Chapter 1

Linear Programming

Introduction.

- Linear programming is a mathematical modeling technique to optimize the usage of limited resources.
- Applications of Linear programming exist in the areas of military, industry, agriculture, transportation, economics, health system, etc.
- The usefulness of the technique is enhanced by the availability of highly efficient computer codes. (Developed by *George Dantzig*in 1940's)

1.1 The General Form of a Linear Programming Model

Maximize $f(x) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$$
$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m$$
$$x_1, x_2, \dots, x_n \ge 0$$

In Matrix form

Maximize $f(x) = c^Tx$

Subject to $Ax \le b$

$x \ge 0$

Hear

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n, \quad c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n, b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m, \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix},$$

wherex_i is a decision variable and a_{ji}, b_j, c_i are constants, for each $i = 1,2,\dots,n$ and $j = 1,2,\dots,m$

Several variations such as minimizing the objective function instead of maximizing and equalities in constrains changing to inequalities are possible. Thus the linear program is the special case of general optimization problem where objective function $f: \mathbb{R}^n \to \mathbb{R}$, is linear and feasible set $S = \{x: Ax = b\}$ can be described in terms of linear equalities/ inequalities.

1.2 Formulating Linear Programming Model

Mathematical modelling is the bridge that converts the real life problem into a mathematical problem. Steps to be applied when formulating a linear programming model of a real world situation are given below.

Step 1:

Study the given situation, find the key decision to be made. Hence, identify the decision variables of the problem (unknown variables to be determined). Usuallydenoted by $x_1, x_2, \dots, x_n$.

Step 2:

Identify the objective function of the LP problem and represent it as a linear function of the decision variables, which is to be maximized or minimized.

Step 3:

Identify all the restrictions or constraints in the problem and express them aslinear equations or inequalities which are linear functions of unknown variables.

Step 4:

Add non-negativity restrictions.

One example in formulating a linear programming model of a real world situation is given below.

Exercise 1

Product Mix Problem

A company produces drugs I and II using machines M_1 and M_2. 1 ton of drugs I requires 1 hour processing on M_1 and 2 hours on M_2. 1 ton of drug II requires 3 hours of processing on M_1 and 1 hour on M_2. 9 hours of processing on M_1 and 8 hours on M_2 are available. Each ton of drug produced (of either type) yields a profit of \$ 1 million. To maximize its profit, how much of each drug should the company produce?

Formulate a linear programming model for this problem.

Exercise 2

Diet Problem

A farmer wishes to choose the least cost diet that will meet the nutritional requirement of his pigs. Pigs require 4, 8, 9 units respectively of nutrients A, B, C per week. There are 4 verities of food available. The nutritional contents of food per Kg are shown below.

Nutrient	Food				Requirement per week
	F1	F2	F3	F4	
A	1	2	1	4	4
B	1	3	0	2	8
C	4	2	6	1	9

If the food cost \$ 5, 6, 7, 9 per weekly respectively, find an optimal weekly diet for the pigs.

Decision variables

x_j =Units of food $j, j = 1,2,3,4$

LP problem

Minimize $5x_1+6x_2 + 7x_3 + 9x_4$
Subject to $x_1+2x_2 + x_3 + 4x_4 \geq 4$
$x_1+3x_2 + 2x_4 \geq 8$
$4x_1+2x_2 + 6x_3 + x_4 \geq 9$
$x_1,x_2, x_3, x_4 \geq 0$
In matrix notation, we can write this as

$$\text{Minimize } (5 \quad 6 \quad 7 \; 9)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\text{Subject to} \begin{pmatrix} 1 & 2 & 14 \\ 1 & 3 & 02 \\ 4 & 2 & 61 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \geq \begin{pmatrix} 4 \\ 8 \\ 9 \end{pmatrix}$$

$$X \geq 0$$

$$\text{Let} = \begin{pmatrix} 5 \\ 6 \\ 7 \\ 9 \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, b = \begin{pmatrix} 4 \\ 8 \\ 9 \end{pmatrix} \text{ and } A = \begin{pmatrix} 1 & 2 & 14 \\ 1 & 3 & 02 \\ 4 & 2 & 61 \end{pmatrix}$$

Matrix form

$$Minimize c^T X$$
$$Subject\ to\ AX \geq b$$
$$X \geq 0$$

Optimization problem

A problem in which we are asked to find the best or optimal solution subject to given conditions is called an optimization problem.

