# Numbers

Mathematical Foundations Lecture Series Part III

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# Numbers?

# It Starts With One

# Then 2, 3, 4,…

- The natural numbers or the counting numbers.
- Typical indexing set.

$$\mathbb{N}$$

And so on…

# Reflection of the Natural Numbers About Zero

- Base, axis, current position: zero, null point, naught.
- Negative numbers -1, -2, -3, -4,…

# The Integers

- closed under the operation of addition and multiplication.
- Associativity.
- Commutativity.
- Existence of an identity.
- Existence of an inverse (additive).

$\mathbb{Z}$

$$\mathbb{Q}$$

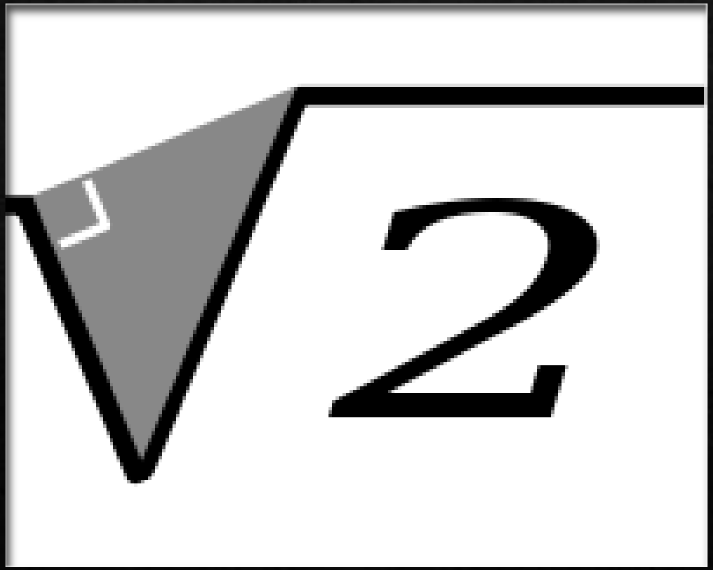
$$\frac{1}{p} + \frac{1}{q} = 1.$$

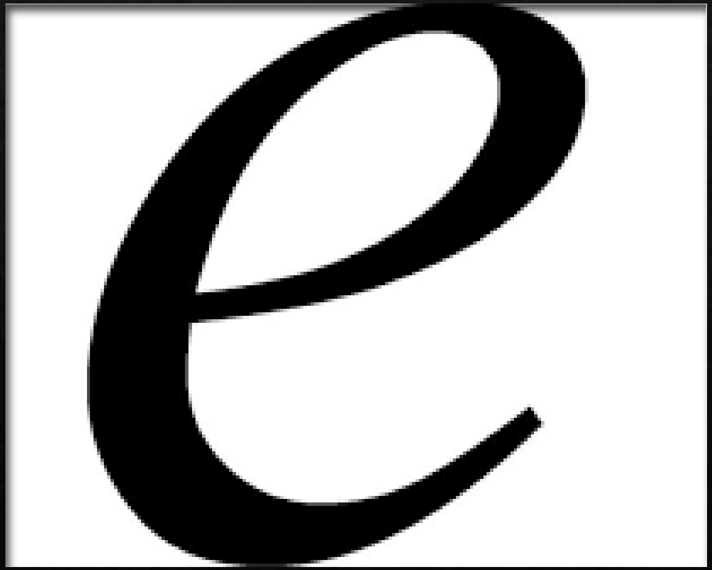
# The Rationals

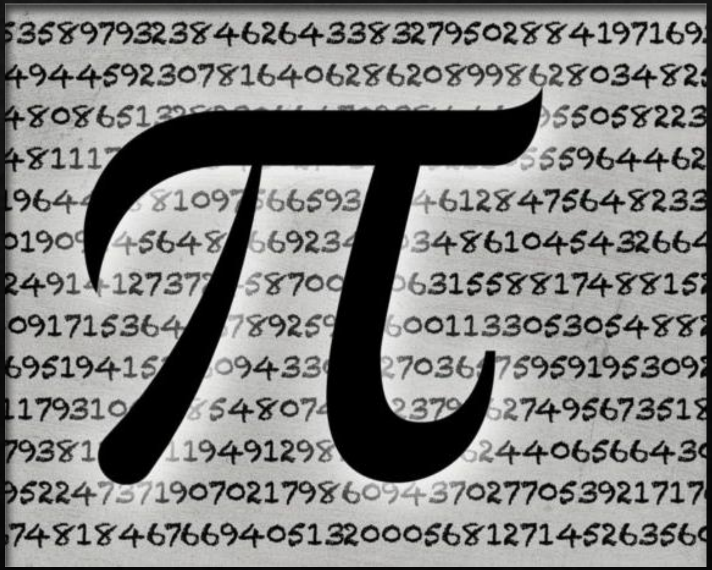
- Ratios of the integers.
- Quotients of integers.
- Terminating decimal sequences.

