# Lab 2 : Review on Bayesian Statistics

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## Competition results

### Speed

As some of you noticed, a slightly more efficient way to compute the variance is to notice that :

$$\mathbb{V}[X] := \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

As a general rule, try using numpy functions and numpy broadcasting as much possible. To compute $\mathbb{E}[X^2]$, we have 2 efficient solutions : `np.dot` and `@` (same as `np.matmul`). For 1D vector dot product, it is recommended to use `np.dot`, while for 2D matrix product you should use `@`.

- Tina Wan : Using `np.dot`
- Yirong Bian and Hanyuan Zhang : Computing using $X.T@X$ (note that it is equivalent to $X@X$ for 1D array)
- Yuhao Zheng : Naive variance computation using `numba`'s `njit` decorator

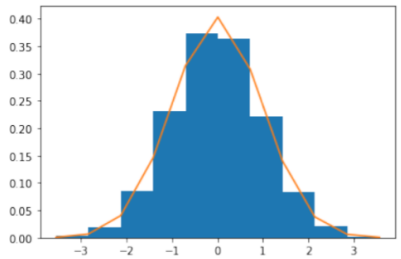
## Competition results

### Originality

- Jiayun Wang : Fitting a normal distribution on this particular dataset

```
mu, std = norm.fit(x_np)
_, bins, _ = plt.hist(x_np, density=True)
plt.plot(bins, norm.pdf(bins, mu, std))
```

```
[<matplotlib.lines.Line2D at 0x7fa6cad6f940>]
```




According to the shape of the histogram we observe that the data is a fair fit of Gaussian distribution, so we can fit a normal curve to the data using scipy to find out the mean and variance.

- Ken Zeng : Defining the mean as $\operatorname{argmin}_{\theta} \mathbb{E}[(X-\theta)^2]$ and minimizing this cost via gradient descent on a linear layer.
- Boyu Hu and Mingxuan Wu : Online mean computation

# Table of Contents

## Introduction

- Experiment : Repeatable operation carried out under controlled conditions, with well defined outcomes.
- Sample space : Set of possible outcomes, often denoted $\Omega$
  Rolling a dice : $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- Event : A subset of the sample space
  Event of obtaining a number greater than $4 = \{5, 6\}$.
- Random variable : Mapping from the possible outcomes in the sample space to a measurable space (often a real number)
  $X =$"The number rolled, mod 2" $\in \{0, 1\}$
- Probability : How likely an event is to occur.
  We often denote $p(x) = \mathbb{P}(X = x)$.

## Independence

### Independence

Two random variables $A, B$ are said to be independent if any of the following is satisfied :

1. $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$
2. $\mathbb{P}(A|B) = \mathbb{P}(A)$

## Mutual independence

### Mutual independence

Let $I$ be an ensemble and $(A_i)_{i \in I}$ a set of events. The $(A_i)_{i \in I}$ are said to be **mutually independent** iif

$$\forall J \in \mathcal{P}(I), \quad \mathbb{P}\left(\bigcap_{j \in J} A_j\right) = \prod_{j \in J} \mathbb{P}(A_j).$$

A set of events is said to be mutually independent if the probability of each event in the set is the same no matter which of the other events has occurred.

## A remark

In particular when $I = J$ :

$$\mathbb{P}\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} \mathbb{P}(A_i).$$

This formula is NOT equivalent to mutual independence.

### Counter example

Assume we are doing coin flips.
A : The first flip is T
B : At most one T in the first 3 flips
C : Same as B
We have $\mathbb{P}(A \cap B \cap C) = \frac{1}{8} = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$ but the events are clearly not mutually independent !

## Computing a probability

There a 3 fundamental theorems for computing probabilities :

- Law of total probability (aka marginalization or sum rule)

$$\mathbb{P}(A) = \sum_{b \in B} \mathbb{P}(A \cap B = b)$$

- Chain rule (aka product rule)

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B|A)\mathbb{P}(C|A, B)$$

- Bayes' theorem

$$\mathbb{P}(A \cap B) = \frac{\mathbb{P}(B \cap A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

## Law of total probability

Marginalization or process of "forgetting irrelevant information".

$$\mathbb{P}(A) = \sum_{b \in B} \mathbb{P}(A \cap B = b).$$

More generally, let $(B_i)_{i \in I}$ a collectively exhaustive family of events, i.e. $\sum_{i \in I} \mathbb{P}(B_i) = 1$,

$$\mathbb{P}(A) = \sum_{i \in I} \mathbb{P}(A \cap B_i).$$

## Chain rule

### Conditional probability

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B|A).$$

### Chain rule

By proceeding recursively, we can expand the formula.
Let $(A_i)_{i \in [\![1,n]\!]}$ a set of random variables, s.t. $\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) \neq 0$

$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) = \mathbb{P}(A_1) \prod_{i=2}^{n} \mathbb{P}\left(A_i \middle| \bigcap_{j=1}^{i-1} A_j\right).$$

## Bayes Theorem

Bayes theorem derives directly from the definition of conditional probabilities.

### Bayes Theorem

Let $A, B$ two events such as $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$. Then,

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

$$\texttt{posterior} = \frac{\texttt{likelihood} \times \texttt{prior}}{\texttt{evidence}}.$$

As we use it to estimate parameters, we sometimes drop the evidence:

$$p(\theta|\mathcal{D}) \propto p(\mathcal{D}|\theta)p(\theta).$$

# Table of Contents

## Bayesian networks

- We can use these three theorems to take advantage of structure in models to make computations cheaper and inference easier
- Probabilistic models with any sort of conditional structure can be described as Bayesian networks
- Formally :

**Bayesian network**

A probabilistic graphical model representing the joint probability of a set of random variables, capturing conditional dependencies via a Directed Acyclic Graph (DAG).