Linear programming problem

An optimization problem in which the objective function can be expressed as a linear function of the variables and in which the constraints can be expressed as linear equations or linear inequalities is called a linear programming problem.

Decision variables

A decision variable is used to represent the level of achievement of a particular course of action. The solution of the linear programming problem will provide the optimum value for each and every decision variable of the model. In the Example 1 (Product Mix Problem) decision variables ($x_1$andx_2) are the production amounts (number of tons) of the products (drug I and drug II).

Objective function coefficient

It is a constant representing the profit per unit or cost per unit of carrying out an activity. Let $c_1, c_2, \dots, c_n$ be the profit (cost) per unit of products $p_1, p_2, \dots, p_n$. Then from our example $c_1$profit per unit of the productp_1 (1 ton of drug I) is \$1 million.

Objective function

It is an expression representing the total profit or cost of carrying out a set of activities at same levels.
Objective function is either maximization type or minimization type. The benefit related objective function will come under maximization type whereas the cost related objective function will come under minimization type. General format of the objective function is,
Maximize or Minimize $f(x) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$
From example Maximize $Z = x_1 + x_2$

Technological coefficient (a_{ij})

The technological coefficient, a_{ij} is the amount of resource i required for the activity j, where i varies from 1 to m and j varies from 1 to n. The generalized format of the technological coefficient matrix is $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$

In our example A=$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$

Resource availability (b_i)

Constant $b_i$amount of resource i available during the planning period.

General format of the resource availability matrix (b) is b=$\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

Machine hours available b=$\begin{pmatrix} 9 \\ 8 \end{pmatrix}$

Set of constraints

A constraint is a kind of restriction on the total amount of particular resource required to carry out the activity at various level.

$x_1 + 3x_2 \leq 9 \; (M_1\; processing)$
$2x_1 + x_2 \leq 8 \; (M_2\; processing)$

Non- negativity constraints

Each and every decision variable in the LP model is non-negative variables. This condition is represented as, $x_1, x_2, \dots, x_n \geq 0$

Product mix problem $x_1, x_2 \geq 0$

Complete Model of the product mix problem

P1

$$\begin{gathered} \text{Maximize } Z = x_1 + x_2 \\ x_1 + 3x_2 \leq 9 \; (M_1\; processing) \\ 2x_1 + x_2 \leq 8 \; (M_2\; processing) \\ x_1, x_2 \geq 0 \end{gathered}$$

1.3 Extra Variables

1.3.1 Slack Variables

We can rewrite the above product mix problem as:

P2

$$\begin{gathered} \text{Maximize } Z = x_1 + x_2 \\ x_1 + 3x_2 + x_3 = 9 \; (M_1\; processing) \\ 2x_1 + x_2 + x_4 = 8 \; (M_2\; processing) \\ x_1, x_2, x_3, x_4 \geq 0 \end{gathered}$$

The variables $x_3$and x_4 are called *slack variables*; x_3 is the unused ,i.e. slack, time on machine M_1. $x_4$is the unused ,i.e. slack, time on machine M_2.

This new problem is equivalent to P1, but he constrains are of the form

$$Ax = b$$
$$x \geq 0$$

1.3.2 Free Variables

Suppose some component of x is unrestricted in sign, e.g. the $x_1 \geq 0$ constraint is omitted. x_1 is then called a *free variable*. We can eliminate this free variable by setting

$$x_1 = u_1 - v_1, \text{ where } u_1, v_1 \geq 0$$

1.3.3 Surplus Variables

To convert "$\geq$" constraints to standard form, a *surplus variable* is subtracted on the left hand side of the constraint. That is number of items produced in excess of the requirement. Non-negativity constraints are also required for surplus variables.

1.4 Standard Form of a Linear Program Model

To solve an LPP algebraically, we first put it in the standard form. This means the objective is maximized, all decision variables are nonnegative and all constraints (other than the non-negativity restrictions) are equations with nonnegative RHS.

