The Chromatic MacMahon Function

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The Chromatic MacMahon Function

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Abstract

We examine colorings of weighted graphs. Firstly, we examine the weighted analogue of Stanley's chromatic symmetric function, and prove that the weighted analogue of Crew's conjecture is not true on it. Secondly, we generalize the chromatic symmetric function to the chromatic MacMahon function, and use that to prove the MacMahon analogue of Crew's conjecture.

Acknowledgements

We thank Ira Gessel and Mercedes Rosas for their help in pointing us to references Lauve & Mastnak (2011) and Rosas (2001) which were very helpful.

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Chapter 1

Background

Abstract

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1.1 Background

The chromatic symmetric function (CSF) is a symmetric function representation of the different ways to color a graph. The CSF is a symmetric function on the graph $G = (V, E)$ defined to be

$$\mathbf{X}_G = \sum_{\text{proper colorings } k} \prod_{v \in V} x_{k(v)},$$

where $w_1, x_2, \ldots$ are commuting indeterminates.

There is a well known open question posed by Stanley Stanley (1995) of whether the CSF is unique on trees up to isomorphism. In the research that has come since, other related functions have also been objects of study. One of these functions is the generalized degree polynomial (GDP) introduced by Crew (Crew, 2020, §4.3), which is a function that carries information on the size of vertex sets, as well as the number of border and internal edges of the vertex set. Tthe GDP on the graph $G = (V, E)$ is defined to be

$$\mathbf{G}_G = \sum_{A \subseteq V} x^{|A|} y^{d(A)} z^{e(A)},$$

where $d(A) = |\{\text{edges of } E \text{ with exactly one endpoint in } A\}|$ and $e(A) = |\{\text{edges of } E \text{ with both endpoints in } A\}|$.

Aliste-Prieto et al. (Aliste-Prieto et al., 2024, Thm. 6) proved Crew's conjecture, that the GDP of a tree is entirely determined by the CSF on that tree. Liu and Tang (Liu & Tang, 2024, Prop. 2.4) later gave another proof of Crew's conjecture using Hopf algebras.

Another way the concept of the CSF has been studied is on weighted graphs. There are weighted analogues of the CSF and the GDP. The weighted case has certain advantages that the non-weighted case does not, particularly the property of deletion-contraction. The weighted CSF was proven to obey a deletion-contraction recurrence in Crew (2022). On the other hand, the weighted CSF is known not to be a complete invariant (Loebl & Sereni, 2019, p.5). A question then emerges, is the weighted GDP entirely determined by the weighted CSF like in the non-weighted case? We prove here that this is untrue by showcasing a specific counterexample.

However, in this paper we discuss a new function called the chromatic MacMahon function (CMF) on weighted graphs that simultaneously generalizes the classical and weighted CSFs. A MacMahon symmetric function will receive a full definition in Def. 3.1.3, but in brief it generalizes the concept of a symmetric function to multiple alphabets, each of which is permuted in the same way. MacMahon symmetric functions have a power sum basis indexed by vector partitions, which generalize integer partitions. This CMF also entirely determines the weighted GDP. This paper discusses the creation of this CMF and discovers some of its properties. We follow a proof from Stanley (Stanley, 1995, Thm. 2.5) to rewrite the CMF in the power sum basis. We use the Hopf coproduct for noncommutative symmetric functions (NCSym) found by Lauve and Mastnak (Lauve & Mastnak, 2011, p.538) and the projection map between NCSym and MacMahon symmetric functions found by Rosas (Rosas, 2001, Def. 1) to find the Hopf coproduct for MacMahon symmetric functions. We then prove this coproduct makes the MacMahon symmetric functions into a Hopf algebra, and find the convolution on the CMF. Finally, we use this convolution to generalize the Liu-Tang proof of Crew's conjecture (Liu & Tang, 2024, Prop. 2.4) to weighted trees. We also present an alternative proof of our main result which generalizes the Aliste-Prieto et al.

proof of Crew's conjecture in (Aliste-Prieto et al., 2024, Thm. 6).

The paper is structured as follows. We begin in section 2.1 by setting up some basic definitions and notations for graphs, the CSF and GDP, as well as the weighted CSF and GDP. Section 2.2 then goes on to disprove the weighted analogue of Crew's conjecture. We continue in section 3.1 by establishing further definitions and notation needed to define and examine the chromatic MacMahon function. Section 3.2 rewrites the CMF in the power sum basis, giving us a far more useful form to use in the following sections' proofs. Section 3.3 proves that the set of MacMahon symmetric functions is a Hopf algebra and, most importantly, what the what the coproduct is. Knowing the coproduct is necessary to define the convolution of two characters, which is a key part of Liu and Tang's argument. In section 3.4, we finally prove the MacMahon analogue of Crew's conjecture, using the convolution found in section 3.3 and the power sum expansion of the CMF found in section 3.2.

