I worked with John Pharo on this worksheet. It took about four hours.

Solving the Poisson Equation

My code for this problem is in `ws12.py`.

First I inspected the data and tried to figure out which column was which. It is clear that column 1 is grid point number, since these are positive integers. The second column monotonically increases and maxes out at $\sim 10^{34}$ – assuming CGS, and assuming this pre-supernova star is very massive ($\sim 100 M_\odot$), this is about the right total mass of the star in grams. I therefore infer that the second column is the enclosed mass of the star. The third column also monotonically increases and maxes out at $\sim 10^{13}$ – about the radius of a very massive star in centimeters. Therefore we can conclude that the third column is radius. The fourth and fifth columns are both reasonable candidates for density, since they each start at about $\sim 10^{10}$, about stellar core density for massive stars in g/cm^3. However, the fourth column does not decrease monotonically, which is unphysical for a system in equilibrium, where volume elements of high density will be pulled by gravity below volume elements of low density. Therefore we conclude that the fifth column is density.

With that established, I then plotted the density as a function of radius in the pre-supernova star in Figure 1. I then interpolated this density onto a equidistant grid with outer radius of 10^9 cm.

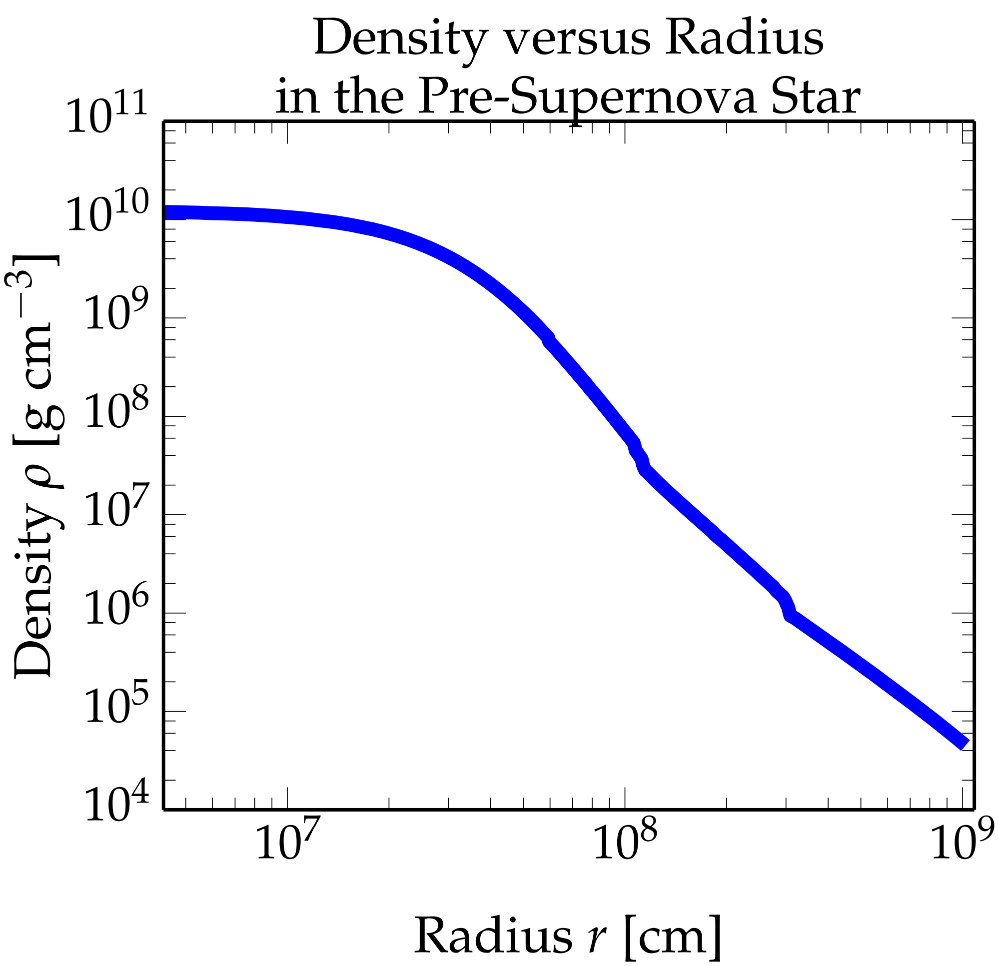

Figure 1: Density as a function of radius in the pre-supernova star model.

