Adaptive MCMC

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Theoretical Part

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See Kallenberg 2021

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Notation

Given a measurable space $(\mathcal{X}, \mathcal{F}_{\mathcal{X}})$, let $\mathbb{M}_1(\mathcal{X})$ denote the set of all probability measures on it. Let P be a transition kernel on $(\mathcal{X}, \mathcal{F}_{\mathcal{X}})$ which defines a linear operator $P \colon \mathbb{M}_1(\mathcal{X}) \to \mathbb{M}_1(\mathcal{X})$ given by

$$\mu P(B) = \int_{\mathcal{X}} P(x, B)\mu(dx), \quad \mu \in \mathbb{M}_1(\mathcal{X}), B \in \mathcal{F}_{\mathcal{X}}.$$

Further, for a measurable function $f \colon \mathcal{X} \to \mathbb{R}$ and $\mu \in \mathbb{M}_1(\mathcal{X})$ we have

$$\mu(f) = \int_{\mathcal{X}} f(x)\mu(dx),$$
$$Pf(x) = \int_{\mathcal{X}} f(y)P(x, dy)$$

whenever well-defined.

1.3 Wasserstein ergodicity

See Douc et al. 2018 and Rudolf and Schweizer 2017

Let d be a metric which is assumed to be lower semi-continuous with respect to the product topology of $\mathcal{X}$. For two probability measures $\nu, \mu \in \mathbb{M}_1(\mathcal{X})$ we define the Wasserstein distance by

$$\mathcal{W}(\nu, \mu) := \inf_{\xi \in \mathcal{C}(\nu, \mu)} \int_{\mathcal{X}^2} d(x, y)\xi(dx, dy),$$

where $\mathcal{C}(\mu_1, \mu_2)$ is the set of all couplings of ν, μ, that is, all probability measures on $\mathcal{X} \times \mathcal{X}$ with marginals ν and μ.

For a measurable function $f\colon \mathcal{X} \to \mathbb{R}$ we denote Lipschitz semi-norm as

$$\|f\|_d = \sup_{\substack{x,y\in\mathcal{X}\\ x\neq y}} \frac{|f(x) - f(y)|}{d(x,y)}$$

Theorem 1 (Kantorovich-Rubenstein duality)**.**

$$\mathcal{W}(\nu,\mu) = \sup_{\|f\|_d\leq 1} |\nu(f) - \mu(f)|\,.$$

Further, we define the following quantities:

1) The eccentricity (see Joulin and Ollivier 2010) is defined as

$$E(x) = \int_{\mathcal{X}} d(x,y)\pi(dy).$$

2) Coarse diffusion coefficient (see Joulin and Ollivier 2010)

$$\text{diff}(x,\gamma) = \int_{\mathcal{X}}\int_{\mathcal{X}} d(x',x'')^2 P_\gamma(x,dx')P_\gamma(x,dx'').$$

3) The Wasserstein contraction coefficient for transition kernel P is

$$\tau(P) = \sup_{\substack{x,y\in\mathcal{X}\\ x\neq y}} \frac{\mathcal{W}(\delta_x P, \delta_y P)}{d(x,y)}.$$

Proposition 2. *For the Wasserstein contraction coefficient one has the properties of*

i) submultiplicativity, that is $\tau(P\tilde{P}) \leq \tau(P)\tau(\tilde{P})$*, and,*

ii) contractivity, that is $\mathcal{W}(\nu P, \mu P) \leq \tau(P)\mathcal{W}(\nu,\mu)$*,*

for any transition kernels P*,* $\tilde{P}$ *and any probability measures* $\nu, \mu \in \mathbb{M}_1(\mathcal{X})$*.*

