Stochastic Calculus
4. Stochastic Calculus

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Abstract

Here will be some meaningful text.

Exercises

Exercise 4.1 Suppose $M(t)$ is a martingale with respect to some filtration $\mathcal{F}(t)$, where $0 \leq t \leq T$. Let $\Delta(t)$ be a simple process adapted to $\mathcal{F}(t)$ (i.e. there is a partition $\Pi = \{t_0, t_1, \dots, t_n\}$ of $[0, T]$ such that, for every j, $\Delta(t_j)$ is $\mathcal{F}(t_j)$-measurable and $\Delta(t)$ is constant in t on each subinterval $[t_j, t_{j+1})$. For $t \in [t_k, t_{k+1})$, define the stochastic integral

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[M(t_{j+1}) - M(t_j)] + \Delta(t_k)[M(t) - M(t_k)].$$

We think of $M(t)$ as the price of an asset at time t and $\Delta(t_j)$ as the number of shares of the asset held by an investor between times t_j and t_{j+1}. Then $I(t)$ is the capital gains that accrue to the investor between times 0 and t. Show that $I(t)$ is a martingale.

soln. It suffices to show that for $0 \leq s < t \leq T$, $\mathbb{E}[I(t)|\mathcal{F}(s)] = I(s)$. We assume that s and t are in different subintervals, i.e. there are partition points t_k and t_l such that $t_l < t_k$, where $s \in [t_l, t_{l+1})$ and $t \in [t_k, t_{k+1})$. Then

$$\begin{aligned}
\mathbb{E}[I(t)|\mathcal{F}(s)] &= \mathbb{E}\left[\sum_{j=0}^{k-1} \Delta(t_j)[M(t_{j+1}) - M(t_j)] + \Delta(t_k)[M(t) - M(t_k)] \middle| \mathcal{F}(s)\right] \\
&= \mathbb{E}\left[\sum_{j=0}^{l-1} \Delta(t_j)[M(t_{j+1}) - M(t_j)] + \Delta(t_l)[M(t_{l+1}) - M(t_l)\right] \\
&\quad + \sum_{j=l+1}^{k-1} \Delta(t_j)[M(t_{j+1}) - M(t_j)] + \Delta(t_k)[M(t) - M(t_k)] \Bigg| \mathcal{F}(s)\Bigg]
\end{aligned}$$

The sum has four parts. We will look at each individually since conditional expectation is linear and it is easier to see.

$$\mathbb{E}\left[\sum_{j=0}^{l-1} \Delta(t_j)[M(t_{j+1}) - M(t_j)] \middle| \mathcal{F}(s)\right] = \sum_{j=0}^{l-1} \Delta(t_j)[M(t_{j+1}) - M(t_j)]$$

For this sum every martingale is $\mathcal{F}(s)$-measurable because the latest time appearing in this sum is t_l and $t_l \leq s$. Therefore, we simply take out what is known. We utilize the same technique of taking out what is known for the second term.

$$\begin{aligned}
\mathbb{E}[\Delta(t_l)[M(t_{l+1}) - M(t_l)]|\mathcal{F}(s)] &= \Delta(t_l)(\mathbb{E}[M(t_{l+1})|\mathcal{F}(s)] - M(t_l)) \\
&= \Delta(t_l)(W(s) - W(t_l))
\end{aligned}$$

Note that the previous result added to this one yields the desire $I(s)$.

$$I(s) = \sum_{j=0}^{l-1} \Delta(t_j)[M(t_{j+1})] - M(t_j) + \Delta(t_l)(W(s) - W(t_l))$$

It remains to show that the other two terms are zero. The summands in the third term are such that $s < t_{l+1} \leq t_j$. This permits us to use iterated conditioning and taking out what is known.

$$\begin{aligned}
\mathbb{E}[\Delta(t_j)[M(t_{j+1}) - M(t_j)]\big|\mathcal{F}(s)] &= \mathbb{E}\bigg(\mathbb{E}\big[\Delta(t_j)[M(t_{j+1}) - M(t_j)]\big|\mathcal{F}(t_j)\big]\bigg|\mathcal{F}(s)\bigg) \\
&= \mathbb{E}\bigg(\Delta(t_j)(\mathbb{E}[M(t_{j+1})\big|\mathcal{F}(t_j)] - M(t_j))]\bigg|\mathcal{F}(s)\bigg) \\
&= \mathbb{E}\bigg(\Delta(t_j)(M(t_j) - M(t_j))]\bigg|\mathcal{F}(s)\bigg) \\
&= 0
\end{aligned}$$

Where we have used the fact that M is a martingale. Because the conditional expectations of the summands of the the third term are zero, the entire sum is zero. The fourth term is treated similarly.

$$\begin{aligned}
\mathbb{E}[\Delta(t_k)[M(t) - M(t_k)]\big|\mathcal{F}(s)] &= \mathbb{E}\bigg(\mathbb{E}\big[\Delta(t_k)[M(t) - M(t_k)]\big|\mathcal{F}(t_k)\big]\bigg|\mathcal{F}(s)\bigg) \\
&= \mathbb{E}\bigg(\Delta(t_k)(\mathbb{E}[M(t)\big|\mathcal{F}(t_k)] - M(t_k))]\bigg|\mathcal{F}(s)\bigg) \\
&= \mathbb{E}\bigg(\Delta(t_j)(M(t_k) - M(t_k))]\bigg|\mathcal{F}(s)\bigg) \\
&= 0
\end{aligned}$$

This completes the proof.