## Conditioning : 3 fundamental patterns


(a) Independent (b) Independent (c) Not independent

Figure 1: Three fundamental patterns

$$p(B, C, D) = p(B)p(C|B)p(D|B)$$
$$p(C, D|B) = p(C|B)p(D|B)$$

## Conditioning : 3 fundamental patterns

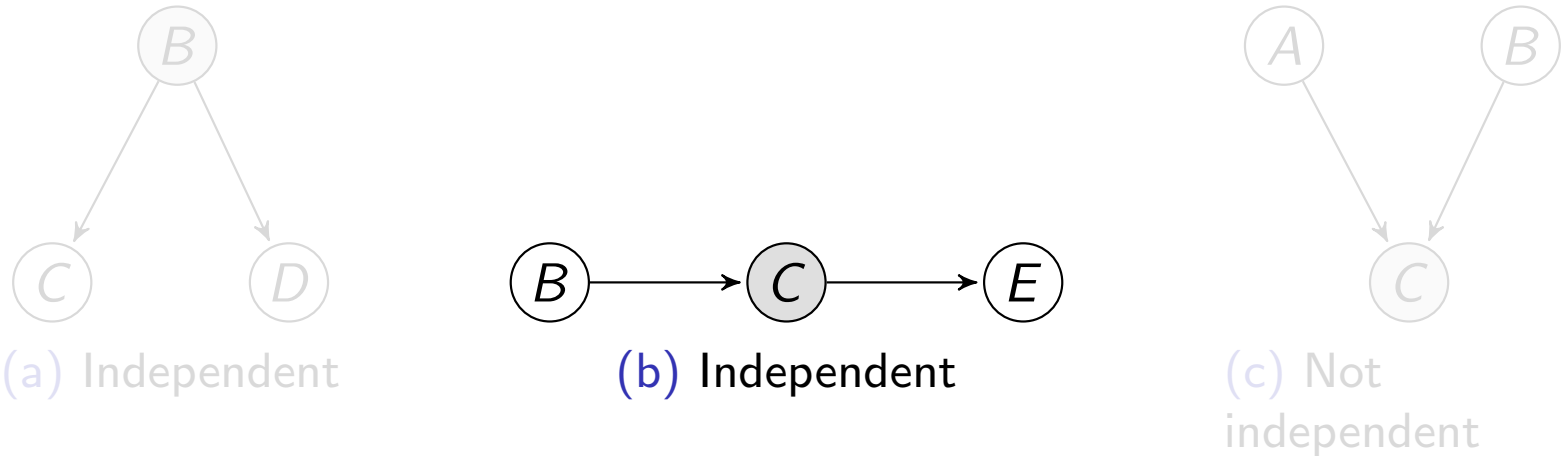
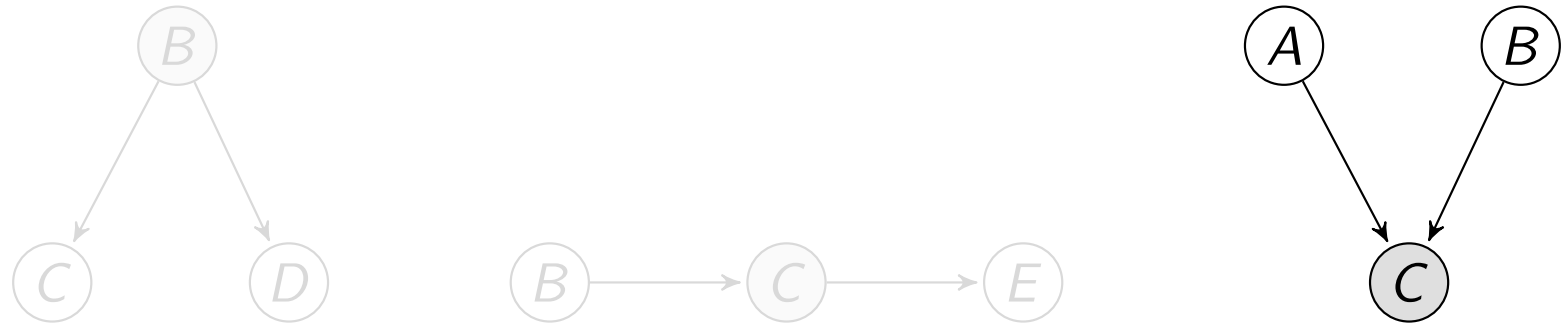



Figure 1: Three fundamental patterns

$$p(B, C, E) = p(B)p(C|B)p(E|C)$$
$$= p(B)\frac{p(B|C)p(C)}{p(B)}p(E|C)$$
$$p(B, E|C) = p(B|C)p(E|C)$$

## Conditioning : 3 fundamental patterns

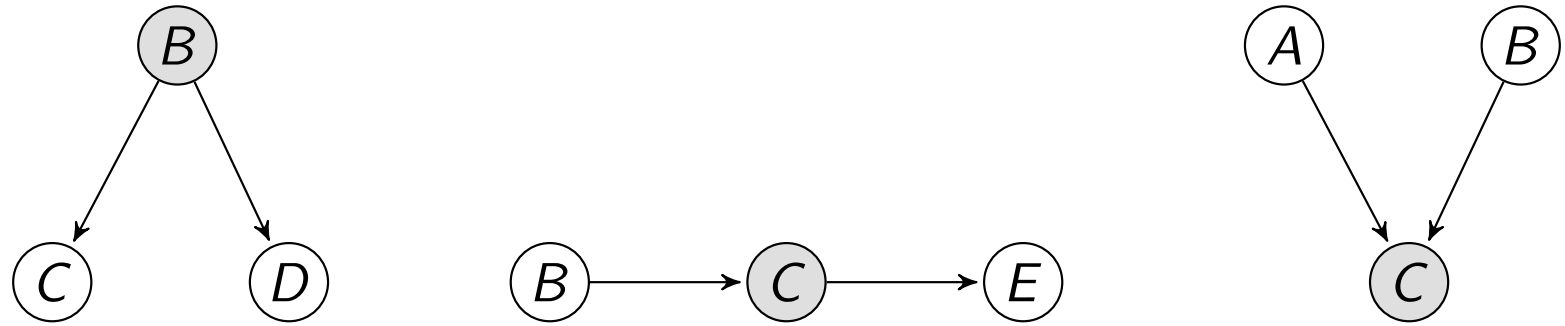



(a) Independent (b) Independent (c) Not independent

Figure 1: Three fundamental patterns

$$p(A,B,C) = p(A)p(B)p(C|A,B)$$
$$= \frac{p(C)^2 p(A|C)p(B|C)}{p(C|A)p(C|B)} p(C|A,B)$$
$$p(A,B|C) = p(A|C)p(B|C)\frac{p(C)p(C|A,B)}{p(C|A)p(C|B)}$$

