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On the Foundations of Combinatorial Theory. VII: Symmetric Functions through the Theory of Distribution and Occupancy*

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1. Introduction

Our purpose in this paper is to derive many of the known results about symmetric functions, and a few new ones, using techniques involving the lattice of partitions of a set. There are a number of advantages to be obtained from this approach. The results come out in a more simple and elegant manner than in more standard approaches, thereby giving, it is hoped, more insight into their meaning. It also gives new interpretations to various formulas and statements. One further possibility is that the present line of attack could be extended to deal with a class of symmetric functions not discussed here, the so-called Schur functions, and in doing this develop the theory of the linear representations of the symmetric group in a very beautiful manner.

In Section 3 we develop the tools to study the monomial symmetric functions, the elementary symmetric functions, and the power sum symmetric functions. Among the results we prove is what is sometimes called the fundamental theorem of symmetric functions, namely that the elementary symmetric functions form a basis for the vector space of all symmetric functions. The standard proof of this fact is an induction argument, but we prove it neatly here by Möbius inversion. We also show that the matrix of coefficients expressing the elementary symmetric functions in terms of the monomial symmetric functions is symmetric.

In Section 4 we extend our techniques to allow us to deal with the complete homogeneous symmetric functions and a new type of symmetric functions which are "dual" to the monomial symmetric functions in the same sense that the elementary symmetric functions are dual to the homogeneous symmetric functions. One of the results proved in this section is one which we believe to be new, namely that under the isometry θ of the space of symmetric functions introduced by Philip Hall, every monomial in the image under θ of a monomial symmetric function appears with the same sign, either positive or negative.

Noting that many of the results obtained in Sections 3 and 4 hold in a stronger form than the statements about symmetric functions they imply, in Section 5 we

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make this precise and thereby set up a system in which we, in the next two sections, study the inner product of Philip Hall and the Kronecker inner product, which arises from the theory of representations of the symmetric group. One of the results obtained can be interpreted to yield an interesting fact about permutation representations of S_n, but this interpretation will not be discussed here.

We would like to thank Dr. Gian-Carlo Rota, whose idea it was to study symmetric functions from the present point of view, for his many helpful suggestions and discussions on the topic.

2. Terminology and assumed results

We assume the reader to have a certain familiarity with symmetric functions, at least to the extent of knowing the definitions of the following basic symmetric functions:

(i) The monomial symmetric functions, denoted k_λ.
(ii) The elementary symmetric functions a_λ.
(iii) The complete homogeneous symmetric functions h_λ.
(iv) The power sum symmetric functions s_λ.

All symmetric functions dealt with have coefficients in Q (the field of rational numbers), involve infinitely many indeterminates $x_1, x_2, \ldots$, and are of a fixed homogeneous degree n. The vector space (over Q) of symmetric functions of degree n is denoted $\mathcal{S}_n$, or simply $\mathcal{S}$. The relevant definitions are to be found in [10] or [19].

Along with the above we assume a knowledge of the definitions and notations used in [19] for partitions of an integer n. A number of these notations are so important that we shall list them here:

(i) $\lambda \vdash n$ denotes that λ is a partition of n.
(ii) $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ or $\lambda = (\lambda_1, \lambda_2, \ldots)$ means that the parts of λ are $\lambda_1, \lambda_2, \ldots$ in non-increasing order, and $\lambda_p > 0$.
(iii) $\lambda = (1^{r_1}2^{r_2}\ldots)$, denotes the partition of n with r_1 parts equal to $1, r_2$ parts equal to 2, etc.

We also use the following notations:

(i) $\lambda! \equiv \lambda_1!\lambda_2!\ldots = 1!^{r_1}2!^{r_2}\ldots$ if $\lambda = (\lambda_1, \lambda_2, \ldots) = (1^{r_1}2^{r_2}\ldots)$
(ii) $|\lambda| \equiv r_1!r_2!\ldots$ if $\lambda = (1^{r_1}2^{r_2}\ldots)$
(iii) $\operatorname{sign}\lambda \equiv (-1)^{r_2+2r_3+3r_4+\cdots}$.

We further assume the reader to be familiar with the notions of (finite) partially ordered sets (posets) and lattices, of segments and direct products of these, and of Möbius inversion over posets (see [18]).

In this paper we deal with the lattice $\prod(D)$ of partitions of a finite set D (also denoted $\prod_n$ if D has n elements). A partition π of D is a family of disjoint subsets $B_1, B_2, \ldots, B_b$, called blocks, whose union is D. The set of partitions of D is ordered by putting $\sigma \leq \pi$ if every block of σ is contained in a block of π. It is easily verified that this is a partial ordering relation and that $\prod(D)$ is in fact a lattice. We assume knowledge of the result that if $\sigma \leq \pi$ in $\prod_n$, the segment $[\sigma, \pi]$ is isomorphic to the direct product of r_1 copies of $\prod_1$, r_2 copies of $\prod_2$, etc., where r_i is the number of blocks of π which are composed of i blocks of σ, and that $\mu(\sigma, \pi) = \prod_i (i-1)!^{r_i}$. To the segment $[\sigma, \pi]$ we assign the partition $\lambda(\sigma, \pi) = (1^{r_1}2^{r_2}\cdots)$, ($\lambda(\sigma, \pi) \vdash m$ for some $m \leq n$), called the type of $[\sigma, \pi]$, and to $\pi \in \prod_n$ we assign the partition $\lambda(\pi) = \lambda(0, \pi)$ of n (where 0 is the minimal element in $\prod(D)$, namely the partition whose blocks are the one point subsets of D), called the type of π. If $\lambda(\pi) = \mu$ we sometimes write $\pi \in \mu$. To $[\sigma, \pi]$ we assign the number sign $(\sigma, \pi) = (-1)^{r_2 + 2r_3 + \cdots} = \operatorname{sign} \lambda(\sigma, \pi)$ and to π we assign the number $\operatorname{sign} \pi = \operatorname{sign}(0, \pi)$. It is not difficult to see that $\operatorname{sign}(\sigma, \pi) = (\operatorname{sign} \sigma)(\operatorname{sign} \pi)$ and that $\mu(\sigma, \pi) = \operatorname{sign}(\sigma, \pi) \cdot |\mu(\sigma, \pi)|$. For more details and some proofs see [2], [6], or [18].

It is useful to know a few simple facts about the symmetric group S_n, the group of permutations of the set $\{1, 2, \ldots, n\}$. Any permutation can be written as a product of disjoint cycles (see [3], page 133, for definitions), and hence to each permutation σ in S_n we can associate the partition $\lambda = (1^{r_1}2^{r_2}\ldots)$ of n, where r_i is the number of cycles of length i in the cycle decomposition of σ. λ is called the type of σ. The number of elements of S_n of type λ, denoted $[{n \atop \lambda}]$, is easily computed to be

$$\frac{n!}{1^{r_1}r_1!2^{r_2}r_2!\cdots}$$

3. The functions k_λ, a_λ, and S_λ

Let D be a set with n elements, $X = \{x_1, x_2, \ldots\}$, and let $F = \{f : D \to X\}$. For $f \in F$, its generating function $\gamma(f)$ is $\prod_{d \in D} f(d)$, i.e., $\prod_i x_i^{|f^{-1}(x_i)|}$. For $T \subset F$, the generating function $\gamma(T)$ is $\sum_{f \in T} \gamma(f)$. To any $f \in F$, we assign a partition $\ker f$ of D, by putting d_1 and d_2 in the same block of $\ker f$ if $f(d_1) = f(d_2)$. $\ker f$ is called the kernel of f.

We now define three types of subsets of F. If $\pi \in \prod(D)$, let

$$\mathscr{K}_\pi = \{f \in F | \ker f = \pi\} \tag{1}$$

$$\mathscr{T}_\pi = \{f \in F | \ker f \geq \pi\} \tag{2}$$

$$\mathscr{A}_\pi = \{f \in F | \ker f \wedge \pi = 0\} \tag{3}$$

Let $k_\pi = \gamma(\mathscr{K}_\pi)$, $s_\pi = \gamma(\mathscr{T}_\pi)$, $a_\pi = \gamma(\mathscr{A}_\pi)$. We now compute k_π, s_π, a_π.

THEOREM 1.

(i) $k_\pi = |\lambda(\pi)| k_{\lambda(\pi)}$, (4)

where k_λ is the monomial symmetric function.