Exercise 4.2 Let $W(t), t \geq 0$, be a Brownian motion, and let $\mathcal{F}(t), t \geq 0$, be an associated filtration. Let $\Delta(t)$ be a *nonrandom* simple process (i.e. there is a partition $\Pi = \{t_0, t_1, \ldots, t_n\}$ of $[0, T]$ such that for every j, $\Delta(t_j)$ is a nonrandom quantity and $\Delta(t) = \Delta(t_j)$ is constant in t on the subinterval $[t_j, t_{j+1})$. For $t \in [t_k, t_{k+1}]$, define the stochastic integral

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)]$$

(i) Show that whenever $0 \leq s < t \leq T$, the increment $I(t) - I(s)$ is independent of $\mathcal{F}(s)$.

soln. Let $0 \leq s < t \leq T$. To show that $I(t) - I(s)$ is independent of $\mathcal{F}(s)$, it suffices to show that $I(t_k) - I(t_l)$ is independent of $\mathcal{F}(t_l)$ whenever t_k and t_l are two partition points points with $t_l < t_k$. We can use the techniques from the previous problem, i.e. applying the linearity of conditional expectation to assess the conditional expectation of the summand and the second term. Independence of Brownian increments combined with the fact that $\Delta(t_j)$ is a nonrandom process gives the desired result since all information after t_l is independent of the information contained at time t_l.

$$\begin{aligned}
\mathbb{E}[I(t_k) - I(t_l)|\mathcal{F}(t_l)] &= \mathbb{E}\left[\sum_{j=l}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)]\bigg|\mathcal{F}(t_l)\right] \\
&= \mathbb{E}\left[\sum_{j=l}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)]\right] \\
&= \mathbb{E}[I(t_k) - I(t_l)]
\end{aligned}$$

(ii) Show that whenever $0 \leq s < t \leq T$, the increment $I(s) - I(t)$ is a normally distributed random variable with mean zero and variance $\int_s^t \Delta^2(u)\ du$

soln. Because Brownian increments are independent and normally distributed $I(s) - I(t)$ is necessarily independent and normally distribute since it is just a linear combination of Brownian increments. We use the fact that $\mathbb{E}[W(t_{j+1}) - W(t_j)] = 0$ to establish the fact the $\mathbb{E}[I(t) - I(s)] = 0$. Using the notation from (i) along with the fact that $\mathrm{Var}(W(t_{j+1}) - W(t_j)) = t_{j+1} - t_j$

gives

$$\begin{aligned}
\mathrm{Var}(I(t_k) - I(t_l)) &= \mathrm{Var}\left[\sum_{j=l}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)]\right] \\
&= \sum_{j=l}^{k-1} \mathrm{Var}\big(\Delta(t_j)[W(t_{j+1}) - W(t_j)]\big) \\
&= \sum_{j=l}^{k-1} \Delta^2(t_j)\mathrm{Var}\big([W(t_{j+1}) - W(t_j)]\big) \\
&= \sum_{j=l}^{k-1} \Delta^2(t_j)[t_{j+1} - t_j] \\
&= \int_{t_l}^{t_k} \Delta^2(u)\ du
\end{aligned}$$

(iii) Use (i) and (ii) to show that $I(t), 0 \leq t \leq T$, is a martingale.

soln. We start with the fact that $I(t) - I(s)$ has mean zero, which was established in (ii) and then apply the independence of increments established in (i). Afterwards we use linearity of conditional expectation and taking out what is known. The computation is as follows

$$\begin{aligned}
\mathbb{E}[I(t) - I(s)] &= 0 \\
\mathbb{E}[I(t) - I(s)|\mathcal{F}(s)] &= 0 \\
\mathbb{E}[I(t)|\mathcal{F}(s)] - \mathbb{E}[I(s)|\mathcal{F}(s)] &= 0 \\
\mathbb{E}[I(t)|\mathcal{F}(s)] &= \mathbb{E}[I(s)|\mathcal{F}(s)] \\
\mathbb{E}[I(t)|\mathcal{F}(s)] &= I(s)
\end{aligned}$$

(iv) Show that $I^2(t) - \int_0^t \Delta^2(u)\ du,\ 0 \leq t \leq T$, is a martingale.

