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MacMahon Symmetric Functions, the Partition Lattice, and Young Subgroups

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A MacMahon symmetric function is a formal power series in a finite number of alphabets that is invariant under the diagonal action of the symmetric group. In this article, we show that the MacMahon symmetric functions are the generating functions for the orbits of sets of functions indexed by partitions under the diagonal action of a Young subgroup of a symmetric group. We define a MacMahon chromatic symmetric function that generalizes Stanley's chromatic symmetric function. Then, we study some of the properties of this new function through its connection with the noncommutative chromatic symmetric function of Gebhard and Sagan. © 2001 Academic Press

1. INTRODUCTION

Doubilet [1] introduced a set of formal objects indexed by set partitions (whose behavior is closely related to the behavior of symmetric functions) that allowed him to make calculations on symmetric functions through properties of the partition lattice and its Möbius function. The aim of this article is to generalize his approach to the MacMahon symmetric functions.

Symmetric functions appear when expressing a monic polynomial in terms of its roots. On the other hand, suppose that the coefficients of a polynomial in two variables can be expressed as a product of linear factors. That is, suppose that $e_{(0,0)} + \cdots + e_{(1,1)}xy + \cdots + e_{(n,n)}x^n y^n$ can be written as $(1+\alpha_1 x+\beta_1 y)\cdots(1+\alpha_n x+\beta_n y)$. Expanding the product of linear factors in the previous equation, we obtain symmetric functions like $e_{(0,0)}=1$, $e_{(1,0)}=\alpha_1+\alpha_2+\cdots+\alpha_n$, and $e_{(0,1)}=\beta_1+\beta_2+\cdots+\beta_n$. But, we also get some things that are different, like $e_{(1,1)}=\alpha_1\beta_2+\alpha_2\beta_1+\cdots+\alpha_{n-1}\beta_n$, and $e_{(2,1)}=\alpha_1\alpha_2\beta_3+\alpha_1\alpha_3\beta_2+\cdots+\alpha_{n-2}\alpha_{n-1}\beta_n$. These objects are invariant under the diagonal action of the symmetric group, but not

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under its full action. (The monomial $\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_k}\beta_{j_1!}\beta_{j_2}\cdots\beta_{j_l}$ is sent to $\alpha_{w(i_1)}\alpha_{w(i_2)}\cdots\alpha_{w(i_k)}\beta_{w(j_1)}\beta_{w(j_2)}\cdots\beta_{w(j_l)}$ by permutation w.)

As noted by MacMahon [5], in order to avoid syzygies between the elementary MacMahon symmetric functions, it is necessary to take the number of linear factors, or equivalently, alphabets $X=\alpha_1+\alpha_2+\cdots$ and $Y=\beta_1+\beta_2+\cdots$, to be infinite. A MacMahon symmetric function is a formal power series of bounded degree, in a finite number of infinite alphabets, that is invariant under the diagonal action of the symmetric group.

2. BASIC DEFINITIONS

Let u be a vector in $\mathbf{N}^k$, where $\mathbf{N}$ is the set of nonnegative integers. A vector partition λ of u, written $\lambda \vdash u$, is an unordered sequence of vectors $(a_1, b_1, ..., c_1)(a_2, b_2, ..., c_2)\cdots$ summing to u. We consider two such sequences equal if they differ by a string of zero vectors. In particular, a partition of a number is consider a vector partition. The nonzero vectors of a vector partition are called the parts of λ, and the number of parts of λ is defined to be the length of λ, written $l(\lambda)$. Sometimes, we write λ using block notation. We write $\lambda=\cdots(a_i, b_i, ..., c_i)^{m_i}\cdots$, where part $(a_i, b_i, ..., c_i)$ appears m_i times in λ and i is running over all different parts of λ. For instance, $\lambda=(2, 1)(2, 1)(1, 1)=(2, 1)^2\,(1, 1)$ is a partition of $(5, 3)$ of length 3. The weight of a vector u, *weight*(u), is the sum of its coordinates.

Let $S_\infty=\bigcup_{i\geqslant 1} S_i$, where the symmetric group S_i acts on the first i letters. The symmetric group S_∞ acts diagonally on f in $\mathbf{Q}[[X, Y, ..., Z]]$ sending f to $f(x_{w(1)}, y_{w(1)}, ..., z_{w(1)}, x_{w(2)}, y_{w(2)}, ..., z_{w(2)}, ...)$. A formal power series f in $\mathbf{Q}[[X, Y, ..., Z]]$ is called a MacMahon symmetric function in k systems of indeterminates if the degree of f is bounded and if it is invariant under the diagonal action of S_∞. We say that f has homogeneous multidegree $(a, b, ..., c)$ if in each monomial term of f there are a letters in alphabet X, b letters in alphabet Y, and so on.

The vector space of homogeneous MacMahon symmetric functions of degree i can be decomposed as the disjoint union of all vector spaces $\mathfrak{M}_u$, where u is a vector of weight i and $\mathfrak{M}_u$ is the vector space of MacMahon symmetric functions of multihomogeneous degree u.

We introduce some basis for the space of MacMahon symmetric functions of multihomogeneous degree u. Let $\lambda=(a_1, b_1, ..., c_1)(a_2, b_2, ..., c_2)\cdots$ be a vector partition of $(a, b, ..., c)$.

(1) Let $\mathbf{x}^\lambda$ be $x_1^{a_1}y_1^{b_1}\cdots z_1^{c_1}x_2^{a_2}y_2^{b_2}\cdots z_2^{c_2}\cdots x_l^{a_l}y_l^{b_l}\cdots z_l^{c_l}$. Then, the monomial MacMahon symmetric function indexed by λ is the sum of all

distinct monomials that can be obtained from $\mathbf{x}^\lambda$ by a permutation w in S_∞ acting diagonally.