Maximize the objective function $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$
subject to problem constraints of the form $c_1x_1 + c_2x_2 + \cdots + c_nx_n = b,\ b \geq 0$
with non-negative constraints

$$x_1, x_2, \dots, x_n \geq 0$$

- If the problem is $min\ Z$, convert it to $max - z$.
  (Minimizing $f(x)$ is equivalent to maximizing $-f(x)$)
- If a constraint is $c_1x_1 + c_2x_2 + \cdots + c_nx_n \leq b$, convert it to an equality constraint by adding a nonnegative slack variable s. The resulting constraint is
  $c_1x_1 + c_2x_2 + \cdots + c_nx_n + s = b$
- If a constraint is $c_1x_1 + c_2x_2 + \cdots + c_nx_n \geq b$, convert it to an equality constraint by subtracting a nonnegative surplus variable s. The resulting constraint is
  $c_1x_1 + c_2x_2 + \cdots + c_nx_n - s = b$
- If some variable x_i is unrestricted in sign (free), replace it everywhere in the formulation by $u_i - v_i$, where $u_i, v_i \geq 0$.

Exercise 3
Convert the following linear programming models in to standard form.

P1 Maximize $f = 4x_1 + 2x_2$
Subject to

$$\begin{aligned} x_1 - 2x_2 &\geq 2 \\ x_1 + 2x_2 &= 8 \\ x_1 - x_2 &\leq 10 \end{aligned}$$

$x_1 \geq 0$, x_2-*unrestricted in sign.*

P2 Minimize $f = 4x_1 - 2x_2$
Subject to

$$\begin{aligned} x_1 - 2x_2 &\geq 2 \\ x_1 + 2x_2 &= 8 \\ 2 \leq x_2 &\leq 4 \\ x_1 &\geq 0 \end{aligned}$$

1.5 Assumptions of Linear Programming

1. Divisibility

Decision variables in a linear programming model are allowed to have any values, including *noninteger*values, that satisfy the constraints. Since each decision variable represents the level of some activity, it is being assumed that the activity can be run at fractional level.

2. Proportionality

The contribution of each activity to the value of the objective function is proportional to the level of the activity x_j. Similarly, the contribution of each activity to the left hand side of each constraint is proportional to the level of the activity x_j.

3. Additivity

Every function in a linear programming model is the sum of the individual contributions of the respective activities.

4. Certainty

The value assigned to each parameter of the linear programming model is assumed to be known constraint.

Chapter 2

The Graphical Method

Introduction.

- The next step after formulation is to solve the problem mathematically to obtain the best possible (optimal) solution.
- This lesson presents graphical solution approaches for solving any LP problem with only two decision variables. Though in practice such small problems are usually not encountered, the graphical procedure is presented to illustrate some of the basic concepts used in solving large Lp problems.

2.1 The graphical procedure includes two steps:

1. Determination of the feasible solution space (or feasible region).
2. Determination of the optimum solution from among all the feasible points in the solution space.

- A *feasible solution* is a solution for which all constraints are satisfied.
- An *infeasible solution* is a solution for which at least one constraint is violated .
- The *feasible region* is the collection of all feasible solutions.
- It is possible for a problem to have no feasible solutions.
- An *optimal solution* is a feasible solution that has the best objective function value (max / min).

Example 1:The Product Mix Problem

Flair Furniture Company produces tables and chairs. Each product must go through a two stage manufacturing process; carpentry and painting. The data for the above problem are given below:

	Tables (per table)	Chairs (per chair)	Hours Available
Profit Contribution	$7 (Rs. 700)	$5 (Rs. 500)	
Carpentry	3 hrs	4 hrs	2400
Painting	2 hrs	1 hr	1000

Other Limitations:

- Make no more than 450 chairs
- Make at least 100 tables

$$\text{Max } 7T + 5C \text{ (profit)}$$

Subject to the constraints:

$$3T + 4C \leq 2400 \text{ (carpentry hrs)}$$

$$2T + 1C \leq 1000 \text{ (painting hrs)}$$

$$C \leq 450 \text{(max number of chairs)}$$

$$T \geq 100 \text{ (min number of tables)}$$

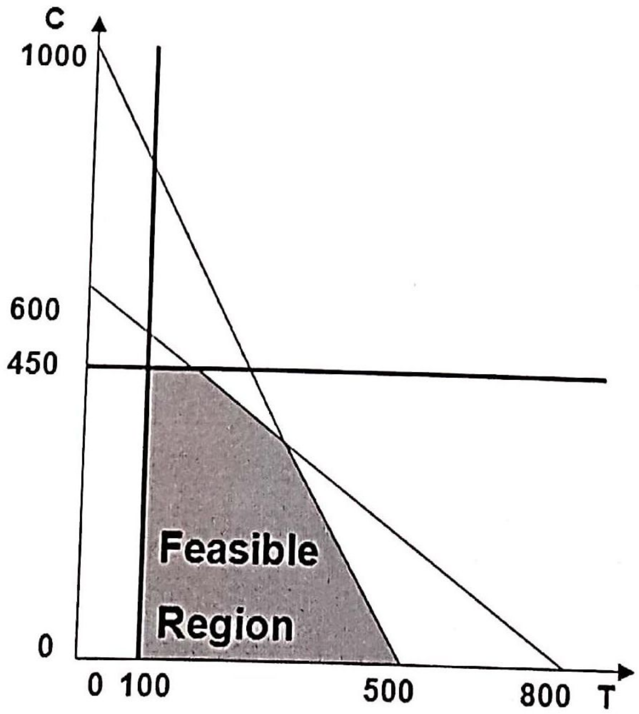
$$T, C \geq 0 \quad \text{(nonnegativity)}$$

Where T= Number of tables to make

C= Number of chairs to make

Graphical Solution:

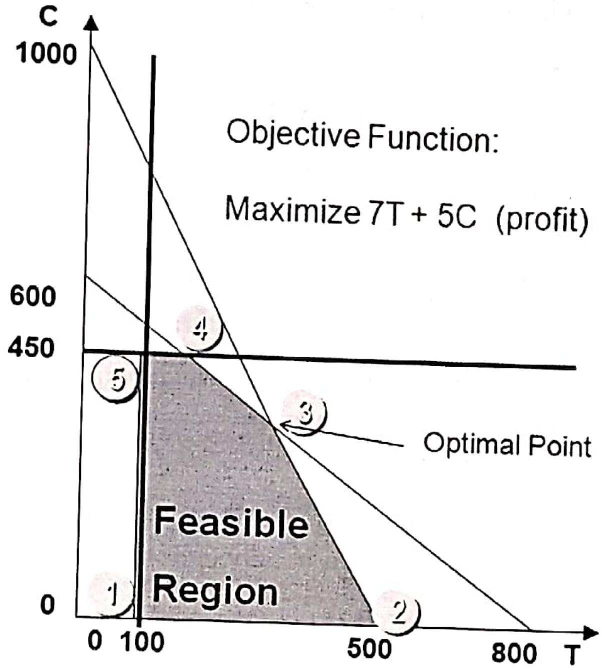
- The *non-negativity* restrictions imply that values of the variables C and T can lie only in the *first quadrant.*
- The effect of the remaining constraints can now be added.



To Find Optimal Solution:

2.2 Method 1: The corner point solution Method

- The corners or vertices of the feasible region are referred to as the extreme points.
- An optimal solution to an LP problem can be found at an extreme point of the feasible region.
- When looking for the optimal solution, you do not have to evaluate all feasible solution points.
- You have to consider only the extreme points of the feasible region.



Point (T ,C)	Profit ($)
1. (100,0)	700
2. (500, 0)	3500
3. (320, 360)	4040
4. (200, 450)	3650
5. (100, 450)	2950

Hence the solution to the problem is T=320, C=360 and maximum Profit = $ 4040.

2.3 Method 2: The Iso-profit (Cost) Function Line Method

According to this method, the optimal solution is found by using the **slope** of the objective function line. An **iso-profit** (or cost) line is a collection of points that designate solutions with the **same value** of objective function.

