Stochastic Calculus
1. General Probability Theory

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Abstract

In this set of exercises, we begin by understanding what constitutes a probability space. We have the sample space Ω, the σ-algebra of subsets of the sample space $\mathcal{F}$, and the probability measure $\mathbb{P}$. We then investigate Borel-measurable subsets of the real line, the σ-algebra of Borel subsets denoted $\beta(\mathbb{R})$, which is the smallest σ-algebra containing all the closed-intervals of the real line. Afterwards, we investigate random variables, which are functions of the form $X : \Omega \to B$, where $B \in \beta(\mathbb{R})$. We conclude by discussing distributions, expectations, the Monotone Convergence Theorem, Dominated Convergence, and change of probability measures.

Exercises

Exercise 1.1 Using properties of Definition 1.1.2 for a probability measure $\mathbb{P}$, show the following.
(i) If $A \in \mathcal{F}, B \in \mathcal{F}$ and $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$.
(ii)If $A \in \mathcal{F}$ and $\{A_n\}_{n=1}^{\infty}$ is a sequence of sets in $\mathcal{F}$ with $\lim_{n\to\infty} \mathbb{P}(A_n) = 0$ and $A \subset A_n$ $\forall n$, then $\mathbb{P}(A) = 0$.

soln. We use the definition of probability measure stated below.

Definition 1. *Let Ω be a nonempty set, and let $\mathcal{F}$ be a σ-algebra of subsets of Ω. A probability measure $\mathbb{P}$ is a function that, to every set $A \in \mathcal{F}$, assigns a number in $[0, 1]$, called the probability of A and written $\mathbb{P}(A)$. We require:*

(i) $\mathbb{P}(\Omega) = 1$
(ii) (countable additivity) whenever $A_1, A_2, \ldots$ is a sequence of disjoint sets in $\mathcal{F}$, then

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n) \tag{1}$$

Let $A \in \mathcal{F}, B \in \mathcal{F}$ and $A \subset B$. Then

$$\begin{aligned}
\mathbb{P}(B) &= \mathbb{P}(A) + \mathbb{P}(A^c \cap B) \\
\mathbb{P}(B) - \mathbb{P}(A) &= \mathbb{P}(A^c \cap B) \geq 0 \\
\mathbb{P}(B) - \mathbb{P}(A) &\geq 0 \\
\mathbb{P}(B) &\geq \mathbb{P}(A).
\end{aligned}$$

Let the antecedent of (ii) hold. By (i), $\mathbb{P}(A) \leq \mathbb{P}(A_n)$. Applying the limit as $n \to \infty$ gives $\mathbb{P}(A) \leq 0$. Since $\mathbb{P} : \Omega \to [0, 1]$, we conclude that $p(A) = 0$.

Exercise 1.2 The infinite coin-toss space Ω_∞ is uncountably infinite. In other words, we cannot list all its elements in a sequence. To see this is impossible, suppose there were such a sequential list of all elements of Ω_∞:

$$\begin{aligned}
\omega^{(1)} &= \omega_1^{(1)} \omega_2^{(1)} \omega_3^{(1)} \omega_4^{(1)} \ldots, \\
\omega^{(2)} &= \omega_1^{(2)} \omega_2^{(2)} \omega_3^{(2)} \omega_4^{(2)} \ldots, \\
\omega^{(3)} &= \omega_1^{(3)} \omega_2^{(3)} \omega_3^{(3)} \omega_4^{(3)} \ldots, \\
&\vdots
\end{aligned}$$

If $\omega_i^{(i)} = H : T$, let $\omega_i = T : H$. Thus, $\omega = \omega_1\omega_2\ldots$ is not in our list. Now consider the set of sequences of coin tosses in which the outcome on each even-numbered toss matches the outcome of the toss preceding it, i.e.,

$$A = \{\omega = \omega_1\omega_2\omega_3\omega_4 : \omega_1 = \omega_2, \omega_3 = \omega_4, \ldots\}$$

(i) Show that A is uncountably infinite.
(ii) Show that, when $0 < p < 1$, we have $\mathbb{P}(A) = 0$.

soln. If we rewrite the sequences in A as $\omega^{(i)} = \omega_1^{(i)}\omega_2^{(i)}\omega_3^{(i)}\omega_4^{(i)}\ldots$, where $(\omega_{2j}, \omega_{2j-1}) \mapsto \omega_j^{(i)}$, $\forall i, j \in \mathbb{N}$. Then the diagonalization argument shows this set is uncountable. When $0 < p < 1$, the probability of any sequence in A is $p^{2k}(1-p)^{2(k-1)}$ or $p^{2(k-1)}(1-p)^{2k}$ as $k \to \infty$, which gives zero since both p and $1-p < 1$.

Exercise 1.3 Consider the set function $\mathbb{P}$ defined for every subset of $[0,1]$ by the formula that $\mathbb{P}(A) = 0$ if A is a finite set and $p(A) = \infty$ if A is an infinite set. Show that $\mathbb{P}$ satisfies (1.1.3)-(1.1.5), but $\mathbb{P}$ does not have the countable additivity property (1). Thus, finite additivity does not imply countable additivity.

soln. List of properties to satisfy:

$$\mathbb{P}(\emptyset) = 0 \tag{2}$$

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) \tag{3}$$

$$\mathbb{P}\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} \mathbb{P}(A_n) \tag{4}$$

Proof. $1 = \mathbb{P}(\Omega) = 1 - \mathbb{P}(\emptyset)$, so $\mathbb{P}(\emptyset) = 0$. Suppose that A and B are both finite. Then $A \cup B$ is finite so that $\mathbb{P}(A \cup B) = 0 = 0 + 0 = \mathbb{P}(A) + \mathbb{P}(B)$. If one or both sets are infinite, then $\mathbb{P}(A \cup B) = \infty = \mathbb{P}(A) + \mathbb{P}(B)$. Property (4) is satisfied much the same as property (3) but with additional terms. Finite sums of probabilities of finite sets gives 0. If one or more of the sets are infinite then the sum of probabilities will be infinite. The countable additivity condition (1) is not satisfied. Consider A_n finite for all $n \in \mathbb{N}$. Then

