Sum of Squares: Part 3

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Goal: *Approximating Graph Expansion.*

Setting

Consider the d-regular graph $G = (V, E)$. Let $|V| = n$.

- Throughout we will consider a d-regular graph (easier to work with).
- Recall the cut polynomial $f_G(\boldsymbol{x}) = \frac{1}{4} \sum_{(i,j) \in E} (\boldsymbol{x}_i - \boldsymbol{x}_j)^2$.
- Maxcut: $\max_{\boldsymbol{x} \in \{-1,1\}^n} f_G(\boldsymbol{x})$ - 0.878-Approx Algo.
- MinCut: $\min_{\boldsymbol{x} \in \{-1,1\}^n} f_G(\boldsymbol{x})$ - 1-Approx Algo.

Today: We will look at "minimum-normalized-cut": Called *expansion*.

Normalized Cut

Definition (Normalized Size of a Cut)

$$\phi_G(S) \stackrel{\text{def}}{=} \frac{\left|E(S, \overline{S})\right|}{\frac{d}{n}\left|S\right|\left(n - \left|S\right|\right)}.$$

Compare the size of the cut defined by S to the size of the cut defined by S in a random-graph of same average-degree.

Definition (Expansion of the Graph)

$$\Phi_G \stackrel{\text{def}}{=} \min_{S \subset V,\ 0 < |S| < n} \phi_G(S).$$

Intuition: Start a random walk from a random vertex in S. What is the chance that it goes out of S in one step $\equiv \phi(S)$. Therefore, Φ_G calculates how "well-connected" the graph is.

Today's Goal

Given a d-regular graph $G = (V, E)$, compute or approximate Φ_G.

Remark(s)

1. *Computing Φ_G is NP-Hard.*
2. *Chawla et al. [Cha+05]: UGC $\implies$ no constant-factor approx for Φ_G.*
3. *Random Cut S:*
$$\mathop{\mathbb{E}}_{S} \phi_G(S) \geq \frac{1}{2},$$
gives no constant approx because this is a minimization problem.

Expansion in Polynomial Form

Recall

$$\phi_G(S) = \frac{\left|E(S,\overline{S})\right|}{\frac{d}{n}\left|S\right|(n-\left|S\right|)}, \quad \text{write this in polynomial form.}$$

$$\frac{\left|E(S,\overline{S})\right|}{\frac{d}{n}\left|S\right|(n-\left|S\right|)} = \frac{f_G(\boldsymbol{x})}{\frac{d}{n}\frac{1}{4}\sum_{i,j}(\boldsymbol{x}_i-\boldsymbol{x}_j)^2}, \quad (S=\{i|\boldsymbol{x}_i=1\})\,.$$

Suppose that,

$$\min_{\boldsymbol{x}\in\{-1,1\}^n}\frac{P(\boldsymbol{x})}{Q(\boldsymbol{x})} = c \implies P(\boldsymbol{x}) - cQ(\boldsymbol{x}) \geq 0\,.$$

Therefore, in our case, find a SoS certificate for the largest c, of the polynomial

$$f_G(\boldsymbol{x}) - c\,\frac{d}{n}\cdot\frac{1}{4}\sum_{i,j}(\boldsymbol{x}_i-\boldsymbol{x}_j)^2\,.$$

Cheeger's Inequality

Theorem (Alon and Milman [AM85] in SoS form)

For all d-regular graph $G = (V, E)$, $|V| = n$,

$$f_G(\boldsymbol{x}) - c \; \frac{d}{4n} \sum_{i,j} (\boldsymbol{x}_i - \boldsymbol{x}_j)^2,$$

has a degree-2 SoS certificate for $c = \frac{1}{2}\Phi_G^2$ (proof of the theorem shows that one can find a cut of size $\mathcal{O}(\sqrt{\Phi_G})$).
Further, given any degree p.d. μ of degree ≥ 2 such that

$$\tilde{\mathbb{E}}_\mu f_G(\boldsymbol{x}) - c \; \tilde{\mathbb{E}}_\mu \frac{d}{4n} \sum_{i,j} (\boldsymbol{x}_i - \boldsymbol{x}_j)^2 \leq 0,$$

we can find a set S with expansion $\phi_G(S) = \mathcal{O}(\sqrt{c})$.