I then wrote an integrator for the gravitational potential $\phi(r)$ and applied it to the case of a sphere of homogeneous density. The numerical result is compared to the analytic result in Figure 2.

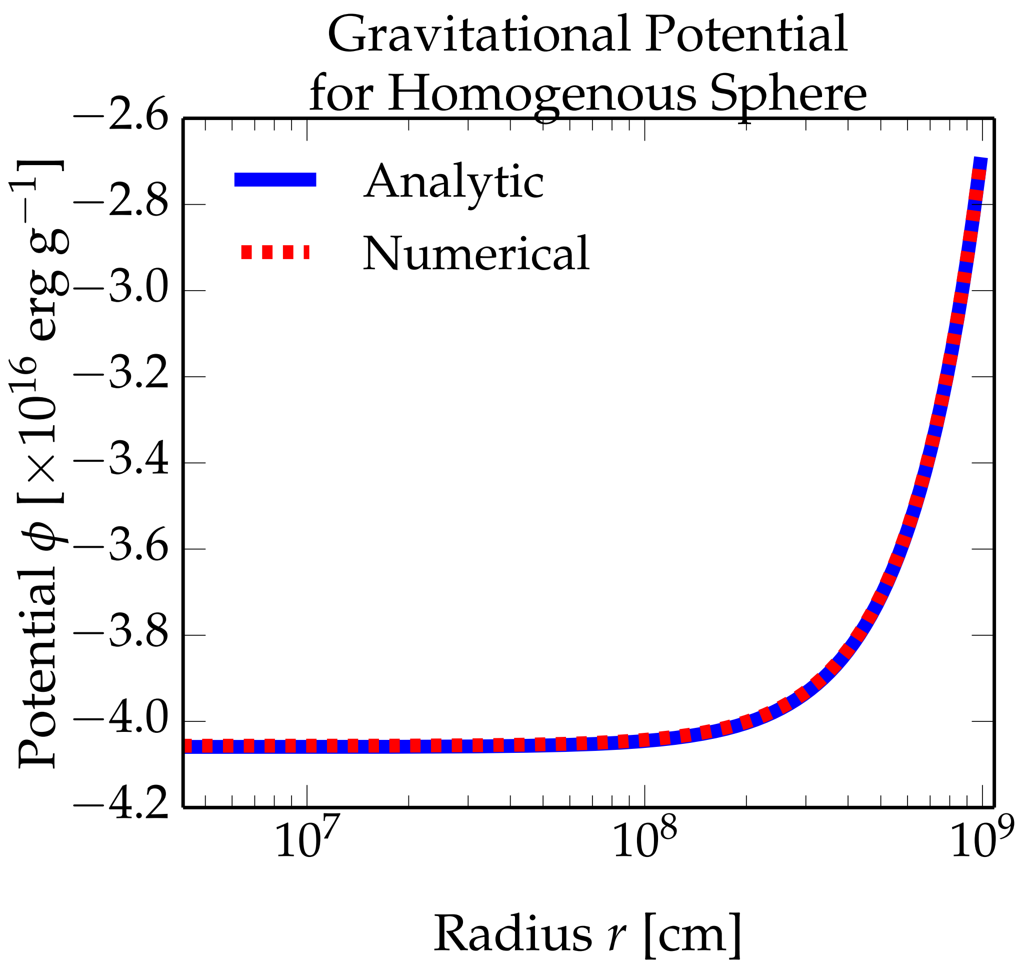

Figure 2: The gravitational potential for a homogenous sphere of uniform density, 10^5 g cm^{-3}. The analytic result of $\phi(r) = \frac{2}{3}\pi G\rho(r^2 - 3r_{\text{max}}^2)$ is plotted along with the numerical solution to the spherically-symmetric Poisson equation with 2000 grid points.

