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The Propositional Calculus

1.1 Propositional Connectives: Truth Tables

Sentences may be combined in various ways to form more complicated sentences. We shall consider only *truth-functional* combinations, in which the truth or falsity of the new sentence is determined by the truth or falsity of its component sentences.

Negation is one of the simplest operations on sentences. Although a sentence in a natural language may be negated in many ways, we shall adopt a uniform procedure: placing a sign for negation, the symbol $\neg$, in front of the entire sentence. Thus, if A is a sentence, then $\neg A$ denotes the negation of A.

The truth-functional character of negation is made apparent in the following *truth table*:

A	$\neg A$
T	F
F	T

When A is true, $\neg A$ is false; when A is false, $\neg A$ is true. We use T and F to denote the *truth values* true and false.

Another common truth-functional operation is the *conjunction*: "and." The conjunction of sentences A and B will be designated by $A \wedge B$ and has the following truth table:

A	B	$A \wedge B$
T	T	T
F	T	F
T	F	F
F	F	F

$A \wedge B$ is true when and only when both A and B are true. A and B are called the *conjuncts* of $A \wedge B$. Note that there are four rows in the table, corresponding to the number of possible assignments of truth values to A and B.

In natural languages, there are two distinct uses of "or": the inclusive and the exclusive. According to the inclusive usage, "A or B" means "A or B or both," whereas according to the exclusive usage, the meaning is "A or B, but not both," We shall introduce a special sign, ∨, for the inclusive connective. Its truth table is as follows:

A	B	$A \vee B$
T	T	T
F	T	T
T	F	T
F	F	F

Thus, $A \vee B$ is false when and only when both A and B are false. "$A \vee B$" is called a *disjunction*, with the *disjuncts* A and B.

Another important truth-functional operation is the *conditional*: "if A, then B." Ordinary usage is unclear here. Surely, "if A, then B" is false when the *antecedent* A is true and the *consequent* B is false. However, in other cases, there is no well-defined truth value. For example, the following sentences would be considered neither true nor false:

1. If $1 + 1 = 2$, then Paris is the capital of France.
2. If $1 + 1 \neq 2$, then Paris is the capital of France.
3. If $1 + 1 \neq 2$, then Rome is the capital of France.

Their meaning is unclear, since we are accustomed to the assertion of some sort of relationship (usually causal) between the antecedent and the consequent. We shall make the convention that "if A, then B" is false when and only when A is true and B is false. Thus, sentences 1–3 are assumed to be true. Let us denote "if A, then B" by "$A \Rightarrow B$." An expression "$A \Rightarrow B$" is called a *conditional*. Then ⇒ has the following truth table:

A	B	$A \Rightarrow B$
T	T	T
F	T	T
T	F	F
F	F	T

This sharpening of the meaning of "if A, then B" involves no conflict with ordinary usage, but rather only an extension of that usage.*

* There is a common non-truth-functional interpretation of "if A, then B" connected with causal laws. The sentence "if this piece of iron is placed in water at time t, then the iron will dissolve" is regarded as false even in the case that the piece of iron is not placed in water at time t—that is, even when the antecedent is false. Another non-truth-functional usage occurs in so-called counterfactual conditionals, such as "if Sir Walter Scott had not written any novels, then there would have been no War Between the States." (This was Mark Twain's contention in *Life on the Mississippi*: "Sir Walter had so large a hand in making Southern character, as it existed before the war, that he is in great measure responsible for the war.") This sentence might be asserted to be false even though the antecedent is admittedly false. However, causal laws and counterfactual conditions seem not to be needed in mathematics and logic. For a clear treatment of conditionals and other connectives, see Quine (1951). (The quotation from *Life on the Mississippi* was brought to my attention by Professor J.C. Owings, Jr.)

A justification of the truth table for ⇒ is the fact that we wish "if *A* and *B*, then *B*" to be true in all cases. Thus, the case in which *A* and *B* are true justifies the first line of our truth table for ⇒, since (*A* and *B*) and *B* are both true. If *A* is false and *B* true, then (*A* and *B*) is false while *B* is true. This corresponds to the second line of the truth table. Finally, if *A* is false and *B* is false, (*A* and *B*) is false and *B* is false. This gives the fourth line of the table. Still more support for our definition comes from the meaning of statements such as "for every x, if x is an odd positive integer, then x^2 is an odd positive integer." This asserts that, for every x, the statement "if x is an odd positive integer, then x^2 is an odd positive integer" is true. Now we certainly do not want to consider cases in which x is not an odd positive integer as counterexamples to our general assertion. This supports the second and fourth lines of our truth table. In addition, any case in which x is an odd positive integer and x^2 is an odd positive integer confirms our general assertion. This corresponds to the first line of the table.

Let us denote "*A* if and only if *B*" by "$A \Leftrightarrow B$." Such an expression is called a *biconditional*. Clearly, $A \Leftrightarrow B$ is true when and only when *A* and *B* have the same truth value. Its truth table, therefore is:

A	B	$A \Leftrightarrow B$
T	T	T
F	T	F
T	F	F
F	F	T

The symbols ¬, ∧, ∨, ⇒, and ⇔ will be called *propositional connectives*.* Any sentence built up by application of these connectives has a truth value that depends on the truth values of the constituent sentences. In order to make this dependence apparent, let us apply the name *statement form* to an expression built up from the *statement letters A*, *B*, *C*, and so on by appropriate applications of the propositional connectives.

1. All statement letters (capital italic letters) and such letters with numerical subscripts† are statement forms.
2. If $\mathscr{B}$ and $\mathscr{C}$ are statement forms, then so are $(\neg\mathscr{B})$, $(\mathscr{B} \wedge \mathscr{C})$, $(\mathscr{B} \vee \mathscr{C})$, $(\mathscr{B} \Rightarrow \mathscr{C})$, and $(\mathscr{B} \Leftrightarrow \mathscr{C})$.

* We have been avoiding and shall in the future avoid the use of quotation marks to form names whenever this is not likely to cause confusion. The given sentence should have quotation marks around each of the connectives. See Quine (1951, pp. 23–27).

† For example, $A_1, A_2, A_{17}, B_{31}, C_2, \ldots$.

3. Only those expressions are statement forms that are determined to be so by means of conditions 1 and 2.* Some examples of statement forms are B, $(\neg C_2)$, $(D_3 \wedge (\neg B))$, $(((\neg B_1) \vee B_2) \Rightarrow (A_1 \wedge C_2))$, and $(((\neg A) \Leftrightarrow A) \Leftrightarrow (C \Rightarrow (B \vee C)))$.

For every assignment of truth values T or F to the statement letters that occur in a statement form, there corresponds, by virtue of the truth tables for the propositional connectives, a truth value for the statement form. Thus, each statement form determines a *truth function*, which can be graphically represented by a truth table for the statement form. For example, the statement form $(((\neg A) \vee B) \Rightarrow C)$ has the following truth table:

A	B	C	$(\neg A)$	$((\neg A) \vee B)$	$(((\neg A) \vee B) \Rightarrow C)$
T	T	T	F	T	T
F	T	T	T	T	T
T	F	T	F	F	T
F	F	T	T	T	T
T	T	F	F	T	F
F	T	F	T	T	F
T	F	F	F	F	T
F	F	F	T	T	F

Each row represents an assignment of truth values to the statement letters A, B, and C and the corresponding truth values assumed by the statement forms that appear in the construction of $(((\neg A) \vee B) \Rightarrow C)$.

The truth table for $((A \Leftrightarrow B) \Rightarrow ((\neg A) \wedge B))$ is as follows:

A	B	$(A \Leftrightarrow B)$	$(\neg A)$	$((\neg A) \wedge B)$	$((A \Leftrightarrow B) \Rightarrow ((\neg A) \wedge B))$
T	T	T	F	F	F
F	T	F	T	T	T
T	F	F	F	F	T
F	F	T	T	F	F

If there are n distinct letters in a statement form, then there are 2^n possible assignments of truth values to the statement letters and, hence, 2^n rows in the truth table.

* This can be rephrased as follows: $\mathscr{C}$ is a statement form if and only if there is a finite sequence $\mathscr{B}_1, \ldots, \mathscr{B}_n$ $(n \geq 1)$ such that $\mathscr{B}_n = \mathscr{C}$ and, if $1 \leq i \leq n$, $\mathscr{B}_i$ is either a statement letter or a negation, conjunction, disjunction, conditional, or biconditional constructed from previous expressions in the sequence. Notice that we use script letters $\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots$ to stand for arbitrary expressions, whereas italic letters are used as statement letters.

A truth table can be abbreviated by writing only the full statement form, putting the truth values of the statement letters underneath all occurrences of these letters, and writing, step by step, the truth values of each component statement form under the principal connective of the form.* As an example, for $((A \Leftrightarrow B) \Rightarrow ((\neg A) \wedge B))$, we obtain

$((A$	$\Leftrightarrow$	$B)$	$\Rightarrow$	$((\neg A)$	$\wedge$	$B))$
T	T	T	F	FT	F	T
F	F	T	T	TF	T	T
T	F	F	T	FT	F	F
F	T	F	F	TF	F	F

Exercises

1.1 Let $\oplus$ designate the exclusive use of "or." Thus, $A \oplus B$ stands for "A or B but not both." Write the truth table for $\oplus$.

1.2 Construct truth tables for the statement forms $((A \Rightarrow B) \vee (\neg A))$ and $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$.

1.3 Write abbreviated truth tables for $((A \Rightarrow B) \wedge A)$ and $((A \vee (\neg C)) \Leftrightarrow B)$.

1.4 Write the following sentences as statement forms, using statement letters to stand for the *atomic sentences*—that is, those sentences that are not built up out of other sentences.

a. If Mr Jones is happy, Mrs Jones is not happy, and if Mr Jones is not happy, Mrs Jones is not happy.

b. Either Sam will come to the party and Max will not, or Sam will not come to the party and Max will enjoy himself.

c. A sufficient condition for x to be odd is that x is prime.

d. A necessary condition for a sequence s to converge is that s be bounded.

e. A necessary and sufficient condition for the sheikh to be happy is that he has wine, women, and song.

f. Fiorello goes to the movies only if a comedy is playing.

g. The bribe will be paid if and only if the goods are delivered.

h. If x is positive, x^2 is positive.

i. Karpov will win the chess tournament unless Kasparov wins today.

* The *principal connective* of a statement form is the one that is applied last in constructing the form.

1.2 Tautologies

A *truth function of n arguments* is defined to be a function of n arguments, the arguments and values of which are the truth values T or F. As we have seen, any statement form containing n distinct statement letters determines a corresponding truth function of n arguments.*

A statement form that is always true, no matter what the truth values of its statement letters may be, is called a *tautology*. A statement form is a tautology if and only if its corresponding truth function takes only the value T, or equivalently, if, in its truth table, the column under the statement form contains only *T*s. An example of a tautology is $(A \vee (\neg A))$, the so-called *law of the excluded middle*. Other simple examples are $(\neg(A \wedge (\neg A)))$, $(A \Leftrightarrow (\neg(\neg A)))$, $((A \wedge B) \Rightarrow A)$, and $(A \Rightarrow (A \vee B))$.

$\mathscr{B}$ is said to *logically imply* $\mathscr{C}$ (or, synonymously, $\mathscr{C}$ is a *logical consequence* of $\mathscr{B}$) if and only if every truth assignment to the statement letters of $\mathscr{B}$ and $\mathscr{C}$ that makes $\mathscr{B}$ true also makes $\mathscr{C}$ true. For example, $(A \wedge B)$ logically implies A, A logically implies $(A \vee B)$, and $(A \wedge (A \Rightarrow B))$ logically implies B.

$\mathscr{B}$ and $\mathscr{C}$ are said to be *logically equivalent* if and only if $\mathscr{B}$ and $\mathscr{C}$ receive the same truth value under every assignment of truth values to the statement letters of $\mathscr{B}$ and $\mathscr{C}$. For example, A and $(\neg(\neg A))$ are logically equivalent, as are $(A \wedge B)$ and $(B \wedge A)$.

* To be precise, enumerate all statement letters as follows: $A, B, \ldots, Z; A_1, B_1, \ldots, Z_1; A_2, \ldots,$. If a statement form contains the $i_{1\text{th}}, \ldots, i_{n\text{th}}$ statement letters in this enumeration, where $i_1 < \cdots < i_n$, then the corresponding truth function is to have $x_{i_1}, \ldots, x_{i_n}$, in that order, as its arguments, where x_{i_j} corresponds to the $i_{j\text{th}}$ statement letter. For example, $(A \Rightarrow B)$ generates the truth function:

x_1	x_2	$f(x_1, x_2)$
T	T	T
F	T	T
T	F	F
F	F	T

whereas $(B \Rightarrow A)$ generates the truth function:

x_1	x_2	$g(x_1, x_2)$
T	T	T
F	T	F
T	F	T
F	F	T

Proposition 1.1

a. $\mathscr{B}$ logically implies $\mathscr{C}$ if and only if $(\mathscr{B} \Rightarrow \mathscr{C})$ is a tautology.
b. $\mathscr{B}$ and $\mathscr{C}$ are logically equivalent if and only if $(\mathscr{B} \Leftrightarrow \mathscr{C})$ is a tautology.

Proof

a. (i) Assume $\mathscr{B}$ logically implies $\mathscr{C}$. Hence, every truth assignment that makes $\mathscr{B}$ true also makes $\mathscr{C}$ true. Thus, no truth assignment makes $\mathscr{B}$ true and $\mathscr{C}$ false. Therefore, no truth assignment makes $(\mathscr{B} \Rightarrow \mathscr{C})$ false, that is, every truth assignment makes $(\mathscr{B} \Rightarrow \mathscr{C})$ true. In other words, $(\mathscr{B} \Rightarrow \mathscr{C})$ is a tautology. (ii) Assume $(\mathscr{B} \Rightarrow \mathscr{C})$ is a tautology. Then, for every truth assignment, $(\mathscr{B} \Rightarrow \mathscr{C})$ is true, and, therefore, it is not the case that $\mathscr{B}$ is true and $\mathscr{C}$ false. Hence, every truth assignment that makes $\mathscr{B}$ true makes $\mathscr{C}$ true, that is, $\mathscr{B}$ logically implies $\mathscr{C}$.
b. $(\mathscr{B} \Leftrightarrow \mathscr{C})$ is a tautology if and only if every truth assignment makes $(\mathscr{B} \Leftrightarrow \mathscr{C})$ true, which is equivalent to saying that every truth assignment gives $\mathscr{B}$ and $\mathscr{C}$ the same truth value, that is, $\mathscr{B}$ and $\mathscr{C}$ are logically equivalent.

By means of a truth table, we have an effective procedure for determining whether a statement form is a tautology. Hence, by Proposition 1.1, we have effective procedures for determining whether a given statement form logically implies another given statement form and whether two given statement forms are logically equivalent.

To see whether a statement form is a tautology, there is another method that is often shorter than the construction of a truth table.

Examples

1. Determine whether $((A \Leftrightarrow ((\neg B) \vee C)) \Rightarrow ((\neg A) \Rightarrow B))$ is a tautology.

Assume that the statement form sometimes is F (line 1). Then $(A \Leftrightarrow ((\neg B) \vee C))$ is T and $((\neg A) \Rightarrow B)$ is F (line 2). Since $((\neg A) \Rightarrow B)$ is F, $(\neg A)$ is T and B is F (line 3). Since $(\neg A)$is T, A is F (line 4). Since A is F and $(A \Leftrightarrow ((\neg B) \vee C))$ is T, $((\neg B) \vee C)$ is F (line 5). Since $((\neg B) \vee C)$ is F, $(\neg B)$ and C are F (line 6). Since $(\neg B)$ is F, B is T (line 7). But B is both T and F (lines 7 and 3). Hence, it is impossible for the form to be false.

$((A$	$\Leftrightarrow$	$((\neg$	$B)$	$\vee$	$C))$	$\Rightarrow$	$((\neg$	$A)$	$\Rightarrow$	$B))$	
						F					1
T									F		2
							T			F	3
F								F			4
				F							5
		F			F						6
			T								7

2. Determine whether $((A \Rightarrow (B \vee C)) \vee (A \Rightarrow B))$ is a tautology.

Assume that the form is F (line 1). Then $(A \Rightarrow (B \vee C))$ and $(A \Rightarrow B)$ are F (line 2). Since $(A \Rightarrow B)$ is F, A is T and B is F (line 3). Since $(A \Rightarrow (B \vee C))$ is F, A is T and $(B \vee C)$ is F (line 4). Since $(B \vee C)$ is F, B and C are F (line 5). Thus, when A is T, B is F, and C is F, the form is F. Therefore, it is not a tautology.