Remarks about the Cheeger's Inequality

Remark(s)

1. *The above polynomial is non-negative for* $c \leq \Phi_G$.
2. *p.d. "pretends" there's a cut of size* c*, and we can find a cut of* $\sqrt{c}$.
3. $c < 1$ *and hence* $\sqrt{c} > c$.
4. *When* ϕ_G *is large, then we have a nice result. But when* $\phi_G = o(1)$*, then this is a bad algorithm.*
5. *If a graph is an expander (*$\Phi_G =$ *const.), then degree-*2 *SoS gives you a certificate that is an expander (losing only constant factor).*
6. *Therefore, aim is to improve on this result when* Φ_G *is small (e.g.,* $\phi_G \ll \frac{1}{\log n}$*).*

Results for Small Φ_G

Theorem (Leighton and Rao [LR99] LP Based Algorithm)

One can find in polynomial time a set S such that

$$\phi_G(S) = \mathcal{O}(\log n)\,\Phi_G\,.$$

Breakthrough Result:

Theorem (Arora, Rao, and Vazirani [ARV09])

One can find in polynomial time a set S such that

$$\phi_G(S) = \mathcal{O}\left(\sqrt{\log n}\right)\Phi_G\,.$$

Stating [ARV09] in SoS form

Theorem (Degree-4 SoS for [ARV09])

Let $G = (V, E)$ be a d-regular graph, and let $|V| = n$. Then there is a degree-4 SoS certificate for

$$f_G(\boldsymbol{x}) - \left(\frac{\Phi_G}{\mathcal{O}(\sqrt{\log n})}\right) \frac{d}{n} \sum_{i,j} (\boldsymbol{x}_i - \boldsymbol{x}_j)^2.$$

Further, for any degree-4 p.d. μ on $\{-1, 1\}^n$, we can find $S \subseteq V$, such that

$$\phi_G(S) \leq \mathcal{O}\left(\sqrt{\log n}\right) \frac{\tilde{\mathbb{E}}_\mu f_G(\boldsymbol{x})}{\tilde{\mathbb{E}}_\mu \frac{d}{n} \sum_{i,j} (\boldsymbol{x}_i - \boldsymbol{x}_j)^2}.$$

Proof

By the time we finish the proof, we will have proved

1. Poly-time algorithm for MinCut.
2. $\mathcal{O}(\log n)$-approximation of [LR99].
3. And finally, $\mathcal{O}(\sqrt{\log n})$-approximation of [ARV09].

Proof

- $\frac{1}{4}\tilde{\mathbb{E}}_\mu(\boldsymbol{x}_i - \boldsymbol{x}_j)^2 \equiv$ "pseudo-probability" that i and j are separated.
- Moreover, $0 \leq \frac{1}{4}\tilde{\mathbb{E}}_\mu(\boldsymbol{x}_i - \boldsymbol{x}_j)^2 \leq 1$.
- $D(i, j) \overset{\text{def}}{=} \frac{1}{4}\tilde{\mathbb{E}}_\mu(\boldsymbol{x}_i - \boldsymbol{x}_j)^2$.

Triangle Inequality

Claim (SoS Triangle Inequality)

For any degree-4 p.d., the following is true:

$$\tilde{\mathbb{E}}_\mu(\boldsymbol{x}_i - \boldsymbol{x}_j)^2 \leq \tilde{\mathbb{E}}_\mu(\boldsymbol{x}_i - \boldsymbol{x}_k)^2 + \tilde{\mathbb{E}}_\mu(\boldsymbol{x}_k - \boldsymbol{x}_j)^2$$

$$\equiv$$

$$D(i,j) \leq D(i,k) + D(k,j)\,.$$

Proof Sketch.

- Open up the brackets, and show that the polynomial is always non-negative using the fact that $\boldsymbol{x}_i, \boldsymbol{x}_j, \boldsymbol{x}_k \in \{-1,1\}^n$.
- Since it is non-negative of degree-2, can represent it as a degree-4 SoS.

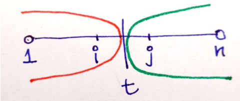
$\square$

Goal: Proof of MinCut

Proof

Let μ be a p.d. of degree-4 such that $(1/4)\tilde{\mathbb{E}}_\mu(x_s - x_t)^2 = 1$, for some vertices s, t.

1. Label the vertices $1, 2, \ldots, n$.
2. Map vertex j to point $D(1, j)$.



3. Choose $t \sim \mathrm{Unif}([0, 1])$.
4. Output $S = \{i \mid D(1, j) \leq t\}$.

Q: What's the chance that any edge (i, j) is cut?