Chapter 2

Adaptive MCMC

2.1 Preliminaries

Taken from Hofstadler et al. 2024 and Laitinen and Vihola 2024
We consider a state space $\mathcal{X}$ and a parameter space $\mathcal{I}$, which are assumed to be Polish with countable generated σ-algebras $\mathcal{F}_{\mathcal{X}}$ and $\mathcal{F}_{\mathcal{I}}$. Given a family of transition kernels $\{P_\gamma\}_{\gamma\in\mathcal{I}}$ on $(\mathcal{X}, \mathcal{F}_{\mathcal{X}})$, let $(X_n, \Gamma_n)_{n\in\mathbb{N}_0}$ be an adaptive MCMC chain evolving in the space $\mathcal{X} \times \mathcal{I}$, where $(\Gamma_n)_{n\in\mathbb{N}_0}$ denotes the sequence of (possibly random) adaptation parameters. The family $\{P_\gamma\}_{\gamma\in\mathcal{I}}$ must satisfy the following non-restrictive regularity condition:

Assumption (A1) (Regularity)**.** *$(\gamma, x) \mapsto P_\gamma(x, A)$ is $\mathcal{F}_{\mathcal{I}} \otimes \mathcal{F}_{\mathcal{X}}$-measurable for all $A \in \mathcal{F}_{\mathcal{X}}$.*

This ensures that $\big((\gamma, x), A\big) \mapsto P_\gamma(x, A)$ defines a probability kernel from $\mathcal{I} \times \mathcal{X}$ to $\mathcal{X}$.

We assume that π is the invariant distribution of $P_\gamma(x, \cdot)$ for any $x \in \mathcal{X}$. Moreover, we assume that any kernel P_γ is ψ-irreducible and aperiodic.

For a fixed test function $\varphi \in L^1(\pi)$, we investigate the convergence properties of the averages

$$\frac{1}{n}\sum_{k=1}^{n} \varphi(X_k) \xrightarrow{n\to\infty} \pi(\varphi), \tag{2.1}$$

2.2 Martingale decomposition

According to Laitinen and Vihola 2024, the following two assumptions are needed:

Assumption (A2) (Iterative algorithm)**.** *There is a filtration $(\mathcal{F}_k)_{k\geq 0}$ such that $(X_k, \Gamma_k)_{k\in\mathbb{N}_0}$ is $(\mathcal{F}_k)_{k\in\mathbb{N}_0}$-adapted, and the following holds for all $k \geq 0$ and all measurable $B \in \mathcal{F}_{\mathcal{X}}$:*

$$\mathbb{P}\left[X_{k+1} \in B \mid \mathcal{F}_k\right] = P_{\Gamma_k}(X_k, B) \qquad a.s.$$

Assumption (A3) (Solutions of the Poisson equation)**.** *There exists a measurable mapping $(x, \gamma) \mapsto u_\gamma(x)$ from $\mathcal{X} \times \mathcal{I}$ to $\mathbb{R}$ which satisfies the following:*

$$u_\gamma(x) - (P_\gamma u_\gamma)(x) = \varphi(x) - \pi(\varphi) \qquad \text{for all } y \in \mathcal{X} \text{ and } \gamma \in \mathcal{I}. \tag{2.2}$$

Our desired convergence (2.1) can be then analysed using the following decomposition:

$$\begin{aligned}
\sum_{k=1}^{n} \varphi(X_k) - \pi(\varphi) &= \sum_{k=1}^{n} u_{\Gamma_k}(X_k) - P_{\Gamma_k} u_{\Gamma_k}(X_k) && (2.3)\\
&= \sum_{k=1}^{n} u_{\Gamma_{k-1}}(X_k) - P_{\Gamma_{k-1}} u_{\Gamma_{k-1}}(X_{k-1}) \qquad \big(=: M_n\big)\\
&+ \sum_{k=1}^{n} u_{\Gamma_k}(X_k) - u_{\Gamma_{k-1}}(X_k) \qquad \big(=: A_n\big)\\
&+ \sum_{k=1}^{n} P_{\Gamma_{k-1}} u_{\Gamma_{k-1}}(X_{k-1}) - P_{\Gamma_k} u_{\Gamma_k}(X_k) \qquad \big(=: R_n\big).
\end{aligned}$$

2.3 The Poisson equation

Look into Douc et al. 2018 for Poisson equation.