## Conditioning : 3 fundamental patterns




(a) Independent (b) Independent (c) Not independent

Figure 1: Three fundamental patterns

The **Markov blanket** for an ensemble $X$ consists in the **children**, **parents** and the **co-parents**.

## Example Bayesian network

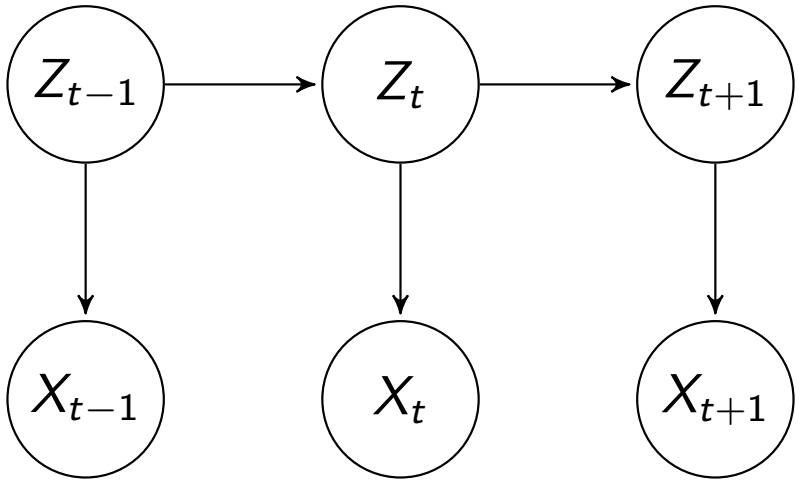
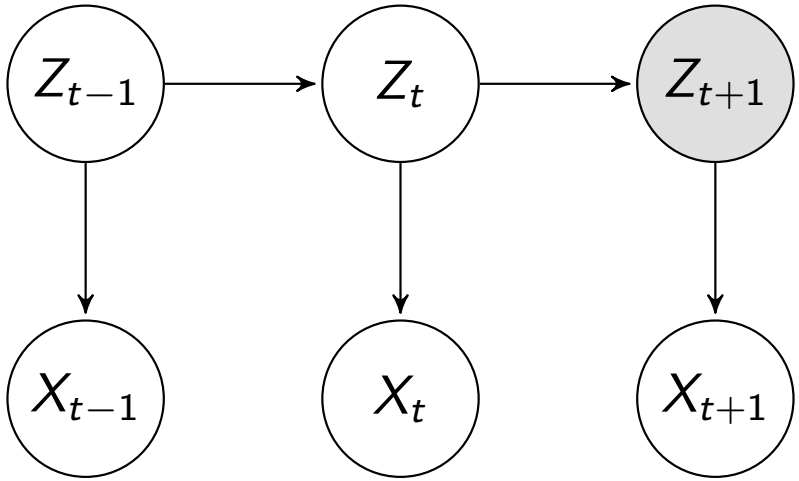



Figure 2: Bayesian Network

### Fact

Each variable is conditionally independent of all its nondescendants in the graph given the value of all its parents.

## Example Bayesian network




$$X_{t+1} \perp\!\!\!\perp Z_{t-1}, X_{t-1}, Z_t, X_t \mid Z_{t+1}$$

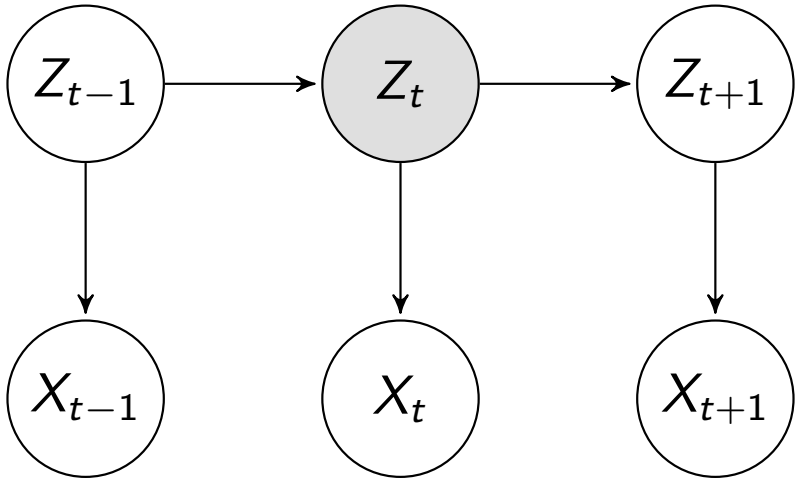
$$X_t \perp\!\!\!\perp Z_{t-1}, X_{t-1}, Z_{t+1}, X_{t+1} \mid Z_t$$

