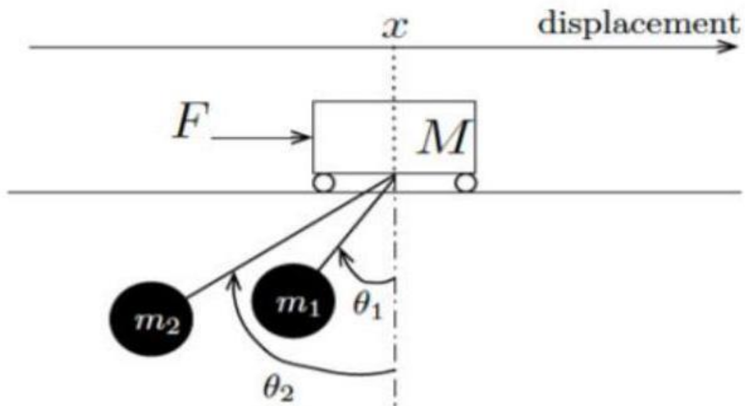
ENPM 667 Final Project

By: Paras Savnani and James Shingler

System of Interest:



Part A:

In the first part of the project, we obtain the equations of motion for the system and corresponding nonlinear state-space representation.

Position vectors for the two pendulums will be:

$$r_1(\mathrm{t}) = (x - l_1 \sin\ \theta_1)\mathrm{i} - l_1 \cos\ \theta_1 j$$
$$r_2(\mathrm{t}) = (x - l_2 \sin\ \theta_2)\mathrm{i} - l_2 \cos\ \theta_2 j \quad (1)\ (2)$$

Differentiating both equations 1 and 2, we get:

$$\dot{r_1}(t) = \left(\dot{x} - l_1 \dot{\theta_1} \cos\ \theta_1\right) i + l_1 \dot{\theta}_1 \sin\ \theta_1 j$$
$$\dot{r_2}(t) = \left(\dot{x} - l_2 \dot{\theta_2} \cos\ \theta_2\right) i + l_2 \dot{\theta}_2 \sin\ \theta_2 j \quad (3)\ (4)$$

We know that kinetic energy of a system is given by:

$$\mathrm{K.E} = \tfrac{1}{2} m v^2 \quad (5)$$

where v can be the net velocity of the system.

Therefore, using the above equations,

$$\mathrm{K.E} = \frac{1}{2}\mathrm{M}\dot{x}^2 + \frac{1}{2} m_1 \left(\dot{x} - l_1 \dot{\theta}_1 \cos\ \theta_1\right)^2 + \frac{1}{2} m_1 \left(\dot{x} - l_2 \dot{\theta}_2 \cos\ \theta_2\right)^2 + \frac{1}{2} m_1 \left(l_1 \dot{\theta}_1 \sin\ \theta_1\right)^2 + \frac{1}{2} m_1 \left(l_2 \dot{\theta}_2 \sin\ \theta_2\right)^2$$
$$\mathrm{P.E} = -m_1 g l_1 \cos\ \theta_1 - m_2 g l_2 \cos\ \theta_2$$

(6)(7)

These terms can be substituted in the Lagrange Equation:

$$\mathrm{L} = \mathrm{K.E} - \mathrm{P.E}$$

$$\mathrm{L} = \tfrac{1}{2}\mathrm{M}\dot{x}^2 + \frac{1}{2}m_1\left(\dot{x} - l_1\dot{\theta}_1\cos\ \theta_1\right)^2 + \frac{1}{2}m_1\left(\dot{x} - l_2\dot{\theta}_2\cos\ \theta_2\right)^2 + \frac{1}{2}m_1\left(l_1\dot{\theta}_1\sin\ \theta_1\right)^2 + \frac{1}{2}m_1\left(l_2\dot{\theta}_2\sin\ \theta_2\right)^2 + m_1gl_1\cos\ \theta_1 + m_2gl_2\cos\ \theta_2 \tag{9}$$

Now,

$$\frac{\partial L}{\partial \dot{x}} = \mathrm{M}\dot{x} + m_1\left(\dot{x} - l_1\dot{\theta}_1\cos\ \theta_1\right) + m_2\left(\dot{x} - l_2\dot{\theta}_2\cos\ \theta_2\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \mathrm{M}\ddot{x} + m_1\left(\ddot{x} - l_1\ddot{\theta}\cos\ \theta_1 + l_1\dot{\theta}_1^2\sin\ \theta_1\right) + m_2\left(\ddot{x} - l_2\ddot{\theta}\cos\ \theta_2 + l_2\dot{\theta}_2^2\sin\ \theta_2\right)$$

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \mathrm{M}\ddot{x} + m_1\left(\ddot{x} - l_1\ddot{\theta}\cos\ \theta_1 + l_1\dot{\theta}_1{}^2\sin\ \theta_1\right) + m_2\left(\ddot{x} - l_2\ddot{\theta}\cos\ \theta_2 + l_2\dot{\theta}_2^2\sin\ \theta_2 = \mathrm{F}\right. \tag{10}$$

For θ_1:

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1\left(\dot{x} - l_1\dot{\theta}_1\cos\ \theta_1\right)(-l_1\cos\ \theta_1) + m_1\left(l_1\dot{\theta}_1\sin\ \theta_1\right)(l_1\sin\ \theta_1)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = -m_1\ddot{x}l_1\cos\ \theta_1 + m_1l_1^2\ddot{\theta}_1 + m_1\dot{x}l_1\ddot{\theta}_1\sin\ \theta_1$$

$$\frac{\partial L}{\partial \theta_1} = m_1l_1^2\dot{\theta}_1 - m_1\dot{x}l_1\cos\ \theta_1$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = -m_1\ddot{x}l_1\cos\theta_1 + m_1l_1^2\ddot{\theta}_1 + m_1\dot{x}l_1\ddot{\theta}_1\sin\theta_1 - m_1l_1^2\dot{\theta}_1 + m_1\dot{x}l_1\cos\theta_1 = 0 \tag{11}$$

For θ_2:

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2\left(x - l_2\dot{\theta}_2\cos\ \theta_2\right)(-l_2\cos\ \theta_2) + m_2\left(l_2\dot{\theta}_2\sin\ \theta_2\right)(l_2\sin\ \theta_2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = -m_2\ddot{x}l_2\cos\ \theta_2 + m_2l_2^2\ddot{\theta}_2 + m_2\dot{x}l_2\ddot{\theta}_2\sin\ \theta_2$$

$$\frac{\partial L}{\partial \theta_2} = m_2l_2^2\dot{\theta}_2 - m_2\dot{x}l_2\cos\ \theta_2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = -m_2\ddot{x}l_2\cos\ \theta_2 + m_2l_2^2\ddot{\theta}_2 + m_2\dot{x}l_2\ddot{\theta}_2\sin\ \theta_2 - m_2l_2^2\dot{\theta}_2 + m_2\dot{x}l_2\cos\ \theta_2 = 0 \tag{12}$$

From above equations,

$$l_1\ddot{\theta}_1 = \ddot{x}\cos\ \theta_1 - g\sin\ \theta_1 \Rightarrow \ddot{\theta}_1 = \frac{\ddot{x}\cos\ \theta_1 - g\sin\ \theta_1}{l_1}$$
$$l_2\ddot{\theta}_2 = \ddot{x}\cos\ \theta_2 - g\sin\ \theta_2 \Rightarrow \ddot{\theta}_2 = \frac{\ddot{x}\cos\ \theta_2 - g\sin\ \theta_2}{l_2}$$

$$(\mathrm{M} + m_2 + m_1)\ddot{x} = m_1(\ddot{x}\cos\ \theta_1 - \mathrm{g}\sin\ \theta_1)\cos\ \theta_1 + m_2(\ddot{x}\cos\ \theta_2 - g\sin\ \theta_2)\cos\ \theta_2 - m_1 l_1\dot{\theta}_1^2\sin\theta_1 - m_2 l_2\dot{\theta}_2^2\sin\theta_2 + \mathrm{F}$$