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \infty \neq 0 = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$$

$\square$

Exercise 1.4 (i) Construct a uniform random variable Z on the probability space $(\Omega_\infty, \mathcal{F}_\infty, \mathbb{P})$ of Ex 1.1.4 (Infinite, independent coin-toss space) under the assumption that the probability for heads is $p = \frac{1}{2}$.
(ii) Construct a standard normal variable Y on the probability space $(\Omega_\infty, \mathcal{F}_\infty, \mathbb{P})$ of Ex 1.1.4 (Infinite, independent coin-toss space) under the assumption that the probability for heads is $p = \frac{1}{2}$
(iii) Define a sequence of random variables $\{Y_n\}_{n=1}^\infty$ on Ω_∞ such that

$$\lim_{n\to\infty} Y_n(\omega) = Y(\omega), \ \forall \omega \in \Omega_\infty$$

and, for each n, Y_n depends only on the first n coin tosses. (This gives us a procedure for approximating a standard normal variable by random variables generated by a finite number of coin tosses, a useful algorithm for Monte Carlo simulation.)

soln. (i) Let $\mathbb{P}$ be the uniform measure on [0,1]. Define

$$X_n(\omega) = \begin{cases} 1, & \text{if } \omega_n = H, \\ 0, & \text{if } \omega_n = T \end{cases}$$

We set

$$Z = \sum_{n=1}^{\infty} \frac{X_n}{2^n}$$

After examining the value of Z for different values of X_n, we conclude that for $0 \le k \le 2^n - 1$, where $k, n \in \mathbb{Z}$, the probability that the interval $[\frac{k}{2^n}, \frac{k+1}{2^n}] \subset [0,1]$ contains Z is $\frac{1}{2^n}$. In terms of the distribution measure μ_Z of Z,

$$\mu_Z\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right] = \frac{1}{2^n}$$

Taking unions of intervals of this form and using the finite additivity of probability measures, we see that whenever $0 \le k \le m \le 2^n$, we have

$$\mu_Z\left[\frac{k}{2^n}, \frac{m}{2^n}\right] = \frac{m}{2^n} - \frac{k}{2^n}$$

Setting $b = \frac{m}{2^n}$ and $a = \frac{k}{2^n}$ we get that $\mu_Z[a,b] = b - a,\ 0 \le a \le b \le 1$. In other words, the distribution measure of Z is uniform on $[0,1]$.

(ii) Let $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ be the standard normal density, and define the cumulative normal distribution function $N(x) = \int_{-\infty}^{x} \varphi(\xi) d\xi$. The function $N(x)$ is a strictly increasing map from $\mathbb{R}$ onto $(0,1)$. Since strictly monotone functions are injective, $N(x)$ has an inverse which is also strictly increasing. In other words $N^{-1}(N(x)) = x \in (0,1)$. Take the uniformly distributed random variable Z from (i) and set $Y = N^{-1}(Z)$. Whenever $-\infty < a \le b < \infty$, we have

$$\begin{aligned}
\mu_Y[a,b] &= \mathbb{P}\{\omega \in \Omega : a \le Y(\omega) \le b\} \\
&= \mathbb{P}\{\omega \in \Omega : a \le N^{-1}(Z(\omega)) \le b\} \\
&= \mathbb{P}\{\omega \in \Omega : N(a) \le N(N^{-1}(Z(\omega))) \le N(b)\} \\
&= \mathbb{P}\{\omega \in \Omega : N(a) \le Z(\omega) \le N(b)\} \\
&= N(b) - N(a) \\
&= \int_a^b \varphi(y) dy
\end{aligned}$$

Since the distribution measure of Y is the standard normal distribution, Y is a standard normal random variable. The book gives another way to construct a standard normal random variable. Take $\Omega = \mathbb{R}, \mathcal{F} = \beta(\mathbb{R})$, where $\beta(\mathbb{R})$ is the set of Borel-measurable subsets of $\mathbb{R}$, and take $\mathbb{P}$ to be the probability measure on $\mathbb{R}$ with $-\infty < a \le b < \infty$ satisfying

$$\mathbb{P}[a,b] = \int_a^b \varphi(x) dx$$

and take $X(\omega) = \omega,\ \forall \omega \in \mathbb{R}$.

(iii) Needs an answer.

Exercise 1.5 When dealing with double Lebesgue integrals, just as with double Riemann integrals, the order of integration can be reversed. The only assumption required is that the function being integrated be either nonnegative or integrable. Here is an application of this fact.

Let X be a nonnegative random variable with cumulative distribution function $F(x) = \mathbb{P}\{X \le x\}$. Show that

$$\mathbb{E}X = \int_0^\infty (1 - F(x))\ dx$$

by showing that

$$\int_\Omega \int_0^\infty \mathbb{I}_{[0,X(\omega))}(x)\ dx\ d\mathbb{P}(\omega)$$

is equal to both $\mathbb{E}X$ and $\int_0^\infty (1 - F(x)) dx$.

soln. First, we cover integration wrt x.

$$\int_\Omega \int_0^\infty \mathbb{I}_{[0,X(\omega))}(x)\ dx\ d\mathbb{P}(\omega) = \int_\Omega \int_0^{X(\omega)} dx\ d\mathbb{P}(\omega) = \int_\Omega X(\omega) d\mathbb{P}(\omega) = \mathbb{E}X$$

Next, we cover integration wrt $\mathbb{P}(\omega)$.

$$\int_0^\infty \int_\Omega \mathbb{I}_{[0,X(\omega))}(x)\ d\mathbb{P}(\omega)\ dx = \int_0^\infty \int_{\{\omega : X(\omega) > x\}} \mathbb{I}_{[0,X(\omega))}(x)\ d\mathbb{P}(\omega)\ dx = \int_0^\infty (1 - F(x))\ dx$$

Thus, $\mathbb{E}X = \int_0^\infty (1 - F(x))\ dx$.