As shown in Hofstadler et al. 2024, (A3) holds under following assumption on Wasserstein contraction:

Assumption (A4). *For each $\gamma \in \mathcal{I}$ we assume that π is the invariant distribution of P_γ and that for all $k \in \mathbb{N}$*

$$\tau(P_\gamma^k) \leq C\tau^k,$$

with $C \in [1, \infty)$ and $\tau \in [0, 1)$ independent of γ.

Proposition 3. *Let $\gamma \in \mathcal{I}$, $\varphi \colon \mathcal{X} \to \mathbb{R}$ be π integrable and Lipschitz with constant $L \in (0, \infty)$ and let $E(x) < \infty$ for any $x \in \mathcal{X}$. If (A4) holds, then the function $u_\gamma \colon \mathcal{X} \to \mathbb{R}$ given via*

$$u_\gamma(x) = \sum_{k=0}^{\infty} P_\gamma^k \left(\varphi - \pi(\varphi)\right)(x) \tag{2.4}$$

is well defined and we have

i) $|u_\gamma(x)| \leq \frac{C\|\varphi\|_d}{1-\tau} E(x)$ *for any* $x \in \mathcal{X}$,

ii) $\|u_\gamma\|_d \leq \frac{C\|\varphi\|_d}{1-\tau}$,

iii) $u_\gamma(x) - P_\gamma u_\gamma(x) = \varphi(x) - \pi(\varphi)$ *for any* $x \in \mathcal{X}$.

We now make an assumption on smoothness w.r.t. parameter γ. For this purpose, we denote the metric on $\mathcal{I}$ as $\tilde{d}$. We aim to have Lipschitz-smoothness of mappings $\gamma \mapsto u_\gamma(x)$ for any $x \in \mathcal{X}$. If we express the condition in terms of Wasserstein distances, the following would suffice:

$$\sum_{n=1}^{\infty} \mathcal{W}(\delta_x P_\gamma^n, \delta_x P_{\gamma'}^n) \le \tilde{d}(\gamma, \gamma') \cdot g(x)$$

for all $\gamma, \gamma' \in \mathcal{I}$, $x \in \mathcal{X}$ and some measurable $g\colon \mathcal{X} \to \mathbb{R}$.

Assumption (A5). *There exists $\tilde{L} < \infty$, s.t. for all $\gamma, \gamma' \in \mathcal{I}$ holds*

$$\sup_{\|f\|_d \le 1} \|(P_\gamma - P_{\gamma'})f\|_d \le \tilde{L} \cdot \tilde{d}(\gamma, \gamma').$$

Lemma 4. *Under (A4) and (A5) for any $x \in \mathcal{X}$, $\gamma, \gamma' \in \mathcal{I}$ and $n \in \mathbb{N}$ it holds*

$$\mathcal{W}(\delta_x P_\gamma^n, \delta_x P_{\gamma'}^n) \le n\tau^{n-1} \cdot C^2 \tilde{L} \tilde{d}(\gamma, \gamma') E(x).$$

Consequently, for u_γ defined in (2.4) *we have for any $x \in \mathcal{X}$:*

$$\sup_{\substack{\gamma, \gamma' \in \mathcal{I} \\ \gamma \ne \gamma'}} \frac{|u_\gamma(x) - u_{\gamma'}(x)|}{\tilde{d}(\gamma, \gamma')} \le \frac{\tilde{L} C^2 \|\varphi\|_d}{(1-\tau)^2} E(x).$$

