1 dominates t_2. It follows that A dominates B. If the literal B does not contain functional terms and B is not unary then the set of its essential arguments is equal to the set of the first k arguments of the term t_1, therefore A dominates B. Finally if B is a unary literal and does not contain function symbols then its only variable argument is an argument of t_1, since C_1 is indecomposable. Therefore A dominates B.
2) Suppose that C_1 does not contain literals with functional terms. Then every literal occurring in C_1 is obtained from some literal with F-prefix $\forall x_1 \ldots \forall x_k$ or $\forall z$ for some z. Let A and B be literals from C_1 where A is obtained from

a literal with F-prefix $\forall x_1 \ldots \forall x_k$. If B is obtained from a literal with the same F-prefix, then A and B are similar. If B is obtained from a literal with F-prefix $\forall z$ then the only variable of B is x_i for some i, $1 \le i \le k$, since C_1 is indecomposable. So in both cases A dominates B.

3) If C_1 consists of literals which are obtained from literals with singular F-prefix then, since C_1 is indecomposable, all literals contain the same variable. If follows that all literals in C_1 are similar.

So we habe shown that in any case C_1 contains some literal which dominates all literals from C_1, i.e. C_1 is a regular component.

Q.E.D.

REGULAR TERMS AND CLAUSES

In the following lemmas we investigate properties of regular terms and clauses.

LEMMA 6.1.: A term t is regular iff it dominates each of its arguments of positive depth.

Proof:
The "if" part follows from the fact that every functional term dominates every constant and every variable argument. The "only if" part is evident.

LEMMA 6.2.: If t is a regular term and t dominates a term s then s is regular too.

Proof:
If $\tau(s) = 0$, then s is trivially regular. Let s be a functional term of the form $g(s_1, \ldots, s_m)$. Since t dominates s, it follows that t is a functional term of the kind $f(s_1, \ldots, s_m, \ldots, s_n)$, where f and g are functional symbols, $s_1, \ldots, s_n$ are terms, $n \ge m > 0$. By lemma 6.1 it is enough to prove that s dominates each of its arguments of positive depth. Let $s_i = h(p_1, \ldots, p_e)$ be some argument of the term s, $i \le m$. Since t is regular, it dominates s_i and it follows that $p_1 = s_1, \ldots, p_e = s_e$. Moreover we have $e < i$, since otherwise $p_i = s_i$, i.e. s_i is equal to one of its argument, which is impossible. Since $e < i$ and $i \le m$ it

holds $e < m$ and each argument of the term s_i is equal to a corresponding argument of the term s. Therefore s dominates s_i which had to be proved.

Corollary 6.1.: Every subterm of a regular term is regular.

Corollary 6.2.: Every regular term dominates each of its subterms.

In general the regularity of terms is not preserved by substitutions. For example, if we substitute the regular term $g(x)$ into a regular term $f(y, z)$ for the variable y then we will obtain the irregular term $f(g(x), z)$. However, a substitution which unifies some set of regular terms preserves regularity. The proof of this fact is given below.

LEMMA 6.3.: Let t be a regular term, and $\sigma = \{t_1/x_1, \dots, t_n/x_n\}$ be a substitution, where $t_1, \dots, t_n$ are constants or variables; then $(t)\sigma$ is a regular term.

Proof:
The substitution σ preserves the dominating relation for terms occurring in t as well as the depth of terms.

LEMMA 6.4.: Let the regular term $t = f(t_1, \dots, t_n)$ dominate the term $s = g(t_1, \dots, t_m)$, $n \geq m \geq 1$, and let σ be a substitution such that all its proper variables occur in s or do not occur in t. If $(s)\sigma$ is regular, then $(t)\sigma$ is regular too.

