Representing the train acceleration/decceleration optimisation problem

Objective

The objective of the problem is to minimize the amount of energy used by a train to move from point A to point B. In order to do this we must figure the optimal time when the train should be accelerating, coasting or braking.

First formulation

In order to do this we will model the train as being in one of three states during the trip.

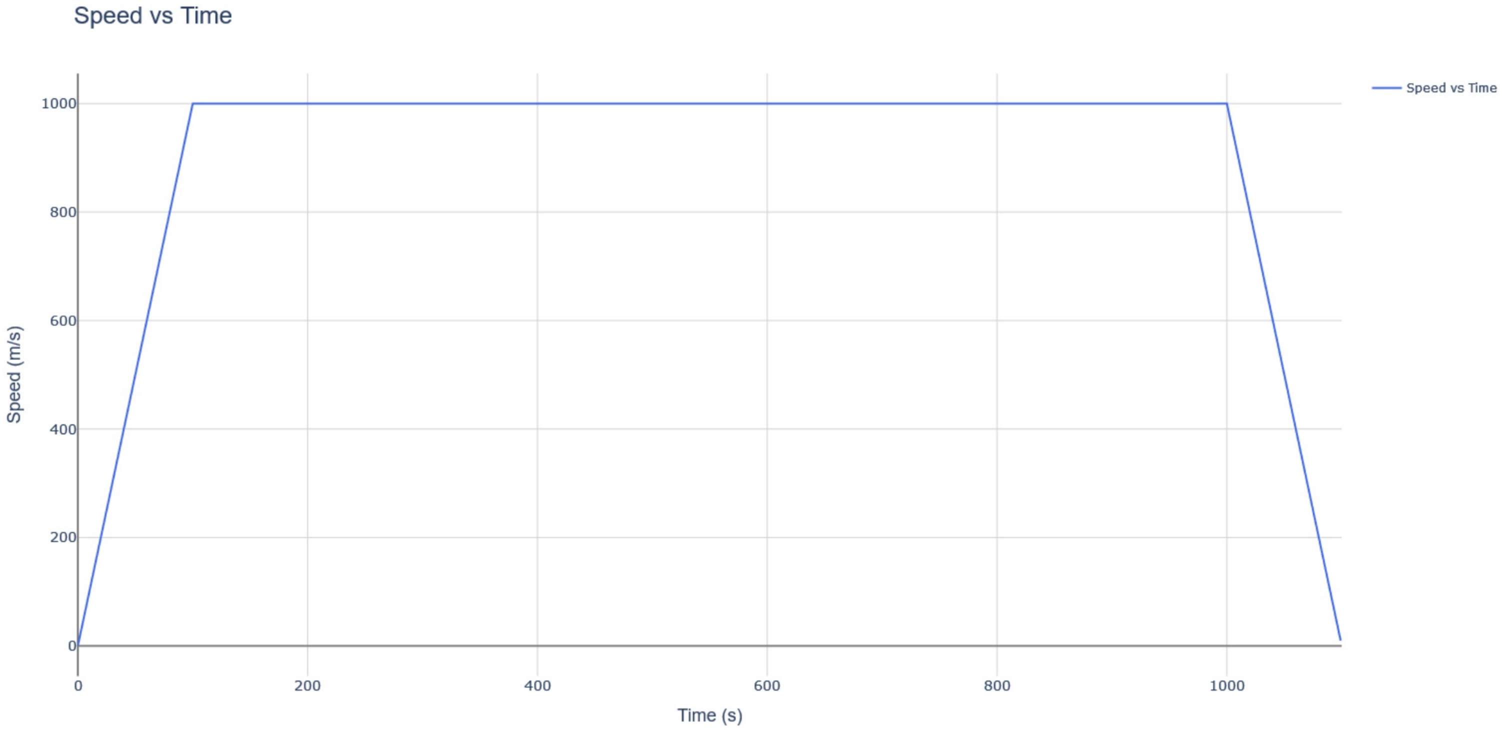
- State 1: Acceleration where: $\vec{a} = a \times \vec{u}$ between t=0 and t=t_1
- State 2: Coasting where: $\vec{a} = 0$ between t=t_1 and t=t_2
- State 3: Braking where: $\vec{a} = -a'$ between t=t_2 and t=T

In this model we have made a few hypothesis:

- There is no friction
- We do not require energy for braking
- There is no regenerative braking
- The efficiency (η) of the motors is constant

This gives us that:

$$v(t) = \begin{cases} at & \text{for } 0 < t < t_1 \\ v_{\max} & \text{for } t_1 < t < t_2 \\ v_{\max} - (t - t_2)a' & \text{for } t_2 < t < T \end{cases}$$



From this we can determine that:

$$E_{total} = E_{State1} + E_{State2} + E_{State3}$$
$$And\ since\ :\ E_{State2} = E_{State3} = 0$$
$$\Rightarrow E_{total} = E_{State1}$$
$$And\ :\ E_{State1} = E(t_1) - E(t = 0) = E(t_1)$$

Considering only the cinetic energy, we need to minimize:

$$E_{total} = \frac{\eta \times m \times v_{max}^2}{2}$$

We will not consider the mass of the train in our problem as it is a constant.

