1 Expanding a power-sum MacMahon symmetric function in the monomial basis

Fix a positive integer r. Throughout, we will work with MacMahon symmetric functions in r alphabets. The complete set of variables is

$$\{x_{i,j} \mid i \in \mathbb{P}, \ j \in [r]\}.$$

Fix vector partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \ldots, \mu_m)$ of the same vector, where

$$\lambda_i = (\lambda_{i,1}, \ldots, \lambda_{i,r}) \in \mathbb{P}^r \setminus \{\vec{0}\},$$
$$\mu_i = (\mu_{i,1}, \ldots, \mu_{i,r}) \in \mathbb{P}^r \setminus \{\vec{0}\}.$$

The monomial MacMahon symmetric function indexed by μ is the sum of all monomials in the diagonal $\mathfrak{S}_n$-orbit of the monomial

$$x_\mu = \prod_{i=1}^{m} \prod_{j=1}^{r} x_{i,j}^{\mu_{i,j}}.$$

The power-sum MacMahon symmetric function indexed by λ is

$$P_\lambda = \prod_{i=1}^{\ell} P_{\lambda_i}$$

$$= \prod_{i=1}^{\ell} \sum_{k \in \mathbb{P}} x_{k,1}^{\lambda_{i,1}} \cdots x_{k,r}^{\lambda_{i,r}}$$

$$= \sum_{(k_1, \ldots, k_\ell) \in \mathbb{P}^\ell} \prod_{i=1}^{\ell} x_{k_i,1}^{\lambda_{i,1}} \cdots x_{k_i,r}^{\lambda_{i,r}}.$$

Thus the coefficient of x_μ in P_λ is

$$\#\left\{(k_1, \ldots, k_\ell) \in \mathbb{P}^\ell \mid \sum_{i: \ k_i = h} \lambda_i = \mu_h \ \forall h \in [m]\right\}.$$

Note that we can replace $\mathbb{P}^\ell$ by m^ℓ.

For example, fix $r = 2$. Let $\mu = (1,1)(1,2)$, so that

$$x_\mu = x_{1,1}^1 x_{1,2}^1 x_{2,1}^1 x_{2,2}^2$$

and let $\lambda = (1,1)(1,0)(0,1)(0,1)$ so that $\ell = 4$ and

$$P_\lambda = \sum_{(k_1, \ldots, k_4)} \prod_{i=1}^{4} x_{k_i,1}^{\lambda_{i,1}} x_{k_i,2}^{\lambda_{i,2}}$$

$$= \sum_{(k_1, \ldots, k_4)} x_{k_1,1}^{\lambda_{1,1}} x_{k_1,2}^{\lambda_{1,2}} x_{k_2,2}^{\lambda_{2,1}} x_{k_2,2}^{\lambda_{2,2}} x_{k_3,1}^{\lambda_{3,1}} x_{k_3,2}^{\lambda_{3,2}} x_{k_4,1}^{\lambda_{4,1}} x_{k_4,2}^{\lambda_{4,2}}$$

$$= \sum_{(k_1, \ldots, k_4)} x_{k_1,1}^{1} x_{k_1,2}^{1} x_{k_2,2}^{1} x_{k_2,2}^{0} x_{k_3,1}^{0} x_{k_3,2}^{1} x_{k_4,1}^{0} x_{k_4,2}^{1}$$

so in order to make this monomial equal to x_μ, we need

$$\sum_{i:\ k_i=1} \lambda_i = \mu_1 = (1,1), \qquad \sum_{i:\ k_i=2} \lambda_i = \mu_2 = (1,2).$$

The solutions for (k_1, k_2, k_3, k_4) and the corresponding expressions for x_μ are

$$
\begin{aligned}
(1,2,2,2) \qquad & x_{1,1}^1 x_{1,2}^1 x_{2,1}^1 x_{2,2}^2 = (x_{1,1}^1 x_{1,2}^1)(x_{2,2}^1 x_{2,2}^0)(x_{2,1}^0 x_{2,2}^1)(x_{2,1}^0 x_{2,2}^1) \\
(2,1,1,2) \qquad & x_{1,1}^1 x_{1,2}^1 x_{2,1}^1 x_{2,2}^2 = (x_{2,1}^1 x_{2,2}^1)(x_{1,2}^1 x_{1,2}^0)(x_{1,1}^0 x_{1,2}^1)(x_{2,1}^0 x_{2,2}^1) \\
(2,1,2,1) \qquad & x_{1,1}^1 x_{1,2}^1 x_{2,1}^1 x_{2,2}^2 = (x_{2,1}^1 x_{2,2}^1)(x_{1,2}^1 x_{1,2}^0)(x_{2,1}^0 x_{2,2}^1)(x_{1,1}^0 x_{1,2}^1)
\end{aligned}
$$

So the coefficient is 3. This could certainly be calculated algorithmically.

