I worked with David Vartanyan (my partner) and John Pharo on this worksheet. It took about five hours total time.

1. Integration via Newton-Cotes Formulae

Part A

I computed $\int_0^\pi \sin x \, \mathrm{d}x$ via the Midpoint, Trapezoidal, and Simpson's Approximations. The analytical answer is 2. Errors and convergence orders are presented below.

Relative errors in approximations at 100 grid points:

Midpoint	$4.19595546677 \times 10^{-5}$
Trapezoidal	$-8.39180530038 \times 10^{-5}$
Simpson's	$3.52111007018 \times 10^{-10}$

Convergence order ($\log_2$ of the factor decrease in error size when resolution is doubled):

Midpoint	2.01456788742	(should be 1)
Trapezoidal	2.01455422188	(should be 2)
Simpson's	4.02909033045	(should be 4)

The Midpoint Approximation has comparable error to the Trapezoidal Approximation; in fact it has one half the error to 4 sig-figs, which is an extremely strange coincidence. It makes sense that these errors should be comparable; it was pointed out in lecture that the Midpoint Approximation "cheats" by having part of the line higher above the function curve and part below, thus canceling error. That said, I do not know why the Midpoint Approximation has a higher convergence order than it should (it should be globally first-order), despite careful inspection of my implementation. It is strange that the convergence order is so close to that of the Trapezoidal Approximation.

Part B

I computed $\int_0^\pi x \sin x \, \mathrm{d}x$ via the Midpoint, Trapezoidal, and Simpson's Approximations. The analytical answer is π. Errors and convergence orders are presented below.

Relative errors in approximations at 100 grid points:

Midpoint	$4.19595546682 \times 10^{-5}$
Trapezoidal	$-8.39180530038 \times 10^{-5}$
Simpson's	$3.5211044459 \times 10^{-10}$

Convergence order ($\log_2$ of the factor decrease in error size when resolution is doubled):

Midpoint	2.0145678874	(should be 1)
Trapezoidal	2.01455422188	(should be 2)
Simpson's	4.02905577134	(should be 4)

Again, I do not know why the Midpoint Approximation has a higher convergence order than it should. Also curious is that my relative errors and convergence orders are almost exactly the same between the two integrands in Parts A and B.

2. Gaussian Quadrature

Part A

First we calculate the coefficient to the integral in Eq. 1:

$$\frac{8\pi \left(k_B T\right)^3}{\left(2\pi\hbar c\right)^3} = 1.05495 \times 10^{35} \text{ cm}^{-3}$$

We set the weight function $W(x) = e^{-x}$ and the function $f(x) = \frac{x^2 e^x}{e^x+1}$. The resulting integrated number density is shown in Table 1 for various numbers of nodes in the Gauss-Laguerre Quadrature.

Number of nodes n	Number density [cm^{-3}]	Change in number density [cm^{-3}]
2	$1.22842960 \times 10^{31}$	-
4	$6.03420893 \times 10^{32}$	$5.91136597 \times 10^{32}$
8	$2.15112647 \times 10^{34}$	$2.09078438 \times 10^{34}$
16	$1.65635419 \times 10^{35}$	$1.44124154 \times 10^{35}$
32	$1.90216416 \times 10^{35}$	$2.45809969 \times 10^{34}$
64	$1.90216489 \times 10^{35}$	$7.34916308 \times 10^{28}$
128	$1.90216489 \times 10^{35}$	0.00000000×10^{0}

Table 1: Approximations to the number density integral of Eq. 1 using the Gauss-Laguerre Quadrature for various numbers of nodes n. Also shown is the change in this approximation as n is varied, demonstrating convergence as this change vanishes for large n.

Part B

Now we must transform the integral $\int_a^b x^2(e^x+1)^{-1}\,\mathrm{d}x$ such that it is over the interval [-1,1]. To do this, let $y = \frac{2}{b-a}(x-a) - 1$. Then

$$x(y) = (y+a)\frac{b-a}{2} + a$$

$$\int_a^b \frac{x^2}{e^x+1}\,\mathrm{d}x = \frac{b-a}{2}\int_{-1}^{1} \frac{x(y)^2}{e^{x(y)}+1}\,\mathrm{d}y$$

This can then be computed with Gauss-Legendre Quadrature. The resulting number densities in each energy bin is depicted in Fig . The sum of the number densities in each bin is $1.85943561534 \times 10^{35}$ cm^{-3}, which is slightly less than the converged answer from Part A of $1.90216489 \times 10^{35}$ cm^{-3}. However, this answer for Part B is invariant to decreasing energy bin width, increasing energy of maximum energy bin, and increasing the number of nodes for each bin's integration. Therefore there is an unknown systematic reason for this 2% discrepancy.

