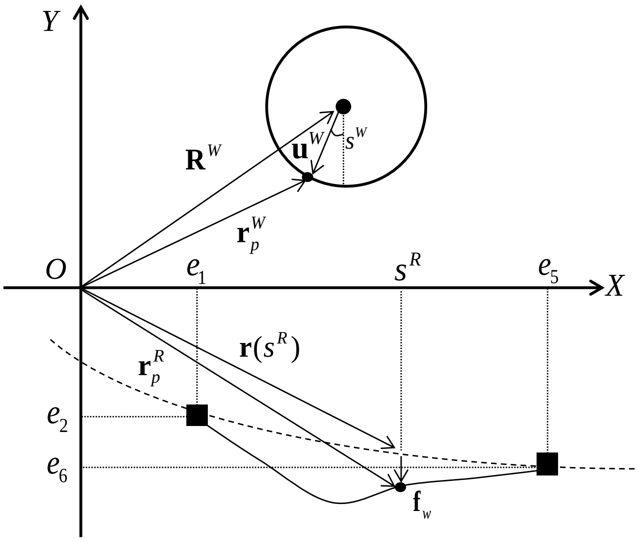
Equation of Motion

1. Wheel and rail contact model



$\mathbf{r}_p^{\mathrm{W}}$ is the contact point on the wheel surface

$\mathbf{r}_p^{\mathrm{R}}$ is the contact point on the rail surface

In order to show contact point, two variables are introduced.

$$\mathbf{s} = \begin{bmatrix} s^W & s^R \end{bmatrix}^T$$

Here, s^W indicate wheel surface and s^R is rail surface respectively

From the wheel perspective. The component of contact point $\mathbf{r}_p^{\mathrm{W}}$ consist of the distance between contact point location of wheel and center gravity of wheel. Which can be shown by this summation vector

$$\begin{aligned} \mathbf{r}_p^{\mathrm{W}} &= \mathbf{R}^w + \mathbf{u}^w \\ &= \begin{bmatrix} x^w \\ y^w \end{bmatrix} + \begin{bmatrix} -r\sin(s^W) \\ -r\cos(s^W) \end{bmatrix} \end{aligned} \tag{1.1}$$

The component of contact point $\mathbf{r}_p^{\mathrm{R}}$ can be shown with this summation vector

$$\begin{aligned} \mathbf{r}_p^{\mathrm{R}} &= \mathbf{r}(s^R) \\ &= \mathbf{Se} \end{aligned} \tag{1.2}$$

$f(s^R)$ is the wear shape function

Furthermore, tangential vector and normal vector for wheel can be defined as

$$\mathbf{t}_p^{\mathrm{W}} = \frac{\partial \mathbf{r}_p^W}{\partial s^W} = \begin{bmatrix} -r\cos(s^W) \\ r\sin(s^W) \end{bmatrix}, \qquad \mathbf{n}_p^{\mathrm{W}} = \mathbf{R}\left(\frac{\pi}{2}\right)\mathbf{t}_p^{\mathrm{W}} = \begin{bmatrix} -r\sin(s^W) \\ -r\cos(s^W) \end{bmatrix} \tag{1.3}$$

Here, $\mathbf{R}\left(\frac{\pi}{2}\right)$ act as the starting point for global coordinate system. Which just transform the matrix to $\frac{\pi}{2}$.

Similarly, tangential and normal vector for the rail can be defines as

$$\mathbf{t}_p^{\mathrm{R}} = \frac{\partial \mathbf{r}_p^R}{\partial s^R} = \frac{\partial \boldsymbol{S}}{\partial s^R}\boldsymbol{e}, \quad \mathbf{n}_p^{\mathrm{R}} = \mathbf{R}\left(\frac{\pi}{2}\right)\mathbf{t}_p^{\mathrm{R}} \tag{1.4}$$

