Faster Spectral Density Estimation and Sparsification in the Nuclear Norm

COLT 2024, Edmonton, Canada

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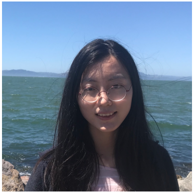
(NYU)



Yujia Jin (Stanford)

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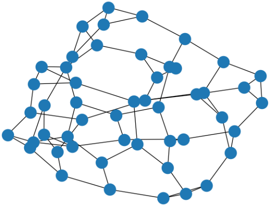
Christopher Musco (NYU)

Aaron Sidford (Stanford)

Problem Setup

- Graph G, with normalized adjacency matrix N.

$$N := D^{-1/2} A D^{-1/2}, \quad D = \mathrm{diag}(\deg(v_1), \dots, \deg(v_n))$$



- Spectrum of $G \equiv$ Eigenvalues of N:

$$-1 \leq \lambda_n \leq \dots \leq \lambda_1 = 1.$$

Goal: Estimate the spectrum of G.

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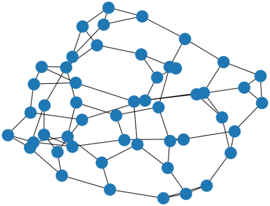

compute in $\lesssim \; n^{2.371}$

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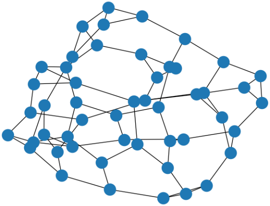

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sublinear time: $o(n^2)$

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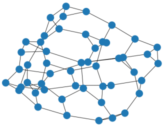

Graph G

Eigenvalues of N

$$-1 \leq \lambda_n \leq \cdots \leq \lambda_1 = 1.$$

Goal: Output $\tilde{\lambda}_n \leq \cdots \leq \tilde{\lambda}_1$ such that

$$\sum_{i=1}^{n} \left|\lambda_i - \tilde{\lambda}_i\right| \leq n\varepsilon.$$

Representing Spectrum as a Distribution

Eigenvalues of N: $-1 \leq \lambda_n \leq \cdots \leq \lambda_1 = 1$

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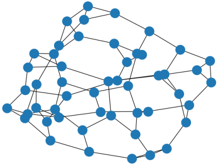

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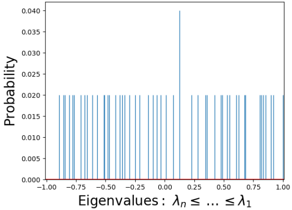

Spectral Density p (pmf)

$p \equiv \mathrm{SpectralDensity}(G)$:

$$p = \mathrm{Unif}(\{\lambda_n, \ldots, \lambda_1\})$$

Spectral Density Estimation (SDE)

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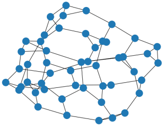

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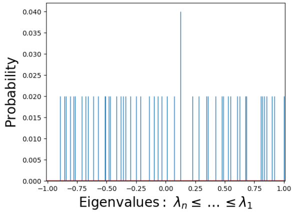

$[p = \mathrm{Unif}(\{\lambda_n, \ldots, \lambda_1\})]$

Spectral Density p (pmf)

Recall Goal: Output $\tilde{\lambda}_n \leq \cdots \leq \tilde{\lambda}_1$ such that $\sum_{i=1}^{n} \left|\lambda_i - \tilde{\lambda}_i\right| \leq n\varepsilon$.

Equivalently: Output q such that Wasserstein-1 distance:

$$W_1(p, q) \leq \varepsilon.$$

Learning Distribution p From Moments

$[X \sim p = \mathrm{Unif}(\{\lambda_n, \ldots, \lambda_1\})]$

Learn p by estimating its moments.

$$m_j = \mathbb{E}[X^j] = \frac{1}{n} \sum_{i=1}^{n} \lambda_i^j$$
$$= \frac{1}{n} \mathrm{Tr}(N^j).$$

Idea:

- Estimate $m_1, \ldots, m_{1/\varepsilon}$ to high accuracy.
- Find a distribution q that has moments close to these estimates.

Existing Results

Graph G with spectral density p

Assume G is stored as adjacency list.

