1 Transitions between bases [Rosas 2001]

First, a standard linear algebra fact. Let V be a vector space of finite dimension n over ground field $\Bbbk$ (it doesn't matter what it is).

Proposition 1. *Suppose that $B = \{v_1, \dots, v_n\}$ and $B^* = \{v_1^*, \dots, v_n^*\}$ are bases for V that are orthogonal, i.e.,*

$$\langle v_i, v_j^* \rangle = \delta_{ij} k_i$$

for some nonzero constants $k_1, \dots, k_n$. Then the unique expansion of u with respect to B is

$$u = \sum_{i=1}^{n} \frac{\langle v_i^*,\, u\rangle}{k_i} v_i. \tag{1}$$

Proof. For every $j \in [n]$ we have

$$\left\langle v_j^*,\, \sum_{i=1}^{n} \frac{\langle v_i^*,\, u\rangle}{k_i} v_i \right\rangle = \sum_{i=1}^{n} \frac{\langle v_i^*,\, u\rangle}{k_i} \langle v_j^*,\, v_i\rangle = \sum_{i=1}^{n} \langle v_i^*,\, u\rangle \delta_{ij} = \langle v_j^*,\, u\rangle.$$

That is, every element of B^* has the same inner product with the left and right-hand sides of (1). This is necessary and sufficient for equality. $\square$

This proposition reduces the problem of expanding a vector in a particular basis B to computing inner products with an orthogonal basis B^*. (The best possible case is that $k_i = 1$ for all i, i.e., B^* is the dual basis to B. However, we're not always that lucky.)

Notation:

- $\mathbb{N}$ = nonnegative integers;
- For $u \in \mathbb{N}^k$, a **vector partition** $\lambda \vdash u$ is an unordered sequence of vectors (**parts**) summing to u; zero vectors can be ignored.
- Weight of a vector partition = sum of all entries in all vectors in it (ex.: $\mathrm{wt}(\{120, 100, 100, 013\}) = 9$)
- $\mathfrak{M}_{\mathbf{u}}$ = MacMahon symmetric functions of homogeneous multidegree u
- $\mathrm{sign}(\lambda) = (-1)^{\text{number of parts with even sum}}$
- $m_\lambda, e_\lambda, h_\lambda, p_\lambda, f_\lambda$: monomial, elementary, homogeneous, power-sum, and forgotten MacMahon symmetric functions (Rosas 327–328)
- For a vector partition λ in which each part v_i occurs with multiplicity m_i, define

$$|\lambda| = \prod_i m_i! \qquad \text{and} \qquad \lambda! = \prod_i \prod_{x \in v_i} (x!)^{m_i}.$$

A vector partition is **unitary** if it is a partition of the all-ones vector $\mathbf{1} \in \mathbb{N}^k$. There is an obvious bijection between unitary vector partitions and set partitions of $[k]$. There is a whole theory of MacMahon symmetric functions of unitary partitions, due to Doubilet. These are the graded pieces $\mathfrak{M}_{(1)^k}$.

The **type** of a set partition $\pi = B_1 | \cdots | B_\ell$, with respect to a vector $u = (u_1, \dots, u_r)$ of weight n, is the vector partition $\mathrm{type}_{\mathbf{u}}(\pi) = \lambda = \lambda_1 \cdots \lambda_\ell$ where λ_k is the vector in $\mathbb{N}^r$ whose ith coordinate is

$$\#\{j \in B_k \mid u_1 + \cdots + u_{i-1} < j \leq i_1 + \cdots + u_i\}.$$

the number of elements of B_k such that in the ith equivalence class.

For a vector $\mathbf{u}$ of weight n and a vector partition λ of weight n, the number of set partitions π of type $\mathrm{type}_{\mathbf{u}}(\pi)$ is

$$\binom{\mathbf{u}}{\lambda} := \frac{\mathbf{u}!}{\lambda!\,|\lambda|}.$$

Define a scalar product on $\mathfrak{M}$:

$$\boxed{\langle h_\lambda,\, m_\mu\rangle = \delta_{\lambda\mu}.} \tag{2}$$

The idea is to use this for basis transitions, along the lines discussed earlier. Specifically, formula (1) says that for any $F \in \mathfrak{M}_{\mathbf{u}}$, we have

$$F = \sum_{\lambda \vdash u} \langle h_\lambda,\, F\rangle m_\lambda \qquad \text{and} \qquad F = \sum_{\lambda \vdash u} \langle m_\lambda,\, F\rangle h_\lambda. \tag{3}$$

Thus we have to know how to compute inner products.