Proof:
It is sufficient to prove that $(t)\sigma$ dominates each of its argument of positive depth (by lemma 6.1). Let $(t_j)\sigma$ be a term of positive depth, $1 \leq j \leq n$. We will consider two cases:

1) t_j is a variable. Then t_j is a proper variable of the substitution σ, since otherwise $(t_j)\sigma = t_j$ which is impossible because $(t_j)\sigma$ is of positive depth. By the assumption of the lemma, t_j occurs in s. The term $(t_j)\sigma$ is a subterm of the term $(s)\sigma$ therefore $(s)\sigma$ dominates $(t_j)\sigma$ (since $(s)\sigma$ is regular). By remark 2 the dominating relation is transitive and is preserved under substituting terms of positive depth. Since t dominates s, $(t)\sigma$ dominates $(s)\sigma$, therefore $(t)\sigma$ dominates $(t_j)\sigma$.

2) t_j is a term of positive depth. It follows that t dominates t_j, since t is regular. Since the dominating relation is preserved under substitution in terms of positive depth, the term $(t)\sigma$ dominates $(t_j)\sigma$.

So we have shown that $(t)\sigma$ dominates each its argument of positive depth and thus is regular.

<u>LEMMA</u> 6.5.: Let $t = f(p_1, \dots, p_n)$ be a regular term which dominates the term $s = g(p_1, \dots, p_m)$, z be a variable which does not occur in s and let σ be the substitution $\{s/z\}$. Then $(t)\sigma$ is regular.

<u>Proof</u>:
Let u be an argument of t and let the depth of $(u)\sigma$ be positive. There are the following cases:

1) u is a variable different from z. Then $(u)\sigma = 0$ which is impossible since the depth of u is equal to 0 but the depth of $(u)\sigma$ is positive.

2) u is equal to z. In this case $(u)\sigma = s$. Since s does not contain z, the terms $p_1, \dots, p_m$ do not contain z too. Therefore

$$(t)\sigma = f(p_1, \dots, p_m, p_{m+1}, \dots, p_n)\sigma = f((p_1)\sigma, \dots, (p_m)\sigma, \dots, (p_n)\sigma) =$$
$$= f(p_1, \dots, p_m, (p_{m+1})\sigma, \dots, (p_n)\sigma).$$

It follows that $(t)\sigma$ dominates $(u)\sigma$.

3) u is a term of positive depth or a constant. Since t is regular, it dominates the term u, therefore $(t)\sigma$ dominates $(u)\sigma$ (according to remark 2).

<u>LEMMA</u> 6.6.: If t and s are regular terms and σ is a m.g.u. for $\{t, s\}$ then $(t)\sigma$ is regular.

<u>Proof</u>:
Let us consider the following cases:

1) One of the terms (s, for example) is a variable. Then t does not contain s, $\sigma = \{t/s\}$ and $(t)\sigma = t$ which is regular.

2) Both of t, s are functional terms of the kind $f(p_1, \dots, p_n)$ and $f(q_1, \dots, q_n)$ respectively (they begin with the same function symbol f because they are unifiable). We will prove the regularity of $(t)\sigma$ by induction on $(n - i)$, where i satisfies the following conditions:

$$p_1 = q_1, \dots, p_i = q_i, \; p_{i+1} \neq q_{i+1}, \; (1 \le i < n) \qquad (1)$$

Since t and s are unifiable, then terms p_{i+1} and q_{i+1} are unifiable too.
The induction base is trivial.
Let $n - i > 0$. In this case one of the terms p_{i+1}, q_{i+1} must be a variable. Assume the contrary and consider the following subcases:

2.1) The depth of both terms is zero. In this case terms p_{i+1} and q_{i+1} are constants. They are different by the condition (1); but then they are not unifiable which contradicts with unifiablility of t and s.

2.2) The depth of p_{i+1} is positive and the depth of s is zero. In this case these terms are not unifiable too.

2.3) Both terms are of positive depth. Assume that $p_{i+1} = g(u_1, \dots, u_m)$, $q_{i+1} = g(v_1, \dots, v_m)$. Since t is regular, it dominates the term p_{i+1}, therefore

$$p_1 = u_1, \dots, p_m = u_m \qquad (2)$$

Since no term contains itself as argument, it follows that $p_{i+1} \neq u_k$ for each k, $k = 1, \dots, m$. Therefore $i + 1 > m$.
Similarly we can show that

$$q_1 = v_1, \dots, q_m = v_m. \qquad (3)$$

So we can conclude from (1) - (3) that $u_k = p_k = q_k = v_k$ for every k, $k = 1, \dots m$. It follows that $p_{i+1} = q_{i+1}$ which contradicts (1).