(ii) $s_\pi = s_{\lambda(\pi)}$ (5)

(iii) $a_\pi = \lambda(\pi)! a_{\lambda(\pi)}$. (6)

Proof. Let $\lambda(\pi) = (1^{r_1}2^{r_2}) \cdots = (\lambda_1, \lambda_2, \ldots)$.

(i) For $f \in \mathscr{K}_\pi$, i.e. $\ker f = \pi$, it is clear that $\gamma(f) = x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots$ for some choice of distinct indices $i_1, i_2, \ldots$.

Further, each such monomial can arise from $r_1!r_2! \cdots$ functions $f \in \mathscr{K}_\pi$.

Therefore $k_\pi = \gamma(\mathscr{K}_\pi) = r_1!r_2! \ldots k_\lambda = |\lambda(\pi)| k_{\lambda(\pi)}$

(ii) $\mathscr{T}_\pi = \{f \in F | \ker f \geq \pi\} = \{f \in F | f$ is a constant on blocks of $\pi\}$.

$$
\begin{aligned}
\therefore\quad \gamma(\mathscr{T}_\pi) &= \sum_{f \in \mathscr{T}_\pi} \gamma(f) = \sum_{f \in \mathscr{T}_\pi} \gamma(f|B_1)\gamma(f|B_2)\cdots \\
&= \sum_{\substack{f: f|B_i \text{ is} \\ \text{constant } \forall i}} \gamma(f|B_1)\gamma(f|B_2)\cdots \\
&= \prod_i \left(\sum_{\substack{f: B_i \to X \\ \text{constant}}} \gamma(f) \right) \\
&= \prod_i (x_1^{|B_i|} + x_2^{|B_i|} + \cdots) \\
&= s_{\lambda(\pi)}.
\end{aligned}
$$

(iii) $\mathscr{A}_\pi = \{f \in F | \ker f \wedge \pi = 0\} = \{f \in F | f \text{ is 1-1 on the blocks of } \pi\}$

$$
\begin{aligned}
\gamma(\mathscr{A}_\pi) &= \sum_{f \in \mathscr{A}_\pi} \gamma(f) = \sum_{f \in \mathscr{A}_\pi} \gamma(f|B_1)\gamma(f|B_2)\ldots \\
&= \sum_{\substack{f \in F: \\ f|B_i \text{ is 1-1 } \forall i}} \gamma(f|B_1)\gamma(f|B_2)\ldots \\
&= \prod_i \left(\sum_{\substack{f: B_i \to X \\ f \text{ 1-1}}} \gamma(f) \right).
\end{aligned}
$$

Now,

$$\sum_{\substack{f: B \to X \\ f \text{ 1-1}}} \gamma(f)$$

is the sum of monomials of $|B|$ distinct terms, and each such monomial can arise from $|B|!$ functions $f: B \to X$. Therefore

$$\sum_{\substack{f: B \to X \\ f \text{ 1-1}}} \gamma(f) = |B|!a_{|B|}.$$

Therefore $a_\pi = \gamma(\mathscr{A}_\pi) = |B_1|!|B_2|! \cdots a_{|B_1|}a_{|B_2|} \cdots = \lambda(\pi)!a_{\lambda(\pi)}$.

By formulas (1), (2), and (3), it follows that

$$s_\pi = \sum_{\sigma \geq \pi} k_\sigma \tag{7}$$

$$a_\pi = \sum_{\sigma: \sigma \wedge \pi = 0} k_\sigma \tag{8}$$

Formula (7) can be inverted by Möbius inversion, as can (8) by the following lemma.

LEMMA. *Let L be a finite lattice on which $\mu(0, x) \neq 0$ for all x in L, where μ is the Möbius function on L. Then*

$$f(x) = \sum_{y: y \wedge x = 0} g(y) \leftrightarrow g(x) = \sum_{y \geq x} \frac{\mu(x, y)}{\mu(0, y)} \sum_{z \leq y} \mu(z, y) f(z) \tag{9}$$

Proof: Let

$$f(x) = \sum_{y: y \wedge x = 0} g(y)$$

Then

$$f(x) = \sum_{y} \left(\sum_{z \leq x \wedge y} \mu(0, z) \right) g(y)$$

$$= \sum_{z \leq x} \mu(0, z) \sum_{y \geq z} g(y) \tag{10}$$

By Möbius inversion of (10),

$$\mu(0, x) \sum_{y \geq x} g(y) = \sum_{z \leq x} \mu(z, x) f(z) \tag{11}$$

Thus

$$\sum_{y \geq x} g(y) = \frac{1}{\mu(0, x)} \sum_{z \leq x} \mu(z, x) f(z) \tag{12}$$

since $\mu(0, x) \neq 0$, so by Möbius inversion again,

$$g(x) = \sum_{y \geq x} \frac{\mu(x, y)}{\mu(0, y)} \sum_{z \leq y} \mu(z, y) f(z)$$

The converse is proved by reversing the steps.

THEOREM 2.

(i) $$k_\pi = \sum_{\sigma \geq \pi} \mu(\pi, \sigma) s_\sigma \tag{13}$$

(ii) $$k_\pi = \sum_{\sigma \geq \pi} \frac{\mu(\pi, \sigma)}{\mu(0, \sigma)} \sum_{\tau \leq \sigma} \mu(\tau, \sigma) a_\tau \tag{14}$$

Proof:
(i) follows from (7) by Möbius inversion
(ii) follows from (8) and the preceding lemma.

COROLLARY. *$\{s_\lambda\}$ and $\{a_\lambda\}$ are bases for $\mathscr{S}_n$, the vector space of symmetric functions of homogeneous degree n.*

Proof: $\{k_\lambda\}$ is a basis for $\mathscr{S}_n$, and Theorem 2 shows that $\{s_\lambda\}$ and $\{a_\lambda\}$ generate $\mathscr{S}_n$. Since all three sets have the same number of elements (namely, the number of partitions of n), the result follows.

We now determine the relationship between the elementary and the power sum symmetric functions.

THEOREM 3.

(i) $$a_\pi = \sum_{\sigma \leq \pi} \mu(0, \sigma) s_\sigma \tag{15}$$

(ii) $$s_\pi = \frac{1}{\mu(0, \pi)} \sum_{\sigma \leq \pi} \mu(\sigma, \pi) a_\sigma \tag{16}$$

(iii) $$1 + \sum_{n \geq 1} a_n t^n = \exp\left(\sum_{n \geq 1} \frac{(-1)^{n-1}}{n} s_n t^n \right) \tag{17}$$

Proof:

(i)
$$\begin{aligned} a_\pi &= \sum_{\sigma:\sigma \wedge \pi = 0} k_\sigma \\ &= \sum_{\sigma} \left(\sum_{\tau \le \sigma \wedge \pi} \mu(0,\tau) \right) k_\sigma \\ &= \sum_{\tau \le \pi} \mu(0,\tau) \sum_{\sigma \ge \tau} k_\sigma \\ &= \sum_{\tau \le \pi} \mu(0,\tau) s_\tau \end{aligned}$$

(ii) Apply Möbius inversion to (15)

(iii) Follows directly from (i) and the corollary to Theorem 4 in [6], since $\mu(0, \sigma)s_\sigma$ is a multiplicative function of σ (in the sense of [6]).

Waring's formula follows from Theorem 3 if we put $\pi = 1$ (the maximum element in $\prod_n$) in (16) using the fact (proved in [6]) that the number of $\pi \in \prod_n$ of type $\lambda = (1^{r_1}2^{r_2}\cdots)$, which we will denote $\binom{n}{\lambda}$, is equal to

$$\frac{n!}{1!^{r_1}r_1!2!^{r_2}r_2!\ldots} = \frac{n!}{\lambda!|\lambda|}$$

If we generalize the formula for a_π in (15) and consider the functions $k_{[\tau,\pi]} = \sum_{\sigma \in [\tau,\pi]} \mu(\tau, \sigma)s_\sigma$ it is not difficult to verify that $k_{[\tau,\pi]} = \sum_{\sigma:\sigma \wedge \pi = \tau} k_\sigma$ and that $k_{[\tau,\pi]}$ is the product $\prod_B k_{\sigma_B}$, where the product is over the blocks B of π and where σ_B is the partition of B whose blocks are those blocks of σ contained in B. We can use this to express the product of monomial symmetric functions as a linear combination of any of the symmetric functions discussed here.

We now prove a well-known result concerning the elementary symmetric functions.

THEOREM 4. *Let* $a_\lambda = \sum_\mu c_{\lambda\mu} k_\mu$. *The* $c_{\lambda\mu} = c_{\mu\lambda}$.