There is a "lifting map" $\hat{\rho}$ [Rosas 2001, Defn. 6] sending MacMahon functions to unitary (Doubilet) symmetric functions. Specifically:

$$\binom{\mathbf{u}}{\lambda} |\lambda|\, m_\lambda \overset{\hat{\rho}}{\longmapsto} \sum_{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda} m_\pi \tag{4a}$$

$$\binom{\mathbf{u}}{\lambda} \lambda!\, h_\lambda \overset{\hat{\rho}}{\longmapsto} \sum_{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda} h_\pi \tag{4b}$$

$$\binom{\mathbf{u}}{\lambda} \lambda!\, e_\lambda \overset{\hat{\rho}}{\longmapsto} \sum_{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda} e_\pi \tag{4c}$$

$$\binom{\mathbf{u}}{\lambda} p_\lambda \overset{\hat{\rho}}{\longmapsto} \sum_{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda} p_\pi \tag{4d}$$

$$\binom{\mathbf{u}}{\lambda} |\lambda|\, f_\lambda \overset{\hat{\rho}}{\longmapsto} \sum_{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda} f_\pi \tag{4e}$$

Moreover, for all $f, g \in \mathfrak{M}_{\mathbf{u}}$ we have

$$\langle f,\, g\rangle = \mathbf{u}!\, \langle \hat{\rho}(f),\, \hat{\rho}(g)\rangle \tag{5}$$

[Rosas, Prop. 7].

Prop. 1 together with the scalar product (2) says in particular that for $F \in \mathfrak{M}_{\mathbf{u}}$ we have

$$F = \sum_{\lambda \vdash \mathbf{u}} \langle m_\lambda,\, F\rangle\, h_\lambda, \qquad F = \sum_{\lambda \vdash \mathbf{u}} \langle h_\lambda,\, F\rangle\, m_\lambda. \tag{6}$$

Doubilet (Appendix 2) calculated the scalar products for all five families of unitary symmetric functions (which requires things like lattice operations and Möbius and zeta functions in the set partition lattice Π_n). Therefore, we should be able to compute scalar products, and thus basis expansions, for all MacMahon functions. Rosas reproduced Doubilet's calculations (in modern notation).

2 Expansions in the m-basis

$$\begin{aligned}
p_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat\rho(h_\lambda),\, \hat\rho(p_\mu) \rangle\, m_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{\binom{\mathbf{u}}{\lambda}\lambda!} \sum_{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda} h_\pi,\ \frac{1}{\binom{\mathbf{u}}{\mu}} \sum_{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu} p_\sigma \right\rangle m_\lambda && \text{(by (4b) and (4d))} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}\lambda!} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}} \langle h_\pi,\, p_\sigma \rangle\, m_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}\lambda!} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}} \zeta(\sigma,\pi) m_\lambda && \text{(by Doubilet formula \#3)} \\
p_\mu &= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}} \sum_{\lambda \vdash \mathbf{u}} \frac{\#\{(\pi,\sigma) \mid \mathrm{type}_{\mathbf{u}}(\pi)=\lambda,\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu,\ \sigma \le \pi\}}{\binom{\mathbf{u}}{\lambda}\lambda!} m_\lambda
\end{aligned}$$

$$\begin{aligned}
h_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat\rho(h_\lambda),\, \hat\rho(h_\mu) \rangle\, m_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{\binom{\mathbf{u}}{\lambda}\lambda!} \sum_{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda} h_\pi,\ \frac{1}{\binom{\mathbf{u}}{\mu}\mu!} \sum_{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu} h_\sigma \right\rangle m_\lambda && \text{(by (4b) twice)} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}\mu!} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}\lambda!} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}} \langle h_\pi,\, h_\sigma \rangle\, m_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}\mu!} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}\lambda!} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}} \lambda(\sigma \wedge \pi)!\, m_\lambda && \text{(by Doubilet formula \#9)}
\end{aligned}$$

$$\begin{aligned}
e_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat\rho(h_\lambda),\, \hat\rho(e_\mu) \rangle\, m_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{\binom{\mathbf{u}}{\lambda}\lambda!} \sum_{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda} h_\pi,\ \frac{1}{\binom{\mathbf{u}}{\mu}\mu!} \sum_{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu} e_\sigma \right\rangle m_\lambda && \text{(by (4b) and (4c))} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}\mu!} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}\lambda!} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}} \langle h_\pi,\, e_\sigma \rangle\, m_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}\mu!} \sum_{\lambda \vdash \mathbf{u}} \frac{\#\{(\pi,\sigma) \mid \mathrm{type}_{\mathbf{u}}(\pi)=\lambda,\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu,\ \pi \wedge \sigma = \hat{\mathbf{0}}\}}{\binom{\mathbf{u}}{\lambda}\lambda!} m_\lambda && \text{(by Doubilet formula \#6)}
\end{aligned}$$