soln. We will use a very similar technique as the previous solution. Establishing

$$\mathbb{E}\left[I^2(t) - \int_0^t \Delta^2(u)\ du \middle| \mathcal{F}(s)\right] = I^2(s) - \int_0^s \Delta^2(u)$$

Amounts to showing that

$$\mathbb{E}\left[I^2(t) - \int_0^t \Delta^2(u)\ du \middle| \mathcal{F}(s)\right] - I^2(s) - \int_0^s \Delta^2(u) = 0$$

We proceed as follows

$$\begin{aligned}
\mathbb{E}\left[I^2(t) - \int_0^t \Delta^2(u)\ du \middle| \mathcal{F}(s)\right] - I^2(s) - \int_0^s \Delta^2(u) &= \mathbb{E}\left[I^2(t) - \int_0^t \Delta^2(u)\ du \middle| \mathcal{F}(s)\right] - \mathbb{E}\left[I^2(s) - \int_0^s \Delta^2(u) \middle| \mathcal{F}(s)\right] \\
&= \mathbb{E}\left[I^2(t) - I^2(s) - \int_0^t \Delta^2(u)\ du + \int_0^s \Delta^2(u) \middle| \mathcal{F}(s)\right] \\
&= \mathbb{E}\left[(I(t) - I(s))^2 + 2I(t)I(s) - 2I^2(s) - \int_s^t \Delta^2(u)\ du \middle| \mathcal{F}(s)\right] \\
&= \mathbb{E}[(I(t) - I(s))^2|\mathcal{F}(s)] + \mathbb{E}[2I(t)I(s) + 2I^2(s)|\mathcal{F}(s)] - \int_s^t \Delta^2(u)\ du \\
&= \int_s^t \Delta^2(u)\ du - \int_s^t \Delta^2(u)\ du \\
&= 0
\end{aligned}$$

Exercise 4.3 We now consider a case in which $\Delta(t)$ in Exercise 4.2 is simple but random. In particular, let $t_0 = 0, t_1 = s$, and $t_2 = t$, and let $\Delta(0)$ be non random and $\Delta(s) = W(s)$. Which of the following assertions is true? Justify your answers.

(i) $I(t) - I(s)$ is independent of $\mathcal{F}(s)$.

(ii) $I(t) - I(s)$ is normally distributed.
(iii) $\mathbb{E}[I(t)|\mathcal{F}(s)] = I(s)$.
(iv) $\mathbb{E}[I^2(t) - \int_0^t \Delta^2(u)\, du|\mathcal{F}(s)] = I^2(s) - \int_0^s \Delta^2(u)$.

Exercise 4.4 (Stratonovich Integral). Let $W(t), t \geq 0$, be a Brownian motion. Let T be a fixed positive number and let $\Pi = \{t_0, t_1, \dots, t_n\}$ be a partition of $[0, T]$ (i.e. $0 = t_0 < t_1 < \cdots < t_n = T$). For each j, define $t_j^* = \frac{t_j + t_{j+1}}{2}$ to be the midpoint of the interval $[t_j, t_{j+1}]$.
(i) Define the *half-sample quadratic variation* corresponding to Π to be

$$Q_{\Pi/2} = \sum_{j=0}^{n-1} \big(W(t_j^*) - W(t_j)\big)^2$$

Show that $Q_{\Pi/2}$ has limit $\frac{1}{2}T$ as $\|\Pi\| \to 0$.

soln. We make use of the hint given in the book which states that, in order to prove this result, it suffices to show that $\mathbb{E}Q_{\Pi/2} = \frac{1}{2}T$ and that $\lim_{\|\Pi\|\to 0} \mathrm{Var}(Q_{\Pi/2}) = 0$. This is done by the following computations

$$\begin{aligned}
\mathbb{E}[Q_{\Pi/2}] &= \mathbb{E}\left[\sum_{j=0}^{n-1} \big(W(t_j^*) - W(t_j)\big)^2\right] \\
&= \sum_{j=0}^{n-1} \mathbb{E}\left[\big(W(t_j^*) - W(t_j)\big)^2\right] \\
&= \sum_{j=0}^{n-1} t_j^* - t_j \\
&= \frac{1}{2}\sum_{j=0}^{n-1} t_{j+1} - t_j \\
&= \frac{1}{2}T
\end{aligned}$$