# The Real Numbers

- The collection of rational numbers and irrational numbers.
- Irrational numbers are nonterminating, nonrepeating decimal sequences.

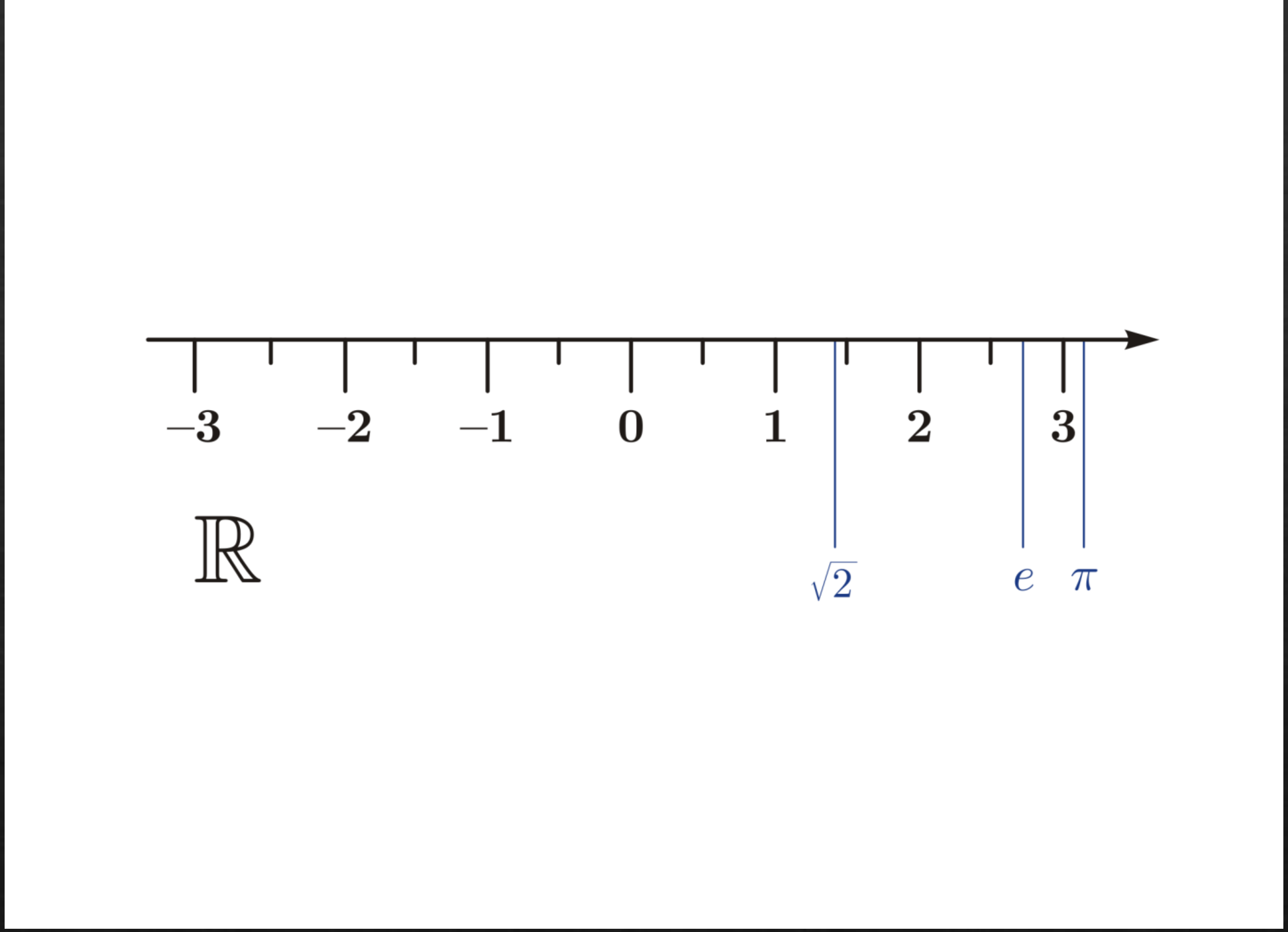






# Real Numbers

- Satisfy the field axioms.
- Satisfy the least upper bound axiom.
- Can be represented geometrically on a line.