(2) The elementary MacMahon symmetric function indexed by vector partition λ is $e_\lambda = e_{(a_1, b_1, ..., c_1)} e_{(a_2, b_2, ..., c_2)} \cdots$, where $e_{(a, b, ..., c)}$ is defined by the generating function:

$$\sum_{a, b, ..., c \geqslant 0} e_{(a, b, ..., c)} s^a t^b \cdots u^c = \prod_{i \geqslant 1} (1 + x_i s + y_i t + \cdots + z_i u).$$

(3) The complete homogeneous MacMahon symmetric function indexed by vector partition λ is $h_\lambda = h_{(a_1, b_1, ..., c_1)} h_{(a_2, b_2, ..., c_2)} \cdots$, where we define $h_{(a, b, ..., c)}$ by the generating function

$$\sum_{a, b, ..., c \geqslant 0} h_{(a, b, ..., c)} s^a t^b \cdots u^c = \prod_{i \geqslant 1} \frac{1}{1 - x_i s - y_i t - \cdots - z_i u}.$$

(4) The power sum MacMahon symmetric function indexed by vector partition λ is $p_\lambda = p_{(a_1, b_1, ..., c_1)} p_{(a_2, b_2, ..., c_2)} \cdots$, where $p_{(a, b, ..., c)}$ is $\sum_{i \geqslant 1} x_i^a y_i^b \cdots z_i^c$.

(5) In the ring of MacMahon Symmetric Functions there is an involution defined by $\omega(e_\lambda) = h_\lambda$. The forgotten MacMahon symmetric functions are defined by $\omega(m_\lambda) = (\text{sign } \lambda)\, f_\lambda$.

To λ we associate the partition of the (number) weight of λ defined by $(a_1 + \cdots + c_1, a_2 + \cdots + c_2, \cdots) = (1^{n_1} 2^{n_2} \cdots)$. Then, the sign of λ is defined as $(-1)^{n_2 + 2n_3 + 3n_4 + \cdots}$.

We are following Doubilet's definition of the forgotten MacMahon symmetric functions [1] instead of the one in Macdonald [4].

A vector partition is unitary if it is a partition of $(1)^n = (1, 1, ..., 1)$. Similarly, a monomial (elementary, etc.) MacMahon symmetric function is unitary if it is indexed by a unitary vector partition. Unitary vector partitions can be identified with set partitions: To $\pi = \{B_1, B_2, ..., B_l\}$ we associate the unitary vector partition $\lambda = \lambda_1 \lambda_2 \cdots \lambda_l$ where λ_j has its ith coordinate 1 if i is in B_j and 0 otherwise.

Let $u = (a, b, ..., c)$ be a vector of weight n. We define a Young subgroup S_u of S_n by $S_u = S_{\{1, 2, ..., a\}} \times S_{\{a+1, a+2, ..., a+b\}} \times \cdots \times S_{\{n-c+1, n-c+2, ..., n\}}$. There is a canonical action of S_u on $[n]$. It partitions $[n]$ into equivalence classes that we order using the smallest element in each of them.

The type of a set partition $\pi = B_1 \mid B_2 \mid \cdots \mid B_l$ under the action of S_u, denoted $\text{type}_u(\pi)$, is the vector partition $\lambda = \lambda_1 \lambda_2 \cdots \lambda_l$, where λ_k is the vector whose i coordinate is the number of elements of B_k in the ith equivalence class. If u equals (n), we may omit the subindex (n).

Let F_n be the set of all functions from $[n]$ to $\mathbf{P}$, the set of positive integers. Each f in F_n defines a set partition $\ker f$, where n_1 and n_2 are in the same block of $\ker f$ if and only if $f(n_1)$ equals $f(n_2)$. We read the expression $f(i)=j$ as saying that ball i has been placed on box j.

Let f be in F_n. Suppose that the Young subgroup S_u is acting on $[n]$. We weight f by $\gamma_u(f)=\prod_{d\in[n]} c(d)_{f(d)}$ where $c(d)$ denotes the equivalence class of d and we use variables $x, y, ..., z$ to denote the equivalence classes. In the particular case where $u=(1)^n$, we denote γ_u by γ. To a set of functions T we associate the generating function: $\gamma_u(T)=\sum_{f\in T}\gamma_u(f)$.

A disposition is an arrangement of the balls (that is, the elements of $[n]$) into the boxes (that is, the positive numbers, $\mathbf{P}$), where we may impose some condition on the way the balls are placed. In particular, a function is a disposition where there is no condition on the way the balls are placed. The underlying function of a disposition p is the function obtained from p if we forget about the extra data condition on the balls. The weight of a disposition is defined as the weight of its underlying function. The kernel of a disposition p, written as $\ker p$, is the kernel of its underlying function.

3. COMBINATORIAL INTERPRETATION OF THE MACMAHON SYMMETRIC FUNCTIONS

In this section we follow Peter Doubilet and define three sets of functions and two sets of dispositions. Then, we show how this sets are related to the MacMahon symmetric functions.

DEFINITION 1 (The Projection Map). Let S_u be a Young subgroup of S_n acting on $[n]$.

Given any function $f\colon [n]\to\mathbf{P}$. Let $\gamma_u(f)$ be defined as $\gamma_u(f_u(i, c(i)))$, where $f_u(i, c(i))=f(i)$, and $c(i)$ is the equivalence class of i under S_u. Given a set of functions T going from $[n]$ to $\mathbf{P}$ we define $\gamma_u(T)$ as $\sum_{f\in T}\gamma_u(f)$.

The map sending $\gamma(T)$ to $\gamma_u(T)$ is called the projection map and denoted ρ_u. Given any set of dispositions, the projection map is defined on their underlying functions.

DEFINITION 2 (Doubilet). Let π be a set partition of $[n]$.

(1) $\mathscr{M}_\pi=\{f\colon f\in F_n, \ker f=\pi\}$, and let m_π be $\gamma(\mathscr{M}_\pi)$.