Therefore, the equation becomes,

$$\ddot{x} = \frac{\mathrm{F} - m_1 g\cos\ \theta_1\sin\ \theta_1 - m_2 g\cos\ \theta_2\sin\ \theta_2 - m_1 l_1\dot{\theta}_1^2\sin\ \theta_1 - m_2 l_2\dot{\theta}_2^2\sin\ \theta_2}{M + m_2\sin\ \theta_2^2 + m_1\sin\ \theta_1^2} \quad (13)$$

With the following state space representation:

$$\mathrm{X} = \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \mathrm{x}_3 \\ \mathrm{x}_4 \\ \mathrm{x}_5 \\ \mathrm{x}_6 \end{bmatrix} = \begin{bmatrix} \mathrm{x} \\ \dot{\mathrm{x}} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} \quad (14)$$

$$\dot{\mathrm{X}} = \begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} \quad (15)$$

Part B:

For the second part of the project, we linearize the system in part A around the equilibrium point $\mathrm{x} = 0,\ \theta_1 = 0,\ \theta_2 = 0$ and obtain the state-space representation of the linearized system.

$$\begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \\ \dot{\theta}_1(t) \\ \ddot{\theta}_1(t) \\ \dot{\theta}_2(t) \\ \ddot{\theta}_2(t) \end{bmatrix} = \begin{bmatrix} \dot{x}(t) \\ \frac{\{\mathrm{F} - m_1 g \cos\,\theta_1 \sin\,\theta_1 - m_2 g \cos\,\theta_2 \sin\,\theta_2 - m_1 l_1 \dot{\theta}_1^2 \sin\,\theta_1 - m_2 l_2 \theta_2^2 \sin\,\theta_2\}}{(M + m_2 \sin\,\theta_2^2 + m_1 \sin\,\theta_1^2)} \\ \dot{\theta}_1(t) \\ \frac{\ddot{x}\cos\,\theta_1 - \mathrm{g}\sin\,\theta_1}{l_1} \\ \dot{\theta}_2(t) \\ \frac{\ddot{x}\cos\,\theta_2 - \mathrm{g}\sin\,\theta_2}{l_2} \end{bmatrix} \tag{16}$$

For small angle at equilibrium point, $\mathrm{x} = 0,\ \theta_1 = 0,\ \theta_2 = 0$

Assume, $\cos\,\theta \approx 1$ and $\sin\,\theta \approx \theta$ and ignoring second order terms.

$$\begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \\ \dot{\theta}_1(t) \\ \ddot{\theta}_1(t) \\ \dot{\theta}_2(t) \\ \ddot{\theta}_2(t) \end{bmatrix} = \begin{bmatrix} \dot{x}(t) \\ \frac{F}{M} - \frac{m_1 g \theta_1}{M} - \frac{m_2 g \theta_2}{M} \\ \dot{\theta}_1(t) \\ \frac{F}{M l_1} - \frac{m_1 g \theta_1}{M l_1} - \frac{m_2 g \theta_2}{M l_1} - \frac{g \theta_1}{l_1} \\ \dot{\theta}_2(t) \\ \frac{F}{M l_2} - \frac{m_1 g \theta_1}{M l_2} - \frac{m_2 g \theta_2}{M l_2} - \frac{g \theta_2}{l_2} \end{bmatrix}$$

The state space can be decomposed as:

$$\begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-m_1 g}{M} & 0 & \frac{-m_2 g}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-m_1 g}{M l_1} - \frac{g}{l_1} & 0 & \frac{-m_2 g}{M l_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-m_1 g}{M l_2} & 0 & \frac{-m_2 g}{M l_2} - \frac{g}{l_2} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \\ 0 \\ 1/M l_1 \\ 0 \\ 1/M l_2 \end{bmatrix} \mathrm{F}$$

Therefore,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-m_1 g}{M} & 0 & \frac{-m_2 g}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-m_1 g}{M l_1} - \frac{g}{l_1} & 0 & \frac{-m_2 g}{M l_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-m_1 g}{M l_2} & 0 & \frac{-m_2 g}{M l_2} - \frac{g}{l_2} & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1/M \\ 0 \\ 1/M l_1 \\ 0 \\ 1/M l_2 \end{bmatrix}$$

NOTE: We obtain the A and B matrices by making small angle approximation. But the state space representation and these matrices can also be obtained by evaluating Jacobian matrices at equilibrium points, where $J = \left[\frac{\partial f_i}{\partial x}\right]$.

Part C:

The given linearized system will be controllable if the controllability matrix **C** is of full rank or in other words its determinant is non-zero.

$$\text{i.e.}\quad \boldsymbol{rank}([\boldsymbol{B}, \boldsymbol{AB}, \boldsymbol{A^2B}, \boldsymbol{A^3B}, \boldsymbol{A^4B}, \boldsymbol{A^5B}\,]) = \mathbf{6} \tag{17}$$

Substituting the values of A and B in above controllability matrix and equating its determinant to find the constraints on the values of M, m_1, m_2, l_1, l_2, g we get:

```
Command Window
>>  syms m1 m2 l1 l2 g M
>> A = [0 1 0 0 0 0;
0 0 (-m1*g)/M 0 (-m2*g)/M 0;
0 0 0 1 0 0;
0 0 ((-m1*g)/(M*l1))-(g/l1) 0 (-m2*g)/(M*l1) 0;
0 0 0 0 0 1;
0 0 (-m1*g)/(M*l2) 0 ((-m2*g)/(M*l2))-(g/l2) 0;]

A =

[0, 1,                      0, 0,                      0, 0]
[0, 0,             -(g*m1)/M, 0,             -(g*m2)/M, 0]
[0, 0,                      0, 1,                      0, 0]
[0, 0, - g/l1 - (g*m1)/(M*l1), 0,         -(g*m2)/(M*l1), 0]
[0, 0,                      0, 0,                      0, 1]
[0, 0,        -(g*m1)/(M*l2), 0, - g/l2 - (g*m2)/(M*l2), 0]

>> B = [0; 1/M; 0; 1/(M*l1); 0; 1/(M*l2)]

B =

       0
     1/M
       0
1/(M*l1)
       0
1/(M*l2)
```

Figure 1. A and B matrix initialization

```
>> cont = horzcat(B, A*B, (A^2)*B, (A^3)*B, (A^4)*B, (A^5)*B)

cont =

[        0,      1/M,                                                0,                      - (g*m1)/(M^
[      1/M,        0,                      - (g*m1)/(M^2*l1) - (g*m2)/(M^2*l2),
[        0, 1/(M*l1),                                                0, - (g/l1 + (g*m1)/(M*l1))/(M*l
[1/(M*l1),         0, - (g/l1 + (g*m1)/(M*l1))/(M*l1) - (g*m2)/(M^2*l1*l2),
[        0, 1/(M*l2),                                                0, - (g/l2 + (g*m2)/(M*l2))/(M*l
[1/(M*l2),         0, - (g/l2 + (g*m2)/(M*l2))/(M*l2) - (g*m1)/(M^2*l1*l2),

>> rank(cont)

ans =

     6

>> det(cont,'Algorithm','minor-expansion')

ans =

-(g^6*l1^2 - 2*g^6*l1*l2 + g^6*l2^2)/(M^6*l1^6*l2^6)

fx >>
```

Figure 2. Check Controllability

$$\boldsymbol{CTRB(A,B)} = \left[\mathbf{B}, \mathbf{AB}, \mathbf{A^2B}, \mathbf{A^3B}, \mathbf{A^4B}, \mathbf{A^5B}\right] = -\frac{g^6(l_1^2 - 2l_1l_2 + l_2^2)}{(Ml_1l_2)^6} = -\frac{g^6(l_1 - l_2)^2}{(Ml_1l_2)^6}$$

Therefore, using the above equation, we can say that the determinant will be zero if $l_1 = l_2$. Thus, for the system to be controllable:

$$\boldsymbol{l_1 \neq l_2}$$

Part D:

