Assessing Rudin's Proof of the Existence of a Continuous Nowhere Differentiable Function: Theorem 7.18 in Principles of Mathematical Analysis

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May 27th, 2020

Abstract

Geometric Brownian motion is an example of a function that is continuous, but nowhere differentiable. I asked my friends if they could come up with such a function off the top of their head or if such a function is even intuitively clear. Needless to say, all were stumped and could not come up with such a function, although they barely put in the effort anyway. The reason that I chose to investigate Rudin's proof is because it uses basic principles from Fourier analysis, such as periodicity and boundedness, as well as a function that is intuitively clear. The idea is that function will oscillate rapidly enough to produce a problem with the difference quotient, but will be convergent, and in turn continuous, for all values in the domain, by way a monotonically decreasing sequence.

The Proof

We restate the theorem and retype the proof here for convenience, pointing out certain insights along the way.

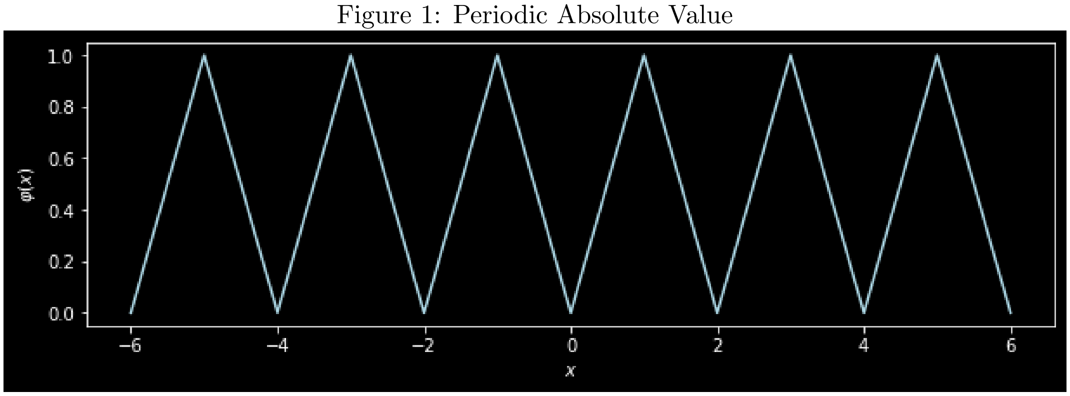
Theorem 1. *There exists a real continuous function on the real line which is nowhere differentiable.*

Proof. Define

$$\varphi(x) = |x| \text{ for } -1 \leq x \leq 1$$

which looks like this
and extend the definition of $\varphi(x)$ to all real x by requiring that $\varphi(x+2) = \varphi(x)$. See Figure 1.

Figure 1: Periodic Absolute Value



Then for all s and t,

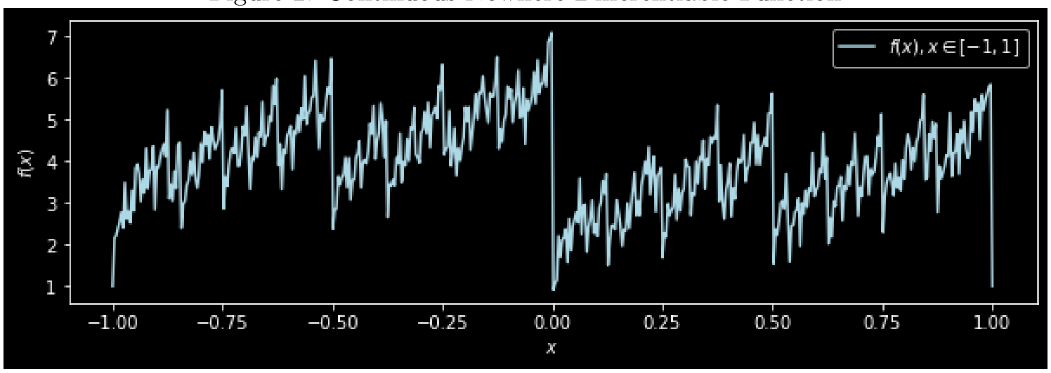
$$|\varphi(s) - \varphi(t)| \leq |s - t|$$

In particular, φ is continuous on $\mathbb{R}$. Define

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x)$$

Figure 2 shows this function for the sum with 500 iterations.