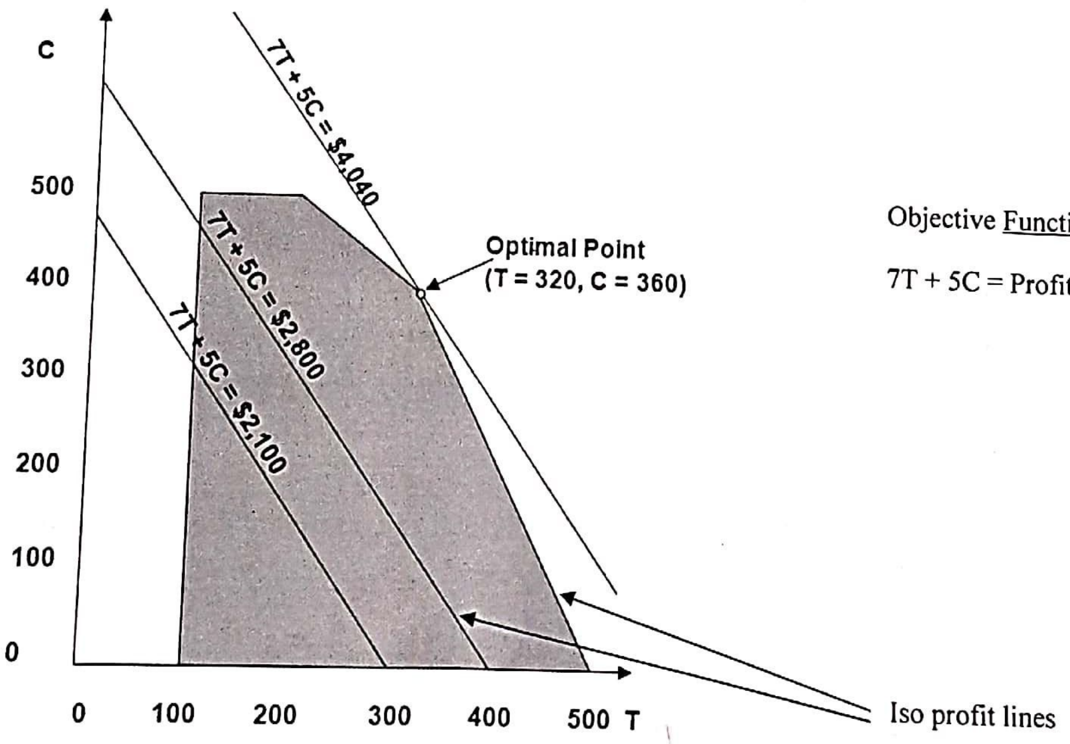
The steps of iso-profit (cost) function method are as follows:

Step 1: Identify the feasible region and extreme points of the feasible region.

Step 2: Draw an iso-profit (iso-cost) line for an arbitrary but small value of the objective function without violating any of the constraints of the given LP problem.

Step 3: Move iso-profit (iso-cost) lines parallel in the direction of increasing (decreasing) objective function values.

- The farthest iso-profit line may intersect only at one corner point providing a single optimal solution.
- Also, this line may coincide with one of the boundary lines of the feasible area. Then at least two optimal solutions must lie on two adjoining corners and others will lie on the boundary connecting them.
- However, if the iso-profit line goes on without limit from the constraints, then an unbounded solution would exist. This usually indicates that an error has been made in formulating the LP model.



2.4 Special Situations in Linear Programming

So far, we have seen that the optimal solution of any linear programming problem occurs at an extreme point of the feasible region and the solution is unique. In this section, we encounter three types of LPs that do not have unique optimal solutions.

- Some LPs have an infinite number of optimal solutions (alternate or multiple optimal solutions).
- Some LPs have no feasible solutions (infeasible LPs).
- Some LPs are unbounded: There are points in the feasible region with arbitrary large (in a max problem) z-values.

1. **Redundant Constraints**

A redundant constraint is one that does affect the feasible region and thus redundancy of any constraint does not cause any difficulty in solving an LP problem graphically.

Example: $x \leq 10$

$x \leq 12$

The second constraint is redundant because it is *less* restrictive.