Proof. We use the following decomposition for the proof:

$$\begin{aligned} P_\gamma^n \varphi(x) - P_{\gamma'}^n \varphi(x) &= \sum_{i=0}^{n-1} \left(P_\gamma^{i+1} P_{\gamma'}^{n-(i+1)} \varphi(x) - P_\gamma^i P_{\gamma'}^{n-i} \varphi(x) \right) \\ &= \sum_{i=0}^{n-1} P_\gamma^i (P_\gamma - P_{\gamma'}) P_{\gamma'}^{n-i-1} \varphi(x). \end{aligned}$$

Denote $\psi(x) := (P_\gamma - P_{\gamma'}) P_{\gamma'}^{n-i-1} \varphi(x)$. Observing that $\pi(\psi) = \pi P_\gamma P_{\gamma'}^{n-i-1}(\varphi) - \pi P_{\gamma'}^{n-i}(\varphi) = 0$ yields

$$\begin{aligned} \left| P_\gamma^i \psi(x) \right| &= \left| \delta_x P_\gamma^i(\psi) - \pi(\psi) \right| \\ &\le \|\psi\|_d \cdot \mathcal{W}(\delta_x P_\gamma^i, \pi) \\ &\le \|\psi\|_d \cdot C\tau^i \mathcal{W}(\delta_x, \pi) \\ &= \|\psi\|_d \cdot C\tau^i E(x). \end{aligned}$$

Assumption (A5) allows to write

$$\begin{aligned} \|\psi\|_d &= \|(P_\gamma - P_{\gamma'}) P_{\gamma'}^{n-i-1} \varphi\|_d \\ &\le \|P_{\gamma'}^{n-i-1} \varphi\|_d \cdot \tilde{L} \tilde{d}(\gamma, \gamma') \\ &\le \|\varphi\|_d \cdot C\tau^{n-i-1} \cdot \tilde{L} \tilde{d}(\gamma, \gamma'). \end{aligned}$$

Hence, we can bound

$$\left|P_\gamma^n \varphi(x) - P_{\gamma'}^n \varphi(x)\right| \le n\tau^{n-1} \cdot \|\varphi\|_d C^2 \tilde{L}\tilde{d}(\gamma,\gamma')E(x).$$

It directly follows that

$$\begin{aligned}\mathcal{W}(\delta_x P_\gamma^n, \delta_x P_{\gamma'}^n) &= \sup_{\|f\|_d \le 1} \left|P_\gamma^n f(x) - P_{\gamma'}^n f(x)\right| \\ &\le n\tau^{n-1} \cdot C^2 \tilde{L}\tilde{d}(\gamma,\gamma')E(x).\end{aligned}$$

Finally,

$$\begin{aligned}|u_\gamma(x) - u_{\gamma'}(x)| &\le \sum_{n=1}^{\infty} \left|P_\gamma^n \varphi(x) - P_{\gamma'}^n \varphi(x)\right| \\ &\le \left(\sum_{n=1}^{\infty} n\tau^{n-1}\right) \cdot \|\varphi\|_d C^2 E(x) \cdot \tilde{L}\tilde{d}(\gamma,\gamma') \\ &= \tilde{d}(\gamma,\gamma') \cdot \frac{\tilde{L}C^2\|\varphi\|_d}{(1-\tau)^2} E(x).\end{aligned}$$

□

2.4 Convergence Analysis

Here we analyze the terms M_n, A_n, R_n as defined in (2.3). We initially assume that (A2), (A4) hold, and u_γ is given by (2.4).

Assumption (A6).

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\mathrm{diff}(X_n, \Gamma_n)\right] = \Lambda < \infty.$$

Lemma 5. *Under (A6), $M_n = \sum_{k=1}^n \Delta_k$ defined in* (2.3) *is a martingale w.r.t.* $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

Proof.

