Stochastic Calculus
2. Information and Conditioning

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Abstract

Here will be some meaningful text.

Exercises

Exercise 2.1 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a general probability space, and suppose a random variable X on this space is measurable with respect to the trivial σ-algebra $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Show that X is not random (i.e. there is a constant c such that $X(\omega) = c$, $\forall \omega \in \Omega$. Such a random variable is called *degenerate*.

soln. Since we are told nothing of the experiment, no finer resolution can be obtained other than each outcome is equally likely. To see this, think of how you would assign probabilities to the outcomes of an experiment you have not yet conducted. Any possibility is just as likely as the other, since there is no information to determine anything, other than the fact that you have a collection of objects. Therefore, $\exists c \in \mathbb{R}$ such that $X(\omega) = c$, $\forall \omega \in \Omega$.

Exercise 2.2 Independence of random variables can be affected by changes of measure. To illustrate this point, consider the space of two coin tosses $\Omega_2 = \{HH, HT, TH, TT\}$, and let stock prices be given by

$$S_0 = 4, S_1(H) = 8, S_1(T) = 2$$
$$S_2(HH) = 16, S_2(HT) = S_2(TH) = 4, S_2(TT) = 1$$

Consider two probability measures given by

$$\tilde{\mathbb{P}}(HH) = \frac{1}{4}, \tilde{\mathbb{P}}(HT) = \frac{1}{4}, \tilde{\mathbb{P}}(TH) = \frac{1}{4}, \tilde{\mathbb{P}}(TT) = \frac{1}{4}$$
$$\mathbb{P}(HH) = \frac{4}{9}, \mathbb{P}(HT) = \frac{2}{9}, \mathbb{P}(TH) = \frac{2}{9}, \mathbb{P}(TT) = \frac{1}{9}$$

Define the random variable

$$X = \begin{cases} 1, & \text{if } S_2 = 4, \\ 0, & \text{if } S_2 \neq 4 \end{cases}$$

(i) List all the sets in $\sigma(X)$.
(ii) List all the sets in $\sigma(S_1)$.
(iii) Show that $\sigma(X)$ and $\sigma(S_1)$ are independent under the probability measure $\tilde{\mathbb{P}}$.
(iv) Show that $\sigma(X)$ and $\sigma(S_1)$ are not independent under the probability measure $\tilde{\mathbb{P}}$.
(v) Under $\mathbb{P}$, we have $\mathbb{P}\{S_1 = 8\} = \frac{2}{3}$ and $\mathbb{P}\{S_1 = 2\} = \frac{1}{3}$. Explain intuitively why, if you are told that $X = 1$, you would want to revise your estimate of the distribution of S_1.

soln. (i) We use the definition of generated σ-algebras.

Definition 1. *Let X be a random variable defined on a nonempty sample space Ω. The σ-algebra generated by X, denoted $\sigma(X)$, is the collection of all subsets of Ω of the form $\{X \in B\}$, where B ranges over the Borel subsets of $\mathbb{R}$.*

In this problem, X is defined on Ω_2. If we take B to be the single number 1, then $\{X \in B\} = \{HT, TH\}$. Next, if we take B to be the single number 0, $\{X \in B\} = \{HH, TT\}$, so that $\sigma(X) = \{\emptyset, \Omega_2, \{HT, TH\}, \{HH, TT\}\}$. The unions and complements of those sets are precisely those sets.

(ii) We perform a similar procedure as in (i) to obtain $\sigma(S_1) = \{\emptyset, \Omega_1, H, T\}$.

(iii) We use definition 2.2.1, which is stated below.

Definition 2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{G}$ and $\mathcal{H}$ be sub-σ-algebras of $\mathcal{F}$ (i.e., the sets in $\mathcal{G}$ and the sets in $\mathcal{H}$ are also in $\mathcal{F}$). We say these two σ-algebras are independent if*

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B), \ \forall A \in \mathcal{G}, \ \forall B \in \mathcal{H}$$

Let X and Y be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. We say these two random variables are independent if the σ-algebras they generate, $\sigma(X)$ and $\sigma(Y)$, are independent. We say that the random variable X is independent of the σ-algebra $\mathcal{G}$ if $\sigma(X)$ and $\mathcal{G}$ are independent.