In this section, we check that the system is controllable and simulate the resulting response with an LQR controller for the given values:

$$M = 1000\ \mathrm{kg}, m_1 = 100\ \mathrm{kg}, m_2 = 100\ \mathrm{kg}, l_1 = 20\ \mathrm{m}, l_2 = 10\ \mathrm{m}, \mathrm{g} = 10\ \mathrm{ms^{-2}}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -0.55 & 0 & -0.05 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -0.1 & 0 & -1.1 & 0 \end{bmatrix}$$

$$B = 0.001 * \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0.05 \\ 0 \\ 0.1 \end{bmatrix}$$

Since $\boldsymbol{l_1 \neq l_2}$ ($20 \neq 30$), we that the system is controllable based on the result in part C. We then check the eigen vales of A matrix to see if the system is stable or not. The eigen values of A will be:

$$\lambda = \begin{bmatrix} 0.0000 + 1.0531\mathrm{i} \\ 0.0000 - 1.0531\mathrm{i} \\ 0.0000 + 0.7356\mathrm{i} \\ 0.0000 - 0.7356\mathrm{i} \\ 0.0000 + 0.0000\mathrm{i} \\ 0.0000 + 0.0000\mathrm{i} \end{bmatrix}$$

We notice that the real part of all eigen vales is zero, thus the system will keep on oscillating without damping. Also, we concluded that the system is controllable so we will design an optimal LQR controller to place the eigen vales to the left half plane to make the system stable.

The LQR controller gives a feedback gain matrix which stabilizes the system. It also allows us to choose the Q and R parameters which are used as a tradeoff between speed of response of controller and control effort. LQR minimizes the following cost function:

$$J = \int_0^\infty (x(\tau)^T \boldsymbol{Q} x(\tau) + u(\tau)^T \boldsymbol{R} u(\tau)) d\tau \tag{19}$$

The "optimal" simulation parameters for the LQR controller of the linearized system at equilibrium point are provided below:

Table 1. LQR Parameters (Part D)

Sr No:	Parameter	Value
1	Initial State y_o	$[0, 0.5, 0, 0.2, 0, 0.2]^T$
2	Q	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10000 \end{bmatrix}$
3	R	0.0001
4	Reference Input	$[0, 0, 0, 0, 0, 0]^T$
5	Time Span	100 seconds

Below are the results of the simulation:

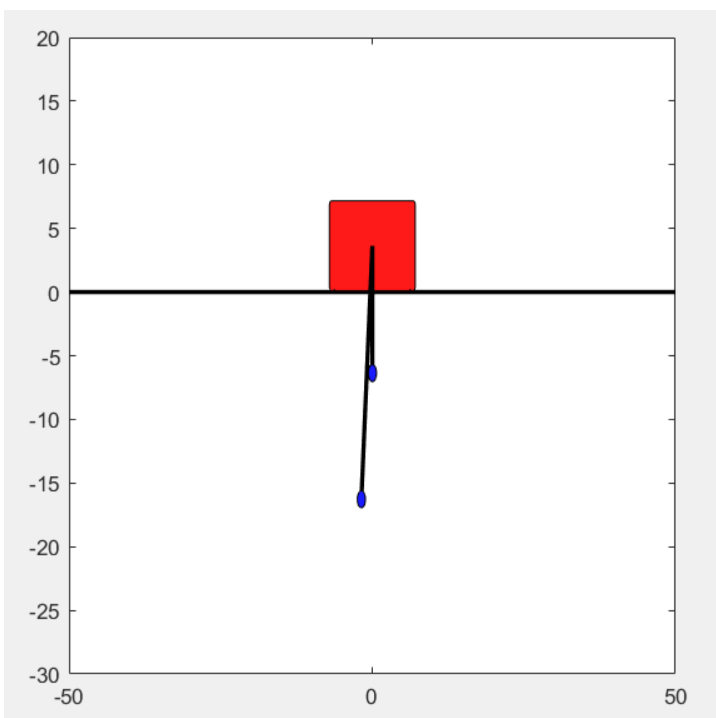


Figure 3. Simulation Illustration

Figure 4. Full State Response (Linear System)

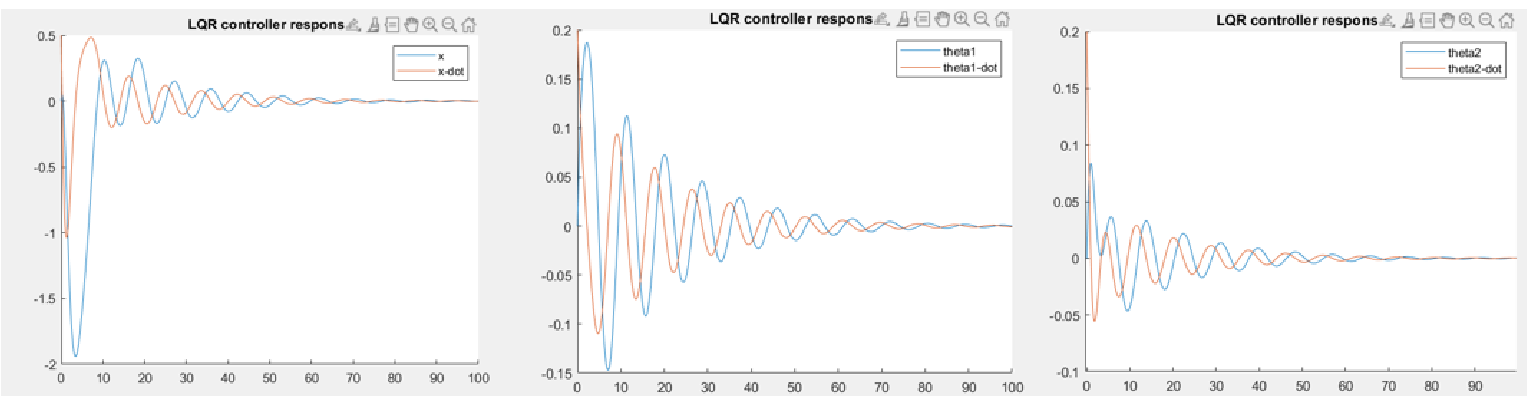


Figure 5. Resulting Response broken down by variable (Linear System)

By iteratively testing Q and R, we found the values in table 1 to be “optimal” for providing a good balance between speed of response and control effort. The closed loop Gain K Matrix derived from the LQR controller is:

$$\boldsymbol{K} = 1.0\text{e} + 03 * [\,0.1000 \quad 0.6101 \quad -0.4944 \quad 4.7422 \quad 6.6067 \quad 7.2896]$$

Now, to check the stability of the system we will use Lyapunov's indirect method to check the eigen vales of the new system dynamics.

$$\dot{X} = AX + BU \qquad \text{where, } U = -KX$$

$$\dot{X} = AX - BKX$$

$$\dot{X} = (A - B * K)X \qquad \text{where, } A_C = A - B * K \tag{20}$$

Therefore, eigen values of this new A_C matrix will be:

$$\lambda = \begin{bmatrix} -0.5383 + 0.9521i \\ -0.5383 - 0.9521i \\ -0.1967 + 0.1999i \\ -0.1967 - 0.1999i \\ -0.0531 + 0.7270i \\ -0.0531 - 0.7270i \end{bmatrix}$$

All the eigen values have negative real part i.e., they lie in the left half plane of phase-space plot, so this concludes that now our system has become stable. Also, the closed loop controller can stabilize the undamped system using state feedback. Here, the system is **locally stable** because we confirmed the stability at the equilibrium point after linearizing the system.

Now we test LQR Response on the original Nonlinear System:

We use ODE45 solver/Simulink to simulate the non-linear system. ODE45 will work with LQR controller because it takes input and the initial state as the parameters. However, for estimation we will have to use Simulink only.