$((A$	$\Rightarrow$	$(B$	$\vee$	$C))$	$\vee$	$(A$	$\Rightarrow$	$B))$	
					F				1
	F						F		2
						T		F	3
T			F						4
		F		F					5

Exercises

1.5 Determine whether the following are tautologies.

a. $(((A \Rightarrow B) \Rightarrow B) \Rightarrow B)$

b. $(((A \Rightarrow B) \Rightarrow B) \Rightarrow A)$

c. $(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)$

d. $(((B \Rightarrow C) \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B))$

e. $((A \vee (\neg(B \wedge C))) \Rightarrow ((A \Leftrightarrow C) \vee B))$

f. $(A \Rightarrow (B \Rightarrow (B \Rightarrow A)))$

g. $((A \wedge B) \Rightarrow (A \vee C))$

h. $((A \Leftrightarrow B) \Leftrightarrow (A \Leftrightarrow (B \Leftrightarrow A)))$

i. $((A \Rightarrow B) \vee (B \Rightarrow A))$

j. $((\neg(A \Rightarrow B)) \Rightarrow A)$

1.6 Determine whether the following pairs are logically equivalent.

a. $((A \Rightarrow B) \Rightarrow A)$ and A

b. $(A \Leftrightarrow B)$ and $((A \Rightarrow B) \wedge (B \Rightarrow A))$

c. $((\neg A) \vee B)$ and $((\neg B) \vee A)$

d. $(\neg(A \Leftrightarrow B))$ and $(A \Leftrightarrow (\neg B))$

e. $(A \vee (B \Leftrightarrow C))$ and $((A \vee B) \Leftrightarrow (A \vee C))$

f. $(A \Rightarrow (B \Leftrightarrow C))$ and $((A \Rightarrow B) \Leftrightarrow (A \Rightarrow C))$

g. $(A \wedge (B \Leftrightarrow C))$ and $((A \wedge B) \Leftrightarrow (A \wedge C))$

1.7 Prove:

a. $(A \Rightarrow B)$ is logically equivalent to $((\neg A) \vee B)$.

b. $(A \Rightarrow B)$ is logically equivalent to $(\neg(A \wedge (\neg B)))$.

1.8 Prove that $\mathscr{B}$ is logically equivalent to $\mathscr{C}$ if and only if $\mathscr{B}$ logically implies $\mathscr{C}$ and $\mathscr{C}$ logically implies $\mathscr{B}$.

1.9 Show that $\mathscr{B}$ and $\mathscr{C}$ are logically equivalent if and only if, in their truth tables, the columns under $\mathscr{B}$ and $\mathscr{C}$ are the same.

1.10 Prove that $\mathscr{B}$ and $\mathscr{C}$ are logically equivalent if and only if $(\neg\mathscr{B})$ and $(\neg\mathscr{C})$ are logically equivalent.

1.11 Which of the following statement forms are logically implied by $(A \wedge B)$?

a. A

b. B

c. $(A \vee B)$

d. $((\neg A) \vee B)$

e. $((\neg B) \Rightarrow A)$

f. $(A \Leftrightarrow B)$

g. $(A \Rightarrow B)$

h. $((\neg B) \Rightarrow (\neg A))$

i. $(A \wedge (\neg B))$

1.12 Repeat Exercise 1.11 with $(A \wedge B)$ replaced by $(A \Rightarrow B)$ and by $(\neg(A \Rightarrow B))$, respectively.

1.13 Repeat Exercise 1.11 with $(A \wedge B)$ replaced by $(A \vee B)$.

1.14 Repeat Exercise 1.11 with $(A \wedge B)$ replaced by $(A \Leftrightarrow B)$ and by $(\neg(A \Leftrightarrow B))$, respectively.

A statement form that is false for all possible truth values of its statement letters is said to be *contradictory*. Its truth table has only Fs in the column under the statement form. One example is $(A \Leftrightarrow (\neg A))$:

A	$(\neg A)$	$(A \Leftrightarrow (\neg A))$
T	F	F
F	T	F

Another is $(A \wedge (\neg A))$.

Notice that a statement form $\mathscr{B}$ is a tautology if and only if $(\neg\mathscr{B})$ is contradictory, and vice versa.

A sentence (in some natural language like English or in a formal theory)* that arises from a tautology by the substitution of sentences for all the statement letters, with occurrences of the same statement letter being replaced by the same sentence, is said to be *logically true* (according to the propositional calculus). Such a sentence may be said to be true by virtue of its truth-functional structure alone. An example is the English sentence, "If it is raining or it is snowing, and it is not snowing, then it is raining," which arises by substitution from the tautology $(((A \vee B) \wedge (\neg B)) \Rightarrow A)$. A sentence that comes from

* By a formal theory we mean an artificial language in which the notions of *meaningful expressions*, *axioms*, and *rules of inference* are precisely described (see page 27).

a contradictory statement form by means of substitution is said to be *logically false* (according to the propositional calculus).

Now let us prove a few general facts about tautologies.

Proposition 1.2

If $\mathscr{B}$ and $(\mathscr{B} \Rightarrow \mathscr{C})$ are tautologies, then so is $\mathscr{C}$.

Proof

Assume that $\mathscr{B}$ and $(\mathscr{B} \Rightarrow \mathscr{C})$ are tautologies. If $\mathscr{C}$ took the value F for some assignment of truth values to the statement letters of $\mathscr{B}$ and $\mathscr{C}$, then, since $\mathscr{B}$ is a tautology, $\mathscr{B}$ would take the value T and, therefore, $(\mathscr{B} \Rightarrow \mathscr{C})$ would have the value F for that assignment. This contradicts the assumption that $(\mathscr{B} \Rightarrow \mathscr{C})$ is a tautology. Hence, $\mathscr{C}$ never takes the value F.

Proposition 1.3

If $\mathscr{T}$ is a tautology containing as statement letters $A_1, A_2, \ldots, A_n$, and $\mathscr{B}$ arises from $\mathscr{T}$ by substituting statement forms $\mathscr{S}_1, \mathscr{S}_2, \ldots, \mathscr{S}_n$ for $A_1, A_2, \ldots, A_n$, respectively, then $\mathscr{B}$ is a tautology; that is, substitution in a tautology yields a tautology.

Example

Let $\mathscr{T}$ be $((A_1 \wedge A_2) \Rightarrow A_1)$, let $\mathscr{S}_1$ be $(B \vee C)$ and let $\mathscr{S}_2$ be $(C \wedge D)$. Then $\mathscr{B}$ is $(((B \vee C) \wedge (C \wedge D)) \Rightarrow (B \vee C))$.

Proof

Assume that $\mathscr{T}$ is a tautology. For any assignment of truth values to the statement letters in $\mathscr{B}$, the forms $\mathscr{S}_1, \ldots, \mathscr{S}_n$ have truth values $x_1, \ldots, x_n$ (where each x_i is T or F). If we assign the values $x_1, \ldots, x_n$ to $A_1, \ldots, A_n$, respectively, then the resulting truth value of $\mathscr{T}$ is the truth value of $\mathscr{B}$ for the given assignment of truth values. Since $\mathscr{T}$ is a tautology, this truth value must be T. Thus, $\mathscr{B}$ always takes the value T.

Proposition 1.4

If $\mathscr{C}_1$ arises from $\mathscr{B}_1$ by substitution of $\mathscr{C}$ for one or more occurrences of $\mathscr{B}$, then $((\mathscr{B} \Leftrightarrow \mathscr{C}) \Rightarrow (\mathscr{B}_1 \Leftrightarrow \mathscr{C}_1))$ is a tautology. Hence, if $\mathscr{B}$ and $\mathscr{C}$ are logically equivalent, then so are $\mathscr{B}_1$ and $\mathscr{C}_1$.

Example
Let $\mathscr{B}_1$ be $(\mathscr{C} \vee D)$, let $\mathscr{B}$ be $\mathscr{C}$, and let $\mathscr{C}$ be $(\neg(\neg\mathscr{C}))$. Then $\mathscr{C}_1$ is $((\neg(\neg\mathscr{C})) \vee D)$. Since $\mathscr{C}$ and $(\neg(\neg\mathscr{C}))$ are logically equivalent, $(\mathscr{C} \vee D)$ and $((\neg(\neg\mathscr{C})) \vee D)$ are also logically equivalent.

Proof

Consider any assignment of truth values to the statement letters. If $\mathscr{B}$ and $\mathscr{C}$ have opposite truth values under this assignment, then $(\mathscr{B} \Leftrightarrow \mathscr{C})$ takes the value F, and, hence, $((\mathscr{B} \Leftrightarrow \mathscr{C}) \Rightarrow (\mathscr{B}_1 \Leftrightarrow \mathscr{C}_1))$ is T. If $\mathscr{B}$ and $\mathscr{C}$ take the same truth values, then so do $\mathscr{B}_1$ and $\mathscr{C}_1$, since $\mathscr{C}_1$ differs from $\mathscr{B}_1$ only in containing $\mathscr{C}$ in some places where $\mathscr{B}_1$ contains $\mathscr{B}$. Therefore, in this case, $(\mathscr{B} \Leftrightarrow \mathscr{C})$ is T, $(\mathscr{B}_1 \Leftrightarrow \mathscr{C}_1)$ is T, and, thus, $((\mathscr{B} \Leftrightarrow \mathscr{C}) \Rightarrow (\mathscr{B}_1 \Leftrightarrow \mathscr{C}_1))$ is T.

Parentheses

It is profitable at this point to agree on some conventions to avoid the use of so many parentheses in writing formulas. This will make the reading of complicated expressions easier.

First, we may omit the outer pair of parentheses of a statement form. (In the case of statement letters, there is no outer pair of parentheses.)

Second, we arbitrarily establish the following decreasing order of strength of the connectives: $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$. Now we shall explain a step-by-step process for restoring parentheses to an expression obtained by eliminating some or all parentheses from a statement form. (The basic idea is that, where possible, we first apply parentheses to negations, then to conjunctions, then to disjunctions, then to conditionals, and finally to biconditionals.) Find the leftmost occurrence of the strongest connective that has not yet been processed.

i. If the connective is $\neg$ and it precedes a statement form $\mathscr{B}$, restore left and right parentheses to obtain $(\neg\mathscr{B})$.
ii. If the connective is a binary connective C and it is preceded by a statement form $\mathscr{B}$ and followed by a statement form $\mathscr{D}$, restore left and right parentheses to obtain $(\mathscr{B}\, C\, \mathscr{D})$.
iii. If neither (i) nor (ii) holds, ignore the connective temporarily and find the leftmost occurrence of the strongest of the remaining unprocessed connectives and repeat (i–iii) for that connective.

Examples
Parentheses are restored to the expression in the first line of each of the following in the steps shown:

1. $A \Leftrightarrow (\neg B) \vee C \Rightarrow A$
   $A \Leftrightarrow ((\neg B) \vee C) \Rightarrow A$
   $A \Leftrightarrow (((\neg B) \vee C) \Rightarrow A)$
   $(A \Leftrightarrow (((\neg B) \vee C) \Rightarrow A))$

2. $A \Rightarrow \neg B \Rightarrow C$
   $A \Rightarrow (\neg B) \Rightarrow C$
   $(A \Rightarrow (\neg B)) \Rightarrow C$
   $((A \Rightarrow (\neg B)) \Rightarrow C)$
3. $B \Rightarrow \neg\neg A$
   $B \Rightarrow \neg(\neg A)$
   $B \Rightarrow (\neg(\neg A))$
   $(B \Rightarrow (\neg(\neg A)))$
4. $A \vee \neg(B \Rightarrow A \vee B)$
   $A \vee \neg(B \Rightarrow (A \vee B))$
   $A \vee (\neg(B \Rightarrow (A \vee B)))$
   $(A \vee (\neg(B \Rightarrow (A \vee B))))$

Not every form can be represented without the use of parentheses. For example, parentheses cannot be further eliminated from $A \Rightarrow (B \Rightarrow C)$, since $A \Rightarrow B \Rightarrow C$ stands for $((A \Rightarrow B) \Rightarrow C)$. Likewise, the remaining parentheses cannot be removed from $\neg(A \vee B)$ or from $A \wedge (B \Rightarrow C)$.

Exercises

1.15 Eliminate as many parentheses as possible from the following forms.

a. $((B \Rightarrow (\neg A)) \wedge C)$
b. $(A \vee (B \vee C))$
c. $(((A \wedge (\neg B)) \wedge C) \vee D)$
d. $((B \vee (\neg C)) \vee (A \wedge B))$
e. $((A \Leftrightarrow B) \Leftrightarrow (\neg(C \vee D)))$
f. $((\neg(\neg(\neg(B \vee C)))) \Leftrightarrow (B \Leftrightarrow C))$
g. $(\neg((\neg(\neg(B \vee C))) \Leftrightarrow (B \Leftrightarrow C)))$
h. $((((A \Rightarrow B) \Rightarrow (C \Rightarrow D)) \wedge (\neg A)) \vee C)$

1.16 Restore parentheses to the following forms.

a. $C \vee \neg A \wedge B$
b. $B \Rightarrow \neg\neg\neg A \wedge C$
c. $C \Rightarrow \neg(A \wedge B \Rightarrow C) \wedge A \Leftrightarrow B$
d. $C \Rightarrow A \Rightarrow A \Leftrightarrow \neg A \vee B$

1.17 Determine whether the following expressions are abbreviations of statement forms and, if so, restore all parentheses.

a. $\neg\neg A \Leftrightarrow A \Leftrightarrow B \vee C$
b. $\neg(\neg A \Leftrightarrow A) \Leftrightarrow B \vee C$
c. $\neg(A \Rightarrow B) \vee C \vee D \Rightarrow B$

d. $A \Leftrightarrow (\neg A \vee B) \Rightarrow (A \wedge (B \vee C)))$

e. $\neg A \vee B \vee C \wedge D \Leftrightarrow A \wedge \neg A$

f. $((A \Rightarrow B \wedge (C \vee D) \wedge (A \vee D))$

1.18 If we write $\neg\mathscr{B}$ instead of $(\neg\mathscr{B})$, $\Rightarrow\mathscr{B}\mathscr{C}$ instead of $(\mathscr{B} \Rightarrow \mathscr{C})$, $\wedge\mathscr{B}\mathscr{C}$ instead of $(\mathscr{B} \wedge \mathscr{C})$, $\vee\mathscr{B}\mathscr{C}$ instead of $(\mathscr{B} \vee \mathscr{C})$, and $\Leftrightarrow\mathscr{B}\mathscr{C}$ instead of $(\mathscr{B} \Leftrightarrow \mathscr{C})$, then there is no need for parentheses. For example, $((\neg A) \wedge (B \Rightarrow (\neg D)))$, which is ordinarily abbreviated as $\neg A \wedge (B \Rightarrow \neg D)$, becomes $\wedge \neg A \Rightarrow B \neg D$. This way of writing forms is called *Polish notation*.

a. Write $((C \Rightarrow (\neg A)) \vee B)$ and $(C \vee ((B \wedge (\neg D)) \Rightarrow C))$ in this notation.

b. If we count $\Rightarrow$, $\wedge$, $\vee$, and $\Leftrightarrow$ each as +1, each statement letter as −1 and $\neg$ as 0, prove that an expression $\mathscr{B}$ in this parenthesis-free notation is a statement form if and only if (i) the sum of the symbols of $\mathscr{B}$ is −1 and (ii) the sum of the symbols in any proper initial segment of $\mathscr{B}$ is nonnegative. (If an expression $\mathscr{B}$ can be written in the form $\mathscr{C}\mathscr{D}$, where $\mathscr{C} \neq \mathscr{B}$, then $\mathscr{C}$ is called a *proper initial segment* of $\mathscr{B}$.)

c. Write the statement forms of Exercise 1.15 in Polish notation.

d. Determine whether the following expressions are statement forms in Polish notation. If so, write the statement forms in the standard way.

