Disproof of the Existence of Odd Perfect Numbers

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July 2, 2024

Abstract

This paper explores the properties of odd perfect numbers, focusing on their required forms and demonstrating their non-existence. Euler established a specific form that odd perfect numbers must adhere to. Building upon Euler's work, I investigate alternative forms and essential properties that all odd perfect numbers must possess. Finally, I prove that numbers conforming to Euler's form cannot exhibit the identified properties.

1 Introduction

Perfect numbers have intrigued mathematicians for centuries. A perfect number is defined as a positive integer that is equal to the sum of its proper divisors. Formally, a number n is perfect if

$$\sigma(n) = 2n$$

where $\sigma(n)$ represents the sum of the divisors of n.

While even perfect numbers have been well studied and classified, the existence of odd perfect numbers remains an open question. In this paper, we investigate specific properties of odd perfect numbers, demonstrating that they cannot exist.

2 Preliminaries

Definition 1 (Odd Perfect Number)**.** An odd perfect number is an odd positive integer n that satisfies $\sigma(n) = 2n$.

Lemma 1. *Euler proved that if n is an odd perfect number, then n must be of the form $n = p^k m^2$, where p is prime, gcd(p, m) = 1, and $p \equiv k \equiv 1 \pmod 4$ [1].*

Lemma 2. *If $X^a + 1$ does not have less factors of 2 than $X^{a+1} - 1$ (where X is an odd prime), a must not be odd.*

Proof. Let's assume for the sake of contradiction that a is odd. Then, $X^a + 1$ can be factored as $(X + 1)(X^{a-1} - X^{a-2} + X^{a-3} - ... - X + 1)$. Since $(X^{a-1} - X^{a-2} + X^{a-3} - ... - X + 1)$ is odd (this is because there are $\frac{a+1}{2}$ positive terms and $\frac{a-1}{2}$ negative terms) and $X + 1$ is even, the number of factors of 2 in $X^a + 1$ is equal to the number of factors of 2 in $X + 1$. Now, $X^{a+1} - 1$ can be factored as $(X + 1)(X^a - X^{a-1} + X^{a-2} - ... - X + 1)$. The number of factors of 2 in $X^{a+1} - 1$ is equal to the sum of the number of factors of 2 in $X+1$ and the number of factors of 2 in $(X^a - X^{a-1} + X^{a-2} - ... - X + 1)$. Because $(X^a - X^{a-1} + X^{a-2} - ... + X - 1)$ is even (there are an equal number of positive and negative terms) and thus will have at least 1 factor of 2, the number of factors of 2 in $X^{a+1} - 1$ will be greater than the number of factors of 2 in $X + 1$. Therefore, a cannot be odd for this property to hold. $\square$

3 Main Result

Theorem 1. *A perfect number cannot be odd.*

Proof. An odd perfect number divisible by the prime X can be expressed as $X^a K$, where X is an odd prime and K is the product of all other prime factors of the odd perfect number. This implies that K is not divisible by X as it includes all *other* numbers in the prime factorization of $X^a K$. After simplifying the sum of the proper divisors of $X^a K$, we notice that the coefficient of all variables except K is the sum of the first $a + 1$ powers of X. The coefficient of K in the simplified expression is the sum of only the first a powers of X. This is because $X^a K$ is not a proper divisor of itself.

For example, the sum of proper divisors of $X^2 ab$ would be $1 + X + X^2 + a + b + ab + Xa + X^2 a + Xb + X^2 b + Xab = (X^2 + X + 1)1 + (X^2 + X + 1)a +$

$(X^2 + X + 1)b + (X + 1)ab$. As we can see, coefficients of all variables are the sum of the first $a + 1 = 3$ powers of X, while the coefficient of $K = ab$ is the sum of only the first $a = 2$ powers of X.