$\underline{\mathcal{O}(1)\text{-time}}$: Access degree of node &
Identity of its i-th neighbour, for any i.

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- [Braverman et al. (2022)] Output q: $W_1(p, q) \leq \varepsilon$ in time $\mathcal{O}\left(n\varepsilon^{-7}\right)$.
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Our Results

- Output q: $W_1(p,q) \leq \varepsilon$ whp in time $\mathcal{O}\left(n\varepsilon^{-3}\right)$.
- Deterministic $\mathcal{O}\left(n/\varepsilon^2\right)$ queries to graph.
- *Simple algorithm via "Nuclear Norm Sparsification"*

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Also works for weighted graphs.
$\mathcal{O}\left(n/\varepsilon^2\right)$ query complexity in random-sampling model.

Our Goal

[Eigenvalues of $N : \lambda_n \leq \cdots \leq \lambda_1$]

[BKM'22] Return q such that $W_1(p, q) \leq \varepsilon$ whp in time $\mathcal{O}\left(\mathrm{nnz}(N) \cdot \frac{1}{\varepsilon}\right)$.

Idea: Construct <u>sparse</u> approximation to N.

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Idea: Construct sparse approximation to N.

Aim:

1. **Sparse**: Find a sparse matrix $\tilde{N}$.
2. **Close Eigenvalues**: $\tilde{\lambda}_1 \leq \cdots \leq \tilde{\lambda}_n$ of $\tilde{N}$ such that

$$\sum_{i=1}^{n} \left|\lambda_i - \tilde{\lambda}_i\right| \leq \varepsilon n.$$

3. **Fast**: Construct $\tilde{N}$ fast.

Sparsification Ideas

[Eigenvalues of N: $\lambda_n \leq \cdots \leq \lambda_1$. Eigenvalues of $\tilde{N}$: $\tilde{\lambda}_n \leq \cdots \leq \tilde{\lambda}_1$]

Recall aim: Sparse, close eigenvalues, and fast to construct

Idea 1: Spectral Sparsification: $(1-\varepsilon)N \preccurlyeq \tilde{N} \preccurlyeq (1+\varepsilon)N$

- $\implies \tilde{\lambda}_i \in (1 \pm \varepsilon)\lambda_i$.

1. Achieves sparsity $o(n^2)$.
2. Eigenvalues are close: $\sum_{i=1}^{n} \left|\lambda_i - \tilde{\lambda}_i\right| \leq \varepsilon n$.
3. [Lee'13] Needs $\Omega(n^2)$ queries to G.

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Idea 2: Additive Spectral Sparsification: $\left\|N - \tilde{N}\right\|_2 \leq \varepsilon$

- $\implies \tilde{\lambda}_i \in \lambda_i \pm \varepsilon$.

1. Achieves sparsity $o(n^2)$.
2. Eigenvalues are close: $\sum_{i=1}^{n} \left|\lambda_i - \tilde{\lambda}_i\right| \leq \varepsilon n$.
3. Randomized procedure to construct $\tilde{N}$ needs $\mathcal{O}\left(n \log n/\varepsilon^2\right)$ queries to G.

- Deterministic procedure requires $\Omega(n^2)$ queries to G.
- Randomized procedure needs $\Omega(n \log n)$ queries to G.

Our Idea: Nuclear Norm Sparsification

[Eigenvalues of N: $\lambda_n \leq \cdots \leq \lambda_1$. Eigenvalues of $\tilde{N}$: $\tilde{\lambda}_n \leq \cdots \leq \tilde{\lambda}_1$]

$$\boxed{\|N\|_* = |\lambda_n| + \cdots + |\lambda_1|.}$$

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By Weyl's Inequality,

$$\left\|N - \tilde{N}\right\|_* \geq \sum_{i=1}^{n} \left|\lambda_i - \tilde{\lambda}_i\right|.$$

$$\left\|N - \tilde{N}\right\|_* \leq \varepsilon n \implies \sum_{i=1}^{n} \left|\lambda_i - \tilde{\lambda}_i\right| \leq \varepsilon n.$$

Nuclear Norm Sparsification

$\left\|N - \tilde{N}\right\|_* \leq \varepsilon n$, and $\tilde{N}$ is sparse

Sparsify N Under adjacency list access to G.