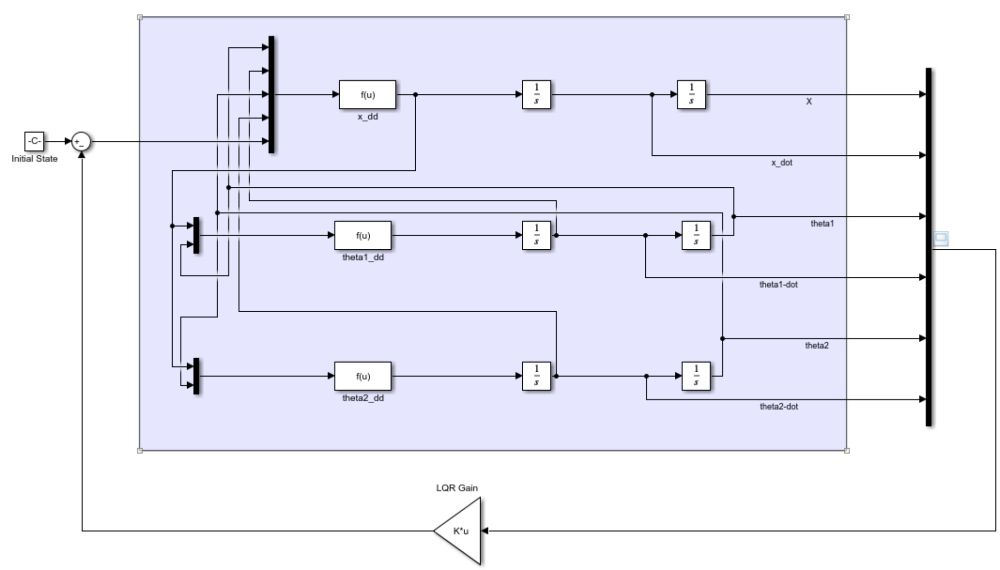


Figure 6. Simulink Model

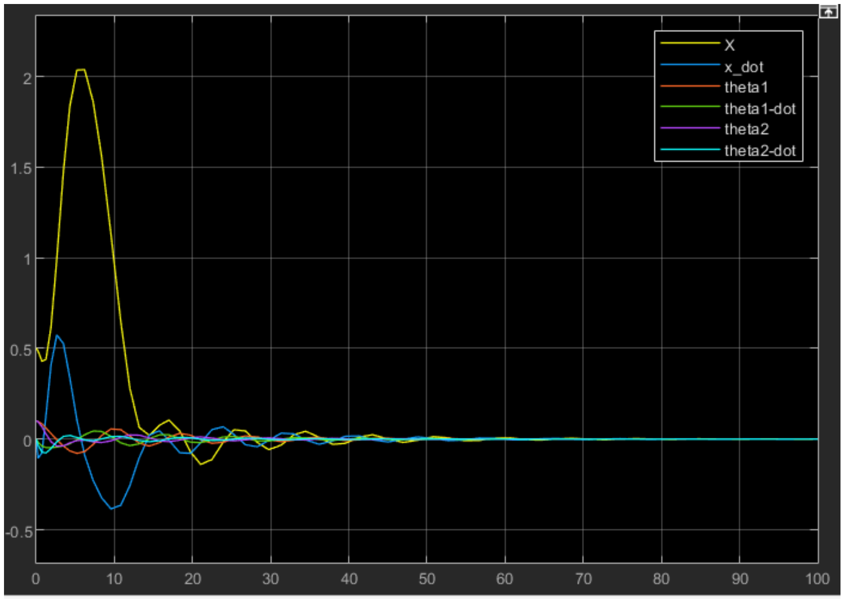


Figure 7. Full State Response (Non-Linear System)

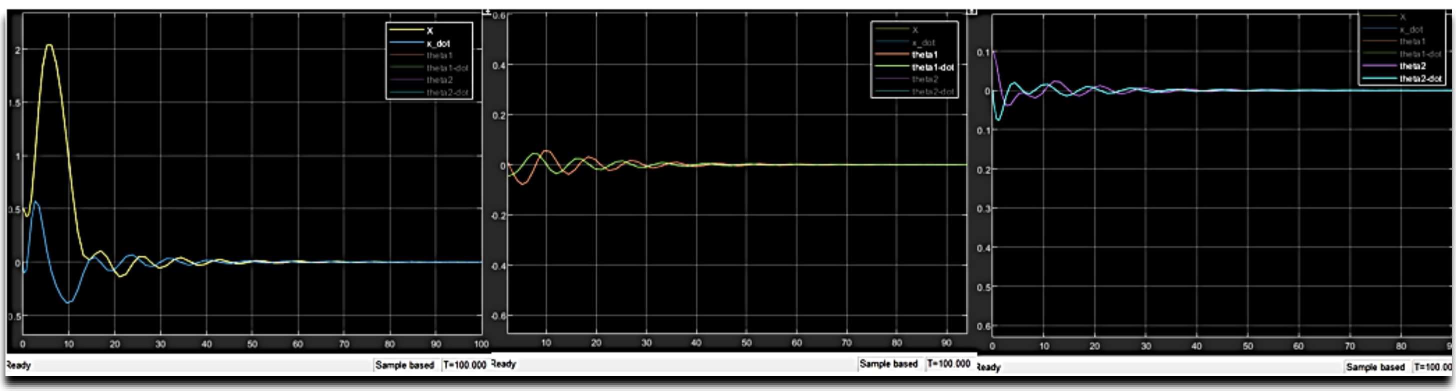


Figure 8. Resulting Response broken down by variable (Non-Linear System)

Part E:

The given linearized system will be observable if the observability matrix **O** is of full rank or in other words its determinant is non-zero.

i.e. $$\boldsymbol{rank}([\boldsymbol{C}, \boldsymbol{AC}, \boldsymbol{A^2C}, \boldsymbol{A^3C}, \boldsymbol{A^4C}, \boldsymbol{A^5C}]^{\boldsymbol{T}}) = \mathbf{6}$$

Therefore, we tested the different output vectors and check if the system is observable for them or not:

For Scenario 1:

$$\text{Output} = x(t)$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We used MATLAB to check the rank of the observability matrix. The rank of $OBSV(A, C)$ is 6 in this case. Hence, the system is observable if information about the displacement of the cart is given.

Scenario 2:

$$\text{Output} = \theta_1(t), \theta_2(t)$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The rank of $\mathrm{OBSV}\,(A, C)$ is 4 in this case. Hence, the system is NOT observable with this limited information i.e., angles of both pendulums.

Scenario 3:

$$\text{Output} = x(t), \theta_2(t)$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The rank of $\mathrm{OBSV}\,(A, C)$ is 6 in this case. Hence, the system is observable with this limited information i.e., displacement of cart and angle of second pendulum.

Scenario 4:

$$\text{Output} = x(t), \theta_1(t), \theta_2(t)$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The rank of $\mathrm{OBSV}\,(A, C)$ is 6 in this case. Hence, the system is observable with this limited information, i.e., displacement of cart and angles of both pendulums.

Part F:

In most practical scenarios we cannot observe the full output state. So, we try to estimate the full state for the closed loop feedback control. We can do this by designing the best Luenberger observer for each observable output vector from the previous question. This observer aims to make the estimated state converge to the actual state and thus reduce the error. We can check the stability of the observer by checking the eigen values of **A-L*C** matrix, and if they lie in the left half plane, the system is stable.