Exercise 1.6 Let u be a fixed number in $\mathbb{R}$, and define the convex function $\varphi(x) = e^{ux},\ \forall x \in \mathbb{R}$. Let X be a normal random variable with mean $\mu = \mathbb{E}X$ and standard deviation $\sigma = [\mathbb{E}(X - \mu)^2]^{\frac{1}{2}}$, i.e., with density

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

(i) Verify that

$$\mathbb{E}e^{uX} = e^{u\mu + \frac{1}{2}u^2\sigma^2}$$

(ii) Verify that Jensen's inequality holds (as it must):

$$\mathbb{E}\varphi(X) \geq \varphi(\mathbb{E}X)$$

soln. (i) This is just the transform associated with a normal random variable. Let $Y \sim N(0,1)$. The density of this variable is given by

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

and the associated transform is

$$\begin{aligned}
\mathbb{E}e^{uY} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} e^{uy} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2} + uy} dy \\
&= e^{\frac{u^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2} + uy - \frac{u^2}{2}} dy \\
&= e^{\frac{u^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{-y^2 + 2uy - u^2}{2}} dy \\
&= e^{\frac{u^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(y-u)^2}{2}} dy \\
&= e^{\frac{u^2}{2}}
\end{aligned}$$

The general normal random variable X is obtained from Y via linear transformation $X = \sigma Y + \mu$. We have $\mathbb{E}e^{uX} = \mathbb{E}[e^{u(\sigma Y + \mu)}] = e^{u\mu}\mathbb{E}[e^{u\sigma Y}] = e^{u\mu + \frac{1}{2}u^2\sigma^2}$ as desired.

(ii) Since the exponential is strictly increasing and $u\mu + \frac{1}{2}u^2\sigma^2 \geq u\mu$, we have $\mathbb{E}\varphi(X) = e^{u\mu + \frac{1}{2}u^2\sigma^2} \geq e^{u\mu} = \varphi(\mathbb{E}X)$. Thus, Jensen's inequality holds.

Exercise 1.7 For each positive integer n, define f_n to be the normal density with mean zero and variance n, i.e.,

$$f_n(x) = \frac{1}{\sqrt{2n\pi}} e^{-\frac{x^2}{2n}}$$

(i) What is the function $f(x) = \lim_{n\to\infty} f_n(x)$?
(ii) What is $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx$?
(iii) Note that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx \neq \int_{-\infty}^{\infty} f(x)dx$$

Explain why this does not violate the Monotone Convergence Theorem.

soln. (i) $f(x) = \lim_{n\to\infty} f_n(x) = 0$.
(ii) $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = 1$.
(iii) Monotone Convergence Theorem is stated below:

Theorem 1. ***(Monotone Convergence).*** *Let $X_1, X_2, X_3, \ldots$ be a sequence of random variables converging almost surely to another random variable X. If*

$$0 \leq X_1 \leq X_2 \leq X_3 \leq \ldots \text{ almost surely,}$$

then

$$\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X$$

Let $f_1, f_2, f_3, \ldots$ be a sequence of Borel-measurable functions on $\mathbb{R}$ converging almost everywhere to a function f. If

$$0 \leq f_1 \leq f_2 \leq f_3 \leq \ldots \text{ almost everywhere,}$$

then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = \int_{-\infty}^{\infty} f(x)dx$$

In order for the integrals of a sequence of functions to converge to the integral of the limiting function, we need the functions to be nonnegative and converge to their limit from below. Thinking of an integral as the area under a curve, the assumption is that farther out we go in the sequence of functions, we keep adding area and never take it away. If we do this, then the area under the limiting function is the limit of the areas under the functions in the sequence. In our case what happens is that instead of having the chain of inequalities as in the theorem, we have the reverse chain

$$0 \geq f_1 \geq f_2 \leq f_3 \geq \ldots \textit{ almost everywhere}$$

which would mean that we take away area at each step. This is not in accordance with the Monotone Convergence Theorem, which is why there is no violation present and the limit of the integrals of a sequence of functions is not equal to the integral of the limiting function.

Exercise 1.8 (Moment-generating function). Let X be a nonnegative random variable, and assume that

$$\varphi(t) = \mathbb{E}e^{tX}$$

is finite $\forall t \in \mathbb{R}$. Assume further that $\mathbb{E}[Xe^{tX}] < \infty$, $\forall t \in \mathbb{R}$. The purpose of this exercise is to show that $\varphi'(t) = \mathbb{E}[Xe^{tx}]$ and, in particular, $\varphi'(0) = \mathbb{E}X$.

We recall the definition of derivative:

$$\varphi'(t) = \lim_{s \to t} \frac{\varphi(t) - \varphi(s)}{t - s} = \lim_{s \to t} \frac{\mathbb{E}e^{tX} - \mathbb{E}e^{sX}}{t - s} = \lim_{s \to t} \mathbb{E}\left[\frac{e^{tX} - e^{sX}}{t - s}\right]$$

The limit above is taken over a continuous variable s, but we can choose a sequence of numbers $\{s_n\}_{n=1}^{\infty}$ converging to t and compute

$$\lim_{s_n \to t} \mathbb{E}\left[\frac{e^{tX} - e^{s_n X}}{t - s_n}\right]$$

where now we are taking a limit of the expectations of the sequence of random variables

$$Y_n = \frac{e^{tX} - e^{s_n X}}{t - s_n}$$

If this limit turns out to be the same, regardless of how we choose the sequence $\{s_n\}_{n=1}^{\infty}$ that converges to t, then this limit is also the same as $\lim_{s \to t} \mathbb{E}\left[\frac{e^{tX} - e^{sX}}{t - s}\right]$ and is $\varphi'(t)$.