Thus we have proved that at least one of the terms p_{i+1} or q_{i+1} (say the first one) is a variable. Let σ_{i+1} be the substitution $\{q_{i+1}/p_{i+1}\}$. If q_{i+1} is a variable, then both $(t)\sigma$ and $(s)\sigma$ are regular by lemma 3.

Let q_{i+1} be of positive depth (it does not contain p_{i+1}). Since s is regular, it dominates the term q_{i+1}. Since the first i arguments of terms t and s are equal by (1), it follows that t dominates q_{i+1} too. By lemma 6.5 the terms

$(t)\sigma_{i+1}$ and $(s)\sigma_{i+1}$ are regular. It follows that the following conditions for the arguments of $(t)\sigma_{i+1}$ and $(s)\sigma_{i+1}$ are valid:

$$(p_1) \quad \sigma_{i+1} = (q_1)\sigma_{i+1}, \ldots$$
$$(p_1) \quad \sigma_{i+1} = (q_i)\sigma_{i+1},$$
$$(p_{i+1}) \quad \sigma_{i+1} = (q_{i+1})\sigma_{i+1} \ldots$$

By the induction hypothesis $((t)\sigma_{i+1})\lambda$ and $((s)\sigma_{i+1})\lambda$ are regular terms, where λ is a m.g.u. for $((t)\sigma_{i+1})$ and $((s)\sigma_{i+1}$.

Q.E.D.

Let us denote by ar(t) the arity of the term t and by minar(t) the minimum arity of nonconstant function symbols occurring in the term t. If t does not contain nonconstant function symbols then let minar(t) = 0.

<u>THEOREM</u> 6.2.: The depth of any regular term t is less or equal to ar(t) - minar(t) + 1.

<u>Proof:</u>
By induction on $\tau(t)$:
For $\tau(t) = 0$ the theorem obviously holds. Let us denote by s the argument of t with maximum depth. It is obvious that

$$\tau(s) + 1 = \tau(t) \text{ and } ar(s) < ar(t) \qquad (4)$$

By corollary 6.1 s is a regular term and by the induction hypothesis it holds $\tau(s) \leq ar(s) - minar(s) + 1$. It follows by (4) that

$$\tau(t) = \tau(s) + 1 \leq ar(s) + 1 - minar(s) + 1 \leq ar(t) - minar(t) + 1,$$

which had to be proved.

Q.E.D.

Now we will prove some properties of regular clauses.

<u>LEMMA</u> 6.7.: Let $A_1 = A(t_1, \ldots, t_n)$ and $A_2 = A(p_1, \ldots, p_n)$ be unifiable literals of type 1. If t_i is a dominating term of A_1, then also p_i must be a dominating term of A_2.

Proof:

Since A_1 is a literal of type 1, its dominating term has maximal arity among other arguments of this literal and is of positive depth. Let p_j be a dominating term of a literal A_2, i. e. p_j is a term of positive depth whose arity is maximal among arities of all arguments of A_2. Let us denote by σ the m.g.u. of A_1 and A_2. There are the following cases:

1) Terms p_i and t_j are variables. Since A_1 is a literal of type 1, the variable t_j is an argument of the dominating term t_i. For the same reason p_i is an argument of the term p_j. The relation "to be an argument" is preserved under substitution, therefore the following statements hold:

   $(t_j)\sigma$ is an argument of $(t_i)\sigma$ and
   $(p_i)\sigma$ is an argument of $(p_j)\sigma$ (5)

   From the unifiability of A_1 and A_2 we conclude that

   $(t_i)\sigma = (p_i)\sigma$ and $(t_j)\sigma = (p_j)\sigma$ (6)

   It follows from (5) and (6) that $(t_j)\sigma$ is an argument of an own argument (and thus a proper subterm of itself), which is impossible.

