Sum of Squares: Part 3

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Goal: *Approximating Graph Expansion.*

Setting

Consider the d-regular graph $G = (V, E)$. Let $|V| = n$.

- Throughout we will consider a d-regular graph (easier to work with).
- Recall the cut polynomial $f_G(\boldsymbol{x}) = \frac{1}{4} \sum_{(i,j) \in E} (\boldsymbol{x}_i - \boldsymbol{x}_j)^2$.
- Maxcut: $\max_{\boldsymbol{x} \in \{-1,1\}^n} f_G(\boldsymbol{x})$ - 0.878-Approx Algo.
- MinCut: $\min_{\boldsymbol{x} \in \{-1,1\}^n} f_G(\boldsymbol{x})$ - 1-Approx Algo.

Today: We will look at "minimum-normalized-cut": Called *expansion*.

Normalized Cut

Definition (Normalized Size of a Cut)

$$\phi_G(S) \stackrel{\text{def}}{=} \frac{\left|E(S, \overline{S})\right|}{\frac{d}{n}\left|S\right|\left(n - \left|S\right|\right)} .$$

Compare the size of the cut defined by S to the size of the cut defined by S in a random-graph of same average-degree.

Definition (Expansion of the Graph)

$$\Phi_G \stackrel{\text{def}}{=} \min_{S \subset V,\ 0 < |S| < n} \phi_G(S) .$$

Intuition: Start a random walk from a random vertex in S. What is the chance that it goes out of S in one step $\equiv \phi(S)$. Therefore, Φ_G calculates how "well-connected" the graph is.

Today's Goal

Given a d-regular graph $G = (V, E)$, compute or approximate Φ_G.

Remark(s)

1. *Computing Φ_G is NP-Hard.*
2. *Chawla et al. [Cha+05]: UGC $\implies$ no constant-factor approx for Φ_G.*
3. *Random Cut S:*

$$\mathbb{E}_S \phi_G(S) \geq \frac{1}{2},$$

gives no constant approx because this is a minimization problem.

Expansion in Polynomial Form

Recall

$$\phi_G(S) = \frac{\left|E(S,\overline{S})\right|}{\frac{d}{n}\left|S\right|(n-\left|S\right|)}, \quad \text{write this in polynomial form.}$$

$$\frac{\left|E(S,\overline{S})\right|}{\frac{d}{n}\left|S\right|(n-\left|S\right|)} = \frac{f_G(\boldsymbol{x})}{\frac{d}{n}\frac{1}{4}\sum_{i,j}(\boldsymbol{x}_i-\boldsymbol{x}_j)^2}, \quad (S = \{i|\boldsymbol{x}_i = 1\}).$$

Suppose that,

$$\min_{\boldsymbol{x}\in\{-1,1\}^n} \frac{P(\boldsymbol{x})}{Q(\boldsymbol{x})} = c \implies P(\boldsymbol{x}) - cQ(\boldsymbol{x}) \geq 0\,.$$

Therefore, in our case, find a SoS certificate for the <u>largest</u> c, of the polynomial

$$f_G(\boldsymbol{x}) - c\,\frac{d}{n}\cdot\frac{1}{4}\sum_{i,j}(\boldsymbol{x}_i-\boldsymbol{x}_j)^2\,.$$

Cheeger's Inequality

Theorem (Alon and Milman [AM85] in SoS form)

For all d-regular graph $G = (V, E)$, $|V| = n$,

$$f_G(\boldsymbol{x}) - c\ \frac{d}{4n} \sum_{i,j} (\boldsymbol{x}_i - \boldsymbol{x}_j)^2,$$

has a degree-2 SoS certificate for $c = \frac{1}{2}\Phi_G^2$.
Further, given any degree p.d. μ of degree ≥ 2 such that

$$\tilde{\mathbb{E}}_\mu f_G(\boldsymbol{x}) - c\ \tilde{\mathbb{E}}_\mu \frac{d}{4n} \sum_{i,j} (\boldsymbol{x}_i - \boldsymbol{x}_j)^2 \geq 0,$$

we can find a set S with expansion $\phi_G(S) = \mathcal{O}(\sqrt{c})$.

