Stochastic Calculus
3. Brownian Motion

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Abstract

Here will be some meaningful text.

Exercises

Exercise 3.1 According to definition 3.3.3(iii), for $0 \le t < u$, the Brownian motion increment $W(u) - W(t)$ is independent of the σ-algebra $\mathcal{F}(t)$. Use this property and property (i) of that definition to show that, for $0 \le t < u_1 < u_2$, the increment $W(u_2) - W(u_1)$ is also independent of $\mathcal{F}(t)$.

soln. Let $0 \le t < u_1 < u_2$. Then

$$\begin{aligned} \mathbb{E}[W(u_2) - W(u_1)|\mathcal{F}(t)] &= \mathbb{E}[W(u_2) - W(t) + W(t) - W(u_1)|\mathcal{F}(t)] \\ &= \mathbb{E}[W(u_2) - W(t)|\mathcal{F}(t)] - \mathbb{E}[W(u_1) - W(t)|\mathcal{F}(t)] \\ &= \mathbb{E}[W(u_2) - W(t)] - \mathbb{E}[W(u_1) - W(t)] \\ &= \mathbb{E}[W(u_2) - W(u_1)] \end{aligned}$$

The second and fourth equalities follow from linearity of expectation, where as the the third equality follows from independence of future increments.

Exercise 3.2 Let $W(t), t \ge 0$, be a Brownian motion, and let $\mathcal{F}(t), t \ge 0$, be a filtration for this Brownian motion. Show that $W^2(t) - t$ is a martingale.

soln. Let $0 \le s \le t$. Then

$$\begin{aligned} \mathbb{E}[W^2(t) - t|\mathcal{F}(s)] &= \mathbb{E}[(W(t) - W(s))^2 + 2W(t)W(s) - W^2(s) - t|\mathcal{F}(s)] \\ &= \mathbb{E}[(W(t) - W(s))^2|\mathcal{F}(s)] + 2W(s)\mathbb{E}[W(t)|\mathcal{F}(s)] - W^2(s) - t \\ &= \mathbb{E}[(W(t) - W(s))^2] + 2W^2(s) - W^2(s) - t \\ &= t - s + W^2(s) - t \\ &= W^2(s) - s \end{aligned}$$

We obtain the first equality from a clever trick. The second equality makes use of linearity of expectation and taking out what is known. The third equality makes use of independence of future increments and the martingale property for Brownian motion. The fourth equality uses the fact that the variance of the normal random variable $W(t) - W(s)$ is simply $t - s$. Thus, proving that $W^2(t) - t$ is a martingale.

Exercise 3.3 (Normal Kurtosis). The *kurtosis* of a random variable is defined to be the ratio of its fourth central moment to the square of its variance. For a normal random variable, the kurtosis is 3. This fact was used to obtain (3.4.7). This exercise verifies this fact.

Let X be a normal random variable with mean μ, so that $X - \mu$ has mean zero. Let the variance of X, which is also the variance of $X - \mu$, be σ^2. In (3.2.13), we computed the moment-generating function of $X - \mu$ to be $\varphi(u) = \mathbb{E}e^{u(X-\mu)} = e^{\frac{1}{2}u^2\sigma^2}$, where u is a real variable. Differentiating this function with respect to u, we obtain

$$\varphi'(u) = \mathbb{E}\Big[(X - \mu)e^{u(X-\mu)}\Big] = \sigma^2 u e^{\frac{1}{2}u^2\sigma^2}$$

and, in particular $\varphi'(0) = \mathbb{E}(X-\mu) = 0$. Differentiating again, we obtain

$$\varphi''(u) = \left[(X-\mu)^2 e^{u(X-\mu)}\right] = (\sigma^2 + \sigma^4 u^2)e^{\frac{1}{2}u^2\sigma^2}$$

and, in particular, $\varphi''(0) = \mathbb{E}[(X-\mu)^2] = \sigma^2$. Differentiate two more times to obtain the normal kurtosis formula $\mathbb{E}[(X-\mu)^4] = 3\sigma^4$.