The Mean Value Theorem from calculus states that if $f(t)$ is a differentiable function, then for any two numbers s and t, there is a number θ between s and t such that

$$f(t) - f(s) = f'(\theta)(t - s)$$

If we fix $\omega \in \Omega$ and define $f(x) = e^{tX(\omega)}$, then this becomes

$$e^{tX(\omega)} - e^{sX(\omega)} = (t - s)X(\omega)e^{\theta(\omega)X(\omega)} \tag{5}$$

where $\theta(\omega)$ is a number depending on ω (i.e., a random variable lying between t and s).

(i) Use the Dominated Convergence Theorem and equation (5) to show that

$$\lim_{n \to \infty} \mathbb{E}Y_n = \mathbb{E}\left[\lim_{n \to \infty} Y_n\right] = \mathbb{E}\left[Xe^{tX}\right] \tag{6}$$

This establishes the desired formula $\varphi'(t) = \mathbb{E}\left[Xe^{tX}\right]$

(ii) Suppose the random variable X can take both positive and negative values and $\mathbb{E}e^{tX} < \infty$ and $\mathbb{E}[|X|e^{tX}] < \infty$, $\forall t \in \mathbb{R}$. Show that once again $\varphi'(t) = \mathbb{E}[Xe^{tX}]$.

soln. (i) The Dominated Convergence Theorem is stated below.

Theorem 2. ***(Dominated Convergence).*** *Let $X_1, X_2, \ldots$ be a sequence of random variables converging almost surely to a random variable X. If there is another random variable Y such that $\mathbb{E}Y < \infty$ and $|X_n| \leq Y$ almost surely for every n, then*

$$\lim_{n \to \infty} \mathbb{E}X_n = \mathbb{E}X$$

Let $f_1, f_2, \ldots$ be a sequence of Borel-measurable functions converging almost everywhere to a function f. If there is another function g such that $\int_{\infty}^{\infty} g(x)dx < \infty$ and $|f_n| \leq g$ almost everywhere for every n, then

$$\lim_{n \to \infty} \int_{\infty}^{\infty} f_n(x)dx = \int_{\infty}^{\infty} f(x)dx$$

We begin by computing

$$\lim_{n\to\infty} \mathbb{E}Y_n = \lim_{n\to\infty} \mathbb{E}\left[\frac{e^{tX(\omega)} - e^{s_n X(\omega)}}{t - s_n}\right] = \lim_{s_n\to t} \mathbb{E}\left[\frac{(t-s_n)X(\omega)e^{\theta(\omega)X(\omega)}}{t - s_n}\right] = \mathbb{E}X(\omega)e^{tX(\omega)}$$

Using the random variable Xe^{tX} along with the fact that $\mathbb{E}[Xe^{tX}] < \infty$, we have that $|Y_n| \leq Xe^{tX}$ since $|Xe^{\theta(\omega)X}| < Xe^{tX}$. The Dominated Convergence Theorem now tells us that limit of the expectations of Y_n converge to the expectation of the limiting random variable $\lim_{n\to\infty} Y_n$. Thus, the equalities in (6) hold.
(ii) For such a random variable, we define the positive and negative parts of X by

$$X^+(\omega) = \max\{X(\omega), 0\},\ X^-(\omega) = \max\{-X(\omega), 0\}$$

Rewriting X as $X^+ - X^-$ we have

$$\lim_{n\to\infty} \mathbb{E}Y_n = \lim_{n\to\infty} \mathbb{E}\left[\frac{e^{t(X^+-X^-)} - e^{s_n(X^+-X^-)}}{t - s_n}\right] = \lim_{n\to\infty} \mathbb{E}\left[\frac{e^{tX^+}e^{-tX^-} - e^{s_nX^+}e^{-s_nX^-}}{t - s_n}\right]$$

so that whenever X is positive we get (i) and whenever X is negative, the minus sign in front of it bring us back to (i).

Exercise 1.9. Suppose X is a random variable on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, A is a set in $\mathcal{F}$, and for every Borel subset B of $\mathbb{R}$, we have

$$\int_A \mathbb{I}_B(X(\omega))d\mathbb{P}(\omega) = \mathbb{P}(A)\cdot\mathbb{P}\{X \in B\} \tag{7}$$

Then we say that X is independent of the event A.

Show that if X is independent of an event A, then

$$\int_A g(X(\omega))d\mathbb{P}(\omega) = \mathbb{P}(A)\cdot\mathbb{E}g(X)$$

for every nonnegative, Borel-measurable function g.

Proof. Let $g(X)$ be an arbitrary nonnegative Borel-measurable function defined on A. $\forall n \in Z^+$ define the sets

$$B_{k,n} = \left\{X : \frac{k}{2^n} \leq g(X) \leq \frac{k+1}{2^n}\right\},\ k = 1, 2, \ldots, 4^n - 1$$

For each fixed n, the sets $B_{0,n}, B_{1,n}, \ldots, B_{4^n-1,n}$ correspond to the partition

$$0 < \frac{1}{2^n} < \frac{2}{2^n} < \cdots < \frac{4^n-1}{2^n} = 2^n$$

At the next stage n+1, the partition points include all those at stage n and new partition points at the midpoints between the old ones. Because of this fact, the simple functions

$$g_n(x) = \sum_{k=0}^{4^n-1} \frac{k}{2^n}\mathbb{I}_{B_{k,n}}(X)$$

satisfy $0 \leq g_1 \leq g_2 \leq \ldots \leq g$. Furthermore, these functions become more and more accurate approximations of g as n becomes larger; indeed, $\lim_{n\to\infty} g_n(X) = g(X),\ \forall X \in A$. Using the Monotone Convergence Theorem,

$$\int_A g(X(\omega))d\mathbb{P}(\omega) = \lim_{n\to\infty}\int_A g_n(X(\omega))d\mathbb{P}(\omega) = \lim_{n\to\infty}\int_A \sum_{k=0}^{4^n-1}\frac{k}{2^n}\mathbb{I}_{B_{k,n}}(X)d\mathbb{P}(\omega) = \mathbb{P}(A)\cdot\sum_{k=0}^{4^n-1}\frac{k}{2^n}\mathbb{P}\{X \in B_{k,n}\} = \mathbb{P}(A)\cdot\mathbb{E}g(X)$$