Scenario 1:

$$\text{Output} = x(t)$$

$$C = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

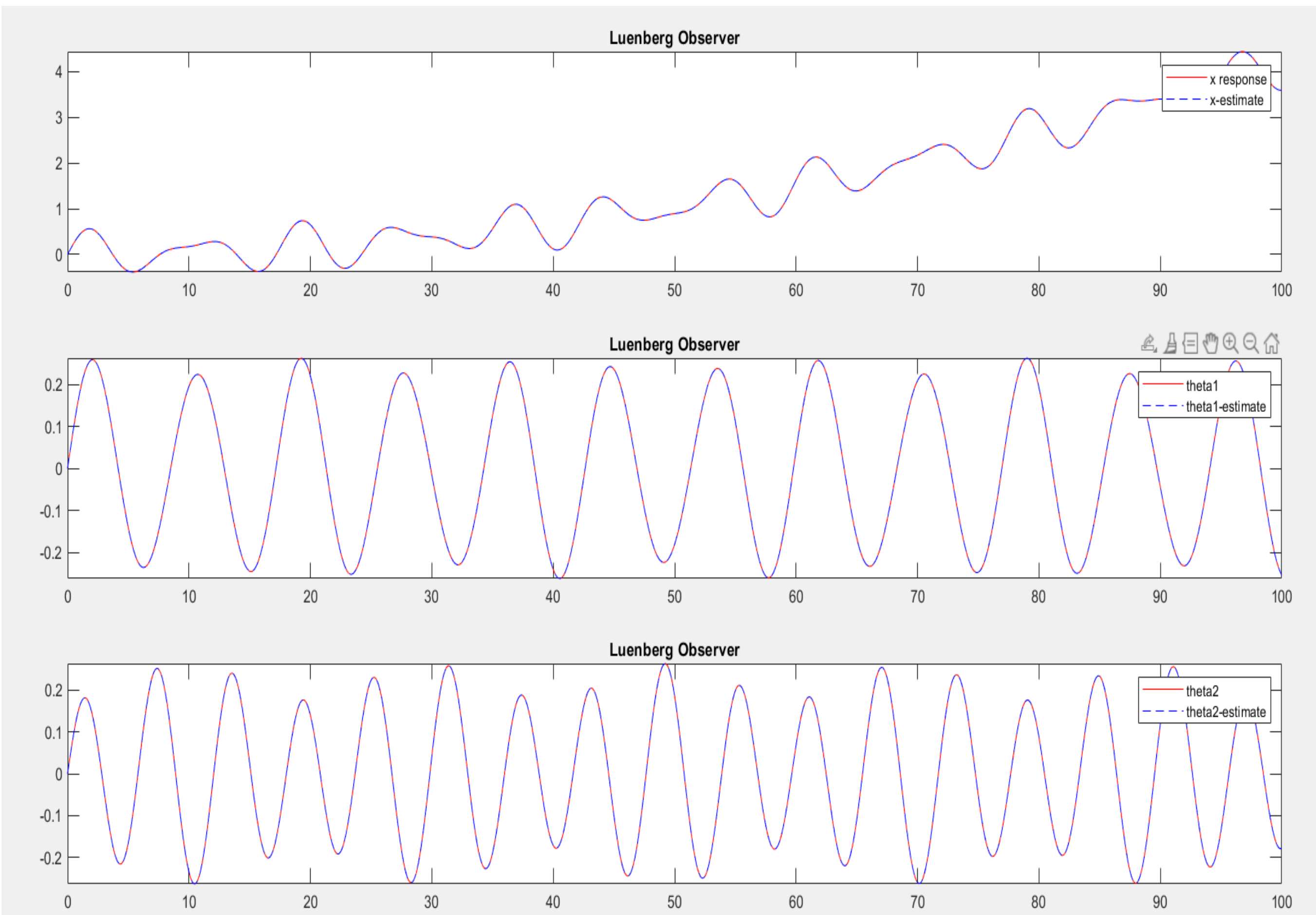


Figure 9. Leunberger observer (Scenario 1)

Scenario 3:

$$\text{Output} = x(t), \theta_2(t)$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

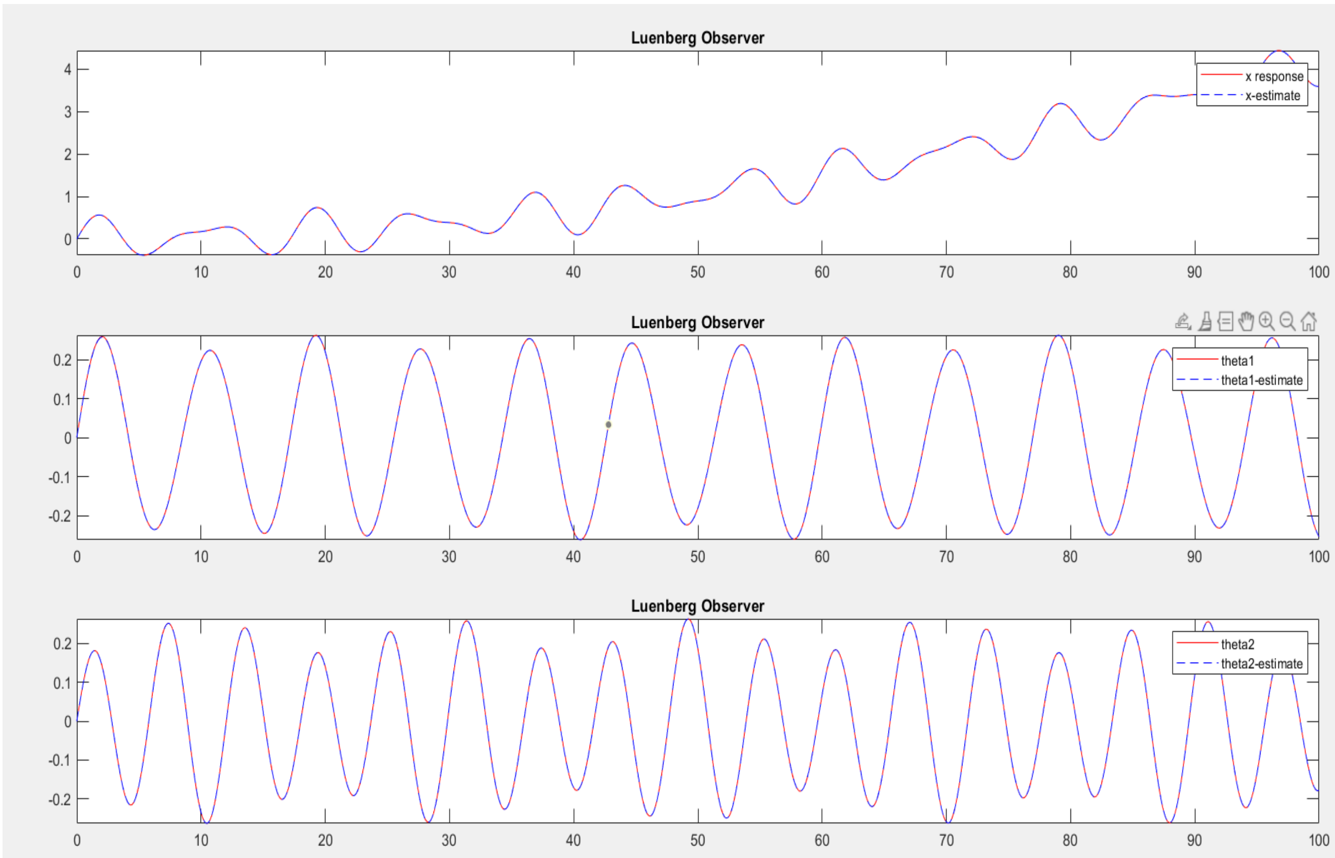


Figure 10. Leunberger observer (Scenario 3)

Scenario 4:

$$\text{Output} = x(t), \theta_1(t), \theta_2(t)$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