Our Results: $\tilde{N}$ with following properties:

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4. Deterministic procedure.

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For spectral density estimation:

- Total Running Time [using [BKM'22]]: $\mathcal{O}\left(n/\varepsilon^2\right) + \mathrm{nnz}(\tilde{N}) \cdot \frac{1}{\varepsilon} = \mathcal{O}\left(n/\varepsilon^3\right)$.
- Total queries to G: $\mathcal{O}\left(n/\varepsilon^2\right)$.

Sparsification Algorithm

$N = D^{-1/2}AD^{-1/2}$. Note: If (v, v') is an edge, then $N_{v,v'} = 1/\sqrt{\deg(v)\deg(v')}$

$\boxed{\tilde{N} \text{ zeroes out } v\text{-th row and column of } N \text{ if } \deg(v) > 1/\varepsilon^2.}$

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Proof Sketch:

- Each row of $\tilde{N}$ has $\leq 1/\varepsilon^2$ non zero entries.
  - $\mathrm{nnz}(\tilde{N}) = n/\varepsilon^2$
- $\tilde{N}$ deletes only small entries of N.
- Prove that $\left\|N - \tilde{N}\right\|_F \leq \varepsilon\sqrt{n}$.
- Implies $\left\|N - \tilde{N}\right\|_* \leq \sqrt{n}\left\|N - \tilde{N}\right\|_F = \varepsilon n$.

Deterministic Sublinear-Time Spectral Density Estimation

Our Results: Deterministic, Sublinear Time, Spectral Density Estimation in time $n \cdot 2^{\tilde{\mathcal{O}}(\varepsilon^{-1})}$.

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1. Construct $\tilde{N}$: $\left\|N - \tilde{N}\right\|_* \leq \varepsilon n$.
2. Calculate moments exactly: $\frac{1}{n} \cdot \mathrm{Tr}(\tilde{N}^j)$, for $j = 1, \ldots, 1/\varepsilon$.
   - Fast to compute because $\tilde{N}$ has sparse rows/columns.
3. Find distribution q that has moments close to the moments calculated.

Sparsification Lower Bound

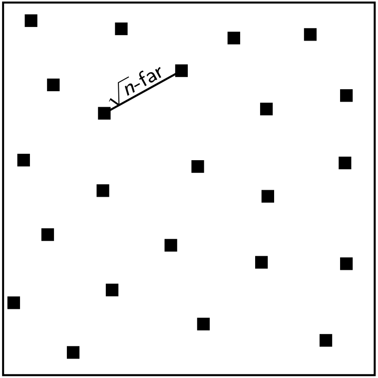
[Graph G, Normalized Adjacency Matrix N]

Our Results: Sparsity $\tilde{\Omega}(n/\varepsilon^2)$ is needed for nuclear norm sparsification.

We will sketch: for $\varepsilon = \Theta(1/\sqrt{n})$.

There is a graph with normalized adjacency N, such that, any $\tilde{N}$ with $\left\|N - \tilde{N}\right\|_* \leq \sqrt{n}$, must have $\mathrm{nnz}(\tilde{N}) = \Omega(n^2/\log n)$.

$\exists N : \left\| N - \tilde{N} \right\|_* \leq \sqrt{n}$ implies $\mathrm{nnz}(\tilde{N}) = \Omega(n^2/\log n)$



There are $\approx \exp(n^2)$ many dense graphs [▪] that are <u>all</u> $\sqrt{n}$ far in $\|\cdot\|_*$.

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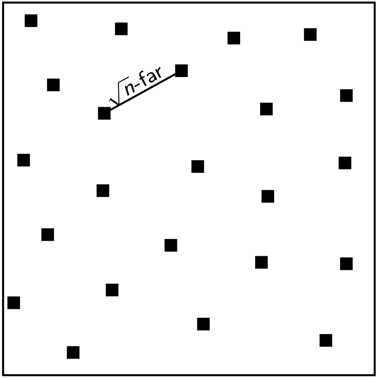

Consider the set $\mathcal{S}$ of s-sparse matrices, i.e., $\mathrm{nnz}(S) = s$.