This establishes the value of the expectation of half-sample quadratic variation. For the limit as the maximum step size approaches of the variance we proceed as follows

$$\begin{aligned}
\lim_{\|\Pi\|\to 0} \mathrm{Var}(Q_{\Pi/2}) &= \lim_{\|\Pi\|\to 0} \mathrm{Var}\left[\sum_{j=0}^{n-1} \big(W(t_j^*) - W(t_j)\big)^2\right] \\
&= \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \mathrm{Var}\left[\big(W(t_j^*) - W(t_j)\big)^2\right] \\
&= \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \mathbb{E}\left[\Big(\big(W(t_j^*) - W(t_j)\big)^2 - (t_j^* - t_j)\Big)^2\right] \\
&= \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \mathbb{E}\left[\big(W(t_j^*) - W(t_j)\big)^4\right] - 2(t_j^* - t_j)\mathbb{E}\left[\big(W(t_j^*) - W(t_j)\big)^2\right] + (t_j^* - t_j)^2
\end{aligned}$$

The fourth moment of a normal random variable with mean zero is three times its variance squared. This if from Exercise 3.3 of Chapter 3. Therefore the above limit amounts to

$$\begin{aligned}
\lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} 3(t_j^* - t_j)^2 - 2(t_j^* - t_j)^2 + (t_j^* - t_j)^2 &= \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} 2(t_j^* - t_j)^2 \\
&\leq \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} 2(t_j^* - t_j)\|\Pi\| = 0
\end{aligned}$$

which is what we set out to prove. Thus, $\lim_{\|\Pi\|\to 0} Q_{\Pi/2} = \frac{1}{2}T$.

(ii) Define the Stratonovich integral of $W(t)$ with respect to $W(t)$ to be

$$\int_0^T W(t) \circ dW(t) = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} W(t_j^*)\big(W(t_{j+1}) - W(t_j)\big)$$

In contrast to the Itô integral $\int_0^T W(t)\ dW(t) = \frac{1}{2}W^2(T) - \frac{1}{2}T$ which evaluates the integrand at the left endpoint of each subinterval $[t_j, t_{j+1}]$, here we evaluate the integrand at the midpoint t_j^*. Show that

$$\int_0^T W(t) \circ dW(t) = \frac{1}{2}W^2(T)$$

soln. We write the approximating sum for the Stratonovich integral as the sum of an approximating sum for the Itô integral and $Q_{\Pi/2}$. We note that the approximating sum for the Itô integral is the one corresponding to the partition $0 = t_0 < t_0^* < t_1 < t_1^* < \cdots < t_{n-1}^* < t_n = T$, as opposed to the partition Π. We will get

$$\begin{aligned}
\sum_{j=0}^{n-1} W(t_j^*)\big(W(t_{j+1}) - W(t_j)\big) &= \sum_{j=0}^{n-1} W(t_j^*)W(t_{j+1}) - W(t_j^*)W(t_j) \\
&= \sum_{j=0}^{n-1} W(t_j^*)W(t_{j+1}) + W(t_j^*)W(t_j) - 2W(t_j^*)W(t_j) \\
&= \sum_{j=0}^{n-1} W(t_j^*)W(t_{j+1}) - W^2(t_j^*) + W(t_j^*)W(t_j) - W^2(t_j) + W^2(t_j^*) - 2W(t_j^*)W(t_j) + W^2(t_j) \\
&= \sum_{j=0}^{n-1} W(t_j^*)\big(W(t_{j+1}) - W(t_j^*)\big) + W(t_j)\big(W(t_j^*) - W(t_j)\big) + \sum_{j=0}^{n-1} \big(W(t_j^*) - W(t_j)\big)^2
\end{aligned}$$

Note that the first sum is the Itô integral corresponding to the partition involving midpoints. The second is the half-sample quadratic variation. Conjoining the first summand to simplify notation and taking the limit $\|\Pi\| \to 0$ gives

$$\begin{aligned}
\sum_{j=0}^{n-1} W(t_j^*)\big(W(t_{j+1}) - W(t_j^*)\big) + W(t_j)\big(W(t_j^*) - W(t_j)\big) &+ \sum_{j=0}^{n-1} \big(W(t_j^*) - W(t_j)\big)^2 \\
&= \sum_{j=0}^{n-1} W(t_j)\big(W(t_{j+1}) - W(t_j)\big) + Q_{\Pi/2} \\
&\to \frac{1}{2}W^2(T) - \frac{1}{2}T + \frac{1}{2}T \\
&= \frac{1}{2}W^2(T)
\end{aligned}$$

Thus, the Stratonovich integral has no additional $\frac{1}{2}T$ term, a result of evaluating the integral at the midpoint as opposed to the left endpoint, which is what we set out to prove.

Exercise 4.5 (Solving the generalized geometric Brownian motion equation). Let $S(t)$ be a positive stochastic process that satisfies the generalized geometric Brownian motion differential equation

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t) \tag{1}$$

where $\alpha(t)$ and $\sigma(t)$ are processes adapted to the filtration $\mathcal{F}(t), t \geq 0$, associated with the Brownian motion $W(t)$. In this exercise, we show that $S(t)$ must be given by the formula

$$S(t) = S(0)e^{X(t)} = S(0)\exp\left\{\int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right)ds\right\} \tag{2}$$

(i.e. this formula provides the only solution to the stochastic differential equation (1)). In the process, we provide a method for solving this equation.