2 General linear algebra

Let V be a vector space of finite dimension n over ground field $\Bbbk$ (it doesn't matter what it is).

Proposition 1. *Suppose that $B = \{v_1, \ldots, v_n\}$ and $B^* = \{v_1^*, \ldots, v_n^*\}$ are bases for V that are orthogonal, i.e.,*

$$\langle v_i, v_j^* \rangle = \delta_{ij} k_i$$

for some nonzero constants $k_1, \ldots, k_n$. Then the unique expansion of u with respect to B is

$$u = \sum_{i=1}^n \frac{\langle v_i^*,\, u \rangle}{k_i} v_i. \tag{1}$$

Proof. For every $j \in [n]$ we have

$$\left\langle v_j^*,\ \sum_{i=1}^n \frac{\langle v_i^*,\, u \rangle}{k_i} v_i \right\rangle = \sum_{i=1}^n \frac{\langle v_i^*,\, u \rangle}{k_i} \langle v_j^*,\, v_i \rangle = \sum_{i=1}^n \langle v_i^*,\, u \rangle \delta_{ij} = \langle v_j^*,\, u \rangle.$$

That is, every element of B^* has the same inner product with the left and right-hand sides of (1). This is necessary and sufficient for equality. $\qquad\square$

(The best possible case is that $k_i = 1$ for all i, i.e., the bases are orthonormal. However, we're not always that lucky.)

3 Transitions between bases, from Rosas 2001

Notation:

- $\mathbb{N}$ = nonnegative integers;
- For $u \in \mathbb{N}^k$, a **vector partition** $\lambda \vdash u$ is an unordered sequence of vectors (**parts**) summing to u; zero vectors can be ignored.
- Weight of a vector partition = sum of all entries in all vectors in it (ex.: $\mathrm{wt}(\{120, 100, 100, 013\}) = 9$)

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- $\mathfrak{M}_u$ = MacMahon symmetric functions of homogeneous multidegree u
- $\mathrm{sign}(\lambda) = (-1)^{\text{number of parts with even sum}}$
- $m_\lambda, e_\lambda, h_\lambda, p_\lambda, f_\lambda$: monomial, elementary, homogeneous, power-sum, and forgotten MacMahon symmetric functions (Rosas 327–328)
- For a vector partition λ in which each part v_i occurs with multiplicity m_i, define

$$|\lambda| = \prod_i m_i! \qquad \text{and} \qquad \lambda! = \prod_i \prod_{x \in v_i} (x!)^{m_i}.$$

A vector partition is **unitary** if it is a partition of the all-ones vector $\mathbf{1} \in \mathbb{N}^k$. There is an obvious bijection between unitary vector partitions and set partitions of $[k]$. There is a whole theory of MacMahon symmetric functions of unitary partitions, due to Doubilet. These are the graded pieces $\mathfrak{M}_{(1)^k}$.

The **type** of a set partition $\pi = B_1 | \cdots | B_\ell$, with respect to a vector $u = (u_1, \ldots, u_r)$ of weight n, is the vector partition $\mathrm{type}_u(\pi) = \lambda = \lambda_1 \cdots \lambda_\ell$ where λ_k is the vector in $\mathbb{N}^r$ whose ith coordinate is

$$\#\{j \in B_k \mid u_1 + \cdots + u_{i-1} < j \leq i_1 + \cdots + u_i\}.$$

the number of elements of B_k such that in the ith equivalence class.

For a vector u of weight n and a vector partition λ of weight n, the number of set partitions π of type $\mathrm{type}_u(\pi)$ is

$$\binom{u}{\lambda} := \frac{u!}{\lambda! \, |\lambda|}.$$

Define a scalar product on $\mathfrak{M}$:

$$\boxed{\langle h_\lambda, \, m_\mu \rangle = \delta_{\lambda\mu}.} \tag{2}$$

The idea is to use this for basis transitions, along the lines discussed earlier. Specifically, formula (1) says that for any $F \in \mathfrak{M}_u$, we have

$$F = \sum_{\lambda \vdash u} \langle h_\lambda, F \rangle m_\lambda \qquad \text{and} \qquad F = \sum_{\lambda \vdash u} \langle m_\lambda, F \rangle h_\lambda. \tag{3}$$

Thus we have to know how to compute inner products.

