Lab 3 : ACF and CCF

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Stationarity

Stationarity is a property of a **process**, not of a time series.
There are two types of stationarity :

- Strong / Strict / Strict-sense stationarity :
  A stochastic process $\{X_t\}_{t\in\mathbb{R}}$ is strongly stationary iif

$$\forall K \in \mathbb{N}, h, t_1, \dots t_K \in \mathbb{R}, \qquad F_{X_{(t_1)},\dots X_{(t_K)}} = F_{X_{(t_1+h)},\dots X_{(t_K+h)}}.$$

  This is a condition on the **distributions**.
- Weak / Wide-sense / covariance stationarity:
  A stochastic process is weakly stationary iif :

$$\mu_X(t) = \mathrm{cst}$$
$$R_X(\tau) = \mathrm{cov}(X_t, X_{t+\tau})$$
$$\mathbb{E}\left[|X(t)|^2\right] < +\infty.$$

  This is a condition on the **moments**.

Examples

A strongly stationary white noise

$$w_t \sim \mathcal{N}(0, \sigma^2), \text{i.i.d.}$$

The i.i.d. assumption is more restrictive than the strong stationarity.

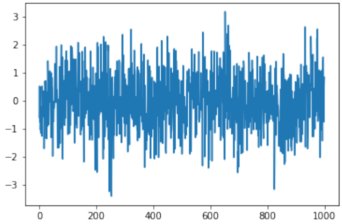

Figure 1: Gaussian noise $\mathcal{N}(0, 1)$

Examples

A weakly stationary white noise

$$w_t = \sin(2\pi t U),$$

where $U \sim \mathcal{U}(0,1)$.
We can show that $\mathbb{E}[w_t] = 0, \mathrm{cov}(w_t, w_{t+h}) = 0$ if $h \neq 0$ and $\mathbb{V}[w_t] = \frac{1}{2}$. However it is not strongly stationary.

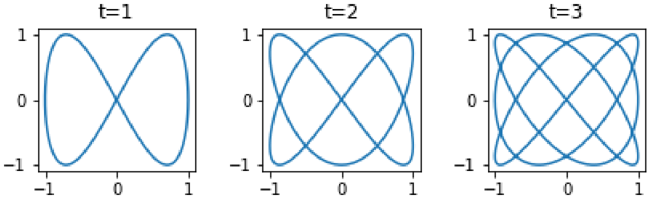

Figure 1: Graph of $w_{t+1} = f(w_t)$ for $u \in [0,1]$

Stationarity: Properties

Remarks

1. Weak stationarity does not imply strong stationarity: a condition on the moments is less restrictive than a condition on the distribution.
   Example : Keep the same first two moments, but change the generating distribution on parity considering independence.
2. Strong stationarity implies weak stationarity if the first two moments exist.
   Example : Let X_t be iid Cauchy. X is then strongly stationary but not weakly stationary.
3. For a Gaussian Process (any subset of variables is gaussian) with a stationary kernel, weak stationarity is equivalent to strong stationarity.

Testing stationarity: (Augmented) Dickey-Fuller (ADF) test

The Dickey-Fuller test tests the null hypothesis that a unit root is present in an AR model.
With an AR(1) model we would have :

$$\begin{aligned} x_t &= \lambda x_{t-1} + w_t \\ \Delta x_t &= (\lambda - 1) x_{t-1} + w_t \\ &= \delta x_{t-1} + w_t. \qquad (\delta := \lambda - 1) \end{aligned}$$

The $t-$statistic $\frac{\hat{\lambda}-\lambda}{\sqrt{1-\lambda^2}}$ does not follow a standard distribution when the null hypothesis holds; instead we compare the value to a Dickey-Fuller table.

Definitions

If we have X, Y **stationary** processes, we can define the following quantities :

$$
\begin{aligned}
\mu_X(t) &= \mathbb{E}[X_t] && \text{(mean)} \\
R_X(t,u) &= \operatorname{cov}(X_t, X_u) && \text{(covariance)} \\
\gamma_X(\tau) &= R_X(t, t+\tau) && \text{(autocovariance)} \\
\rho_X(t,u) &= \frac{R_X(t,u)}{\sqrt{R_X(t,t)R_X(u,u)}} && \text{(autocorrelation)} \\
R_{X,Y}(t,u) &= \operatorname{cov}(X_t, Y_u) && \text{(cross-covariance)} \\
\rho_{X,Y}(t,u) &= \frac{R_X(t,u)}{\sqrt{R_X(t,t)R_X(u,u)}} && \text{(cross-correlation)}
\end{aligned}
$$

Definitions

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\gamma_X(\tau) &= R_X(t, t+\tau) && \text{(autocovariance)} \\
\rho_X(\tau) &= \frac{\gamma_X(\tau)}{\sigma_X^2} && \text{(autocorrelation)} \\
R_{X,Y}(t,u) &= \operatorname{cov}(X_t, Y_u) && \text{(cross-covariance)} \\
\rho_{X,Y}(t,u) &= \frac{R_X(t,u)}{\sigma_X \sigma_Y} && \text{(cross-correlation)}
\end{aligned}
$$

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Backshift

The backshift operator B acting on a serie $(x_t)_{t \in \mathbb{N}}$ is defined as :

$$Bx_t = x_{t-1},$$

with $x_{-1} = 0$.
This helps to identify parameter redundancy. An ARMA process can be written as :

$$\phi(B)x_t = \theta(B)w_t,$$

where ϕ, θ are two coprime polynomials (no common roots).