$$
\begin{aligned}
f_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat\rho(h_\lambda),\, \hat\rho(f_\mu) \rangle\, m_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{\binom{\mathbf{u}}{\lambda}\lambda!} \sum_{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda} h_\pi,\ \frac{1}{\binom{\mathbf{u}}{\mu}|\mu|} \sum_{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu} f_\sigma \right\rangle m_\lambda && \text{(by (4b) and (4d))} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}|\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}\lambda!} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}} \langle h_\pi,\, f_\sigma \rangle\, m_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}|\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}\lambda!} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu \\ \sigma \le \pi}} \lambda(\sigma,\pi)!\, m_\lambda && \text{(by Doubilet formula \#13)}
\end{aligned}
$$

3 Expansions in the h-basis

$$
\begin{aligned}
m_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat\rho(m_\lambda),\, \hat\rho(m_\mu) \rangle\, h_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda} m_\pi,\ \frac{1}{\binom{\mathbf{u}}{\mu}|\mu|} \sum_{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu} m_\sigma \right\rangle h_\lambda && \text{(by (4a) twice)} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}|\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}} \langle m_\pi,\, m_\sigma \rangle\, h_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}|\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}} \sum_{\tau \ge \pi \vee \sigma} \frac{\mu(\pi,\tau)\mu(\sigma,\tau)}{\mu(\hat{\mathbf{0}},\tau)}\, h_\lambda && \text{(by Doubilet formula \#7)}
\end{aligned}
$$

$$
\begin{aligned}
p_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat\rho(m_\lambda),\, \hat\rho(p_\mu) \rangle\, h_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda} m_\pi,\ \frac{1}{\binom{\mathbf{u}}{\mu}} \sum_{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu} p_\sigma \right\rangle h_\lambda && \text{(by (4a) and (4d))} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}} \langle m_\pi,\, p_\sigma \rangle\, h_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu \\ \pi \le \sigma}} \frac{\mu(\pi,\sigma)}{|\mu(\hat{\mathbf{0}},\sigma)|}\, h_\lambda && \text{(by Doubilet formula \#10)}
\end{aligned}
$$

$$
\begin{aligned}
e_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat{\rho}(m_\lambda),\, \hat{\rho}(e_\mu) \rangle \, h_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda} m_\pi,\ \frac{1}{\binom{\mathbf{u}}{\mu}\mu!} \sum_{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu} e_\sigma \right\rangle h_\lambda && \text{(by (4a) and (4c))} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}\mu!} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}} \langle m_\pi,\, e_\sigma \rangle \, h_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}\mu!} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu \\ \pi \le \sigma}} \mathrm{sign}(\pi)\lambda(\pi,\sigma)!\, h_\lambda && \text{(by Doubilet formula \#8)}
\end{aligned}
$$

$$
\begin{aligned}
f_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat{\rho}(m_\lambda),\, \hat{\rho}(f_\mu) \rangle \, h_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda} m_\pi,\ \frac{1}{\binom{\mathbf{u}}{\mu}|\mu|} \sum_{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu} f_\sigma \right\rangle h_\lambda && \text{(by (4a) and (4e))} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}|\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}} \langle m_\pi,\, f_\sigma \rangle \, h_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}|\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}} \sum_{\tau \ge \sigma \vee \pi} \frac{|\mu(\sigma,\tau)| \cdot \mu(\pi,\tau)}{\mu(\hat{\mathbf{0}},\tau)}\, h_\lambda && \text{(by Doubilet formula \#11)}
\end{aligned}
$$

4 Expansions in the p-basis

The power-sum basis is orthogonal (though not orthonormal). A remark in Rosas (p.333) says that for $\lambda \vdash \mathbf{u}$,

$$
\langle p_\lambda,\, p_\mu \rangle = \frac{\delta_{\lambda,\mu}|\lambda|\,\lambda!}{\mu(\hat{\mathbf{0}},\lambda)} \tag{7}
$$

where $\mu(\hat{\mathbf{0}},\lambda)$ means $\mu(\hat{\mathbf{0}},\pi)$ for any set partition π such that $\mathrm{type}_u(\pi) = \lambda$.