soln. We obtain the third derivative using the product rule. We have

$$\begin{aligned} \xi &= \sigma^2 + \sigma^4 u^2 & \xi' &= 2\sigma^4 u \\ \eta &= e^{\frac{1}{2}u^2\sigma^2} & \eta' &= \sigma^2 u e^{\frac{1}{2}u^2\sigma^2} \end{aligned}$$

so that

$$\varphi^{(3)}(u) = \xi\eta' + \xi'\eta = (3\sigma^4 u + \sigma^6 u^3)e^{\frac{1}{2}u^2\sigma^2}$$

Finally, for the fourth derivative, we have

$$\begin{aligned} \gamma &= 3\sigma^4 u + \sigma^6 u^3 & \gamma' &= 3\sigma^4 + 3\sigma^6 u^2 \\ \lambda &= e^{\frac{1}{2}u^2\sigma^2} & \lambda' &= \sigma^2 u e^{\frac{1}{2}u^2\sigma^2} \end{aligned}$$

so that

$$\varphi^4(u) = \gamma\lambda' + \gamma'\lambda = (3\sigma^4 + 6\sigma^6 u^2 + \sigma^8 u^4)e^{\frac{1}{2}u^2\sigma^2}$$

In particular, $\varphi^{(4)}(0) = \mathbb{E}[(X-\mu)^4] = 3\sigma^4$, which is the normal kurtosis formula.

Exercise 3.4 (Other variations of Brownian motion). Theorem 3.4.3 asserts that if T is a positive number and we choose a partition Π with points $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$, then as the number n of partition points approaches infinity and the length of the longest subinterval $\|\Pi\|$ approaches zero, the sample quadratic variation

$$\sum_{j=0}^{n-1}[W(t_{j+1}) - W(t_j)]^2$$

approaches T for almost every path of the Brownian motion W. In Remark 3.4.5, we further showed that $\sum_{j=0}^{n-1}[W(t_{j+1}) - W(t_j)](t_{j+1}-t_j)$ and $\sum_{j=0}^{n-1}(t_{j+1}-t_j)^2$ have limit zero. We summarize these facts by the multiplication rules

$$dW(t)dW(t) = dt,\ dW(t)dt = 0,\ dtdt = 0$$

(i) Show that as that as the number m of partition points approaches infinity and the length of the longest subinterval approaches zero, the sample first variation

$$\sum_{j=0}^{n-1}|W(t_{j+1}) - W(t_j)|$$

approaches ∞ for almost every path of the Brownian motion W.

(ii) Show that as the number n of partition points approaches infinity and the length of the longest subinterval approaches zero, the sample cubic variation

$$\sum_{j=0}^{n-1}|W(t_{j+1}) - W(t_j)|^3$$

approaches zero for almost every path of the Brownian motion W.

soln. (i) Notice that

$$\sum_{j=0}^{n-1}[W(t_{j+1}) - W(t_j)]^2 \le \max_{0\le k\le n-1}|W(t_{k+1}) - W(t_k)| \cdot \sum_{j=0}^{n-1}|W(t_{j+1}) - W(t_j)|$$

or better

$$\frac{\sum_{j=0}^{n-1}[W(t_{j+1}) - W(t_j)]^2}{\max_{0\le k\le n-1}|W(t_{k+1}) - W(t_k)|} \le \sum_{j=0}^{n-1}|W(t_{j+1}) - W(t_j)|$$

When $m \to \infty$ and $\|\Pi\| \to 0$, the denominator on the left side approaches zero so that the sum on the right becomes infinitely large.

(ii) Showing that the sample cubic variation is zero makes use of a similar inequality

$$\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^3 \leq \max_{0 \leq k \leq n-1} |W(t_{k+1}) - W(t_k)| \cdot \sum_{j=0}^{n-1} [W(t_{j+1}) - W(t_j)]^2$$