Parameter redundancy : Intuition

Informally we could express the recursion using formal series.
For example :

$$
\begin{aligned}
x_t &= 0.8x_{t-1} - 0.16x_{t-2} + w_t - w_{t-1} + 0.24w_t \\
z^t(x_t - 0.8x_{t-1} + 0.16x_{t-2}) &= z^t(w_t - w_{t-1} + 0.24w_{t-2}) \\
(1 - 0.8z + 0.16z^2)\sum_{t=1}^{+\infty} x_t z^t &= (1 - z + 0.24z^2)\sum_{t=1}^{+\infty} w_t z^t \\
(1 - 0.4z)^2\sum_{t=1}^{+\infty} x_t z^t &= (1 - 0.4z)(1 - 0.6z)\sum_{t=1}^{+\infty} w_t z^t \\
(1 - 0.4z)\sum_{t=1}^{+\infty} x_t z^t &= (1 - 0.6z)\sum_{t=1}^{+\infty} w_t z^t
\end{aligned}
$$

By unicity of formal series, both recursion relations are the same.

Causality

Causal

A process is said to be **causal** if $\exists(\psi_j)_{j\in\mathbb{N}}$, s.t. $\left|(\psi_j)_{j\in\mathbb{N}}\right|_{L^1} < +\infty$ and

$$x_t = \sum_{j\in\mathbb{N}} \psi_j w_{t-j}.$$

In words, it means that x_t depends only on past values.

An ARMA process is causal iif

$$\phi(x) = 0 \implies |x| > 1.$$

Invertibility

Invertibility

A process is said to be **invertible** if $\exists(\pi_j)_{j\in\mathbb{N}}$, s.t. $|(\pi_j)_{j\in\mathbb{N}}|_{L^1} < +\infty$ and

$$w_t = \sum_{j\in\mathbb{N}} \pi_j x_{t-j}.$$

In words, the current error is a function of past observations.

An ARMA process is invertible iif

$$\theta(x) = 0 \implies |x| > 1.$$

Invertibility is mostly defined for making models identifiable (i.e. unique solution).

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Part I: Autocorrelation function (ACF)

A) implement ACF

Do your own implementation of the ACF function. Your implementation will be checked against statsmodels.tsa.stattools.acf.

```
def acf_impl(x, nlags):
    """
    TODO
    @param x: a 1-d numpy array (data)
    @param nlags: an integer indicating how far back to compute the ACF
    @return a 1-d numpy array with (nlags+1) elements.
            Where the first element denotes the acf at lag = 0 (1.0 by definition).
    """
    #TODO: replace the template code with your code here. This part will be graded.
    return np.zeros(nlags+1)
```

Figure 2: Part I A)

Part I: Autocorrelation function (ACF)

B) ACF of White Noise

$$w_t \sim N(0, \sigma^2)$$

1. Set $\sigma = 1$, sample n = 500 points from the process above
2. Plot the white noise
3. Plot the sample ACF up to lag = 20.
4. Compare sample ACF with analytical ACF.
5. Compare your sample ACF with true sample ACF.
6. What trend/observation can you find in the ACF plot?
7. Change n to 50, compare the new ACF plot (n=50) to the old ACF plot (n=500). What causes the difference?

Figure 3: Part I B)

Part I: Autocorrelation function (ACF)

```python
## 1.
n = 500
mean = 0
std = 1
lag = 20

# create white noise
w_t = np.random.normal(mean, std, size=n)

# create subplots
fig, (ax1, ax2, ax3) = plt.subplots(1, 3, figsize=(15, 5))

## 2.
# plot white noise
ax1.plot(w_t, label="w_t")
ax1.set_title("Signal")
ax1.legend()

## 3.
# calculate acf
acf_val = acf(x=w_t, nlags=lag)
## 4.
acf_analytic = np.zeros(lag)
acf_analytic[0] = std**2
plot_acf(x=w_t, lags=lag, title="ACF w_t", ax=ax2, label="Sample ACF")
ax2.stem(acf_analytic, markerfmt='r.', linefmt='r--', label="Analytic ACF")
ax2.set_ylim([1.3*min(acf_val), 1.1*(std**2)])
ax2.legend()

## 5.
# your implementation:
acf_val_impl = acf_impl(x=w_t, nlags=lag)
ax3.plot(acf_val, 'or', label='True sample acf')
ax3.plot(acf_val_impl, 'xb', label='Own sample acf')
ax3.legend();
ax3.set_title('your sample ACF impl against true sample ACF');
```