i. $\neg\Rightarrow ABC \vee AB \neg C$

ii. $\Rightarrow\Rightarrow AB \Rightarrow\Rightarrow BC \Rightarrow\neg AC$

iii. $\vee \wedge \vee \neg A \neg BC \wedge \vee AC \vee \neg C \neg A$

iv. $\vee \wedge B \wedge BBB$

1.19 Determine whether each of the following is a tautology, is contradictory, or neither.

a. $B \Leftrightarrow (B \vee B)$

b. $((A \Rightarrow B) \wedge B) \Rightarrow A$

c. $(\neg A) \Rightarrow (A \wedge B)$

d. $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$

e. $(A \Leftrightarrow \neg B) \Rightarrow A \vee B$

f. $A \wedge (\neg(A \vee B))$

g. $(A \Rightarrow B) \Leftrightarrow ((\neg A) \vee B)$

h. $(A \Rightarrow B) \Leftrightarrow \neg(A \wedge (\neg B))$

i. $(B \Leftrightarrow (B \Leftrightarrow A)) \Rightarrow A$

j. $A \wedge \neg A \Rightarrow B$

1.20 If A and B are true and C is false, what are the truth values of the following statement forms?

a. $A \vee C$

b. $A \wedge C$

c. $\neg A \wedge \neg C$

d. $A \Leftrightarrow \neg B \vee C$

e. $B \vee \neg C \Rightarrow A$

f. $(B \vee A) \Rightarrow (B \Rightarrow \neg C)$

g. $(B \Rightarrow \neg A) \Leftrightarrow (A \Leftrightarrow C)$

h. $(B \Rightarrow A) \Rightarrow ((A \Rightarrow \neg C) \Rightarrow (\neg C \Rightarrow B))$

1.21 If $A \Rightarrow B$ is T, what can be deduced about the truth values of the following?

a. $A \vee C \Rightarrow B \vee C$

b. $A \wedge C \Rightarrow B \wedge C$

c. $\neg A \wedge B \Leftrightarrow A \vee B$

1.22 What further truth values can be deduced from those shown?

a. $\neg A \vee (A \Rightarrow B)$

F

b. $\neg(A \wedge B) \Leftrightarrow \neg A \Rightarrow \neg B$

T

c. $(\neg A \vee B) \Rightarrow (A \Rightarrow \neg C)$

F

d. $(A \Leftrightarrow B) \Leftrightarrow (C \Rightarrow \neg A)$

F T

1.23 If $A \Leftrightarrow B$ is F, what can be deduced about the truth values of the following?

a. $A \wedge B$

b. $A \vee B$

c. $A \Rightarrow B$

d. $A \wedge C \Leftrightarrow B \wedge C$

1.24 Repeat Exercise 1.23, but assume that $A \Leftrightarrow B$ is T.

1.25 What further truth values can be deduced from those given?

a. $(A \wedge B) \Leftrightarrow (A \vee B)$

F F

b. $(A \Rightarrow \neg B) \Rightarrow (C \Rightarrow B)$

F

1.26 a. Apply Proposition 1.3 when $\mathscr{T}$ is $A_1 \Rightarrow A_1 \vee A_2$, $\mathscr{S}_1$ is $B \wedge D$, and $\mathscr{S}_2$ is $\neg B$.

b. Apply Proposition 1.4 when $\mathscr{B}_1$ is $(B \Rightarrow C) \wedge D$, $\mathscr{B}$ is $B \Rightarrow C$, and $\mathscr{C}$ is $\neg B \vee C$.

1.27 Show that each statement form in column I is logically equivalent to the form next to it in column II.

	I	II	
a.	$A \Rightarrow (B \Rightarrow C)$	$(A \wedge B) \Rightarrow C$	
b.	$A \wedge (B \vee C)$	$(A \wedge B) \vee (A \wedge C)$	(Distributive law)
c.	$A \vee (B \wedge C)$	$(A \vee B) \wedge (A \vee C)$	(Distributive law)
d.	$(A \wedge B) \vee \neg B$	$A \vee \neg B$	
e.	$(A \vee B) \wedge \neg B$	$A \wedge \neg B$	
f.	$A \Rightarrow B$	$\neg B \Rightarrow \neg A$	(Law of the contrapositive)
g.	$A \Leftrightarrow B$	$B \Leftrightarrow A$	(Biconditional commutativity)
h.	$(A \Leftrightarrow B) \Leftrightarrow C$	$A \Leftrightarrow (B \Leftrightarrow C)$	(Biconditional associativity)
i.	$A \Leftrightarrow B$	$(A \wedge B) \vee (\neg A \wedge \neg B)$	
j.	$\neg(A \Leftrightarrow B)$	$A \Leftrightarrow \neg B$	
k.	$\neg(A \vee B)$	$(\neg A) \wedge (\neg B)$	(De Morgan's law)
l.	$\neg(A \wedge B)$	$(\neg A) \vee (\neg B)$	(De Morgan's law)
m.	$A \vee (A \wedge B)$	A	
n.	$A \wedge (A \vee B)$	A	
o.	$A \wedge B$	$B \wedge A$	(Commutativity of conjunction)
p.	$A \vee B$	$B \vee A$	(Commutativity of disjunction)
q.	$(A \wedge B) \wedge C$	$A \wedge (B \wedge C)$	(Associativity of conjunction)
r.	$(A \vee B) \vee C$	$A \vee (B \vee C)$	(Associativity of disjunction)
s.	$A \oplus B$	$B \oplus A$	(Commutativity of exclusive "or")
t.	$A \oplus B) \oplus C$	$A \oplus (B \oplus C)$	(Associativity of exclusive "or")
u.	$A \wedge (B \oplus C)$	$(A \wedge B) \oplus (A \wedge C)$	(Distributive law)

1.28 Show the logical equivalence of the following pairs.

a. $\mathscr{T} \wedge \mathscr{B}$ and $\mathscr{B}$, where $\mathscr{T}$ is a tautology.

b. $\mathscr{T} \vee \mathscr{B}$ and $\mathscr{T}$, where $\mathscr{T}$ is a tautology.

c. $\mathscr{F} \wedge \mathscr{B}$ and $\mathscr{F}$, where $\mathscr{F}$ is contradictory.

d. $\mathscr{F} \vee \mathscr{B}$ and $\mathscr{B}$, where $\mathscr{F}$ is contradictory.

1.29 a. Show the logical equivalence of $\neg(A \Rightarrow B)$ and $A \wedge \neg B$.

b. Show the logical equivalence of $\neg(A \Leftrightarrow B)$ and $(A \wedge \neg B) \vee (\neg A \wedge B)$.

c. For each of the following statement forms, find a statement form that is logically equivalent to its negation and in which negation signs apply only to statement letters.

i. $A \Rightarrow (B \Leftrightarrow \neg C)$

ii. $\neg A \vee (B \Rightarrow C)$

iii. $A \wedge (B \vee \neg C)$

1.30 (*Duality*)

a. If $\mathscr{B}$ is a statement form involving only ¬, ∧, and ∨, and $\mathscr{B}'$ results from $\mathscr{B}$ by replacing each ∧ by ∨ and each ∨ by ∧, show that $\mathscr{B}$ is a tautology if and only if $\neg\mathscr{B}'$ is a tautology. Then prove that, if $\mathscr{B} \Rightarrow \mathscr{C}$ is a tautology, then so is $\mathscr{C}' \Rightarrow \mathscr{B}'$, and if $\mathscr{B} \Leftrightarrow \mathscr{C}$ is a tautology, then so is $\mathscr{B}' \Leftrightarrow \mathscr{C}'$. (Here $\mathscr{C}$ is also assumed to involve only ¬, ∧, and ∨.)

b. Among the logical equivalences in Exercise 1.27, derive (c) from (b), (e) from (d), (l) from (k), (p) from (o), and (r) from (q).

c. If $\mathscr{B}$ is a statement form involving only ¬, ∧, and ∨, and $\mathscr{B}^*$ results from $\mathscr{B}$ by interchanging ∧ and ∨ and replacing every statement letter by its negation, show that $\mathscr{B}^*$ is logically equivalent to $\neg\mathscr{B}$. Find a statement form that is logically equivalent to the negation of $(A \vee B \vee C) \wedge (\neg A \vee \neg B \vee D)$, in which ¬ applies only to statement letters.

1.31 a. Prove that a statement form that contains ⇔ as its only connective is a tautology if and only if each statement letter occurs an even number of times.

b. Prove that a statement form that contains ¬ and ⇔ as its only connectives is a tautology if and only if ¬ and each statement letter occur an even number of times.

1.32 (Shannon, 1938) An electric circuit containing only on–off switches (when a switch is on, it passes current; otherwise it does not) can be represented by a diagram in which, next to each switch, we put a letter representing a necessary and sufficient condition for the switch to be on (see Figure 1.1). The condition that a current flows through this network can be given by the statement form $(A \wedge B) \vee (C \wedge \neg A)$. A statement form representing the circuit shown in Figure 1.2 is $(A \wedge B) \vee ((C \vee A) \wedge \neg B)$, which is logically equivalent to each of the following forms by virtue of the indicated logical equivalence of Exercise 1.27.

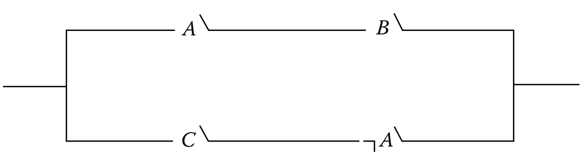

FIGURE 1.1

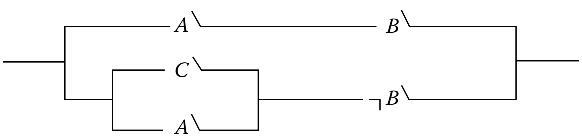

FIGURE 1.2

$$((A \wedge B) \vee (C \vee A)) \wedge ((A \wedge B) \vee \neg B) \quad \text{(c)}$$

$$((A \wedge B) \vee (C \vee A)) \wedge (A \vee \neg B) \quad \text{(d)}$$

$$((A \wedge B) \vee (A \vee C)) \wedge (A \vee \neg B) \quad \text{(p)}$$

$$(((A \wedge B) \vee A) \vee C) \wedge (A \vee \neg B) \quad \text{(r)}$$

$$(A \vee C) \wedge (A \vee \neg B) \quad \text{(p), (m)}$$

$$A \vee (C \wedge \neg B) \quad \text{(c)}$$

Hence, the given circuit is equivalent to the simpler circuit shown in Figure 1.3. (Two circuits are said to be *equivalent* if current flows through one if and only if it flows through the other, and one circuit is *simpler* if it contains fewer switches.)

a. Find simpler equivalent circuits for those shown in Figures 1.4 through 1.6.

b. Assume that each of the three members of a committee votes *yes* on a proposal by pressing a button. Devise as simple a circuit as you can that will allow current to pass when and only when at least two of the members vote in the affirmative.

c. We wish a light to be controlled by two different wall switches in a room in such a way that flicking either one of these switches will turn the light on if it is off and turn it off if it is on. Construct a simple circuit to do the required job.

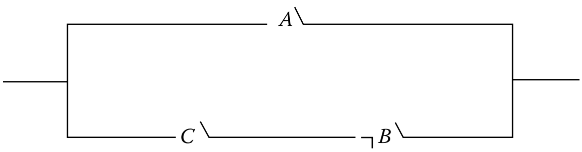

FIGURE 1.3

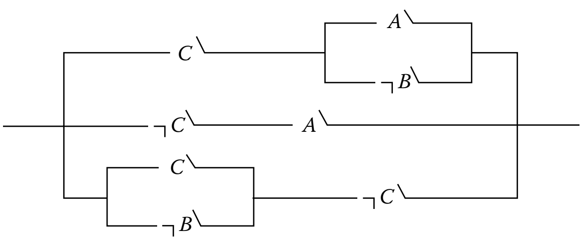

FIGURE 1.4

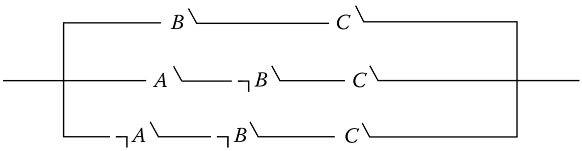

FIGURE 1.5

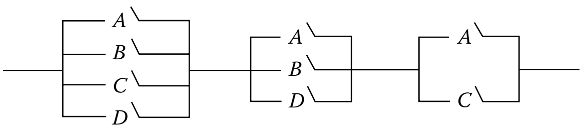

FIGURE 1.6

1.33 Determine whether the following arguments are logically correct by representing each sentence as a statement form and checking whether the conclusion is logically implied by the conjunction of the assumptions. (To do this, assign T to each assumption and F to the conclusion, and determine whether a contradiction results.)

a. If Jones is a communist, Jones is an atheist. Jones is an atheist. Therefore, Jones is a communist.

b. If the temperature and air pressure remained constant, there was no rain. The temperature did remain constant. Therefore, if there was rain, then the air pressure did not remain constant.

c. If Gorton wins the election, then taxes will increase if the deficit will remain high. If Gorton wins the election, the deficit will remain high. Therefore, if Gorton wins the election, taxes will increase.

d. If the number x ends in 0, it is divisible by 5. x does not end in 0. Hence, x is not divisible by 5.

e. If the number x ends in 0, it is divisible by 5. x is not divisible by 5. Hence, x does not end in 0.

f. If $a = 0$ or $b = 0$, then $ab = 0$. But $ab \neq 0$. Hence, $a \neq 0$ and $b \neq 0$.

g. A sufficient condition for f to be integrable is that g be bounded. A necessary condition for h to be continuous is that f is integrable. Hence, if g is bounded or h is continuous, then f is integrable.

h. Smith cannot both be a running star and smoke cigarettes. Smith is not a running star. Therefore, Smith smokes cigarettes.

i. If Jones drove the car, Smith is innocent. If Brown fired the gun, then Smith is not innocent. Hence, if Brown fired the gun, then Jones did not drive the car.

1.34 Which of the following sets of statement forms are satisfiable, in the sense that there is an assignment of truth values to the statement letters that makes all the forms in the set true?

a. $A \Rightarrow B$
   $B \Rightarrow C$
   $C \vee D \Leftrightarrow \neg B$

b. $\neg(\neg B \vee A)$
   $A \vee \neg C$
   $B \Rightarrow \neg C$

c. $D \Rightarrow B$
   $A \vee \neg B$
   $\neg(D \wedge A)$
   D

1.35 Check each of the following sets of statements for consistency by representing the sentences as statement forms and then testing their conjunction to see whether it is contradictory.

a. Either the witness was intimidated or, if Doherty committed suicide, a note was found. If the witness was intimidated, then Doherty did not commit suicide. If a note was found, then Doherty committed suicide.

b. The contract is satisfied if and only if the building is completed by 30 November. The building is completed by 30 November if and only if the electrical subcontractor completes his work by 10 November. The bank loses money if and only if the contract is not satisfied. Yet the electrical subcontractor completes his work by 10 November if and only if the bank loses money.

1.3 Adequate Sets of Connectives

Every statement form containing n statement letters generates a corresponding truth function of n arguments. The arguments and values of the function are T or F. Logically equivalent forms generate the same truth function. A natural question is whether all truth functions are so generated.

Proposition 1.5

Every truth function is generated by a statement form involving the connectives $\neg$, $\wedge$, and $\vee$.

Proof

(Refer to Examples 1 and 2 below for clarification.) Let $f(x_1, \ldots, x_n)$ be a truth function. Clearly f can be represented by a truth table of 2^n rows, where each row represents some assignment of truth values to the variables $x_1, \ldots, x_n$, followed by the corresponding value of $f(x_1, \ldots, x_n)$. If $1 \le i \le 2^n$, let C_i be the conjunction $U_1^i \wedge U_2^i \wedge \ldots \wedge U_n^i$, where U_j^i is A_j if, in the ith row of the truth table, x_j takes the value T, and U_j^i is $\neg A_j$ if x_j takes the value F in that row. Let D be the disjunction of all those C_is such that f has the value T for the ith row of the truth table. (If there are no such rows, then f always takes the value F, and we let D be $A_1 \wedge \neg A_1$, which satisfies the theorem.) Notice that D involves only $\neg$, $\wedge$, and $\vee$. To see that D has f as its corresponding truth function, let there be given an assignment of truth values to the statement letters $A_1, \ldots, A_n$, and assume that the corresponding assignment to the variables $x_1, \ldots, x_n$ is row k of the truth table for f. Then C_k has the value T for this assignment, whereas every other C_i has the value F. If f has the value T for row k, then C_k is a disjunct of D. Hence, D would also have the value T for this assignment. If f has the value F for row k, then C_k is not a disjunct of D and all the disjuncts take the value F for this assignment. Therefore, D would also have the value F. Thus, D generates the truth function f.