Proof: By (8) and Theorem 1, the coefficient of $|\mu| k_\mu$ in $\lambda! a_\lambda$ is $\sum_{\sigma \in \mu} \delta(o, \sigma \wedge \pi)$ where π is some fixed partition of type λ. Thus the coefficient of k_μ in a_λ is

$$\begin{aligned} \frac{|\mu|}{\lambda!} \sum_{\sigma \in \mu} \delta(0, \sigma \wedge \pi) &= \frac{|\mu|}{\lambda!} \frac{1}{\binom{n}{\lambda}} \sum_{\sigma \in \mu, \pi \in \lambda} \delta(0, \sigma \wedge \pi) \\ &= \frac{|\lambda||\mu|}{n!} \sum_{\sigma \in \mu, \pi \in \lambda} \delta(0, \sigma \wedge \pi) \end{aligned}$$

which is symmetric in λ and μ.

4. The functions h_λ and f_λ

We now generalize the notions introduced at the beginning of Section 3 in a way that will allow us to obtain the complete homogeneous symmetric functions h_n. Consider the domain set D to be a set of "balls", and $x_1, x_2, \ldots$ to be "boxes". By a placing p we mean an arrangement of the balls in the boxes in which the balls in each box may be placed in some configuration. The kernel ker p is the partition

$\pi \in \prod(D)$ such that d_1 and d_2 are in the same block of π if they are in the same box. The generating function $\gamma(p)$ of a placing p is the monomial $\prod_i x_i$ (number of balls in x_i), or equivalently $\gamma(p) = \prod_{d \in D}$(box in which d lies). The generating function $\gamma(P)$ of a set P of placings is $\sum_{p \in P} \gamma(p)$.

The notion of a placing is a generalization of a function, since to a function $f:D \to X$ we can associate the placing $\bar{f}$ in which ball d is in box $f(d)$, and the balls in each box are in no special configuration. (Actually placings are similar to reluctant functions, a generalization of functions defined in [15]). It is important to note that $\ker f = \ker \bar{f}$ and $\gamma(f) = \gamma(\bar{f})$. If p is a placing, by the "underlying function" of p we mean the mapping $f:D \to X$ given by: $f(d) = x_i$ if ball d is in box x_i in the placing p.

Using this terminology, the definitions at the beginning of Section 3 can be restated as follows:

$\mathscr{K}_\pi$ = {placings p with no configuration and kernel π}
= {ways of placing the blocks of π into distinct boxes}

$\mathscr{A}_\pi$ = {placings p with no configuration such that no two balls from the same block of π go into the same box}

$\mathscr{T}_\pi$ = {placings p with no configuration such that balls from the same block of π go into the same box}.

We now define two new families:

$\mathscr{H}_\pi$ = {placings p such that within each box the balls from the same block of π are linearly ordered.}.

$\mathscr{F}_\pi$ = {ways of placing the blocks of π into the boxes and within each box linearly ordering the blocks appearing}.

Put $h_\pi = \gamma(\mathscr{H}_\pi)$, $f_\pi = \gamma(\mathscr{F}_\pi)$. We now determine h_π.

THEOREM 5.

$$h_\pi = \lambda(\pi)! h_{\lambda(\pi)} \tag{18}$$

Proof: Since a placing $p \in \mathscr{H}_\pi$ can be obtained by first placing the balls from B_1 into the boxes and linearly ordering within each box, then placing the balls from B_2 and linearly ordering again within each box (independently of how the balls from B_1 are ordered), etc. (where $B_1, B_2, \ldots$ are the blocks of π in some order), it follows that h_π is the product $\prod_{\text{blocks } B} \gamma(\mathscr{H}_B)$, where $\mathscr{H}_B$ is the set of placings of the balls from B into the boxes and linearly ordering them within each box.

In $\gamma(\mathscr{H}_B)$, each monomial $x_{i_1}^{u_1} x_{i_2}^{u_2} \cdots$ of degree $|B|$ arises $|B|!$ times, since there is a one-one correspondence between $\{p \in \mathscr{H}_B | \gamma(p) = x_{i_1}^{u_1} x_{i_2}^{u_2} \cdots\}$ and linear orderings of B, namely to each such placing associate the linear ordering obtained by taking first the balls in x_{i_1} in the given order, then those in x_{i_2}, and so on.

Hence

$$\gamma(\mathscr{H}_B) = |B|! h_{|B|}, \qquad \text{so } \gamma(\mathscr{H}_\pi) = |B_1|! |B_2|! \cdots h_{|B_1|} h_{|B_2|} \cdots$$

i.e.,

$$h_\pi = \lambda(\pi)! h_{\lambda(\pi)}.$$

The relationship between the h's and the k's is given by the following.

THEOREM 6.

$$h_\pi = \sum_\sigma \lambda(\sigma \wedge \pi)!k_\sigma. \tag{19}$$

Proof: For each function $f: D \to X$ with kernel σ, the number of $p \in \mathscr{H}_\pi$ with underlying function f is the number of ways of putting balls D in boxes X as prescribed by f and then linearly ordering within each box the elements from the same block of π, i.e., the number of ways of independently linearly ordering the elements within each block of $\sigma \wedge \pi$. This number is just $\lambda(\sigma \wedge \pi)!$. Since this depends only on σ (for fixed π) and not on f, we have

$$h_\pi = \sum_\sigma \lambda(\sigma \wedge \pi)!k_\sigma.$$

COROLLARY. *If $h_\lambda = \sum_\mu d_{\lambda\mu}k_\mu$, then $d_{\lambda\mu} = d_{\mu\lambda}$.*

Proof: same as the proof of Theorem 4, replacing the function $\delta(0, \sigma \wedge \pi)$ with $\lambda(\sigma \wedge \pi)!$, both of which are symmetric in σ and π.

Formula (19) can be better dealt with using the following result.

LEMMA.

$$\sum_{\sigma \in [\tau, \pi]} |\mu(\tau, \sigma)| = \lambda(\tau, \pi)! \tag{20}$$

Proof. Let

$$f(\tau, \pi) = \sum_{\sigma \in [\tau, \pi]} |\mu(\tau, \sigma)|.$$

It is clear that $f(\tau, \pi) = \prod_i \left(\sum_{\sigma \in \Pi(B_i)} |\mu(0, \sigma)|\right)$ where $B_1, B_2, \ldots$ are the relative blocks of $[\tau, \pi]$. Hence it suffices to show that $\sum_{\sigma \in \Pi_n} |\mu(0, \sigma)| = n!$ But

$$\begin{aligned}
\sum_{\sigma \in \pi} |\mu(0, \sigma)| &= \sum_{\lambda \vdash n} \binom{n}{\lambda} 0!^{r_1} 1!^{r_2} \cdots (i-1)!^{r_i} \cdots \\
&= \sum_{\lambda \vdash n} \frac{n!}{1^{r_1} r_1! 2^{r_2} r_2! \cdots} \\
&= \sum_{\lambda \vdash n} (\#\text{ of permutations } \sigma \text{ in the symmetric group } S_n \text{ of type } \lambda) \\
&= |S_n| \quad (\text{where } |A| = \text{number of elements in } A) \\
&= n!
\end{aligned}$$

Note: The lemma also follows easily from the results on the lattice of partitions in [6].

We now obtain the relationship between the h's and the k's, a's, and s's.

THEOREM 7.

$$\text{(i) } k_\pi = \sum_{\sigma \geq \pi} \frac{\mu(\pi, \sigma)}{|\mu(0, \sigma)|} \sum_{\tau \leq \sigma} \mu(\tau, \sigma) h_\tau \tag{21}$$

$$\text{(ii) } h_\pi = \sum_{\sigma \leq \pi} |\mu(0, \sigma)| s_\sigma \tag{22}$$

(iii) $s_\pi = \dfrac{1}{|\mu(0,\pi)|} \sum_{\sigma \leq \pi} \mu(\sigma,\pi) h_\sigma$ (23)

(iv) $a_\pi = \sum_{\sigma \leq \pi} (\text{sign } \sigma)\lambda(\sigma,\pi)! h_\sigma$ (24)

(v) $h_\pi = \sum_{\sigma \leq \pi} (\text{sign } \sigma)\lambda(\sigma,\pi)! a_\sigma$ (25)

(vi) $1 + \sum_{n \geq 1} h_n t^n = \exp\left(\sum_{n \geq 1} \dfrac{1}{n} s_n t^n\right).$ (26)

Proof:

(i)
$$h_\pi = \sum_\sigma \lambda(\sigma \wedge \pi)! k_\sigma$$
$$= \sum_\sigma \left(\sum_{\tau \leq \sigma \wedge \pi} |\mu(0,\tau)| \right) k_\sigma$$
$$= \sum_{\tau \leq \pi} |\mu(0,\tau)| \sum_{\sigma \geq \tau} k_\sigma$$

Invert twice to obtain (21).