(i) Using (1) and the Itô-Doeblin formula, compute $d\log(S(t))$. Simplify so that you have a formula for $d\log(S(t))$ that does not involve $S(t)$.

soln. We use the Itô-Doeblin formula with $f(t, S(t)) = \log(S(t))$. It is helpful now to recall that Brownian motion accumaltes quadratic variation at rate one per unit time, i.e. $dW(t)dW(t) = dt$, and that $dtdt = 0$ along with $dtdW(t) = dW(t)dt = 0$.

$$
\begin{aligned}
df(t, S(t)) &= f_t(t, S(t))dt + f_s(t, S(t))dS(t) + \frac{1}{2}f_{ss}(t, S(t))dS(t)dS(t) \\
&= \frac{dS(t)dt}{S(t)} + \frac{dS(t)}{S(t)} - \frac{dS(t)dS(t)}{2S^2(t)} \\
&= \frac{2S^2(t)dS(t)dt + 2S(t)dS(t) - dS(t)dS(t)}{2S^2(t)} \\
&= \frac{2S^2(t)(\alpha(t)S(t)dt + \sigma(t)S(t)dW(t))dt + 2S(t)(\alpha(t)S(t)dt + \sigma(t)S(t)dW(t)) - (\alpha(t)S(t)dt + \sigma(t)S(t)dW(t))^2}{2S^2(t)} \\
&= \frac{2\alpha(t)S^2(t)dt + 2\sigma(t)S^2(t)dW(t) - \sigma(t)S^2(t)dt}{2S^2(t)} \\
&= \alpha(t)dt + \sigma(t)dW(t) - \frac{1}{2}\sigma(t)dt
\end{aligned}
$$

(ii) Integrate the formula you obtained in (i), and then exponentiate the answer to obtain obtain (2).

soln. We proceed as follows

$$
\begin{aligned}
d\log S(t) &= \alpha(t)dt + \sigma(t)dW(t) - \frac{1}{2}\sigma(t)dt \\
\log S(t) &= \log S(0) + \int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right)ds \\
S(t) &= S(0)\exp\left\{\int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right)ds\right\}
\end{aligned}
$$

which is (2).

Exercise 4.6. Let $S(t) = S(0)\exp\{\sigma W(t) + (\alpha - \frac{1}{2}\sigma^2)t\}$ be a geometric Brownian motion. Let p be a positive constant. Compute $d(S^p(t))$, the differential of $S(t)$ raised to the power p.

soln.

$$
\begin{aligned}
d(S^p(t)) &= d(S(t) \cdot S^{p-1}(t)) \\
&= S^{p-1}(t)dS(t) + S(t)dS^{p-1}(t) \\
&= \alpha(t)S^p(t)dt + \sigma(t)S^p(t)dW(t) + S(t)(S^{p-2}(t)dS(t) + S(t)dS^{p-2}(t))
\end{aligned}
$$

Notice that repeated iterations of the differential operator results in $S(t)$ being raised to the same exponent p so that the result is then

$$d(S^p(t)) = p\alpha(t)S^p(t)dt + p\sigma(t)S^p(t)dW(t) = pS^{p-1}(t)dS(t)$$

Exercise 4.7. (i) Compute $dW^4(t)$ and then write $W^4(T)$ as the sum of an ordinary (Lebesgue) integral with respect to time and an Itô integral.
(ii) Take expectations on both sides of the formula you obtained in (i), use the fact that $\mathbb{E}W^2(t) = t$, and derive the formula $\mathbb{E}W^4(T) = 3T^2$.
(iii) Use the method of (i) and (ii) to derive a formula for $\mathbb{E}W^6(T)$.

Exercise 4.8 (Solving the Vasicek equation). The Vasicek interest rate stochastic differential equation is

$$dR(t) = (\alpha - \beta R(t))dt + \sigma dW(t) \tag{3}$$

where α, β, and σ are positive constants. The solution to this equation is given by

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t}\int_0^t e^{\beta s}dW(s) \tag{4}$$

This exercise shows how to derive this solution.