(2) $\mathscr{P}_\pi=\{f\colon f\in F_n, \ker f\geqslant\pi\}$, and let p_π be $\gamma(\mathscr{P}_\pi)$.

(3) $\mathscr{E}_\pi=\{f\colon f\in F_n, \ker f\wedge\pi=\hat{0}\}$, and let e_π be $\gamma(\mathscr{E}_\pi)$.

(4) Let $\mathscr{H}_\pi$ be the set of dispositions such that within each box the balls from the same block of π are linearly ordered, and let h_π be $\gamma(\mathscr{H}_\pi)$.

(5) Let $\mathscr{F}_\pi$ be the set of dispositions such that balls from the same block of π go into the same box, and within each box the blocks appearing are linearly ordered, and let f_π be $\gamma(\mathscr{F}_\pi)$.

For any vector partition $\lambda=(a_1, b_1, ..., c_1)\cdots(a_l, b_l, ..., c_l)$, (or $\lambda=\cdots(a_i, b_i, ..., c_i)^{m_i}\cdots$ when written in block notation), define $|\lambda|=\prod_i m_i!$ and $\lambda!=a_1!\,b_1!\cdots c_1!\,a_2!\,b_2!\cdots c_2!\cdots a_l!\,b_l!\cdots c_l!$

THEOREM 3. *Let S_u be a Young subgroup of S_n. Let π be a set partition of $[n]$ and let λ be the type π under S_u. Then, under the projection map ρ_u*

$$m_\pi \mapsto |\lambda|\, m_\lambda, \qquad p_\pi \mapsto p_\lambda, \qquad e_\pi \mapsto \lambda!\, e_\lambda$$

$$h_\pi \mapsto \lambda!\, h_\lambda, \qquad f_\pi \mapsto |\lambda|\, f_\lambda.$$

In particular, $\rho_u\colon \mathfrak{M}_{(1)^n} \to \mathfrak{M}_u$.

Proof. If f is in $\mathscr{M}_\pi$, then, $\gamma_u(f)=x_{i_1}^{a_1} y_{i_1}^{b_1}\cdots z_{i_1}^{c_1} x_{i_2}^{a_2}\cdots z_{i_2}^{c_2}\cdots x_{i_l}^{a_l} y_{i_l}^{b_l}\cdots z_{i_l}^{c_l}$. Therefore,

$$\rho_u(m_\pi)=\sum_{\substack{i_1, i_2, ..., i_l\geqslant 1\\ \text{different}}} x_{i_1}^{a_1} y_{i_1}^{b_1}\cdots z_{i_1}^{c_1}\cdot x_{i_2}^{a_2} y_{i_2}^{b_2}\cdots z_{i_2}^{c_2}\cdots x_{i_l}^{a_l} y_{i_l}^{b_l}\cdots z_{i_l}^{c_l}.$$

Any monomial appears $m_1!\,m_2!\cdots m_l!$ times. Hence, $\rho_u(m_\pi)=|\lambda|\, m_\lambda$.

We can rewrite $\mathscr{P}_\pi$ as

$$\mathscr{P}_\pi=\{f\colon f\in F \text{ and is constant in blocks } B_1, B_2, ... \text{ of } \pi\},$$

Therefore,

$$\begin{aligned}
\rho_u(p_\pi) &= \sum_{f\in P_\pi} \gamma_u(f)=\sum_{f\in P_\pi} \gamma_u(f\mid B_1)\,\gamma_u(f\mid B_2)\cdots\\
&= \sum_{\substack{f\in P_\pi\\ f\mid B_i \text{ constant}}} \gamma_u(f\mid B_1)\,\gamma_u(f\mid B_2)\cdots=\prod_i \sum_{\substack{f\colon B_i\to \mathbf{P}\\ f \text{ constant}}} \gamma_u(f)\\
&=\prod_i (x_1^{a_i} y_1^{b_i}\cdots z_1^{c_i}+x_2^{a_i} y_2^{b_i}\cdots z_2^{c_i}+\cdots)\\
&= \sum_{i_1, i_2, \cdots} x_{i_1}^{a_{i_1}} y_{i_1}^{b_{i_1}}\cdots z_{i_1}^{c_{i1}} x_{i_2}^{a_{i_2}} y_{i_2}^{b_{i_2}}\cdots z_{i_2}^{c_{i_2}}\cdots\\
&= p_\lambda.
\end{aligned}$$

By definition $\mathscr{E}_\pi$ is the set of functions from $[n]$ to $\mathbf{P}$ that are injective on the blocks of π. Hence, $\rho_u(e_\pi)$ equals

$$\sum_{f \in E_\pi} \gamma_u(f) = \sum_{\substack{f \in F \\ f \mid B_i \text{ injective}}} \gamma_u(f \mid B_1)\, \gamma_u(f \mid B_2) \cdots = \prod_i \Bigg(\sum_{\substack{f\colon B_i \to \mathbf{P} \\ f \text{ injective}}} \gamma_u(f) \Bigg).$$

Note that $\gamma_u(f)$ is a monomial without repeated factor. Moreover, each such monomial arises from $a_i!\, b_i! \cdots c_i!$ different functions $f\colon B_i \to \mathbf{P}$. Therefore,

$$\sum_{\substack{f\colon B_i \to \mathbf{P} \\ f \text{ injective}}} \mu\gamma_u(f) = a_i!\, b_i! \cdots c_i!\, e_{\lambda_i}$$

and $\rho_u(e_\pi) = a_1!\, b_1! \cdots c_1!\, a_2!\, b_2! \cdots c_2! \cdots e_{\lambda_1 \lambda_2 \cdots} = \lambda!\, e_\lambda$.