There are two addendums, which cover additional details. Addendum 1 in section A explains how the coproduct for the Hopf algebra of MacMahon symmetric functions was found. Addendum 2 in section B goes over an alternative proof of the MacMahon analogue of Crew's conjecture, modeled after the proof of Crew's conjecture in Aliste-Prieto et al. (2024).

Chapter 2

Disproving the weighted analogue of Crew's conjecture

Abstract

Use the chapterabstract environment, not the abstract environment, if you want to plant an abstract at the top of the chapter.

2.1 Basic definitions and notation

Definition 2.1.1. For a graph $G = (V,E)$, we write $n = |V| = |G|$, $e(G) = |E|$, and $c(G) =$ the number of connected components of G.

A **tree** is a graph $T = (V,E)$ such that T is connected and acyclic. Equivalently, T is a tree if and only if T is connected and

$$e(T) = n - 1. \tag{2.1.1}$$

A **forest** is a graph $F = (V,E)$ that is acyclic, but not necessarily connected. A forest can be thought of as a collection of trees. Equivalently, F is a forest if and only if

$$e(F) = n - c(F). \tag{2.1.2}$$

We will sometimes write a graph as $G = (C_1, \ldots, C_{c(G)})$ where C_i is a connected component of G.

Definition 2.1.2. A **weighted graph** $G = (V,E,w)$ is a graph $G = (V,E)$ along with some weight function w.

A **weight function** $w : V \to_+$ is a function that maps each vertex in H to a positive integer. We write $w = w(G) = \sum_{v_i \in V} w(v_i)$ for the total weight of G.

We will sometimes write a weighted graph as $G = (C_1, \ldots, C_{c(G)}, w)$ where the C_i are the connected components of G.

Definition 2.1.3. The **type** of a graph $G = (C_1, \ldots, C_{c(G)})$, $\text{type}(G) = (n_1, \ldots, n_{c(G)})$ is the partition of $n(G)$ whose parts are $|C_1|, \ldots, |C_{c(G)}|$. If $G = (V, E)$ is a graph and $S \subseteq E$, we set $\text{type}(S) = \text{type}((V, S))$.

The **weighted type** of a weighted graph $G = (C_1, \ldots, C_{c(G)}, w)$, $\text{wtp}(G) = (w_1, \ldots, w_{c(G)})$, is the partition of $w(G)$ whose parts are $w(C_1), \ldots, w(C_{c(G)})$. If $G = (V, E, w)$ is a graph and $S \subseteq E$, we set $\text{wtp}(S) = \text{wtp}((V, S, w))$.

Definition 2.1.4. Given a graph $G = (V, E)$, the **chromatic symmetric function**, or **CSF**, of G is defined to be

$$\mathbf{X}_G = \sum_{\substack{k \in K(G) \\ \text{proper}}} \prod_{v \in V} x_{k(v)},$$

where $K(G)$ is defined to be the set of colorings, k, on the graph G. Stanley (Stanley, 1995, Thm. 2.5) showed that the CSF could be rewritten in the power sum basis in the following way,

$$\mathbf{X}_G = \sum_{S \subseteq E} (-1)^{|S|} p_{\text{type}(S)}. \tag{2.1.3}$$

When the graph is a tree $T = (V, E)$, (Stanley, 1995, Cor. 2.8) the CSF can be written in the power sum basis in the following way,

$$\mathbf{X}_T = \sum_{\lambda \vdash n} b_\lambda(T)(-1)^{n-\ell(\lambda)} p_\lambda$$

where $b_\lambda(T) = |\{S \subseteq E : \text{ type}(S) = \lambda\}|$.

Definition 2.1.5. Given a graph $G = (V,E)$, the **generalized degree polynomial**, or **GDP**, of G is defined to be

$$\mathbf{G}_G = \sum_{A \subseteq V} x^{|A|} y^{d(A)} z^{e(A)}$$

where $d(A) = |\{\text{edges of } E \text{ with exactly one endpoint in } A\}|$ and $e(A) = |\{\text{edges of } E \text{ with both endpoints in } A\}|$. We also call the edges counted by $d(A)$ the **external** edges of A, and the edges counted by $e(A)$ the **internal** edges of A. We sometimes also write the GDP as $\mathbf{G}_G(x,y,z)$. The definition of this function comes from (Crew, 2020, §4.3).

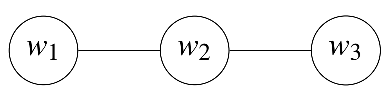
Definition 2.1.6. Given a weighted graph $G = (V,E,w)$, the **weighted chromatic symmetric function**, or **wCSF**, of G is defined to be

$$\mathbf{X}_{G,w} = \sum_{\substack{k \in K(G) \\ \text{proper}}} x_{k(v)}^{w(v)}.$$

Similarly to equation 2.1.3, the wCSF can be written in the power sum basis as

$$\mathbf{X}_{G,w} = \sum_{S \subseteq E} (-1)^{|S|} p_{\mathrm{wtp}(S)}. \tag{2.1.4}$$

Example 2.1.7. We calculate $\mathbf{X}_{T,w}$ for the following weighted tree, T.



