Shreve Function

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Abstract

Here is a very interesting function that was moderately difficult to integrate. The reason I find it interesting is because the function argument has an absolute value which convolutes the integral. I actually worked backwards, starting with the marginal densities and getting to the desired integral. Here I present it in a forward fashion along with the plot of the function, which is rather intriguing.

The Function

Exercise 2.5 Let (X, Y) be a pair of random variables with joint density function

$$f_{X,Y}(x,y) = \begin{cases} \frac{2|x|+y}{\sqrt{2\pi}} \exp\left\{ -\frac{(2|x|+y)^2}{2} \right\}, & \text{if } y \geq -|x|, \\ 0, & \text{if } y < -|x| \end{cases}$$

Show that X and Y are standard normal random variables and that they are uncorrelated but not independent.

soln. We use the fact that the marginal distribution of X can be obtained from integrating the joint distribution with respect to y. Similarly, the marginal distribution of Y can be obtained from integrating the joint distribution with respect to x. We fix x to obtain

$$f_X(x) = \int_{-|x|}^{\infty} \frac{2|x|+y}{\sqrt{2\pi}} \exp\left\{ -\frac{(2|x|+y)^2}{2} \right\} dy$$

We make the substitution $u = 2|x| + y$ so that

$$\int_{-|x|}^{\infty} \frac{2|x|+y}{\sqrt{2\pi}} \exp\left\{ -\frac{(2|x|+y)^2}{2} \right\} dy = \int_{|x|}^{\infty} \frac{u}{\sqrt{2\pi}} \exp\left\{ \frac{-u^2}{2} \right\} du$$

We make the substitution $v = \frac{-u^2}{2}$ to so that

$$\int_{|x|}^{\infty} \frac{u}{\sqrt{2\pi}} \exp\left\{ \frac{-u^2}{2} \right\} du = -\frac{1}{\sqrt{2\pi}} \int_{-\frac{x^2}{2}}^{-\infty} e^v dv = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

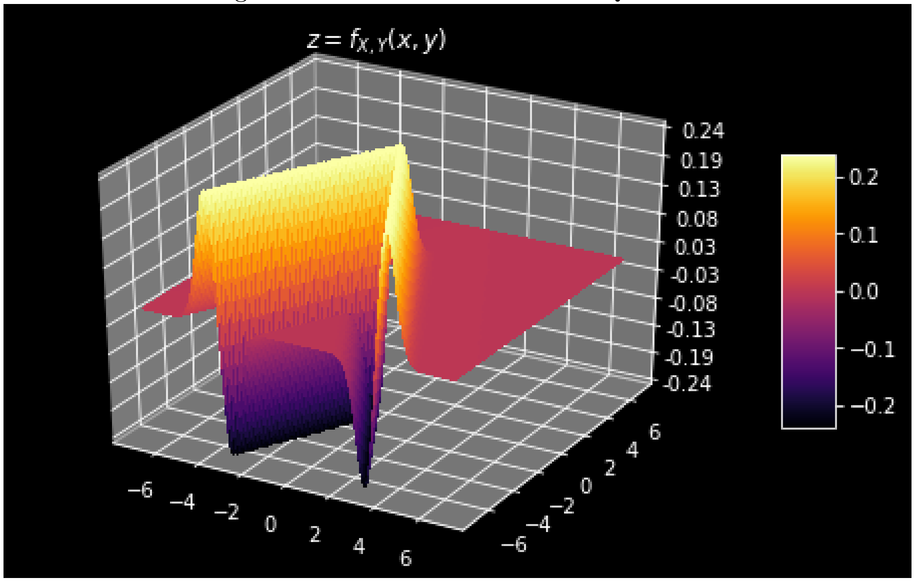
This shows that X is standard normal. We next fix y so that

$$f_Y(y) = \int_{-\infty}^{\infty} \frac{2|x|+y}{\sqrt{2\pi}} \exp\left\{ -\frac{(2|x|+y)^2}{2} \right\} dx$$

The integral is nonzero whenever $-|x| \leq y$ and zero otherwise. We break down the first inequality in two cases,

$$y \geq -|x| \iff \begin{cases} y \geq -x, & \text{if } x \geq 0 \\ y \geq x, & \text{if } x < 0 \end{cases}$$

Figure 1: Plot of the Function in Question



This gives us our bounds of integration. We then have the following

$$\int_{-\infty}^{-y} \frac{2x+y}{\sqrt{2\pi}} \exp\left\{-\frac{(2x+y)^2}{2}\right\} dx + \int_{y}^{\infty} \frac{-2x+y}{\sqrt{2\pi}} \exp\left\{-\frac{(-2x+y)^2}{2}\right\} dx$$

Two substitutions are in order, $u = 2x + y, \frac{1}{2}du = dx$, with $u(-\infty, y) = -\infty$ and $u(-y, y) = -y$. Then $v = -2x + y, -\frac{1}{2}dv = dx$, with $v(y, y) = -y$ and $v(\infty, y) = -\infty$. We'll then get

$$\begin{aligned}
&\int_{-\infty}^{-y} \frac{1}{2}\frac{u}{\sqrt{2\pi}} \exp\left\{-\frac{u^2}{2}\right\} du + \int_{-y}^{-\infty} -\frac{1}{2}\frac{v}{\sqrt{2\pi}} \exp\left\{-\frac{v^2}{2}\right\} dv \\
&= -\frac{1}{2}\int_{-y}^{-\infty} \frac{u}{\sqrt{2\pi}} \exp\left\{-\frac{u^2}{2}\right\} du - \frac{1}{2}\int_{-y}^{-\infty} \frac{v}{\sqrt{2\pi}} \exp\left\{-\frac{v^2}{2}\right\} dv \\
&= \frac{1}{2}\int_{-\infty}^{-\frac{y^2}{2}} \frac{1}{\sqrt{2\pi}} \exp\left\{\theta_u\right\} d\theta_u + \frac{1}{2}\int_{-\infty}^{-\frac{y^2}{2}} \frac{1}{\sqrt{2\pi}} \exp\left\{\theta_v\right\} d\theta_v \\
&= \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{\theta_u}\Big|_{\theta_u=-\infty}^{-\frac{y^2}{2}} + \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{\theta_v}\Big|_{\theta_v=-\infty}^{-\frac{y^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}
\end{aligned}$$

This shows that X and Y are standard normal random variables. To show that these two variables are uncorrelated we show that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

$$\begin{aligned}
\mathbb{E}[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^2(x, y)\varphi(x)\varphi(y) \\
&= \int_{-\infty}^{\infty} h(x, y)\varphi(x)dx \int_{-\infty}^{\infty} h(x, y)\varphi(y)dy \\
&= \mathbb{E}[X]\mathbb{E}[Y]
\end{aligned}$$

where $\varphi(n)$ is the standard normal density. We know that X and Y are not independent, since their joint density does not factor.