Suppose that in $[n]/S_u$ there are a elements in the first equivalence class, b on the second, and so on. There are $n!$ different ways of linearly ordering the balls of $[n]$ and we can make $\binom{n}{a,\, b,\, ...,\, c}$ sequences of words (with the same underlying function) out of these letters. Therefore there are $a!\, b! \cdots c!$ repetitions. We obtain that

$$\rho_u(h_\pi) = a_1!\, b_1! \cdots c_1!\, a_2!\, b_2! \cdots c_2! \cdots h_{\lambda_1 \lambda_2 \cdots} = \lambda!\, h_\lambda.$$

In [1] Doubilet showed that $\omega(m_\pi) = \operatorname{sign}(\pi)\, f_\pi$ when π is a unitary vector partition. Therefore, we have that $(\operatorname{sign} \pi)\, f_\pi = \omega(m_\pi) \mapsto \omega(|\lambda|\, m_\lambda) = |\lambda|\, \omega(m_\lambda) = |\lambda|\, (\operatorname{sign} \lambda)\, f_\lambda$. Since $\operatorname{sign} \lambda = \operatorname{sign} \pi$ for any π of type λ, we have that $f_\pi \mapsto |\lambda|\, f_\lambda$. ∎

Although the notation $|\lambda|$ may be misleading, we have decided to follow Doubilet [1] and Gebhard and Sagan [3], and adopt it.

Corollary 4 (Doubilet). *Let S_n be the symmetric group, and π be in a set partition Π_n. In this case $\text{type}_{(n)}(\pi)$ is a partition of a number, say λ.*

Then, $\rho_n\colon \mathfrak{M}_{(1)^n} \to \mathfrak{M}_n$, the vector space of symmetric functions of homogeneous degree n.

$$m_\pi \mapsto |\lambda|\, m_\lambda, \qquad p_\pi \mapsto p_\lambda, \qquad e_\pi \mapsto \lambda!\, e_\lambda$$

$$h_\pi \mapsto \lambda!\, h_\lambda, \qquad f_\pi \mapsto |\lambda|\, f_\lambda,$$

where m_λ, p_λ, e_λ, h_λ, and f_λ are symmetric functions.

4. THE SCALAR AND KRONECKER PRODUCTS AND THE TRANSITION MATRICES

Following Gessel [2], we define a scalar product of MacMahon symmetric functions by $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu}$.

For any MacMahon symmetric functions f and for any vector partition λ, the scalar product $\langle h_\lambda, f \rangle$ gives the coefficient of $x_1^{a_1} \cdots z_1^{c_1} x_2^{a_2} \cdots z_2^{c_2} \cdots$ in f. The Kronecker product of MacMahon symmetric functions is defined by $p_\lambda * p_\mu = \langle p_\lambda, p_\mu \rangle\, p_\lambda$, and extended by linearity. If λ is a partition of a number, then the Kronecker product gives the multiplicity of the irreducible representations of the symmetric group in the tensor product of their irreducible representations. An analogous interpretation for the Kronecker product for the MacMahon symmetric functions is unknown.

Proposition 5. *Let u be a vector weight u. The number of set partitions π in Π_n of the type λ under S_u is*

$$\binom{u}{\lambda} = \frac{u!}{\lambda!\,|\lambda|}.$$

Definition 6 (The Lifting Map). Let λ be a vector partition of u. Let $M_\lambda = \binom{u}{\lambda}\,|\lambda|\, m_\lambda$. Then define the lifting map $\hat{\rho}_u\colon \mathfrak{M}_u \to \mathfrak{M}_{(1)^n}$ by

$$\hat{\rho}_u(M_\lambda) = \sum_{\mathrm{type}_u(\pi) = \lambda} m_\pi.$$

Similarly, we define $E_\lambda = \binom{u}{\lambda}\,\lambda!\, e_\lambda$, $P_\lambda = \binom{u}{\lambda}\, p_\lambda$, $H_\lambda = \binom{u}{\lambda}\,\lambda!\, h_\lambda$, and $F_\lambda = \binom{u}{\lambda}\,|\lambda|\, f_\lambda$. It is easy to show that $E_\lambda \mapsto \sum_{\mathrm{type}_u(\pi)=\lambda} e_\pi$, $P_\lambda \mapsto \sum_{\mathrm{type}_u(\pi)=\lambda} p_\pi$, $H_\lambda \mapsto \sum_{\mathrm{type}_u(\pi)=\lambda} h_\pi$, and $F_\lambda \mapsto \sum_{\mathrm{type}_u(\pi)=\lambda} f_\pi$.

Proposition 7. *The lifting map $\hat{\rho}_u$ has the property that $\rho_u \hat{\rho}_u = \mathbf{1}$. Moreover, for all f, $g \in \mathfrak{M}_u$,*

$$\langle f, g \rangle = u!\,\langle \hat{\rho}_u(f), \hat{\rho}_u(g) \rangle.$$

Proof. We show that $\rho_u \hat{\rho}_u(M_\lambda) = M_\lambda$,

$$\rho_u \hat{\rho}_u(M_\lambda) = \rho_u\left(\sum_{\mathrm{type}_u(\pi)=\lambda} m_\pi\right) = |\lambda| \sum_{\mathrm{type}_u(\pi)=\lambda} m_\lambda = |\lambda| \binom{u}{\lambda} m_\lambda = M_\lambda.$$

Moreover, $\langle H_\lambda, M_\lambda\rangle = u!\,\langle \hat{\rho}_u(H_\lambda), \hat{\rho}_u(M_\lambda)\rangle$. Because, on the one hand,

$$\begin{aligned}
\langle \hat{\rho}_u(H_\lambda), \hat{\rho}_u(M_\mu)\rangle &= \Big\langle \sum_{\text{type}_u(\pi)=\lambda} h_\pi, \sum_{\text{type}_u(\sigma)=\mu} m_\sigma \Big\rangle \\
&= \sum_{\text{type}_u(\pi)=\lambda} \sum_{\text{type}_u(\sigma)=\mu} \langle h_\pi, m_\sigma\rangle \\
&= \sum_{\text{type}_u(\pi)=\lambda} \delta_{\lambda,\mu} = \delta_{\lambda,\mu}\binom{u}{\lambda}.
\end{aligned}$$