□

Exercise 1.10. Let $\mathbb{P}$ be the uniform (Lebesgue) measure on $\Omega = [0, 1]$. Define

$$Z(\omega) = \begin{cases} 0, & \text{if } 0 \leq \omega < \frac{1}{2}, \\ 2, & \text{if } \frac{1}{2} \leq \omega < 1 \end{cases}$$

For $A \in \beta[0, 1]$, define

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega)d\mathbb{P}(\omega)$$

(i) Show that $\tilde{\mathbb{P}}$ is a probability measure.
(ii) Show that if $\mathbb{P}(A) = 0$, then $\tilde{\mathbb{P}}(A) = 0$. We say that $\tilde{\mathbb{P}}$ is absolutely continuous with respect to $\mathbb{P}$.
(iii) Show that there is a set A for which $\tilde{\mathbb{P}}(A) = 0$ but $p(A) > 0$. In other words $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are not equivalent.

soln. We must show that $\tilde{\mathbb{P}}(\Omega) = 1$ and that countable additivity is satified. Defined in the problem is the equality

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega)$$

Taking $A = \Omega$ we have $\forall \omega \in A$ and

$$\tilde{\mathbb{P}}(\Omega) = \int_\Omega Z(\omega) d\mathbb{P}(\omega) = \mathbb{E}Z = 0 \cdot \frac{1}{2} + 2 \cdot \left(1 - \frac{1}{2}\right) = 2 \cdot \frac{1}{2} = 1$$

We now show countable additivity. Suppose that $\{A_k\}_{k=1}^{\infty}$ is a sequence of disjoint sets in $\mathcal{F}$. Define $S_n = \cup_{k=1}^{n} A_k$ and $S_\infty = \cup_{k=1}^{\infty} A_k$. If we consider indicator functions over A_k, then we have the following monotone increasing sequence of functions

$$\mathbb{I}_{S_1} \le \mathbb{I}_{S_2} \le \mathbb{I}_{S_3} \le \dots$$

By definition, we have that $\lim_{n\to\infty} \mathbb{I}_{S_n} = \mathbb{I}_{S_\infty}$. Using the Monotone Convergence Theorem with Borel-measurable functions, we have that

$$\lim_{n\to\infty} \int_\Omega \mathbb{I}_{S_n}(\omega) Z(\omega) d\mathbb{P}(\omega) = \int_\Omega \lim_{n\to\infty} \mathbb{I}_{S_n}(\omega) Z(\omega) d\mathbb{P}(\omega) = \int_\Omega \mathbb{I}_{S_\infty}(\omega) Z(\omega) d\mathbb{P}(\omega) = \tilde{\mathbb{P}}(S_\infty)$$

Since the indicator function at the nth subset is taken over a union of n disjoint sets we have that $\mathbb{I}_{S_n} = \sum_{k=1}^{n} \mathbb{I}_{A_k}$. After substitution of this finite sum, as well as rearrangement of some terms in the train of equalities, we have

$$\tilde{\mathbb{P}}(S_\infty) = \tilde{\mathbb{P}}\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{n\to\infty} \int_\Omega \sum_{k=1}^{n} \mathbb{I}_{A_k}(\omega) Z(\omega) d\mathbb{P}(\omega) = \lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{P}(A_k) = \sum_{k=1}^{\infty} \mathbb{P}(A_k)$$

which shows that we have countable additivity.

(ii) Suppose that $\mathbb{P}(A) = 0$. Then, $\mathbb{P}(\mathbb{I}_A Z) = 0$ almost surely. It follows that

$$\tilde{\mathbb{P}}(A) = \int_{\beta[0,1]} \mathbb{I}_A(\omega) Z(\omega) d\mathbb{P}(\omega) = 0$$

(iii) Consider the subset $A = \left[0, \frac{1}{2}\right)$. We have that

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega) = 0$$

But, since $\mathbb{P}$ is the uniform measure, it is easily seen that $\mathbb{P}(A) = \frac{1}{2} > 0$. Thus, $\mathbb{P} \not\sim \tilde{\mathbb{P}}$.

Exercise 1.11. In Example 1.6.6, we began with a standard normal random variable X under a measure $\mathbb{P}$. According to Exercise 1.6, this random variable has the moment-generating function

$$\mathbb{E}e^{uX} = e^{\frac{1}{2}u^2}, \ \forall u \in \mathbb{R}$$

The moment-generating function of a random variable determines its distribution. In particular, any random variable that has moment-generating function $e^{\frac{1}{2}u^2}$ must be standard normal.

In Example 1.6.6, we also defined $Y = X + \theta$, where θ is a constant, we set $Z = e^{-\theta X - \frac{1}{2}\theta^2}$, and we defined $\tilde{\mathbb{P}}$ by the formula

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega), \ \forall A \in \mathcal{F}$$

We showed by considering its cumulative distribution function that Y is a standard normal random variable under $\tilde{\mathbb{P}}$. Give another proof that Y is standard normal under $\tilde{\mathbb{P}}$ by verifying the moment-generating function formula

$$\tilde{\mathbb{E}}e^{uY} = e^{\frac{1}{2}u^2}, \ \forall u \in \mathbb{R}$$

soln. We begin by observing that e^{uY} and Z are nonnegative. From Theorem 1.6.1, which we state below,

Theorem 3. ***(Probability Measure).*** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let Z be an almost surely nonegative random variable with $\mathbb{E}Z = 1$. For $A \in \mathcal{F}$, define*