Examples

1.

x_1	x_2	$f(x_1, x_2)$
T	T	F
F	T	T
T	F	T
F	F	T

D is $(\neg A_1 \wedge A_2) \vee (A_1 \wedge \neg A_2) \vee (\neg A_1 \wedge \neg A_2)$.

2.

x_1	x_2	x_3	$g(x_1, x_2, x_3)$
T	T	T	T
F	T	T	F
T	F	T	T
F	F	T	T
T	T	F	F
F	T	F	F
T	F	F	F
F	F	F	T

$$D \text{ is } (A_1 \wedge A_2 \wedge A_3) \vee (A_1 \wedge \neg A_2 \wedge A_3) \vee (\neg A_1 \wedge \neg A_2 \wedge A_3) \vee (\neg A_1 \wedge \neg A_2 \wedge \neg A_3).$$

Exercise

1.36 Find statement forms in the connectives ¬, ∧, and ∨ that have the following truth functions.

x_1	x_2	x_3	$f(x_1, x_2, x_3)$	$g(x_1, x_2, x_3)$	$h(x_1, x_2, x_3)$
T	T	T	T	T	F
F	T	T	T	T	T
T	F	T	T	T	F
F	F	T	F	F	F
T	T	F	F	T	T
F	T	F	F	F	T
T	F	F	F	T	F
F	F	F	T	F	T

Corollary 1.6

Every truth function can be generated by a statement form containing as connectives only ∧ and ¬, or only ∨ and ¬, or only ⇒ and ¬.

Proof

Notice that $\mathscr{B} \vee \mathscr{C}$ is logically equivalent to $\neg(\neg\mathscr{B} \wedge \neg\mathscr{C})$. Hence, by the second part of Proposition 1.4, any statement form in ∧, ∨, and ¬ is logically equivalent to a statement form in only ∧ and ¬ [obtained by replacing all expressions $\mathscr{B} \vee \mathscr{C}$ by $\neg(\neg\mathscr{B} \wedge \neg\mathscr{C})$]. The other parts of the corollary are similar consequences of the following tautologies:

$$\mathscr{B} \wedge \mathscr{C} \Leftrightarrow \neg(\neg\mathscr{B} \vee \neg\mathscr{C})$$

$$\mathscr{B} \vee \mathscr{C} \Leftrightarrow (\neg\mathscr{B} \Rightarrow \mathscr{C})$$

$$\mathscr{B} \wedge \mathscr{C} \Leftrightarrow \neg(\mathscr{B} \Rightarrow \neg\mathscr{C})$$

We have just seen that there are certain pairs of connectives—for example, ∧ and ¬—in terms of which all truth functions are definable. It turns out that there is a single connective, ↓ (joint denial), that will do the same job. Its truth table is

A	B	$A \downarrow B$
T	T	F
F	T	F
T	F	F
F	F	T

$A \downarrow B$ is true when and only when neither A nor B is true. Clearly, $\neg A \Leftrightarrow (A \downarrow A)$ and $(A \wedge B) \Leftrightarrow ((A \downarrow A) \downarrow (B \downarrow B))$ are tautologies. Hence, the adequacy of $\downarrow$ for the construction of all truth functions follows from Corollary 1.6.

Another connective, $|$ (alternative denial), is also adequate for this purpose. Its truth table is

A	B	$A \vert B$
T	T	F
F	T	T
T	F	T
F	F	T

$A|B$ is true when and only when not both A and B are true. The adequacy of $|$ follows from the tautologies $\neg A \Leftrightarrow (A|A)$ and $(A \vee B) \Leftrightarrow ((A|A)|(B|B))$.

Proposition 1.7

The only binary connectives that alone are adequate for the construction of all truth functions are $\downarrow$ and $|$.

Proof

Assume that $h(A, B)$ is an adequate connective. Now, if $h(T, T)$ were T, then any statement form built up using h alone would take the value T when all its statement letters take the value T. Hence, $\neg A$ would not be definable in terms of h. So, $h(\text{T}, \text{T}) = \text{F}$. Likewise, $h(\text{F}, \text{F}) = \text{T}$. Thus, we have the partial truth table:

A	B	$h(A, B)$
T	T	F
F	T	
T	F	
F	F	T

If the second and third entries in the last column are F, F or T, T, then h is $\downarrow$ or $|$. If they are F, T, then $h(A, B) \Leftrightarrow \neg B$ is a tautology; and if they are T, F, then $h(A, B) \Leftrightarrow \neg A$ is a tautology. In both cases, h would be definable in terms of $\neg$.

But $\neg$ is not adequate by itself because the only truth functions of one variable definable from it are the identity function and negation itself, whereas the truth function that is always T would not be definable.

Exercises

1.37 Prove that each of the pairs $\Rightarrow, \vee$ and $\neg, \Leftrightarrow$ is not alone adequate to express all truth functions.

1.38 a. Prove that $A \vee B$ can be expressed in terms of $\Rightarrow$ alone.

b. Prove that $A \wedge B$ cannot be expressed in terms of $\Rightarrow$ alone.

c. Prove that $A \Leftrightarrow B$ cannot be expressed in terms of $\Rightarrow$ alone.

1.39 Show that any two of the connectives $\{\wedge, \Rightarrow, \Leftrightarrow\}$ serve to define the remaining one.

1.40 With one variable A, there are four truth functions:

A	$\neg A$	$A \vee \neg A$	$A \wedge \neg A$
T	F	T	F
F	T	T	F

a. With two variable A and B, how many truth functions are there?

b. How many truth functions of n variables are there?

1.41 Show that the truth function h determined by $(A \vee B) \Rightarrow \neg C$ generates all truth functions.

1.42 By a *literal* we mean a statement letter or a negation of a statement letter. A statement form is said to be in *disjunctive normal form* (dnf) if it is a disjunction consisting of one or more disjuncts, each of which is a conjunction of one or more literals—for example, $(A \wedge B) \vee (\neg A \wedge C)$, $(A \wedge B \wedge \neg A) \vee (C \wedge \neg B) \vee (A \wedge \neg C)$, A, $A \wedge B$, and $A \vee (B \vee C)$. A form is in *conjunctive normal form* (cnf) if it is a conjunction of one or more conjuncts, each of which is a disjunction of one or more literals—for example, $(B \vee C) \wedge (A \vee B)$, $(B \vee \neg C) \wedge (A \vee D)$, $A \wedge (B \vee A) \wedge (\neg B \vee A)$, $A \vee \neg B$, $A \wedge B$, A. Note that our terminology considers a literal to be a (degenerate) conjunction and a (degenerate) disjunction.

a. The proof of Proposition 1.5 shows that every statement form $\mathscr{B}$ is logically equivalent to one in disjunctive normal form. By applying this result to $\neg\mathscr{B}$, prove that $\mathscr{B}$ is also logically equivalent to a form in conjunctive normal form.

b. Find logically equivalent dnfs and cnfs for $\neg(A \Rightarrow B) \vee (\neg A \wedge C)$ and $A \Leftrightarrow ((B \wedge \neg A) \vee C)$. [*Hint*: Instead of relying on Proposition 1.5, it is usually easier to use Exercise 1.27(b) and (c).]

c. A dnf (cnf) is called *full* if no disjunct (conjunct) contains two occurrences of literals with the same letter and if a letter that occurs in one disjunct (conjunct) also occurs in all the others. For example, $(A \wedge \neg A \wedge B) \vee (A \wedge B)$, $(B \wedge B \wedge C) \vee (B \wedge C)$ and $(B \wedge C) \vee B$ are not full, whereas $(A \wedge B \wedge \neg C) \vee (A \wedge B \wedge C) \vee (A \wedge \neg B \wedge \neg C)$ and $(A \wedge \neg B) \vee (B \wedge A)$ are full dnfs.

   i. Find full dnfs and cnfs logically equivalent to $(A \wedge B) \vee \neg A$ and $\neg(A \Rightarrow B) \vee (\neg A \wedge C)$.

   ii. Prove that every noncontradictory (nontautologous) statement form $\mathscr{B}$ is logically equivalent to a full dnf (cnf) $\mathscr{C}$, and, if $\mathscr{C}$ contains exactly n letters, then $\mathscr{B}$ is a tautology (is contradictory) if and only if $\mathscr{C}$ has 2^n disjuncts (conjuncts).

d. For each of the following, find a logically equivalent dnf (cnf), and then find a logically equivalent full dnf (cnf):

   i. $(A \vee B) \wedge (\neg B \vee C)$

   ii. $\neg A \vee (B \Rightarrow \neg C)$

   iii. $(A \wedge \neg B) \vee (A \wedge C)$

   iv. $(A \vee B) \Leftrightarrow \neg C$

e. Construct statement forms in $\neg$ and $\wedge$ (respectively, in $\neg$ and $\vee$ or in $\neg$ and $\Rightarrow$) logically equivalent to the statement forms in (d).

1.43 A statement form is said to be *satisfiable* if it is true for some assignment of truth values to its statement letters. The problem of determining the satisfiability of an arbitrary cnf plays an important role in the theory of computational complexity; it is an example of a so-called $\mathscr{NP}$*-complete* problem (see Garey and Johnson, 1978).

a. Show that $\mathscr{B}$ is satisfiable if and only if $\neg\mathscr{B}$ is not a tautology.

b. Determine whether the following are satisfiable:

   i. $(A \vee B) \wedge (\neg A \vee B \vee C) \wedge (\neg A \vee \neg B \vee \neg C)$

   ii. $((A \Rightarrow B) \vee C) \Leftrightarrow (\neg B \wedge (A \vee C))$

c. Given a disjunction $\mathscr{D}$ of four or more literals: $L_1 \vee L_2 \vee \ldots \vee L_n$, let $C_1, \ldots, C_{n-2}$ be statement letters that do not occur in D, and construct the cnf $\mathscr{E}$:

$$(L_1 \vee L_2 \vee C_1) \wedge (\neg C_1 \vee L_3 \vee C_2) \wedge (\neg C_2 \vee L_4 \vee C_3) \wedge \ldots$$
$$\wedge (\neg C_{n-3} \vee L_{n-1} \vee C_{n-2}) \wedge (\neg C_{n-2} \vee L_n \vee \neg C_1)$$

Show that any truth assignment satisfying $\mathscr{D}$ can be extended to a truth assignment satisfying $\mathscr{E}$ and, conversely, any truth assignment satisfying $\mathscr{E}$ is an extension of a truth assignment satisfying $\mathscr{D}$. (This permits the reduction of the problem of satisfying cnfs to the corresponding problem for cnfs with each conjunct containing at most three literals.)

d. For a disjunction $\mathscr{D}$ of three literals $L_1 \vee L_2 \vee L_3$, show that a form that has the properties of $\mathscr{E}$ in (c) cannot be constructed, with $\mathscr{E}$ a cnf in which each conjunct contains at most two literals (R. Cowen).

1.44 (*Resolution*) Let $\mathscr{B}$ be a cnf and let C be a statement letter. If C is a disjunct of a disjunction $\mathscr{D}_1$ in $\mathscr{B}$ and $\neg C$ is a disjunct of another disjunction $\mathscr{D}_2$ in $\mathscr{B}$, then a nonempty disjunction obtained by eliminating C from $\mathscr{D}_1$ and $\neg\mathscr{C}$ from $\mathscr{D}_2$ and forming the disjunction of the remaining literals (dropping repetitions) is said to be obtained from $\mathscr{B}$ by *resolution on C*. For example, if $\mathscr{B}$ is

$$(A \vee \neg C \vee \neg B) \wedge (\neg A \vee D \vee \neg B) \wedge (C \vee D \vee A)$$

the first and third conjuncts yield $A \vee \neg B \vee D$ by resolution on C. In addition, the first and second conjuncts yield $\neg C \vee \neg B \vee D$ by resolution on A, and the second and third conjuncts yield $D \vee \neg B \vee C$ by resolution on A. If we conjoin to $\mathscr{B}$ any new disjunctions obtained by resolution on all variables, and if we apply the same procedure to the new cnf and keep on iterating this operation, the process must eventually stop, and the final result is denoted $\mathscr{Res}(\mathscr{B})$. In the example, $\mathscr{Res}(\mathscr{B})$ is

$$(A \vee \neg C \vee \neg B) \wedge (\neg A \vee D \vee \neg B) \wedge (C \vee D \vee A) \wedge (\neg C \vee \neg B \vee D)$$
$$\wedge (D \vee \neg B \vee C) \wedge (A \vee \neg B \vee D) \wedge (D \vee \neg B)$$

Notice that we have not been careful about specifying the order in which conjuncts or disjuncts are written, since any two arrangements will be logically equivalent.)

a. Find $\mathscr{Res}(\mathscr{B})$ when B is each of the following:

   i. $(A \vee \neg B) \wedge B$

   ii. $(A \vee B \vee C) \wedge (A \vee \neg B \vee C)$

   iii. $(A \vee C) \wedge (\neg A \vee B) \wedge (A \vee \neg C) \wedge (\neg A \vee \neg B)$

b. Show that $\mathscr{B}$ logically implies $\mathscr{Res}(\mathscr{B})$.

c. If $\mathscr{B}$ is a cnf, let $\mathscr{B}_C$ be the cnf obtained from $\mathscr{B}$ by deleting those conjuncts that contain C or $\neg C$. Let $r_C(\mathscr{B})$ be the cnf that is the conjunction of $\mathscr{B}_C$ and all those disjunctions obtained from $\mathscr{B}$ by resolution on C. For example, if $\mathscr{B}$ is the cnf in the example above, then $r_C(\mathscr{B})$ is $(\neg A \vee D \vee \neg B) \wedge (A \vee \neg B \vee D)$. Prove that, if $r_C(\mathscr{B})$ is satisfiable, then so is $\mathscr{B}$ (R. Cowen).

d. A cnf $\mathscr{B}$ is said to be a *blatant contradiction* if it contains some letter C and its negation $\neg C$ as conjuncts. An example of a blatant contradiction is $(A \vee B) \wedge B \wedge (C \vee D) \wedge \neg B$. Prove that if $\mathscr{B}$ is unsatisfiable, then $\mathscr{R}es(\mathscr{B})$ is a blatant contradiction. [*Hint*: Use induction on the number n of letters that occur in $\mathscr{B}$. In the induction step, use (c).]

e. Prove that $\mathscr{B}$ is unsatisfiable if and only if $\mathscr{R}es(\mathscr{B})$ is a blatant contradiction.

1.45 Let $\mathscr{B}$ and $\mathscr{D}$ be statement forms such that $\mathscr{B} \Rightarrow \mathscr{D}$ is a tautology.

a. If $\mathscr{B}$ and $\mathscr{D}$ have no statement letters in common, show that either $\mathscr{B}$ is contradictory or $\mathscr{D}$ is a tautology.

b. (*Craig's interpolation theorem*) If $\mathscr{B}$ and $\mathscr{D}$ have the statement letters $B_1, \ldots, B_n$ in common, prove that there is a statement form $\mathscr{C}$ having $B_1, \ldots, B_n$ as its only statement letters such that $\mathscr{B} \Rightarrow \mathscr{C}$ and $\mathscr{C} \Rightarrow \mathscr{D}$ are tautologies.

c. Solve the special case of (b) in which $\mathscr{B}$ is $(B_1 \Rightarrow A) \wedge (A \Rightarrow B_2)$ and $\mathscr{D}$ is $(B_1 \wedge C) \Rightarrow (B_2 \wedge C)$.

1.46 a. A certain country is inhabited only by *truth-tellers* (people who always tell the truth) and *liars* (people who always lie). Moreover, the inhabitants will respond only to *yes or no* questions. A tourist comes to a fork in a road where one branch leads to the capital and the other does not. There is no sign indicating which branch to take, but there is a native standing at the fork. What yes or no question should the tourist ask in order to determine which branch to take? [*Hint*: Let A stand for "You are a truth-teller" and let B stand for "The left-hand branch leads to the capital." Construct, by means of a suitable truth table, a statement form involving A and B such that the native's answer to the question as to whether this statement form is true will be *yes* when and only when B is true.]

b. In a certain country, there are three kinds of people: *workers* (who always tell the truth), *businessmen* (who always lie), and *students* (who sometimes tell the truth and sometimes lie). At a fork in the road, one branch leads to the capital. A worker, a businessman and a student are standing at the side of the road but are not identifiable in any obvious way. By asking two yes or no questions, find out which fork leads to the capital (Each question may be addressed to any of the three.)