On the other hand,

$$\langle H_\lambda, M_\mu\rangle = \Big\langle \binom{u}{\lambda}\lambda!\, h_\lambda, \binom{u}{\mu}|\mu|\, m_\mu \Big\rangle = \binom{u}{\lambda}^2 \lambda!\,|\lambda|\,\delta_{\lambda,\mu} = u!\,\delta_{\lambda,\mu}\binom{u}{\lambda}.$$

Proposition 8, together with the Theorem 4 and Doubilet's calculations [1], allows us to compute the transition matrices and the scalar product between the different basis of $\mathfrak{M}_u$. For instance, to compute the scalar product of power sum MacMahon symmetric functions assume that $\lambda=\mu$. (Otherwise their scalar product is 0.) Then, $\langle P_\lambda, P_\lambda\rangle = u!\,\langle \hat{\rho}_u(P_\lambda), \hat{\rho}_u(P_\lambda)\rangle = u!\,\langle \sum_{\text{type}_u(\pi)=\lambda} p_\pi, \sum_{\text{type}_u(\sigma)=\lambda} p_\sigma\rangle = u!\sum_{\text{type}_u(\pi)=\lambda}\langle p_\pi, p_\pi\rangle = u!\,\binom{u}{\lambda}(1/|\mu(\hat{0},\pi)|)$. We have obtained that $\langle p_\lambda, p_\lambda\rangle = |\lambda|\,\lambda!/|\mu(\hat{0},\lambda)|$, where $\mu(\hat{0},\lambda)$ is defined to be $\mu(\hat{0},\pi)$ for a set partition π of type λ under the action of S_u.

PROPOSITION 8. *Let f and g be functions in $\mathfrak{M}_u$. Then*

(1) *The lifting map $\hat{\rho}_u$ satisfies $\hat{\rho}_u(f*g) = u!\,\hat{\rho}_u(f)*\hat{\rho}_u(g)$*

(2) *The Kronecker product on $\mathfrak{M}_{(1)^n}$ and $\mathfrak{M}_u$ are related by*

$$f*g = u!\,\rho_u(\hat{\rho}_u(f)*\hat{\rho}_u(g))$$

(3) *For all $f, g\in\mathfrak{M}_u$, involution ω satisfies*

$$\omega(f)*\omega(g) = f*g$$

Proof. (1) On the one hand,

$$\begin{aligned}
\hat{\rho}_u(P_\lambda * P_\mu) &= \hat{\rho}_u\left(\binom{u}{\lambda}\binom{u}{\mu}\langle p_\lambda, p_\mu\rangle\, p_\lambda\right) \\
&= \binom{u}{\lambda}\frac{|\lambda|\,\lambda!}{|\mu(\hat{0},\pi)|}\delta_{\lambda,\mu}\,\hat{\rho}_u(P_\lambda) = \frac{u!\,\delta_{\lambda,\mu}}{|\mu(\hat{0},\pi)|}\Big(\sum_{\text{type}_u(\pi)=\lambda} p_\lambda\Big).
\end{aligned}$$

On the other hand,

$$\hat{\rho}_u(P_\lambda) * \hat{\rho}_u(P_\mu) = \sum_{\text{type}_u(\pi)=\lambda} p_\pi * \sum_{\text{type}_u(\sigma)=\mu} p_\sigma = \sum_{\substack{\text{type}_u(\pi)=\lambda \\ \text{type}_u(\sigma)=\mu}} p_\pi * p_\sigma$$

$$= \sum_{\substack{\text{type}_u(\pi)=\lambda \\ \text{type}_u(\sigma)=\mu}} \langle p_\pi, p_\sigma \rangle \, p_\pi = \frac{\delta_{\lambda,\mu}}{|\mu(\hat{0}, \pi)|} \left(\sum_{\text{type}_u(\pi)=\lambda} p_\lambda \right).$$

(2) This follows from (1). Since, $f * g = \rho_u(\hat{\rho}_u(f * g)) = \rho_u(u! \, \hat{\rho}_u(f) * \hat{\rho}_u(g)) = u! \, \rho_u(\hat{\rho}_u(f) * \hat{\rho}_u(g))$.

(3) It is enough to show that $\omega(p_\pi) * \omega(p_\sigma) = p_\pi * p_\sigma$. Then, $\omega(p_\pi) * \omega(p_\sigma) = \text{sign}(\pi) \, p_\pi * \text{sign}(\sigma) \, p_\sigma = \text{sign}(\pi\sigma) \; p_\pi * p_\sigma = p_\pi * p_\sigma$

Proposition 9 together with the Theorem 4 and Doubilet's calculations [1] allows us to compute the Kroneckerproduct of two functions in $\mathfrak{M}^{(k)}$. For instance,

$$h_\lambda * h_\mu = \frac{1}{\lambda! \binom{u}{\lambda} \mu! \binom{u}{\mu}} H_\lambda * H_\mu = \frac{1}{\lambda! \binom{u}{\lambda} \mu! \binom{u}{\mu}} \rho_u \left(\hat{\rho}_u(H_\lambda * H_\mu) \right)$$

$$= \frac{|\lambda| \; |\mu|}{u!} \rho_u \left(\sum_{\text{type}_u(\pi)=\lambda} h_\pi * \sum_{\text{type}_u(\sigma)=\mu} h_\sigma \right) = \frac{|\lambda| \; |\mu|}{u!} \sum_{\substack{\text{type}_u(\pi)=\lambda \\ \text{type}_u(\sigma)=\mu}} \rho_u(h_{\pi \wedge \sigma})$$

$$= \frac{|\lambda| \; |\mu|}{u!} \sum_{\substack{\text{type}_u(\pi)=\lambda \\ \text{type}_u(\sigma)=\mu}} \text{type}(\pi \wedge \sigma)! \; h_{\text{type}(\pi \wedge \sigma)}.$$

5. THE CHROMATIC MACMAHON SYMMETRIC FUNCTION

We extend Stanley's definition of chromatic symmetric function of a graph G [9] obtaining a MacMahon symmetric function. Moreover, we show that the techniques developed by Gebhard and Sagan to study noncommutative chromatic symmetric function [3] can be used to study the MacMahon chromatic symmetric function.