$$Z_{t+1} \perp\!\!\!\perp Z_{t-1}, X_{t-1}, X_t \mid Z_t$$

$$X_{t-1} \perp\!\!\!\perp Z_t, X_t, Z_{t+1}, X_{t+1} \mid Z_{t-1}$$

$$Z_t \perp\!\!\!\perp X_{t-1} \mid Z_{t-1}$$

## Example Bayesian network

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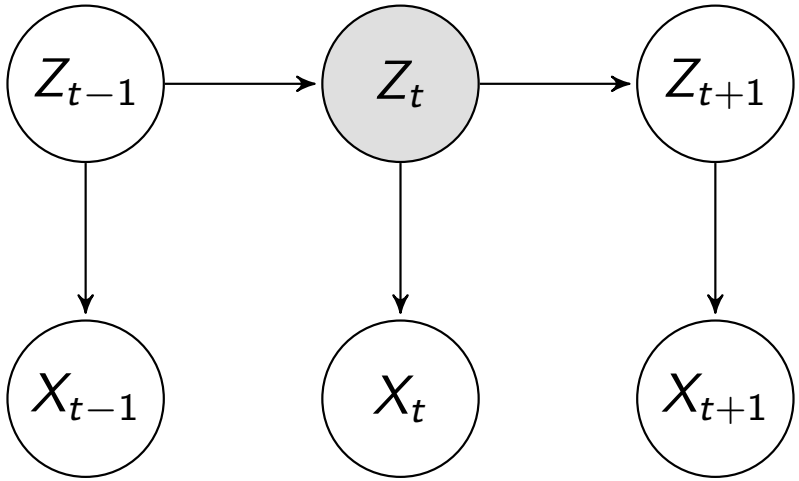
## Example Bayesian network

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$$X_{t-1} \perp\!\!\!\perp Z_t, X_t, Z_{t+1}, X_{t+1} \mid Z_{t-1}$$
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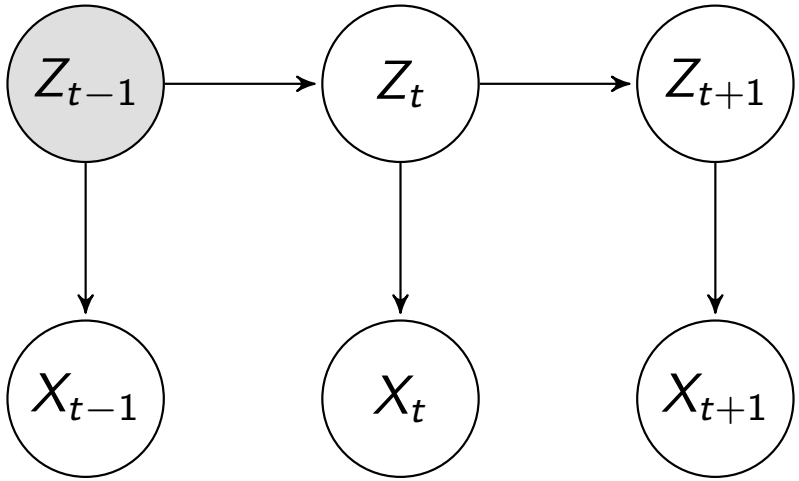
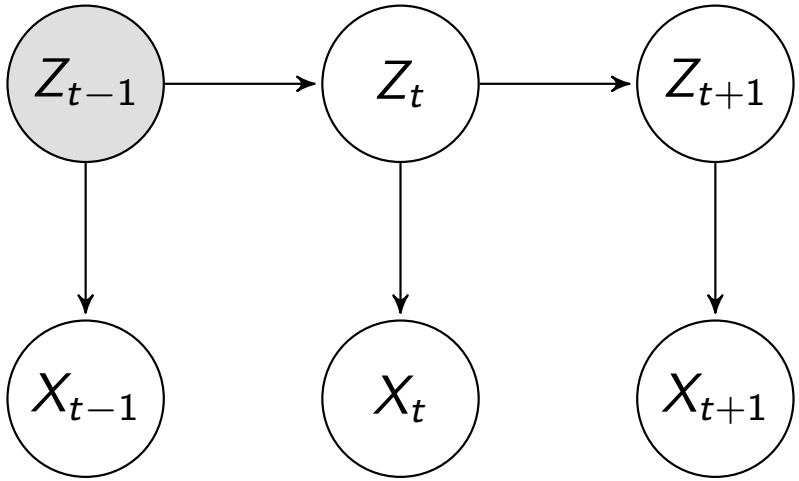
## Example Bayesian network




$$
\begin{aligned}
X_{t+1} &\perp\!\!\!\perp Z_{t-1}, X_{t-1}, Z_t, X_t &&| Z_{t+1} \\
X_t &\perp\!\!\!\perp Z_{t-1}, X_{t-1}, Z_{t+1}, X_{t+1} &&| Z_t \\
Z_{t+1} &\perp\!\!\!\perp Z_{t-1}, X_{t-1}, X_t &&| Z_t \\
X_{t-1} &\perp\!\!\!\perp Z_t, X_t, Z_{t+1}, X_{t+1} &&| Z_{t-1} \\
Z_t &\perp\!\!\!\perp X_{t-1} &&| Z_{t-1}
\end{aligned}
$$

## Example Bayesian network




$$
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X_{t+1} &\perp\!\!\!\perp Z_{t-1}, X_{t-1}, Z_t, X_t &&| Z_{t+1} \\
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Z_{t+1} &\perp\!\!\!\perp Z_{t-1}, X_{t-1}, X_t &&| Z_t \\
X_{t-1} &\perp\!\!\!\perp Z_t, X_t, Z_{t+1}, X_{t+1} &&| Z_{t-1} \\
Z_t &\perp\!\!\!\perp X_{t-1} &&| Z_{t-1}
\end{aligned}
$$