Remarks about the Cheeger's Inequality

Remark(s)

1. *The above polynomial is non-negative for* $c \leq \Phi_G$.
2. *p.d. "pretends" there's a cut of size* c*, and we can find a cut of* $\sqrt{c}$.
3. $c < 1$ *and hence* $\sqrt{c} > c$.
4. *When* ϕ_G *is large, then we have a nice result. But when* $\phi_G = o(1)$*, then this is a bad algorithm.*
5. *If a graph is an expander (*$\Phi_G =$ *const.), then degree-*2 *SoS gives you a certificate that is an expander (losing only constant factor).*
6. *Therefore, aim is to improve on this result when* Φ_G *is small (e.g.,* $\phi_G << \frac{1}{\log n}$*).*

Results for Small Φ_G

Theorem (Leighton and Rao [LR99] LP Based Algorithm)

One can find in polynomial time a set S such that

$$\phi_G(S) = \mathcal{O}(\log n)\, \Phi_G \,.$$

Breakthrough Result:

Theorem (Arora, Rao, and Vazirani [ARV09])

One can find in polynomial time a set S such that

$$\phi_G(S) = \mathcal{O}\left(\sqrt{\log n}\right) \Phi_G \,.$$

Stating [ARV09] in SoS form

Theorem (Degree-4 SoS for [ARV09])

Let $G = (V, E)$ be a d-regular graph, and let $|V| = n$. Then there is a degree-4 SoS certificate for

$$f_G(\boldsymbol{x}) - \left(\frac{\Phi_G}{\mathcal{O}(\sqrt{\log n})}\right) \frac{d}{n} \sum_{i,j} (\boldsymbol{x}_i - \boldsymbol{x}_j)^2.$$

Further, for any degree-4 p.d. μ on $\{-1, 1\}^n$, we can find $S \subseteq V$, such that

$$\phi_G(S) \leq \mathcal{O}\left(\sqrt{\log n}\right) \frac{\tilde{\mathbb{E}}_\mu f_G(\boldsymbol{x})}{\tilde{\mathbb{E}}_\mu \frac{d}{n} \sum_{i,j} (\boldsymbol{x}_i - \boldsymbol{x}_j)^2}.$$

Proof

By the time we finish the proof, we will have proved

1. Poly-time algorithm for MinCut.
2. $\mathcal{O}(\log n)$-approximation of [LR99].
3. And finally, $\mathcal{O}(\sqrt{\log n})$-approximation of [ARV09].

Proof

- $\frac{1}{4}\tilde{\mathbb{E}}_\mu(\boldsymbol{x}_i - \boldsymbol{x}_j)^2 \equiv$ "pseudo-probability" that i and j are separated.
- Moreover, $0 \leq \frac{1}{4}\tilde{\mathbb{E}}_\mu(\boldsymbol{x}_i - \boldsymbol{x}_j)^2 \leq 1$.
- $D(i,j) \stackrel{\text{def}}{=} \frac{1}{4}\tilde{\mathbb{E}}_\mu(\boldsymbol{x}_i - \boldsymbol{x}_j)^2$.

Triangle Inequality

Claim (SoS Triangle Inequality)

For any degree-4 p.d., the following is true:

$$\tilde{\mathbb{E}}_\mu(\boldsymbol{x}_i - \boldsymbol{x}_j)^2 \leq \tilde{\mathbb{E}}_\mu(\boldsymbol{x}_i - \boldsymbol{x}_k)^2 + \tilde{\mathbb{E}}_\mu(\boldsymbol{x}_k - \boldsymbol{x}_j)^2$$
$$\equiv$$
$$D(i,j) \leq D(i,k) + D(k,j)\,.$$

Proof Sketch.

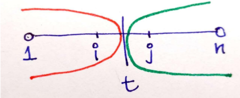
- Open up the brackets, and show that the polynomial is always non-negative using the fact that $\boldsymbol{x}_i, \boldsymbol{x}_j, \boldsymbol{x}_k \in \{-1, 1\}^n$.
- Since it is non-negative of degree-2, can represent it as a degree-4 SoS.

□

Goal: Proof of MinCut

Proof

1. Label the vertices $1, 2, \ldots, n$.
2. Map vertex j to point $D(1, j)$.



3. Choose $t \sim \mathrm{Unif}([0, \max_j D(1, j)])$.
4. Output $\color{red}{S} = \{i \mid D(1, j) \leq t\}$.

Q: What's the chance that any edge (i, j) is cut?

A: $|D(1, j) - D(1, i)| \;\leq\; D(i, j)$, (by triangle ineq.)

. $D(i, j) \equiv$ chance of a "pseudo-cut".

Therefore,

$$\mathop{\mathbb{E}}_{\color{red}{S}} f_G(\color{red}{S}) \leq \tilde{\mathbb{E}}_\mu f_G(\boldsymbol{x}) \,. \quad \square$$

Generalizing This Procedure

Let $A \subset V$ and define $D(i, A) = \min_{j \in A} D(i, j)$.

Claim

The same analysis works if we start the line-embedding with $D(j, A)$: The distance from set A.

Proof.

Probability that edge (i, j) is cut:

$$|D(i, A) - D(j, A)| \leq D(i, j).$$

$\square$

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