2) One of the terms t_j or p_i (say, t_j) is a variable and the other one is a constant or is of positive depth. Then t_j is an argument of t_i and p_j dominates the term p_i, therefore $ar(p_i) \leq ar(p_j)$. Since, for non-variable terms, the dominating relation is preserved under substitution, the following statements hold:

   $(t_j)\sigma$ is an argument of $(t_i)\sigma$ and
   $(p_j)\sigma$ dominates $(p_i)\sigma$ (7)

   From (6) and (7) we conclude that $(t_j)\sigma$ is an argument of some term which is dominated by $(p_j)\sigma$. Therefore $(t_j)\sigma$ is an argument of $(p_j)\sigma$, which is impossible.

3) Both terms p_i and t_j are not variables. Then for their arities we get:

   $ar(t_i) \geq ar(t_j)$ since t_i is a dominating term for A_1
   $ar(p_j) \geq ar(p_i)$ since p_j is a dominating term for A_2
   $ar(t_i) = ar(p_i)$ since t_i and p_i are unifiable,
   $ar(t_j) = ar(p_j)$ since t_j and p_j are unifiable.

These statements imply that $ar(t_i) = ar(t_j) = ar(p_i) = ar(p_j)$. It follows that p_i is a dominating term for A_2 too.

Q.E.D.

<u>LEMMA</u> 6.8.: Assume that A is a regular literal of the type 1, the term t is a dominating term of A and α is a substitution such that $(t)\alpha$ is regular. Then $(A)\alpha$ is a regular literal.

<u>Proof</u>:
Let us show that $(t)\alpha$ dominates each argument of $(A)\alpha$. Assume that u is some argument of A. There are the following cases:

1) u is a variable. In this case u is an argument of the dominating term t since A is a regular literal. It follows that $(u)\alpha$ is an argument of $(t)\alpha$. Since $(t)\alpha$ is a regular term, it dominates each of its arguments, particularly the term $(u)\alpha$.

2) u is a constant or a term of positive depth. Then t dominates u. In this case the dominating relation is preserved under substitution. Therefore $(t)\alpha$ dominates the term $(u)\alpha$.

Q.E.D.

<u>LEMMA</u> 6.9.: Assume that $A_1 = A(t_1, \dots, t_n)$ and $A_2 = A(p_1, \dots, p_n)$ are regular literals and α is a m.g.u. of A_1 and A_2. Then $(A_1)\alpha$ is regular.

<u>Proof</u>:
We will consider the following cases:

1) A_1 and A_2 are both literals of type 1. Assume that t_i is a dominating term for A_1. By lemma 6.6 the dominating term for A_2 is the term p_i. Let Θ be the m.g.u. for t_i and p_i. By lemma 6.6 the terms $(t_i)\Theta$ and $(p_i)\Theta$ are regular. By lemma 6.8 the literals $(A_1)\Theta$ $(A_2)\Theta$ are regular and their dominating terms $(t_i)\Theta$ and $(p_i)\Theta$ are equal.

   Now we use induction on the number of nonequal arguments in $(A_1)\Theta$ and $(A_2)\Theta$ to prove the lemma.
   The induction base is trivial. Let $(t_j)\Theta$ differ from $(p_j)\Theta$ for some j, $1 \leq j \leq n$,

and let α_j be a m.g.u. for these terms. Since $(t_i)\Theta$ is regular and dominates the term $(t_j)\Theta$, this last term is regular too by lemma 6.2. For the same reason also $(p_j)\Theta$ is regular. The term $((t_i)\Theta)\alpha_j$ is regular by lemma 6.4. Therefore the literal $((A_1)\Theta)\alpha_j$ is regular by lemma 6.8. Similarly we can prove that the literal $((A_2)\Theta)\alpha_j$ is regular too. By the induction hypothesis the lemma holds for the literals $((A_1)\Theta)\alpha_j$ and $((A_2)\Theta)\alpha_j$.