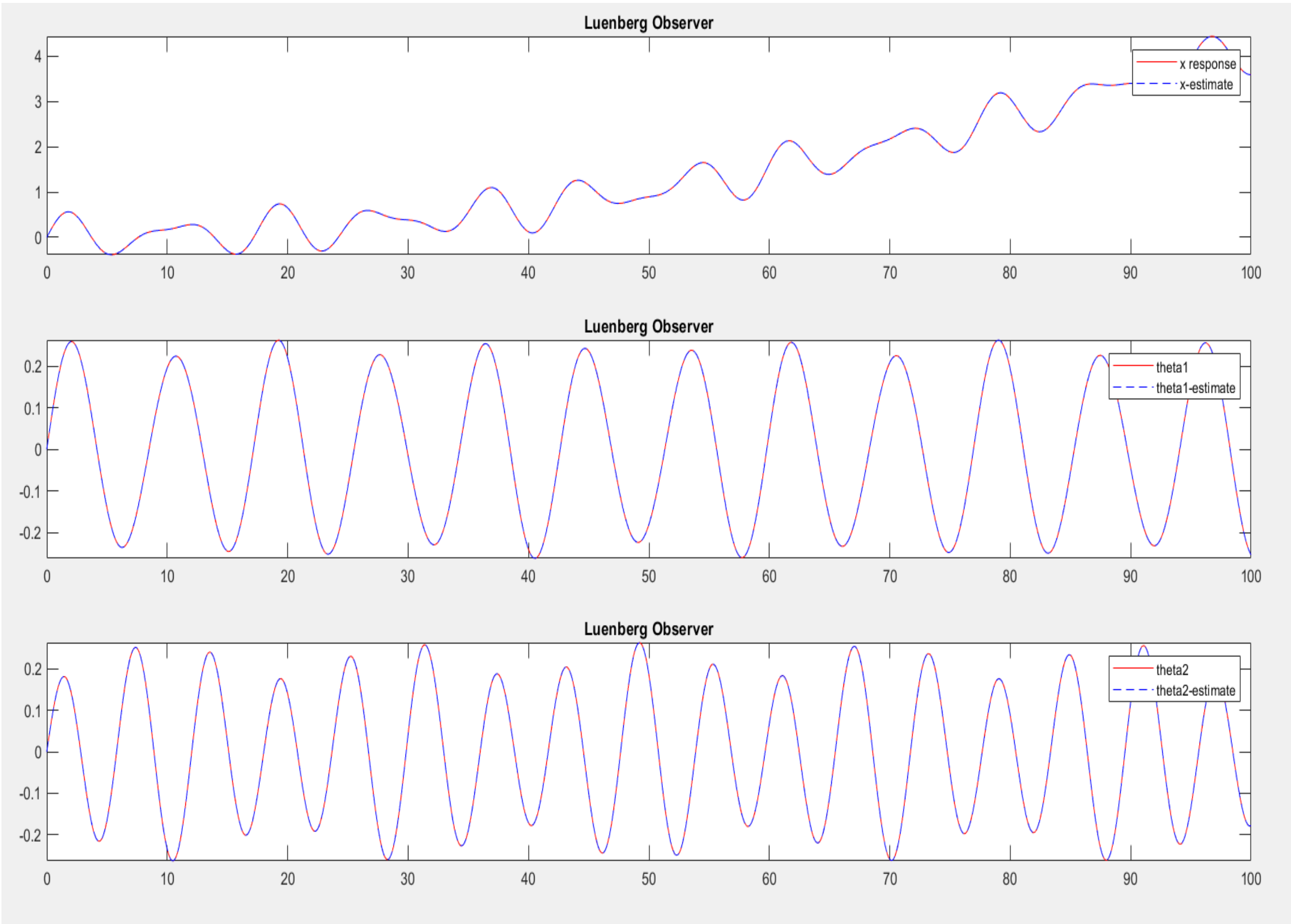


Figure 11. Leunberger observer (Scenario 4)

The Observer can also be used to estimate the states for non-linear system using the following Simulink design:

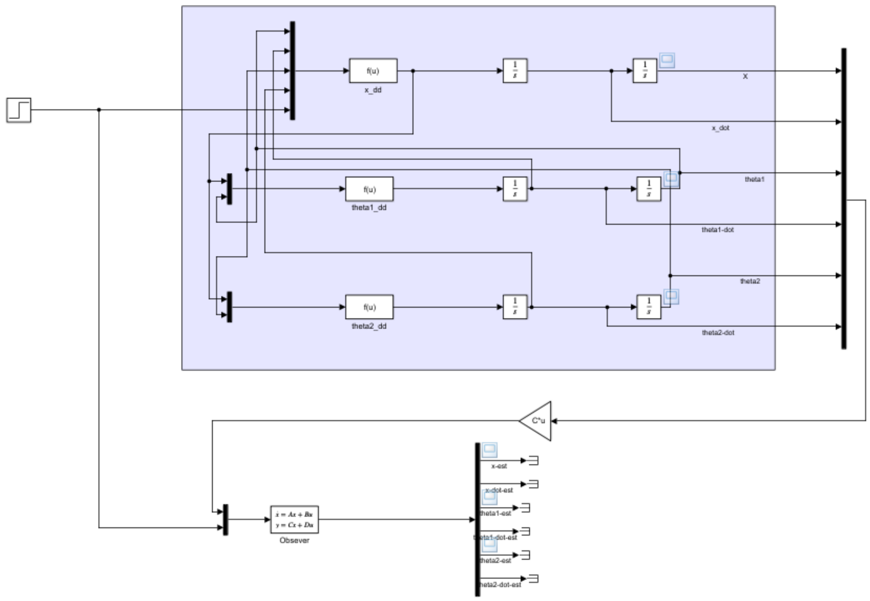


Figure 12. Non-linear Simulink Design with Observer

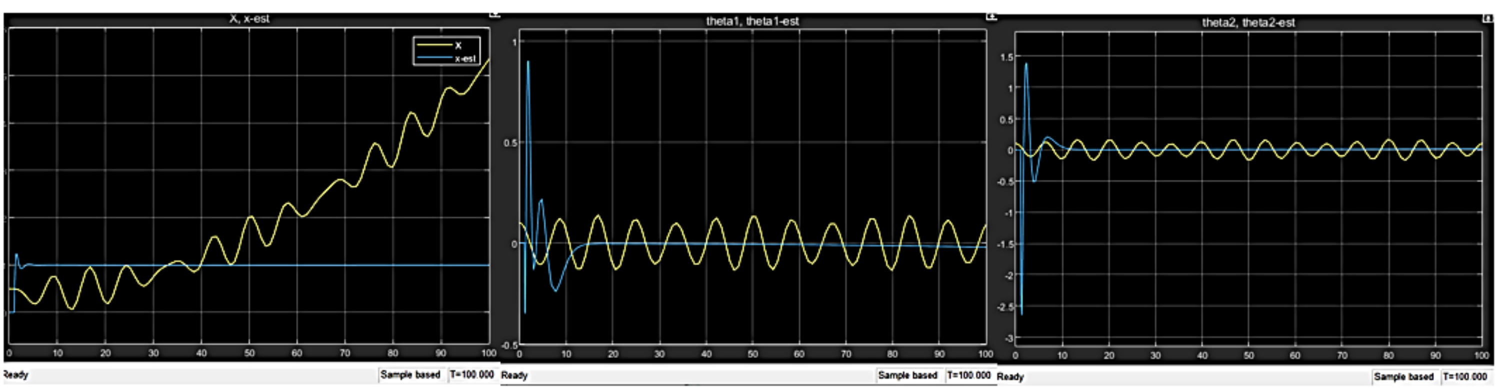


Figure 13. Leunberger observer (x, theta1, theta2) (Non-linear Scenario 1)

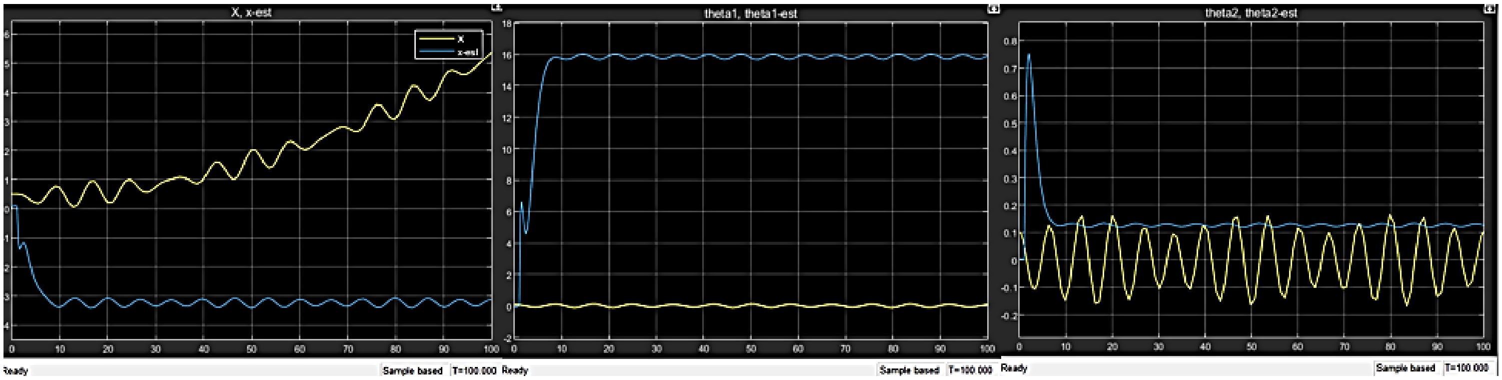


Figure 14. Leunberger observer(x, theta1, theta2) (Non-linear Scenario 3)