Let G be a graph with vertex set $[n]$ and edge set E. A function κ: $[n] \to \mathbf{P}$ is called a proper coloring of G if $\kappa(u) \neq \kappa(v)$ whenever u and v are vertices of an edge of G. A stable partition π of G is a set partition of $[n]$ such that each block of π is totally disconnected. Let G be a simple graph with vertex set $[n]$ and edge set E. Stanley's chromatic symmetric function [9] is defined as

$$_G = \sum_{\kappa \text{ proper coloring}} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)}.$$

Let G be a simple graph (without loops or multiple edges) with vertex set $[n]$ and edge set E. Let $S(G)$ be the set of all stable partitions of G. Let $\mathscr{F}_n(G)$ be $\bigcup_{\pi \in S(G)} \mathscr{M}_\pi$. Stanley [9] showed that $\rho_n(\gamma(\mathscr{F}_n(G))) = X_G$.

A set composition $[C_1 \mid C_2 \mid \cdots \mid C_l]$ of $[n]$ is an ordered partition of $[n]$, where some parts may be empty. The vector associated to C has ith coordinate equal to the cardinality of C_i. Then S_u acts on $[n]$ as $S_{C_1} \times S_{C_2} \cdots \times S_{C_l}$.

Definition 9. Let G be a graph (or directed graph) with vertex set $[n]$. Let $C = [C_1 \mid C_2 \mid \cdots \mid C_l]$ be a set composition of $[n]$. The chromatic MacMahon symmetric function according to C in alphabets X_i (with $i = 1, 2, ..., l$) is defined by

$$\bar{X}_{G,C} = \sum_{\kappa \text{ proper coloring}} x_{c(1), \kappa_{(1)}} x_{c(2), \kappa_{(2)}} \cdots x_{c(n), \kappa_{(n)}},$$

where $X_i = \{x_{i,1}, x_{i,2}, ..., x_{i,n}\}$, and $c(i)$ denotes that vertex i belongs to C_i.

For instance, if C_i is defined as the set of vertices of degree i, then $\bar{X}_{g,C}$ is an invariant of the graph under relabelings, called the MacMahon chromatic symmetric function according to the degree, and denoted $\bar{X}_{g, \text{degree}}$.

If $C = [1 \mid 2 \mid \cdots \mid n]$, then $\bar{X}_{G,C}$ is called the MacMahon chromatic symmetric function according to the vertices and denoted $\bar{X}_{g, \text{vertices}}$. It is an unitary MacMahon symmetric functions equivalent to the noncommutative chromatic symmetric function of Gebhard and Sagan. Moreover, it satisfies a deletion-contraction property, and specializes to all the invariant chromatic symmetric function described in this section. However, it is not invariant under relabelings of the vertex set.

If $C = [12 \cdots n]$, then $\bar{X}_{G,C}$ is the chromatic symmetric function introduced by Stanley, and denoted by X_G. Stanley showed that the chromatic symmetric function reduces to the chromatic polynomial χ_G under a certain specialization of variables. Then, he proved various theorem generalizing results about the chromatic polynomial, as well as some new ones that did not make sense for χ_G [9].

We produce an invariant chromatic MacMahon symmetric functions associated to graph (or directed graph) G if we construct the set composition C according to an invariant property of G. For instance, C can be obtained by partitioning the vertex set according its connected components, according to the number of different closed walks starting at each vertex, according to its in degree, and so on.

PROPOSITION 10. *Let G be a graph with vertex set $[n]$. Let C be a set composition of $[n]$ with associated vector u. Then $\bar{X}_{G,C}$ is a MacMahon symmetric function of homogeneous multidegree u.*

Moreover, we can recover Stanley's chromatic symmetric function X_G from $\bar{X}_G$ through the projection map $\rho_n(\bar{X}_{G,C}) = X_G$ where $\rho_n(\bar{X}_{G,C})$ is the image under ρ_n of any preimage of $\bar{X}_{G,C}$ under ρ_u.

PROPOSITION 11. *Let G be a graph with vertex set $[n]$. Let C be a set composition of $[n]$ with associated vector u. Then*

$$\bar{X}_{G,C} = \rho_u\left(\sum_{\pi \in S(G)} \mathscr{M}_\pi\right) = \sum_{\lambda \vdash u} a_\lambda\, |\lambda|\, m_\lambda,$$

where a_λ is the number of set partitions π in $S(G)$ such that $\text{type}_u(\pi) = \lambda$.

As an illustration, we look at the MacMahon chromatic symmetric function according to the degree, denoted $\bar{X}_{G,\text{degree}}$. In [9] Stanley showed that the chromatic symmetric function for graphs G and H (see Fig. 1) does not distinguish between them.

On the other hand, the chromatic MacMahon symmetric function of G and H are different because $X_{G,\text{degree}} \in \mathfrak{M}_{(0,4,0,1)}$ and $X_{H,\text{degree}} \in \mathfrak{M}_{(1,1,3)}$. This is not too surprising because their degree sequences are different. The remarkable point is that the MacMahon chromatic symmetric function encloses more information than what we can obtain by having both the chromatic symmetric function and the degree sequence of a graph, while being of the same complexity.

