Linear Algebra Homework 6

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11a) Prove that if a square matrix A is similar to a scalar matrix λI, then $A = \lambda I$

Proof. Let $A \in M_{n\times n}(F)$ be a matrix similar to a scalar matrix λI, for some scalar λ. Let Q be an invertible matrix of size $n \times n$. Then,

$$\begin{aligned} A &= Q^{-1}(\lambda I)Q \\ &= (Q^{-1}\lambda)(IQ) \\ &= (Q^{-1}\lambda)Q \\ &= \lambda(Q^{-1}Q) \\ &= \lambda I \end{aligned}$$

Thus, $A = \lambda I$. □

11b) Show that a diagonalizable matrix having only one eigenvalue is a scalar matrix.

Proof. Suppose that $A \in M_{n\times n}(F)$ is a diagonalizable matrix having only one eigenvalue. Then, A is similar to some diagonal matrix. This diagonal matrix will have the egeinvalues of A, call them $\lambda \in F$, along its diagonal. Since λ is the only eigenvalue for A, the diagonal matrix has the form λI. Since A is similar to λI, by part (a), $A = \lambda I$, so A is a scalar matrix. □

11c) Prove that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

Proof. $\det\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}\right) = 0$
$\iff (1-\lambda)^2 = 0 \iff \lambda = 1$ with multiplicity 2. There is only one eigenvalue for this matrix and it is not a scalar matrix, which means that is is not diagonalizable. Thus, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable. □

15a) Let T be a linear operator on a vector space V, and let x be an eigenvector of T corresponding to the eigenvalue λ. For any positive integer m, prove that x is an eigenvecotr of T^m corresponding to the eigenvalue λ^m.

Proof. Let T be a linear operator on a vector space V, and let x be an eigenvector of T corresponding to the eigenvalue λ.

Base case: $m = 1$. $T(x) = \lambda x$. This is true since x is an eigenvector corresponding to the eigenvalue λ.

Next, assume $T^m(x) = \lambda^m x$. Then,

$$\begin{aligned} T^m(x) &= \lambda^m x \\ \lambda T^m(x) &= \lambda\lambda^m x \\ T^m(\lambda x) &= \lambda^{m+1} x \\ T^{m+1}(x) &= \lambda^{m+1} x \end{aligned}$$

This completes the inductive step. $\square$

20) Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \ldots + a_1 t + a_0$$

Prove that $f(0) = a_0 = \det(A)$. Deduce that A is invertible if and only if $a_0 \neq 0$.

Proof. $f(t) = \det(A - tI_n)$. $f(0) = \det(A - 0I_n) = \det(A - 0) = \det(A)$. But, $f(0) = a_0$, therefore $\det(A) = a_0$ as well.

Suppose that A is invertible. Then, $\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{a_0}$ is well defined, meaning $a_0 \neq 0$. Next, Suppose $a_0 \neq 0$. Then, $\det(A) \neq 0$, which means that A is row reducible to I_n, thus A is invertible. $\square$

21a) Let A and $f(t)$ be as in Exercise 20. Prove that $f(t) = (A_{11}-t)(A_{22}-t)\cdots(A_{nn}-t)+q(t)$, where $q(t)$ is a polynomial of degree at most $n-2$.

Proof. We check the base case $n = 2$. Then

$$f(t) = \det(A - tI_2) = \begin{vmatrix} a_{11} - t & a_{12} \\ a_{21} & a_{22} - t \end{vmatrix} = (a_{11} - t)(a_{22} - t) - a_{12}a_{21}$$

Here, $q(t) = -a_{12}a_{21}$. So, $\deg(q(t)) \leq 0 = n - 2$. Next, assume that $f(t) = (A_{11} - t)(A_{22} - t)\cdots(A_{nn} - t) + q(t)$ where $q(t)$ is a polynomial of degree at most $n - 2$ is true. Consider an $(n+1) \times (n+1)$ matrix A. It's characteristic polynomial is given by $f(t) = \det(A - tI_{n+1n+1})$. This is given by,

$$\begin{vmatrix} a_{11} - t & a_{12} & \cdots & a_{1n} & a_{1n+1} \\ a_{21} & a_{22} - t & \cdots & a_{2n} & a_{2n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - t & a_{nn+1} \\ a_{n+11} & a_{n+12} & \cdots & a_{n+1n} & a_{n+1n+1} - t \end{vmatrix}$$

By the deleting the first row and the first column this becomes,

$$(a_{11} - t)\begin{vmatrix} a_{22} - t & \cdots & a_{2n} & a_{2n+1} \\ \vdots & & \vdots & \vdots \\ a_{n2} & \cdots & a_{nn} - t & a_{nn+1} \\ a_{n+12} & \cdots & a_{n+1n} & a_{n+1n+1} - t \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & \cdots & a_{2n} & a_{2n+1} \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} - t & a_{nn+1} \\ a_{n+11} & \cdots & a_{n+1n} & a_{n+1n+1} - t \end{vmatrix} + \ldots$$

$$+(-1)^{n+2}a_{1n+1}\begin{vmatrix} a_{21} & a_{22} - t & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - t \\ a_{n+11} & a_{n+12} & \cdots & a_{n+1n} \end{vmatrix} \tag{1}$$