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega)$$

Then $\tilde{\mathbb{P}}$ is a probability measure. Furthermore, if X is a nonnegative random variable, then

$$\tilde{\mathbb{E}} = \mathbb{E}[XZ]$$

If Z is almost surely strictly positive, we also have

$$\mathbb{E}Y = \tilde{\mathbb{E}}\left[\frac{Y}{Z}\right]$$

for every nonnegative random variable Y.

we have that

$$\tilde{\mathbb{E}}[e^{uY}] = \mathbb{E}[e^{uY} Z]$$

By substituting $Y = X + \theta$, this becomes,

$$\mathbb{E}\left[\exp\left\{uX + u\theta - \theta X - \frac{1}{2}\theta^2\right\}\right] = \mathbb{E}\left[\exp\left\{(u-\theta)X + \left(u - \frac{1}{2}\theta\right)\theta\right\}\right]$$

We now consider the following theorem,

Theorem 4. ***(Computation of Expectation).*** *Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let g be a Borel-measurable function on $\mathbb{R}$. Suppose that X has a denisty f. Then*

$$\mathbb{E}|g(X)| = \int_{-\infty}^{\infty} |g(x)| f(x) dx$$

and if this quantity is finite, then

$$\mathbb{E}g(X) = \int_{-\infty}^{\infty} g(x) f(x) dx$$

We are aware that $g(x) = e^x$ is a Borel-measurable function on $\mathbb{R}$ and that X has normal density $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$. Therefore, we can apply the theorem to the expectation we are trying to compute. We have

$$\begin{aligned}
\mathbb{E}\left[\exp\left\{(u-\theta)X + \left(u - \frac{1}{2}\theta\right)\theta\right\}\right] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{(u-\theta)X + \left(u - \frac{1}{2}\theta\right)\theta\right\} \exp\left\{-\frac{1}{2}x^2\right\} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\left(\frac{x^2 - 2(u-\theta)x - 2u\theta + \theta^2}{2}\right)\right\} dx \\
&= e^{\frac{1}{2}u^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\left(\frac{x^2 - 2(u-\theta)x + \theta^2 - 2u\theta + u^2}{2}\right)\right\} dx \\
&= e^{\frac{1}{2}u^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\left(\frac{x^2 - 2(u-\theta)x + (u-\theta)^2}{2}\right)\right\} dx \\
&= e^{\frac{1}{2}u^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\left(\frac{(x-u+\theta)^2}{2}\right)\right\} dx
\end{aligned}$$

We make the change of variable $X + \theta = Y$ so that we get the integral of a standard normal variable with mean u and variance 1 which equals

$$= e^{\frac{1}{2}u^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(y-u)^2}{2}} dy$$

The integral of a standard normal variable with the normalization constant in front yields 1, so

$$\tilde{\mathbb{E}}[e^{uY}] = e^{\frac{1}{2}u^2}$$

which is the desired result.

Exercise 1.12. In Example 1.6.6, we began with a standard normal random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and defined the random variable $Y = X + \theta$, where θ is a constant. We also defined $Z = e^{-\theta X - \frac{1}{2}\theta^2}$ and used Z as the Radon-Nikodym derivative to construct the probability measure $\tilde{\mathbb{P}}$ by the formula

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega), \ \forall A \in \mathcal{F}$$

Under $\tilde{\mathbb{P}}$, the random variable Y was shown to be standard normal.

We now have a standard normal random variable Y on the probability space $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$, and X is related to Y by $X = Y - \theta$. By what we have just stated, with X replaced by Y and θ replaced by $-\theta$, we could define $\hat{Z} = e^{\theta Y - \frac{1}{2}\theta^2}$ and then use $\hat{Z}$ as a Radon-Nikodym derivative to construct a probability measure $\hat{\mathbb{P}}$ by the formula

$$\hat{\mathbb{P}}(A) = \int_A \hat{Z}(\omega) d\hat{\mathbb{P}}(\omega), \ \forall A \in \mathcal{F}$$

so that, under $\hat{\mathbb{P}}$, the random variable X is standard normal. Show that $\hat{Z} = \frac{1}{Z}$ and $\hat{\mathbb{P}} = \mathbb{P}$.

soln. Showing the first equality is not so trying given the way in which defined $\hat{Z}$.

$$\begin{aligned}
\frac{1}{Z} &= e^{-(-\theta X - \frac{1}{2}\theta^2)} \\
&= e^{\theta X + \frac{1}{2}\theta^2} \\
&= e^{\theta Y - \theta^2 + \frac{1}{2}\theta^2} \\
&= e^{\theta Y - \frac{1}{2}\theta^2} \\
&= \hat{Z}
\end{aligned}$$

To show that the probability measures are equal, they must agree on which sets have probability zero. Consider $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 0$. We use the definition of hp as well as Theorem 1.6.1 to establish

$$\begin{aligned}
\hat{\mathbb{P}}(A) &= \int_A \hat{Z}(\omega) d\hat{\mathbb{P}}(\omega) \\
&= \int_\Omega \mathbb{I}_A \hat{Z}(\omega) d\hat{\mathbb{P}}(\omega) \\
&= \hat{\mathbb{E}} \mathbb{I}_A \hat{Z} \\
&= \hat{\mathbb{E}}[\mathbb{I}_A \frac{1}{Z}] \\
&= \mathbb{E}[\mathbb{I}_A] \\
&= \mathbb{P}(A)
\end{aligned}$$

The two measures agree on sets that yield zero probability, therefore we can conclude $\hat{\mathbb{P}} = \mathbb{P}$.