As $m \to \infty$ and $\|\Pi\| \to 0$ we obtain

$$0 \leq \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^3 \leq 0 \cdot T = 0$$

or simply

$$\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^3 = 0$$

Exercise 3.5 (Black-Scholes-Merton formula). Let the interest rate r and the volatility $\sigma > 0$ be constant. Let

$$S(t) = S(0)e^{(r - \frac{1}{2}\sigma^2)t + \sigma W(t)}$$

be a geometric Brownian motion with mean rate of return r, where the initial stock price $S(0)$ is positive. Show that, for $T > 0$,

$$\mathbb{E}\left[e^{-rT}\Big(S(T) - K\Big)^+\right] = S(0)N\big(d_+(T, S(0))\big) - Ke^{-rT}N\big(d_-(T, S(0))\big)$$

where

$$d_\pm = \frac{1}{\sigma\sqrt{T}}\left[\log\frac{S(0)}{K} + \left(r \pm \frac{\sigma^2}{2}\right)T\right]$$

and N is the cumulative standard normal distribution function

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{1}{2}z^2} = \frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty} e^{-\frac{1}{2}z^2}$$

soln. We proceed by integrating the option price against the standard normal density

$$\frac{e^{-rT}}{\sqrt{2\pi}}\int e^{-\frac{1}{2}x^2}\Big(S(0)e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}x} - K\Big)^+ dx$$

The integrand is nonzero when

$$S(0)e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}x} \geq K$$

or

$$rT - \frac{1}{2}\sigma^2 T + \sigma\sqrt{T}x \geq \log\frac{K}{S(0)}$$

From this equation, we obtain our bounds of integration, i.e.

$$x \geq \frac{\log\frac{K}{S(0)} + \frac{1}{2}\sigma^2 T + rT}{\sigma\sqrt{T}}$$

Denote the right hand side by Γ. Substituting this as the lower bound gives

$$\frac{e^{-rT}}{\sqrt{2\pi}}\left[\int_{\Gamma}^{\infty} e^{-\frac{1}{2}x^2} S(0)e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}x} dx - \int_{\Gamma}^{\infty} e^{-\frac{1}{2}x^2} K dx\right]$$

Here,

$$\int_{\Gamma}^{\infty} e^{-\frac{1}{2}x^2} K dx = \sqrt{2\pi}KN\left(\frac{\log\frac{S(0)}{K} - \frac{1}{2}\sigma^2 T + rT}{\sigma\sqrt{T}}\right)$$

Next, we perform the change of variables, $x = y + \sigma\sqrt{t}$ for the first integral term to obtain

$$\int_{\Gamma - \sigma\sqrt{T}}^{\infty} e^{-\frac{1}{2}y^2} S(0)e^{rT} dy = \sqrt{2\pi}S(0)e^{rT}N\left(\frac{\log\frac{S(0)}{K} + \frac{1}{2}\sigma^2 T + rT}{\sigma\sqrt{T}}\right)$$

Subtracting these two and multiplying by the constant out front gives

$$S(0)N\big(d_+(T,S(0))\big) - Ke^{-rT}N\big(d_-(T,S(0))\big)$$

as desired.

Exercise 3.6 Let $W(t)$ be a Brownian motion and let $\mathcal{F}(t), t \geq 0$, be an associated filtration.

(i) For $\mu \in \mathbb{R}$, consider the *Brownian motion with drift* μ:

$$X(t) = \mu t + W(t)$$

Show that for any Borel-measurable function $f(y)$, and for any $0 \leq s < t$, the function

$$g(x) = \frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{\infty} f(y)\exp\bigg\{-\frac{(y-x-\mu(t-s))^2}{2(t-s)}\bigg\}dy$$

satisfies $\mathbb{E}[f(X(t))|\mathcal{F}(s)] = g(X(s))$, and hence X has the Markov property. We may rewrite $g(x)$ as $g(x) = \int_{-\infty}^{\infty} f(y)p(\tau,x,y)dy$, where $\tau = t-s$ and

$$p(\tau,x,y) = \frac{1}{\sqrt{2\pi t}}\exp\bigg\{-\frac{(y-x-\mu\tau)^2}{2\tau}\bigg\}$$

is the *transition density* for Brownian motion with drift μ.