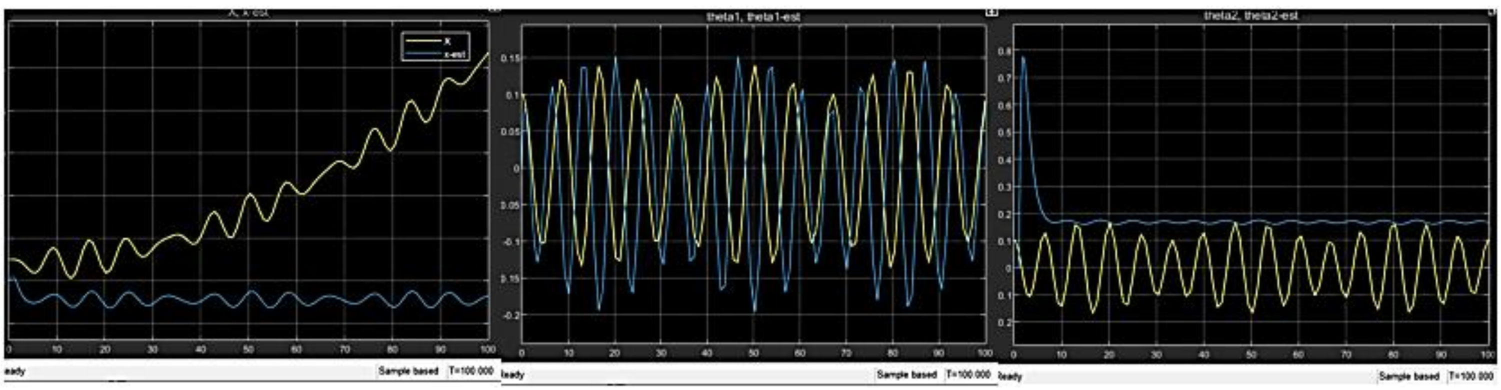


Figure 15. Leunberger observer (x, theta1, theta2) (Non-linear Scenario 4)

Part G:

In the final section, we designed an output feedback controller using the observer generated in the last section. The following state-space equations were chosen for this system. We will use the separation principle to design an optimal controller i.e. LQR and an optimal state estimator i.e. Kalman Filter to achieve optimal full state feedback known as Linear Quadratic Gaussian (LQG). The state space for LQG with gaussian process noise **v(t)** and gaussian measurement noise **w(t)** is:

$$\begin{gathered} \dot{\mathrm{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t) + \mathbf{v}(t) \\ \mathbf{y}(t) = C(t)\mathbf{x}(t) + \mathbf{w}(t) \end{gathered} \tag{21}$$

Following equations define the Kalman Filter estimator/observer:

$$\begin{gathered} \hat{x} = (A - \mathrm{L} * C)\hat{x} + [B\mathrm{L}]\begin{bmatrix} u \\ y \end{bmatrix} \\ \hat{y} = C * \hat{x} \end{gathered} \tag{22}$$

Finally, we combine the optimal controller and estimator to derive the following optimal equations for full-state feedback LQG:

$$\begin{bmatrix}\dot{x}\\ \dot{e}\end{bmatrix} = \begin{bmatrix}A - B*\mathrm{K} & B*\mathrm{K}\\ 0 & A - \mathrm{L}*C\end{bmatrix}\begin{bmatrix}x\\ e\end{bmatrix} + \begin{bmatrix}B\\ 0\end{bmatrix}\begin{bmatrix}u & y\end{bmatrix}$$
$$y = \begin{bmatrix}C & 0\end{bmatrix}x \tag{23}$$

The non-linear design for the feedback controller system is shown in Simulink below:

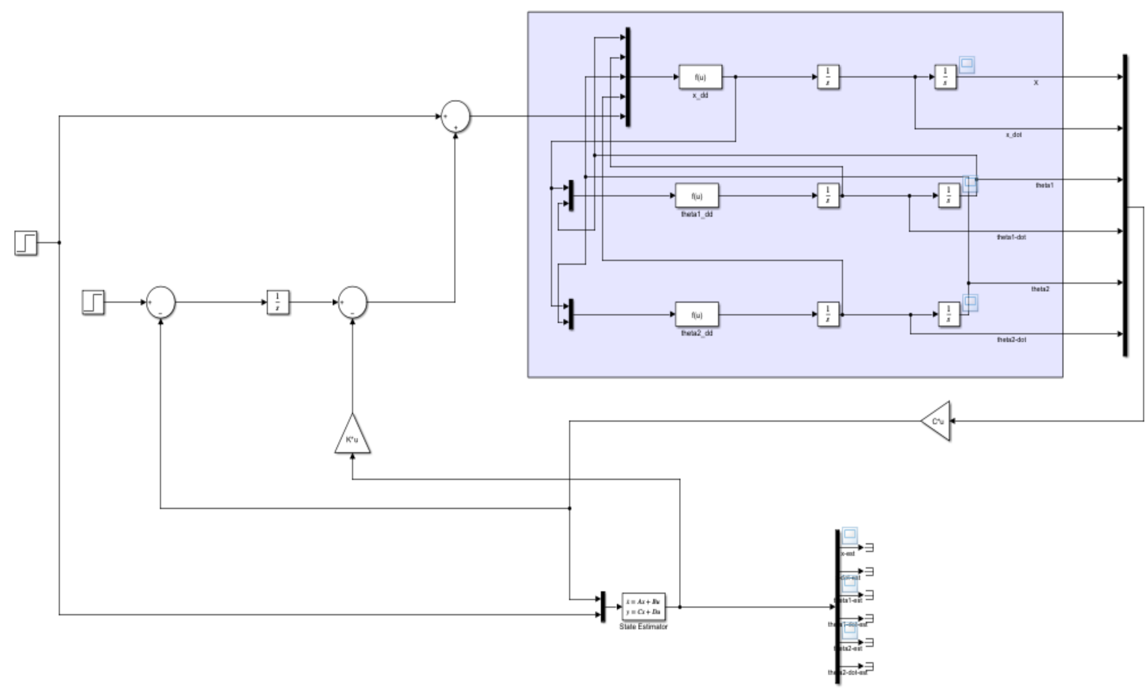


Figure 16. Non-linear Simulink Design

The following two figures show the predicted states from the linear equation and a comparison of the non-linear resulting response to the predicted values.