Labeling the edges from left to right as e_1 and e_2, the edge sets are $\varnothing, \{e_1\}, \{e_2\}, \{e_1, e_2\}$. Therefore when setting $w_1 = 2$, $w_2 = 1$, and $w_3 = 3$,

$$\mathbf{X}_{T,w} = p_{321} - p_{33} - p_{24} + p_{6}.$$

When setting $w_1 = w_2 = w_3 = 1$,

$$\mathbf{X}_{T,w} = p_{111} - 2p_{21} + p_3.$$

This is the same as $\mathbf{X}_T$. Therefore setting $w_1 = w_2 = w_3 = 1$ recovers the CSF.

Definition 2.1.8. Given a weighted graph $G = (V, E, w)$, the **weighted generalized degree polynomial**, or **wGDP**, of G is defined to be

$$\mathbf{G}_{G,w} = \sum_{A \subseteq V} x^{w(A)} y^{d(A)} z^{e(A)}.$$

We sometimes also write the weighted GDP as

$$\mathbf{G}_G(x,y,z) = \sum_{a,b,c} g_T(a,b,c) x^a y^b z^c.$$

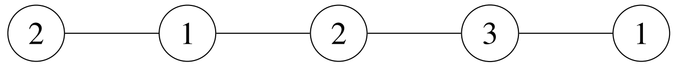
Here $g_T(a,b,c) = |\{A \subseteq V : w(A) = a, d(A) = b, e(A) = c\}|$ is the coefficient of $x^a y^b z^c$ in $\mathbf{G}_{G,w}$.

2.2 Disproving the weighted analogue of Crew's conjecture

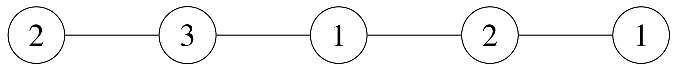
As shown in (Aliste-Prieto et al., 2024, Thm. 6), the GDP is determined by the CSF on trees. In this section we show that the same does not hold true for the wGDP and the wCSF. To prove this we will show a direct counterexample. We will then go on to provide the intuition for why the wCSF could never give you the wGDP.

Proposition 2.2.1. *There exist two trees T_1 and T_2 with the same wCSF but different wGDPs.*

Proof. Consider the two weighted graphs drawn below, which we will call T_1 and T_2 respectively.



(2.2.1)



(2.2.2)

As remarked in (Loebl & Sereni, 2019, p.5), T_1 and T_2 have the same wCSF, despite not being isomorphic. This can be calculated by hand, and if one does so they will see that $\mathbf{X}_{T_1,w} = \mathbf{X}_{T_2,w}$ and they are both equal to

$$p_{32211} - 2p_{3321} - p_{5211} - p_{4221} + 2p_{531} + 2p_{432} + 2p_{621} - p_{81} - p_{54} - p_{63} - p_{72} + p_9.$$

However, the wGDP on T_1 and T_2 is not the same. Again this can be calculated by hand. However, doing this would be tedious, and it would be difficult to parse where the functions differ. We will instead identify a particular monomial where each function has a different coefficient. The monomial we will examine will be x^4y^3, corresponding to the number of vertex sets with total weight 4, external edge count 3, and internal edge count 0. For each graph, we label the vertices from left to right as a,b,c,d, and e.

In T_1, there are 5 vertex sets of weight 4, $\{a,b,e\},\{b,c,e\},\{a,c\},\{b,d\},\{d,e\}$. Of these, only $\{a,c\}$ has 3 external edges and no internal edges.

In T_2, there are still 5 vertex sets of total weight 4, $\{a,c,e\},\{a,d\},\{c,d,e\},\{b,c\},\{b,e\}$. This time, however, 2 of them have 3 external edges and 0 internal edges, $\{a,d\},\{b,e\}$. Therefore, the coefficients of x^4y^3 in each function are different. Therefore, the wCSF of a tree can never entirely determine the wGDP of that tree. $\square$

Thus, in contrast to the situation for unweighted trees, there can exist no map sending $X_{T,w}$ to $G_{T,w}$ for every tree. But why does this happen? For one, weighted graphs simply contain more information that unweighted graphs. Yet, the wCSF does not contain more information than the CSF, it just contains information about the weights instead of about the vertices. This is part of why it is so easy to construct an example where the wCSF does not distinguish trees. More than that, though, the wCSF contains less valuable information than the CSF. This comes down to the fact that we are specifically examining trees. The reason trees are the object of study is that they have a very useful property, seen in equation 2.1.1 that the number of edges is equal to the number of vertices minus 1. This means knowing information about the vertices gives you information about the edges. However, there is no similar rule for weights. The wCSF cannot extract information about edges because knowing information about weights tells you very little about the edges.