(ii) For $\nu \in \mathbb{R}$ and $\sigma > 0$, consider the *geometric Brownian motion*

$$S(t) = S(0)e^{\sigma W(t)+\nu t}$$

Set $\tau = t-s$ and

$$p(\tau,x,y) = \frac{1}{\sigma y\sqrt{2\pi\tau}}\exp\bigg\{-\frac{(\log\frac{x}{y}-\nu\tau)^2}{2\sigma^2\tau}\bigg\}$$

Show that for any Borel-measurable function $f(y)$ and for any $0 \leq s < t$ the function $g(x) = \int_0^\infty h(y)p(\tau,x,y)$ satisfies $\mathbb{E}[f(S(t))|\mathcal{F}(s)] = g(S(s))$ and hence S the Markov property and $p(\tau,x,y)$ is its transition density.

soln. (i) We begin by rewriting the conditional expectation in such a way as to allow us to use of the ideas developed in the previous chapter.

$$\mathbb{E}[f(X(t))|\mathcal{F}(s)] = \mathbb{E}[f((X(t)-X(s))+X(s))|\mathcal{F}(s)]$$

The random variable $X(t)-X(s)$ is independent of $\mathcal{F}(s)$, and the random variable $X(s)$ is $\mathcal{F}(s)$ measurable. This permits us to apply the Independence Lemma, which is stated below.

Lemma 1. (Independence). *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, and let $\mathcal{G}$ be a sub σ-algebra of $\mathcal{F}$. Suppose the random variables $X_1,\dots,X_K$ are $\mathcal{G}$-measurable and the random variables $Y_1,\dots,Y_L$ are independent of $\mathcal{G}$. Let $f(x_1,\dots,x_K,y_1,\dots,y_L)$ be a function of the dummy variables $x_1,\dots,x_k$ and $y_1,\dots,y_L$, and define*

$$g(x_1,\dots,x_K) = \mathbb{E}f(x_1,\dots,x_K,Y_1,\dots,Y_L)$$

Then

$$\mathbb{E}[f(X_1,\dots,X_K,Y_1,\dots,Y_L)|\mathcal{G}] = g(X_1,\dots,X_K)$$

The idea is to replae $X(s)$ by a dummy variable x, hold it constant, and evaluate the unconditional expectation of the remaining random variables, i.e. $g(x) = \mathbb{E}[f(X(t)-X(s)+x)]$. But, $X(t)-X(s)$ is normally distributed with mean $\mu(t-s)$ and variance $t-s$ so that

$$g(x) = \frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{\infty} f(w+x)\exp\bigg\{-\frac{(w-\mu(t-s))^2}{2(t-s)}\bigg\}dw$$

The Independence Lemma states that if we now take the function $g(x)$ and replace the dummy variable with x with $X(s)$ then $\mathbb{E}[f(X(t))|\mathcal{F}(s)] = g(X(s))$ holds. We then make the change of variables $y = w+x$ and $\tau = t-s$ to obtain

$$g(x) = \int_{-\infty}^{\infty} f(y)p(\tau,x,y)dy, \text{ where } p(\tau,x,y) = \frac{1}{\sqrt{2\pi t}}\exp\bigg\{-\frac{(y-x-\mu\tau)^2}{2\tau}\bigg\}$$

Thus, X has the Markov property and g is written in the desired form.

(ii) We perform a similar procedure, this time with

$$\mathbb{E}[f(S(t))|\mathcal{F}(s)] = \mathbb{E}[f((S(t)-S(s))+S(s))|\mathcal{F}(s)]$$

To use the Independence Lemma we make the dummy variable $x = S(s)$, hold it constant, and evaluate the unconditional expectation $\mathbb{E}[f(S(t) - S(s) + x)|\mathcal{F}(s)]$, which results in the following equation for g,

$$g(x) = \frac{1}{\sigma\sqrt{2\pi\tau}} \int_0^\infty h(w + \log x) \exp\left\{ - \frac{-(w - \nu t)^2}{2\sigma^2\tau} \right\} dw$$

The Independence Lemma states that $\mathbb{E}[f(S(t))|\mathcal{F}(s)] = g(S(s))$. The $\log x$ term appears since we are accessing the exponent of $S(t) - S(s) + x$. We perform the change of variables $\log(y) = w + \log(x)$ to obtain

$$g(x) = \frac{1}{\sigma\sqrt{2\pi\tau}} \int_0^\infty \frac{h(y)}{y} \exp\left\{ - \frac{-(\log \frac{y}{x} - \nu t)^2}{2\sigma^2\tau} \right\} dy$$

Thus, S has the Markov property and g is written in the desired form. Note that the bounds of integration are from zero to infinity since the underlying $S(t)$ is defined by an exponential which is strictly positive and $S(0)$ was taken to be nonnegative.

