Math 131B HW 7

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Exercise 1.5.9 Show that a metric space (X, d) is compact if and only if every sequence in X has at least one limit point.

Proof. Suppose that X is compact. Let $(x^{(n)})_{n=1}^{\infty}$ be a sequence in X. Since X is compact, there exists at least one convergent subsequence $(x^{(n_j)})_{j=1}^{\infty} \longrightarrow L \in X$. By Proposition 1.4.5, L is a limit point of $(x^{(n)})_{n=1}^{\infty}$. Thus, there is at least one limit point for every sequence.

Suppose that every sequence $(x^{(n)})_{n=1}^{\infty}$ in X has at least one limit point. Call it $L \in X$. By Proposition 1.4.5 there exists a convergent subsequence $(x^{(n_j)})_{j=1}^{\infty} \longrightarrow L$. This shows that X is compact. □

Exercise 1.5.11 Let X have the property that every open cover of X has a finite subcover. Show that X is compact.

Proof. Suppose that X is not compact. By Exercise 1.5.9 a sequence $(x^{(n)})_{n=1}^{\infty}$ in X has no limit points. So, $\forall x \in X$, there exists a ball of radius $\epsilon > 0$ around x, i.e. $B(x, \epsilon)$, containing x which contains at most finitely many elements of this sequence. Since the sequence has infinitely many terms, you would need infinitely many such $B(x, \epsilon)$ to cover the sequence. Hence, there is an open cover of X which does not contain a finite subcover. □

Exercise 3.2.3 Let (X, d_X) be a metric space, and for every integer $n \geq 1$, let $f_n : X \to \mathbb{R}$ be a real-valued function. Suppose that f_n converges pointwise to another function $f : X \to \mathbb{R}$ on X. Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that the functions $h \circ f_n$ converge pointwise to $h \circ f$ on X, where $h \circ f_n : X \to \mathbb{R}$ is the function $h \circ f_n(x) := h(f_n(x))$, and similarly for $h \circ f$.

Proof. Let $x \in X$. Since f_n converges pointwise to another function f, we have that

$$\lim_{n \to \infty} f_n(x) = f(x)$$

Since h is continuous on $\mathbb{R}$, we know that h is continuous at $f(x)$. Thus,

$$\lim_{n \to \infty} h(f_n(x)) = h(f(x))$$

So $h \circ f_n$ converges pointwise to $h \circ f$ on X. □

Exercise 3.2.4 Let $f_n : X \to Y$ be a sequence of bounded functions. Suppose that f_n converges uniformly to another function $f : X \to Y$. Suppose that f is a bounded function. Show that the sequence f_n is uniformly bounded.

Proof. Let $f_n : X \to Y$ be a sequence of bounded functions. Suppose that f is a bounded function. Suppose that f_n converges uniformly to another function $f : X \to Y$. This means

$$\forall \epsilon > 0,\ \exists N > 0 : d_Y(f_n(x), f(x)) < \epsilon,\ \forall x \in X,\ \forall n > N$$

Let $N > 0$. Then $n > N$ implies

$$f_n(x) \in B(f(x), \epsilon) \subseteq B(y_0, R) \subseteq Y$$

for some $R > 0$ and $y_0 \in Y$ (assumption that f is bounded $\forall x$). □

Exercise 3.3.2 Prove Proposition 3.3.3.

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces with Y complete. Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from $E \subseteq X$ to Y and suppose that this sequence converges uniformly to $f : E \to Y$, i.e.

$$\forall \epsilon > 0,\ \exists N > 0 : d_Y(f^{(n)}(x), f(x)) < \epsilon,\ \forall x \in E,\ \forall n > N \tag{1}$$

Let $x_0 \in X$ be an adherent point of E and suppose that for each n

$$\lim_{x \to x_0; x \in E} f^{(n)}(x) \tag{2}$$

exists. Call this limit $L^{(n)}$. We also know that

$$\forall n,\ \exists \delta > 0 : \forall x \in E,\ d_X(x, x_0) < \delta_n \implies d_Y(f^{(n)}(x), L^{(n)}) < \epsilon \tag{3}$$

