Rings

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Definition. A **ring** is a triple $(R, +, \times)$, with operations called addition and multiplication. Properties:

1. $(R, +)$ is an abelian group with identity 0_R.
2. $\times$ is an associative binary operation on R with an identity 1_R.
3. Multiplication distributes over addition.

A ring R is **commutative** if $\times$ is commutative.

Definition. The **trivial ring** is the ring with only one element, where $0_R = 1_R$.

Fact. *For any ring R with $r \in R$, $r \cdot 0_R$ and $r * (-1_R) = -r$.*

Definition. The **Gaussian integers** are the complex numbers with integer real and imaginary parts.

$$Z[i] = \{a + bi : a, b \in \mathbb{Z}\}$$

Definition. Let R, S be rings. Then the **product ring**, denoted $R \times S$ is defined as ordered pairs (r, s) with both operations done component-wise.

Definition. A **unit** of a ring R is an element $u \in R$ which is invertible. That is, for some $x \in R$, $ux = 1_R$.

Definition. A nontrivial commutative ring is a **field** if all of its nonzero elements are units.

Remark. Colloquially, a field is a structure where you can add, subtract, multiply, and divide.

Definition. Let R be a ring and $a, b \in R$ such that $ab = 0$ and $a, b \neq 0$. Then a and b are the **zero divisors** of R. Example: $\mathbb{Z}_{15}$ has this property, where $3, 5$ are the zero divisors because $3 \cdot 5 \equiv 0 \pmod{15}$.

Definition. A nontrivial commutative ring with no zero divisors is an **integral domain**.

Definition. Let $R = (R, +_R, \times_R)$ and $S = (S, +_S, \times_S)$ be rings. A **ring homomorphism** is a map $\phi : R \to S$ such that

1. $\forall x, y \in R, \phi(x +_R y) = \phi(x) +_S \phi(y)$.
2. $\forall x, y \in R, \phi(x \times_R y) = \phi(x) \times_S \phi(y)$.
3. $\phi(1_R) = 1_S$.

If ϕ is a bijection, then ϕ is an **isomorphism** and R and S are **isomorphic**.

Definition. The **kernel** of a ring homomorphism $\phi : R \to S$, denoted $\ker\phi$, is defined as follows:

$$\ker\phi = \{r \in R : \phi(r) = 0\}$$

Fact. *If $\phi(x) = \phi(y) = 0$, then $\phi(x + y) = \phi(x) + \phi(y) = 0$, so $\ker\phi$ should be closed under addition.*

Fact. *If $\phi(x) = 0$, then $\forall r \in R, \phi(rx) = \phi(r)\phi(x) = 0$, so for $x \in \ker\phi$ and any $r \in R$, $rx \in \ker\phi$.*

Definition. A nonempty subset $I \subseteq R$ is an **ideal** if it is closed under addition, and $\forall x \in I, \forall r \in R, rx \in I$.

Remark. This definition is true if we assume the ring is commutative. If the ring is not commutative, we must also add the condition that $xr \in I$.

Theorem. *Let R be a ring with $I \subset R$. Then I is the kernel of some homomorphism if and only if I is an ideal.*

Definition. Let R be a ring and I be an ideal. Then the **quotient ring** is given by

$$R/I = \{r + I : r \in R\}$$

R/I is when we declare all elements of I are zero, i.e. we "mod out by elements of I".

Theorem. *The only ideals of $\mathbb{Z}$ are those of the form $n\mathbb{Z}$ where n is an integer.*