General Form of a Solution to a Linear Equation

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A linear differential equation of order n has the general form,

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

This equation is linear because all of the coefficients are functions of x, and because y and all of its derivatives are raised to the power of 1. For $n = 1$ we obtain a *linear equation*,

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{1}$$

By dividing (1) with $a_1(x)$ and replacing $\frac{a_0(x)}{a_1(x)}$ with $P(x)$ for simplicity, we obtain the *standard form*,

$$\frac{dy}{dx} + P(x)y = f(x) \tag{2}$$

With a little bit of algebra, we can rewrite (2) as

$$dy + \big[P(x)y - f(x)\big]dx = 0 \tag{3}$$

An interesting and useful property of linear equations is that there is always an integrating factor $\mu(x)$, that when multiplied into (3) yields an exact differential equation of the form,

$$\mu(x)dy + \mu(x)\big[P(x)y - f(x)\big]dx = 0 \tag{4}$$

Recall that an exact differential equation of the form $P(x, y)dx + Q(x, y)dy$ has a theorem associated with it that states that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Since (4) is exact we can say,

$$\frac{\partial}{\partial y}\mu(x)\big[P(x)y - f(x)\big] = \frac{\partial}{\partial x}\mu(x) \tag{5}$$

Evaluating this we obtain,

$$\mu(x)P(x) = \frac{d\mu}{dx} \tag{6}$$

which is separable and can be written as,

$$\frac{d\mu}{\mu(x)} = P(x)dx \tag{7}$$

By integrating both sides and raising each to the power of e, the integrating factor is obtained.

$$\mu = e^{\int P(x)dx} \tag{8}$$

Putting (8) into (4) gives

$$e^{\int P(x)dx}dy + e^{\int P(x)dx}\big[P(x)y - f(x)\big]dx = 0 \tag{9}$$

$$e^{\int P(x)dx}dy + e^{\int P(x)dx}P(x)ydx = e^{\int P(x)dx}f(x)dx \tag{10}$$

the left side of which can be reduced to the derivative of the product between the integrating factor and y,

$$d\big[e^{\int P(x)dx}y\big] = e^{\int P(x)dx}f(x)dx \tag{11}$$

and after integrating both sides,

$$e^{\int P(x)dx}y = \int e^{\int P(x)dx}f(x)dx + c \tag{12}$$

or

$$y = e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)dx + ce^{-\int P(x)dx} \tag{13}$$

which is our general one parameter family of solutions to a linear equation written in standard form.

Now that we've obtained the general form for our solution, it is natural to either admire such a result, or to be completely daunted by it. The way to appease both parties is by a practical example.

Example 1: Suppose that we have an RL circuit with inductance L, resistance R, and potential difference E, all constant, with variable current i. We wish to find a solution to the differential equation,

$$L\frac{di}{dt} + Ri = E$$

subject to the initial condition $i(0) = i_0$.

Solution: We have before us an initial value problem involving a linear equation. The first step is to convert the equation into standard form by dividing out L from the first term to obtain,

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$$

or

$$di + \frac{R}{L}i\,dt = \frac{E}{L}dt$$

Next, we recognize that i is linear in t and that our equation can be exact when multiplied by an integrating factor, which is of the form,

$$e^{\int \frac{R}{L}dt} = e^{\frac{R}{L}t}$$

Where $\tau = \frac{R}{L}$. Multiplying our equation with this integrating factor results in,

$$e^{\frac{R}{L}t}di + e^{\frac{R}{L}t}\frac{R}{L}i\,dt = e^{\frac{R}{L}t}\frac{E}{L}dt$$

or

$$d\left[e^{\frac{R}{L}t}i\right] = e^{\frac{R}{L}t}\frac{E}{L}dt$$

Integrating both sides we obtain,

$$e^{\frac{R}{L}t}i = \frac{E}{R}e^{\frac{R}{L}t} + c$$

so,

$$i = \frac{E}{R} + ce^{-\frac{R}{L}t}$$

Using our initial condition $i(0) = i_0$ implies that $c = i_0 - E/R$ and the solution to our equation is then,

$$i(t) = \frac{E}{R} + \left(i_0 - \frac{E}{R}\right)e^{-\frac{R}{L}t}$$

and we are done!