9. $k_\pi = \sum_{\sigma \geq \pi} \operatorname{sign}(\pi, \sigma)\lambda(\pi, \sigma)!f_\sigma \qquad f_\pi = \sum_{\sigma \geq \pi} \lambda(\pi, \sigma)!k_\sigma$

10. $a_\pi = \sum_{\sigma \leq \pi} (\operatorname{sign} \sigma)\lambda(\sigma, \pi)!h_\sigma \qquad h_\pi = \sum_{\sigma \leq \pi} (\operatorname{sign} \sigma)\lambda(\sigma, \pi)!a_\sigma$

Appendix 2: The inner product on $\tilde{\mathscr{S}}$

1. $(\tilde{h}_\pi, \tilde{k}_\sigma) = n!\delta_{\pi\sigma}$

2. $(\tilde{s}_\pi, \tilde{s}_\sigma) = \delta_{\pi\sigma}\dfrac{n!}{|\mu(0, \pi)|}$

3. $(\tilde{h}_\pi, \tilde{s}_\sigma) = n!\zeta(\sigma, \pi)$, where $\zeta(\sigma, \pi) = \begin{cases} 1 & \text{if } \sigma \leq \pi \\ 0 & \text{if not} \end{cases}$

4. $(\tilde{a}_\pi, \tilde{s}_\sigma) = n!(\operatorname{sign} \sigma)\zeta(\sigma, \pi)$

5. $(\tilde{a}_\pi, \tilde{a}_\sigma) = n!\lambda(\sigma \wedge \pi)!$

6. $(\tilde{a}_\pi, \tilde{h}_\sigma) = \begin{cases} n! & \text{if } \pi \wedge \sigma = 0 \\ 0 & \text{if } \pi \wedge \sigma \neq 0 \end{cases}$

7. $(\tilde{k}_\pi, \tilde{k}_\sigma) = n! \sum_{\tau \geq \pi \vee \sigma} \dfrac{\mu(\pi, \tau)\mu(\sigma, \tau)}{|\mu(0, \tau)|}$

8. $(\tilde{a}_\pi, \tilde{k}_\sigma) = n!(\operatorname{sign} \sigma)\lambda(\sigma, \pi)!\zeta(\sigma, \pi)$

9. $(\tilde{h}_\pi, \tilde{h}_\sigma) = n!\lambda(\sigma \wedge \pi)!$

10. $(\tilde{k}_\pi, \tilde{s}_\sigma) = n!\dfrac{\mu(\pi, \sigma)}{|\mu(0, \sigma)|}\zeta(\pi, \sigma)$

11. $(\tilde{f}_\pi, \tilde{k}_\sigma) = n! \sum_{\tau \geq \pi \vee \sigma} \dfrac{|\mu(\pi, \tau)|\mu(\sigma, \tau)}{|\mu(0, \tau)|}$

12. $(\tilde{f}_\pi, \tilde{f}_\sigma) = (\operatorname{sign} \pi)(\operatorname{sign} \sigma)n! \sum_{\tau \geq \pi \vee \sigma} \dfrac{\mu(\pi, \tau)\mu(\sigma, \tau)}{|\mu(0, \tau)|}$

13. $(\tilde{f}_\pi, \tilde{h}_\sigma) = n!\lambda(\pi, \sigma)!\zeta(\pi, \sigma)$

14. $(\tilde{f}_\pi, \tilde{s}_\sigma) = (\operatorname{sign} \pi)(\operatorname{sign} \sigma)n!\dfrac{\mu(\pi, \sigma)}{|\mu(0, \sigma)|}\zeta(\pi, \sigma)$

15. $(\tilde{f}_\pi, \tilde{a}_\sigma) = n!(\operatorname{sign} \pi)\delta_{\pi\sigma}$

Doubilet's notation	Standard notation
k	m
a	e
h	h
s	p
f	f

\pi and \sigma are set partitions of [n].
\mu is the Mobius function in ∏_n

Appendix 3: The Kronecker inner product on $\tilde{\mathscr{S}}$

1. $[\tilde{s}_\pi, \tilde{s}_\sigma] = \dfrac{n!}{|\mu(0, \pi)|}\delta_{\pi\sigma}\tilde{s}_\pi$

2. $[\tilde{h}_\pi, \tilde{s}_\sigma] = n!\zeta(\sigma, \pi)\tilde{s}_\sigma$

3. $[\tilde{a}_\pi, \tilde{s}_\sigma] = n!(\operatorname{sign} \sigma)\zeta(\sigma, \pi)\tilde{s}_\sigma$