A: $|D(1, j) - D(1, i)| \;\leq\; D(i, j)$, (by triangle ineq.)

. $D(i, j) \equiv$ chance of a "pseudo-cut".

Therefore,

$$\mathop{\mathbb{E}}_{S} f_G(S) \leq \tilde{\mathbb{E}}_\mu f_G(\boldsymbol{x}) \,. \quad \square$$

Generalizing This Procedure

Let $A \subset V$ and define $D(i, A) = \min_{j \in A} D(i, j)$.

Claim

The same analysis works if we start the line-embedding with $D(j, A)$: The distance from set A.

Proof.

Probability that edge (i, j) is cut:

$$|D(i, A) - D(j, A)| \leq D(i, j).$$

$\square$

Note: We need to argue that the cut is non-trivial.

Analysing this Procedure for Expansion

Goal: Find a distribution μ' on $\boldsymbol{x}' \in \{-1, 1\}^n$ so that

$$\frac{\mathbb{E}_{\mu'} f_G(\boldsymbol{x}')}{\frac{d}{4n}\sum_{i,j}\mathbb{E}_{\mu'}(\boldsymbol{x}'_i - \boldsymbol{x}'_j)^2}, \text{ is } \underline{\text{small}}.$$

We will reason that it is small by comparing it with

$$\tilde{m} \stackrel{\text{def}}{=} \frac{\tilde{\mathbb{E}}_{\mu} f_G(\boldsymbol{x})}{\frac{d}{4n}\sum_{i,j}\tilde{\mathbb{E}}_{\mu}(\boldsymbol{x}_i - \boldsymbol{x}_j)^2}, \text{ which is just a number}.$$

Suppose $\exists$ distribution μ' on $\boldsymbol{x}' \in \{-1, 1\}^n$

$$\frac{\mathbb{E}_{\mu'} f_G(\boldsymbol{x}')}{\frac{d}{4n}\sum_{i,j}\mathbb{E}_{\mu'}(\boldsymbol{x}'_i - \boldsymbol{x}'_j)^2} \leq \frac{\tilde{m}}{\Delta} \implies \mathop{\mathbb{E}}_{\mu'} f_G(\boldsymbol{x}') - \left(\frac{\tilde{m}}{\Delta}\right)\frac{d}{4n}\sum_{i,j}\mathop{\mathbb{E}}_{\mu'}(\boldsymbol{x}'_i - \boldsymbol{x}'_j)^2 \leq 0.$$

This implies that the normalized cut is smaller than $\left(\frac{\tilde{m}}{\Delta}\right)$, and the aim is to make this quantity as small as possible.

Therefore, we need to simultaneously make the numerator small and denominator large. We can handle the numerator (MinCut), but for denominator we need to ensure that the size of the cut is "balanced".

$$\mathop{\mathbb{E}}_{\mu'} f_G(\boldsymbol{x}') \leq \tilde{\mathbb{E}}_\mu f_G(\boldsymbol{x}) \qquad \text{(Easy: MinCut)}$$

$$\sum_{i,j} \mathop{\mathbb{E}}_{\mu'} (\boldsymbol{x}_i' - \boldsymbol{x}_j')^2 \geq \left(\frac{1}{\Delta}\right) \sum_{i,j} \tilde{\mathbb{E}}_\mu (\boldsymbol{x}_i - \boldsymbol{x}_j)^2 \quad \text{(Hard: balanced-cut)}$$

Idea: Looking at MinCut Closely

Definition (Large Δ-Separated Sets)

A, B are large Δ-separated sets if for all $i \in A$, $j \in B$, $D(i, j) \geq \Delta$, and $|A| \cdot |B| = \Omega\left(n^2\right)$.

Idea:

1. Suppose we have A, B as the *large Δ-separated sets*.
2. Do the "line-embedding" according to A.
3. Any threshold t we choose will put at least all vertices of A is one set.
4. If Δ is large-enough, then A will be on one side of cut and B on the other side $\equiv$ non-trivial cut ("balanced-cut").

Idea Cont...