For the last three problems, I will be answering each problem statement immediately after typing it out, as opposed to stating the entire problem, and then writing the entire solution. This is because the next three problems are all four parts or more. We begin by presenting Theorem 3.6.2, which is instrumental to exercise 3.7.

Theorem 1. *For $m \in \mathbb{R}$, the first passage time of Brownian motion to level m is finite almost surely, and the Laplace transform of its distribution is given by*

$$\mathbb{E}e^{-\alpha\tau_m} = e^{-|m|\sqrt{2\alpha}}, \ \forall \alpha > 0$$

Exercise 3.7. Theorem 3.6.2, labeled Theorem 1 here, provides the Laplace transform of the first passage time for Brownian motion. This problem derives the analogous formula for Brownian motions with drift. Let W be a Brownian motion. Fix $m > 0$ and $\mu \in \mathbb{R}$. For $0 \le t < \infty$, define

$$\begin{aligned} X(t) &= \mu t + W(t) \\ \tau_m &= \min\{t \ge 0 : X(t) = m\} \end{aligned}$$

As usual, we set $\tau_m = \infty$ if $X(t)$ never reaches the level m. Let σ be a positive number and set

$$Z(t) = \exp\left\{ \sigma X(t) - \left(\sigma\mu + \frac{1}{2}\sigma^2 \right) t \right\}$$

(i) Show that $Z(t), t \ge 0$ is a martingale.

soln. We need to show that for $0 \le s < t$, $\mathbb{E}[Z(t)|\mathcal{F}(s)] = Z(s)$.

$$\begin{aligned} \mathbb{E}[Z(t)|\mathcal{F}(s)] &= \mathbb{E}\left[\exp\left\{ \sigma X(t) - \left(\sigma\mu + \frac{1}{2}\sigma^2 \right) t \right\} \middle| \mathcal{F}(s) \right] \\ &= \mathbb{E}\left[\exp\{\sigma(X(t) - X(s))\} \exp\left\{ \sigma X(s) - \left(\sigma\mu + \frac{1}{2}\sigma^2 \right) t \right\} \middle| \mathcal{F}(s) \right] \\ &= \mathbb{E}[\exp\{\sigma(X(t) - X(s))\}] \exp\left\{ \sigma X(s) - \left(\sigma\mu + \frac{1}{2}\sigma^2 \right) t \right\} \\ &= \exp\left\{ \left(\sigma\mu + \frac{1}{2}\sigma^2 \right)(t - s) \right\} \exp\left\{ \sigma X(s) - \left(\sigma\mu + \frac{1}{2}\sigma^2 \right) t \right\} \\ &= \exp\left\{ \sigma X(s) - \left(\sigma\mu + \frac{1}{2}\sigma^2 \right) s \right\} \\ &= Z(s) \end{aligned}$$

(ii) Use (i) to conclude that

$$\mathbb{E}\left[\exp\left\{ \sigma X(t \wedge \tau_m) - \left(\sigma\mu + \frac{1}{2}\sigma^2 \right)(t \wedge \tau_m) \right\} \right] = 1, \ t \ge 0$$

soln. τ_m is the first time the Brownian motion reaches level m. If the Brownian motion never reaches level m, we set $\tau_m = \infty$. A martingale that is stopped at a stopping time is still a martingale, but must have constant expectation. Because of this fact,

$$1 = Z(0) = \mathbb{E}[Z(t \wedge \tau_m)] = \mathbb{E}\left[\exp\left\{ \sigma X(t \wedge \tau_m) - \left(\sigma\mu + \frac{1}{2}\sigma^2 \right)(t \wedge \tau_m) \right\} \right]$$

where $t \wedge \tau_m = \min\{t, \tau_m\}$.

(iii) Now suppose $\mu \geq 0$. Show that, for $\sigma > 0$,

$$\mathbb{E}\left[\exp\left\{\sigma m - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\mathbb{I}_{\{\tau_m<\infty\}}\right] = 1$$

Use this fact to show $\mathbb{P}\{\tau_m < \infty\} = 1$ and to obtain the Laplace transform

$$\mathbb{E}e^{-\alpha\tau_m} = e^{m\mu - m\sqrt{2\alpha+\mu^2}},\ \forall\alpha > 0$$

soln. With $m > 0$, $\mu \geq 0$ and $\sigma > 0$, the Brownina motion is always at or below level m for $t \leq \tau_m$ and so

$$0 \leq \exp\{X(t \wedge \tau_m)\} \leq e^{\sigma m}$$

We have two cases.