We now perform several easy calculations,

$$\tilde{\mathbb{P}}(X \in 1 \cap S_1 \in 8) = \tilde{\mathbb{P}}(\{HT, TH\} \cap \{1^{st} \text{ toss} = H\}) = \tilde{\mathbb{P}}(HT) = \frac{1}{4}$$
$$\tilde{\mathbb{P}}(X \in 1) \cdot \tilde{\mathbb{P}}(S_1 \in 8) = \tilde{\mathbb{P}}(\{HT, TH\}) \cdot \tilde{\mathbb{P}}(\{1^{st} \text{ toss} = H\}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

which gives $\tilde{\mathbb{P}}(X \in 1 \cap S_1 \in 8) = \tilde{\mathbb{P}}(X \in 1) \cdot \tilde{\mathbb{P}}(S_1 \in 8)$.

$$\tilde{\mathbb{P}}(X \in 0 \cap S_1 \in 8) = \tilde{\mathbb{P}}(\{HH, TT\} \cap \{1^{st} \text{ toss} = H\}) = \tilde{\mathbb{P}}(HH) = \frac{1}{4}$$
$$\tilde{\mathbb{P}}(X \in 0) \cdot \tilde{\mathbb{P}}(S_1 \in 8) = \tilde{\mathbb{P}}(\{HH, TT\}) \cdot \tilde{\mathbb{P}}(\{1^{st} \text{ toss} = H\}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

which gives $\tilde{\mathbb{P}}(X \in 0 \cap S_1 \in 8) = \tilde{\mathbb{P}}(X \in 0) \cdot \tilde{\mathbb{P}}(S_1 \in 8)$.

$$\tilde{\mathbb{P}}(X \in 1 \cap S_1 \in 2) = \tilde{\mathbb{P}}(\{HT, TH\} \cap \{1^{st} \text{ toss} = T\}) = \tilde{\mathbb{P}}(TH) = \frac{1}{4}$$
$$\tilde{\mathbb{P}}(X \in 1) \cdot \tilde{\mathbb{P}}(S_1 \in 2) = \tilde{\mathbb{P}}(\{HT, TH\}) \cdot \tilde{\mathbb{P}}(\{1^{st} \text{ toss} = T\}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

which gives $\tilde{\mathbb{P}}(X \in 1 \cap S_1 \in 2) = \tilde{\mathbb{P}}(X \in 1) \cdot \tilde{\mathbb{P}}(S_1 \in 2)$.

$$\tilde{\mathbb{P}}(X \in 0 \cap S_1 \in 2) = \tilde{\mathbb{P}}(\{HH, TT\} \cap \{1^{st} \text{ toss} = T\}) = \tilde{\mathbb{P}}(TT) = \frac{1}{4}$$
$$\tilde{\mathbb{P}}(X \in 0) \cdot \tilde{\mathbb{P}}(S_1 \in 2) = \tilde{\mathbb{P}}(\{HH, TT\}) \cdot \tilde{\mathbb{P}}(\{1^{st} \text{ toss} = T\}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

which gives $\tilde{\mathbb{P}}(X \in 0 \cap S_1 \in 2) = \tilde{\mathbb{P}}(X \in 0) \cdot \tilde{\mathbb{P}}(S_1 \in 2)$. We also have that $\mathbb{P}(\Omega_2 \cap \Omega_1) = 1\mathbb{P}(\Omega_2) \cdot \mathbb{P}(\Omega_1)$. The calculations with the empty set are also trivial.

(iv) We only need to show there exist sets in each σ-algebra that contradict equality.

$$\mathbb{P}(X \in 1 \cap S_1 \in 8) = \mathbb{P}(\{HT, TH\} \cap \{1^{st} \text{ toss} = H\}) = \mathbb{P}(HT) = \frac{2}{9}$$
$$\mathbb{P}(X \in 1) \cdot \tilde{\mathbb{P}}(S_1 \in 8) = \mathbb{P}(\{HT, TH\}) \cdot \tilde{\mathbb{P}}(\{1^{st} \text{ toss} = H\}) = \frac{4}{9} \cdot \frac{2}{3} = \frac{8}{27}$$

Clearly $\frac{2}{9} \neq \frac{8}{27}$, so we do not have independence under $\mathbb{P}$.

(v) Intuitively, this is because X independent of the first toss and only depends on the second toss. You would want to revise your estimate of the distribution of S_1 so that when we know the value of S_1 we already know the value of S_2. (Please check if this is correct!)