We want to show that $L^{(n)} = \lim_{x \to x_0; x \in E} f^{(n)}(x)$ is Cauchy. Then $forallj, k \geq N, x \in E \cap B(x_0, min\{\delta_k, \delta_j\})$. Then

$$\begin{aligned} d_Y(L^{(j)}, L^{(k)}) &\leq d_Y(L^{(j)}, f^{(j)}(x)) + d_Y(f^{(j)}(x), f(x)) + d_Y(f(x), f^{(k)}(x)) + d_Y(f^{(k)}(x), L^{(k)}) \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} \\ &= \epsilon \end{aligned}$$

Now we want to show

$$\lim_{x \to x_0; x \in E} f(x) = \lim_{n \to \infty} L^{(n)} =: L$$

Let $\epsilon > 0$. Then we have the following statements:

$$\exists N_1 : \forall n \geq N_1,\ d_Y(L, L^{(n)}) < \epsilon \tag{4}$$

$$\exists N_2 : \forall n \geq N_2,\ \forall x \in E, d_Y(f(x), f^{(n)}(x)) < \epsilon \tag{5}$$

Now, set $N = \max\{N_1, N_2\}$. Then

$$\exists \delta > 0 : \forall x \in E d_X(x, x_0) < \delta \implies d_Y(f^{(N)}, L^{(N)}) < \epsilon$$

Then $\forall x \in E$ such that $d_X(x, x_0) < \delta$

$$\begin{aligned} d_Y(f(x), L) &\leq d_Y(f^{(N)}, f(x)) + d_Y(f^{(N)}, L^{(N)}) + d_Y(L, L^{(N)}) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \qquad\qquad = \epsilon \end{aligned}$$

□

Exercise 3.3.3 Comparing it against example 1.2.8, we see that prop 3.3.3 is different since the sequence of functions $f^{(n)} = x^n$ does not converge uniformly so the interchange of limits does not hold

Exercise 3.3.6 Prove proposition 3.3.6. Discuss how this proposition differs from Exercise 3.2.4.

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from X to Y, where the functions $f^{(n)}$ are bounded on X for each n, and suppose that this sequence converges uniformly to $f : X \to Y$, i.e.

$$\forall \epsilon > 0,\ \exists N > 0 : d_Y(f^{(n)}(x), f(x)) < \epsilon,\ \forall x \in X,\ \forall n > N \tag{6}$$

This means that

$$f(x) \in B(f^{(n)}(x), \epsilon)$$

But since $f^{(n)}$ bounded $\forall n$ we know that $B(f^{(n)}(x), \epsilon) \subseteq B(y_i, R_i)$ where $y_i \in Y$ and $R_i > 0$ for $i \in [1, n]$.

The difference between this problem and Exercise 3.2.4 is that in this problem you show the limiting function is bounded provided the $f^{(n)}$ are bounded. In Exercise 3.2.4 you suppose the limiting function bounded and prove that the $f^{(n)}$ are uniformly bounded. □

Exercise 3.3.8 Let (X, d) be a metric space, $\forall n \in \mathbb{Z}^+$ let $f_n : X \to \mathbb{R}$ and $g_n : X \to \mathbb{R}$. Suppose that $(f_n)_{n=1}^\infty$ converges uniformly to $f : X \to \mathbb{R}$ and $(g_n)_{n=1}^\infty$ converges uniformly to $g : X \to \mathbb{R}$. Suppose $(g_n)_{n=1}^\infty$, $(f_n)_{n=1}^\infty$ are uniformly bounded $\forall n \in \mathbb{Z}^+$ and $x \in X$. Prove that $f_n g_n \to fg$ uniformly.

Proof. f_n uniformly continuous means

$$\forall \epsilon_f > 0,\ \exists N_f > 0 : d(f_n(x), f(x)) < \epsilon_f,\ \forall x \in X,\ \forall n > N_f \tag{7}$$

g_n uniformly continuous means

$$\forall \epsilon_g > 0,\ \exists N_g > 0 : d(g_n(x), g(x)) < \epsilon_g,\ \forall x \in X,\ \forall n > N_g \tag{8}$$

Let $\epsilon > 0$. Let $M > 0$. Let $\epsilon_f = \epsilon_g = \frac{\epsilon}{2M}$. Let $N = \max\{N_f, N_g\}$.

$$\begin{aligned}
d(f_n(x)g_n(x), f(x)g(x)) &= |f_n(x)g_n(x) - f(x)g(x)| \\
&= |f_n(x)g_n(x) - f(x)g_n(x) + f(x)g_n(x) - f(x)g(x)| \\
&= |(f_n(x) - f(x))g_n(x) + (g_n(x) - g(x))f(x)| \\
&\leq |(f_n(x) - f(x))||g_n(x)| + |(g_n(x) - g(x))||f(x)| \\
&< \frac{\epsilon}{2M}M + \frac{\epsilon}{2M}M \\
&= \epsilon
\end{aligned}$$