Therefore, Proposition 1 says that for every $F \in \mathfrak{M}$,

$$
F = \sum_\lambda \frac{\langle p_\lambda,\, F\rangle \mu(\hat{\mathbf{0}},\lambda)}{|\lambda|\,\lambda!} p_\lambda = \sum_\lambda \frac{\langle p_\lambda,\, F\rangle \mu(\hat{\mathbf{0}},\lambda)\binom{\mathbf{u}}{\lambda}}{\mathbf{u}!} p_\lambda \tag{8}
$$

So, as before:

$$
\begin{aligned}
m_\mu &= \sum_\lambda \left\langle \binom{\mathbf{u}}{\lambda} \hat\rho(p_\lambda),\ \hat\rho(m_\mu) \right\rangle \mu(\hat{\mathbf{0}}, \lambda) p_\lambda && \text{(by (8) and (5))} \\
&= \frac{1}{\binom{\mathbf{u}}{\mu}|\mu|} \sum_\lambda \mu(\hat{\mathbf{0}}, \lambda) \sum_{\substack{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\lambda \\ \pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\mu}} \langle p_\sigma,\ m_\pi \rangle\, p_\lambda && \text{(by (4d) and (4a) and bilinearity)} \\
&= \frac{1}{\binom{\mathbf{u}}{\mu}|\mu|} \sum_\lambda \left[\mu(\hat{\mathbf{0}}, \lambda) \sum_{\substack{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\lambda \\ \pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\mu \\ \pi \le \sigma}} \frac{\mu(\pi, \sigma)}{|\mu(\hat{\mathbf{0}}, \sigma)|} \right] p_\lambda && \text{(by Doubilet formula \#10)}
\end{aligned}
$$

$$
\begin{aligned}
h_\mu &= \sum_\lambda \langle \hat\rho(p_\lambda),\ \hat\rho(h_\mu) \rangle \mu(\hat{\mathbf{0}}, \lambda) \binom{\mathbf{u}}{\lambda} p_\lambda && \text{(by (8) and (5))} \\
&= \frac{1}{\binom{\mathbf{u}}{\mu}\mu!} \sum_\lambda \mu(\hat{\mathbf{0}}, \lambda) \sum_{\substack{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\lambda \\ \pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\mu}} \langle p_\sigma,\ h_\pi \rangle\, p_\lambda && \text{(by (4d) and (4b) and bilinearity)} \\
&= \frac{1}{\binom{\mathbf{u}}{\mu}\mu!} \sum_\lambda \left[\mu(\hat{\mathbf{0}}, \lambda) \sum_{\substack{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\lambda \\ \pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\mu \\ \sigma \le \pi}} \right] p_\lambda && \text{(by Doubilet formula \#3)}
\end{aligned}
$$

$$
\begin{aligned}
e_\mu &= \sum_\lambda \langle \hat\rho(p_\lambda),\ \hat\rho(e_\mu) \rangle \mu(\hat{\mathbf{0}}, \lambda) \binom{\mathbf{u}}{\lambda} p_\lambda && \text{(by (8) and (5))} \\
&= \frac{1}{\binom{\mathbf{u}}{\mu}\mu!} \sum_\lambda \mu(\hat{\mathbf{0}}, \lambda) \sum_{\substack{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\lambda \\ \pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\mu}} \langle p_\sigma,\ e_\pi \rangle\, p_\lambda && \text{(by (4d) and (4c) and bilinearity)} \\
&= \frac{1}{\binom{\mathbf{u}}{\mu}\mu!} \sum_\lambda \left[\mu(\hat{\mathbf{0}}, \lambda) \sum_{\substack{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\lambda \\ \pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\mu \\ \sigma \le \pi}} \mathrm{sign}(\sigma) \right] p_\lambda && \text{(by Doubilet formula \#4)}
\end{aligned}
$$

$$
\begin{aligned}
f_\mu &= \sum_\lambda \langle \hat\rho(p_\lambda),\ \hat\rho(f_\mu)\rangle \mu(\hat{\mathbf{0}},\lambda)\binom{\mathbf{u}}{\lambda} p_\lambda && \text{(by (8) and (5))}\\
&= \frac{1}{\binom{\mathbf{u}}{\mu}|\mu|}\sum_\lambda \mu(\hat{\mathbf{0}},\lambda) \sum_{\substack{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\lambda\\ \pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\mu}} \langle p_\sigma,\ f_\pi\rangle\, p_\lambda && \text{(by (4d) and (4e) and bilinearity)}\\
&= \frac{1}{\binom{\mathbf{u}}{\mu}|\mu|}\sum_\lambda \left[\mu(\hat{\mathbf{0}},\lambda) \sum_{\substack{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\lambda\\ \pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\mu\\ \pi\le\sigma}} \mathrm{sign}(\pi)\,\mathrm{sign}(\sigma)\,\frac{\mu(\pi,\sigma)}{|\mu(\hat{\mathbf{0}},\sigma)|}\right] p_\lambda && \text{(by Doubilet formula \#14)}
\end{aligned}
$$