To check convergence, I calculated

$$Q_{\text{inner}} \equiv \frac{\phi_{20}(r=0) - \phi(r=0)}{\phi_{40}(r=0) - \phi(r=0)} = \left(\frac{40}{20}\right)^n,$$

where n is the convergence rate and ϕ_N is the numerical solution to ϕ with N grid points.

Since the Forward Euler Method, a first-order method, was used to integrate the Poisson equation, we expect the convergence rate of our algorithm to be 1, resulting in a Q_{inner} of 2. In fact, the measured Q_{inner} was 2.03869180587, close to the expected convergence rate.

I then applied my integrator to the pre-supernova model; the potential as a function of radius is shown in Figure 3.

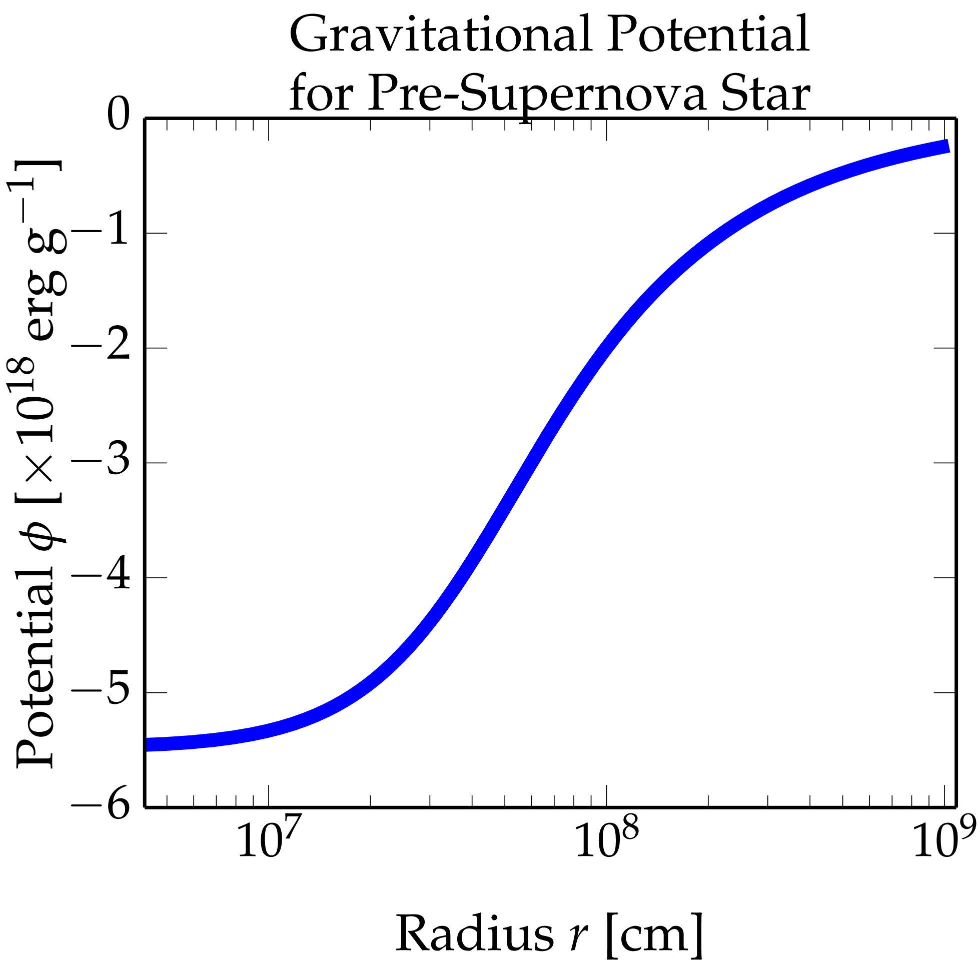

Figure 3: The gravitational potential for the pre-supernova model, solved by numerical integration of the Poisson equation with 2000 grid points.