More puzzles of this kind may be found in Smullyan (1978, Chapter 3; 1985, Chapters 2, 4 through 8).

1.4 An Axiom System for the Propositional Calculus

Truth tables enable us to answer many of the significant questions concerning the truth-functional connectives, such as whether a given statement form is a tautology, is contradictory, or neither, and whether it logically implies or is logically equivalent to some other given statement form. The more complex parts of logic we shall treat later cannot be handled by truth tables or by any other similar effective procedure. Consequently, another approach, by means of formal axiomatic theories, will have to be tried. Although, as we have seen, the propositional calculus surrenders completely to the truth table method, it will be instructive to illustrate the axiomatic method in this simple branch of logic.

A formal theory $\mathscr{S}$ is defined when the following conditions are satisfied:

1. A countable set of symbols is given as the symbols of $\mathscr{S}$.* A finite sequence of symbols of $\mathscr{S}$ is called an *expression* of $\mathscr{S}$.
2. There is a subset of the set of expressions of $\mathscr{S}$ called the set of *well-formed formulas* (wfs) of $\mathscr{S}$. There is usually an effective procedure to determine whether a given expression is a wf.
3. There is a set of wfs called the set of *axioms* of $\mathscr{S}$. Most often, one can effectively decide whether a given wf is an axiom; in such a case, $\mathscr{S}$ is called an *axiomatic* theory.
4. There is a finite set $R_1, \ldots, R_n$ of relations among wfs, called *rules of inference*. For each R_i, there is a unique positive integer j such that, for every set of j wfs and each wf $\mathscr{B}$, one can effectively decide whether the given j wfs are in the relation R_i to $\mathscr{B}$, and, if so, $\mathscr{B}$ is said to *follow from* or to be a *direct consequence of* the given wfs by virtue of R_i.†

A *proof* in $\mathscr{S}$ is a sequence $\mathscr{B}_1, \ldots, \mathscr{B}_k$ of wfs such that, for each i, either $\mathscr{B}_i$ is an axiom of $\mathscr{S}$ or $\mathscr{B}_i$ is a direct consequence of some of the preceding wfs in the sequence by virtue of one of the rules of inference of $\mathscr{S}$.

A *theorem* of $\mathscr{S}$ is a wf $\mathscr{B}$ of $\mathscr{S}$ such that $\mathscr{B}$ is the last wf of some proof in $\mathscr{S}$. Such a proof is called a *proof of* $\mathscr{B}$ *in* $\mathscr{S}$.

Even if $\mathscr{S}$ is axiomatic—that is, if there is an effective procedure for checking any given wf to see whether it is an axiom—the notion of "theorem" is not necessarily effective since, in general, there is no effective procedure for determining, given any wf $\mathscr{B}$, whether there is a proof of $\mathscr{B}$. A theory for which there is such an effective procedure is said to be *decidable*; otherwise, the theory is said to be *undecidable*.

* These "symbols" may be thought of as arbitrary objects rather than just linguistic objects. This will become absolutely necessary when we deal with theories with uncountably many symbols in Section 2.12.

† An example of a rule of inference will be the rule *modus ponens* (MP): $\mathscr{C}$ follows from $\mathscr{B}$ and $\mathscr{B} \Rightarrow \mathscr{C}$. According to our precise definition, this rule is the relation consisting of all ordered triples $\langle \mathscr{B}, \mathscr{B} \Rightarrow \mathscr{C}, \mathscr{C} \rangle$, where $\mathscr{B}$ and $\mathscr{C}$ are arbitrary wfs of the formal system.

From an intuitive standpoint, a decidable theory is one for which a machine can be devised to test wfs for theoremhood, whereas, for an undecidable theory, ingenuity is required to determine whether wfs are theorems.

A wf $\mathscr{C}$ is said to be a *consequence* in $\mathscr{S}$ of a set Γ of wfs if and only if there is a sequence $\mathscr{B}_1, \ldots, \mathscr{B}_k$ of wfs such that $\mathscr{C}$ is $\mathscr{B}_k$ and, for each *i*, either $\mathscr{B}_i$ is an axiom or $\mathscr{B}_i$ is in Γ, or $\mathscr{B}_i$ is a direct consequence by some rule of inference of some of the preceding wfs in the sequence. Such a sequence is called a *proof* (or *deduction*) *of* $\mathscr{C}$ *from* Γ. The members of Γ are called the *hypotheses* or *premisses* of the proof. We use $\Gamma \vdash \mathscr{C}$ as an abbreviation for "$\mathscr{C}$ is a consequence of Γ". In order to avoid confusion when dealing with more than one theory, we write $\Gamma \vdash_{\mathscr{S}} \mathscr{C}$, adding the subscript $\mathscr{S}$ to indicate the theory in question.

If Γ is a finite set $\{\mathscr{H}_1, \ldots, \mathscr{H}_m\}$, we write $\mathscr{H}_1, \ldots, \mathscr{H}_m \vdash \mathscr{C}$ instead of $\{\mathscr{H}_1, \ldots, \mathscr{H}_m\} \vdash \mathscr{C}$. If Γ is the empty set ∅, then $\varnothing \vdash \mathscr{C}$ if and only if C is a theorem. It is customary to omit the sign "∅" and simply write $\vdash \mathscr{C}$. Thus, $\vdash \mathscr{C}$ is another way of asserting that $\mathscr{C}$ is a theorem.

The following are simple properties of the notion of consequence:

1. If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \mathscr{C}$, then $\Delta \vdash \mathscr{C}$.
2. $\Gamma \vdash \mathscr{C}$ if and only if there is a finite subset Δ of Γ such that $\Delta \vdash \mathscr{C}$.
3. If $\Delta \vdash \mathscr{C}$, and for each $\mathscr{B}$ in Δ, $\Gamma \vdash \mathscr{B}$, then $\Gamma \vdash \mathscr{C}$.

Assertion 1 represents the fact that if $\mathscr{C}$ is provable from a set Γ of premisses, then, if we add still more premisses, $\mathscr{C}$ is still provable. Half of 2 follows from 1. The other half is obvious when we notice that any proof of $\mathscr{C}$ from Γ uses only a finite number of premisses from Γ. Proposition 1.3 is also quite simple: if $\mathscr{C}$ is provable from premisses in Δ, and each premiss in Δ is provable from premisses in Γ, then $\mathscr{C}$ is provable from premisses in Γ.

We now introduce a formal axiomatic theory L for the propositional calculus.

1. The symbols of L are ¬, ⇒, (,), and the letters A_i with positive integers i as subscripts: $A_1, A_2, A_3, \ldots$. The symbols ¬ and ⇒ are called *primitive connectives*, and the letters A_i are called *statement letters*.

2. a. All statement letters are wfs.
   b. If B and C are wfs, then so are (¬B) and (B ⇒C).* Thus, a wf of L is just a statement form built up from the statement letters A_i by means of the connectives ¬ and ⇒.

3. If $\mathscr{B}$, $\mathscr{C}$, and $\mathscr{D}$ are wfs of L, then the following are axioms of L:

(A1) $(\mathscr{B} \Rightarrow (\mathscr{C} \Rightarrow \mathscr{B}))$

(A2) $((\mathscr{B} \Rightarrow (\mathscr{C} \Rightarrow \mathscr{D})) \Rightarrow ((\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow (\mathscr{B} \Rightarrow \mathscr{D})))$

(A3) $(((\neg \mathscr{C}) \Rightarrow (\neg \mathscr{B})) \Rightarrow (((\neg \mathscr{C}) \Rightarrow \mathscr{B}) \Rightarrow \mathscr{C}))$

* To be precise, we should add the so called extremal clause: (c) an expression is a wf if and only if it can be shown to be a wf on the basis of clauses (a) and (b). This can be made rigorous using as a model the definition of *statement form* in the footnote on page 4.

4. The only rule of inference of L is *modus ponens*: $\mathscr{C}$ is a direct consequence of $\mathscr{B}$ and $(\mathscr{B} \Rightarrow \mathscr{C})$. We shall abbreviate applications of this rule by MP.*

We shall use our conventions for eliminating parentheses.

Notice that the infinite set of axioms of L is given by means of three axiom schemas (A1)–(A3), with each schema standing for an infinite number of axioms. One can easily check for any given wf whether or not it is an axiom; therefore, L is axiomatic. In setting up the system L, it is our intention to obtain as theorems precisely the class of all tautologies.

We introduce other connectives by definition:

$$(\text{D1})\ (\mathscr{B} \wedge \mathscr{C}) \text{ for } \neg(\mathscr{B} \Rightarrow \neg\mathscr{C})$$

$$(\text{D2})\ (\mathscr{B} \vee \mathscr{C}) \text{ for } (\neg\mathscr{B}) \Rightarrow \mathscr{C}$$

$$(\text{D3})\ (\mathscr{B} \Leftrightarrow \mathscr{C}) \text{ for } (\mathscr{B} \Rightarrow \mathscr{C}) \wedge (\mathscr{C} \Rightarrow \mathscr{B})$$

The meaning of (D1), for example, is that, for any wfs $\mathscr{B}$ and $\mathscr{C}$, "$(\mathscr{B} \wedge \mathscr{C})$" is an abbreviation for "$\neg(\mathscr{B} \Rightarrow \neg\mathscr{C})$".

Lemma 1.8: $\vdash_L \mathscr{B} \Rightarrow \mathscr{B}$ for all wfs $\mathscr{B}$.

Proof†

We shall construct a proof in L of $\mathscr{B} \Rightarrow \mathscr{B}$.

1. $(\mathscr{B} \Rightarrow ((\mathscr{B} \Rightarrow \mathscr{B}) \Rightarrow \mathscr{B})) \Rightarrow ((\mathscr{B} \Rightarrow (\mathscr{B} \Rightarrow \mathscr{B})) \Rightarrow (\mathscr{B} \Rightarrow \mathscr{B}))$ Instance of axiom schema (A2)

* A common English synonym for modus ponens is the *detachment rule*.

† The word "proof" is used in two distinct senses. First, it has a precise meaning defined above as a certain kind of finite sequence of wfs of L. However, in another sense, it also designates certain sequences of the English language (supplemented by various technical terms) that are supposed to serve as an argument justifying some assertion about the language L (or other formal theories). In general, the language we are studying (in this case, L) is called the *object language*, while the language in which we formulate and prove statements about the object language is called the *metalanguage*. The metalanguage might also be formalized and made the subject of study, which we would carry out in a metametalanguage, and so on. However, we shall use the English language as our (unformalized) metalanguage, although, for a substantial part of this book, we use only a mathematically weak portion of the English language. The contrast between object language and metalanguage is also present in the study of a foreign language; for example, in a Sanskrit class, Sanskrit is the object language, while the metalanguage, the language we use, is English. The distinction between *proof* and *metaproof* (i.e., a proof in the metalanguage) leads to a distinction between theorems of the object language and *metatheorems* of the metalanguage. To avoid confusion, we generally use "proposition" instead of "metatheorem." The word "metamathematics" refers to the study of logical and mathematical object languages; sometimes the word is restricted to those investigations that use what appear to the metamathematician to be constructive (or so-called finitary) methods.

2. $\mathscr{B} \Rightarrow ((\mathscr{B} \Rightarrow \mathscr{B}) \Rightarrow \mathscr{B})$ — Axiom schema (A1)
3. $(\mathscr{B} \Rightarrow (\mathscr{B} \Rightarrow \mathscr{B})) \Rightarrow (\mathscr{B} \Rightarrow \mathscr{B})$ — From 1 and 2 by MP
4. $\mathscr{B} \Rightarrow (\mathscr{B} \Rightarrow \mathscr{B})$ — Axiom schema (A1)
5. $\mathscr{B} \Rightarrow \mathscr{B}$ — From 3 and 4 by MP*

Exercise

1.47 Prove:

a. $\vdash_L (\neg\mathscr{B} \Rightarrow \mathscr{B}) \Rightarrow \mathscr{B}$
b. $\mathscr{B} \Rightarrow \mathscr{C}, \mathscr{C} \Rightarrow \mathscr{D} \vdash_L \mathscr{B} \Rightarrow \mathscr{D}$
c. $\mathscr{B} \Rightarrow (\mathscr{C} \Rightarrow \mathscr{D}) \vdash_L \mathscr{C} \Rightarrow (\mathscr{B} \Rightarrow \mathscr{D})$
d. $\vdash_L (\neg\mathscr{C} \Rightarrow \neg\mathscr{B}) \Rightarrow (\mathscr{B} \Rightarrow \mathscr{C})$

In mathematical arguments, one often proves a statement $\mathscr{C}$ on the assumption of some other statement $\mathscr{B}$ and then concludes that "if $\mathscr{B}$, then $\mathscr{C}$" is true. This procedure is justified for the system L by the following theorem.

Proposition 1.9 (Deduction Theorem)†

If Γ is a set of wfs and $\mathscr{B}$ and $\mathscr{C}$ are wfs, and $\Gamma, \mathscr{B} \vdash \mathscr{C}$, then $\Gamma \vdash \mathscr{B} \Rightarrow \mathscr{C}$. In particular, if $\mathscr{B} \vdash \mathscr{C}$, then $\vdash \mathscr{B} \Rightarrow \mathscr{C}$ (Herbrand, 1930).

Proof

Let $\mathscr{C}_1, \ldots, \mathscr{C}_n$ be a proof of $\mathscr{C}$ from $\Gamma \cup \{\mathscr{B}\}$, where $\mathscr{C}_n$ is $\mathscr{C}$. Let us prove, by induction on j, that $\Gamma \vdash \mathscr{B} \Rightarrow \mathscr{C}_j$ for $1 \leq j \leq n$. First of all, $\mathscr{C}_1$ must be either in Γ or an axiom of L or $\mathscr{B}$ itself. By axiom schema (A1), $\mathscr{C}_1 \Rightarrow (\mathscr{B} \Rightarrow \mathscr{C}_1)$ is an axiom. Hence, in the first two cases, by MP, $\Gamma \vdash \mathscr{B} \Rightarrow \mathscr{C}_1$. For the third case, when $\mathscr{C}_1$ is $\mathscr{B}$, we have $\vdash \mathscr{B} \Rightarrow \mathscr{C}_1$ by Lemma 1.8, and, therefore, $\Gamma \vdash \mathscr{B} \Rightarrow \mathscr{C}_1$. This takes care of the case $j = 1$. Assume now that $\Gamma \vdash \mathscr{B} \Rightarrow \mathscr{C}_k$ for all $k < j$. Either $\mathscr{C}_j$ is an axiom, or $\mathscr{C}_j$ is in Γ, or $\mathscr{C}_j$ is $\mathscr{B}$, or $\mathscr{C}_j$ follows by modus ponens from some $\mathscr{C}_\ell$ and $\mathscr{C}_m$, where $\ell < j$, $m < j$, and $\mathscr{C}_m$ has the form $\mathscr{C}_\ell \Rightarrow \mathscr{C}_j$. In the first three cases, $\Gamma \vdash \mathscr{B} \Rightarrow \mathscr{C}_j$ as in the case $j = 1$ above. In the last case, we have, by inductive hypothesis, $\Gamma \vdash \mathscr{B} \Rightarrow \mathscr{C}_\ell$ and $\Gamma \vdash \mathscr{B} \Rightarrow (\mathscr{C}_\ell \Rightarrow \mathscr{C}_j)$. But, by axiom schema (A2), $\vdash (\mathscr{B} \Rightarrow (\mathscr{C}_\ell \Rightarrow \mathscr{C}_j)) \Rightarrow ((\mathscr{B} \Rightarrow \mathscr{C}_\ell) \Rightarrow (\mathscr{B} \Rightarrow \mathscr{C}_j))$. Hence, by MP, $\Gamma \vdash (\mathscr{B} \Rightarrow \mathscr{C}_\ell) \Rightarrow (\mathscr{B} \Rightarrow \mathscr{C}_j)$, and, again by MP, $\Gamma \vdash \mathscr{B} \Rightarrow \mathscr{C}_j$. Thus, the proof by induction is complete. The case $j = n$ is the desired result. [Notice that, given a deduction of $\mathscr{C}$ from Γ and

* The reader should not be discouraged by the apparently unmotivated step 1 of the proof. As in most proofs, we actually begin with the desired result, $\mathscr{B} \Rightarrow \mathscr{B}$, and then look for an appropriate axiom that may lead by MP to that result. A mixture of ingenuity and experimentation leads to a suitable instance of axiom (A2).