Proposition

If we round according to the line-embedding $D(j, A)$, then

$$\sum_{i,j} \mathop{\mathbb{E}}_{\mu'}(\boldsymbol{x}'_i - \boldsymbol{x}'_j)^2 \geq \sum_{i \in A, j \in B} D(j, A) \geq \Delta\, |A| \cdot |B| = \Omega\left(\Delta n^2\right)\,, \tag{1}$$

$$\mathop{\mathbb{E}}_{\mu'} f_G(\boldsymbol{x}') \leq \tilde{\mathbb{E}}_{\mu} f_G(\boldsymbol{x})\,. \tag{2}$$

Note:

$$\sum_{i,j} \tilde{\mathbb{E}}_{\mu}(\boldsymbol{x}_i - \boldsymbol{x}_j)^2 \leq n^2 \implies \sum_{i,j} \mathop{\mathbb{E}}_{\mu'}(\boldsymbol{x}'_i - \boldsymbol{x}'_j)^2 \geq \Omega(\Delta) \sum_{i,j} \tilde{\mathbb{E}}_{\mu}(\boldsymbol{x}_i - \boldsymbol{x}_j)^2\,.$$

$$\Phi_G \leq \frac{\mathbb{E}_{\mu'} f_G(\boldsymbol{x}')}{\frac{d}{4n}\sum_{i,j} \mathbb{E}_{\mu'}(\boldsymbol{x}'_i - \boldsymbol{x}'_j)^2} \leq \left(\frac{1}{\Delta}\right) \frac{\tilde{\mathbb{E}}_{\mu} f_G(\boldsymbol{x})}{\frac{d}{4n}\sum_{i,j} \tilde{\mathbb{E}}_{\mu}(\boldsymbol{x}_i - \boldsymbol{x}_j)^2} \leq \left(\frac{1}{\Delta}\right) \Phi_G\,.$$

$\therefore$ (1) and (2) $\implies$ rounding with approx-ratio $\frac{1}{\Theta(\Delta)}$.

Towards large Δ-separated sets

Theorem (Global Structure Theorem [ARV09])

Let $G = (V, E)$ be a d-regular graph, μ a degree-4 p.d., and $D(i,j) = \frac{1}{4}\tilde{\mathbb{E}}_\mu(\boldsymbol{x}_i - \boldsymbol{x}_j)^2$ satisfies triangle inequality. Suppose

$$\frac{1}{n^2}\sum_{i,j} D(i,j) \geq 0.1, \qquad \left(\text{Extra Hypothesis: least expanding set is large}\right).$$

Then there exists sets A, B (can be found in polynomial time) such that $|A| \cdot |B| \geq \Omega(n^2)$, and

$$\forall i \in A, j \in B, \quad D(i,j) \geq \underbrace{\Omega\left(\frac{1}{\sqrt{\log n}}\right)}_{=\Delta}.$$

Proof of a Weaker Structure Theorem ($\Delta = 1/\Theta(\log n)$)

Assume the extra-hypothesis: $\frac{1}{n^2}\sum_{i,j} D(i,j) \geq 0.1$.
Goal: Constructing Δ-separated large sets-

1. Let $\boldsymbol{g} \sim \mathcal{N}(\mathbf{0}, \tilde{\mathbb{E}}_\mu \boldsymbol{x}\boldsymbol{x}^\top)$, (Note: $\mathbb{E}\,\boldsymbol{g}_i^2 = 1$, $\mathbb{E}\,\boldsymbol{g}_i = 0$).
2. Note $D(i,j) = \tilde{\mathbb{E}}_\mu(\boldsymbol{x}_i - \boldsymbol{x}_j)^2 = \mathbb{E}(\boldsymbol{g}_i - \boldsymbol{g}_j)^2$. So maybe thresholding on $\boldsymbol{g}_i, \boldsymbol{g}_j$ works.
3. Let $A^0 = \{i \mid \boldsymbol{g}_i \leq -1\}$, and $B^0 = \{j \mid \boldsymbol{g}_j \geq 1\}$.

Claim

$$\mathbb{E}\left|A^0\right| \cdot \left|B^0\right| \geq \Omega(n^2)\,.$$

Proof.

$\mathbb{P}\{\boldsymbol{g}_i \leq -1 \text{ and } \boldsymbol{g}_j \geq 1\} \geq c \cdot D(i,j) \implies \mathbb{E}\left|A^0\right| \cdot \left|B^0\right| \geq \Omega(n^2)$ (by extra-hypothesis). □

Constructing Δ-Separated Set

Proof of Leighton and Rao [LR99].