Exercise 2.3 (Rotating the axes). Let X and Y be independent standard normal random variables. Let θ be a constant, and define random variables

$$V = X\cos\theta + Y\sin\theta \text{ and } W = -X\sin\theta + Y\cos\theta$$

Show that V and W are independent normal standard random variables.

soln. The random variables V and W are independent and standard normal if they have the joint density

$$f_{V,W}(v, w) = \frac{1}{2\pi} e^{-\frac{1}{2}(v^2 + w^2)}, \ \forall v \in \mathbb{R}, w \in \mathbb{R}$$

Consider what happens when we calculate v^2 and w^2, and then add them up.

$$\begin{aligned} v^2 &= (x\cos\theta + y\sin\theta)^2 = x^2\cos^2\theta + 2xy\cos\theta\sin\theta + y^2\sin^2\theta \\ +w^2 &= (-x\sin\theta + y\cos\theta)^2 = x^2\sin^2\theta - 2xy\cos\theta\sin\theta + y^2\cos^2\theta \\ &= x^2 + y^2 \end{aligned}$$

The joint density of (V, W) is the same as the joint density of (X, Y), which were assumed to be independent and standard normal. This implies that the pair V and W are independent normal standard random variables. This is useful since special properties remain invariant under rotation.

Exercise 2.4 In Example 2.2.10, X is a standard normal random variable and Z is an independent random variable satisfying

$$\mathbb{P}\{Z = 1\} = \mathbb{P}\{Z = -1\} = \frac{1}{2}$$

We defined $Y = XZ$ and showed that Y is standard normal. We established that although X and Y are uncorrelated, they are not independent. In this exercise, we use moment generating functions to show that Y is standard normal and X and Y are not independent.

(i) Establish the joint moment-generating function formula

$$\mathbb{E}e^{uX+vY} = e^{\frac{1}{2}(u^2+v^2)} \cdot \frac{e^{uv} + e^{-uv}}{2}$$

(ii) Use the formula above to show that $\mathbb{E}e^{vY} = e^{\frac{1}{2}v^2}$. This is the moment-generating function for a standard normal random variable, and thus Y must be a standard normal random variable.

(iii) Use the formula in (i) and Theorem 2.2.7(iv) to show that X and Y are not independent.

soln. (i) In example 2.2.10, we established that the pair (X, Y) does not have a joint density. This is seen by considering the absolute value function and applying it to both X and Y, to obtain $|X| = |Y|$. From here, we note that the pair (X, Y) takes values only in the set $C = \{(x, y) : x = \pm y\}$. The joint distribution measure is then $\mu_{X,Y}(C) = 1$. But, C has zero area. It follows that for any nonnegative function f,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbb{I}_C(x, y) f(x, y) \; dy \; dx = 0$$

There cannot be a joint density for (X, Y) due to the fact that with our choice of C, the joint distribution measure is 1, but the double integral above is 0. This means that we cannot just substitute normal densities in our computation of the expectation, since we do not have a joint density function. Instead we use the fact that $\mathbb{P}\{Z = 1\} = \mathbb{P}\{Z = -1\} = \frac{1}{2}$ and the result from Example 1.6.6 of Chapter 1 to obtain the following computation,

$$\begin{aligned}
\mathbb{E}e^{uX+vY} &= \mathbb{E}[e^{(u+v)X}] \cdot \mathbb{P}\{Z = 1\} + \mathbb{E}[e^{(u-v)X}] \cdot \mathbb{P}\{Z = -1\} \\
&= \frac{1}{2} e^{\frac{1}{2}(u+v)^2}] + \frac{1}{2} e^{\frac{1}{2}(u-v)^2} \\
&= e^{\frac{1}{2}(u^2+v^2)} \cdot \frac{e^{uv} + e^{-uv}}{2}
\end{aligned}$$

(ii) This is just (i) but with $u = 0$.

(iii) Theorem 2.2.7(iv) states that X and Y are independent if and only if the joint moment-generating function factors. This would mean that

$$\mathbb{E}e^{uX+vY} = \mathbb{E}e^{uX} \cdot \mathbb{E}e^{vY} = e^{\frac{1}{2}(u^2+v^2)}$$

which is not the case here, since our computation in (i) resulted in an additional $\cosh(uv)$ term multiplied in.