2. Contact Condition

The contact condition for the wheel and the rail is simple. Which is, tangential vector had to be orthogonal with the normal vector. Hence, the condition became

$$\mathbf{E}(\mathbf{s}) = \begin{bmatrix} E_{\mathbf{1}} \\ E_2 \end{bmatrix} = \begin{bmatrix} \mathbf{t}_{\boldsymbol{p}}^W \cdot \mathbf{n}_{\boldsymbol{p}}^R \\ \mathbf{t}_{\boldsymbol{p}}^R \cdot (\mathbf{r}_{\boldsymbol{p}}^W - \mathbf{r}_{\boldsymbol{p}}^R) \end{bmatrix} = \mathbf{0} \tag{2.1}$$

The solution for equation above can be solved with Newton Raphson.
For the unknown variable $\mathbf{S}$ step function i+1

$$\mathbf{S}^{(i+1)} = \mathbf{S}^{(i)} + \Delta\mathbf{S}^{(i)} \tag{2.2}$$

$$\Delta\mathbf{S}^{(i)} = -(\mathbf{E}_{\mathbf{S}}^{(i)})^{-1}\mathbf{E}^{(i)} \tag{2.3}$$

Here, $\mathbf{E}_s$ can be solve with partial derivative of $\mathbf{E}$ with respect to $\mathbf{s}$

$$\mathbf{E}_s = \frac{\partial \mathbf{E}}{\partial \mathbf{s}} = \begin{bmatrix} \frac{\partial \mathbf{E_1}}{\partial \mathbf{s}^W} & \frac{\partial \mathbf{E_1}}{\partial \mathbf{s}^{\boldsymbol{R}}} \\ \frac{\partial \mathbf{E_2}}{\partial \mathbf{s}^W} & \frac{\partial \mathbf{E_2}}{\partial \mathbf{s}^R} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{t}_{\boldsymbol{p}}^W}{\partial \mathbf{s}^W} \cdot \mathbf{n}_{\boldsymbol{p}}^R & \mathbf{t}_{\boldsymbol{p}}^W \cdot \frac{\partial \mathbf{n}_{\boldsymbol{p}}^R}{\partial \mathbf{s}^{\boldsymbol{R}}} \\ \mathbf{t}_{\boldsymbol{p}}^R \cdot \mathbf{t}_{\boldsymbol{p}}^W & \frac{\partial \mathbf{t}_{\boldsymbol{p}}^R}{\partial \mathbf{s}^R} \cdot \left(\mathbf{r}_{\boldsymbol{p}}^W - \mathbf{r}_{\boldsymbol{p}}^R\right) - \mathbf{t}_{\boldsymbol{p}}^R \cdot \mathbf{t}_{\boldsymbol{p}}^R \end{bmatrix} \tag{2.4}$$

3. Equation of motion

a. Equation of motion for the wheel

$$\mathbf{M}^W \ddot{\mathbf{q}}^W = \mathbf{Q}_N^W + \mathbf{Q}_g^W \tag{3.1}$$

Where $\mathbf{M}^W = \begin{bmatrix} m_w & 0 & 0 \\ 0 & m_w & 0 \\ 0 & 0 & \frac{1}{2} m_w r^2 \end{bmatrix}$, $\mathbf{q}^W = [\mathbf{x}^W \quad \mathbf{y}^W \quad \theta^w]$

$\mathbf{Q}_N^W$, and $\mathbf{Q}_g^W$ is normal force and gravitational force that working on the wheel, which is

$$\mathbf{Q}_g^W = \begin{pmatrix} 0 \\ -\boldsymbol{m}_g^W \\ 0 \end{pmatrix} \tag{3.2}$$

$$\mathbf{Q}_N^W = N \hat{n}_p^R \tag{3.3}$$

K_c is the spring hertz constant and N is the contact force that working on the normal direction can be defined as

$$N = -K_c {\delta_n}^{3/2} \tag{3.4}$$

Where δ_n is the contact measure. Which can be defined as the projection of normal vector at the point between the difference of the point location of the wheel and rail.