† For the remainder of the chapter, unless something is said to the contrary, we shall omit the subscript L in $\vdash_L$. In addition, we shall use $\Gamma, \mathscr{B} \vdash \mathscr{C}$ to stand for $\Gamma \cup \{\mathscr{B}\} \vdash \mathscr{C}$. In general, we let $\Gamma, \mathscr{B}_1, \ldots, \mathscr{B}_n \vdash \mathscr{C}$ stand for $\Gamma \cup \{\mathscr{B}_1, \ldots, \mathscr{B}_n\} \vdash \mathscr{C}$.

$\mathscr{B}$, the proof just given enables us to construct a deduction of $\mathscr{B} \Rightarrow \mathscr{C}$ from Γ. Also note that axiom schema (A3) was not used in proving the deduction theorem.]

Corollary 1.10

a. $\mathscr{B} \Rightarrow \mathscr{C}, \mathscr{C} \Rightarrow \mathscr{D} \vdash \mathscr{B} \Rightarrow \mathscr{D}$
b. $\mathscr{B} \Rightarrow (\mathscr{C} \Rightarrow \mathscr{D}), \mathscr{C} \vdash \mathscr{B} \Rightarrow \mathscr{D}$

Proof

For part (a):

1.	$\mathscr{B} \Rightarrow \mathscr{C}$	Hyp (abbreviation for "hypothesis")
2.	$\mathscr{C} \Rightarrow \mathscr{D}$	Hyp
3.	$\mathscr{B}$	Hyp
4.	$\mathscr{C}$	1, 3, MP
5.	$\mathscr{D}$	2, 4, MP

Thus, $\mathscr{B} \Rightarrow \mathscr{C}, \mathscr{C} \Rightarrow \mathscr{D}, \mathscr{B} \vdash \mathscr{D}$. So, by the deduction theorem, $\mathscr{B} \Rightarrow \mathscr{C}, \mathscr{C} \Rightarrow \mathscr{D} \vdash \mathscr{B} \Rightarrow \mathscr{D}$.

To prove (b), use the deduction theorem.

Lemma 1.11

For any wfs $\mathscr{B}$ and $\mathscr{C}$, the following wfs are theorems of L.

a. $\neg\neg\mathscr{B} \Rightarrow \mathscr{B}$
b. $\mathscr{B} \Rightarrow \neg\neg\mathscr{B}$
c. $\neg\mathscr{B} \Rightarrow (\mathscr{B} \Rightarrow \mathscr{C})$
d. $(\neg\mathscr{C} \Rightarrow \neg\mathscr{B}) \Rightarrow (\mathscr{B} \Rightarrow \mathscr{C})$
e. $(\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow (\neg\mathscr{C} \Rightarrow \neg\mathscr{B})$
f. $\mathscr{B} \Rightarrow (\neg\mathscr{C} \Rightarrow \neg(\mathscr{B} \Rightarrow \mathscr{C}))$
g. $(\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow ((\neg\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow \mathscr{C})$

Proof

a. $\vdash \neg\neg\mathscr{B} \Rightarrow \mathscr{B}$

1.	$(\neg\mathscr{B} \Rightarrow \neg\neg\mathscr{B}) \Rightarrow ((\neg\mathscr{B} \Rightarrow \neg\mathscr{B}) \Rightarrow \mathscr{B})$	Axiom (A3)
2.	$\neg\mathscr{B} \Rightarrow \neg\mathscr{B}$	Lemma 1.8*

* Instead of writing a complete proof of $\neg\mathscr{B} \Rightarrow \neg\mathscr{B}$, we simply cite Lemma 1.8. In this way, we indicate how the proof of $\neg\neg\mathscr{B} \Rightarrow \mathscr{B}$ could be written if we wished to take the time and space to do so. This is, of course, nothing more than the ordinary application of previously proved theorems.

3. $(\neg\mathscr{B} \Rightarrow \neg\neg\mathscr{B}) \Rightarrow \mathscr{B}$	1, 2, Corollary 1.10(b)
4. $\neg\neg\mathscr{B} \Rightarrow (\neg\mathscr{B} \Rightarrow \neg\neg\mathscr{B})$	Axiom (A1)
5. $\neg\neg\mathscr{B} \Rightarrow \mathscr{B}$	3, 4, Corollary 1.10(a)

b. $\vdash \mathscr{B} \Rightarrow \neg\neg\mathscr{B}$

1. $(\neg\neg\neg\mathscr{B} \Rightarrow \neg\mathscr{B}) \Rightarrow ((\neg\neg\neg\mathscr{B} \Rightarrow \mathscr{B}) \Rightarrow \neg\neg\mathscr{B})$	Axiom (A3)
2. $\neg\neg\neg\mathscr{B} \Rightarrow \neg\mathscr{B}$	Part (a)
3. $(\neg\neg\neg\mathscr{B} \Rightarrow \mathscr{B}) \Rightarrow \neg\neg\mathscr{B}$	1, 2, MP
4. $\mathscr{B} \Rightarrow (\neg\neg\neg\mathscr{B} \Rightarrow \mathscr{B})$	Axiom (A1)
5. $\mathscr{B} \Rightarrow \neg\neg\mathscr{B}$	3, 4, Corollary 1.10(a)

c. $\vdash \neg\mathscr{B} \Rightarrow (\mathscr{B} \Rightarrow \mathscr{C})$

1. $\neg\mathscr{B}$	Hyp
2. $\mathscr{B}$	Hyp
3. $\mathscr{B} \Rightarrow (\neg\mathscr{C} \Rightarrow \mathscr{B})$	Axiom (A1)
4. $\neg\mathscr{B} \Rightarrow (\neg\mathscr{C} \Rightarrow \neg\mathscr{B})$	Axiom (A1)
5. $\neg\mathscr{C} \Rightarrow \mathscr{B}$	2, 3, MP
6. $\neg\mathscr{C} \Rightarrow \neg\mathscr{B}$	1, 4, MP
7. $(\neg\mathscr{C} \Rightarrow \neg\mathscr{B}) \Rightarrow ((\neg\mathscr{C} \Rightarrow \mathscr{B}) \Rightarrow \mathscr{C})$	Axiom (A3)
8. $(\neg\mathscr{C} \Rightarrow \mathscr{B}) \Rightarrow \mathscr{C}$	6, 7, MP
9. $\mathscr{C}$	5, 8, MP
10. $\neg\mathscr{B}, \mathscr{B} \vdash \mathscr{C}$	1–9
11. $\neg\mathscr{B} \vdash \mathscr{B} \Rightarrow \mathscr{C}$	10, deduction theorem
12. $\vdash \neg\mathscr{B} \Rightarrow (\mathscr{B} \Rightarrow \mathscr{C})$	11, deduction theorem

d. $\vdash (\neg\mathscr{C} \Rightarrow \neg\mathscr{B}) \Rightarrow (\mathscr{B} \Rightarrow \mathscr{C})$

1. $\neg\mathscr{C} \Rightarrow \neg\mathscr{B}$	Hyp
2. $(\neg\mathscr{C} \Rightarrow \neg\mathscr{B}) \Rightarrow ((\neg\mathscr{C} \Rightarrow \mathscr{B}) \Rightarrow \mathscr{C})$	Axiom (A3)
3. $\mathscr{B} \Rightarrow (\neg\mathscr{C} \Rightarrow \mathscr{B})$	Axiom (A1)
4. $(\neg\mathscr{C} \Rightarrow \mathscr{B}) \Rightarrow \mathscr{C}$	1, 2, MP
5. $\mathscr{B} \Rightarrow \mathscr{C}$	3, 4, Corollary 1.10(a)
6. $\neg\mathscr{C} \Rightarrow \neg\mathscr{B} \vdash \mathscr{B} \Rightarrow \mathscr{C}$	1–5
7. $\vdash (\neg\mathscr{C} \Rightarrow \neg\mathscr{B}) \Rightarrow (\mathscr{B} \Rightarrow \mathscr{C})$	6, deduction theorem

e. $\vdash (\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow (\neg\mathscr{C} \Rightarrow \neg\mathscr{B})$

1. $\mathscr{B} \Rightarrow \mathscr{C}$	Hyp
2. $\neg\neg\mathscr{B} \Rightarrow \mathscr{B}$	Part (a)
3. $\neg\neg\mathscr{B} \Rightarrow \mathscr{C}$	1, 2, Corollary 1.10(a)
4. $\mathscr{C} \Rightarrow \neg\neg\mathscr{C}$	Part (b)

5. $\neg\neg\mathscr{B} \Rightarrow \neg\neg\mathscr{C}$	3, 4, Corollary 1.10(a)
6. $(\neg\neg\mathscr{B} \Rightarrow \neg\neg\mathscr{C}) \Rightarrow (\neg\mathscr{C} \Rightarrow \neg\mathscr{B})$	Part (d)
7. $\neg\mathscr{C} \Rightarrow \neg\mathscr{B}$	5, 6, MP
8. $\mathscr{B} \Rightarrow \mathscr{C} \vdash \neg\mathscr{C} \Rightarrow \neg\mathscr{B}$	1–7
9. $\vdash (\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow (\neg\mathscr{C} \Rightarrow \neg\mathscr{B})$	8, deduction theorem

f. $\vdash \mathscr{B} \Rightarrow (\neg\mathscr{C} \Rightarrow \neg(\mathscr{B} \Rightarrow \mathscr{C}))$

Clearly, $\mathscr{B}, \mathscr{B} \Rightarrow \mathscr{C} \vdash \mathscr{C}$ by MP. Hence, $\vdash \mathscr{B} \Rightarrow ((\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow \mathscr{C}))$ by two uses of the deduction theorem. Now, by (e), $\vdash((\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow \mathscr{C}) \Rightarrow (\neg\mathscr{C} \Rightarrow \neg(\mathscr{B} \Rightarrow \mathscr{C}))$. Hence, by Corollary 1.10(a), $\vdash \mathscr{B} \Rightarrow (\neg\mathscr{C} \Rightarrow \neg(\mathscr{B} \Rightarrow \mathscr{C}))$.

g. $\vdash (\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow ((\neg\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow \mathscr{C})$

1. $\mathscr{B} \Rightarrow \mathscr{C}$	Hyp
2. $\neg\mathscr{B} \Rightarrow \mathscr{C}$	Hyp
3. $(\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow (\neg\mathscr{C} \Rightarrow \neg\mathscr{B})$	Part (e)
4. $\neg\mathscr{C} \Rightarrow \neg\mathscr{B}$	1, 3, MP
5. $(\neg\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow (\neg\mathscr{C} \Rightarrow \neg\neg\mathscr{B})$	Part (e)
6. $\neg\mathscr{C} \Rightarrow \neg\neg\mathscr{B}$	2, 5, MP
7. $(\neg\mathscr{C} \Rightarrow \neg\neg\mathscr{B}) \Rightarrow ((\neg\mathscr{C} \Rightarrow \neg\mathscr{B}) \Rightarrow \mathscr{C})$	Axiom (A3)
8. $(\neg\mathscr{C} \Rightarrow \neg\mathscr{B}) \Rightarrow \mathscr{C}$	6, 7, MP
9. $\mathscr{C}$	4, 8, MP
10. $\mathscr{B} \Rightarrow \mathscr{C}, \neg\mathscr{B} \Rightarrow \mathscr{C} \vdash \mathscr{C}$	1–9
11. $\mathscr{B} \Rightarrow \mathscr{C} \vdash (\neg\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow \mathscr{C}$	10, deduction theorem
12. $\vdash(\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow ((\neg\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow \mathscr{C})$	11, deduction theorem

Exercises

1.48 Show that the following wfs are theorems of L.

a. $\mathscr{B} \Rightarrow (\mathscr{B} \vee \mathscr{C})$

b. $\mathscr{B} \Rightarrow (\mathscr{C} \vee \mathscr{B})$

c. $\mathscr{C} \vee \mathscr{B} \Rightarrow \mathscr{B} \vee \mathscr{C}$

d. $\mathscr{B} \wedge \mathscr{C} \Rightarrow \mathscr{B}$

e. $\mathscr{B} \wedge \mathscr{C} \Rightarrow \mathscr{C}$

f. $(\mathscr{B} \Rightarrow \mathscr{D}) \Rightarrow ((\mathscr{C} \Rightarrow \mathscr{D}) \Rightarrow (\mathscr{B} \vee \mathscr{C} \Rightarrow \mathscr{D}))$

g. $((\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow \mathscr{B}) \Rightarrow \mathscr{B}$

h. $\mathscr{B} \Rightarrow (\mathscr{C} \Rightarrow (\mathscr{B} \wedge \mathscr{C}))$

1.49 Exhibit a complete proof in L of Lemma 1.11(c). [*Hint*: Apply the procedure used in the proof of the deduction theorem to the demonstration given earlier of Lemma 1.11(c).] Greater fondness for the deduction theorem will result if the reader tries to prove all of Lemma 1.11 without using the deduction theorem.

It is our purpose to show that a wf of L is a theorem of L if and only if it is a tautology. Half of this is very easy.

Proposition 1.12

Every theorem of L is a tautology.

Proof

As an exercise, verify that all the axioms of L are tautologies. By Proposition 1.2, modus ponens leads from tautologies to other tautologies. Hence, every theorem of L is a tautology.

The following lemma is to be used in the proof that every tautology is a theorem of L.

Lemma 1.13

Let $\mathscr{B}$ be a wf and let $B_1, \ldots, B_k$ be the statement letters that occur in $\mathscr{B}$. For a given assignment of truth values to $B_1, \ldots, B_k$, let B_j' be B_j if B_j takes the value T; and let B_j' be $\neg B_j$ if B_j takes the value F. Let $\mathscr{B}'$ be $\mathscr{B}$ if $\mathscr{B}$ takes the value T under the assignment, and let $\mathscr{B}'$ be $\neg\mathscr{B}$ if B takes the value F. Then $B_1', \ldots, B_k' \vdash \mathscr{B}'$.

For example, let $\mathscr{B}$ be $\neg(\neg A_2 \Rightarrow A_5)$. Then for each row of the truth table

A_2	A_5	$\neg(\neg A_2 \Rightarrow A_5)$
T	T	F
F	T	F
T	F	F
F	F	T

Lemma 1.13 asserts a corresponding deducibility relation. For instance, corresponding to the third row there is $A_2, \neg A_5 \vdash \neg\neg(\neg A_2 \Rightarrow A_5)$, and to the fourth row, $\neg A_2, \neg A_5 \vdash \neg(\neg A_2 \Rightarrow A_5)$.

Proof

The proof is by induction on the number n of occurrences of $\neg$ and $\Rightarrow$ in $\mathscr{B}$. (We assume $\mathscr{B}$ written without abbreviations.) If $n = 0$, $\mathscr{B}$ is just a statement letter B_1, and then the lemma reduces to $B_1 \vdash B_1$ and $\neg B_1 \vdash \neg B_1$. Assume now that the lemma holds for all $j < n$.

Case 1: $\mathscr{B}$ is $\neg\mathscr{C}$. Then $\mathscr{C}$ has fewer than n occurrences of $\neg$ and $\Rightarrow$.

Subcase 1a: Let $\mathscr{C}$ take the value T under the given truth value assignment. Then $\mathscr{B}$ takes the value F. So, $\mathscr{C}'$ is $\mathscr{C}$ and $\mathscr{B}'$ is $\neg\mathscr{B}$. By the inductive hypothesis applied to $\mathscr{C}$, we have $B_1', \ldots, B_k' \vdash \mathscr{C}$. Then, by Lemma 1.11(b) and MP, $B_1', \ldots, B_k' \vdash \neg\neg\mathscr{C}$. But $\neg\neg\mathscr{C}$ is $\mathscr{B}'$.

Subcase 1b: Let $\mathscr{C}$ take the value F. Then $\mathscr{B}$ takes the value T. So, $\mathscr{C}'$ is $\neg\mathscr{C}$ and $\mathscr{B}'$ is $\mathscr{B}$, By inductive hypothesis, $B_1', \ldots, B_k' \vdash \neg\mathscr{C}$. But $\neg\mathscr{C}$ is $\mathscr{B}'$.

Case 2: $\mathscr{B}$ is $\mathscr{C} \Rightarrow \mathscr{D}$. Then $\mathscr{C}$ and $\mathscr{D}$ have fewer occurrences of $\neg$ and $\Rightarrow$ than $\mathscr{B}$. So, by inductive hypothesis, $B_1', \ldots, B_k' \vdash \neg\mathscr{C}'$ and $B_1', \ldots, B_k' \vdash \neg\mathscr{D}'$.

