I worked with David Vartanyan (my partner) and John Pharo on this worksheet. It took about 10^{14} transitions between the two energy levels of the cesium-133 atom (4 hours).

Root Finding: Eccentricity Anomality

(a) I implemented the bisection method to solve $0 = E - \omega t - e \sin E$ for E for different values of t. For this part, the eccentricity $e = 0.0167$. The fractional error in E from one computation to the next is set to be less than 10^{-10} before the computation terminates. Because E is an angle $\in [0, 2\pi]$, I set the initial interval for the bisection method to $\in [0, 2\pi]$. The resulting values of E can then be used to compute $x(E)$ and $y(E)$. The results are as follows:

Day	E	x [AU]	y [AU]	Iterations required
91	1.582	-0.011	1.000	55
182	3.131	-1.000	0.011	52
273	4.679	-0.033	-0.999	53

The bisection method takes many more iterations to find E for this eccentricity than Newton's method does (about 4).

(b) Now setting $e = 0.99999$ while keeping everything else the same, we achieve the following results:

Day	E	x [AU]	y [AU]	Iterations required
91	2.307	-0.671	0.003	54
182	3.136	-1.000	0.000	52
273	3.964	-0.681	-0.003	53

Since e is very close to 1, b/a is very close to 0, indicating that the orbit is an ellipse flattened in the y-direction. Therefore the planet will be close to the x-axis at all times, corresponding to $E \simeq 0$ or π. Since $E = 0$ when $t = 0$, our initial guess for E is π. If we were using Newton's method, we would choose this for our initial iteration point. However, since we are using bisection method, we can only try to decrease the interval size around this guess. Therefore, we use the initial interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$ to try to speed up the computation. The results are as follows:

Day	E	x [AU]	y [AU]	Iterations required
91	2.307	-0.671	0.003	53
182	3.136	-1.000	0.000	51
273	3.964	-0.681	-0.003	52

Indeed, this sped up our computation: by 1 iteration! This would drastically decrease the number of iterations required for Newton's method, but it doesn't for the bisection method. The reason for this is that the bisection method never stops until the relative error between the function values at the boundaries of our final interval is less than 10^{-10}. For this to be the case (except for extreme coincidence), the interval size must be extremely small. This requires halving the initial interval many times. All we have done by changing the starting interval to $[\frac{\pi}{2}, \frac{3\pi}{2}]$ was halve the interval once already, which should decrease the required number of iterations by 1, and indeed it does.