2. **Infeasibility** – when no feasible solution exists (there is no feasible region)

Example: $x \leq 10$

$x \geq 15$

3. **Alternate Optimal Solutions** – when there is more than one optimal solution

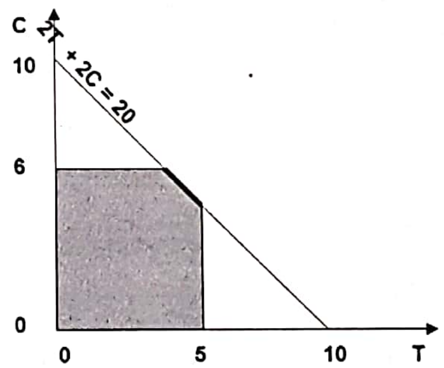
Max $2T + 2C$

Subject to: $T + C \leq 10$

$T \leq 5$

$C \leq 6$

$T, C \geq 0$

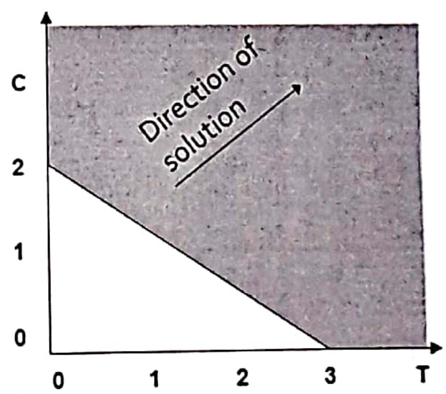


4. **Unbounded Solutions** – when nothing prevents the solution from becoming infinitely large

Max $2T + 2C$

Subject to: $2T + 3C > 6$

$T, C > 0$



Chapter 3

The Simplex Method

Introduction.

- The Simplex Method, developed by Prof. George Dantzig, can be used to solve any LP problem involving any number of variables and constraints.
- The simplex method is the basic building block for all other methods. We can solve a Linear programming problem in standard form by using Simplex method.

3.1The simplex algorithm proceeds as follows:

Step 1: Convert the LP to standard form.

Step 2: Obtain a basic feasible solution (bfs) (if possible) from the standard form.

Basic feasible Solution

If the system has a unique solution, this solution is called a basic solution. If further if it is feasible, it is called a Basic Feasible Solution (BFS)

Step 3: Determine whether the current bfs is optimal.

Optimality condition

For maximization problem, if all the coefficients of the objective function are less or equal to zero,then optimality is reached.

Step 4:If the current bfs is not optimal, then determine which non-basic variable should become a basic variable and which basic variable should become a non-basic variable to find a new bfs with better objective function value.

Basic variable

A variable is said to be a basic variable if it has a unit coefficient in one of the constraints and zero coefficient in the remaining constraints.

- Select the variable with most positive coefficient as the entering variable column–*pivot column.* Corresponding variable is the *entering variable.*
- In each row, find the ratio between the value column and pivot column.
- Then select the pivot row with respect to the minimum ratio. Corresponding variable is the *leaving variable.* The value at the intersection of the pivot row and column is called *the pivot element.* (Only positive real values are considered for minimum ratio)

Step 5: Use row operations to find newbfs with the better objective function value. Go back to step 3.

Rule 1: If all variables have a non-positive (<= 0) coefficient in Objective Row , the current basic feasible solution (bfs) is optimal.Otherwise, pick a variable with a positive coefficient in Objective Row.

The variable chosen by Rule 1 is called the entering variable.

Rule 2: For each constraint Row, where there is a strictly positive entering variable coefficient, compute the ratio of the RHS to the "entering variable coefficient".

Choose the pivot row as being the one with MINIMUM ratio.

3.2 Basic Variables and Basic Feasible Solutions:

Consider an LPP (in standard form) with m constraints and n variables:

$$\text{Maximize } f(x) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$
$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$
$$x_1, x_2, \ldots, x_n \geq 0$$

We assume $m \leq n$. We choose $n - m$ variables and set them equal to zero. Thus we will be left with a system of m equations in n variables. If this $m \times n$ square system has a unique solution, this solution is called a *basic solution.* A basic solution that also satisfies the non-negativity condition is called a *Basic Feasible Solution (BFS).*

The $n - m$ variables set to zero are called *non-basic* and the m variables which we are solving for are known as *basic variables.* Thus a basic solution is of the form $x = (x_1, x_2, \ldots, x_n)$ where $n - m$ "components" are zero and the remaining m components form the unique solution of the square system (formed by the m constraint equations).