(i) Use (3) and the Itô-Doeblin formula to compute $d(e^{\beta t}R(t))$. Simplify it so that you have a formula for $d(e^{\beta t}R(t))$ that does not involve $R(t)$.

soln. We will use the Itô-Doeblin formula with $f(t,R(t)) = e^{\beta t}R(t)$. The computation is as follows

$$\begin{aligned}
d(e^{\beta t}R(t)) &= \left[\beta e^{\beta t}R(t) + e^{\beta t}\frac{dR(t)}{dt}\right]dt + e^{\beta t}dR(t) \\
&= 2\alpha e^{\beta t}dt + 2\sigma e^{\beta t}dW(t) - \beta e^{\beta t}R(t)dt \\
&= 2\alpha e^{\beta t}dt + 2\sigma e^{\beta t}dW(t) - \frac{d}{dt}e^{\beta t}\cdot R(t)dt \\
2d(e^{\beta t}R(t)) &= 2\alpha e^{\beta t}dt + 2\sigma e^{\beta t}dW(t) \\
\\
d(e^{\beta t}R(t)) &= \alpha e^{\beta t}dt + \sigma e^{\beta t}dW(t)
\end{aligned}$$

(ii) Integrate the equation you obtained in (i) and solve for $R(t)$ to obtain (4).

soln. We proceed as follows

$$\begin{aligned}
e^{\beta t}R(t) &= R(0) + \int_0^t \alpha e^{\beta s}ds + \int_0^t \sigma e^{\beta s}dW(s) \\
R(t) &= e^{-\beta t}R(0) + \frac{\alpha}{\beta e^{\beta t}}(e^{\beta t}-1) + \sigma e^{-\beta t}\int_0^t e^{\beta s}dW(s) \\
&= e^{-\beta t}R(0) + \frac{\alpha}{\beta}(1-e^{-\beta t}) + \sigma e^{-\beta t}\int_0^t e^{\beta s}dW(s)
\end{aligned}$$

which is the desired result.

Exercise 4.9. For a European call expiring at time T with strike price K, the Black-Scholes-Merton price at time t, if the time-t stock price is x, is

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x))$$

where

$$\begin{aligned}
d_+(\tau,x) &= \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r + \frac{1}{2}\sigma^2\right)\tau\right] \\
d_-(\tau,x) &= d_+(\tau,x) - \sigma\sqrt{\tau}
\end{aligned}$$

and $N(y)$ is the cumulative standard normal distribution

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{\frac{-z^2}{2}}dz = \frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty} e^{\frac{-z^2}{2}}dz$$

The purpose of this exercise is to show that the function c satisfies the Black-Scholes-Merton partial differential equation

$$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2x^2c_{xx}(t,x) = rc(t,x),\ 0 \le t < T,\ x > 0 \tag{5}$$

the *terminal condition*

$$\lim_{t\uparrow T} c(t,x) = (x-K)^+,\ x>0,\ x\neq K \tag{6}$$

and the *boundary conditions*

$$\lim_{x\downarrow 0} c(t,x) = 0,\ \lim_{x\to\infty}\left[c(t,x) - (x - e^{-r(T-t)}K)\right] = 0,\ 0\le t<T \tag{7}$$

Equation (6) and the first part of (7) are usually written more simply but less precisely as

$$c(T,x) = (x-K)^+,\ x \ge 0$$

and

$$c(t,0) = 0,\ 0 \le t \le T$$

For this exercise, we abbreviate $c(t,x)$ as simply c and $d_\pm(T-t,x)$ as simply $d_\pm$.

(i) Verify first the equation

$$Ke^{-r(T-t)}N'(d_-) = xN'(d_+)$$

soln. We first observe that $\frac{\partial}{\partial x}N(d_+) = N'(d_+)\frac{\partial d_+}{\partial x}$ where $N'(d_+) = \frac{1}{\sqrt{2\pi}}e^{\frac{-d_+^2}{2}}$. We must show that

$$\exp\left\{\frac{d_+^2 - d_-^2}{2}\right\} = \frac{xe^{r(T-t)}}{K}$$

Let us first expand the argument of the exponential function.

$$\begin{aligned}
d_+^2 - d_-^2 &= d_+^2 - (d_+ - \sigma\sqrt{T-t})^2 \\
&= 2d_+\sigma\sqrt{T-t} - \sigma^2(T-t) \\
&= 2\left[\log\frac{x}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right] - \sigma^2(T-t) \\
&= 2\log\frac{x}{K} + 2r(T-t) \\
&= 2\log\frac{xe^{r(T-t)}}{K}
\end{aligned}$$

The equality is then obtained by dividing by two and then taking the exponential of both sides.

(ii) Show that $c_x = N(d_+)$. This is the *delta* of the option.

soln. The delta of the call option is the derivative of the call option with respect to the spot price, which in this exercise, is appropriately labeled x. We proceed as follows

$$\begin{aligned}
c_x &= N(d_+) + xN'(d_+)\frac{\partial d_+}{\partial x} - Ke^{-r(T-t)}N'(d_-)\frac{\partial d_-}{\partial x} \\
&= N(d_+) + \left(xN'(d_+) - Ke^{-r(T-t)}N'(d_-)\right)\frac{\partial d_+}{\partial x} \\
&= N(d_+)
\end{aligned}$$

Where we have used the result obtained in (i) as well as the fact that the $\sigma\sqrt{\tau}$ term in d_- is negligible in the evaluation of its partial derivative with respect to x.