Converting the problem to a QUBO problem

What are QUBO problems

The QUBO (Quadratic Unconstrained Binary Optimization) framework is a mathematical formulation used in optimization problems, particularly in quantum computing when the objective is to minimize a quadratic function of binary variables.

Particularities

1. **Quadratic Objective Function**: The objective function is expressed as a quadratic polynomial of the binary variables. It typically has the form:

$$f(x) = \sum_i a_i x_i + \sum_{i<j} b_{ij} x_i x_j$$

Where:

- a_i are linear coefficients
- b_{ij} are quadratic coefficients
- x_i are the binary variables.

2. **Unconstrained**: QUBO problems do not have explicit constraints on the variables, although constraints can often be embedded into the objective function.

Converting our problem to QUBO

Defining a solution

We will be working with speed and not energy in order to simplify our problem. Firstly, we start by discretising our functions from $f(t)$ to f_n. We then consider our three states (as defined earlier) encoded into two bits x_i, where:

$$v_{i+1} = \begin{cases} v_i & \text{if } x_i = 00 \\ v_i + \Delta v & \text{if } x_i = 01 \\ v_i - \Delta v & \text{if } x_i = 10 \\ v_i & \text{if } x_i = 11 \end{cases}$$

Where $\Delta v = v_{i+1} - v_i$

We then consider a system with N discrete time-steps the i^{th} time step having a corresponding two-bit decision variable $x_{i,a}$;$x_{i,b}$.

We consider a solution being an $x \in \{00, 01, 10, 11\}^N$

Defining the cost function

We use a constant energy model, why hypothesises the fact that the energy needed to go from one speed to another is always constant. We also consider that the efficiency η is proportional to the ratio of the average velocity (between initial and final speed) up to a constant C, ie. $\eta = C \times \frac{v_{avg,i}}{\Delta v_i}$.
And with:

$$E_{i+1} = \frac{1}{2\eta}(v_{i+1}^2 - v_i^2) = \frac{1}{\eta}\Delta v_i \; v_{avg,i}$$

Without braking regeneration

Our cost function is:

$$E(x) = \sum_{i=1}^{N}(\Delta v^2 \times x_{i,b})$$

With braking regeneration

Our cost function is:

$$E(x) = \sum_{i=1}^{N}(\Delta v^2 \times x_{i,b} - \alpha \times \Delta v^2 \times x_{i,a})$$

Where α is the efficiency of the energy recuperation system.

Converting the constraints

As explained earlier, QUBO problems are unconstrained, we therefore have to turn constraints into penalties our resolution will heavily try to reduce.

No simultaneous braking and acceleration

Our train cannot accelerate and brake at the same time, we therefore apply a penalty when both our bits are 1.

$$P_1 = \lambda_1 \sum_{i=1}^{N} x_{i,a} \times x_{i,b}$$

As we want to force that value towards 0.

Distance constraint

Another constraint we have to respect is that we have to have arrived at our destination, therefore having only gone through a distance of D.

$$P_2 = \lambda_2 \, [\sum_{i=1}^{N}((N-i)\Delta v \times x_{i,b} - (N-i)\Delta v \times x_{i,a}) - D]^{\,2}$$

As we want to force our distance towards D.

Net-Zero speed constraint

Another constraint we have to respect is that we have to have stopped at our destination therefore braking as many times as we have accelerated.

$$P_3 = \lambda_3 \, [\sum_{i=1}^{N}(x_{i,a} - x_{i,b})]^{\,2}$$

As we want to force that value towards 0.

Maximum Speed Constraint

The last contraint we have to respect is a maximum speed constraint.

$$P_4 = \lambda_4 \, [(\Delta v \times \sum_{i=1}^{N} x_{i,b}) - v_{max}]^2$$

Where λ_i are multipliers used to accentuate the importance of these penalties.

The final QUBO problem

We want to minimise:

$$E(x) = \Delta v^2 \sum_{i=1}^{N}(x_{i,a} - \alpha \, x_{i,b}) + \lambda_1 \sum_{i=1}^{N} x_{i,a} \times x_{i,b} + \lambda_2 \, [\sum_{i=1}^{N}((N-i)\Delta v \times x_{i,a} - (N-i)\Delta v \times x_{i,b}) - D]^{\,2} + \lambda_3 \, [\sum_{i=1}^{N}(x_{i,a} - x_{i,b})]^{\,2} + \lambda_4[(\Delta v \sum_{i=1}^{N} x_{i,b}) - v_{max}]^2$$