Subcase 2a: $\mathscr{C}$ takes the value F. Then $\mathscr{B}$ takes the value T. So, $\mathscr{C}'$ is $\neg\mathscr{C}$ and $\mathscr{B}'$ is $\mathscr{B}$. Hence, $B_1', \ldots, B_k' \vdash \neg\mathscr{C}$. By Lemma 1.11(c) and MP, $B_1', \ldots, B_k' \vdash \mathscr{C} \Rightarrow \mathscr{D}$. But $\mathscr{C} \Rightarrow \mathscr{D}$ is $\mathscr{B}'$.

Subcase 2b: $\mathscr{D}$ takes the value T. Then $\mathscr{B}$ takes the value T. So, $\mathscr{D}'$ is $\mathscr{D}$ and $\mathscr{B}'$ is $\mathscr{B}$. Hence, $B_1', \ldots, B_k' \vdash \mathscr{D}$. Then, by axiom (A1) and MP, $B_1', \ldots, B_k' \vdash \mathscr{C} \Rightarrow \mathscr{D}$. But $\mathscr{C} \Rightarrow \mathscr{D}$ is $\mathscr{B}'$.

Subcase 2c: $\mathscr{C}$ takes the value T and $\mathscr{D}$ takes the value F. Then $\mathscr{B}$ takes the value F. So, $\mathscr{C}'$ is $\mathscr{C}$, $\mathscr{D}'$ is $\neg\mathscr{D}$, and $\mathscr{B}'$ is $\neg\mathscr{B}$. Therefore, $B_1', \ldots, B_k' \vdash \mathscr{C}$ and $B_1', \ldots, B_k' \vdash \neg\mathscr{D}$. Hence, by Lemma 1.11(f) and MP, $B_1', \ldots, B_k' \vdash \neg\left(\mathscr{C} \Rightarrow \mathscr{D}\right)$. But $\neg(\mathscr{C} \Rightarrow \mathscr{D})$ is $\mathscr{B}'$.

Proposition 1.14 (Completeness Theorem)

If a wf $\mathscr{B}$ of L is a tautology, then it is a theorem of L.

Proof

(Kalmár, 1935) Assume $\mathscr{B}$ is a tautology, and let $B_1, \ldots, B_k$ be the statement letters in B. For any truth value assignment to $B_1, \ldots, B_k$, we have, by Lemma 1.13, $B_1', \ldots, B_k' \vdash \mathscr{B}$. ($\mathscr{B}'$ is $\mathscr{B}$ because $\mathscr{B}$ always takes the value T.) Hence, when B_k' is given the value T, we obtain $B_1', \ldots, B_{k-1}', B_k \vdash \mathscr{B}$, and, when B_k is given the value F, we obtain $B_1', \ldots, B_{k-1}', \neg B_k \vdash \mathscr{B}$. So, by the deduction theorem, $B_1', \ldots, B_{k-1}' \vdash \mathrm{B}_k \Rightarrow \mathscr{B}$ and $B_1', \ldots, B_{k-1}', \vdash \neg B_k \Rightarrow \mathscr{B}$. Then by Lemma 1.11(g) and MP, $B_1', \ldots, B_{k-1}' \vdash \mathscr{B}$. Similarly, B_{k-1} may be chosen to be T or F and, again applying the deduction theorem, Lemma 1.11(g) and MP, we can eliminate B_{k-1}' just as we eliminated B_k'. After k such steps, we finally obtain $\vdash \mathscr{B}$.

Corollary 1.15

If $\mathscr{C}$ is an expression involving the signs $\neg$, $\Rightarrow$, $\wedge$, $\vee$, and $\Leftrightarrow$ that is an abbreviation for a wf $\mathscr{B}$ of L, then $\mathscr{C}$ is a tautology if and only if $\mathscr{B}$ is a theorem of L.

Proof

In definitions (D1)–(D3), the abbreviating formulas replace wfs to which they are logically equivalent. Hence, by Proposition 1.4, $\mathscr{B}$ and $\mathscr{C}$ are logically equivalent, and $\mathscr{C}$ is a tautology if and only if B is a tautology. The corollary now follows from Propositions 1.12 and 1.14.

Corollary 1.16

The system L is consistent; that is, there is no wf $\mathscr{B}$ such that both $\mathscr{B}$ and $\neg\mathscr{B}$ are theorems of L.

Proof

By Proposition 1.12, every theorem of L is a tautology. The negation of a tautology cannot be a tautology and, therefore, it is impossible for both $\mathscr{B}$ and $\neg\mathscr{B}$ to be theorems of L.

Notice that L is consistent if and only if not all wfs of L are theorems. In fact, if L is consistent, then there are wfs that are not theorems (e.g., the negations of theorems). On the other hand, by Lemma 1.11(c), $\vdash_L \neg\mathscr{B} \Rightarrow (\mathscr{B} \Rightarrow \mathscr{C})$, and so, if L were inconsistent, that is, if some wf $\mathscr{B}$ and its negation $\neg\mathscr{B}$ were provable, then by MP any wf $\mathscr{C}$ would be provable. (This equivalence holds for any theory that has *modus ponens* as a rule of inference and in which Lemma 1.11(c) is provable.) A theory in which not all wfs are theorems is said to be *absolutely consistent*, and this definition is applicable even to theories that do not contain a negation sign.

Exercise

1.50 Let $\mathscr{B}$ be a statement form that is not a tautology. Let L^+ be the formal theory obtained from L by adding as new axioms all wfs obtainable from $\mathscr{B}$ by substituting arbitrary statement forms for the statement letters in $\mathscr{B}$, with the same form being substituted for all occurrences of a statement letter. Show that L^+ is inconsistent.

1.5 Independence: Many-Valued Logics

A subset Y of the set of axioms of a theory is said to be *independent* if some wf in Y cannot be proved by means of the rules of inference from the set of those axioms not in Y.

Proposition 1.17

Each of the axiom schemas (A1)–(A3) is independent.

Proof

To prove the independence of axiom schema (A1), consider the following tables:

A	$\neg A$	A	B	$A \Rightarrow B$
0	1	0	0	0
1	1	1	0	2
2	0	2	0	0
		0	1	2
		1	1	2
		2	1	0
		0	2	2
		1	2	0
		2	2	0

For any assignment of the values 0, 1, and 2 to the statement letters of a wf $\mathscr{B}$, these tables determine a corresponding value of $\mathscr{B}$. If $\mathscr{B}$ always takes the value 0, $\mathscr{B}$ is called *select*. Modus ponens preserves selectness, since it is easy to check that, if $\mathscr{B}$ and $\mathscr{B} \Rightarrow \mathscr{C}$ are select, so is $\mathscr{C}$. One can also verify that all instances of axiom schemas (A2) and (A3) are select. Hence, any wf derivable from (A2) and (A3) by modus ponens is select. However, $A_1 \Rightarrow (A_2 \Rightarrow A_1)$, which is an instance of (A1), is not select, since it takes the value 2 when A_1 is 1 and A_2 is 2.

To prove the independence of axiom schema (A2), consider the following tables:

A	$\neg A$	A	B	$A \Rightarrow B$
0	1	0	0	0
1	0	1	0	0
2	1	2	0	0
		0	1	2
		1	1	2
		2	1	0
		0	2	1
		1	2	0
		2	2	0

Let us call a wf that always takes the value 0 according to these tables *grotesque*. Modus ponens preserves grotesqueness and it is easy to verify that all instances of (A1) and (A3) are grotesque. However, the instance $(A_1 \Rightarrow (A_2 \Rightarrow A_3)) \Rightarrow ((A_1 \Rightarrow A_2) \Rightarrow (A_1 \Rightarrow A_3))$ of (A2) takes the value 2 when A_1 is 0, A_2 is 0, and A_3 is 1 and, therefore, is not grotesque.

The following argument proves the independence of (A3). Let us call a wf $\mathscr{B}$ *super* if the wf $h(\mathscr{B})$ obtained by erasing all negation signs in $\mathscr{B}$ is a tautology. Each instance of axiom schemas (A1) and (A2) is super. Also, modus ponens preserves the property of being super; for if $h(\mathscr{B} \Rightarrow \mathscr{C})$ and $h(\mathscr{B})$ are tautologies, then $h(\mathscr{C})$ is a tautology. (Just note that $h(\mathscr{B} \Rightarrow \mathscr{C})$ is $h(\mathscr{B}) \Rightarrow h(\mathscr{C})$ and use Proposition 1.2.) Hence, every wf $\mathscr{B}$ derivable from (A1) and (A2) by modus ponens is super. But $h((\neg A_1 \Rightarrow \neg A_1) \Rightarrow ((\neg A_1 \Rightarrow A_1) \Rightarrow A_1))$ is $(A_1 \Rightarrow A_1) \Rightarrow ((A_1 \Rightarrow A_1) \Rightarrow A_1)$, which is not a tautology. Therefore, $(\neg A_1 \Rightarrow \neg A_1) \Rightarrow ((\neg A_1 \Rightarrow A_1) \Rightarrow A_1)$, an instance of (A3), is not super and is thereby not derivable from (A1) and (A2) by modus ponens.

The idea used in the proof of the independence of axiom schemas (A1) and (A2) may be generalized to the notion of a *many-valued logic*. Select a positive integer n, call the numbers $0, 1, \ldots, n$ *truth values*, and choose a number m such that $0 \le m < n$. The numbers $0, 1, \ldots, m$ are called *designated values*. Take a finite number of "truth tables" representing functions from sets of the form $\{0, 1, \ldots, n\}^k$ into $\{0, 1, \ldots, n\}$. For each truth table, introduces a sign, called the corresponding *connective*. Using these connectives and statement letters, we may construct "statement forms," and every such statement form containing j distinct letters determines a "truth function" from $\{0, 1, \ldots, n\}^j$ into $\{0, 1, \ldots, n\}$. A statement form whose corresponding truth function takes only designated values is said to be *exceptional*. The numbers m and n and the basic truth tables are said to define a (finite) *many-valued logic* M. A formal theory involving statement letters and the connectives of M is said to be *suitable* for M if and only if the theorems of the theory coincide with the exceptional statement forms of M. All these notions obviously can be generalized to the case of an infinite number of truth values. If $n = 1$ and $m = 0$ and the truth tables are those given for $\neg$ and $\Rightarrow$ in Section 1.1, then the corresponding two-valued logic is that studied in this chapter. The exceptional wfs in this case were called tautologies. The system L is suitable for this logic, as proved in Propositions 1.12 and 1.14. In the proofs of the independence of axiom schemas (A1) and (A2), two three-valued logics were used.

Exercises

1.51 Prove the independence of axiom schema (A3) by constructing appropriate "truth tables" for $\neg$ and $\Rightarrow$.

1.52 (McKinsey and Tarski, 1948) Consider the axiomatic theory P in which there is exactly one binary connective $*$, the only rule of inference is modus ponens (that is, $\mathscr{C}$ follows from $\mathscr{B}$ and $\mathscr{B} * \mathscr{C}$), and the axioms are all wfs of the form $\mathscr{B} * \mathscr{B}$. Show that P is not suitable for any (finite) many-valued logic.

1.53 For any (finite) many-valued logic M, prove that there is an axiomatic theory suitable for M.

Further information about many-valued logics can be found in Rosser and Turquette (1952), Rescher (1969), Bolc and Borowik (1992), and Malinowski (1993).

1.6 Other Axiomatizations

Although the axiom system L is quite simple, there are many other systems that would do as well. We can use, instead of ¬ and ⇒, any collection of primitive connectives as long as these are adequate for the definition of all other truth-functional connectives.

Examples

L_1: ∨ and ¬ are the primitive connectives. We use $\mathscr{B} \Rightarrow \mathscr{C}$ as an abbreviation for $\neg\mathscr{B} \vee \mathscr{C}$. We have four axiom schemas: (1) $\mathscr{B} \vee \mathscr{B} \Rightarrow \mathscr{B}$; (2) $\mathscr{B} \Rightarrow \mathscr{B} \vee \mathscr{C}$; (3) $\mathscr{B} \vee \mathscr{C} \Rightarrow \mathscr{C} \vee \mathscr{B}$; and (4) $(\mathscr{C} \Rightarrow \mathscr{D}) \Rightarrow (\mathscr{B} \vee \mathscr{C} \Rightarrow \mathscr{B} \vee \mathscr{D})$. The only rule of inference is modus ponens. Here and below we use the usual rules for eliminating parentheses. This system is developed in Hilbert and Ackermann (1950).

L_2: ∧ and ¬ are the primitive connectives. $\mathscr{B} \Rightarrow \mathscr{C}$ is an abbreviation for $\neg(\mathscr{B} \wedge \neg\mathscr{C})$. There are three axiom schemas: (1) $\mathscr{B} \Rightarrow (\mathscr{B} \wedge \mathscr{B})$; (2) $\mathscr{B} \wedge \mathscr{C} \Rightarrow \mathscr{B}$; and (3) $(\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow (\neg(\mathscr{C} \wedge \mathscr{D}) \Rightarrow \neg(\mathscr{D} \wedge \mathscr{B}))$. Modus ponens is the only rule of inference. Consult Rosser (1953) for a detailed study.

L_3: This is just like our original system L except that, instead of the axiom schemas (A1)–(A3), we have three specific axioms: (1) $A_1 \Rightarrow (A_2 \Rightarrow A_1)$; (2) $(A_1 \Rightarrow (A_2 \Rightarrow A_3)) \Rightarrow ((A_1 \Rightarrow A_2) \Rightarrow (A_1 \Rightarrow A_3))$; and (3) $(\neg A_2 \Rightarrow \neg A_1) \Rightarrow ((\neg A_2 \Rightarrow A_1) \Rightarrow A_2)$. In addition to modus ponens, we have a substitution rule: we may substitute any wf for all occurrences of a statement letter in a given wf.

L_4: The primitive connectives are ⇒, ∧, ∨, and ¬. Modus ponens is the only rule, and we have 10 axiom schemas: (1) $\mathscr{B} \Rightarrow (\mathscr{C} \Rightarrow \mathscr{B})$; (2) $(\mathscr{B} \Rightarrow (\mathscr{C} \Rightarrow \mathscr{D})) \Rightarrow ((\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow (\mathscr{B} \Rightarrow \mathscr{D}))$; (3) $\mathscr{B} \wedge \mathscr{C} \Rightarrow \mathscr{B}$; (4) $\mathscr{B} \wedge \mathscr{C} \Rightarrow \mathscr{C}$; (5) $\mathscr{B} \Rightarrow (\mathscr{C} \Rightarrow (\mathscr{B} \wedge \mathscr{C}))$; (6) $\mathscr{B} \Rightarrow (\mathscr{B} \vee \mathscr{C})$; (7) $\mathscr{C} \Rightarrow (\mathscr{B} \vee \mathscr{C})$; (8) $(\mathscr{B} \Rightarrow \mathscr{D}) \Rightarrow ((\mathscr{C} \Rightarrow \mathscr{D}) \Rightarrow (\mathscr{B} \vee \mathscr{C} \Rightarrow \mathscr{D}))$; (9) $(\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow ((\mathscr{B} \Rightarrow \neg\mathscr{C}) \Rightarrow \neg\mathscr{B})$; and (10) $\neg\neg\mathscr{B} \Rightarrow \mathscr{B}$. This system is discussed in Kleene (1952).

Axiomatizations can be found for the propositional calculus that contain only one axiom schema. For example, if ¬ and ⇒ are the primitive connectives and modus ponens the only rule of inference, then the axiom schema

$$\left[\left(\left(\left(\mathscr{B} \Rightarrow \mathscr{C}\right) \Rightarrow \left(\neg\mathscr{D} \Rightarrow \neg\mathscr{E}\right)\right) \Rightarrow \mathscr{D}\right) \Rightarrow \mathscr{F}\right] \Rightarrow \left[\left(\mathscr{F} \Rightarrow \mathscr{B}\right) \Rightarrow \left(\mathscr{E} \Rightarrow \mathscr{B}\right)\right]$$

is sufficient (Meredith, 1953). Another single-axiom formulation, due to Nicod (1917), uses only alternative denial $|$. Its rule of inference is: $\mathscr{D}$ follows from $\mathscr{B} | (\mathscr{C} | \mathscr{D})$ and $\mathscr{B}$, and its axiom schema is

$$(\mathscr{B} | (\mathscr{C} | \mathscr{D})) | \{[\mathscr{E} | (\mathscr{E} | \mathscr{E})] | [(\mathscr{F} | \mathscr{C}) | ((\mathscr{B} | \mathscr{F}) | (\mathscr{B} | \mathscr{F}))]\}$$

Further information, including historical background, may be found in Church (1956) and in a paper by Lukasiewicz and Tarski in Tarski (1956, IV).