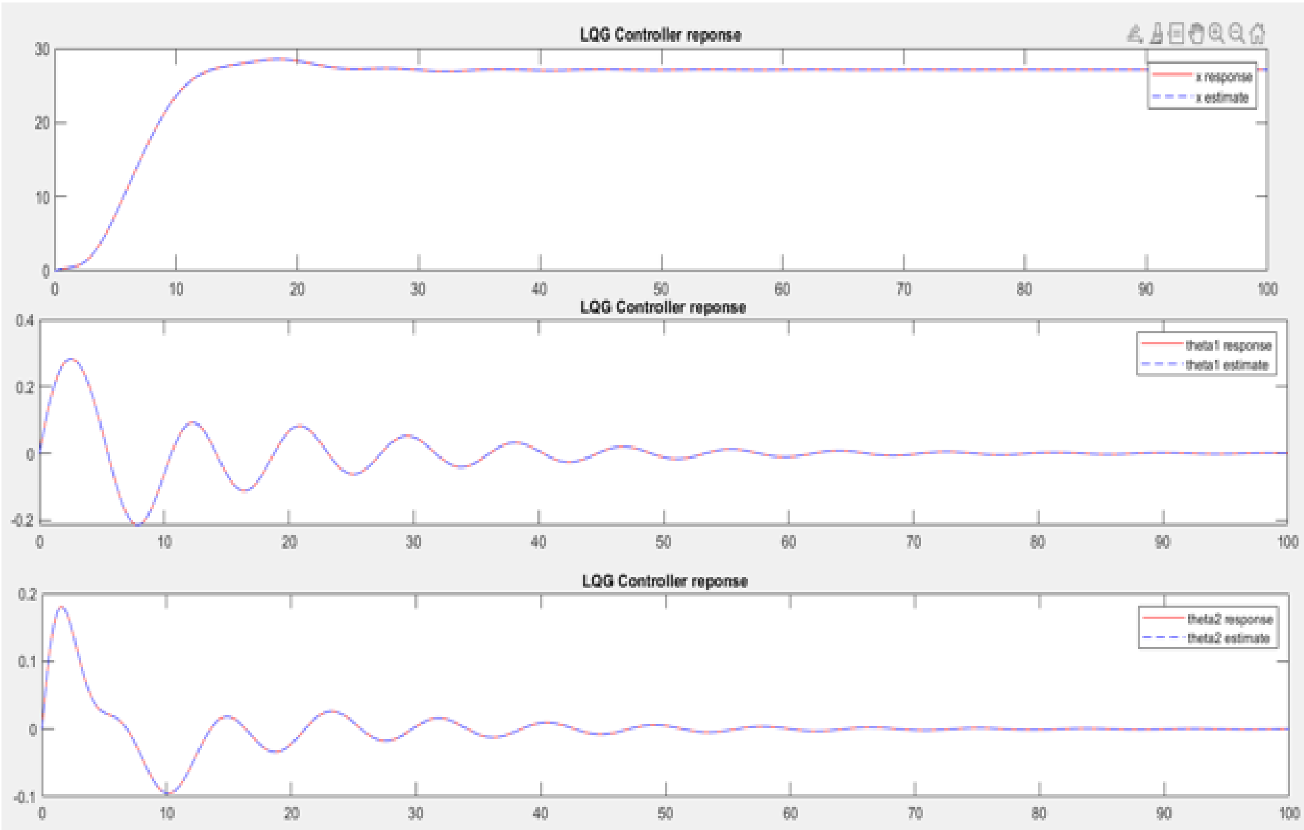


Figure 17. Linearized LQG Controller response and estimate

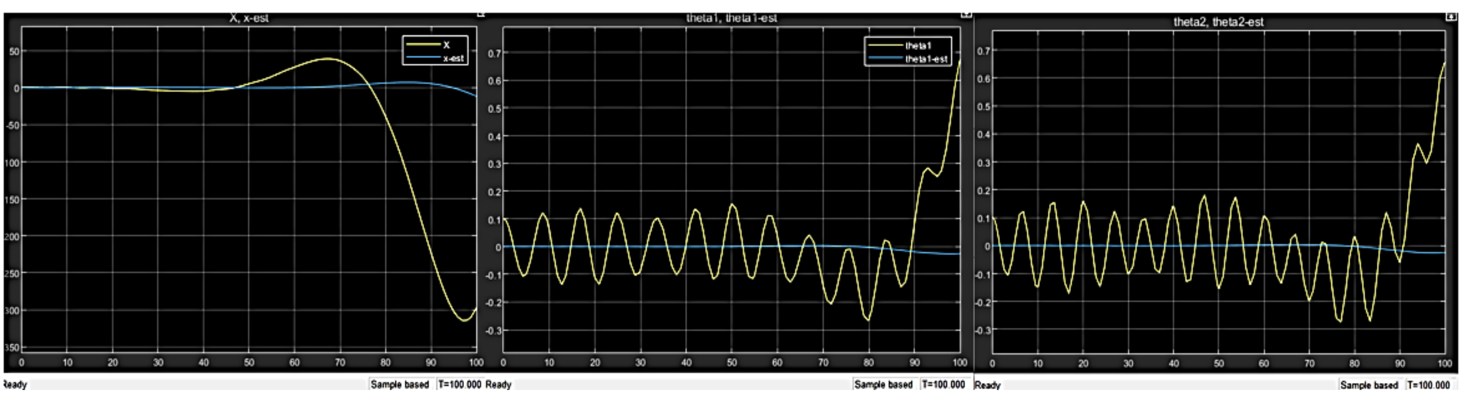


Figure 18. Non-Linear Resulting Response and Estimate

To track a constant reference, we have added an **integral gain** to the full state feedback with the LQG to remove the accumulating error in the system.

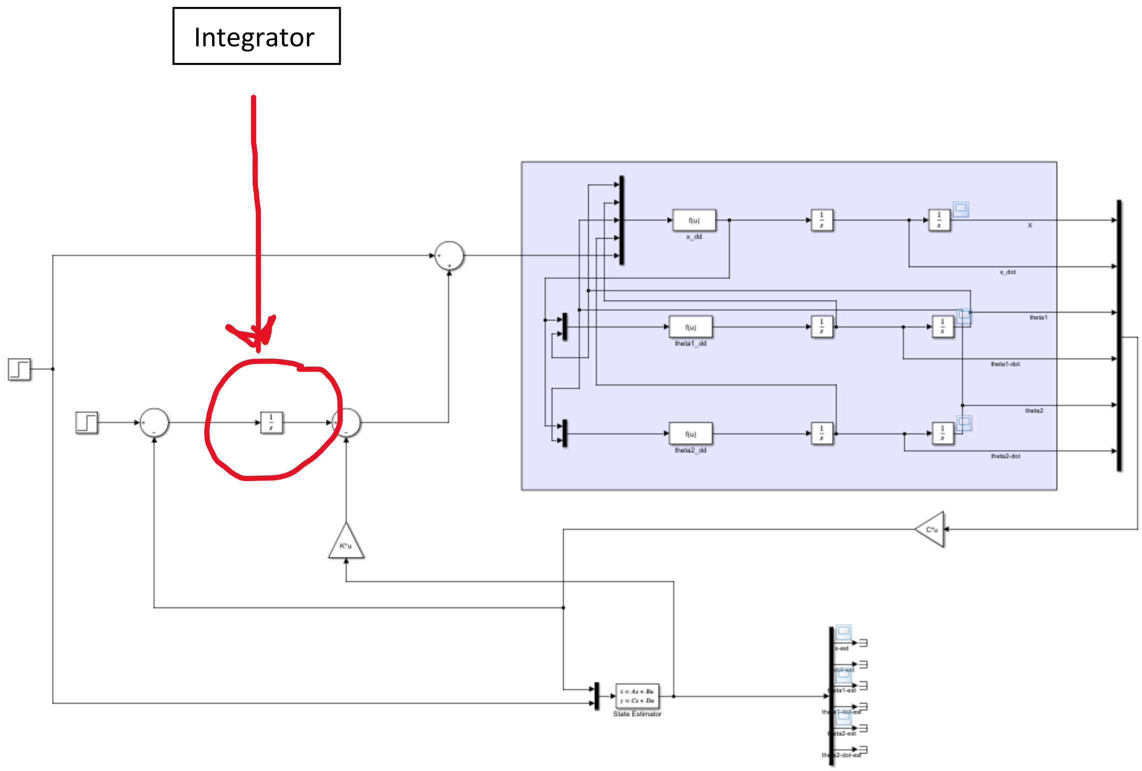


Figure 17. Non-linear LQG + integral controller

The LQG controller relies on estimated states and is prone to internal and external noise and disturbances due to state estimation. So, this design might not be able to reject constant force disturbances applied to the system. But, if we can completely observe the system without estimator like LQR, then we might be able to reject the external disturbances and noise.

References:

[1] Control Bootcamp - Steven Brunton – University of Washington

[2] Nonlinear Control Systems

[3] Linear Quadratic Gaussian Design

[4] Linear Quadratic Controller Design

[5] Kalman Filter Tutorials – Dr. Michel van Biezen