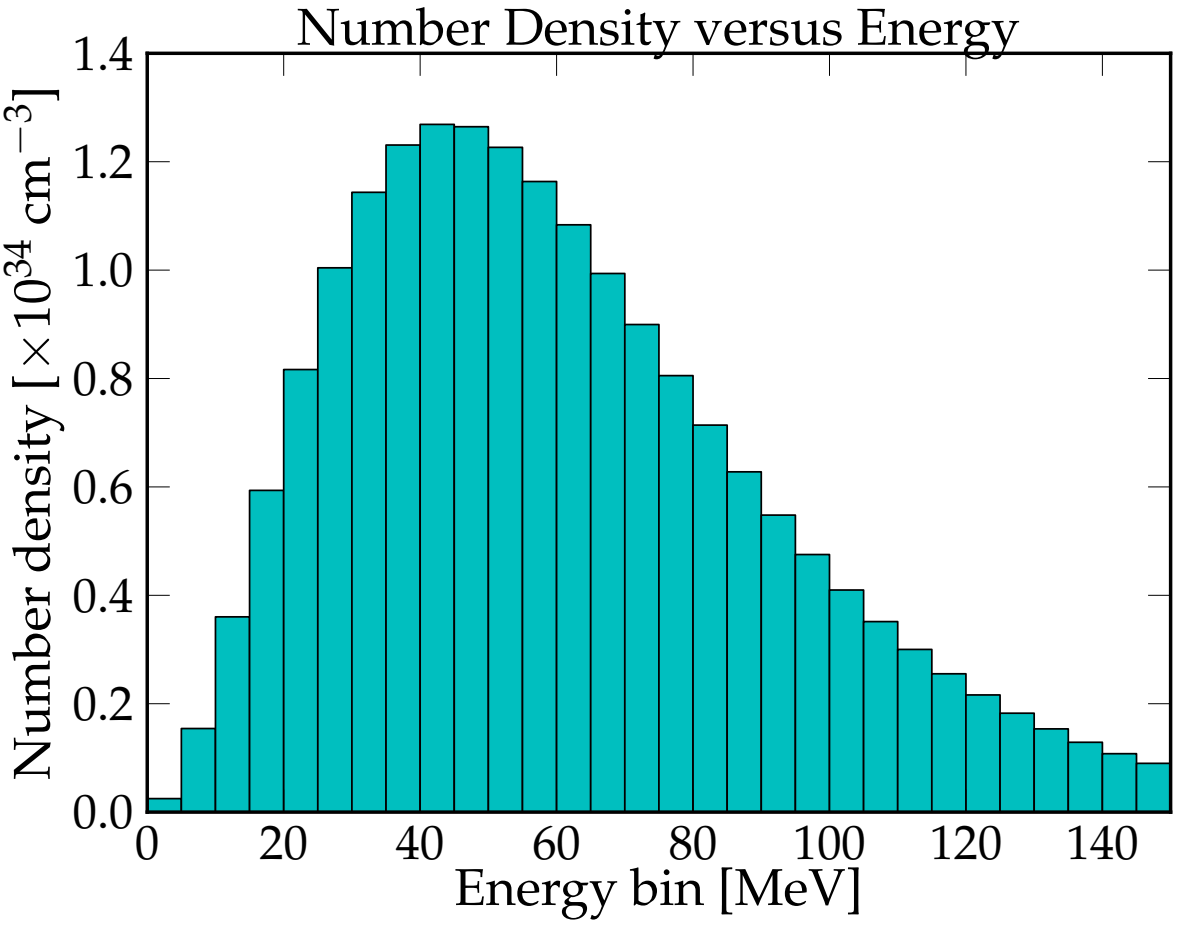

Figure 1: A plot of the ratios of the differencing errors between $h_1 = 1$ and $h_2 = 2$, for both forward and central differencing. We expect these ratios to be equal to the convergence factor $\left(\frac{h_2}{h_1}\right)^n = 2^n$, where n is the convergence order of the finite differencing: $n = 1$ for forward differencing and $n = 2$ for central differencing. Indeed, we see that the ratio is approximately 2 for forward differencing and 4 for central differencing as expected.