(iii) Show that

$$c_t = -rKe^{-r(T-t)}N(d_-) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+)$$

This is the *theta* of the option.

soln. We will use the equality established in (i). The computation for the theta of the option is as follows

$$\begin{aligned}
c_t &= xN'(d_+)\frac{\partial d_+}{\partial t} - rKe^{-r(T-t)}N(d_-) - Ke^{-r(T-t)}N'(d_-)\frac{\partial d_+}{\partial t} && (8) \\
&= (xN'(d_+) - Ke^{-r(T-t)}N'(d_-))\frac{\partial d_+}{\partial t} - rKe^{-r(T-t)}N(d_-) - \frac{\sigma}{2\sqrt{T-t}}Ke^{-r(T-t)}N'(d_-) && (9) \\
&= -rKe^{-r(T-t)}N(d_-) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+) && (10)
\end{aligned}$$

(iv) Use the formulas above to show that c satisfies (5).

soln. We need to obtain the second partial derivative of the option with respect to x. This is known as the *gamma* of the option and is computed as follows

$$\begin{aligned}
c_{xx} &= N'(d_+)\frac{\partial d_+}{\partial x} \\
&= \frac{N'(d_+)}{x\sigma\sqrt{T-t}}
\end{aligned}$$

We can now proceed with the verification of equation (5).

$$\begin{aligned}
c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2x^2c_{xx}(t,x) &= -rKe^{-r(T-t)}N(d_-) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+) + rxN(d_+) + \frac{1}{2}\sigma^2x^2\frac{N'(d_+)}{x\sigma\sqrt{T-t}} \\
&= rxN(d_+) - rKe^{-r(T-t)}N(d_-) \\
&= r(xN(d_+) - Ke^{-r(T-t)}N(d_-)) \\
&= rc(t,x)
\end{aligned}$$

(v) Show that for $x > K$, $\lim_{t\uparrow T} d_\pm = \infty$, but for $0 < x < K$, $\lim_{t\uparrow T} d_\pm = -\infty$. Use these equations to verify the terminal condition (6).

soln. When $x > K$, $\log \frac{x}{K} > 0$. As $t \uparrow T$ the denominator of d_+ will become close to zero, leading to a limit of ∞. When $0 < x < K$, $\log \frac{x}{K} < 0$, so that there is a minus sign in front of the limit ∞. Because $N(\infty)$ is 1 and $N(-\infty)$ is 0 we conclude that $\lim_{t\uparrow T} c(t,x) = (x-K)^+$.

(vi) Show that for $0 \le t < T$, $\lim_{x\downarrow 0} d_\pm = -\infty$. Use this fact to verify the first part of boundary condition (7) as $x \downarrow 0$.

soln. Suppose $0 \le t < T$. Then $\lim_{x\downarrow 0} d_\pm = -\infty$ because $\lim_{x\downarrow 0} \log \frac{x}{K} = -\infty$.We then have that $\lim_{x\downarrow 0} c(t,x) = 0$ because the cdf of the normal variable evaluated at $-\infty$ is 0.

(vii) Show that for $0 \le t < T$, $\lim_{x\to\infty} d_\pm = \infty$. Use this fact to verify the second part of boundary condition (7) as $x \to \infty$.

soln. We first show that for $0 \le t < T$, $\lim_{x\to\infty} d_\pm = \infty$. This is obvious since the logarithm will increase without bound under these conditions. Next we'll verify $\lim_{x\to\infty}[c(t,x) - (x - e^{-r(T-t)}K)] = 0$. This amounts to showing that

$$\lim_{x\to\infty} \frac{N(d_+) - 1}{x^{-1}} = 0$$

Notice that $\lim_{x\to\infty} \frac{N(d_+)-1}{x^{-1}}$ is an indeterminate form $\frac{0}{0}$. We use L'Hôpital's rule and the fact that

$$x = K \exp\left\{\sigma\sqrt{T-t}d_+ - (T-t)\left(r + \frac{1}{2}\sigma^2\right)\right\}$$

to obtain

$$\begin{aligned}
\lim_{x\to\infty} \frac{\frac{d}{dx}N(d_+)}{\frac{d}{dx}x^{-1}} &= \lim_{x\to\infty} \frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{d_+^2}{2}}\frac{1}{x\sigma\sqrt{T-t}}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{xe^{-\frac{d_+^2}{2}}}{\sigma\sqrt{2\pi(T-t)}} \\
&= \lim_{d_+\to\infty} \frac{K\exp\left\{\sigma\sqrt{T-t}d_+ - (T-t)\left(r + \frac{1}{2}\sigma^2\right) - \frac{d_+^2}{2}\right\}}{\sigma\sqrt{2\pi(T-t)}}
\end{aligned}$$

$-d_+^2$ will overtake the rest of the values in the expression as $d_+ \to \infty$. Since there is a minus sign in front and it is in the argument of an exponential, the limit will approach 0. Thus

$$\lim_{d_+\to\infty} \frac{K\exp\left\{\sigma\sqrt{T-t}d_+ - (T-t)\left(r + \frac{1}{2}\sigma^2\right) - \frac{d_+^2}{2}\right\}}{\sigma\sqrt{2\pi(T-t)}} = 0$$

which implies that $\lim_{x\to\infty} \frac{N(d_+)-1}{x^{-1}} = 0$.