There is a "lifting map" $\hat{\rho}$ [Rosas 2001, Defn. 6] sending MacMahon functions to unitary (Doubilet) symmetric functions. Specifically:

$$\binom{u}{\lambda} |\lambda| \, m_\lambda \overset{\rho}{\longmapsto} \sum_{\pi : \, \mathrm{type}_u(\pi) = \lambda} m_\pi \tag{4a}$$

$$\binom{u}{\lambda} \lambda! \, e_\lambda \overset{\rho}{\longmapsto} \sum_{\pi : \, \mathrm{type}_u(\pi) = \lambda} e_\pi \tag{4b}$$

$$\binom{u}{\lambda} \lambda! \, h_\lambda \overset{\rho}{\longmapsto} \sum_{\pi : \, \mathrm{type}_u(\pi) = \lambda} h_\pi \tag{4c}$$

$$\binom{u}{\lambda} p_\lambda \overset{\rho}{\longmapsto} \sum_{\pi : \, \mathrm{type}_u(\pi) = \lambda} p_\pi \tag{4d}$$

$$\binom{u}{\lambda} |\lambda| \, f_\lambda \overset{\rho}{\longmapsto} \sum_{\pi : \, \mathrm{type}_u(\pi) = \lambda} f_\pi \tag{4e}$$

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Moreover, for all $f, g \in \mathfrak{M}_u$ we have

$$\langle f,\, g \rangle = u!\, \langle \hat{\rho}(f),\, \hat{\rho}(g) \rangle \tag{5}$$

[Rosas, Prop. 7]. For instance, Doubilet's Formula #1 says that for set partitions π, σ of $[n]$ we have

$$\langle m_\pi,\, h_\sigma \rangle = n!\delta_{\pi\sigma} \tag{6}$$

Therefore, if λ and μ are vector partitions of u of weight n, then

$$
\begin{aligned}
\langle m_\lambda,\, h_\mu \rangle &= u!\, \langle \hat{\rho}(m_\lambda),\, \hat{\rho}(h_\mu) \rangle && \text{(by (5))} \\
&= u! \frac{1}{\binom{u}{\lambda}|\lambda|\binom{u}{\mu}\mu!} \left\langle \sum_{\pi:\ \mathrm{type}_u(\pi)=\lambda} m_\pi,\ \sum_{\sigma:\ \mathrm{type}_u(\sigma)=\mu} h_\sigma \right\rangle && \text{(by (4a) and (4c))} \\
&= \delta_{\lambda\mu} \frac{u!\, n!\, \#\{\pi:\ \mathrm{type}_u(\pi) = \lambda\}}{\binom{u}{\lambda}^2 |\lambda|\, \lambda!} && \text{(by linearity and (6))}
\end{aligned}
$$

Since Doubilet (Appendix 2) calculated the scalar products for all five families of unitary symmetric functions, we can lift them to MacMahon functions. (We will need the lattice operations and Möbius and zeta functions in the set partition lattice Π_n

Prop. 1 together with the scalar product (2) says in particular that for $F \in \mathfrak{M}_\mathbf{u}$ we have

$$F = \sum_{\lambda \vdash \mathbf{u}} \langle m_\lambda,\, F \rangle\, h_\lambda, \qquad\qquad F = \sum_{\lambda \vdash \mathbf{u}} \langle h_\lambda,\, F \rangle\, m_\lambda. \tag{7}$$

In particular (I am going to be very repetitive here),

$$m_\mu = \sum_{\lambda \vdash \mathbf{u}} \langle m_\lambda,\, m_\mu \rangle\, h_\lambda, \qquad\qquad m_\mu = \sum_{\lambda \vdash \mathbf{u}} \langle h_\lambda,\, m_\mu \rangle\, m_\lambda, \tag{8}$$

$$h_\mu = \sum_{\lambda \vdash \mathbf{u}} \langle m_\lambda,\, h_\mu \rangle\, h_\lambda, \qquad\qquad h_\mu = \sum_{\lambda \vdash \mathbf{u}} \langle h_\lambda,\, h_\mu \rangle\, m_\lambda, \tag{9}$$

$$e_\mu = \sum_{\lambda \vdash \mathbf{u}} \langle m_\lambda,\, e_\mu \rangle\, h_\lambda, \qquad\qquad e_\mu = \sum_{\lambda \vdash \mathbf{u}} \langle h_\lambda,\, e_\mu \rangle\, m_\lambda, \tag{10}$$

$$p_\mu = \sum_{\lambda \vdash \mathbf{u}} \langle m_\lambda,\, p_\mu \rangle\, h_\lambda, \qquad\qquad p_\mu = \sum_{\lambda \vdash \mathbf{u}} \langle h_\lambda,\, p_\mu \rangle\, m_\lambda, \tag{11}$$

$$f_\mu = \sum_{\lambda \vdash \mathbf{u}} \langle m_\lambda,\, f_\mu \rangle\, h_\lambda, \qquad\qquad f_\mu = \sum_{\lambda \vdash \mathbf{u}} \langle h_\lambda,\, f_\mu \rangle\, m_\lambda. \tag{12}$$

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