Part I: Autocorrelation function (ACF)

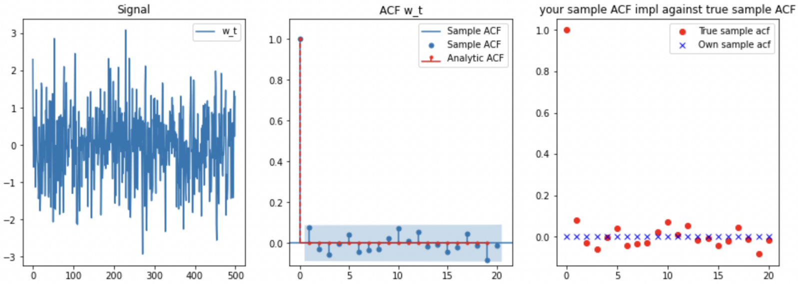

Figure 5: Part I B)

Part I: Autocorrelation function (ACF)

C) ACF of Moving Average

$$v_t = \frac{1}{3}(w_t + w_{t+1} + w_{t+2})$$

1. Sample n+2 white noise from N(0,1)
2. Add code to compute the moving average v_t.
3. Plot both w_t and v_t and compare the two time series.
4. Derive the analytical ACF
5. Compare sample ACF up to lag 20 with the analytical ACF.
6. Compare your sample ACF with true sample ACF.

Figure 6: Part I C)

Part I: Autocorrelation function (ACF)

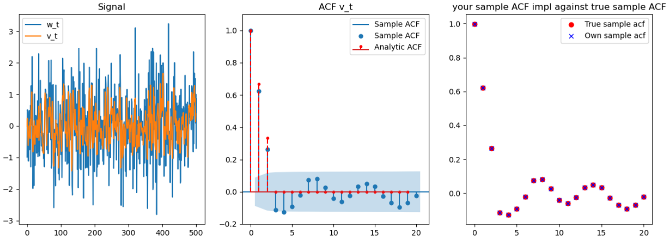

Figure 7: Part I C)

Part I: Autocorrelation function (ACF)

D) ACF of signal in noise

$$v_t = 2cos(\frac{2\pi t}{50} + 0.6\pi) + w_t$$

1. Sample white noise of length n from $N(0, 1)$
2. Add code to compute v_t.
3. Plot both w_t and v_t. Compare the two plots.
4. Plot the sample ACF of v_t. What's the pattern? What causes the observed pattern?

[Optional]: derive and compare to the analytical ACF (hint, use cosine trig identity)

Figure 8: Part I D)

Part II: Cross-correlation function (CCF)

Part II: Cross-correlation Function

A) CCF of signal with noise

Synthetic Data

$$x_t \sim N(0, \sigma_x^2)$$

$$y_t = 2x_{t-5} + w_t$$

$$w_t \sim N(0, \sigma_x^2)$$

- In this example, we created two processes with a lag of 5.
- Plot both samples and verify the lag.
- Plot the empirical ACF for both samples.
- Plot the empirical CCF. What information can you conclude from the CCF plot?

Figure 9: Part II A)

Part II: Cross-correlation function (CCF)

B) CCF of data

Southern Oscillation Index (SOI) v.s. Recruitment (Rec)

- Replicate the procedure in the previous section.
- What information can you tell from the CCF plot.
- In this example, our procedure is actually flawed. Unlike the previous example, we can not tell if the cross-correlation estimate is significantly different from zero by looking at the CCF. Why is that? What can we do to address this issue?

```
soi = np.array(pd.read_csv("soi.csv")["x"])
rec = np.array(pd.read_csv("rec.csv")["x"])
#TODO: This part will be graded.
# plot data
# plot acf
# plot ccf
```

Figure 10: Part II B)

Part III: Moving Average (MA)

Part III

Moving Average

A)

$$x_t = 0.5x_{t-1} - 0.5w_{t-1} + w_t$$

$$w_t \sim N(0, \sigma^2)$$

Is x_t same as white noise w_t? Think about ACF.

Then use code below to assess and verify your guess.

Figure 11: Part III A)

Part III: Moving Average (MA)

B)

$$x_t = w_t + \frac{1}{5}w_{t-1}, w_t \sim N(0, 25)$$

$$y_t = v_t + 5v_{t-1}, v_t \sim N(0, 1)$$

Are x_t and y_t the same? Think about ACF.

Then use code below to assess and verify your guess.

Figure 12: Part III B)