Exercise 4.9 (Self-financing trading). The fundamental idea behind no-arbitrage pricing is to reproduce the payoff of a derivative security by trading in the underlying asset (which we call a stock) and the money market account. In discrete time, we let X_k denote the value of the hedging portfolio at time k and let Δ_k denote the number of shares of stock held between times k and $k+1$. Then, at time k, after rebalancing (i.e. moving from a position of Δ_{k-1} to a position Δ_k in the stock), the amount in the money market account is $X_k - S_k\Delta_k$. The value of the portfolio at time $k+1$ is

$$X_{k+1} = \Delta_k S_{k+1} + (1+r)(X_k - \Delta_k S_k) \tag{11}$$

This formula can be rearranged to become

$$X_{k+1} - X_k = \Delta_k(S_{k+1} - S_k) + r(X_k - \Delta_k S_k) \tag{12}$$

which says that the gain between time k and time $k+1$ is the sum of the capital gain on the stock holdings, $\Delta_k(S_{k+1} - S_k)$, and the interest earnings on the money market account, $r(X_k - \Delta_k S_k)$. The continuous time analogue of (12) is

$$dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt \tag{13}$$

Alternatively, one could define the value of a share of the money market account at time k to be

$$M_k = (1+r)^k \tag{14}$$

and formulate the discrete-time model with two processes, Δ_k as before and Γ_k denoting the number of shares of the money market account held at time k after rebalancing. Then

$$X_k = \Delta_k S_k + \Gamma_k M_k \tag{15}$$

so that (11) becomes

$$X_{k+1} = \Delta_k S_{k+1} + (1+r)\Gamma_k M_k = \Delta_k S_k + \Gamma_k M_{k+1} \tag{16}$$

Subtracting (15) from (16), we obtain in place of (12) the equation

$$X_{k+1} - X_k = \Delta_k (S_{k+1} - S_k) + \Gamma_k (M_{k+1} - M_k) \tag{17}$$

which says that the gain between time k and time $k+1$ is the sum of the capital gain on stock holdings $\Delta_k(S_{k+1} - S_k)$, and the earnings from the money market investment $\Gamma_k(M_{k+1} - M_k)$.

But Δ_k and Γ_k cannot be chosen arbitrarily. The agent arrives at time $k+1$ with some portfolio of Δ_k shares of stock and Γ_k shares of the money market account and then rebalances. In terms of Δ_k and Γ_k, the value of the portfolio upon arrival at time $k+1$ is by (16). After rebalancing, it is

$$X_{k+1} = \Delta_{k+1} S_{k+1} + \Gamma_{k+1} M_{k+1} \tag{18}$$

Setting these two values equal, we obtain the *discrete-time self-financing condition*

$$S_{k+1}(\Delta_{k+1} - \Delta_k) + M_{k+1}(\Gamma_{k+1} - \Gamma_k) = 0 \tag{19}$$

The first term is the cost of rebalancing in the stock, and the second is the cost of rebalancing in the money market account. If the sum of these two terms is not zero, then money must either be put into the position or can be taken out as a by-product of rebalancing. The point is that when the two processes Δ_k and Γ_k are used to desribe the evolution of the portfolio value X_k, then the two equations (17) and (19), are require rather than the single equation (12) when only the process Δ_k is used.

Finally, we note that we may rewrite the *discrete-time self-financing condition* (19) as

$$S_k(\Delta_{k+1} - \Delta_k) + (S_{k+1} - S_k)(\Delta_{k+1} - \Delta_k) + M_k(\Gamma_{k+1} - \Gamma_k) + M_{k+1}(\Gamma_{k+1} - \Gamma_k) = 0 \tag{20}$$

This is suggestive of the *continuous-time self-financing condition*

$$S(t)d\Delta(t) + dS(t)d\Delta(t) + M(t)d\Gamma(t) + dM(t)d\Gamma(t) = 0 \tag{21}$$

which we derive below.

(i) In continuous time, let $M(t) = e^{rt}$ be the price of a share of the money market account at time t, let $\Delta(t)$ denote the number of shares of stock held at time t, and let $\Gamma(t)$ denote the number of shares of the money market account held at time t, so that the total portfolio value at time t is

$$X(t) = \Delta(t)S(t) + \Gamma(t)M(t) \tag{22}$$

Using (22) and (13), derive the continuous-time self-financing condition (21).