Exercise 1.13. (Change of measure for a normal random variable). A nonrigorous but informative derivation of the formula for the Radon-Nikodym derivative $Z(\omega)$ in Example 1.6.6 is provided by this exercise. As in that example, let X be a standard normal random variabl on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $Y = X + \theta$. Our goal is to define a strictly positive random variable $Z(\omega)$ so that when we set

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega), \ \forall A \in \mathcal{F}$$

the random variable Y under $\tilde{\mathbb{P}}$ is standard normal. If we fix $\bar{\omega} \in \Omega$ and choose a set A that contain $\bar{\omega}$ and is 'small', then the equality above gives

$$\tilde{\mathbb{P}}(A) \approx Z(\bar{\omega}) \mathbb{P}(A)$$

where the symbol $\approx$ means 'is approximately equal to.' Dividing by $\mathbb{P}(A)$ we see that

$$\frac{\tilde{\mathbb{P}}(A)}{\mathbb{P}(A)} \approx Z(\bar{\omega})$$

for 'small' sets A containing $\bar{\omega}$. We use this observation to identify $Z(\bar{\omega})$.

With $\bar{\omega}$ fixed let $x = X(\bar{\omega})$. For $\epsilon > 0$, we define $B(x, \epsilon) = [x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}]$ to be the closed interval centered at x and length ϵ. Let $y = x + \theta$ and $B(y, \epsilon) = [y - \frac{\epsilon}{2}, y + \frac{\epsilon}{2}]$

(i) Show that

$$\frac{1}{\epsilon} \mathbb{P}\{X \in B(x, \epsilon)\} \approx \frac{1}{\sqrt{2\pi}} \exp\left\{ - \frac{X^2(\bar{\omega})}{2} \right\}$$

(ii) In order for Y to be a standard normal random variable under $\tilde{\mathbb{P}}$, show that we must have

$$\frac{1}{\epsilon} \tilde{\mathbb{P}}\{Y \in B(y, \epsilon)\} \approx \frac{1}{\sqrt{2\pi}} \exp\left\{ - \frac{Y^2(\bar{\omega})}{2} \right\}$$

(iii) Show that $\{X \in B(x,\epsilon)\}$ and $\{Y \in B(y,\epsilon)\}$ are the same set, which we call $A(\bar{\omega},\epsilon)$. This set contains $\bar{\omega}$ and is 'small' when ϵ is small.

(iv) Show that

$$\frac{\tilde{\mathbb{P}}(A)}{\mathbb{P}(A)} \approx \exp\bigg\{ -\theta X(\bar{\omega}) - \frac{1}{2}\theta^2 \bigg\}$$

The right-hand side is the value we obtained for $Z(\bar{\omega})$ in Example 1.6.6.

soln. (i) We use the notion of a 'small' set A defined in the problem which results in the following implication

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega)d\mathbb{P}(\omega),\ \forall A \in \mathcal{F} \Longrightarrow \tilde{\mathbb{P}}(A) \approx Z(\bar{\omega})\mathbb{P}(A)$$

We evaluate the probability using the standard normal density $\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$ and the 'small' set $B(x,\epsilon)$, which gives

$$\begin{aligned}
\frac{1}{\epsilon}\mathbb{P}\{X \in B(x,\epsilon)\} &= \frac{1}{\epsilon}\int_{B(x,\epsilon)} \varphi(x)dx \\
&\approx \frac{1}{\epsilon}\varphi(x)\mathbb{P}(B(x,\epsilon)) \\
&= \frac{1}{\epsilon}\varphi(X(\bar{\omega}))\mathbb{P}\bigg[X(\bar{\omega}) - \frac{\epsilon}{2}, X(\bar{\omega}) + \frac{\epsilon}{2}\bigg] \\
&= \frac{1}{\sqrt{2\pi}}\exp\bigg\{ -\frac{X^2(\bar{\omega})}{2} \bigg\}
\end{aligned}$$

which is the desired result.

(ii) The procedure is similiar to that of part (i).

$$\begin{aligned}
\frac{1}{\epsilon}\tilde{\mathbb{P}}\{Y \in B(y,\epsilon)\} &\approx \frac{1}{\epsilon}\varphi(y)\tilde{\mathbb{P}}(B(y,\epsilon)) \\
&= \frac{1}{\epsilon}\varphi(x)Z(\bar{\omega})\mathbb{P}(B(y,\epsilon)) \\
&= \frac{1}{\epsilon}\varphi(Y(\bar{\omega}))\mathbb{P}\bigg[Y(\bar{\omega}) - \frac{\epsilon}{2}, Y(\bar{\omega}) + \frac{\epsilon}{2}\bigg] \\
&= \frac{1}{\sqrt{2\pi}}\exp\bigg\{ -\frac{Y^2(\bar{\omega})}{2} \bigg\}
\end{aligned}$$

(iii) Consider a generic element in $\{Y \in B(y,\epsilon)\}$. Then we have

$$\begin{aligned}
y - \frac{\epsilon}{2} &\le Y \le y + \frac{\epsilon}{2} \\
x + \theta - \frac{\epsilon}{2} &\le X + \theta \le x + \theta + \frac{\epsilon}{2} \\
x - \frac{\epsilon}{2} &\le X \le x + \frac{\epsilon}{2}
\end{aligned}$$

Thus, $\{Y \in B(y,\epsilon)\} = \{X \in B(x,\epsilon)\}$.

(iv) Once more, we use the fact that $Y = \theta + X$ to obtain

$$\begin{aligned}
\frac{\tilde{\mathbb{P}}(A)}{\mathbb{P}(A)} &\approx \frac{\frac{1}{\sqrt{2\pi}}\exp\Big\{ -\frac{Y^2(\bar{\omega})}{2} \Big\}}{\frac{1}{\sqrt{2\pi}}\exp\Big\{ -\frac{X^2(\bar{\omega})}{2} \Big\}} \\
&= \frac{\exp\Big\{ \frac{-X^2(\bar{\omega}) - 2\theta X(\bar{\omega}) - \theta^2}{2} \Big\}}{\exp\Big\{ -\frac{X^2(\bar{\omega})}{2} \Big\}} \\
&= \exp\bigg\{ \frac{-2\theta X(\bar{\omega}) - \theta^2}{2} \bigg\} \\
&= \exp\bigg\{ -\theta X(\bar{\omega}) - \frac{1}{2}\theta^2 \bigg\}
\end{aligned}$$