Figure 2: Continuous Nowhere Differentiable Function



Consider the following convergence theorem:

Theorem 2. *Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose*

$$|f_n(x)| \leq M_n, \;\; x \in E, \;\; n = 1, 2, 3, \dots$$

Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

By taking $M_n = \left(\frac{3}{4}\right)^n$ and noting that $0 \leq \varphi \leq 1$, the series converges uniformly on $E = \mathbb{R}$ by Theorem 2. The following theorem asserts that f is continuous on $\mathbb{R}$.

Theorem 3. *If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then f is continuous on E.*

Now fix $x \in \mathbb{R}$ and $m \in \mathbb{Z}^+$. Put

$$\delta_m = \pm \frac{1}{2} \cdot 4^{-m}$$

where the sign is chosen so that no integer lies between $4^m x$ and $4^m(x + \delta_m)$. This can be done since $4^m|\delta_m| = \frac{1}{2}$. Define

$$\gamma_n = \frac{\varphi(4^n(x + \delta_m)) - \varphi(4^n x)}{\delta_m}$$

There are two cases:

(I) $n > m$. In this case $4^n \delta_m \in 2\mathbb{Z}$, so that $\gamma_n = 0$.

(II) $0 \leq n \leq m$. Continuity of φ implies that $|\gamma_n| \leq 4^n$, with equality when $n = m$. Since $|\gamma_m| = 4^m$, we conclude that

$$\begin{aligned} \left| \frac{f(x + \delta_m) - f(x)}{\delta_m} \right| &= \left| \sum_{n=0}^{m} \left(\frac{3}{4}\right)^n \gamma_n \right| \\ &\geq 3^m - \sum_{n=0}^{m-1} 3^n \\ &= \frac{1}{2}(3^m + 1) \end{aligned}$$

As $m \to \infty$, $\delta_m \to 0$. It follows that f is not differentiable at x. □

Analysis

So, what happened? The conclusion of the proof is that the difference quotient approaches ∞ and $-\infty$ as δ_m approached zero.

At each point the slope is essentially vertical but diametrically opposed on an abitrarily small interval, which violates the differentiability condition. Note that the function in Figure 2 exhibits the same pathological behavior as a geometric Brownian motion.

Now, we'll take a look at how the inequality in case (II) was obtained. Proceed as follows,

$$\begin{aligned}\left|\frac{f(x+\delta_m)-f(x)}{\delta_m}\right| &= \left|\sum_{n=0}^{m}\left(\frac{3}{4}\right)^n \gamma_n\right| \\ &\leq \sum_{n=0}^{m}\left(\frac{3}{4}\right)^n |\gamma_n| \\ &= 3^m + \sum_{n=0}^{m-1}\left(\frac{3}{4}\right)^n |\gamma_n|\end{aligned}$$

which implies that

$$3^m - \sum_{n=0}^{m-1}\left(\frac{3}{4}\right)^n \gamma_n \leq \left|\sum_{n=0}^{m}\left(\frac{3}{4}\right)^n \gamma_n\right| \leq 3^m + \sum_{n=0}^{m-1}\left(\frac{3}{4}\right)^n \gamma_n$$

Since $|\gamma_n| \leq 4^n$ we have, from the leftmost inequality,

$$3^m - \sum_{n=0}^{m-1} 3^n \leq 3^m - \sum_{n=0}^{m-1}\left(\frac{3}{4}\right)^n \gamma_n \leq \left|\sum_{n=0}^{m}\left(\frac{3}{4}\right)^n \gamma_n\right|$$

So that now that concluding portion of the proof has been justified.