$$\begin{aligned}\mathbb{E}\left[|\Delta_k|\right] &= \mathbb{E}\left[|u_{\Gamma_{k-1}}(X_k) - P_{\Gamma_{k-1}} u_{\Gamma_{k-1}}(X_{k-1})|\right] \\ &\le \mathbb{E}\left[\int_{\mathcal{X}} |u_{\Gamma_{k-1}}(X_k) - u_{\Gamma_{k-1}}(x)| P_{\Gamma_{k-1}}(X_{k-1}, dx)\right] \\ &\le \frac{C\|\varphi\|_d}{1-\tau} \mathbb{E}\left[\int_{\mathcal{X}} d(X_k, x) P_{\Gamma_{k-1}}(X_{k-1}, dx)\right] \\ &= \frac{C\|\varphi\|_d}{1-\tau} \mathbb{E}\left[\int_{\mathcal{X}}\int_{\mathcal{X}} d(x', x) P_{\Gamma_{k-1}}(X_{k-1}, dx') P_{\Gamma_{k-1}}(X_{k-1}, dx)\right] \\ &\le \frac{C\|\varphi\|_d}{1-\tau} \sqrt{\mathbb{E}\left[\int_{\mathcal{X}}\int_{\mathcal{X}} d(x', x)^2 P_{\Gamma_{k-1}}(X_{k-1}, dx') P_{\Gamma_{k-1}}(X_{k-1}, dx)\right]} \\ &\le \frac{C\|\varphi\|_d}{1-\tau} \sqrt{\Lambda} < \infty,\end{aligned}$$

where Jensen's inequality is used in the end

Disintegration Thm (See Thm. 8.5 in Kallenberg 2021) yields

$$\begin{aligned}\mathbb{E}\left[\Delta_k \mid \mathcal{F}_{k-1}\right] &= \mathbb{E}\left[u_{\Gamma_{k-1}}(X_k) \mid \mathcal{F}_{k-1}\right] - P_{\Gamma_{k-1}} u_{\Gamma_{k-1}}(X_{k-1}) \\ &\overset{\text{a.s}}{=} P_{\Gamma_{k-1}} u_{\Gamma_{k-1}}(X_{k-1}) - P_{\Gamma_{k-1}} u_{\Gamma_{k-1}}(X_{k-1}) \\ &= 0\end{aligned}$$

□

Lemma 6. *Under (A6) for M_n defined in 2.3 it holds*

$$\frac{M_n}{n} \xrightarrow{a.s.} 0.$$

Proof. Consider the martingale $V_n = \sum_{k=1}^{n} \frac{\Delta_k}{k}$. The differences Δ_k are uniformly bounded in L^2:

$$\begin{aligned}\mathbb{E}\left[\Delta_k^2\right] &\leq \frac{C^2\|\varphi\|_d^2}{(1-\tau)^2}\mathbb{E}\left[\int_{\mathcal{X}}\int_{\mathcal{X}} d(x',x)^2 P_{\Gamma_{k-1}}(X_{k-1},dx')P_{\Gamma_{k-1}}(X_{k-1},dx)\right] \\ &\leq \frac{C^2\|\varphi\|_d^2}{(1-\tau)^2}\Lambda.\end{aligned}$$

Because martingale differences are orthogonal, we have

$$\begin{aligned}\mathbb{E}\left[V_n^2\right] &= \sum_{k=1}^{n} \frac{\mathbb{E}\left[\Delta_k^2\right]}{k^2} \\ &\leq \frac{C^2\|\varphi\|_d^2}{(1-\tau)^2}\Lambda \cdot \sum_{k=1}^{\infty}\frac{1}{k^2} \\ &= \frac{C^2\|\varphi\|_d^2}{(1-\tau)^2}\Lambda \cdot \frac{\pi^2}{6}.\end{aligned}$$

That is, V_n is a L^2-bounded martingale, which converges to an a.s. finite $V_\infty = \sum_{k=1}^{\infty} \frac{\Delta_k}{k}$ almost surely (see Corollary E.3.5 in Douc et al. 2018). Whenever V_∞ is finite, Kronecker's lemma implies that

$$\frac{M_n}{n} = \frac{1}{n}\sum_{k=1}^{n} k\frac{\Delta_k}{k} \xrightarrow{n\to\infty} 0 \quad \text{(a.s.)}.$$

□

Assumption (A7).