5 The omega involution, and expansions in the e- and f-bases

There is an automorphism $\omega\colon \mathfrak{M}\to\mathfrak{M}$ satisfying $\omega^2=\mathrm{Id}$. Rosas states that

$$\omega(e_\lambda) = h_\lambda$$

(that is the definition), and that

$$\omega(m_\lambda) = \mathrm{sign}(\lambda) f_\lambda.$$

I would like to know what ω does to p_λ. For symmetric functions, we have $\omega(p_\lambda)=\mathrm{sign}(\lambda)p_\lambda$ (where λ is a partition), and for unitary MacMahon functions, we have $\omega(p_\pi)=\mathrm{sign}(\pi)p_\pi$ (where π is a set partition). It would be natural to hope that $\omega(p_\lambda)=\mathrm{sign}(\lambda)p_\lambda$ in general, though Rosas does not state that. Can we prove it? And can we prove that ω is an isometry?

Or should we just assume all of these things? If so, then applying ω to the conversions to the h-basis gives

$$f_\mu = \mathrm{sign}(\mu)\frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}|\mu|}\sum_{\lambda\vdash\mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda\\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}} \sum_{\tau\ge\pi\vee\sigma} \frac{\mu(\pi,\tau)\mu(\sigma,\tau)}{\mu(\hat{\mathbf{0}},\tau)}\, e_\lambda$$

$$p_\mu = \mathrm{sign}(\mu)\frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}}\sum_{\lambda\vdash\mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda\\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu\\ \pi\le\sigma}} \frac{\mu(\pi,\sigma)}{|\mu(\hat{\mathbf{0}},\sigma)|}\, e_\lambda$$

$$h_\mu = \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}\mu!}\sum_{\lambda\vdash\mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda\\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu\\ \pi\le\sigma}} \mathrm{sign}(\pi)\lambda(\pi,\sigma)!\, e_\lambda$$

$$m_\mu = \operatorname{sign}(\mu)\frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}|\mu|}\sum_{\lambda\vdash\mathbf{u}}\frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|}\sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda\\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}}\sum_{\tau\geq\sigma\vee\pi}\frac{|\mu(\sigma,\tau)|\cdot\mu(\pi,\tau)}{\mu(\hat{\mathbf{0}},\tau)}\, e_\lambda$$

and likewise applying ω to the conversions to the m-basis gives

$$p_\mu = \operatorname{sign}(\mu)\frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}}\sum_{\lambda\vdash\mathbf{u}}\operatorname{sign}(\lambda)\frac{\#\{(\pi,\sigma)\mid \mathrm{type}_{\mathbf{u}}(\pi)=\lambda,\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu,\ \sigma\leq\pi\}}{\binom{\mathbf{u}}{\lambda}\lambda!}f_\lambda$$

$$e_\mu = \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}\mu!}\sum_{\lambda\vdash\mathbf{u}}\operatorname{sign}(\lambda)\frac{1}{\binom{\mathbf{u}}{\lambda}\lambda!}\sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda\\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}}\lambda(\sigma\wedge\pi)!\, f_\lambda$$

$$h_\mu = \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}\mu!}\sum_{\lambda\vdash\mathbf{u}}\operatorname{sign}(\lambda)\frac{\#\{(\pi,\sigma)\mid \mathrm{type}_{\mathbf{u}}(\pi)=\lambda,\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu,\ \pi\wedge\sigma=\hat{\mathbf{0}}\}}{\binom{\mathbf{u}}{\lambda}\lambda!}f_\lambda$$

$$m_\mu = \operatorname{sign}(\mu)\frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}|\mu|}\sum_{\lambda\vdash\mathbf{u}}\operatorname{sign}(\lambda)\frac{1}{\binom{\mathbf{u}}{\lambda}\lambda!}\sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda\\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu\\ \sigma\leq\pi}}\lambda(\sigma,\pi)!f_\lambda$$