2) One of the literals, A_2 for example, is of term depth 0 and the other one is a literal of the type 1 with a dominating term t_i. In this case we can prove the lemma by induction on the number of pairs of nonequal terms (t_e, t_j) where e and j are numbers with $p_e = p_j$, $1 \leq e, j \leq n$.
The induction base is trivial because the unifier is a match.

Assume that $p_e = p_j$ and $t_e \neq t_j$. Since A_1 and A_2 are unifiable t_e and t_j are unifiable too. Let μ be a m.g.u. for t_e and t_j. By lemma 6.4 the term $(t_i)\mu$ is regular. By lemma 6.8. the literal $(A_1)\mu$ is regular. For the literals $(A_1)\mu$ and $(A_2)\mu$ the lemma holds by the induction hypothesis.

3) The literals A_1 and A_2 are both of term depth zero. The proof is evident since $(A_1)\sigma$ and $(A_2)\sigma$ do not contain functional terms and therefore are regular.

Q.E.D.

LEMMA 6.10.: Let M be a regular set of literals, A be a dominating literal for M and t be a dominating term for A. Then t dominates each argument of each literal from M.

Proof:
Let s be some argument of some literal B from M. Since A is a dominating literal, it dominates B i.e. there exists some term in A which dominates s. Since A is regular, t dominates each argument of A, therefore dominates s, which was to show.

Q.E.D.

LEMMA 6.11.: If a regular term t is a dominating term for a regular set M of literals, then every literal A which contains t is a dominating literal for M.

The *proof* follows immediately from the definition of the dominating relation for literals.

<u>LEMMA</u> 6.12.: If a regular term t dominates the term s and σ is a substitution such that $(t)\sigma$ is regular, then $(t)\sigma$ dominates $(s)\sigma$.

<u>Proof</u>:

The case $t = s$ is trivial. Suppose now that t and s are different. Since t dominates s, the depth of t is nonzero. Assume that the depth of s is zero. If s is constant, the proof is evident. Let s be a variable. Then it is an argument of the term t. The relation "to be an argument" is preserved substitution, therefore $(s)\sigma$ is an argument of $(t)\sigma$. Since $(t)\sigma$ is regular, it dominates each of its arguments, in particular it dominates $(s)\sigma$. Assume now that the depth of s is greater than zero. Since the dominating relation is preserved under substitution in terms of positive depth, $(t)\sigma$ dominates $(s)\sigma$.

Q.E.D.

<u>LEMMA</u> 6.13.: Let F be a formula, t be a regular F-term and let k be the minimal arity of function symbols occurring in F. Then the first k arguments of the term t are of depth 0.

<u>Proof</u>.

Let $t = f(t_1, \dots, t_n)$, $n \geq k$. Assume that the depth of t_i is nonzero for some i, $1 \leq i \leq k$, i.e. $t_i = g(u_1, \dots, u_m)$ for some m, $k \leq m \leq n$. Since the term t is regular, it dominates t_i, therefore $u_j = t_j$ for every j, $1 \leq j \leq m$. It follows that $t_i = g(t_1, \dots, t_m)$ for some i, $1 \leq i \leq k \leq m$, which is impossible.

Q.E.D.

<u>LEMMA</u> 6.14.: Assume that $\{A, B\}$ is a k-regular indecomposable set of F-literals for a formula F, such that A dominates B and σ is a substitution such that $(A)\sigma$ is regular; suppose further that k is not greater than the minimal arity of function symbols occurring in $(A)\sigma$. Then $(A)\sigma$ dominates $(B)\sigma$ and the set $\{A, B\}\sigma$ is k-regular.

<u>Proof</u>.

Let us consider following cases:

1) Both A and B are literals of type 1.