Exercises

1.54 (Hilbert and Ackermann, 1950) Prove the following results about the theory L_1.

a. $\mathscr{B} \Rightarrow \mathscr{C} \vdash_{L1} \mathscr{D} \vee \mathscr{B} \Rightarrow \mathscr{D} \vee \mathscr{C}$
b. $\vdash_{L1} (\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow ((\mathscr{D} \Rightarrow \mathscr{B}) \Rightarrow (\mathscr{D} \Rightarrow \mathscr{C}))$
c. $\mathscr{D} \Rightarrow \mathscr{B}, \mathscr{B} \Rightarrow \mathscr{C} \vdash_{L1} \mathscr{D} \Rightarrow \mathscr{C}$
d. $\vdash_{L1} \mathscr{B} \Rightarrow \mathscr{B}$ (i.e., $\vdash_{L1} \neg\mathscr{B} \vee \mathscr{B}$)
e. $\vdash_{L1} \mathscr{B} \vee \neg\mathscr{B}$
f. $\vdash_{L1} \mathscr{B} \Rightarrow \neg\neg\mathscr{B}$
g. $\vdash_{L1} \neg\mathscr{B} \Rightarrow (\mathscr{B} \Rightarrow \mathscr{C})$
h. $\vdash_{L1} \mathscr{B} \vee (\mathscr{C} \vee \mathscr{D}) \Rightarrow ((\mathscr{C} \vee (\mathscr{B} \vee \mathscr{D})) \vee \mathscr{B})$
i. $\vdash_{L1} (\mathscr{C} \vee (\mathscr{B} \vee \mathscr{D})) \vee \mathscr{B} \Rightarrow \mathscr{C} \vee (\mathscr{B} \vee \mathscr{D})$
j. $\vdash_{L1} \mathscr{B} \vee (\mathscr{C} \vee \mathscr{D}) \Rightarrow \mathscr{C} \vee (\mathscr{B} \vee \mathscr{D})$
k. $\vdash_{L1} (\mathscr{B} \Rightarrow (\mathscr{C} \Rightarrow \mathscr{D})) \Rightarrow (\mathscr{C} \Rightarrow (\mathscr{B} \Rightarrow \mathscr{D}))$
l. $\vdash_{L1} (\mathscr{D} \Rightarrow \mathscr{B}) \Rightarrow ((\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow (\mathscr{D} \Rightarrow \mathscr{C}))$
m. $\mathscr{B} \Rightarrow (\mathscr{C} \Rightarrow \mathscr{D}), \mathscr{B} \Rightarrow \mathscr{C} \vdash_{L1} \mathscr{B} \Rightarrow (\mathscr{B} \Rightarrow \mathscr{D})$
n. $\mathscr{B} \Rightarrow (\mathscr{C} \Rightarrow \mathscr{D}), \mathscr{B} \Rightarrow \mathscr{C} \vdash_{L1} \mathscr{B} \Rightarrow \mathscr{D}$
o. If $\Gamma, \mathscr{B} \vdash_{L1} \mathscr{C}$, then $\Gamma \vdash_{L1} \mathscr{B} \Rightarrow \mathscr{C}$ (Deduction theorem)
p. $\mathscr{C} \Rightarrow \mathscr{B}, \neg\mathscr{C} \Rightarrow \mathscr{B} \vdash_{L1} \mathscr{B}$
q. $\vdash_{L1} \mathscr{B}$ if and only if $\mathscr{B}$ is a tautology.

1.55 (Rosser, 1953) Prove the following facts about the theory L_2.

a. $\mathscr{B} \Rightarrow \mathscr{C}, \mathscr{C} \Rightarrow \mathscr{D} \vdash_{L2} \neg(\neg\mathscr{D} \wedge \mathscr{B})$
b. $\vdash_{L2} \neg(\neg\mathscr{B} \wedge \mathscr{B})$
c. $\vdash_{L2} \neg\neg\mathscr{B} \Rightarrow \mathscr{B}$
d. $\vdash_{L2} \neg(\mathscr{B} \wedge \mathscr{C}) \Rightarrow (\mathscr{C} \Rightarrow \neg\mathscr{B})$
e. $\vdash_{L2} \mathscr{B} \Rightarrow \neg\neg\mathscr{B}$
f. $\vdash_{L2} (\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow (\neg\mathscr{C} \Rightarrow \neg\mathscr{B})$

g. $\neg\mathscr{B} \Rightarrow \neg\mathscr{C} \vdash_{L2} \mathscr{C} \Rightarrow \mathscr{B}$
h. $\mathscr{B} \Rightarrow \mathscr{C} \vdash_{L2} \mathscr{D} \land \mathscr{B} \Rightarrow \mathscr{C} \land \mathscr{D}$
i. $\mathscr{B} \Rightarrow \mathscr{C}, \mathscr{C} \Rightarrow \mathscr{D}, \mathscr{D} \Rightarrow \mathscr{E} \vdash_{L2} \mathscr{B} \Rightarrow \mathscr{E}$
j. $\vdash_{L2} \mathscr{B} \Rightarrow \mathscr{B}$
k. $\vdash_{L2} \mathscr{B} \land \mathscr{C} \Rightarrow \mathscr{C} \land \mathscr{B}$
l. $\mathscr{B} \Rightarrow \mathscr{C}, \mathscr{C} \Rightarrow \mathscr{D} \vdash_{L2} \mathscr{B} \Rightarrow \mathscr{D}$
m. $\mathscr{B} \Rightarrow \mathscr{C}, \mathscr{D} \Rightarrow \mathscr{E} \vdash_{L2} \mathscr{B} \land \mathscr{D} \Rightarrow \mathscr{C} \land \mathscr{E}$
n. $\mathscr{C} \Rightarrow \mathscr{D} \vdash_{L2} \mathscr{B} \land \mathscr{C} \Rightarrow \mathscr{B} \land \mathscr{D}$
o. $\vdash_{L2} (\mathscr{B} \Rightarrow (\mathscr{C} \Rightarrow \mathscr{D})) \Rightarrow ((\mathscr{B} \land \mathscr{C}) \Rightarrow \mathscr{D})$
p. $\vdash_{L2} ((\mathscr{B} \land \mathscr{C}) \Rightarrow \mathscr{D}) \Rightarrow (\mathscr{B} \Rightarrow (\mathscr{C} \Rightarrow \mathscr{D}))$
q. $\mathscr{B} \Rightarrow \mathscr{C}, \mathscr{B} \Rightarrow (\mathscr{C} \Rightarrow \mathscr{D}) \vdash_{L2} \mathscr{B} \Rightarrow \mathscr{D}$
r. $\vdash_{L2} \mathscr{B} \Rightarrow (\mathscr{C} \Rightarrow \mathscr{B} \land \mathscr{C})$
s. $\vdash_{L2} \mathscr{B} \Rightarrow (\mathscr{C} \Rightarrow \mathscr{B})$
t. If $\Gamma, \mathscr{B} \vdash_{L2} \mathscr{C}$, then $\Gamma \vdash_{L2} \mathscr{B} \Rightarrow \mathscr{C}$ (Deduction theorem)
u. $\vdash_{L2} (\neg\mathscr{B} \Rightarrow \mathscr{B}) \Rightarrow \mathscr{B}$
v. $\mathscr{B} \Rightarrow \mathscr{C}, \neg\mathscr{B} \Rightarrow \mathscr{C} \vdash_{L2} \mathscr{C}$
w. $\vdash_{L2} \mathscr{B}$ if and only if $\mathscr{B}$ is a tautology.

1.56 Show that the theory L_3 has the same theorems as the theory L.

1.57 (Kleene, 1952) Derive the following facts about the theory L_4.

a. $\vdash_{L4} \mathscr{B} \Rightarrow \mathscr{B}$
b. If $\Gamma, \mathscr{B} \vdash_{L4} \mathscr{C}$, then $\Gamma \vdash_{L4} \mathscr{B} \Rightarrow \mathscr{C}$ (deduction theorem)
c. $\mathscr{B} \Rightarrow \mathscr{C}, \mathscr{C} \Rightarrow \mathscr{D} \vdash_{L4} \mathscr{B} \Rightarrow \mathscr{D}$
d. $\vdash_{L4} (\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow (\neg\mathscr{C} \Rightarrow \neg\mathscr{B})$
e. $\mathscr{B}, \neg\mathscr{B} \vdash_{L4} \mathscr{C}$
f. $\vdash_{L4} \mathscr{B} \Rightarrow \neg\neg\mathscr{B}$
g. $\vdash_{L4} \neg\mathscr{B} \Rightarrow (\mathscr{B} \Rightarrow \mathscr{C})$
h. $\vdash_{L4} \mathscr{B} \Rightarrow (\neg\mathscr{C} \Rightarrow \neg(\mathscr{B} \Rightarrow \mathscr{C}))$
i. $\vdash_{L4} \neg\mathscr{B} \Rightarrow (\neg\mathscr{C} \Rightarrow \neg(\mathscr{B} \lor \mathscr{C}))$
j. $\vdash_{L4} (\neg\mathscr{C} \Rightarrow \mathscr{B}) \Rightarrow ((\mathscr{C} \Rightarrow \mathscr{B}) \Rightarrow \mathscr{B})$
k. $\vdash_{L4} \mathscr{B}$ if and only if $\mathscr{B}$ is a tautology.

1.58[D] Consider the following axiomatization of the propositional calculus $\mathscr{L}$ (due to Lukasiewicz). $\mathscr{L}$ has the same wfs as our system L. Its only rule of inference is modus ponens. Its axiom schemas are:

a. $(\neg\mathscr{B} \Rightarrow \mathscr{B}) \Rightarrow \mathscr{B}$
b. $\mathscr{B} \Rightarrow (\neg\mathscr{B} \Rightarrow \mathscr{C})$
c. $(\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow ((\mathscr{C} \Rightarrow \mathscr{D}) \Rightarrow (\mathscr{B} \Rightarrow \mathscr{D}))$

Prove that a wf $\mathscr{B}$ of $\mathscr{L}$ is provable in $\mathscr{L}$ if and only if $\mathscr{B}$ is a tautology. [*Hint*: Show that L and $\mathscr{L}$ have the same theorems. However, remember that none of the results proved about L (such as Propositions 1.8–1.13) automatically carries over to $\mathscr{L}$. In particular, the deduction theorem is not available until it is proved for $\mathscr{L}$.]

1.59 Show that axiom schema (A3) of L can be replaced by the schema $(\neg\mathscr{B} \Rightarrow \neg\mathscr{C}) \Rightarrow (\mathscr{C} \Rightarrow \mathscr{B})$ without altering the class of theorems.

1.60 If axiom schema (10) of L_4 is replaced by the schema (10)′: $\neg\mathscr{B} \Rightarrow (\mathscr{B} \Rightarrow \mathscr{C})$, then the new system L_I is called the *intuitionistic* propositional calculus.* Prove the following results about L_I.

a. Consider an $(n + 1)$-valued logic with these connectives: $\neg\mathscr{B}$ is 0 when $\mathscr{B}$ is n, and otherwise it is n; $\mathscr{B} \wedge \mathscr{C}$ has the maximum of the values of $\mathscr{B}$ and $\mathscr{C}$, whereas $\mathscr{B} \vee \mathscr{C}$ has the minimum of these values; and $\mathscr{B} \Rightarrow \mathscr{C}$ is 0 if $\mathscr{B}$ has a value not less than that of $\mathscr{C}$, and otherwise it has the same value as $\mathscr{C}$. If we take 0 as the only designated value, all theorems of L_I are exceptional.

b. $A_1 \vee \neg A_1$ and $\neg\neg A_1 \Rightarrow A_1$ are not theorems of L_I.

c. For any m, the wf

$$(A_1 \Leftrightarrow A_2) \vee \ldots \vee (A_1 \Leftrightarrow A_m) \vee (A_2 \Leftrightarrow A_3) \vee \ldots \\ \vee (A_2 \Leftrightarrow A_m) \vee \ldots \vee (A_{m-1} \Leftrightarrow A_m)$$

is not a theorem of L_I

d. (Gödel, 1933) L_I is not suitable for any finite many-valued logic.

e. i. If $\Gamma, \mathscr{B} \vdash_{LI} \mathscr{C}$, then $\Gamma \vdash_{LI} \mathscr{B} \Rightarrow \mathscr{C}$ (deduction theorem)

ii. $\mathscr{B} \Rightarrow \mathscr{C}, \mathscr{C} \Rightarrow \mathscr{D} \vdash_{LI} \mathscr{B} \Rightarrow \mathscr{D}$

iii. $\vdash_{LI} \mathscr{B} \Rightarrow \neg\neg\mathscr{B}$

iv. $\vdash_{LI} (\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow (\neg\mathscr{C} \Rightarrow \neg\mathscr{B})$

v. $\vdash_{LI} \mathscr{B} \Rightarrow (\neg\mathscr{B} \Rightarrow \mathscr{C})$

vi. $\vdash_{LI} \neg\neg(\neg\neg\mathscr{B} \Rightarrow \mathscr{B})$

vii. $\neg\neg(\mathscr{B} \Rightarrow \mathscr{C}), \neg\neg\mathscr{B} \vdash_{LI} \neg\neg\mathscr{C}$

viii. $\vdash_{LI} \neg\neg\neg\mathscr{B} \Rightarrow \neg\mathscr{B}$

f[D]. $\vdash_{LI} \neg\neg\mathscr{B}$ if and only if $\mathscr{B}$ is a tautology.

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* The principal origin of intuitionistic logic was L.E.J. Brouwer's belief that classical logic is wrong. According to Brouwer, $\mathscr{B} \vee \mathscr{C}$ is proved only when a proof of $\mathscr{B}$ or a proof of $\mathscr{C}$ has been found. As a consequence, various tautologies, such as $\mathscr{B} \vee \neg\mathscr{B}$, are not generally acceptable. For further information, consult Brouwer (1976), Heyting (1956), Kleene (1952), Troelstra (1969), and Dummett (1977). Jaśkowski (1936) showed that L_I is suitable for a many-valued logic with denumerably many values.

g. $\vdash_{LI} \neg\mathscr{B}$ if and only if $\neg\mathscr{B}$ is a tautology.

h.[D] If $\mathscr{B}$ has $\vee$ and $\neg$ as its only connectives, then $\vdash_{LI} \mathscr{B}$ if and only if $\mathscr{B}$ is a tautology.

1.61[A] Let $\mathscr{B}$ and $\mathscr{C}$ be in the relation R if and only if $\vdash_L \mathscr{B} \Leftrightarrow \mathscr{C}$. Show that R is an equivalence relation. Given equivalence classes $[\mathscr{B}]$ and $[\mathscr{C}]$, let $[\mathscr{B}] \cup [\mathscr{C}] = [\mathscr{B} \vee \mathscr{C}]$, $[\mathscr{B}] \cap [\mathscr{C}] = [\mathscr{B} \wedge \mathscr{C}]$, and $[\overline{\mathscr{B}}] = [\neg\mathscr{B}]$. Show that the equivalence classes under R form a Boolean algebra with respect to $\cap$, $\cup$, and $^-$, called the *Lindenbaum algebra* L* determined by L. The element 0 of L* is the equivalence class consisting of all contradictions (i.e., negations of tautologies). The unit element 1 of L* is the equivalence class consisting of all tautologies. Notice that $\vdash_L \mathscr{B} \Rightarrow \mathscr{C}$ if and only if $[\mathscr{B}] \le [\mathscr{C}]$ in L*, and that $\vdash_L \mathscr{B} \Leftrightarrow \mathscr{C}$ if and only if $[\mathscr{B}] = [\mathscr{C}]$. Show that a Boolean function f (built up from variables, 0, and 1, using $\cup$, $\cap$, and $^-$) is equal to the constant function 1 in all Boolean algebras if and only if $\vdash_L f^{\#}$, where $f^{\#}$ is obtained from f by changing $\cup$, $\cap$, $^-$, 0, and 1 to $\vee$, $\wedge$, $\neg$, $A_1 \wedge \neg A_1$, and $A_1 \vee \neg A_1$, respectively.