(ii) We showed in (i) that $h_\pi = \sum_{\tau \leq \pi} |\mu(0,\tau)| \sum_{\sigma \geq \tau} k_\sigma$, and since $\sum_{\sigma \geq \tau} k_\sigma = s_\tau$, the result follows.

(iii) Apply Möbius inversion to (ii).

(iv)
$$a_\pi = \sum_{\sigma \leq \pi} \mu(0,\sigma) s_\sigma$$
$$= \sum_{\sigma \leq \pi} \mu(0,\sigma) \cdot \frac{1}{|\mu(0,\sigma)|} \sum_{\tau \leq \sigma} \mu(\tau,\sigma) h_\tau$$
$$= \sum_{\sigma \leq \pi} (\text{sign } \sigma) \sum_{\tau \leq \sigma} \mu(\tau,\sigma) h_\tau$$
$$= \sum_{\tau \leq \pi} \left(\sum_{\sigma \in [\tau,\pi]} (\text{sign } \sigma) \mu(\tau,\sigma) \right) h_\tau$$
$$= \sum_{\tau \leq \pi} (\text{sign } \tau) \left(\sum_{\sigma \in [\tau,\pi]} |\mu(\tau,\sigma)| \right) h_\tau$$
$$= \sum_{\tau \leq \pi} (\text{sign } \tau) \lambda(\tau,\pi)! h_\tau.$$

(v) Proved in the same way as (iv).

(vi) Follows from (ii) and the corollary to theorem 4 in [6].

COROLLARY 1. *$\{h_\lambda\}$ is a basis for $\mathcal{S}_n$.*

COROLLARY 2. *The mapping $\theta : \mathcal{S}_n \to \mathcal{S}_n$ given by $\theta(a_\lambda) = h_\lambda$ satisfies*

(i) *θ is an involution, i.e., $\theta^2 = I$.*

(ii) *The s_λ's are the eigenvectors of θ, with $\theta(s_\lambda) = (\text{sign } \lambda) \cdot s_\lambda$.*

Proof:

(i) This is equivalent to showing that if $a_\lambda = \sum_\mu c_{\lambda\mu} h_\mu$ then $h_\lambda = \sum_\mu c_{\lambda\mu} a_\mu$. But this follows from (iv) and (v) of Theorem 7.

(ii) Let $\pi \in \lambda$. Then

$$\begin{aligned}\theta(s_\lambda) = \theta(s_\pi) &= \theta\left(\frac{1}{\mu(0,\pi)}\sum_{\sigma\le\pi}\mu(\sigma,\pi)a_\sigma\right)\\ &= \frac{1}{\mu(0,\pi)}\sum_{\sigma\le\pi}\mu(\sigma,\pi)h_\sigma\\ &= \operatorname{sign}\pi\frac{1}{|\mu(0,\pi)|}\sum_{\sigma\le\pi}\mu(\sigma,\pi)h_\sigma\\ &= (\operatorname{sign}\pi)s_\pi\\ &= (\operatorname{sign}\lambda)s_\lambda.\end{aligned}$$

It is now time to determine what $f_\pi = \sigma(\mathscr{F}_\pi)$ is. If $p \in \mathscr{F}_\pi$, then clearly $\ker p \ge \pi$. For $\tau \ge \pi$ and $f: D \to X$ with kernel τ, the number of $p \in \mathscr{F}_\pi$ with underlying function f is the number of ways of placing the blocks of π in the boxes as prescribed by f (which makes sense since $\ker f \ge \pi$) and then linearly ordering the blocks in each box. But the number of blocks of π in the various boxes is $\lambda_1, \lambda_2, \ldots, \lambda_p$, where $\lambda(\pi,\tau) = (\lambda_1, \lambda_2, \ldots, \lambda_p)$. Hence the number of $p \in \mathscr{F}_\pi$ with underlying function f is $\lambda_1!\lambda_2!\ldots\lambda_p! = \lambda(\pi,\tau)!$. Thus

$$f_\pi = \sum_{\tau\ge\pi}\lambda(\pi,\tau)!k_\tau. \tag{27}$$

Hence,

$$\begin{aligned}f_\pi &= \sum_{\tau\ge\pi}\lambda(\pi,\tau)!k_\tau\\ &= \sum_{\tau\ge\pi}\left(\sum_{\sigma\in[\pi,\tau]}|\mu(\pi,\sigma)|\right)k_\tau\\ &= \sum_{\sigma\ge\pi}|\mu(\pi,\sigma)|\sum_{\tau\ge\sigma}k_\tau\\ &= \sum_{\sigma\ge\pi}|\mu(\pi,\sigma)|s_\sigma.\end{aligned} \tag{28}$$

Thus

$$\begin{aligned}\theta(k_\pi) &= \theta\left(\sum_{\sigma\ge\pi}\mu(\pi,\sigma)s_\sigma\right)\\ &= \sum_{\sigma\ge\pi}\mu(\pi,\sigma)(\operatorname{sign}\sigma)s_\sigma\\ &= (\operatorname{sign}\pi)\sum_{\sigma\ge\pi}|\mu(\pi,\sigma)|s_\sigma\\ &= (\operatorname{sign}\pi)f_\pi.\end{aligned}$$

Thus we have proved that the f_π's are the images under θ of the k's. Defining $f_\lambda = 1/|\lambda| f_\pi$ (where $\pi \in \lambda$), which makes sense since $\gamma(f_\pi)$ depends only on the type of π, we have proved the following theorem.

THEOREM 8.
(i) $\theta(k_\lambda) = (\text{sign } \lambda) f_\lambda$
(ii) $f_\pi = \sum_{\tau \geq \pi} \lambda(\pi, \tau)! k_\tau$
(iii) $f_\pi = \sum_{\sigma \geq \pi} |\mu(\pi, \sigma)| s_\sigma$.

COROLLARY. *In the image of* k_λ *under* θ, *all monomials appearing have coefficients of the same sign, namely the sign of* λ.

Most of the relationships among the k_π's, s_π's, a_π's, h_π's, and f_π's have been stated by this point. All others (for example the h's in terms of the f's) can easily be obtained from these, and are listed in Appendix 1.

5. The vector space $\tilde{\mathscr{S}}$

It is interesting to note that for most of the results proved so far, no use is made of the fact that k_π, a_π, etc. are really symmetric functions and hence that $k_\pi = k_\sigma$ for π, σ of the same type. For example, not only is the matrix $(c_{\lambda\mu})$ given by $a_\lambda = \sum_\mu c_{\lambda\mu} k_\mu$ symmetric, but so is the matrix $c_{\pi\sigma}$ given by $a_\pi = \sum_\sigma c_{\pi\sigma} k_\sigma$, namely $c_{\pi\sigma} = \delta(0, \pi \wedge \sigma)$. Another example is the fact that $\theta(s_\pi) = (\text{sign } \pi) s_\pi$, which implies that $\theta(s_\lambda) = (\text{sign } \lambda) s_\lambda$, but which is a stronger result (for example, if we showed that $\theta(s_\pi) = \frac{1}{2}((\text{sign } \sigma) s_\sigma + (\text{sign } \tau) s_\tau)$ with π, σ, and τ all of type λ, it would follow that $\theta(s_\lambda) = (\text{sign } \lambda) s_\lambda$).