The first determinant in the sum is that of size $n \times n$, with the $a_{ii} - t$ $(2 \le i \le n)$ terms along the entire diagonal. By assumption, this equals $(a_{22}-t)(a_{33}-t)\cdots(a_{n+1n+1}-t)+q(t)$, where $\deg(q(t)) \le n-2$. So, (1) can be rewritten as,

$$(a_{11}-t)\Big[(a_{22}-t)(a_{33}-t)\cdots(a_{n+1n+1}-t)+q(t)\Big]-a_{12}\begin{vmatrix} a_{21} & \cdots & a_{2n} & a_{2n+1} \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn}-t & a_{nn+1} \\ a_{n+11} & \cdots & a_{n+1n} & a_{n+1n+1}-t \end{vmatrix}+\dots$$

$$+(-1)^n a_{1n+1}\begin{vmatrix} a_{21} & a_{22}-t & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-t \\ a_{n+11} & a_{n+12} & \cdots & a_{n+1n} \end{vmatrix}$$

which after distributing $a_{11}-t$ becomes,

$$(a_{11}-t)(a_{22}-t)(a_{33}-t)\cdots(a_{n+1n+1}-t)+(a_{11}-t)q(t)-a_{12}\begin{vmatrix} a_{21} & \cdots & a_{2n} & a_{2n+1} \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn}-t & a_{nn+1} \\ a_{n+11} & \cdots & a_{n+1n} & a_{n+1n+1}-t \end{vmatrix}+\dots$$

$$+\begin{vmatrix} a_{21} & a_{22}-t & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-t \\ a_{n+11} & a_{n+12} & \cdots & a_{n+1n} \end{vmatrix} \tag{2}$$

The remaining determinants in (2) form polynomials of degree at most $n-1$ since each determinant is of size $n \times n$ and missing at least one $a_{ii}-t$ term. $(a_{11}-t)q(t)$ is a polynomial of degree at most $n-1$. So, let

$$p(t) = (a_{11}-t)q(t) - a_{12}\begin{vmatrix} a_{21} & \cdots & a_{2n} & a_{2n+1} \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn}-t & a_{nn+1} \\ a_{n+11} & \cdots & a_{n+1n} & a_{n+1n+1}-t \end{vmatrix}+\dots$$

$$+\begin{vmatrix} a_{21} & a_{22}-t & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-t \\ a_{n+11} & a_{n+12} & \cdots & a_{n+1n} \end{vmatrix}$$

Then, $f(t) = (a_{11}-t)(a_{22}-t)(a_{33}-t)\cdots(a_{n+1n+1}-t)+p(t)$, where $p(t)$ is of degree at most $n-1$. $\square$

21b) Show that $\mathrm{tr}(A) = (-1)^{n-1}a_{n-1}$.

Proof. We proceed with induction on n. We first check the base case, $n = 1$.

$$f(t) = (a_{11}-t) = -t + a_{11}$$

Here, $\mathrm{tr}(A) = a_{11} = (-1)^0 a_0$, where $a_0 = a_{11}$ is the constant term in the above equation. Next, assume that $\mathrm{tr}(A) = (-1)^{n-1}a_{n-1}$ for holds n. This can be rewritten as $(-1)^{1-n}\mathrm{tr}(A) = a_{n-1}$. Then, $(-1)^{1-n}\mathrm{tr}(A) = (-1)^{1-n}(a_{11}+a_{22}+\ldots+a_{nn}) = a_{n-1}$. By part (a) we can write,

$$f(t) = (a_{11}-t)(a_{22}-t)\cdots(a_{nn}-t)+q(t) = (-1)^n t^n + a_{n-1}t^{n-1}+\ldots+a_1 t + a_0$$

where $q(t)$ is of degree at most $n-2$. Using substitution,

$$(a_{11}-t)(a_{22}-t)\cdots(a_{nn}-t)+q(t)=(-1)^n t^n+(-1)^{1-n}(a_{11}+a_{22}+\ldots+a_{nn})t^{n-1}+\ldots+a_1t+a_0$$

Let $p=n+1$. Multiplying both sides of this equation by $(a_{pp}-t)$ yields,

$$(a_{11}-t)(a_{22}-t)\cdots(a_{nn}-t)(a_{pp}-t)+q(t)(a_{pp}-t)=$$
$$(a_{pp}-t)\Big[(-1)^n t^n+(-1)^{1-n}(a_{11}+a_{22}+\ldots+a_{nn})t^{n-1}+\ldots+a_1t+a_0\Big]$$

Through distribution you obtain,

$$a_{pp}(-1)^n t^n+(-1)^{1-n}(a_{pp})(a_{11}+a_{22}+\ldots+a_{nn})t^{n-1}+\ldots+a_{pp}a_1t+a_{pp}a_0$$
$$(-1)^p t^p+(-1)^{2-n}(a_{11}+a_{22}+\ldots+a_{nn})t^n-\ldots-a_1t^2-a_0t$$

Rearranging terms gives

$$(-1)^{n+1}t^{n+1}+(-1)^{2-n}(a_{11}+a_{22}+\ldots+a_{nn})t^n+a_{n+1n+1}(-1)^n t^n+p(t) \tag{3}$$

where $p(t)$ is the rest of the polynomial. Notice that $(-1)^{2-n}=(-1)^{-n}=(-1)^n,\ \forall n\in\mathbb{N}$. Then, (3) can be further reduced to

$$(-1)^{n+1}t^{n+1}+(-1)^n(a_{11}+a_{22}+\ldots+a_{nn}+a_{n+1n+1})t^n+p(t)$$

So, the coefficient of t^n, which we'll denote a_n, is equal to $(-1)^n(tr(A'))$ where A' is a matrix of size $(n+1)\times(n+1)$. So, $a_n=(-1)^n(tr(A'))\iff tr(A')=(-1)^n a_n$. $\square$