Exercise 2.5 Let (X, Y) be a pair of random variables with joint density function

$$f_{X,Y}(x, y) = \begin{cases} \frac{2|x|+y}{\sqrt{2\pi}} \exp\Big\{ -\frac{(2|x|+y)^2}{2} \Big\}, & \text{if } y \geq -|x|, \\ 0, & \text{if } y < -|x| \end{cases}$$

Show that X and Y are standard normal random variables and that they are uncorrelated but not independent.

soln. We use the fact that the marginal distribution of X can be obtained from integrating the joint distribution with respect to y. Similarly, the marginal distribution of Y can be obtained from integrating the joint distribution with respect to x. We fix x to obtain

$$f_X(x) = \int_{-|x|}^{\infty} \frac{2|x| + y}{\sqrt{2\pi}} \exp\Big\{ -\frac{(2|x| + y)^2}{2} \Big\} \; dy$$

We make the substitution $u = 2|x| + y$ so that

$$\int_{-|x|}^{\infty} \frac{2|x| + y}{\sqrt{2\pi}} \exp\left\{ -\frac{(2|x| + y)^2}{2} \right\} dy = \int_{|x|}^{\infty} \frac{u}{\sqrt{2\pi}} \exp\left\{ \frac{-u^2}{2} \right\} du$$

We make the substitution $v = \frac{-u^2}{2}$ to so that

$$\int_{|x|}^{\infty} \frac{u}{\sqrt{2\pi}} \exp\left\{ \frac{-u^2}{2} \right\} du = -\frac{1}{\sqrt{2\pi}} \int_{-\frac{x^2}{2}}^{-\infty} e^v dv = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

This shows that X is standard normal. We next fix y so that

$$f_Y(y) = \int_{-\infty}^{\infty} \frac{2|x| + y}{\sqrt{2\pi}} \exp\left\{ -\frac{(2|x| + y)^2}{2} \right\} dx$$

The integral is nonzero whenever $-|x| \leq y$ and zero otherwise. We break down the first inequality in two cases,

$$y \geq -|x| \iff \begin{cases} y \geq -x, & \text{if } x \geq 0 \\ y \geq x, & \text{if } x < 0 \end{cases}$$

This gives us our bounds of integration. We then have the following

$$\int_{-\infty}^{-y} \frac{2x + y}{\sqrt{2\pi}} \exp\left\{ -\frac{(2x + y)^2}{2} \right\} dx + \int_{y}^{\infty} \frac{-2x + y}{\sqrt{2\pi}} \exp\left\{ -\frac{(-2x + y)^2}{2} \right\} dx$$

Two substitutions are in order, $u = 2x+y$, $\frac{1}{2}du = dx$, with $u(-\infty, y) = -\infty$ and $u(-y, y) = -y$. Then $v = -2x+y$, $-\frac{1}{2}dv = dx$, with $v(y, y) = -y$ and $v(\infty, y) = -\infty$. We'll then get

$$\begin{aligned}
&\int_{-\infty}^{-y} \frac{1}{2} \frac{u}{\sqrt{2\pi}} \exp\left\{ -\frac{u^2}{2} \right\} du + \int_{-y}^{-\infty} -\frac{1}{2} \frac{v}{\sqrt{2\pi}} \exp\left\{ -\frac{v^2}{2} \right\} dv \\
&= -\frac{1}{2} \int_{-y}^{-\infty} \frac{u}{\sqrt{2\pi}} \exp\left\{ -\frac{u^2}{2} \right\} du - \frac{1}{2} \int_{-y}^{-\infty} \frac{v}{\sqrt{2\pi}} \exp\left\{ -\frac{v^2}{2} \right\} dv \\
&= \frac{1}{2} \int_{-\infty}^{-\frac{y^2}{2}} \frac{1}{\sqrt{2\pi}} \exp\left\{ \theta_u \right\} d\theta_u + \frac{1}{2} \int_{-\infty}^{-\frac{y^2}{2}} \frac{1}{\sqrt{2\pi}} \exp\left\{ \theta_v \right\} d\theta_v \\
&= \frac{1}{2} \frac{1}{\sqrt{2\pi}} e^{\theta_u} \Big|_{\theta_u = -\infty}^{-\frac{y^2}{2}} + \frac{1}{2} \frac{1}{\sqrt{2\pi}} e^{\theta_v} \Big|_{\theta_v = -\infty}^{-\frac{y^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}
\end{aligned}$$