Since A dominates B, the dominating term t_1 of the literal A dominates each argument of the literal B by lemma 6.10. Since the literal $(A)\sigma$ is regular, the term $(t_1)\sigma$ is regular and dominates each argument of the literal $(B)\sigma$ by lemma 6.12. It follows that $(A)\sigma$ dominates $(B)\sigma$. Now let us prove that $(B)\sigma$ is a regular literal. The term t_1 dominates the dominating term t_2 of the literal B by lemma 6.10., therefore $(t_1)\sigma$ dominates $(t_2)\sigma$ by lemma 6.12. Since $(t_1)\sigma$ is regular, $(t_2)\sigma$ is regular too by lemma 6.2; moreover the term $(t_2)\sigma$ dominates each argument of the literal $(B)\sigma$ by lemma 6.12. It follows that $(B)\sigma$ is regular. So we have shown that $(A)\sigma$ dominates $(B)\sigma$ and both literals are regular, i.e. the set $\{(A)\sigma, (B)\sigma\}$ is a k-regular set by definition.

2) A is a literal of type 1, B is a literal of type 2.

Then the set of arguments of the literal B is similar to the set consisting of the first k arguments of the term t_1. Since $(A)\sigma$ is regular, $(t_1)\sigma$ is regular and its first k arguments are terms of depth zero by lemma 6.13. Therefore the set of arguments of $(B)\sigma$ consists of the first k arguments of the term $(t_1)\sigma$ and may be of constants. It follows that $(B)\sigma$ is regular and $\{(A)\sigma, (B)\sigma\}$ is a k-regular set.

3) A is a literal of type 1. B is aliteral of the type 3.

Let x be the only variable occurring in B. Then x is an argument of the dominating term t_1 of the literal A since $\{A, B\}$ is an indecomposable set. Since $(t_1)\sigma$ is regular, it dominates $(x)\sigma$ by lemma 6.12. Therefore $(x)\sigma$ is regular, which implies that $(B)\sigma$ is regular. So we have shown that $(A)\sigma$ dominates $(B)\sigma$ and both literals are regular, i.e. the set $\{(A)\sigma, (B)\sigma\}$ is a k-regular set by defintion of k-regularity.

4) Both A and B are literals of type 2.

Since $\{A, B\}$ is regular, A and B are similar and the sets of their arguments differ in constants only. Since $(A)\sigma$ is regular. $(B)\sigma$ is regular too. So we have shown that $\{A, B\}\sigma$ is a k-regular set.

5) A is a literal of type 2, B is a literal of type 3.

In this case the variable x, occurring in B, occurs in A too, since $\{A,B\}$ is indecomposable. The literal $(A)\sigma$ is regular, therfore $(x)\sigma$ is regular too. It follows that the literal $(B)\sigma$ is regular. It is also obvious that $(A)\sigma$ dominates $(B)\sigma$.

6) Both A and B are literals of type 3. Similarly to the previous case.

Q.E.D.

Corollary 6. 3.: If M is a regular indecomposable set of literals, A is a dominating literal and $(A)\sigma$ is regular, then $(M)\sigma$ is regular.

LEMMA 6. 15.: Assume that $\{A, B\}$ is a regular set of literals, A is a literal of type 1 and σ is a substitution such that $A\sigma$ regular and each of its proper variables either occurs in A or does not occur in B. Then $\{A, B\}\sigma$ is regular.

Proof:

The case where A dominates B is considered in lemma 6. 14. Let B dominate A. It means that the dominating term t_2 of the literal B dominates the dominating term t_1 of the literal A. Since $(A)\sigma$ is regular, $(t_1)\sigma$ is regular too. By lemma 6. 4 the term $(t_2)\sigma$ is regular. By lemma 6. 8 $(B)\sigma$ is a regular literal. Since the dominating relation for terms of positive depth terms is preserved under substitution, $(t_2)\sigma$ dominates $(t_1)\sigma$. It follows that $(B)\sigma$ dominates $(A)\sigma$, i.e. $\{A, B\}\sigma$ is a regular set, which was to show.

Q.E.D.

LEMMA 6. 16.: Assume that $(C_1 \cup \{B\})$ and $(C_2 \cup \{\neg B\})$ are indecomposable, regular clauses and that the clause $C_1 \cup C_2$ is the resolvent of these clauses with empty m.g.u.. Then if the literals B and $\neg B$ are dominating literals for their clauses, then $C_1 \cup C_2$ is quasiregular.