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[E(X_n)^2\right] = K < \infty.$$

Lemma 7. *Under (A7) for R_n defined in* (2.3) *it holds*

$$\frac{R_n}{n} \xrightarrow{a.s.} 0, \qquad \frac{R_n}{\sqrt{n}} \xrightarrow{P} 0.$$

Proof. For $p = 1, \frac{1}{2}$ we have

$$\begin{aligned}\left|\frac{R_n}{n^p}\right| &= \frac{1}{n^p}\left|P_{\Gamma_n}u_{\Gamma_n}(X_n) - P_{\Gamma_0}u_{\Gamma_0}(X_0)\right| \\ &\leq \frac{1}{n^p}\left|P_{\Gamma_n}u_{\Gamma_n}(X_n)\right| + \frac{1}{n^p}\left|P_{\Gamma_0}u_{\Gamma_0}(X_0)\right|\end{aligned}$$

Under (A1), $P_{\Gamma_0}u_{\Gamma_0}(X_0)$ is finite for each $\omega \in \Omega$ and hence the second term converges a.s. and, hence, in probability.

For the first term we bound

$$\begin{aligned}\mathbb{P}\left[\frac{1}{n^p}\left|P_{\Gamma_n}u_{\Gamma_n}(X_n)\right| > \varepsilon\right] &\leq \frac{1}{n^{2p}\varepsilon^2}\mathbb{E}\left[P_{\Gamma_n}u_{\Gamma_n}(X_n)^2\right] \\ &= \frac{1}{n^{2p}\varepsilon^2}\mathbb{E}\left[\mathbb{E}\left[u_{\Gamma_n}(X_{n+1}) \mid \mathcal{F}_n\right]^2\right] \\ &\leq \frac{1}{n^{2p}\varepsilon^2}\mathbb{E}\left[\mathbb{E}\left[u_{\Gamma_n}(X_{n+1})^2 \mid \mathcal{F}_n\right]\right] \\ &\leq \left(\frac{C\|\varphi\|_d}{1-\tau}\right)^2 \cdot \frac{1}{n^{2p}\varepsilon^2}\mathbb{E}\left[\ E(X_{n+1})^2\right] \\ &\leq \left(\frac{C\|\varphi\|_d}{1-\tau}\right)^2 \cdot \frac{K}{n^{2p}\varepsilon^2}.\end{aligned}$$

This shows $\sum_{n=1}^{\infty}\mathbb{P}\left[\frac{1}{n}\left|P_{\Gamma_n}u_{\Gamma_n}(X_n)\right| > \varepsilon\right] < \infty$. By Borel-Cantelli lemma it follows that

$$\frac{1}{n}\left|P_{\Gamma_n}u_{\Gamma_n}(X_n)\right| \xrightarrow{\text{a.s.}} 0.$$

We also have $\mathbb{P}\left[\frac{1}{\sqrt{n}}\left|P_{\Gamma_n}u_{\Gamma_n}(X_n)\right| > \varepsilon\right] \xrightarrow{n\to\infty} 0$, yielding

$$\frac{1}{\sqrt{n}}\left|P_{\Gamma_n}u_{\Gamma_n}(X_n)\right| \xrightarrow{P} 0.]$$

□

Denote $D_k := \|\tilde{d}(\Gamma_{k-1}, \Gamma_k)\|_2 = \sqrt{\mathbb{E}\left[\tilde{d}(\Gamma_{k-1}, \Gamma_k)^2\right]}$.