For instance, L and M (See Fig. 2) have the same degree sequence and the same chromatic symmetric function, but their chromatic MacMahon symmetric functions according to the degree are different. It is not known

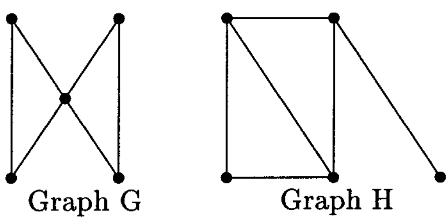

FIGURE 1

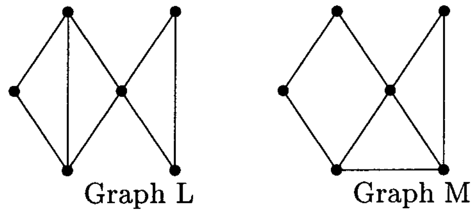

FIGURE 2

an example of two non isomorphic graphs with the same MacMahon chromatic symmetric function according to the degree.

In general, the MacMahon chromatic symmetric function do not satisfy a Deletion-Contraction law. This was the main motivation (in the symmetric functions case) for Gebhard and Sagan [3] to introduce their noncommutative chromatic symmetric function Y_G, having the nice properties that it obeys such a law, and that it specializes to X_G when we allow the variables to commute.

Noncommutative symmetric functions [3] are indexed by set partitions. Let $\{u_1, u_2, \cdots\}$ be a set of non commuting letters. The noncommutative monomial symmetric function $\tilde{m}_\pi$ is defined as $\tilde{m}_\pi = \sum_{i_1, i_2 \cdots, i_d} u_{i_1} u_{i_2} \cdots u_{i_d}$ where the sum is taken over all sequences $i_1, i_2 \cdots, i_d$ of positive integers such that $i_j = i_k$ if and only if j and k are in the same block of π. For example, $\tilde{m}_{12\,|\,3} = u_1u_1u_2 + u_2u_2u_1 + u_1u_1u_3 + u_3u_3u_1 \cdots$. The noncommutative power sums and the noncommutative elementary symmetric functions are defined through the identities in Appendix 6.1.

There is a canonical isomorphism between the vector space of noncommutative symmetric functions and the vector space of unitary MacMahon symmetric functions sending the noncommutative monomial $u_{i_1} u_{i_2} \cdots u_{i_n}$ to the monomial $x_{i_1} y_{i_2} \cdots z_{i_n}$. Under this isomorphism $\tilde{m}_\pi$ goes to the unitary MacMahon symmetric functions m_π.

For any graph G, with vertices $v_1, v_2, ..., v_d$ in a fixed order, the noncommutative chromatic symmetric function Y_G is defined as the sum over all proper coloring of G of $u_{\kappa(1)} u_{\kappa(2)} \cdots u_{\kappa(d)}$. (Under the canonical isomorphism just described Y_G corresponds to the MacMahon chromatic symmetric function according to the vertices.)

The noncommutative chromatic symmetric function Y_G satisfies a deletion-contraction property (and so does the $X_{G,\,\text{vertices}}$). This remarkable fact allows inductive proofs of noncommutative versions of some results of Stanley [9], making straightforward to generalize them to the chromatic MacMahon symmetric function. For instance, the following theorems holds.

THEOREM 12. *Let G be a graph with vertex set $[n]$. Let C be any set composition of $[n]$ with associated vector u. Then*

$$X_{G,C} = \sum_{S \subseteq E} (-1)^{|S|}\, p_{\mathrm{type}_u(\pi(S))},$$

where $\pi(S)$ denotes the partition of $[d]$ associated with the vertex partition of the spanning subgraph of G induced by S.

All other results of Gebhard and Sagan [3] generalize to the MacMahon chromatic symmetric function in a similar straightforward manner that is left to the reader. (In the proof of Lemma 6.3 appears a product of non-commutative symmetric functions. It corresponds to the product of unitary MacMahon symmetric functions because the labels of the vertex sets are disjoints.)

A final remark, the MacMahon chromatic symmetric function according to the vertices distinguish among all graph G with no loops or multiple edges [3]. But, it is not an invariant of G.

6. APPENDIX

We reproduce the Appendix 1 and 2 in [1]. The definition of the scalar product used differs from the one in [1] by a constant factor.

6.1. *The Matrices of Change of Basis*

$$p_\pi = \sum_{\sigma \geqslant \pi} m_\sigma, \qquad m_\pi = \sum_{\sigma \geqslant \pi} \mu(\pi, \sigma)\, p_\sigma,$$

$$e_\pi = \sum_{\sigma : \sigma \wedge \pi = \hat{0}} m_\sigma, \qquad m_\pi = \sum_{\tau \geqslant \pi} \frac{\mu(\pi, \tau)}{\mu(\hat{0}, \tau)} \sum_{\sigma \leqslant \tau} \mu(\sigma, \tau)\, e_\sigma,$$

$$e_\pi = \sum_{\sigma \leqslant \pi} \mu(\hat{0}, \sigma)\, p_\sigma, \qquad p_\pi = \frac{1}{\mu(\hat{0}, \pi)} \sum_{\sigma \leqslant \pi} \mu(\sigma, \pi)\, e_\sigma,$$

$$p_\pi = \sum_{\sigma \geqslant \pi} \mathrm{sign}(\pi, \sigma)\, f_\sigma, \qquad f_\pi = \sum_{\sigma \geqslant \pi} |\mu(\pi, \sigma)|\, p_\sigma,$$

$$h_\pi = \sum_{\sigma : \sigma \wedge \pi = \hat{0}} \mathrm{sign}(\sigma)\, f_\sigma, \qquad f_\pi = \sum_{\tau \geqslant \pi} \frac{|\mu(\pi, \tau)|}{|\mu(\hat{0}, \tau)|} \sum_{\sigma \leqslant \tau} \mu(\sigma, \tau)\, h_\sigma,$$