Proof:

From the assumptions of the lemma it follows that B dominates C_2 as well. Therefore the set M of all literals which occur in the clause $\{B\} \cup C_1 \cup C_2$ is regular. Now we will prove that the set M_1 obtained from M by removing B is quasiregular.

Let M_1 contain some literal of type 1 and A be a literal containing a term with maximum arity. The literal A is regular since it occurs in a regular set. Let M_2

be the set of all literals from M_1 which contain no variable from A. Then A is a dominating literal for $M_1 - M_2$. In fact, the dominating term t_1 for A dominates each term of any literal of types 1 and 2 since it has a maximum arity among all terms of the set $M_1 - M_2$; by lemma 6.1 t_1 also dominates each argument of any literal of type 3 from $M_1 - M_2$. The set M_2 consists of literals of type 3 only and therefore is quasiregular.

In the case where M_1 contains no literal of type 1, but contains some literal of type 2, we can take A to be a literal of type 2. If M_1 consists of literals of the type 3 only, we can take an arbitrary literal from M_1 for A. So we have shown that M_1 is decomposable in two components M_2 and $M_1 - M_2$, one of which is regular, and the second one is quasiregular, which was to show.

Q.E.D.

LEMMA 6.17.: Assume that $C_1 \cup \{A\}$ and $C_2 \cup \{B\}$ are regular indecomposable clauses and the clause $(C_1 \cup C_2)\sigma$ is the resolvent of these clauses by resolution upon A and B. Then if the literals α and B are dominating for their clauses, then $(C_1 \cup C_2)\sigma$ is quasiregular.

Proof:

According to lemma 6.9 the literals $A\sigma$, $B\sigma$ are regular. By lemma 6.14 the clauses $(C_1 \cup \{A\})\sigma$ and $(C_2 \cup \{B\})\sigma$ are quasiregular (note that $A^d\sigma = B\sigma$). We conlude that the clause $(C_1 \cup C_2)\sigma$ is quasiregular by lemma 6.16.

Q.E.D.

Corollary 6.4.: The lemma holds if holds if the parents of the resolvent are quasiregular.

6.2 A DECISION PROCEDURE FOR THE CLASS KS

We will use a π - ordering refinement (see chapter 4) to decide KS. Recall that for π - orderings we consider clauses as lists rather than sets of literals. Here we define a specific π - ordering in the following way:

Let F be a formula from KS and HF its Herbrand expansion. Let the term list $t_1, \dots, t_n, \dots$ be a list of all terms from HF satisfying the following conditions:

1) If $\tau(t_j) < \tau(t_i)$ then $i > j$.
2) If t_i dominates t_j then $i > j$.
3) If the terms t_i and t_j are similar (i.e. they dominate each other) then the ordering is lexicographical.

Now we can define the ordering $L_1, L_2, \dots, L_n, \dots$ for all occurrences of literals in HF in the following way:

1) If the maximal term (in the sense of the term ordering above) of L_i is greater than the maximal term of L_j then $i > j$.
2) If the maximal terms of L_i and L_j are equal and the set of essential arguments of L_j is a proper subset of the analogous set of L_i then $i > j$.
3) If the sets of essential arguments of literals L_i and L_j are equal, L_i is an instance of some literal of type 2 and L_j is an instance of some literal of type 3, then $i > j$.

Any clause $L_{i_1}, \dots, L_{i_k}$ of literals from the ordered list $L_1, \dots, L_n, \dots$ will be called π - ordered if for all $e, m \in N$ $e < m$ implies $L_e < L_m$. If the formula F is unsatisfiable then (by Herbrand's theorem) there exists a finite part HF' of HF which is unsatisfiable as well.

According to the completeness of π - ordering refinements (proved in chapter 4) there exists a π - ordered refutation R of HF' (i.e. there exists a list of π - ordered clauses, each of which is either in HF' or derived from HF' by the π-ordering refinement of resolution). Note that, in this case, the acyclicity of the π-ordering is guaranteed by the fact that HF' is propositional.