- Note that A^0 and B^0 are random sets. We want to understand how $D(i,j)$ behaves for $i \in A^0, j \in B^0$ (extra-hypothesis implies $D(i,j)$ well-spread on average).
- What's the failure chance (sets are not Δ separated): $\exists i \in A^0, j \in B^0$, s.t., $\mathbb{E}(\boldsymbol{g}_i - \boldsymbol{g}_j)^2 \leq \Delta$, (recall $\boldsymbol{g}_j - \boldsymbol{g}_i \geq 2$).
- Since $(\boldsymbol{g}_j - \boldsymbol{g}_i)$ is also a Gaussian, we get that $\mathbb{P}\{\boldsymbol{g}_j - \boldsymbol{g}_i \geq 2\} \leq \exp(-c/\Delta)$. For small Δ this is tiny.
- We want to find what's the probability that no such (i,j) exists. Using union bound,

$$\mathbb{P}\left\{A^0 \text{ \& } B^0 \text{ are } \Delta\text{-Separated}\right\} \geq 1 - n^2 \exp\left(\frac{-c}{\Delta}\right).$$

- Choosing $\Delta \ll \frac{1}{\log n}$, makes A^0, B^0 are large Δ-Separated sets.

□

Improving Upon Leighton and Rao [LR99]

Notation: Let H be a *short-edge* graph.

$$E(H) = \left\{(i,j) \in [n]^2 \;\middle|\; D(i,j) \leq \Delta\right\}.$$

Goal: Find $\Omega(n)$ size sets A, B that is a vertex separator in H.
Idea: Remove pairs that are "close-by" but go across A^0, B^0.
Algorithm:

1. Take A^0, B^0 as before.
2. Find a maximal matching M greedily in $E(H) \cap A^0 \times B^0$, *i.e., fix the traversal order independently of* $\mathbf{g}$*, and traverse on the edges. If it goes across* A^0, B^0*, then remove the edge and corresponding vertices in* A^0, B^0*, and continue.*
3. *Output* $A = A^0 \setminus V(M)$, $B = B^0 \setminus V(M)$.

Obervation(s):

1. A, B are Δ-Separated sets.
2. A, B are vertex-separator in H.

To Prove: A, B are large

Remarks:

1. M is a random quantity ($\because A^0, B^0$ dependent on $\boldsymbol{g}$).
2. M has "directed" edges, i.e., if $(i, j) \in M$, then $\boldsymbol{g}_j - \boldsymbol{g}_i \geq 2$.
3. Since $\boldsymbol{g}$ has $\mathbf{0}$-mean, we get that $\boldsymbol{g}, -\boldsymbol{g}$ have the same distribution.

   . If we instead worked with $-\boldsymbol{g}$ then A^0, B^0 would be interchanged, which implies

$$\mathbb{P}\{\text{vertex } i \text{ has an incoming edge in } M\} = \mathbb{P}\{\text{vertex } i \text{ has an outgoing edge in } M\}$$

4. If $|A| \cdot |B| = o(n^2)$, then $\mathbb{E}\,|M| \geq \Omega(n)$: only way we fail, is if we remove too many edges from A^0, B^0.

To Prove: A, B are large

We will show that $\mathbb{E}\,|M| \geq \Omega(n)$ is a rare event.

Intuition:

1. We already saw that for each $(i,j) \in M$, $\mathbb{E}(\boldsymbol{g}_i - \boldsymbol{g}_j)^2 \leq \Delta$, and $\boldsymbol{g}_j - \boldsymbol{g}_i \geq 2$: and probability of this happening was "rare".
2. One way many of these rare-events can occur together is if suppose one of the $\boldsymbol{g}_j \geq \mathcal{O}\left(\sqrt{\log n}\right)$. Then constant fraction of edges attached to it will gave small values of $\boldsymbol{g}_i$. Therefore, many rare-events happen due to one vertex being huge.
3. But, since we took a matching M, a single vertex contributes only one edge.
4. Therefore, in some sense, these give us "less" correlated set of rare-events, all occurring together, the probability of which should be small.

We will now formally analyze the expected size of the matching.

A, B large $\implies$ [ARV09]

Lemma (Large Matching $\equiv$ Large Expected Max of Gaussians)

$$\frac{\Omega(1)}{\Delta} \cdot \left(\frac{\mathbb{E}\,|M|}{n}\right)^3 \leq \mathbb{E} \max_{i,j\in[n]} \boldsymbol{g}_j - \boldsymbol{g}_i \leq \mathcal{O}\left(\sqrt{\log n}\right) .$$

Proof of: Lemma $\implies$ [ARV09].