Consider the LP

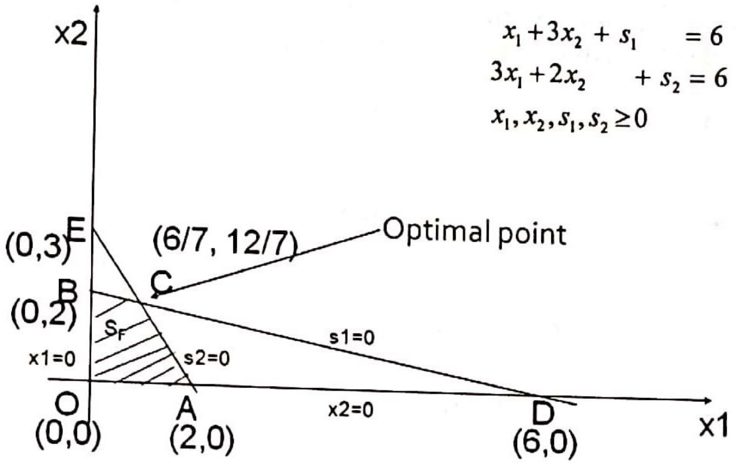
$$\begin{aligned} \text{Maximize} \quad & f = 2x_1 + 3x_2 \\ \text{Subject to} \quad & x_1 + 3x_2 \leq 6 \\ & 3x_1 + 2x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

In standard form

$$\begin{aligned} \text{Maximize} \quad & f = 2x_1 + 3x_2 \\ \text{Subject to} \quad & x_1 + 3x_2 + s_1 = 6 \\ & 3x_1 + 2x_2 + s_2 = 6 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$

s_1 and s_2 are slack variables

Graphical solution of the above LP



Point (x_1, x_2)	$f = 2x_1 + 3x_2$
1. (0,0)	0
2. (2, 0)	4
3. $(\frac{6}{7}, \frac{12}{7})$	$\frac{48}{7}$
4. (2,0)	6

Hence the solution to the problem is $x_1 = \frac{6}{7}$, $x_2 = \frac{12}{7}$ and maximum value is $f = \frac{48}{7}$.

Nonbasic (zero) variables	Basic variable	Basic solution (x_1, x_2, s_1, s_2)	Associated corner point	Feasible	Objective value $Z=2x_1 + 3x_2$
(x_1, x_2)	(s_1, s_2)	(0,0,6,6)	O	Yes	0
(x_1, s_1)	(x_2, s_2)	(0,2,0,2)	B	Yes	6
(x_1, s_2)	(x_2, s_1)	(0,3,-3,0)	E	No	-
(x_2, s_1)	(x_1, s_2)	(6,0,0, -12)	D	No	-
(x_2, s_2)	(x_1, s_1)	(2,0,4,0)	A	Yes	4
(s_1, s_2)	(x_1, x_2)	$(\frac{6}{7}, \frac{12}{7}, 0, 0)$	C	Yes	48/7 Optimal

Thus every Basic Feasible Solution corresponds to a corner (=vertex) of the setof all feasible solutions.

Consider the linear programming problem

$$\text{Maximize } Z = x_1 + x_2$$
$$x_1 + 3x_2 \le 9$$
$$2x_1 + x_2 \le 8$$
$$x_1, x_2 \ge 0$$

In standard form

$$\text{Maximize } Z = x_1 + x_2$$
$$x_1 + 3x_2 + s_1 = 9$$
$$2x_1 + x_2 + s_2 = 8$$
$$x_1, x_2, s_1, s_2 \ge 0$$

Initial Tableau

Basic	x_1	x_2	S_1	S_2	**RHS**	**Ratio**
S_1	1	3	1	0	9	9/1
S_2	2	1	0	1	8	8/2
z	1	1	0	0	0	

Initial Basic Feasible Solution: basic $s_1=9$, $s_2=8$, nonbasic $x_1=x_2=0$; $z=0$

Second Tableau

Basic	x_1	x_2	S_1	S_2	**RHS**	**Ratio**
S_1	0	5/2	1	-1/2	5	2
x_1	1	1/2	0	1/2	4	8
	0	1/2	0	-1/2	-4	

Basic Solution: basic $x_1=4$, $s_1=5$nonbasic $x_2=s_2=0$; $z - 4=0 \Rightarrow z=4$