These considerations lead us to define a vector space $\tilde{\mathscr{S}}$, the vector space over Q freely generated by the symbols $\{\tilde{k}_\pi | \pi \in \prod_n\}$. We then define elements $\tilde{a}_\pi$, $\tilde{s}_\pi$, $\tilde{h}_\pi$, $\tilde{f}_\pi$, in $\tilde{\mathscr{S}}$ by

$$\tilde{a}_\pi = \sum_{\sigma : \sigma \wedge \pi = 0} \tilde{k}_\sigma \tag{29}$$

$$\tilde{s}_\pi = \sum_{\sigma \geq \pi} \tilde{k}_\sigma \tag{30}$$

$$\tilde{h}_\pi = \sum_\sigma \lambda(\sigma \wedge \pi)! \tilde{k}_\sigma \tag{31}$$

$$\tilde{f}_\pi = \sum_{\sigma \geq \pi} \lambda(\pi, \sigma)! \tilde{k}_\sigma. \tag{32}$$

Since all other formulas obtained in the previous two sections and in Appendix 1 can be obtained from these by Möbius inversion and similar techniques, they all hold in $\tilde{\mathscr{S}}$. Hence $\{\tilde{a}_\pi\}$, $\{\tilde{s}_\pi\}$, $\{\tilde{h}_\pi\}$, and $\{\tilde{f}_\pi\}$ are bases of $\tilde{\mathscr{S}}$. Also, if we define the mapping $\tilde{\theta} : \tilde{\mathscr{S}} \to \tilde{\mathscr{S}}$ by $\tilde{\theta}(\tilde{a}_\pi) = \tilde{h}_\pi$, using formulas (24), (25), and the same method as in the proof of part (ii) of Corollary 2 to Theorem 7, it follows that $\tilde{\theta}$ is an involution with eigenvectors $\tilde{s}_\pi$.

Summarizing the results obtained so far, we have

THEOREM 9.
(i) $\{\tilde{a}_\pi\}$, $\{\tilde{h}_\pi\}$, $\{\tilde{s}_\pi\}$, *and* $\{\tilde{f}_\pi\}$ *are bases of* $\tilde{\mathscr{S}}$.
(ii) *The matrices* $(c_{\pi\sigma})$ *and* $(d_{\pi\sigma})$ *given by* $\tilde{a}_\pi = \sum_\sigma c_{\pi\sigma} \tilde{k}_\sigma$ *and* $\tilde{h}_\pi = \sum_\sigma d_{\pi\sigma} \tilde{k}_\sigma$ *are symmetric.*
(iii) $\tilde{\theta}$ *is an involution*
(iv) $\tilde{\theta}(\tilde{s}_\pi) = (\text{sign } \pi) \tilde{s}_\pi$
(v) $\tilde{\theta}(\tilde{k}_\pi) = (\text{sign } \pi) \tilde{f}_\pi$.

Now we define a mapping $\phi:\tilde{\mathscr{S}} \to \mathscr{S}$ by $\phi(\tilde{k}_\pi) = k_\pi(= |\lambda(\pi)| k_{\lambda(\pi)})$. By the results of the previous two sections, we have

THEOREM 10. $\phi(\tilde{a}_\pi) = a_\pi$, $\phi(\tilde{s}_\pi) = s_\pi$, $\phi(\tilde{h}_\pi) = h_\pi$, $\phi(\tilde{f}_\pi) = f_\pi$.

The interesting fact of this section, and one that will be useful in the next two sections is that there is a map $\phi^*:\mathscr{S} \to \tilde{\mathscr{S}}$ such that $\phi\phi^* = I =$ identity map on $\mathscr{S}$. To see this, we define elements $\tilde{K}_\lambda, \tilde{A}_\lambda, \tilde{S}_\lambda, \tilde{H}_\lambda$, and $\tilde{F}_\lambda$ in $\tilde{\mathscr{S}}$ by $\tilde{K}_\lambda = \sum_{\pi\in\lambda} \tilde{k}_\pi$, $\tilde{A}_\lambda = \sum_{\pi\in\lambda} \tilde{a}_\pi$, etc., and elements $K_\lambda, A_\lambda, S_\lambda, H_\lambda$, and F_λ in $K_\lambda = \sum_{\pi\in\lambda} k_\pi = \binom{n}{\lambda}|\lambda| k_\lambda$ (where we recall that $\binom{n}{\lambda} = \dfrac{n!}{\lambda!|\lambda|}$ is the number of $\pi \in \prod_n$ of type λ), $A_\lambda = \sum_{\pi\in\lambda} a_\pi = \binom{n}{\lambda}\lambda! a_\lambda$, etc. Let $\phi^*:\mathscr{S} \to \tilde{\mathscr{S}}$ be defined by $\phi^*(K_\lambda) = \tilde{K}_\lambda$, and let "$\mathscr{S}$" be the image under ϕ^* of $\mathscr{S}$ (i.e., "$\mathscr{S}$" is the subspace of $\tilde{\mathscr{S}}$ spanned by $\{\tilde{K}_\lambda\}$). We then have the following theorem.

THEOREM 11.
(i) $\tilde{A}_\lambda, \tilde{S}_\lambda, \tilde{H}_\lambda, \tilde{F}_\lambda$ *are all in* "$\mathscr{S}$".
(ii) $\phi^*(A_\lambda) = \tilde{A}_\lambda, \phi^*(S_\lambda) = \tilde{S}_\lambda$, *etc.*
(iii) $\phi\phi^* = I =$ *identity map on* $\mathscr{S}$.

Proof:

(i) $$\tilde{A}_\lambda = \sum_{\pi\in\lambda} \tilde{a}_\pi = \sum_{\pi\in\lambda}\left(\sum_{\sigma:\sigma\wedge\pi=0} \tilde{k}_\sigma\right)$$
$$= \sum_\sigma (\#\text{ of } \pi\in\lambda \text{ s.t. } \sigma\wedge\pi = 0)\tilde{k}_\sigma.$$

But clearly ($\#$ of $\pi\in\lambda$ s.t. $\sigma\wedge\pi = 0$) depends only on the type μ of σ (for fixed λ), so call this number $c_{\lambda\mu}$.
Then $\tilde{A}_\lambda = \sum_\mu c_{\lambda\mu}(\sum_{\sigma\in\mu} \tilde{k}_\sigma) = \sum_\mu c_{\lambda\mu}\tilde{K}_\mu$.
Therefore $\tilde{A}_\lambda \in$ "$\mathscr{S}$"
Similarly for $\tilde{S}_\lambda, \tilde{H}_\lambda$, and $\tilde{F}_\lambda$.

(ii) $\tilde{A}_\lambda = \sum_\mu c_{\lambda\mu}\tilde{K}_\mu$ with $c_{\lambda\mu}$ as above.
Therefore $\phi(\tilde{A}_\lambda) = \sum_\mu c_{\lambda\mu}\phi(\tilde{K}_\mu) = \sum_\mu c_{\lambda\mu}K_\mu$.
Therefore $A_\lambda = \sum_\mu c_{\lambda\mu}K_\mu$.
Applying ϕ^*, $\phi^*(A_\lambda) = \sum_\mu c_{\lambda\mu}\phi^*(K_\mu) = \sum_\mu c_{\lambda\mu}\tilde{K}_\mu = \tilde{A}_\lambda$.
Similarly for $\tilde{S}_\lambda, \tilde{H}_\lambda$, and $\tilde{F}_\lambda$.

(iii) $\phi\phi^*(K_\lambda) = \phi(\tilde{K}_\lambda) = K_\lambda$, so the result holds since $\{K_\lambda\}$ is a basis for $\mathscr{S}$.

6. Hall's inner product

In this section we define an inner product on $\tilde{\mathscr{S}}$ in such a way that $\phi^*:\mathscr{S} \to \tilde{\mathscr{S}}$ is an isometry with respect to the inner product on $\mathscr{S}$ defined by Philip Hall (see [10]). Again in this section, many familiar results holding for Hall's inner product hold in their stronger form for $\tilde{\mathscr{S}}$, and also certain computations of inner products in $\mathscr{S}$ can be facilitated by translating them, via ϕ^*, to computations in $\tilde{\mathscr{S}}$ which are usually simpler.

Hall's inner product on $\mathscr{S}$ is defined by taking $(h_\lambda, k_\mu) = \delta_{\lambda\mu}$ (and extending by bilinearity). We define an inner product on $\tilde{\mathscr{S}}$ by putting $(\tilde{h}_\pi, \tilde{k}_\sigma) = n!\delta_{\pi\sigma}$. The notation (,) for both inner products is the same, but this should cause no confusion.

A number of important results about this inner product are given by the following theorem.

THEOREM 12.

(i) *This inner product is symmetric, i.e.,* $(\tilde{f}, \tilde{g}) = (\tilde{g}, \tilde{f})$ *for* $\tilde{f}, \tilde{g} \in \tilde{\mathscr{S}}$.

(ii) $\phi^* : \mathscr{S} \to \tilde{\mathscr{S}}$ is an isometry, i.e., $(\phi^*(f), \phi^*(g)) = (f, g)$ *for* $f, g \in \mathscr{S}$.

(iii) $(\tilde{s}_\pi, \tilde{s}_\sigma) = \delta_{\pi\sigma} \cdot n!/|\mu(0, \pi)|$.