The lemma above implies that $\mathbb{E}\,|M| \leq n\left(\Delta\sqrt{\log n}\right)^{1/3}$. This gives us that if we want to ensure that $|A| \cdot |B| = \Omega(n^2)$, choosing $\Delta \ll \frac{1}{\sqrt{\log n}}$ suffices. $\square$

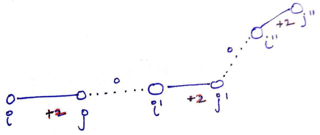
Towards Proof of Lemma

Notation:

- $H^k(i) \stackrel{\text{def}}{=}$ vertices within k steps from i in H.
- $Y_i^k \stackrel{\text{def}}{=} \max_{j \in H^k(i)} \boldsymbol{g}_j - \boldsymbol{g}_i$.

(very) **Rough Idea:**

- $(i, j) \in M \implies \boldsymbol{g}_j - \boldsymbol{g}_i \geq 2$.
- Form a chain $H^k(i)$
  (chain of length k).



- Hope that constant fraction of edges in chain belong to the matching M.
- Then these long paths could imply that the maximum is large, e.g., $(\boldsymbol{g}_j - \boldsymbol{g}_i), (\boldsymbol{g}_{j'} - \boldsymbol{g}_{i'}), (\boldsymbol{g}_{j''} - \boldsymbol{g}_{i''}), \cdots \geq 2$

Formalizing the Rough Idea

Definition

$$\Phi(k) = \sum_{i=1}^{n} \mathbb{E}\, Y_i^k\, .$$

Notice,

$$\frac{\Phi(k)}{n} \leq \mathbb{E} \max_{i,j \in [n]} \boldsymbol{g}_j - \boldsymbol{g}_i .$$

Claim (Potential Functions Grows with k)

$$\Phi(k+1) - \Phi(k) \geq \mathbb{E}\,|M| - \mathcal{O}(n) \max_{\substack{i \in [n], \\ j \in H^{k+1}(i)}} \left(\mathbb{E}(\boldsymbol{g}_j - \boldsymbol{g}_i)^2\right)^{1/2} .$$

Claim $\Longrightarrow$ Lemma

Proof

$$\max_{i\in[n],j\in H^{k+1}(i)} \mathbb{E}(\boldsymbol{g}_j - \boldsymbol{g}_i)^2 \leq (k+1)\Delta\,.$$

Therefore, RHS of the Claim gives that

$$\Phi(k+1) - \Phi(k) \geq \mathbb{E}\,|M| - \mathcal{O}(n)\,\sqrt{k\Delta}\,.$$

Let $k_0 = c\left(\frac{\mathbb{E}|M|}{n}\right)^2/\Delta$. We choose this k_0 because, $\sqrt{k_0\Delta} = c'\,\mathbb{E}\,|M|$. Therefore $\forall k \leq k_0$,

$$\Phi(k+1) - \Phi(k) \geq \frac{1}{2}\,\mathbb{E}\,|M|\;.$$

(we will use this inductively).

Proof Cont...

Then,

$$\begin{aligned}
\max_{i,j\in[n]} \boldsymbol{g}_j - \boldsymbol{g}_i &\geq \frac{\Phi(k_0)}{n} \\
&\geq \frac{1}{2} k_0 \frac{\mathbb{E}\,|M|}{n} && \text{(Inductively)} \\
&= \frac{\Theta(1)}{\Delta} \left(\frac{\mathbb{E}\,|M|}{n}\right)^3 .
\end{aligned}$$

$\square$

All we are left with is proof of the Claim:

$$\Phi(k+1) - \Phi(k) \geq \mathbb{E}\,|M| - \mathcal{O}(n)\sqrt{k\Delta}\,.$$

Proof of Claim

Proof

- $Y_i^{k+1} \geq Y_j^k + \boldsymbol{g}_j - \boldsymbol{g}_i$, for all $(i,j) \in E(H)$.

- If $(i,j) \in M$, then $\boldsymbol{g}_j - \boldsymbol{g}_i \geq 2 \implies Y_i^{k+1} \geq Y_j^k + 2$ (huge matching will help us increase it).
- Let $N \in [n] \times [n]$ be an arbitrary matching of vertices not in M.
- $\forall (i,j) \in N$, $\frac{1}{2} Y_i^{k+1} + \frac{1}{2} Y_j^{k+1} \geq \frac{1}{2} Y_i^k + \frac{1}{2} Y_j^k$.
- $\forall (i,j) \in M$, $Y_i^{k+1} \geq Y_j^k + 2$.