$$h_\pi = \sum_{\sigma \leqslant \pi} |\mu(\hat{0}, \sigma)|\, p_\sigma, \qquad p_\pi = \frac{1}{|\mu(\hat{0}, \pi)|} \sum_{\sigma \leqslant \pi} \mu(\sigma, \pi)\, h_\sigma,$$

$$h_\pi = \sum_\sigma \operatorname{type}(\pi \wedge \sigma)!\, m_\sigma, \qquad m_\pi = \sum_{\tau \geqslant \pi} \frac{\mu(\pi, \tau)}{|\mu(\hat{0}, \tau)|} \sum_{\sigma \leqslant \tau} \mu(\sigma, \tau)\, h_\sigma,$$

$$e_\pi = \sum_\sigma \operatorname{sign}(\sigma) \operatorname{type}(\sigma \wedge \pi)!\, f_\sigma, \qquad f_\pi = \sum_{\tau \geqslant \pi} \frac{|\mu(\pi, \tau)|}{\mu(\hat{0}, \tau)} \sum_{\sigma \leqslant \tau} \mu(\sigma, \tau)\, e_\sigma,$$

$$m_\pi = \sum_{\sigma \geqslant \pi} \operatorname{sign}(\pi, \sigma) \operatorname{type}(\pi, \sigma)!\, f_\sigma, \qquad f_\pi = \sum_{\sigma \geqslant \pi} \operatorname{type}(\pi, \sigma)!\, m_\sigma,$$

$$e_\pi = \sum_{\sigma \leqslant \pi} \operatorname{sign}(\pi) \operatorname{type}(\sigma, \pi)!\, h_\sigma, \qquad h_\pi = \sum_{\sigma \leqslant \pi} \operatorname{sign}(\sigma) \operatorname{type}(\sigma, \pi)!\, e_\sigma.$$

6.2. *The Scalar Product*

$$\langle h_\pi, m_\sigma \rangle = \delta_{\pi, \sigma}, \qquad \langle p_\pi, p_\sigma \rangle = \frac{\delta_{\pi, \sigma}}{|\mu(\hat{0}, \pi)|},$$

$$\langle h_\pi, p_\sigma \rangle = \zeta(\sigma, \pi), \qquad \langle e_\pi, p_\sigma \rangle = \operatorname{sign}(\sigma)\, \zeta(\sigma, \pi),$$

$$\langle e_\pi, e_\sigma \rangle = \operatorname{type}(\sigma \wedge \pi)!, \qquad \langle e_\pi, h_\sigma \rangle = \delta_{\pi \wedge \sigma, \hat{0}},$$

$$\langle e_\pi, m_\sigma \rangle = \operatorname{sign}(\sigma) \operatorname{type}(\sigma, \pi)!\, \zeta(\sigma, \pi), \qquad \langle h_\pi, h_\sigma \rangle = \operatorname{type}(\sigma \wedge \pi)!,$$

$$\langle m_\pi, p_\sigma \rangle = \frac{\mu(\pi, \sigma)}{|\mu(\hat{0}, \sigma)|}\, \zeta(\pi, \sigma), \qquad \langle f_\pi, e_\sigma \rangle = \operatorname{sign}(\pi)\, \delta_{\pi, \sigma},$$

$$\langle f_\pi, h_\sigma \rangle = \operatorname{type}(\pi, \sigma)!\, \zeta(\pi, \sigma),$$

$$\langle f_\pi, p_\sigma \rangle = \underline{\operatorname{sign}(\pi\sigma)}\, \frac{\mu(\pi, \sigma)}{|\mu(\hat{0}, \sigma)|}\, \zeta(\pi, \sigma),$$

That should be sign(pi) sign(sigma)

$$\langle f_\pi, m_\sigma \rangle = \sum_{\tau \geqslant \pi \vee \sigma} \frac{|\mu(\pi, \tau)|\, \mu(\sigma, \tau)}{|\mu(\hat{0}, \tau)|},$$

$$\langle m_\pi, m_\sigma \rangle = \sum_{\tau \geqslant \pi \vee \sigma} \frac{\mu(\pi, \tau)\, \mu(\sigma, \tau)}{|\mu(\hat{0}, \tau)|},$$

where ζ is the zeta function and $\delta_{\pi, \sigma}$ is the Kronecker symbol.

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REFERENCES

1. P. Doubilet, On the foundations of combinatorial theory. VII: Symmetric functions through the theory of distribution and occupancy, *Stud. Appl. Math.* **51**, No. 4 (1972), 377–396; reprint, *in* "Gian-Carlo Rota in Combinatorics" (J. P. Kung, Ed.), pp. 403–422, Birkhäuser, Basel, 1995.
2. I. M. Gessel, Enumerative applications of symmetric functions, *in* "Actes 17e Séminaire Lotharingien," Vol. 348, pp. 5–17, Publ. I.R.M.A., Strasbourg, 1988.
3. D. D. Gebhard and B. E. Sagan, "A Noncommutative Chromatic Symmetric Function," preprint.
4. I. G. Macdonald, "Symmetric Functions and Hall Polynomials," 2nd ed., Oxford Univ. Press, 1995.
5. P. A. MacMahon, "Combinatory Analysis," Vols. 1–2, Cambridge Univ. Press, Cambridge, UK, 1915, 1916; Reprint, Chelsea, New York, 1960.
6. M. H. Rosas, "A Combinatorial Overview of the Theory of MacMahon Symmetric Functions and a Study of the Kronecker Product of Schur Functions," Ph.D. thesis, Brandeis University, 2000.
7. M. H. Rosas, Specializations of MacMahon symmetric functions and the Stirling numbers, *Discrete Math.*, in press.
8. G.-C. Rota and J. A. Stein, Plethystic algebras and vector symmetric functions, *Proc. Nat. Acad. Sci. U.S.A.* **91** (1994), 13062–13066.
9. R. Stanley, A symmetric function generalization of the chromatic polynomial of a graph, *Adv. Math.* **111** (1995), 166–194.