## Computational costs in Bayesian networks

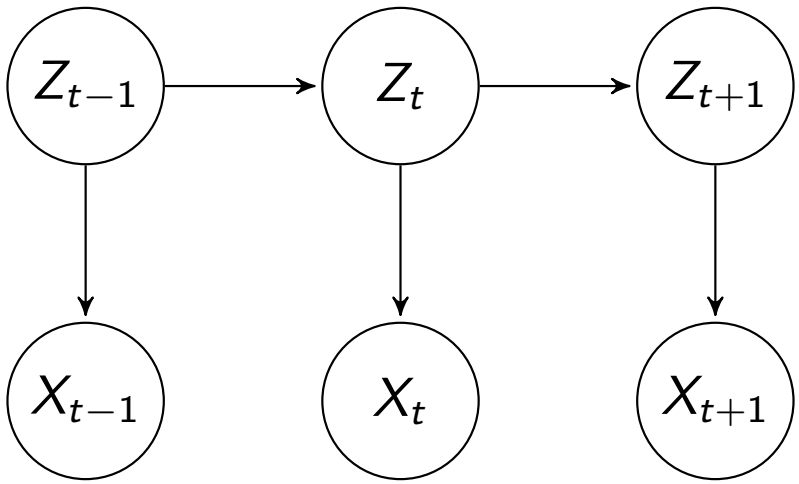
- We can take advantage of the structure in Bayesian networks to simplify calculations and reduce the computational cost.
- The probability $p(X_{t+1}, X_t, X_{t-1})$ (the evidence) is often one of interest.

$$p(X_{t+1}, X_t, X_{t-1}) = \sum_{Z_{t+1}} \sum_{Z_t} \sum_{Z_{t-1}} p(X_{t+1}, X_t, X_{t-1}, Z_{t+1}, Z_t, Z_{t-1})$$

- Assuming $Z$ can take one of 3 values, and not using the conditional structure of the model, this has a computational cost of $3^3$

## Using the conditional structure of Bayesian networks

- Using the conditional structure, we can rewrite the $p(X_{t+1}, X_t, X_{t-1}, Z_{t+1}, Z_t, Z_{t-1})$ as the product of conditionals:
$p(X_{t+1}|Z_{t+1})p(Z_{t+1}|Z_t)p(X_t|Z_t)p(Z_t|Z_{t-1})p(X_{t-1}|Z_{t-1})p(Z_{t-1})$



## Using the conditional structure of Bayesian networks

- Using the conditional structure, we can rewrite the $p(X_{t+1}, X_t, X_{t-1}, Z_{t+1}, Z_t, Z_{t-1})$ as the product of conditionals:
  $p(X_{t+1}|Z_{t+1})p(Z_{t+1}|Z_t)p(X_t|Z_t)p(Z_t|Z_{t-1})p(X_{t-1}|Z_{t-1})p(Z_{t-1})$
- Moving the terms that don't appear out of each sum, we can simplify the computation

$$p(X_{t+1}, X_t, X_{t-1}) = \sum_{Z_{t+1}} \Bigg( p(X_{t+1}|Z_{t+1}) \sum_{Z_t} \Bigg( p(Z_{t+1}|Z_t)p(X_t|Z_t) \sum_{Z_{t-1}} p(Z_t|Z_{t-1})p(X_{t-1}|Z_{t-1})p(Z_{t-1}) \Bigg) \Bigg)$$

## Computational cost of computing the evidence

- To compute the inner most sum, we must create a table of $Z_{t-1}$ and $Z_t$ values, and sum over $Z_t$, this has a cost of $3^2$

| $p(Z_t \mid Z_{t-1})$ | $Z_t = 0$ | $Z_t = 1$ | $Z_t = 2$ |
|---|---|---|---|
| $Z_{t-1} = 0$ | $p(Z_t = 0 \mid Z_{t-1} = 0)$ | $p(Z_t = 1 \mid Z_{t-1} = 0)$ | $p(Z_t = 2 \mid Z_{t-1} = 0)$ |
| $Z_{t-1} = 1$ | $p(Z_t = 0 \mid Z_{t-1} = 1)$ | $p(Z_t = 1 \mid Z_{t-1} = 1)$ | $p(Z_t = 2 \mid Z_{t-1} = 1)$ |
| $Z_{t-1} = 2$ | $p(Z_t = 0 \mid Z_{t-1} = 2)$ | $p(Z_t = 1 \mid Z_{t-1} = 2)$ | $p(Z_t = 2 \mid Z_{t-1} = 2)$ |

$\times$

| $p(X_{t-1} \mid Z_{t-1})$ | $Z_t = 0$ | $Z_t = 1$ | $Z_t = 2$ |
|---|---|---|---|
| $Z_{t-1} = 0$ | $p(X_{t-1} \mid Z_{t-1} = 0)$ | $p(X_{t-1} \mid Z_{t-1} = 0)$ | $p(X_{t-1} \mid Z_{t-1} = 0)$ |
| $Z_{t-1} = 1$ | $p(X_{t-1} \mid Z_{t-1} = 1)$ | $p(X_{t-1} \mid Z_{t-1} = 1)$ | $p(X_{t-1} \mid Z_{t-1} = 1)$ |
| $Z_{t-1} = 2$ | $p(X_{t-1} \mid Z_{t-1} = 2)$ | $p(X_{t-1} \mid Z_{t-1} = 2)$ | $p(X_{t-1} \mid Z_{t-1} = 2)$ |

$\times$

| $p(Z_{t-1})$ | $Z_t = 0$ | $Z_t = 1$ | $Z_t = 2$ |
|---|---|---|---|
| $Z_{t-1} = 0$ | $p(Z_{t-1} = 0)$ | $p(Z_{t-1} = 0)$ | $p(Z_{t-1} = 0)$ |
| $Z_{t-1} = 1$ | $p(Z_{t-1} = 1)$ | $p(Z_{t-1} = 1)$ | $p(Z_{t-1} = 1)$ |
| $Z_{t-1} = 2$ | $p(Z_{t-1} = 2)$ | $p(Z_{t-1} = 2)$ | $p(Z_{t-1} = 2)$ |

Figure 3: Product of inner sum

## Computational cost of computing the evidence

- To compute the inner most sum, we must create a table of $Z_{t-1}$ and $Z_t$ values, and sum over $Z_t$, this has a cost of $3^2$