$$\delta_n = \hat{n}_p^R \cdot (r_p^W - r_p^R) \tag{3.5}$$

Furthermore, $\hat{n}_p^R$ is the normal unit vector of the contact point on the rail. Which can be shown as

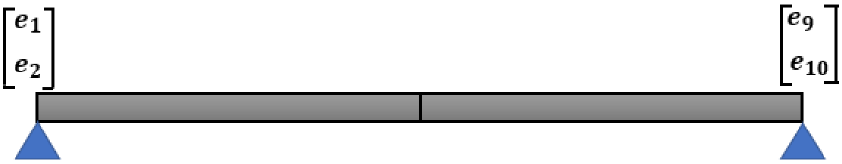
$$\hat{n}_p^R = \frac{n_p^R}{|n_p^R|} \tag{3.6}$$

b. Equation of motion for simply supported beam

$$\boldsymbol{M}^{R}\ddot{\boldsymbol{q}}^{R} = \boldsymbol{Q}_{N}^{R} + \boldsymbol{Q}_{g}^{R} + \boldsymbol{Q}_{k} \tag{3.7}$$

Then, augmented formulation is used to solved $\ddot{\boldsymbol{q}}^{R}$

$$\begin{bmatrix} \begin{bmatrix} \mathbf{M}_e^1 & \\ & \mathbf{M}_e^2 \end{bmatrix} & \begin{bmatrix} \boldsymbol{\Phi}_{\boldsymbol{q}}^{T} \end{bmatrix} \\ \begin{bmatrix} \boldsymbol{\Phi}_{\boldsymbol{q}} \end{bmatrix} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{Q}_{N}^{R} + \boldsymbol{Q}_{g}^{R} + \boldsymbol{Q}_{k} \\ \boldsymbol{\gamma} \end{bmatrix} \tag{3.8}$$



The component of the equation 1.1 can be shown below

Where, $\mathbf{M}_e$ is the element mass of ANCF

$$\mathbf{M}_e = m\begin{bmatrix} \frac{13}{35} & 0 & \frac{11l}{210} & 0 & \frac{9}{70} & 0 & \frac{-13l}{420} & 0 \\ 0 & \frac{13}{35} & 0 & \frac{11l}{210} & 0 & \frac{9}{70} & 0 & \frac{-13l}{420} \\ \frac{11l}{210} & 0 & \frac{l^2}{105} & 0 & \frac{13l}{420} & 0 & \frac{-l^2}{140} & 0 \\ 0 & \frac{11l}{210} & 0 & \frac{l^2}{105} & 0 & \frac{13l}{420} & 0 & \frac{-l^2}{140} \\ \frac{9}{70} & 0 & \frac{13l}{420} & 0 & \frac{13}{35} & 0 & \frac{-11l}{210} & 0 \\ 0 & \frac{9}{70} & 0 & \frac{13l}{420} & 0 & \frac{13}{35} & 0 & \frac{-11l}{210} \\ \frac{-13l}{420} & 0 & \frac{-l^2}{140} & 0 & \frac{-11l}{210} & 0 & \frac{l^2}{105} & 0 \\ 0 & \frac{-13l}{420} & 0 & \frac{-l^2}{140} & 0 & \frac{-11l}{210} & 0 & \frac{l^2}{105} \end{bmatrix} \tag{3.9}$$

Φ_q is the constraint condition, which fixed the beam into x and y axis of the supports. And $\mathbf{q}^R$ is the generalized coordinate of the rail

$$\Phi_q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{1.5}$$

$$\boldsymbol{q}^R = [e_1 \quad e_2 \quad e_3 \quad e_4 \quad e_5 \quad e_6 \quad e_7 \quad e_8 \quad e_9 \quad e_{10} \quad e_{11} \quad e_{12}]^T \tag{1.6}$$