Proof of Claim

Proof Cont...

- Summing up: $\sum_{i=1}^{n} Y_i^{k+1} \cdot L_i \geq \sum_{j=1}^{n} Y_j^{k} \cdot R_j + 2\,|M|$, where

$$L_i/(R_i) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } i \text{ has outgoing/ (incoming) edge in } M \\ 0, & \text{if } i \text{ has incoming/ (outgoing) edge in } M \\ \frac{1}{2}, & \text{if } M \text{ is unmatched.} \end{cases}$$

- Summing up over edges in M and we gather $+2$ for every matched edge.
- Recall i has equal chance of an incoming and outgoing edge in M, therefore,

$$\mathbb{E}\, L_i = \mathbb{E}\, R_i = 1/2.$$

Proof of Claim

Proof Cont...

- We now want to take expectation of $\sum_{i=1}^{n} Y_i^{k+1} \cdot L_i \geq \sum_{j=1}^{n} Y_j^k \cdot R_j + 2|M|$. We shall use the following:

$$\begin{aligned}
\left|\mathbb{E}\left[Y_i^{k+1} \cdot L_i\right] - \mathbb{E}\, Y_i^{k+1} \cdot \mathbb{E}\, L_i\right| &= \left|\mathbb{E}\left[(Y_i^{k+1} - \mathbb{E}\, Y_i^{k+1})(L_i - \mathbb{E}\, L_i)\right]\right| \\
&\leq \sqrt{\mathbb{E}(Y_i^{k+1} - \mathbb{E}\, Y_i^{k+1})^2} \\
&\quad \cdot \sqrt{\mathbb{E}(L_i - \mathbb{E}\, L_i)^2} \\
&= \sqrt{\operatorname{Var}\left(Y_i^{k+1}\right)} \cdot \underbrace{\sqrt{\operatorname{Var}\left(L_i\right)}}_{\leq 1}.
\end{aligned}$$

Proof of Claim

Proof Cont...

- Using max of Gaussians result, we get

$$\mathrm{Var}\left(Y_i^{k+1}\right) \leq \mathcal{O}(1) \max_{j \in H_i^{k+1}} \mathrm{Var}\left(\boldsymbol{g}_j - \boldsymbol{g}_i\right) = \mathcal{O}(1) \max_{j \in H_i^{k+1}} \mathbb{E}(\boldsymbol{g}_i - \boldsymbol{g}_j)^2 \,.$$

- So,

$$\left| \mathbb{E}\, Y_i^{k+1} \cdot L_i - \mathbb{E}\, Y_i^{k+1} \cdot \mathbb{E}\, L_i \right| \leq \mathcal{O}(1) \sqrt{\max_{j \in H^{k+1}(i)} \mathbb{E}(\boldsymbol{g}_j - \boldsymbol{g}_i)^2}\,, \;\; \&,$$

$$\left| \mathbb{E}\, Y_j^{k} \cdot R_j - \mathbb{E}\, Y_j^{k} \cdot \mathbb{E}\, R_j \right| \leq \mathcal{O}(1) \sqrt{\max_{i \in H^{k}(j)} \mathbb{E}(\boldsymbol{g}_i - \boldsymbol{g}_j)^2}\,.$$

- Using this, we can now replace expectation of product by product of expectations.

Proof of Claim

Proof Cont...

- Recall: $\sum_{i=1}^{n} Y_i^{k+1} \cdot L_i \geq \sum_{j=1}^{n} Y_j^{k} \cdot R_j + 2\,|M|$.
- Now, taking expectations (product of expectation), we get $\sum_{i=1}^{n} \mathbb{E}\, Y_i^{k+1} \cdot \mathbb{E}\, L_i = \frac{1}{2}\Phi(k+1)$, $\sum_{j=1}^{n} \mathbb{E}\, Y_j^{k} \cdot \mathbb{E}\, R_j = \frac{1}{2}\Phi(k)$.
- Therefore, we get

$$\Phi(k+1) - \Phi(k) \geq 4\,\mathbb{E}\,|M| - \mathcal{O}(n) \cdot \max_{\substack{i \in [n], \\ j \in H^{k+1}(i)}} \sqrt{\mathbb{E}(\boldsymbol{g}_j - \boldsymbol{g}_i)^2}\,.$$

$\square$

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