$=$

| $p(Z_t \mid Z_{t-1})p(X_{t-1} \mid Z_{t-1})p(Z_{t-1})$ | $Z_t = 0$ | $Z_t = 1$ | $Z_t = 2$ |
|---|---|---|---|
| $Z_{t-1} = 0$ | $A$ | $B$ | $C$ |
| $Z_{t-1} = 1$ | $D$ | $E$ | $F$ |
| $Z_{t-1} = 2$ | $G$ | $H$ | $I$ |

Figure 4: Result of product

## Computational cost of computing the evidence

- To compute the inner most sum, we must create a table of $Z_{t-1}$ and $Z_t$ values, and sum over $Z_t$, this has a cost of $3^2$

| $p(Z_t \mid Z_{t-1})p(X_{t-1} \mid Z_{t-1})p(Z_{t-1})$ | $Z_t = 0$ | $Z_t = 1$ | $Z_t = 2$ |
|---|---|---|---|
| $Z_{t-1} = 0$ | $A$ | $B$ | $C$ |
| $Z_{t-1} = 1$ | $D$ | $E$ | $F$ |
| $Z_{t-1} = 2$ | $G$ | $H$ | $I$ |

$$\sum_{Z_{t-1}} \rightarrow$$

| | $Z_t = 0$ | $Z_t = 1$ | $Z_t = 2$ |
|---|---|---|---|
| $\sum_{Z_{t-1}} p(Z_t \mid Z_{t-1})p(X_{t-1} \mid Z_{t-1})p(Z_{t-1})$ | $A + D + G$ | $B + E + H$ | $C + F + I$ |

Figure 5: Summing $Z_{t-1}$

## Computational cost of computing the evidence

- Now do the same thing for the second sum, where the sum we computed on the previous slide is equal to $p(X_{t-1}, Z_t)$. The cost is again $3^2$.

| $p(Z_{t+1}\mid Z_t)$ | $Z_{t+1}=0$ | $Z_{t+1}=1$ | $Z_{t+1}=2$ |
|---|---|---|---|
| $Z_t=0$ | $p(Z_{t+1}=0\mid Z_t=0)$ | $p(Z_{t+1}=0\mid Z_t=1)$ | $p(Z_{t+1}=0\mid Z_t=2)$ |
| $Z_t=1$ | $p(Z_{t+1}=1\mid Z_t=0)$ | $p(Z_{t+1}=1\mid Z_t=1)$ | $p(Z_{t+1}=1\mid Z_t=2)$ |
| $Z_t=2$ | $p(Z_{t+1}=2\mid Z_t=0)$ | $p(Z_{t+1}=2\mid Z_t=1)$ | $p(Z_{t+1}=2\mid Z_t=2)$ |

$\times$

| $p(X_t\mid Z_t)$ | $Z_{t+1}=0$ | $Z_{t+1}=1$ | $Z_{t+1}=2$ |
|---|---|---|---|
| $Z_t=0$ | $p(X_t\mid Z_t=0)$ | $p(X_t\mid Z_t=0)$ | $p(X_t\mid Z_t=0)$ |
| $Z_t=1$ | $p(X_t\mid Z_t=1)$ | $p(X_t\mid Z_t=1)$ | $p(X_t\mid Z_t=1)$ |
| $Z_t=2$ | $p(X_t\mid Z_t=2)$ | $p(X_t\mid Z_t=2)$ | $p(X_t\mid Z_t=2)$ |

$\times$

| $p(X_{t-1}, Z_t)$ | $Z_{t+1}=0$ | $Z_{t+1}=1$ | $Z_{t+1}=2$ |
|---|---|---|---|
| $Z_t=0$ | $A+D+G$ | $A+D+G$ | $A+D+G$ |
| $Z_t=1$ | $B+E+H$ | $B+E+H$ | $B+E+H$ |
| $Z_t=2$ | $C+F+I$ | $C+F+I$ | $C+F+I$ |

Figure 6: Product of second sum

## Computational cost of computing the evidence

- Now do the same thing for the second sum, where the sum computed two slides ago is equal to $p(X_{t-1}, Z_t)$. The cost is again $3^2$.

| $p(Z_{t+1}\|Z_t)p(X_t\|Z_t)p(X_{t-1}, Z_t)$ | $Z_{t+1}=0$ | $Z_{t+1}=1$ | $Z_{t+1}=2$ |
|---|---|---|---|
| $Z_t=0$ | $J$ | $K$ | $L$ |
| $Z_t=1$ | $M$ | $N$ | $O$ |
| $Z_t=2$ | $P$ | $Q$ | $R$ |

$$\sum_{Z_t} \rightarrow$$

| | $Z_{t+1}=0$ | $Z_{t+1}=1$ | $Z_{t+1}=2$ |
|---|---|---|---|
| $\sum_{Z_t} p(Z_{t+1}\|Z_t)p(X_t\|Z_t)p(X_{t-1}, Z_t)$ | $J+M+P$ | $K+N+Q$ | $L+O+R$ |

Figure 7: Summing $Z_t$

## Computational cost of computing the evidence

- For the final sum, we only have two tables with only one row (for $Z_{t+1}$), so the cost for this will only be $3^1$. The sum on the previous slide is now $p(X_t, X_{t-1}, Z_{t+1})$

| | $Z_{t+1}=0$ | $Z_{t+1}=1$ | $Z_{t+1}=2$ |
|---|---|---|---|
| $p(X_{t+1}\mid Z_{t+1})$ | $p(X_{t+1}\mid Z_{t+1}=0)$ | $p(X_{t+1}\mid Z_{t+1}=1)$ | $p(X_{t+1}\mid Z_{t+1}=2)$ |

$\times$

| | $Z_{t+1}=0$ | $Z_{t+1}=1$ | $Z_{t+1}=2$ |
|---|---|---|---|
| $p(X_t, X_{t-1}, Z_{t+1})$ | $J+M+P$ | $K+N+Q$ | $L+O+R$ |