$\boldsymbol{Q}_N^R, \boldsymbol{Q}_g^R,$ and $\boldsymbol{Q}_k$ respectively is the normal force, gravitational force, and elastic force. Which can be shown as

$$\boldsymbol{Q}_g^R = \left[0 \quad -\frac{\rho_R l a_R g}{2} \quad 0 \quad -\frac{l(\rho_R l a_R g)}{8} \quad 0 \quad -\frac{\rho_R l a_R g}{2} \quad 0 \quad \frac{l(\rho_R l a_R g)}{8}\right]^T \tag{1.7}$$

$$\boldsymbol{Q}_N^R = S(\xi)^T(-N\hat{n}_p^R) \tag{1.8}$$

The elastic force can be calculated as

$$\boldsymbol{Q}_k = \boldsymbol{Q}_l + \boldsymbol{Q}_t$$

Where $\boldsymbol{Q}_l$ and $\boldsymbol{Q}_t$ are

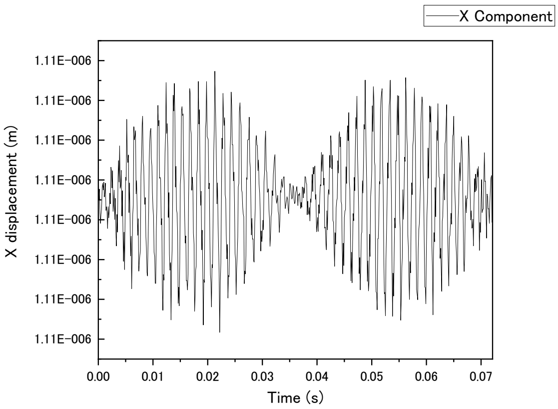
$$\mathbf{Q}_l = (\frac{\partial U_l}{\partial \mathbf{e}})^T = \mathbf{K}_l \mathbf{e} \tag{1.9}$$

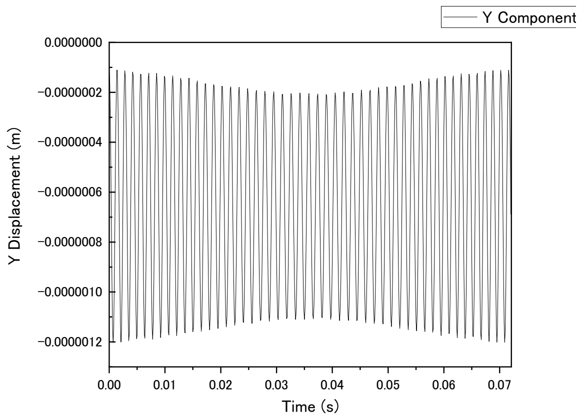
$$\mathbf{Q}_t = \left(\frac{\partial U_t}{\partial \mathbf{e}}\right)^T = \mathbf{K}_t \mathbf{e} \tag{1.10}$$

4. Result

a. Parameter

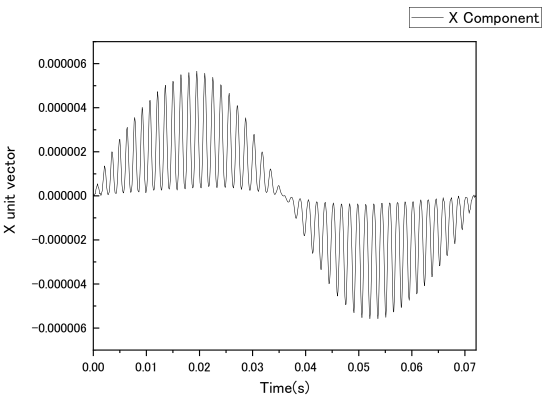
Disk Radius	0.043 m
Speed	4 km/h
Disk mass	1.346 Kg
Contact stiffness (接触剛性) / K_c	2.2785×10^{10}
Sleeper	2