<u>case I:</u> $\tau_m < \infty$. The term $\exp\{-(\sigma\mu + \frac{1}{2}\sigma^2)(t \wedge \tau_m)\}$ is equal to $\exp\{-(\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\}$ for large enough t.
<u>case II:</u> $\tau_m = \infty$. The term $\exp\{-(\sigma\mu + \frac{1}{2}\sigma^2)(t \wedge \tau_m)\}$ is equal to $\exp\{-(\sigma\mu + \frac{1}{2}\sigma^2)t\}$, and as $t \to \infty$, this converges to 0.

We capture these two cases by writing

$$\lim_{t\to\infty} \exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t \wedge \tau_m)\right\} = \mathbb{I}_{\{\tau_m<\infty\}} \exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}$$

We consider the two cases within the context of this development.

<u>case I:</u> $\tau_m < \infty$. The term $\exp\{-(\sigma\mu + \frac{1}{2}\sigma^2)X(t \wedge \tau_m)\} = \exp\{-(\sigma\mu + \frac{1}{2}\sigma^2)X(\tau_m)\} = e^{\sigma m}$ for large enough t.
<u>case II:</u> $\tau_m = \infty$. The term is bounded as $t \to \infty$ since $0 \leq \exp\{\sigma X(t \wedge \tau_m)\} \leq e^{\sigma m}$. This bound is enough to ensure that the product of $\exp\{\sigma X(t \wedge \tau_m)\}$ and $\exp\{-(\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\}$ has limit zero.

In conclusion, we have

$$\lim_{t\to\infty} \exp\left\{\sigma X(t \wedge \tau_m) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t \wedge \tau_m)\right\} = \mathbb{I}_{\{\tau_m<\infty\}} \exp\left\{\sigma m - \left(\sigma + \mu\frac{1}{2}\sigma^2\right)\tau_m\right\}$$

We can now take the limit of the result in (ii) to obtain

$$\mathbb{E}\left[\exp\left\{\sigma m - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\mathbb{I}_{\{\tau_m<\infty\}}\right] = 1$$

or, equivalently,

$$\mathbb{E}\left[\mathbb{I}_{\{\tau_m<\infty\}} \exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\right] = e^{-\sigma m}$$

This equation holds when m and σ are positive. Taking the limit of both sides as $\sigma \downarrow 0$ yields $\mathbb{E}[\mathbb{I}_{\{\tau_m<\infty\}}] = 1$ or, equivalently, $\mathbb{P}\{\tau_m < \infty\} = 1$. Because τ_m is finite almost surely, i.e. with probability one, we may drop the indicator of this event to obtain

$$\mathbb{E}\left[\exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\right] = e^{-\sigma m}$$

To obtain the Laplace transform, let $\alpha > 0$ and set $\sigma = -\mu + \sqrt{2\alpha + \mu^2}$, so that $\alpha = \sigma\mu + \frac{1}{2}\sigma^2$. Then

$$\mathbb{E}\left[\exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\right] = e^{-\sigma m} \text{ implies}$$

$$\mathbb{E}[e^{-\alpha\tau_m}] = e^{m\mu - m\sqrt{2\alpha+\mu^2}}$$

(iv) Show that if $\mu > 0$, then $\mathbb{E}\tau_m < \infty$. Obtain a formula for $\mathbb{E}\tau_m$.

soln.

$$\begin{aligned}\frac{d}{d\alpha}\mathbb{E}[e^{-\alpha\tau_m}] &= \frac{d}{d\alpha}e^{m\mu - m\sqrt{2\alpha+\mu^2}} \\ &= e^{m\mu - m\sqrt{2\alpha+\mu^2}} \cdot \frac{m}{\sqrt{2\alpha}}\end{aligned}$$

Letting $\alpha \downarrow 0$, we obtain $\mathbb{E}\tau_m = \infty$ so long as $m \neq 0$.