□

Exercise 3.5.1 Let $f^{(1)}, f^{(2)}, \ldots, f^{(N)}$ be a finite sequence of bounded functions from a metric space (X, d_X) to $\mathbb{R}$. Show that $\sum_{i=1}^N f^{(i)}$ is also bounded. Prove a similar claim when "bounded" is replaced by "continuous. What if "continuous" was replaced by "uniformly continuous"?

Proof. Bounded. Since $f^{(i)}$ is bounded for $i \in [1, N]$, $\exists M > 0$ such that $|f^{(i)}| \leq M$. So

$$\left|\sum_{i=1}^N f^{(i)}\right| \leq \sum_{i=1}^N |f^{(i)}| \leq \sum_{i=1}^N M$$

This last sum is convergent since it is a finite sum of real numbers. Say that it equals $K \in \mathbb{R}$. Then $|\sum_{i=1}^N f^{(i)}| \leq K$, so the sum is bounded. □

Proof. Continuous. Since $f^{(i)}$ is continuous for $i \in [1, N]$, $\exists \delta > 0$ such that $d_X(x, x_0) < \delta$ implies that $f^{(i)}$ is continuous. Let δ be so chosen. Let $\epsilon > 0$. Then

$$\begin{aligned}
d\left(\sum_{i=1}^N f^{(i)}(x), \sum_{i=1}^N f^{(i)}(x_0)\right) &= |f^{(1)}(x) + \ldots + f^{(N)}(x) - (f^{(1)}(x_0) + \ldots + f^{(N)}(x_0))| \\
&= |f^{(1)}(x) - f^{(1)}(x_0) + \ldots + f^{(N)}(x) - f^{(N)}(x_0)| \\
&\leq |f^{(1)}(x) - f^{(1)}(x_0)| + \ldots + |f^{(N)}(x) - f^{(N)}(x_0)| \\
&< \frac{\epsilon}{N} + \ldots + \frac{\epsilon}{N} \\
&= \epsilon
\end{aligned}$$

Thus, $\sum_{i=1}^N f^{(i)}$ is continuous. □

Proof. Uniformly Continuous. The proof is the same as that of continuity except you let $x, x_0 \in X$ both be arbitrary with $d(x, x_0) < \delta$. The rest of the work is the same. □

Exercise 3.6.1 Let $[a, b]$ be an interval, and let $(f^{(n)})_{n=1}^\infty$ be a sequence of functions on $[a, b]$ such that the series $\sum_{n=1}^\infty f^{(n)}$ is uniformly convergent. Then we have

$$\sum_{n=1}^\infty \int_{[a,b]} f^{(n)} = \int_{[a,b]} \sum_{n=1}^\infty f^{(n)}$$

Proof. Since $\sum_{n=1}^{\infty} f^{(n)}$ is uniformly convergent, we have that

$$\sum_{n=1}^{N} f^{(n)} \longrightarrow \sum_{n=1}^{\infty} f^{(n)} = f \text{ as } N \to \infty$$

Where f is some function. By Theorem 3.6.1, f is Riemann integrable on $[a,b]$. So,

$$\begin{aligned}
\int_{[a,b]} f &= \lim_{N\to\infty} \int_{[a,b]} \sum_{n=1}^{N} f^{(n)} \\
&= \lim_{N\to\infty} \left[\int_{[a,b]} f^{(1)} + \ldots + \int_{[a,b]} f^{(N)} \right] \\
&= \int_{[a,b]} f^{(1)} + \int_{[a,b]} f^{(2)} + \ldots \\
&= \sum_{n=1}^{\infty} \int_{[a,b]} f^{(n)}
\end{aligned}$$

But, we know that $\sum_{n=1}^{\infty} f^{(n)} = f$, so

$$\int_{[a,b]} \sum_{n=1}^{\infty} f^{(n)} = \int_{[a,b]} f$$

Thus,

$$\sum_{n=1}^{\infty} \int_{[a,b]} f^{(n)} = \int_{[a,b]} \sum_{n=1}^{\infty} f^{(n)}$$

$\square$