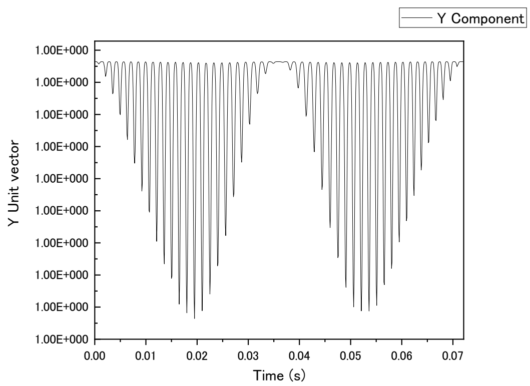




X and Y component of $(r_p^W - r_p^R)$

figure 4.11 : X component of $(r_p^W - r_p^R)$ figure 4.12 : Y component of $(r_p^W - r_p^R)$





X and Y component of $\hat{n}_p^R$

figure 4.21 : X component of $\hat{n}_p^R$ figure 4.22 : Y component of $\hat{n}_p^R$

Recalling equation 3.5, Dot product are used to calculate contact measure/接触量 (δ_n)

$$\delta_n = \hat{n}_p^R \cdot (r_p^W - r_p^R) \tag{3.5}$$

Hence δ_n graph can be shown below

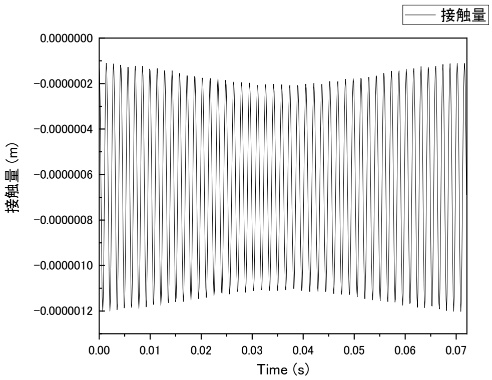


figure 4.3：contact measure /接触量

Furthermore, equation 3.4 is used to calculated contact force which is

$$N = -K_c {\delta_n}^{3/2} \tag{3.4}$$