(iv) $\tilde{\theta} : \tilde{\mathscr{S}} \to \tilde{\mathscr{S}}$ *is an isometry.*

Proof:

(i) Follows from Theorem 9 (ii), together with the following easily proved fact from linear algebra: If V is any vector space, $\{v_i\}$ and $\{w_i\}$ two bases such that $v_i = \sum_j c_{ij} w_j$, then the inner $(\,,\,)$ on V defined by $(v_i, w_j) = \delta_{ij}$ is symmetric if and only if the matrix (c_{ij}) is symmetric.

(ii) It suffices to show that $(\phi^*(b_\lambda), \phi^*(b'_\mu)) = (b_\lambda, b'_\mu)$ for some pair of bases $\{b_\lambda\}, \{b'_\lambda\}$ of $\mathscr{S}$. Now,

$$
\begin{aligned}
(\phi^*(H_\lambda), \phi^*(K_\mu)) = (\tilde{H}_\lambda, \tilde{K}_\mu) &= \left(\sum_{\pi\in\lambda} \tilde{h}_\pi, \sum_{\sigma\in\mu} \tilde{k}_\sigma\right) \\
&= \sum_{\substack{\pi\in\lambda \\ \sigma\in\mu}} \delta_{\pi\sigma} n! \\
&= n!\,\delta_{\lambda\mu} \cdot (\#\text{ of } \pi \in \lambda) \\
&= n! \binom{n}{\lambda} \delta_{\lambda\mu}.
\end{aligned}
$$

And

$$
\begin{aligned}
(H_\lambda, K_\mu) &= \left(\binom{n}{\lambda} \lambda!\, h_\lambda, \binom{n}{\mu} |\mu| k_\mu\right) \\
&= \delta_{\lambda\mu} \binom{n}{\lambda} \cdot \binom{n}{\mu} \lambda!\,|\mu| \\
&= \delta_{\lambda\mu} \binom{n}{\lambda}^2 \lambda!\,|\lambda| \\
&= \delta_{\lambda\mu} \cdot \binom{n}{\lambda} \cdot \frac{n!}{\lambda!\,|\lambda|} \cdot \lambda!\,|\lambda| \\
&= \delta_{\lambda\mu} \binom{n}{\lambda} \cdot n!
\end{aligned}
$$

Therefore $(\phi^*(H_\lambda), \phi^*(K_\mu)) = (H_\lambda, K_\mu)$.

(iii)
$$
\begin{aligned}
(\tilde{s}_\pi, \tilde{s}_\sigma) &= \left(\frac{1}{|\mu(0, \pi)|} \sum_{\tau\leq\pi} \mu(\tau, \pi) \tilde{h}_\tau, \sum_{\nu\geq\sigma} \tilde{k}_\nu\right) \\
&= \frac{1}{|\mu(0, \pi)|} \sum_{\substack{\tau\leq\pi \\ \nu\geq\sigma}} \mu(\tau, \pi)(\tilde{h}_\tau, \tilde{k}_\nu)
\end{aligned}
$$

$$= \frac{n!}{|\mu(0,\pi)|} \sum_{\tau \in [\sigma,\pi]} \mu(\tau,\pi)$$

$$= \delta_{\sigma\pi} \cdot \frac{n!}{|\mu(0,\pi)|}.$$

(iv) $(\tilde{\theta}(\tilde{s}_\pi), \tilde{\theta}(\tilde{s}_\sigma)) = ((\text{sign }\pi)\tilde{s}_\pi, (\text{sign }\sigma)\tilde{s}_\sigma)$

$$= (\text{sign }\pi)(\text{sign }\sigma)(\tilde{s}_\pi, \tilde{s}_\sigma)$$

$$= (\tilde{s}_\pi, \tilde{s}_\sigma) \quad (\text{since } (\tilde{s}_\pi, \tilde{s}_\sigma) = 0 \text{ if } \pi \neq \sigma).$$

COROLLARY:
(i) *Hall's inner product on* $\mathcal{S}$ *is symmetric*
(ii) $(s_\lambda, s_\mu) = \delta_{\lambda\mu} 1^{r_1} r_1! 2^{r_2} r_2! \ldots$, where $\lambda = (1^{r_1} 2^{r_2} \cdots)$.
(iii) $\theta : \mathcal{S} \to \mathcal{S}$ is an isometry.
Proof:
(i) $(f, g) = (\phi^*(f), \phi^*(g)) = (\phi^*(g), \phi^*(f)) = (g, f)$

(ii) $(s_\lambda, s_\mu) = \dfrac{1}{\binom{n}{\lambda}\binom{n}{\mu}} (S_\lambda, S_\mu) = \dfrac{1}{\binom{n}{\lambda}\binom{n}{\mu}} (\tilde{S}_\lambda, \tilde{S}_\mu)$

$$= \frac{1}{\binom{n}{\lambda}\binom{n}{\mu}} \left(\sum_{\pi \in \lambda} \tilde{s}_\pi, \sum_{\sigma \in \mu} \tilde{s}_\sigma \right)$$

$$= \delta_{\lambda\mu} \frac{1}{\binom{n}{\lambda}^2} \sum_{\pi \in \lambda} (\tilde{s}_\pi, \tilde{s}_\pi)$$

$$= \delta_{\lambda\mu} \frac{1}{\binom{n}{\lambda}^2} \cdot \binom{n}{\lambda} \cdot \frac{n!}{|\mu(0,\pi)|} \quad (\text{where } \pi \in \lambda)$$

$$= \delta_{\lambda\mu} \cdot \frac{n!}{\binom{n}{\lambda} |\mu(0,\pi)|}$$

$$= \delta_{\lambda\mu} \cdot 1^{r_1} r_1! 2^{r_2} r_2! \ldots$$

(iii) $(\theta(s_\lambda), \theta(s_\mu)) = ((\text{sign }\lambda) s_\lambda, (\text{sign }\mu) s_\mu)$

$$= (\text{sign }\lambda)(\text{sign }\mu)(s_\lambda, s_\mu)$$

$$= (s_\lambda, s_\mu) \quad (\text{since } (s_\lambda, s_\mu) = 0 \text{ if } \lambda \neq \mu)$$

All inner products involving the members of the bases studied so far are easily calculated, and are listed in Appendix 2.

7. The Kronecker inner product

The Kronecker inner product on $\mathscr{S}$ is a mapping from $\mathscr{S} \times \mathscr{S} \to \mathscr{S}$ which reflects the Kronecker inner product of representations of S_n (the symmetric group of degree n) in the same way as Hall's inner product mirrors the inner product of representations. Since we are assuming no knowledge of group theory, we can define the Kronecker inner product, which we denote [,], as follows:

$$\left[\frac{1}{n!}\sum_{\lambda\vdash n}\begin{bmatrix} n \\ \lambda \end{bmatrix} c_\lambda s_\lambda, \frac{1}{n!}\sum_{\lambda\vdash n}\begin{bmatrix} n \\ \lambda \end{bmatrix} d_\lambda s_\lambda\right] = \frac{1}{n!}\sum_{\lambda\vdash n}\begin{bmatrix} n \\ \lambda \end{bmatrix}(c_\lambda d_\lambda) s_\lambda \tag{33}$$

where $c_\lambda, d_\lambda \in Q$ and

$$\begin{bmatrix} n \\ \lambda \end{bmatrix} = \frac{n!}{1^{r_1}r_1!\,2^{r_2}r_2!\cdots} = \binom{n}{\lambda}\cdot|\mu(0,\pi)| \qquad (\text{for } \pi \in \lambda).$$

This mapping is bilinear, and it is easily verified that

$$[s_\lambda, s_\mu] = \delta_{\lambda\mu}\cdot\frac{n!}{\begin{bmatrix} n \\ \lambda \end{bmatrix}} s_\lambda = (s_\lambda, s_\mu)s_\lambda. \tag{34}$$

In $\tilde{\mathscr{S}}$, we define $[\,,\,]: \tilde{\mathscr{S}} \times \tilde{\mathscr{S}} \to \tilde{\mathscr{S}}$ to be the bilinear map with

$$[\tilde{s}_\pi, \tilde{s}_\sigma] = (\tilde{s}_\pi, \tilde{s}_\sigma)\cdot\tilde{s}_\pi = \delta_{\pi\sigma}\cdot\frac{n!}{|\mu(0,\pi)|}\tilde{s}_\pi.$$

Parallelling Theorem 11, we have

THEOREM 13:

(i) $[\,,\,]$ *is symmetric on* $\tilde{\mathscr{S}}$, *i.e.,* $\tilde{\mathscr{S}}$ *with product given by* $[\,,\,]$ *is a commutative algebra.*

(ii) $\phi^*[f,g] = [\phi^*(f), \phi^*(g)]$ *for* $f, g \in \mathscr{S}$, *i.e.,* $\phi^*: \mathscr{S} \to \tilde{\mathscr{S}}$ *is an algebra homomorphism.*

(iii) $[f,g] = \phi[\phi^*(f), \phi^*(g)]$ *for* $f, g \in \mathscr{S}$.