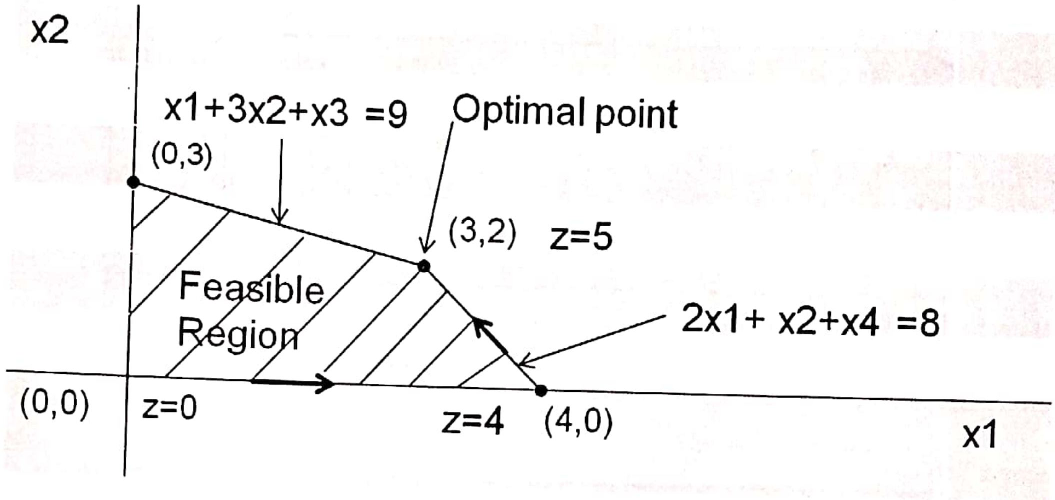
Third Tableau

Basic	x_1	x_2	S_1	S_2	**RHS**	**Ratio**
x_2	0	1	2/5	-1/5	2	
x_1	1	0	-1/5	3/5	3	
	0	0	-1/5	-2/5	-5	

Optimality is reached, optimal value is $Z^* = 5$. The optimal solution is basic $x_1=3$, $x_2=2$nonbasic $s_1=s_2=0$

3.3 Graphical Interpretation

Since the above example has only 2 variables, it is interesting to interpret the steps of the simplex method graphically.



The simplex method starts in the corner point ($x_1=0$, $x_2=0$) with z=0.

Then it discovers that z can increase by increasing, say x_1, (Rule 1). Since we keep $x_2=0$, this means we move along the x-axis. How far can we go? Only until we hit a constraint: If we went any further, the solution would become infeasible. That is exactly what Rule 2 of simplex method does:

The minimum ratio rule identifies the first constraint that will be encountered. And when the constraint is reached, its slack s_2 becomes zero. So, after the first pivot, we are at the point $x_1=4$, $x_2=0$. Rule 1 discovers that z can be increased by increasing x_2 while keeping $s_1=0$. This means that we are moving along the boundary of the feasible region $2x_1 + x_2 = 8$ until we reach another constraint! After pivoting, we reach the optimal point $x_1=3$, $x_2=2$.

Exercise

Solve the following LPP by Simplex Method

1. $Max\ Z = 2x_1 + 3x_2$
   $Subject\ to\ 0.25x_1 + 0.5x_2 \leq 40$
   $0.4x_1 + 0.2x_2 \leq 40$
   $0.8x_2 \leq 40$
   $x_1, x_2 \geq 0$

2. $Max\ Z = 5x_1 + 3x_2$
   $Subject\ to\ x_1 + x_2 \leq 2$
   $5x_1 + 2x_2 \leq 10$
   $3x_1 + 8x_2 \leq 12$
   $x_1, x_2 \geq 0$

3. $Max\ Z = 6x_1 + 8x_2 + 5x_3 + 9x_4$
   $Subject\ to\ 2x_1 + x_2 + x_3 + 3x_4 \leq 5$
   $x_1 + 3x_2 + x_3 + 2x_4 \leq 3$
   $x_1, x_2, x_3, x_4 \geq 0$

4. $Max\ Z = 3x_1 + 2x_2$
   $Subject\ to\ x_1 + x_2 \leq 4$
   $x_1 - x_2 \leq 2$
   $x_1, x_2 \geq 0$