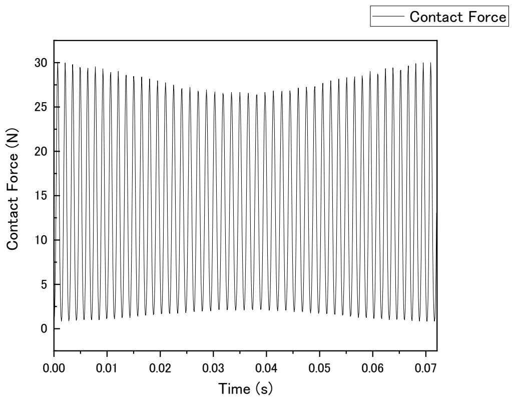


figure 4.4：contact Force

Finally, the equation for normal force $\mathbf{Q}_N^W$ can be calculated as

$$\mathbf{Q}_N^W = N\hat{n}_p^R \tag{3.3}$$

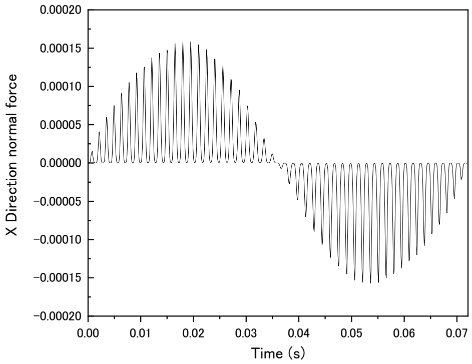


figure 4.51：X component of $\mathbf{Q}_N^W$

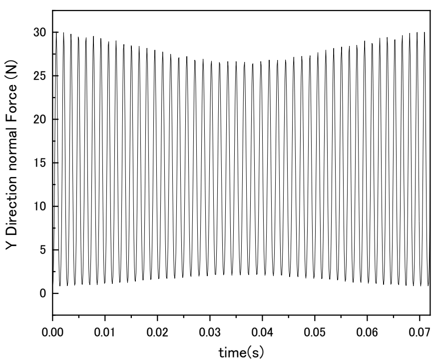


figure 4.52：Y component of $\mathbf{Q}_N^W$

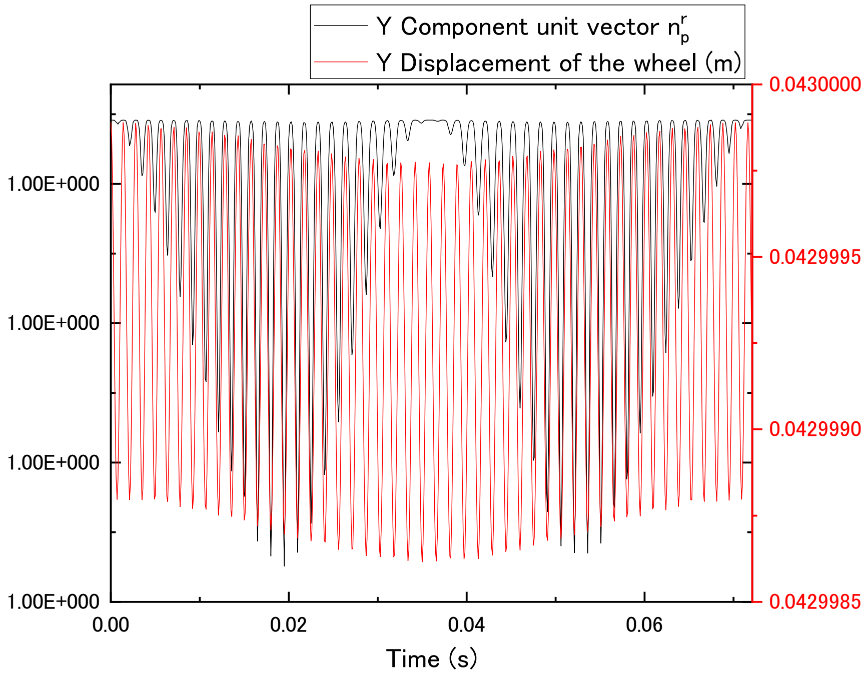


figure 4.6 : Comparison between Y displacement and Y component of unit vector $\hat{n}_p^R$

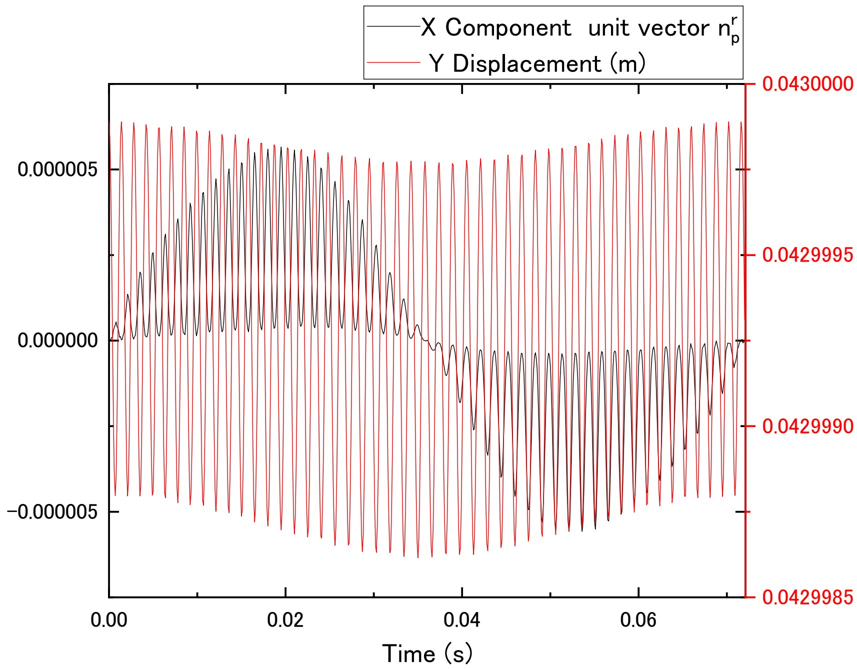


figure 4.7 : Comparison between Y displacement and X component of unit vector $\hat{n}_p^R$