$=$

| | $Z_{t+1}=0$ | $Z_{t+1}=1$ | $Z_{t+1}=2$ |
|---|---|---|---|
| $p(X_{t+1}\mid Z_{t+1})\ p(X_t, X_{t-1}, Z_{t+1})$ | $S$ | $T$ | $U$ |

Figure 8: Product of final sum

## Computational cost of computing the evidence

- As a the final result, we get that $\sum_{Z_{t+1}} p(X_{t+1}|Z_{t+1})p(X_t, X_{t-1}, Z_{t+1}) = S + T + U = p(X_{t+1}, X_t, X_{t-1})$
- The total cost of this computation is $2 \times 3^2 + 3^1$

## Generalizing the model

- If instead of only having $\{Z_i, X_i | i \in \{t-1, t, t+1\}\}$, we extend the model to $\{Z_i, X_i | i \in \{1, \ldots, T\}\}$ and we allow the dimensionality of $Z_i$ to be generalized to $d$, we find that the computational cost is $T \times d^2 + d^1$
- The cost of computing the evidence naively is $d^T$
- We have thus reduced the cost from being exponential in time (where time is defined as the value of $T$) and polynomial in dimension ($d$) to linear in time and quadratic in dimension

# Table of Contents

## Gaussian identities

- Gaussians are the preferred distributions data scientists work with due to their nice properties
- The two most important identities are that the sum and product of two Gaussians is another Gaussian
- These give rise to the fact that the marginal of a Gaussian over two variables is another Gaussian
- This means inference can be done exactly with analytical solutions since the normalization constants can be computed exactly.
- https://cs.nyu.edu/∼roweis/notes/gaussid.pdf (replace the ∼ with the one from your keyboard for link to work)

## Two important identities (1/2)

Let $\mathbf{z}$ :

$$\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \sim \mathcal{N}\left( \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^\top & \mathbf{B} \end{bmatrix} \right).$$

Then

$$\mathbf{x}|\mathbf{y} \sim \mathcal{N}\left( \mathbf{a} + \mathbf{C}\mathbf{B}^{-1}(\mathbf{y} - \mathbf{b}), \mathbf{A} - \mathbf{C}\mathbf{B}^{-1}\mathbf{C}^\top \right)$$

$$\mathbf{y}|\mathbf{x} \sim \mathcal{N}\left( \mathbf{b} + \mathbf{C}^\top\mathbf{A}^{-1}(\mathbf{x} - \mathbf{a}), \mathbf{B} - \mathbf{C}^\top\mathbf{A}^{-1}\mathbf{C} \right)$$

## Two important identities (2/2)

Let $\mathbf{x}, \mathbf{y}$ :

$$p(\mathbf{x}) = \mathcal{N}(\mathbf{x}|\mathbf{\Lambda}^{-1})$$
$$p(\mathbf{y}|\mathbf{x}) = \mathcal{N}(\mathbf{y}|\mathbf{A}\mathbf{x} + \mathbf{b}, \mathbf{L}^{-1}).$$

Then,

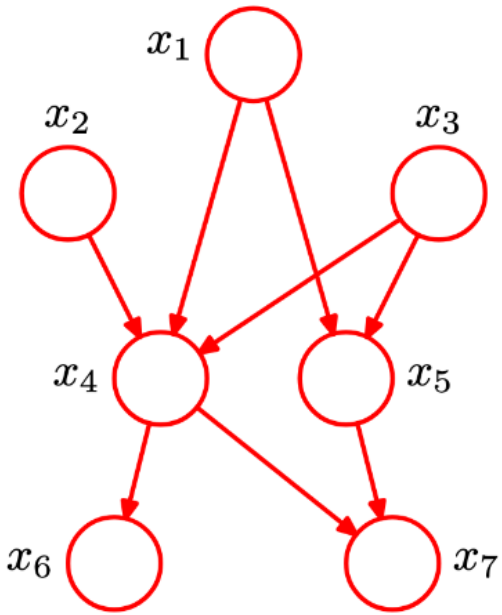
$$p(\mathbf{y}) = \mathcal{N}(\mathbf{y}|\mathbf{A}\mu + \mathbf{b}, \mathbf{L}^{-1} + \mathbf{A}\mathbf{\Lambda}^{-1}\mathbf{A}^{\top})$$
$$p(\mathbf{x}|\mathbf{y}) = \mathcal{N}(\mathbf{x}|\mathbf{\Sigma}\left[\mathbf{A}^{\top}\mathbf{L}(\mathbf{y} - \mathbf{b}) + \mathbf{\Lambda}\mu\right], \mathbf{\Sigma}),$$

where

$$\mathbf{\Sigma} = \left(\mathbf{\Lambda} + \mathbf{A}^{\top}\mathbf{L}\mathbf{A}\right)^{-1}.$$