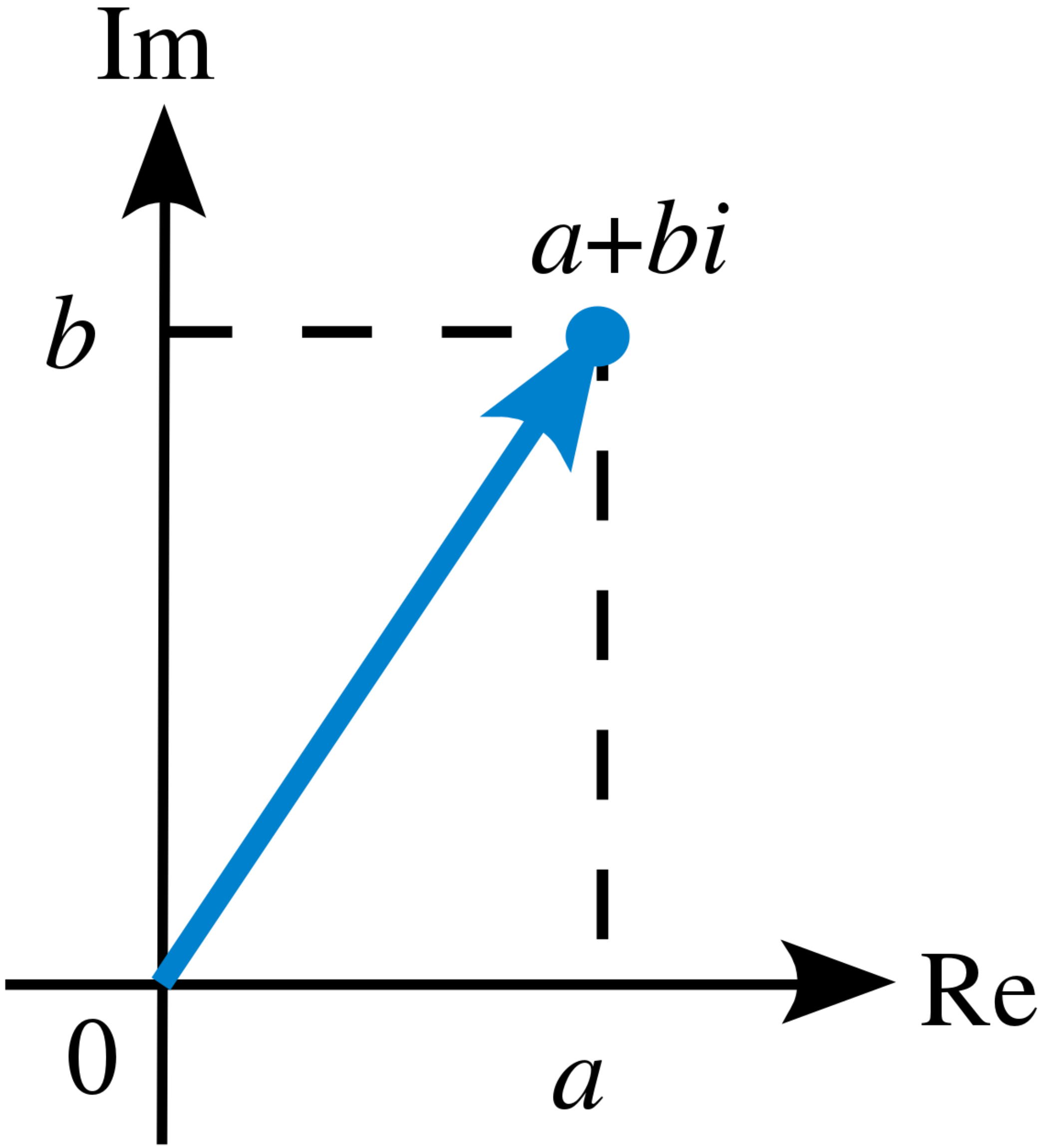


# The Archimedean Property

- For every real number there exists a larger positive integer.

# Complex Numbers

Numbers of the form a + bi where a and b are real numbers and i is the imaginary component



# Imaginary Numbers Are Real Things

$$i^2 = -1$$

# Useful Algebraic Interpretation

- Real polynomials modulo x^2 + 1 are isomorphic to the complex numbers.

$$\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$$

# Factorials

- $n! = n\ (n-1)\ (n-2)\ \dots\ 2\ 1$

| $N$ | $N!$ |
|---|---|
| 0 | 1 |
| 1 | 1 |
| 2 | 2 |
| 3 | 6 |
| 4 | 24 |
| 5 | 120 |
| 6 | 720 |
| 7 | 5,040 |
| 8 | 40,320 |
| 9 | 362,880 |
| 10 | 3,628,800 |

# What's So Special About Factorials?

- They're products of consecutive lists of integers
- Exercise: There are arbitrarily large gaps between consecutive primes.  (In other words, it is possible to find arbitrarily large sets of consecutive non-prime numbers.)
- How to solve? With the factorial.