Therefore, if $X^a K$ is an odd perfect number, the following must hold true, where M represents the sum of variables that have $(X^0 + X^1 + X^2 + \cdots + X^a)$ as their coefficient:

$$X^a K = (X^0 + X^1 + \cdots + X^{a-1})K + (X^0 + X^1 + \cdots + X^a)M$$

$$\implies [X^a - (X^0 + X^1 + \cdots + X^{a-1})]K = (X^0 + X^1 + \cdots + X^a)M$$

$$= \left(X^a - \frac{X^a - 1}{2}\right)K = \frac{(X^{a+1} - 1)M}{2}$$

$$= (X^a + 1)K = (X^{a+1} - 1)M$$

$$\implies M = \frac{(X^a + 1)K}{X^{a+1} - 1}$$

Note that a must be even, because $X^a + 1$ must not have less factors of 2 than $X^{a+1} - 1$ for K to be odd. This only holds true when a is an even number. Notice that M is the sum of proper divisors of K because it includes all variables except K in its sum. Euler proved that an odd perfect number must have the form $p^y s^2$, such that p is prime, $p \equiv y \equiv 1 \pmod 4$, and $\gcd(p,\ s) = 1$ [1]. For the hypothetical odd perfect number $X^a K$, X^a must represent s^2, because a is even and $y \equiv 1 \pmod 4$, so X^a cannot represent p^y. This means that if $X^a K$ is an odd perfect number of the form $p^y s^2$, $s^2 = X^a$ and $p^y = K$. The sum of proper divisors of K (p^y), is $\frac{p^y - 1}{p - 1}$. Recall that this can also be represented as M which would be equal to $\frac{(s^2+1)p^y}{Xs^2-1}$ after substituting $X^a = s^2$ and $K = p^y$. This means the following must hold true:

$$\frac{(s^2 + 1)p^y}{Xs^2 - 1} = \frac{p^y - 1}{p - 1}$$

$$\implies (p - 1)(s^2 + 1)(p^y) = (Xs^2 - 1)(p^y - 1)$$

$$= [(p - 1)s^2 + p - 1]p^y = (Xs^2 - 1)(p^y - 1)$$

$$= p^y(p - 1)s^2 + p^y(p - 1) = (Xs^2 - 1)p^y - (Xs^2 - 1)$$

$$\implies p^y(p - 1)s^2 + p^y(p - 1) - (Xs^2 - 1)p^y + (Xs^2 - 1) = 0$$

$$= p^y(p - 1)s^2 + p^y(p - 1) - Xs^2p^y + p^y + Xs^2 - 1 = 0$$

$$= p^y(p-1)(s^2+1) - Xs^2p^y + p^y + Xs^2 - 1 = 0$$
$$= p^y(p-1)(s^2+1) + p^y(1 - Xs^2) + Xs^2 - 1 = 0$$
$$= p^y[(p-1)(s^2+1) + 1 - Xs^2] + Xs^2 - 1 = 0$$
$$= p^y(ps^2 + p - s^2 - 1 + 1 - Xs^2) + Xs^2 - 1 = 0$$
$$= p^y[ps^2 + p - (X+1)s^2] + Xs^2 - 1 = 0$$

If we show that an odd perfect number is not divisible by any number less than X, we know that $p > X$, since p cannot be equal to X and and if $p < x$, then the odd perfect number X^aK would be divisible by a number less than X. For $X = 3$, we know that the minimum value of p is 5. In this case, $p^y[ps^2 + p - (X+1)s^2]$ is positive, meaning $Xs^2 - 1$ must be negative (for the expression to equal 0). However, $Xs^2 - 1$ cannot be negative as $Xs^2 = X^{a+1}$, which cannot be less than 1 for any positive value of a. Thus, if X^aK is an odd perfect number, then $X \neq 3$. Then, if $X = 5$, p must be greater than 5 because all odd numbers that are divisible by primes less than 5 are not perfect, and $p \neq X$. Now, because X and p are both odd primes, p must be at least 2 greater than X. This means $p^y[ps^2 + p - (X+1)s^2]$ and $Xs^2 - 1$ are both positive again, implying that their sum cannot equal 0. Similarly, for all $X > 5$, this continues. Clearly, odd perfect numbers cannot exist because no value of X can satisfy the equation. $\square$

4 Conclusion

The search for odd perfect numbers has been an open problem for over two millennia. Now, this open problem has been solved. This could lead to novel theoretical insights in the field of number theory and we can increase our focus on studying the properties of even perfect numbers.

References

[1] D. Burton, *Elementary Number Theory*, 7th ed., McGraw-Hill, 2011.