Exercise 3.8. This problem presents the convergence of the distribution of stock prices in a sequence of binomial models to the distribution of geometric Brownian motion. In contrast to the analysis of Subsection 3.2.7, here we allow the interest rate to be different from zero.

Let $\sigma > 0$ and $r \geq 0$ be given. For each positive integer n, we consider a binomial model taking n steps per unit time. In this model, the interest rate per period is $\frac{r}{n}$, the up factor is $u_n = e^{\sigma/\sqrt{n}}$, and the down factor is $d_n = e^{-\sigma/\sqrt{n}}$. The risk-neutral probabilities are then

$$\tilde{p_n} = \frac{\frac{r}{n} + 1 - e^{-\sigma/\sqrt{n}}}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}}, \qquad \tilde{q_n} = \frac{e^{\sigma/\sqrt{n}} - \frac{r}{n} - 1}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}},$$

Let t be an arbitrary positive rational number, and for each positive integer n for which nt is an integer, define

$$M_{nt,n} = \sum_{k=1}^{nt} X_{k,n}$$

where $X_{1,n}, \ldots, X_{n,n}$ are independent and identically distributed random variables with

$$\tilde{\mathbb{P}}\{X_{k,n} = 1\} = \tilde{p_n}, \quad \tilde{\mathbb{P}}\{X_{k,n} = -1\} = \tilde{q_n}, \quad k = 1, \ldots, n$$

The stock price at time t in this binomial model, which is the result of nt steps from the initial time, is given by

$$\begin{aligned}
S_n(t) &= S(0) u_n^{\frac{1}{2}(nt + M_{nt,n})} d_n^{\frac{1}{2}(nt - M_{nt,n})} \\
&= S(0) \exp\left\{ \frac{\sigma}{2\sqrt{n}}(nt + M_{nt,n}) \right\} \exp\left\{ -\frac{\sigma}{2\sqrt{n}}(nt - M_{nt,n}) \right\} \\
&= S(0) \exp\left\{ \frac{\sigma}{\sqrt{n}} M_{nt,n} \right\}
\end{aligned}$$

This problem shows that as $n \to \infty$, the distribution of the sequence of random variables $\frac{\sigma}{\sqrt{n}} M_{nt,n}$ appearing in the exponent above converges to the normal distribution with mean $(r - \frac{1}{2}\sigma^2)t$ and variance $\sigma^2 t$. Therefore, the limiting distribution of $S_n(t)$ is the same as the distribution of the geometric Brownian motion $S(0)\exp\{\sigma W(t) + (r - \frac{1}{2}\sigma^2)t\}$ at time t.

(i) Show that the moment generating function $\varphi_n(u)$ of $\frac{1}{\sqrt{n}} M_{nt,n}$ is given by

$$\varphi_n(u) = \left[e^{\frac{u}{\sqrt{n}}} \left(\frac{\frac{r}{n} + 1 - e^{-\sigma/\sqrt{n}}}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}} \right) - e^{-\frac{u}{\sqrt{n}}} \left(\frac{\frac{r}{n} + 1 - e^{\sigma/\sqrt{n}}}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}} \right) \right]^{nt}$$

soln.

$$\begin{aligned}
\varphi_n(u) &= \mathbb{E} e^{\frac{u}{\sqrt{n}} M_{nt,n}} \\
&= \mathbb{E} \exp\left\{ \frac{u}{\sqrt{n}} M_{nt,n} \right\} \\
&= \mathbb{E} \exp\left\{ \frac{u}{\sqrt{n}} \sum_{j=1}^{nt} X_{j,n} \right\} \\
&= \mathbb{E} \prod_{j=1}^{nt} \exp\left\{ \frac{u}{\sqrt{n}} X_{j,n} \right\} \\
&= \prod_{j=1}^{nt} \mathbb{E} \exp\left\{ \frac{u}{\sqrt{n}} X_{j,n} \right\} \text{ (since } X_{k,n} \text{ are i.i.d.)} \\
&= \prod_{j=1}^{nt} \left(\tilde{p_n} e^{\frac{u}{\sqrt{n}}} + \tilde{q_n} e^{\frac{u}{\sqrt{n}}} \right) \\
&= \left[e^{\frac{u}{\sqrt{n}}} \left(\frac{\frac{r}{n} + 1 - e^{-\sigma/\sqrt{n}}}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}} \right) - e^{-\frac{u}{\sqrt{n}}} \left(\frac{\frac{r}{n} + 1 - e^{\sigma/\sqrt{n}}}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}} \right) \right]^{nt}
\end{aligned}$$