# Proof

1.28) Show that for every positive integer $k$, there exist $k$ consecutive composite integers. Thus, there are arbitrarily large gaps between primes.

*Proof.* This proof provides a new way of looking at $n!$. Consider $(k+1)!$. We can see that

$$\begin{aligned}
(k+1)! + 2 &= [(k+1)(k)(k-1)\cdots(4)(3)+1]\cdot 2\\
(k+1)! + 3 &= [(k+1)(k)(k-1)\cdots(4)(2)+1]\cdot 3\\
(k+1)! + 4 &= [(k+1)(k)(k-1)\cdots(3)(2)+1]\cdot 4\\
&\vdots\\
(k+1)! + k &= [(k+1)(k-1)\cdots(3)(2)+1]\cdot k\\
(k+1)! + (k+1) &= [(k)(k-1)\cdots(3)(2)+1]\cdot (k+1)
\end{aligned}$$

so that $2 \mid (k+1)!+2,\ 3 \mid (k+1)!+3,\ \ldots,\ k+1 \mid (k+1)!+k+1$. This is a sequence of $k$ composite consecutive integers. $n!$ provides all the integers needed to make this possible since it contains $n-1$ consecutive factors. $\square$

# Famous Problem

- What is the sum of the first 100 consecutive integers? Try to think of a way to do it without counting your fingers. Refer to Gauss if you get stuck.

# Math Induction

- A method of a proof where you first prove a base case n=0 or 1.
- Assume that it is true for n.
- Prove the statement to be true for n+1.

# The Trolley Problem

- In a purely abstracted sense the decision is the same either way.

$$i^n = \begin{cases} 1 & \text{if } n \equiv 0 \bmod 4 \\ i & \text{if } n \equiv 1 \bmod 4 \\ -1 & \text{if } n \equiv 2 \bmod 4 \\ -i & \text{if } n \equiv 3 \bmod 4 \end{cases}$$

$$\Longrightarrow \quad i^1 = i^5$$

# More To It Than Just Numbers

# From Numbers to Commutative Diagrams

**Proposition 1.** *Given a commutative diagram of groups and homomorphisms*

$$\begin{tikzcd}
G_1 \arrow[r, "\theta_1"] \arrow[d, "\psi_1"] & G_2 \arrow[r, "\theta_2"] \arrow[d, "\psi_2"] & G_3 \arrow[r, "\theta_3"] \arrow[d, "\psi_3"] & G_4 \arrow[r, "\theta_4"] \arrow[d, "\psi_4"] & G_5 \arrow[d, "\psi_5"] \\
H_1 \arrow[r, "\phi_1"] & H_2 \arrow[r, "\phi_2"] & H_3 \arrow[r, "\phi_3"] & H_4 \arrow[r, "\phi_4"] & H_5
\end{tikzcd}$$

*in which the rows are exact sequences, and $\psi_2, \psi_4$ are isomorphisms, $\psi_1$ is onto and $\psi_5$ is $(1-1)$, then $\psi_3$ is an isomorphism.*

*Proof.* To show that $\psi_3$ is $(1-1)$, consider an element $x \in G_3$ such that $\psi_3(x) = 1$. Then $\psi_4\theta_3(x) = \phi_3\psi_3(x) = 1$.

$$\begin{tikzcd}
G_1 \arrow[r, "\theta_1"] \arrow[d, "\psi_1"] & G_2 \arrow[r, "\theta_2"] \arrow[d, "\psi_2"] & G_3 \arrow[r, "\theta_3"] \arrow[d, dashed, "\psi_3"] & G_4 \arrow[r, "\theta_4"] \arrow[d, "\psi_4"] & G_5 \arrow[d, "\psi_5"] \\
H_1 \arrow[r, "\phi_1"] & H_2 \arrow[r, "\phi_2"] & H_3 \arrow[r, "\phi_3"] & H_4 \arrow[r, "\phi_4"] & H_5
\end{tikzcd}$$

so that $\theta_3(x) = 1$ since $\psi_4$ is an isomorphism. By exactness, $x = \theta_2(y)$, $y \in G_2$; and then $\phi_2\psi_2(y) = \psi_3\theta_2(y) = 1$. By exactness again, $\psi_2(y) = \phi_1(z)$, $z \in H_1$; and $z = \psi_1(w)$, $w \in G_1$ since $\psi_1$ is onto. Thus, $\psi_2\theta_1(w) = \phi_1\psi_1(w) = 1 = \psi_2(y)$, so that $\theta_1(w) = y$; but then $x = \theta_2(y) = \theta_2\theta_1(w) = 1$ $\square$