This shows that X and Y are standard normal random variables. To show that these two variables are uncorrelated we show that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

$$\begin{aligned}
\mathbb{E}[XY] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h^2(x, y)\varphi(x)\varphi(y) \\
&= \int_{-\infty}^{\infty} h(x, y)\varphi(x)dx \int_{-\infty}^{\infty} h(x, y)\varphi(y)dy \\
&= \mathbb{E}[X]\mathbb{E}[Y]
\end{aligned}$$

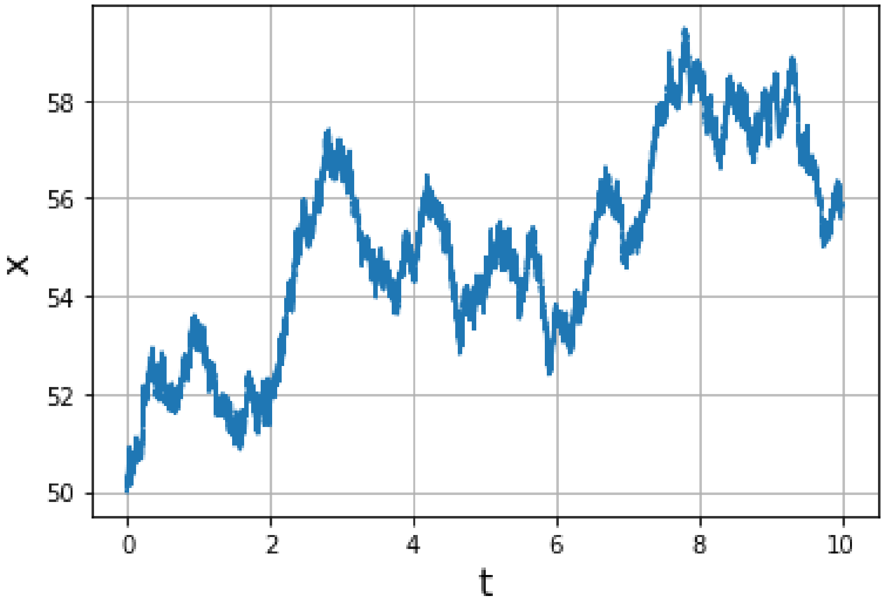
where $\varphi(n)$ is the standard normal density. We know that X and Y are not independent, since their joint density does not factor.

Exercise 2.6 Consider a probability space Ω with four elements, which we call $a, b, c,$ and d (i.e. $\Omega = \{a, b, c, d\}$). The σ-algebra $\mathcal{F}$ is the collection of all subsets of Ω; i.e., the sets in $\mathcal{F}$ are

$$\begin{aligned}
&\Omega,\ \{a, b, c\},\ \{a, b, d\},\ \{a, c, d\},\ \{b, c, d\} \\
&\{a, b\},\ \{a, c\},\ \{a, d\},\ \{b, c\},\ \{b, d\},\ \{c, d\} \\
&\{a\},\ \{b\},\ \{c\},\ \{d\},\ \emptyset
\end{aligned}$$

We define a probability measure $\mathbb{P}$ by specifying that

$$\mathbb{P}\{a\} = \frac{1}{6},\ \mathbb{P}\{b\} = \frac{1}{3},\ \mathbb{P}\{c\} = \frac{1}{4},\ \mathbb{P}\{d\} = \frac{1}{4}$$



and, as usual, the probability of every other set in $\mathcal{F}$ is the sum of probabilities of the elements in the set, e.g., $\mathbb{P}\{a,b,c\} = \mathbb{P}\{a\} + \mathbb{P}\{b\} + \mathbb{P}\{c\} = \frac{3}{4}$

We next define two random variables, X and Y by the formulas

$$X(a) = 1, X(b) = 1, X(c) = -1, X(d) = -1$$
$$Y(a) = 1, X(b) = -1, Y(c) = 1, X(d) = -1$$

We then define $Z = X + Y$.

(i) List the sets in $\sigma(X)$.

(ii) Determine $\mathbb{E}[Y|X]$ (i.e. specify the values of this random variable for a, b, c and d). Verify that the partial-averaging property is satisfied.

(iii) Deteremine $\mathbb{E}[Z|X]$. Again, verify the partial averaging property.

(iv) Compute $\mathbb{E}[Z|X] - \mathbb{E}[Y|X]$. Citing the appropriate properties of conditional expectation from Theorem 2.3.2, explain why you get X.

soln.