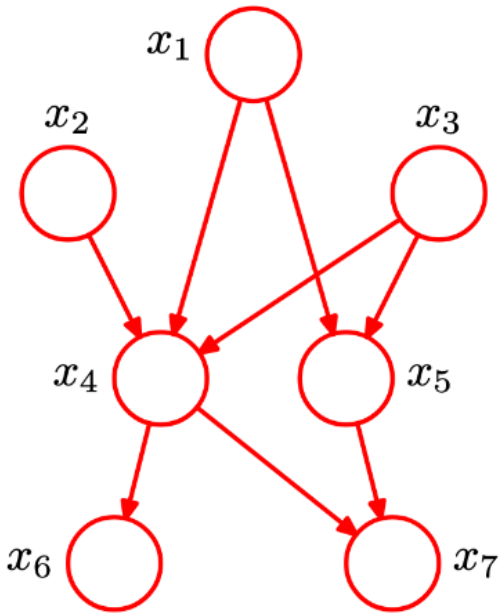
# Table of Contents

## From Bishop (Chapter 8)



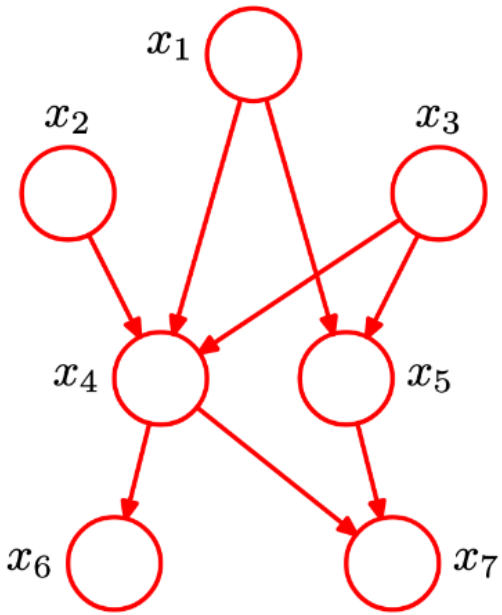

1. Markov blanket of $x_5$?
2. Joint distribution ?

## From Bishop (Chapter 8)



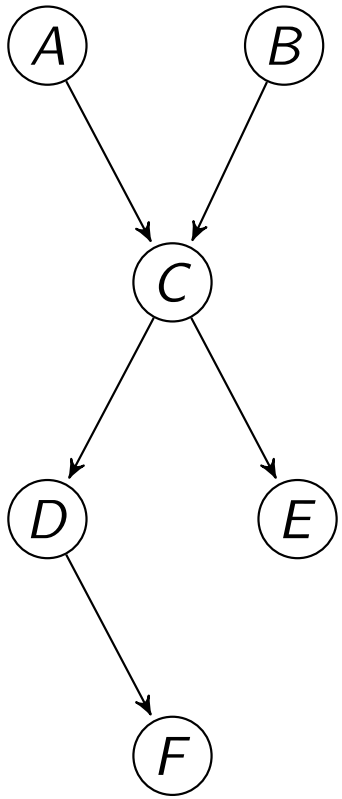

1. Markov blanket of $x_5$? $\{x_7, x_1, x_3, x_4\}$
2. Joint distribution ?

## From Bishop (Chapter 8)




1. Markov blanket of $x_5$? $\{x_7, x_1, x_3, x_4\}$
2. Joint distribution ?
   $p(x_1)p(x_2)p(x_3)p(x_4|x_1, x_2, x_3)p(x_5|x_1, x_3)p(x_6|x_4)p(x_7|x_4, x_5)$

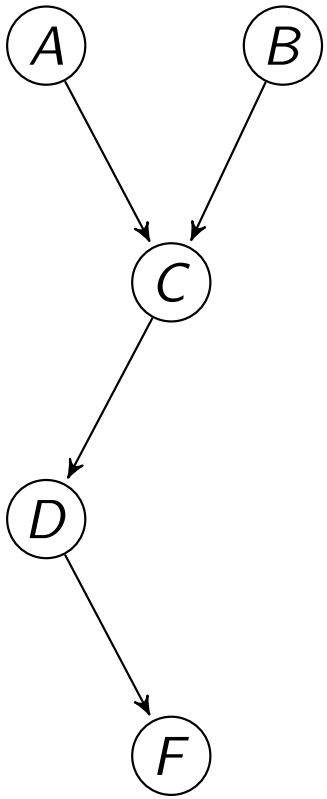
## From http://web.mit.edu/jmn/www/6.034/d-separation.pdf



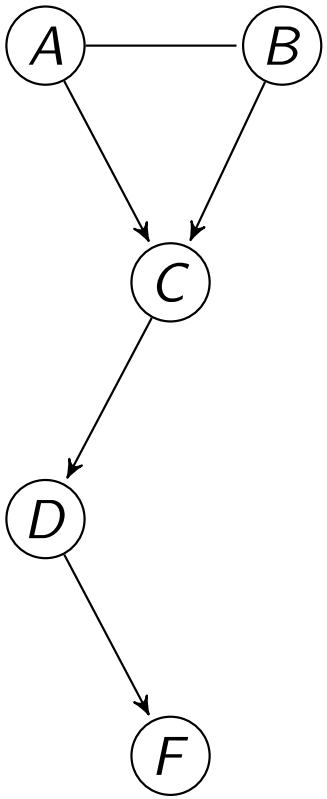

1. Are $A$ and $B$ conditionally independent given $D$ and $F$ ?
2. Are $D$ and $E$ conditionally independent given $C$?
3. Are $D$ and $E$ conditionally independent given $A$ and $B$ ?
4. $P(D|CEF) \stackrel{?}{=} P(D|C)$

## From http://web.mit.edu/jmn/www/6.034/d-separation.pdf

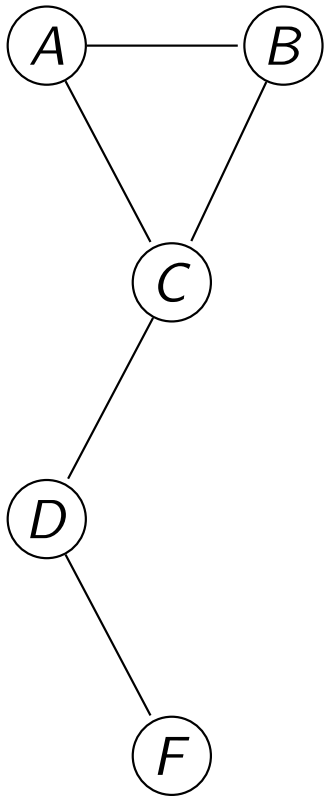
Are $A$ and $B$ conditionally independent given $D$ and $F$ ? **No.**



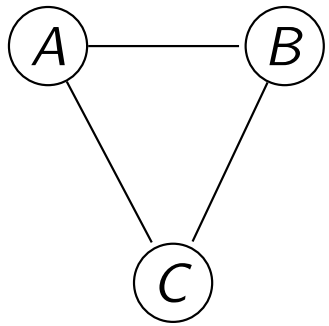

(a) Ancestral graph




(b) Moralize




(c) Disorient




(d) Delete givens