Assumption (A8). *Suppose the following series converges:*

$$\sum_{k=1}^{\infty}\frac{1}{k}\sqrt{\mathbb{E}\left[\tilde{d}(\Gamma_{k-1}, \Gamma_k)^2\right]} = \sum_{k=1}^{\infty}\frac{1}{k}D_k =: D < \infty.$$

Assumption (A9). *Suppose for some $p > 0$ it holds*

$$\frac{1}{n^p}\sum_{k=1}^{n}D_k \xrightarrow{n\to\infty} 0.$$

Lemma 8. *Under (A5), (A7) and (A8) it holds*

$$\frac{A_n}{n} \xrightarrow{a.s.} 0.$$

Proof. (A5) yields

$$\begin{aligned}\left|\frac{A_n}{n}\right| &\leq \frac{1}{n}\sum_{k=1}^{n}\left|u_{\Gamma_k}(X_k) - u_{\Gamma_{k-1}}(X_k)\right| \\ &\leq \frac{\tilde{L}C^2\|\varphi\|_d}{(1-\tau)^2}\cdot\frac{1}{n}\sum_{k=1}^{n}\tilde{d}(\Gamma_{k-1},\Gamma_k)E(X_k).\end{aligned}$$

Consider $Y_n := \sum_{k=1}^{n}\frac{1}{k}\tilde{d}(\Gamma_{k-1},\Gamma_k)E(X_k)$.

We apply Cauchy–Schwarz and use (A7) and **??**:

$$\begin{aligned}\mathbb{E}\left[Y_n\right] &\leq \sum_{k=1}^{n}\frac{1}{k}\sqrt{\mathbb{E}\left[\tilde{d}(\Gamma_{k-1},\Gamma_k)^2\right]\mathbb{E}\left[E(X_k)^2\right]} \\ &\leq \sqrt{K}\sum_{k=1}^{n}\frac{1}{k}\sqrt{\mathbb{E}\left[\tilde{d}(\Gamma_{k-1},\Gamma_k)^2\right]} \\ &= \sqrt{K}\sum_{k=1}^{n}\frac{1}{k}D_k \\ &\leq \sqrt{K}\cdot D < \infty.\end{aligned}$$

By monotone convergence, the pointwise limit Y_∞ satisfies $\mathbb{E}\left[Y_\infty\right] \leq \sqrt{K}\cdot D$ and hence Y_∞ is a.s. finite. Finally, Kronecker's lemma implies that

$$\frac{1}{n}\sum_{k=1}^{n}\tilde{d}(\Gamma_{k-1},\Gamma_k)E(X_k) = \frac{1}{n}\sum_{k=1}^{n}k\cdot\frac{1}{k}\tilde{d}(\Gamma_{k-1},\Gamma_k)E(X_k) \xrightarrow{\text{a.s.}} 0.$$

□

Lemma 9. *Under (A5), (A7) and (A9) (for some $p > 0$) it holds:*

$$\frac{A_n}{n^p} \xrightarrow{L_1} 0.$$

As consequence, convergence in probability also holds.

Proof.

$$\begin{aligned}\mathbb{E}\left[\left|\frac{A_n}{n^p}\right|\right] &\leq \frac{\tilde{L}C^2\|\varphi\|_d}{(1-\tau)^2}\cdot\frac{1}{n^p}\sum_{k=1}^{n}\mathbb{E}\left[\tilde{d}(\Gamma_{k-1},\Gamma_k)E(X_k)\right] \\ &\leq \frac{\tilde{L}C^2\|\varphi\|_d}{(1-\tau)^2}\cdot\frac{1}{n^p}\sum_{k=1}^{n}\sqrt{\mathbb{E}\left[\tilde{d}(\Gamma_{k-1},\Gamma_k)^2\right]\mathbb{E}\left[E(X_k)^2\right]} \\ &\leq \frac{\tilde{L}C^2\|\varphi\|_d}{(1-\tau)^2}\sqrt{K}\cdot\frac{1}{n^p}\sum_{k=1}^{n}D_k \xrightarrow{n\to\infty} 0.\end{aligned}$$

□

Chapter 3

Numerical Experiments

3.1 Benchmarks

See Magnusson et al. 2024

3.2 Adaptive Metropolis

3.3 AIR

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