(ii) We want to compute

$$\lim_{n \to \infty} \varphi_n(u) = \lim_{x \downarrow 0} \varphi_{\frac{1}{x^2}}(u)$$

where we have made the change of variable $x = \frac{1}{\sqrt{n}}$. To do this, we will compute $\log \varphi_{\frac{1}{x^2}}(u)$ and then take the limit as $x \downarrow \infty$. Show that

$$\log \varphi_{\frac{1}{x^2}}(u) = \frac{t}{x^2} \log \left[\frac{(rx^2+1)\sinh ux + \sinh(\sigma - u)x}{\sinh \sigma x} \right]$$

(the definitions are $\sinh z = \frac{e^z - e^{-z}}{2}$, $\cosh z = \frac{e^z + e^{-z}}{2}$)), and use the formula

$$\sinh(A - B) = \sinh A \cosh B - \cosh A sinh B$$

to rewrite this as

$$\log \varphi_{\frac{1}{x^2}}(u) = \frac{t}{x^2} \log \left[\cosh ux + \frac{(rx^2 + 1 - \cosh \sigma x)\sinh ux}{\sinh \sigma x} \right]$$

soln. Making the change of variable $x = \frac{1}{\sqrt{n}}$ and writing out the linear combinations of exponentionals as hyperbolic trig functions results in

$$\begin{aligned}
\varphi_{\frac{1}{x^2}}(u) &= \left[e^{ux}\left(\frac{rx^2+1-e^{-\sigma x}}{e^{\sigma x} - e^{-\sigma x}} \right) - e^{-ux}\left(\frac{rx^2+1-e^{\sigma x}}{e^{\sigma x} - e^{-\sigma x}} \right) \right]^{\frac{t}{x^2}} \\
\log \varphi_{\frac{1}{x^2}}(u) &= \frac{t}{x^2} \log \left[e^{ux}\left(\frac{rx^2+1-e^{-\sigma x}}{e^{\sigma x} - e^{-\sigma x}} \right) - e^{-ux}\left(\frac{rx^2+1-e^{\sigma x}}{e^{\sigma x} - e^{-\sigma x}} \right) \right] \\
&= \frac{t}{x^2} \log \left[\frac{rx^2(e^{ux} - e^{-ux}) + (e^{ux} - e^{-ux}) + (e^{(\sigma-u)x} - e^{-(\sigma-u)x})}{e^{\sigma x} - e^{-\sigma x}} \right] \\
\\
&= \frac{t}{x^2} \log \left[\frac{\frac{1}{2}(rx^2(e^{ux} - e^{-ux}) + (e^{ux} - e^{-ux}) + (e^{(\sigma-u)x} - e^{-(\sigma-u)x}))}{\frac{1}{2}(e^{\sigma x} - e^{-\sigma x})} \right] \\
&= \frac{t}{x^2} \log \left[\frac{(rx^2+1)\sinh ux + \sinh(\sigma-u)x}{\sinh \sigma x} \right] \text{ (first result)} \\
&= \frac{t}{x^2} \log \left[\frac{(rx^2+1)\sinh ux + \sinh \sigma x \cosh ux - \cosh \sigma x \sinh ux}{\sinh \sigma x} \right] \\
&= \frac{t}{x^2} \log \left[\cosh ux + \frac{(rx^2+1-\cosh \sigma x)\sinh ux}{\sinh \sigma x} \right] \text{ (second result)}
\end{aligned}$$

(iii) Use the Taylor series expansions

$$\cosh z = 1 + \frac{1}{2}z^2 + O(z^4), \quad \sinh z = z + O(z^3)$$

to show that

$$\cosh ux + \frac{(rx^2+1-\cosh \sigma x)\sinh ux}{\sinh \sigma x} = 1 + \frac{1}{2}u^2x^2 + \frac{rux^2}{\sigma} - \frac{1}{2}ux^2\sigma + O(x^4)$$