(iv) $\tilde{\theta}: \tilde{\mathscr{S}} \to \tilde{\mathscr{S}}$ *satisfies* $[\tilde{\theta}(\tilde{f}), \tilde{\theta}(\tilde{g})] = [\tilde{f}, \tilde{g}]$ *for* $\tilde{f}, \tilde{g} \in \tilde{\mathscr{S}}$, *i.e.,* $\tilde{\theta}$ *is an algebra homomorphism.*

Proof:

$$\begin{aligned}
\text{(i) } [\tilde{s}_\pi, \tilde{s}_\sigma] &= (\tilde{s}_\pi, \tilde{s}_\sigma)\tilde{s}_\pi \\
&= (\tilde{s}_\pi, \tilde{s}_\sigma)\tilde{s}_\sigma \quad (\text{since } (\tilde{s}_\pi, \tilde{s}_\sigma) = 0 \text{ if } \pi \neq \sigma) \\
&= (\tilde{s}_\sigma, \tilde{s}_\pi)\tilde{s}_\sigma \\
&= [\tilde{s}_\sigma, \tilde{s}_\pi]
\end{aligned}$$

The result follows by bilinearity.

(ii) It suffices, by bilinearity, to show that

$$\phi^*[S_\lambda, S_\mu] = [\phi^*(S_\lambda), \phi^*(S_\mu)],$$

Now

$$\phi^*[S_\lambda, S_\mu] = \phi^*\left(\binom{n}{\lambda}\binom{n}{\mu}(s_\lambda, s_\mu)s_\lambda\right)$$

$$= \phi^*\left(\delta_{\lambda\mu}\binom{n}{\lambda}^2 \frac{n!}{\binom{n}{\lambda}|\mu(0,\pi)|}s_\lambda\right) \qquad (\text{for } \pi \in \lambda)$$

$$= \phi^*\left(\delta_{\lambda\mu}\frac{n!}{|\mu(0,\pi)|}S_\lambda\right)$$

$$= \delta_{\lambda\mu}\cdot\frac{n!}{|\mu(0,\pi)|}\tilde{S}_\lambda$$

and

$$[\phi^*(S_\lambda), \phi^*(S_\mu)] = [\tilde{S}_\lambda, \tilde{S}_\mu]$$

$$= \left[\sum_{\pi\in\lambda}\tilde{s}_\pi, \sum_{\sigma\in\mu}\tilde{s}_\sigma\right]$$

$$= \sum_{\pi\in\lambda,\sigma\in\mu}[\tilde{s}_\pi, \tilde{s}_\sigma]$$

$$= \delta_{\lambda\mu}\sum_{\pi\in\lambda}(\tilde{s}_\pi, \tilde{s}_\pi)\tilde{s}_\pi$$

$$= \delta_{\lambda\mu}\cdot\frac{n!}{|\mu(0,\pi)|}\sum_{\pi\in\lambda}\tilde{s}_\pi$$

$$= \delta_{\lambda\mu}\frac{n!}{|\mu(0,\pi)|}\tilde{S}_\lambda \qquad (\text{for } \pi \in \lambda),$$

which proves (ii).

(iii) Apply ϕ to (ii) and use the fact that $\phi\phi^* = I$.

(iv) $[\tilde{\theta}(\tilde{s}_\pi), \tilde{\theta}(\tilde{s}_\sigma)] = [(\text{sign }\pi)\tilde{s}_\pi, (\text{sign }\sigma)\tilde{s}_\sigma] = (\text{sign }\pi)(\text{sign }\sigma)[\tilde{s}_\pi, \tilde{s}_\sigma] = [\tilde{s}_\pi, \tilde{s}_\sigma]$, since $[\tilde{s}_\pi, \tilde{s}_\sigma] = 0$ if $\pi \neq \sigma$.

Part (iii) of the preceding theorem can be used to translate computations of the Kronecker inner product in $\mathscr{S}$ to a computation in $\tilde{\mathscr{S}}$, which is usually easier. As an example, let us compute $[h_\lambda, h_\mu]$:

First,

$$[\tilde{h}_\pi, \tilde{h}_\sigma] = \left[\sum_{\tau\leq\pi}|\mu(0,\tau)|\tilde{s}_\tau, \sum_{\nu\leq\sigma}|\mu(0,\nu)|\tilde{s}_\nu\right]$$

$$= \sum_{\tau\leq\pi,\nu\leq\sigma}|\mu(0,\tau)|\cdot|\mu(0,\nu)|[\tilde{s}_\tau, \tilde{s}_\nu]$$

$$= \sum_{\tau\leq\pi,\sigma}|\mu(0,\tau)|^2[\tilde{s}_\tau, \tilde{s}_\tau] \qquad (\text{since } [\tilde{s}_\tau, \tilde{s}_\nu] = 0 \quad \text{if } \tau \neq \nu)$$

$$= \sum_{\tau\leq\pi\wedge\sigma}|\mu(0,\tau)|^2\frac{n!}{|\mu(0,\tau)|}\tilde{s}_\tau$$

$$= n!\sum_{\tau\leq\pi\wedge\sigma}|\mu(0,\tau)|\tilde{s}_\tau$$

$$= n!\tilde{h}_{\pi\wedge\sigma}.$$

Thus,

$$
\begin{aligned}
[h_\lambda, h_\mu] &= \frac{1}{\lambda!\binom{n}{\lambda}\mu!\binom{n}{\mu}}[H_\lambda, H_\mu] \\
&= \frac{1}{\lambda!\binom{n}{\lambda}\mu!\binom{n}{\mu}}\phi[\phi^*(H_\lambda), \phi^*(H_\mu)] \\
&= \frac{|\lambda|\cdot|\mu|}{(n!)^2}\phi\left[\sum_{\pi\in\lambda}\tilde{h}_\pi, \sum_{\sigma\in\mu}\tilde{h}_\sigma\right] \\
&= \frac{|\lambda||\mu|}{(n!)}\sum_{\pi\in\lambda,\sigma\in\mu}\phi(\tilde{h}_{\pi\wedge\sigma}) \\
&= \frac{|\lambda||\mu|}{(n!)}\sum_{\pi\in\lambda,\sigma\in\mu}\lambda(\pi\wedge\sigma)!h_{\lambda(\pi\wedge\sigma)}
\end{aligned}
$$

This shows that $[h_\lambda, h_\mu]$ is a positive linear combination of h_ρ's, which can be interpreted to say something interesting about permutation representations of S_n. Also, since

$$[\tilde{a}_\pi, \tilde{a}_\sigma] = [\tilde{\theta}(\tilde{a}_\pi), \tilde{\theta}(\tilde{a}_\sigma)] = [\tilde{h}_\pi, \tilde{h}_\sigma] = n!\tilde{h}_{\pi\wedge\sigma},$$

it follows that $[a_\lambda, a_\mu]$ is a positive linear combination of h_ρ's, again a fact of interest.

A complete list of Kronecker products of the various bases is to be found in Appendix 3.