Let R' be the lifted refutation obtained from R and let σ be the substituion with $R'\sigma = R$. We then have:

<u>LEMMA</u> 6.18.: Let C be a clause in the refutation R'. If C is regular then the resolved literal is dominating for C.

Proof:

We assume to the contrary that the resolved literal A from C is not dominating and that B is a dominating literal of C. It follows that there exists some term t_B in B which is not dominated by any term in A. Let us consider the following cases:

1) The literal B contains a functional term.
Then its dominating term t_B dominates each term in A but is not dominated by any term in A. It follows that each term in $A\sigma$ either is dominated by $t_B\sigma$ or is a proper subterm of $t_B\sigma$. In this case the literal $B\sigma$ must be greater than $A\sigma$ according to point 1) of the definition of literal ordering, but this contradicts the fact that A is the resolved literal.

2) The literal B does not contain function symbols.
Then there exists a variable x s.t. $x \in V(B) - V(A)$. If follows that B is of type 2 and A is of type 3. If $B\sigma$ contains at least one essential term which does not occur in $A\sigma$ then $B\sigma$ must be greater than $A\sigma$, according to point 2) of the literal ordering and we get a contradiction. If the sets of essential arguments of$A\sigma$ and $B\sigma$ are equal then $B\sigma$ must be greater than $A\sigma$, according to point 3) of the definition of the literal ordering; again we get a contradiction.

Thus we have shown that the resolved literal in a regular clause must be dominating.

Q.E.D.

THEOREM 6.3.: The class KS is decidable.

Proof:

Let F be a formula in KS. We have shown that, in case F is unsatisfiable, there exists a π-refutation of HF. Using induction on the length of R we can prove that every clause in a lifted refutation R' is quasiregular (by theorem 6.1 and lemma 6.17). It remains to show that the set of clauses which are π-derivable from a set of clauses in KS is always finite. Let S be a set of clauses in KS and $C \in S$. It is obvious that the number of variables in C does not exceed the number of universal quantified variables in the prefix of the K-formula corresponding to S.
Let ϑ be the number of variables, m the maximal arity of function symbols and φ be the number of different function symbols of the formula F. Then the number r of all regular F-terms cannot exceed $\varphi(\vartheta + \varphi)^m$. Indeed, if we

consider the short form of a regular term we can choose one of φ function symbols, and one of ϑ variables or one of φ function symbols for each of m arguments.

Now let p be the number of different predicate symbols and n be the maximal arity of predicate symbols in the formula F. Then the number of atoms in a regular clause cannot exceed $p \cdot r^n$. Indeed we can choose one of p predicate symbols as leading predicate symbol and any of r regular terms for each of the n arguments. Thus the maximal length of a regular clause (corresponding to F) is less or equal to $p \cdot r^n$.

Let $\alpha = p \cdot r^n$; then the number c of regular clauses cannot exceed 3^α. Thus c is the maximal number of literals in an indecomposable component of a quasi-regular clause and consequently the number of quasiregular clauses is less or equal 3^c. We conclude that the number of π - derivable clauses is finite.

Q.E.D.

It is easy to show that the class defined by building conjunctions of formulas from K is decidable (in the same way).

It was shown in lemma 6.18 that lifting a ground refutation (based on the specific ground ordering defined in this section) the resolved literals on the general level are always dominating. Thus we always find a refutation where no strictly dominated literal is resolved. Formally we may define the following ordering:

<u>DEFINITION</u> 6.17.: A $>_k$ B iff A dominates B, but B does not dominate A.

It is interesting to note that $>_k$ itself is not a π - ordering because property (D) from chapter 4 is not fulfilled (for a detailed discussion of such orderings see chapter 4.4).

We thus get the following resolution decision procedure for K:

a) Split the set of clauses into sets of sets of regular clauses.

b) Apply the $>_k$ - ordering refinement to each set of clauses.

By completeness and termination of the $>_k$ - refinement (on sets of regular clauses) a), b) indeed gives a decision procedure for K.