$f(n) = \mathcal{O}(g(n))$

1) $2^{n+1} = \mathcal{O}(2^n)$

$f(n) = \mathcal{O}(g(n))$ si $\exists\ c_i > 0, n_0 \in \mathbb{N}$ t.q.

$0 \leq f(n) \leq c \cdot g(n), \ \forall n \geq n_0.$

Sean $f(n) = 2^{n+1}$ y $g(n) = 2^n$

$2^{n+1} = O(2^n) \Rightarrow \exists\ c, n_0$ t.q. $0 \leq 2^{n+1} \leq c \cdot 2^n \ \forall n \geq n_0$

$\Rightarrow \log_2 2^{n+1} \leq \log_2 (c 2^n)$

$\Rightarrow n+1 \leq \log_2 c + n$

$\Rightarrow \log_2 c \geq 1$

Con $c = 2$ y $n_0 = 1$ se cumple la desigualdad, por lo tanto

$2^{n+1} \in O(2^n)$

2) $2^{2n} = \mathcal{O}(2^n)$

Sean $f(n) = 2^{2n}$ y $g(n) = 2^n$

$2^{2n} = \mathcal{O}(2^n) \Rightarrow \exists\ c, n_0$ t.q. $0 \leq 2^{2n} \leq c 2^n \ \forall n \geq n_0$

$\Rightarrow \log_2(2^{2n}) \leq \log_2(c \cdot 2^n)$

$\Rightarrow 2n \leq \log_2 c + n$

$\Rightarrow n \leq \log_2 c$

$\Rightarrow \frac{n}{\log_2 c} \leq 1$ no se cumple la desigualdad $\forall n \geq n_0$

y por lo tanto $2^{2n} \notin O(2^n)$

(3) $3^n = O(2^{n^2})$

Sean $f(n) = 3^n$ y $g(n) = 2^{n^2}$

$3^n = O(2^{n^2}) \Rightarrow \exists\, C \in \mathbb{R}^+, n_0 \in \mathbb{N}$ t.q. $0 \leq 3^n \leq C 2^{n^2}$

$\forall\, n \geq n_0$

$\Rightarrow C \geq \dfrac{3^n}{2^{n^2}} \Rightarrow \log_2 C \geq n \log_2 3 - n^2$

$\lim\limits_{n\to\infty} \log_2 C \geq \lim\limits_{n\to\infty} n \log_2 3 - n^2 \Leftrightarrow C \geq 1$

con $C = 2$ y $n_0 = 1$ se cumple $0 \leq 3^n \leq C 2^{n^2}$ $\forall\, n \geq 1$

Entonces concluimos que $3^n \in O(2^{n^2})$

(4) $2^n + O(n) = \Theta(2^n)$

Sean $f(n) = 2^n + O(n)$ y $g(n) = 2^n$

$2^n + O(n) = \Theta(2^n) \Rightarrow \exists\, c \in \mathbb{R}^+, n_0 \in \mathbb{N}$ t.q.

$$C_3 2^n \leq 2^n + O(n) = 2^n + C_1 n \leq C_2 2^n$$

$$C_3 \leq 1 + \frac{C_1 n}{2^n} \leq C_2$$

Con $n_0 = 1$, $C_1 = 1$, $C_3 = 1$, $C_2 = 2$ se cumple la desigualdad y concluimos que $2^n + O(n) \in \Theta(2^n)$