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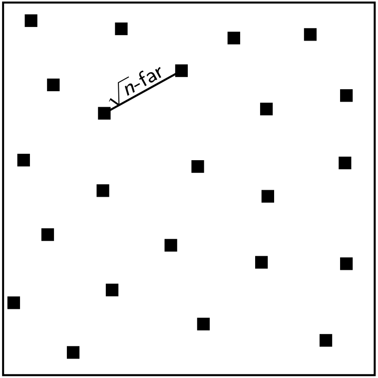

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Construct a net $\mathcal{N}_\delta[\ \circ\ ]$ of size $\exp\left(\mathcal{O}\left(s \log n\right)\right)$ that approximates $\mathcal{S}$ very well.

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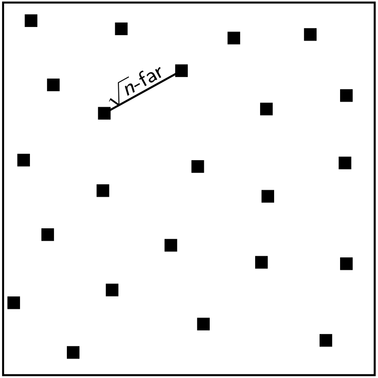

Construct a net $\mathcal{N}_\delta[\circ]$ of size $\exp\left(\mathcal{O}\left(s \log n\right)\right)$ that approximates $\mathcal{S}$ very well.
If $s = o(n^2/\log n)$, then size of net $|\mathcal{N}_\delta[\circ]| = \exp(o(n^2))$.

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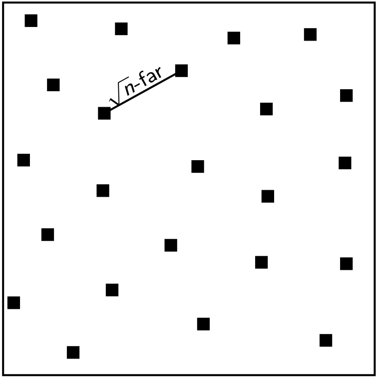

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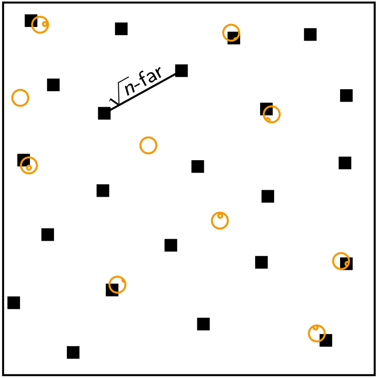

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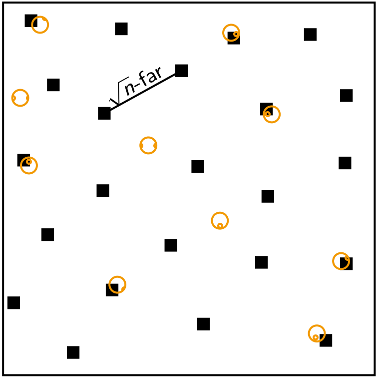

Can extend it to a larger range of ε.

Open Problems

- Can we further improve the running time?
  - We achieve a running time of $\mathcal{O}\left(n/\varepsilon^3\right)$.
  - [CKSV'18] achieve a running time of $2^{\mathcal{O}(1/\varepsilon)}$.
  - Something in between? Sublinear in n and polynomial in $1/\varepsilon$.
- Lower bound for spectral density estimation under adjacency list access to G and stronger access models?

Summary of Our Results

SDE: Spectral Density Estimation

1. Sublinear time SDE (randomized) in $\mathcal{O}\left(n/\varepsilon^3\right)$ time.
2. Small nuclear norm implies small W_1 distance.
3. Deterministic $\mathcal{O}\left(n/\varepsilon^2\right)$ time algorithm for nuclear norm sparsification.
4. Achieve optimal sparsity and query complexity, up to $\log n$ factors.
5. Deterministic, sublinear time SDE in $n2^{\tilde{\mathcal{O}}(1/\varepsilon)}$.