Exercise 1.14. (Change of measure for an exponential random variable). Let X be a nonnegative random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the *exponential distribution*, which is

$$\mathbb{P}\{X \leq a\} = 1 - e^{\lambda a},\ a \geq 0$$

where λ is a positive constant. Let $\bar{\lambda}$ be another positive constant, and define

$$Z = \frac{\bar{\lambda}}{\lambda} e^{-(\bar{\lambda}-\lambda)X}$$

Define $\tilde{\mathbb{P}}$ by

$$\tilde{\mathbb{P}}(A) = \int_A Z d\mathbb{P},\ \forall A \in \mathcal{F}$$

(i) Show that $\tilde{\mathbb{P}}(\Omega) = 1$.
(ii) Compute the cumulative distribution function

$$\tilde{\mathbb{P}}\{X \leq a\} \text{ for } a \geq 0$$

for a random variable X under the probability measure $\tilde{\mathbb{P}}$.

soln. (i) To show that $\tilde{\mathbb{P}}(\Omega) = 1$ we use the exponential probability density function

$$\eta(x) = \begin{cases} \lambda e^{\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise} \end{cases}$$

We have,

$$\begin{aligned}
\tilde{\mathbb{P}}(\Omega) &= \int_\Omega Z(\omega) d\mathbb{P}(\omega) \\
&= \frac{\bar{\lambda}}{\lambda} \int_\Omega e^{-(\bar{\lambda}-\lambda)X(\omega)} d\mathbb{P}(\omega) \\
&= \frac{\bar{\lambda}}{\lambda} \int_{-\infty}^{\infty} e^{-(\bar{\lambda}-\lambda)x} \eta(x) dx \\
&= \frac{\bar{\lambda}}{\lambda} \int_0^{\infty} e^{-\bar{\lambda}x+\lambda x} \lambda e^{-\lambda x} dx \\
&= \bar{\lambda} \int_0^{\infty} e^{-\bar{\lambda}x} dx \\
&= \lim_{M\to\infty} -e^{-\bar{\lambda}M} - (-e^0) \\
&= 0 - (-1) = 1
\end{aligned}$$

(ii) The computation is straightforward:

$$\begin{aligned}
\tilde{\mathbb{P}}\{X \leq a\} &= \int_{\{\omega : X(\omega) \leq a\}} Z(\omega) d\mathbb{P}(\omega) \\
&= \int_\Omega \mathbb{I}_{\{\omega : X(\omega) \leq a\}} Z(\omega) d\mathbb{P}(\omega) \\
&= \frac{\bar{\lambda}}{\lambda} \int_{-\infty}^{\infty} \mathbb{I}_{\{x \leq a\}} e^{-(\bar{\lambda}-\lambda)x} \eta(x) dx \\
&= \bar{\lambda} \int_0^a e^{-\bar{\lambda}x} dx \\
&= 1 - e^{-\bar{\lambda}a}
\end{aligned}$$

Exercise 1.15. (Provided by Alexander Ng). Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and assume X has a density function $f(x)$ that is positive for every $x \in \mathbb{R}$. Let g be a strictly increasing, differentiable function satisfying

$$\lim_{y\to-\infty} g(y) = -\infty,\ \lim_{y\to\infty} g(y) = \infty$$

and define the random variable $Y = g(X)$.

Let $h(y)$ be an arbitrary nonnegative function satisfying $\int_{-\infty}^{\infty} h(y)dy = 1$. We want to change the probability measure so that $h(y)$ is the density function for the random variable Y. To do this, we define

$$Z = \frac{h(g(X))g'(X)}{f(X)}$$

(i) Show that Z is nonnegative and $\mathbb{E}Z = 1$.

Now define $\tilde{\mathbb{P}}$ by

$$\tilde{\mathbb{P}}(A) = \int_A Z d\mathbb{P}, \ \forall A \in \mathcal{F}$$

(ii) Show that Y has density h under $\tilde{\mathbb{P}}$.

soln. Z is nonnegative since h is nonnegative, f is positive and $g' > 0$ (due to the fact that g is strictly increasing). We now compute the expectation of Z.

$$\begin{aligned}
\mathbb{E}Z &= \int_{-\infty}^{\infty} \frac{h(g(x))g'(x)}{f(x)} f(x)dx \\
&= \int_{-\infty}^{\infty} h(g(x))g'(x)dx \\
&= \int_{-\infty}^{\infty} h(y)dy \\
&= 1
\end{aligned}$$

We have made the substitution $y = g(x)$. The bounds of the integral stay the same because of the limits defined for g in the problem. The last equality follows by the assumption made on h.
(ii) To show that Y has probability density function h, we show that

$$\tilde{\mathbb{P}}\{Y \leq a\} = \int_{-\infty}^{a} h(y)dy$$

This is achieved by the following computation

$$\begin{aligned}
\tilde{\mathbb{P}}\{Y \leq a\} &= \tilde{\mathbb{P}}\{g(X) \leq a\} \\
&= \int_{\{\omega : g(X(\omega)) \leq a\}} Z(\omega)d\mathbb{P}(\omega) \\
&= \int_{\Omega} \mathbb{I}_{\{\omega : g(X(\omega)) \leq a\}} Z(\omega)d\mathbb{P}(\omega) \\
&= \int_{-\infty}^{\infty} \mathbb{I}_{\{g(x) \leq a\}} \frac{h(g(x))g'(x)}{f(x)} f(x)dx \\
&= \int_{-\infty}^{\infty} \mathbb{I}_{\{g(x) \leq a\}} h(g(x))g'(x)dx \\
&= \int_{-\infty}^{a} h(y)dy
\end{aligned}$$

When making the substitution $y = g(x)$ the bounds are $-\infty$ to $g(a)$, however, since g is strictly increasing, we can use a as the top bound. Thus, h is the probability density function for Y, completing the statistical recentering of Y.