Appendix 1: Connections between bases

1. $s_\pi = \sum_{\sigma\geq\pi} k_\sigma$ $\qquad$ $k_\pi = \sum_{\sigma\geq\pi}\mu(\pi,\sigma)s_\sigma$

2. $a_\pi = \sum_{\sigma:\sigma\wedge\pi=0} k_\sigma$ $\qquad$ $k_\pi = \sum_{\tau\geq\pi}\frac{\mu(\pi,\tau)}{\mu(0,\tau)}\sum_{\sigma\leq\tau}\mu(\sigma,\tau)a_\sigma$

3. $a_\pi = \sum_{\sigma\leq\pi}\mu(0,\sigma)s_\sigma$ $\qquad$ $s_\pi = \frac{1}{\mu(0,\pi)}\sum_{\sigma\leq\pi}\mu(\sigma,\pi)a_\sigma$

4. $s_\pi = \sum_{\sigma\geq\pi}\operatorname{sign}(\pi,\sigma)f_\sigma$ $\qquad$ $f_\pi = \sum_{\sigma\geq\pi}|\mu(\pi,\sigma)|s_\sigma$

5. $h_\pi = \sum_{\sigma:\sigma\wedge\pi=0}(\operatorname{sign}\sigma)f_\sigma$ $\qquad$ $f_\pi = \sum_{\tau\geq\pi}\frac{|\mu(\pi,\tau)|}{|\mu(0,\tau)|}\sum_{\sigma\leq\tau}\mu(\sigma,\tau)h_\sigma$

6. $h_\pi = \sum_{\sigma\leq\pi}|\mu(0,\sigma)|s_\sigma$ $\qquad$ $s_\pi = \frac{1}{|\mu(0,\pi)|}\sum_{\sigma\leq\pi}\mu(\sigma,\pi)h_\sigma$

7. $h_\pi = \sum_{\sigma}\lambda(\pi\wedge\sigma)!k_\sigma$ $\qquad$ $k_\pi = \sum_{\tau\geq\pi}\frac{\mu(\pi,\tau)}{|\mu(0,\tau)|}\sum_{\sigma\leq\tau}\mu(\sigma,\tau)h_\sigma$

8. $a_\pi = \sum_{\sigma}(\operatorname{sign}\sigma)\lambda(\pi\wedge\sigma)!f_\sigma$ $\qquad$ $f_\pi = \sum_{\tau\geq\pi}\frac{|\mu(\pi,\tau)|}{\mu(0,\tau)}\sum_{\sigma\leq\tau}\mu(\sigma,\tau)a_\sigma$

9. $k_\pi = \sum_{\sigma \geq \pi} \operatorname{sign}(\pi, \sigma)\lambda(\pi, \sigma)!f_\sigma \qquad f_\pi = \sum_{\sigma \geq \pi} \lambda(\pi, \sigma)!k_\sigma$

10. $a_\pi = \sum_{\sigma \leq \pi} (\operatorname{sign} \sigma)\lambda(\sigma, \pi)!h_\sigma \qquad h_\pi = \sum_{\sigma \leq \pi} (\operatorname{sign} \sigma)\lambda(\sigma, \pi)!a_\sigma$

Appendix 2: The inner product on $\tilde{\mathscr{S}}$

1. $(\tilde{h}_\pi, \tilde{k}_\sigma) = n!\delta_{\pi\sigma}$
2. $(\tilde{s}_\pi, \tilde{s}_\sigma) = \delta_{\pi\sigma}\dfrac{n!}{|\mu(0, \pi)|}$
3. $(\tilde{h}_\pi, \tilde{s}_\sigma) = n!\zeta(\sigma, \pi)$, where $\zeta(\sigma, \pi) = \begin{cases} 1 & \text{if } \sigma \leq \pi \\ 0 & \text{if not} \end{cases}$
4. $(\tilde{a}_\pi, \tilde{s}_\sigma) = n!(\operatorname{sign} \sigma)\zeta(\sigma, \pi)$
5. $(\tilde{a}_\pi, \tilde{a}_\sigma) = n!\lambda(\sigma \wedge \pi)!$
6. $(\tilde{a}_\pi, \tilde{h}_\sigma) = \begin{cases} n! & \text{if } \pi \wedge \sigma = 0 \\ 0 & \text{if } \pi \wedge \sigma \neq 0 \end{cases}$
7. $(\tilde{k}_\pi, \tilde{k}_\sigma) = n! \sum_{\tau \geq \pi \vee \sigma} \dfrac{\mu(\pi, \tau)\mu(\sigma, \tau)}{|\mu(0, \tau)|}$
8. $(\tilde{a}_\pi, \tilde{k}_\sigma) = n!(\operatorname{sign} \sigma)\lambda(\sigma, \pi)!\zeta(\sigma, \pi)$
9. $(\tilde{h}_\pi, \tilde{h}_\sigma) = n!\lambda(\sigma \wedge \pi)!$
10. $(\tilde{k}_\pi, \tilde{s}_\sigma) = n!\dfrac{\mu(\pi, \sigma)}{|\mu(0, \sigma)|}\zeta(\pi, \sigma)$
11. $(\tilde{f}_\pi, \tilde{k}_\sigma) = n! \sum_{\tau \geq \pi \vee \sigma} \dfrac{|\mu(\pi, \tau)|\mu(\sigma, \tau)}{|\mu(0, \tau)|}$
12. $(\tilde{f}_\pi, \tilde{f}_\sigma) = (\operatorname{sign} \pi)(\operatorname{sign} \sigma)n! \sum_{\tau \geq \pi \vee \sigma} \dfrac{\mu(\pi, \tau)\mu(\sigma, \tau)}{|\mu(0, \tau)|}$
13. $(\tilde{f}_\pi, \tilde{h}_\sigma) = n!\lambda(\pi, \sigma)!\zeta(\pi, \sigma)$
14. $(\tilde{f}_\pi, \tilde{s}_\sigma) = (\operatorname{sign} \pi)(\operatorname{sign} \sigma)n!\dfrac{\mu(\pi, \sigma)}{|\mu(0, \sigma)|}\zeta(\pi, \sigma)$
15. $(\tilde{f}_\pi, \tilde{a}_\sigma) = n!(\operatorname{sign} \pi)\delta_{\pi\sigma}$

Appendix 3: The Kronecker inner product on $\tilde{\mathscr{S}}$

1. $[\tilde{s}_\pi, \tilde{s}_\sigma] = \dfrac{n!}{|\mu(0, \pi)|}\delta_{\pi\sigma}\tilde{s}_\pi$
2. $[\tilde{h}_\pi, \tilde{s}_\sigma] = n!\zeta(\sigma, \pi)\tilde{s}_\sigma$
3. $[\tilde{a}_\pi, \tilde{s}_\sigma] = n!(\operatorname{sign} \sigma)\zeta(\sigma, \pi)\tilde{s}_\sigma$

4. $[\tilde{f}_\pi, \tilde{s}_\sigma] = n! \dfrac{|\mu(\pi, \sigma)|}{|\mu(0, \sigma)|} \zeta(\pi, \sigma) \tilde{s}_\sigma$

5. $[\tilde{k}_\pi, \tilde{s}_\sigma] = n! \dfrac{\mu(\pi, \sigma)}{|\mu(0, \sigma)|} \zeta(\pi, \sigma) \tilde{s}_\sigma$

6. $[\tilde{h}_\pi, \tilde{h}_\sigma] = n! \tilde{h}_{\pi \wedge \sigma}$

7. $[\tilde{a}_\pi, \tilde{a}_\sigma] = n! \tilde{h}_{\pi \wedge \sigma}$

8. $[\tilde{a}_\pi, \tilde{h}_\sigma] = n! \tilde{a}_{\pi \wedge \sigma}$

9. $[\tilde{k}_\pi, \tilde{h}_\sigma] = n! \sum_{\tau \in [\pi, \sigma]} \mu(\pi, \tau) \tilde{s}_\tau$

10. $[\tilde{f}_\pi, \tilde{h}_\sigma] = n! \sum_{\tau \in [\pi, \sigma]} |\mu(\pi, \tau)| \tilde{s}_\tau$

11. $[\tilde{k}_\pi, \tilde{a}_\sigma] = n! (\operatorname{sign} \pi) \sum_{\tau \in [\pi, \sigma]} |\mu(\pi, \tau)| \tilde{s}_\tau$

12. $[\tilde{k}_\pi, \tilde{k}_\sigma] = n! \sum_{\tau \geq \pi \vee \sigma} \dfrac{\mu(\pi, \tau) \mu(\sigma, \tau)}{|\mu(0, \tau)|} \tilde{s}_\tau$

13. $[\tilde{f}_\pi, \tilde{f}_\sigma] = n! \sum_{\tau \geq \pi \vee \sigma} \dfrac{|\mu(\pi, \tau)| \, |\mu(\sigma, \tau)|}{|\mu(0, \tau)|} \tilde{s}_\tau$

14. $[\tilde{k}_\pi, \tilde{f}_\sigma] = n! \sum_{\tau \geq \pi \vee \sigma} \dfrac{\mu(\pi, \tau) |\mu(\sigma, \tau)|}{|\mu(0, \tau)|} \tilde{s}_\tau$

15. $[\tilde{f}_\pi, \tilde{a}_\sigma] = n! (\operatorname{sign} \pi) \sum_{\tau \in [\pi, \sigma]} \mu(\pi, \tau) \tilde{s}_\tau$

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