The reason the original proof of Crew's conjecture works is taking advantage of equation 2.1.1. Since the GDP contains information about vertices and edges, the CSF carries enough information to determine the GDP, since it carries information about vertices, which in a tree also give information about edges. However, the wGDP contains information about the vertices, edges, and weights, and the wCSF can never determine all that information.

To create a version of Crew's conjecture that holds for weighted trees, you would need a generalization of the CSF and wCSF that contains information about both weights and vertices. This would no longer be a symmetric function, as you would need some variables to represent weights and some variables to represent vertices, and these variables could not be permuted without changing the function. Therefore we need to construct a new function that uses two different alphabets of variables. This is the motivation for the chromatic MacMahon function.

Chapter 3

Proving the MacMahon analogue of Crew's conjecture

Abstract

Chapter 3 Abstract: Code and figure example.

3.1 Further definitions and notation

Definition 3.1.1. A **bialgebra** is a vector space A endowed with a multiplication operation, $m : A \otimes A \to A$, a comultiplication operation, $\Delta : A \to A \otimes A$, a multiplicative unit, I, and a comultiplicative unit, ε, such that m is associative, Δ is coassociative, and the two functions are compatible. We call m the product and Δ the coproduct.

The function Δ is **coassociative** if the below diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \otimes A \arrow[d, "\Delta\otimes I"] \\ A \otimes A \arrow[r, "I\otimes\Delta"] & A \otimes A \otimes A \end{tikzcd} \tag{3.1.1}$$

The functions m and Δ are **compatible** if the following diagram commutes:

$$\begin{tikzcd} A \otimes A \arrow[r, "m"] \arrow[d, "\Delta\otimes\Delta"'] & A \arrow[d, "\Delta"] \\ A \otimes A \otimes A \otimes A \arrow[r, "m_{1,3}\otimes m_{2,4}"] & A \otimes A \end{tikzcd} \tag{3.1.2}$$

where $m_{1,3} \otimes m_{2,4}$ means to multiply the first and third coordinate together, and tensor with the second and fourth coordinate multiplied together.

A bialgebra A is **graded** if there is a well defined notion of degree upon elements in A such that the following properties hold. We define A_n to be set of all elements in A of degree n.

$$A = \bigoplus_{n \geq 0} A_n \tag{3.1.3}$$

$$A_k \cdot A_{n-k} \subseteq A_n \text{ for every } k \leq n \tag{3.1.4}$$

$$\Delta A_n \subseteq \bigoplus_{k \leq n} A_k \otimes A_{n-k} \text{ for every } n \in \tag{3.1.5}$$

A graded bialgebra A is **connected** if A_0 has dimension 1 as a vector space.

A bialgebra A with a well defined antipode is a **Hopf algebra**. For our purposes, however, we will be using the result from Grinberg & Reiner (2014) that states any bialgebra that is connected and graded is a Hopf algebra.

Example 3.1.2. An example of a Hopf algebra is the Hopf algebra over symmetric functions. Multiplication is defined in the standard way, and comultiplication can be defined on the power sum basis as the unique linear map that satisfies $\Delta(p_n) = 1 \otimes p_n + p_n \otimes 1$ (Liu & Tang, 2024, p. 3). The comultiplication is then forced to be

$$\Delta(p_\lambda) = \sum_{K \subseteq [\ell(\lambda)]} p_{\lambda|_K} \otimes p_{\lambda|_{\overline{K}}},$$

where $\lambda|_K =$ the partition containing only the parts of λ indexed by K.

Definition 3.1.3. Let $m \in$ and let $\{x_{j,k} \colon j \in, \ k \in [m]\}$ be commuting indeterminates. Let R be the ring of formal power series in these variables. When fixing a particular $i \in [m]$, we call the set $\{x_{j,i} \colon j \in\}$ the i^{th} **alphabet**. The permutation $\sigma \in \mathfrak{S}_\infty$, the group of permutations of , acts on R by sending $x_{j,k}$ to $x_{\sigma(j),k}$. We call this action the **diagonal action**. Notice that the diagonal action permutes each alphabet in the same way.

A **MacMahon symmetric function** is a formal power series in R that is invariant with respect to the diagonal action.

Example 3.1.4. For example, $x_1y_1^2 + x_2y_2^2 + x_3y_3^2 + \cdots$. Here we use x and y for the different alphabets as opposed to subscripts for readability. This is not a symmetric function, as you can't permute the x's with the y's without changing the function. You also cannot permute the alphabet of x's and the alphabet of y's in different ways without changing the function. But as long as you permute the x's with each other and the y's with each other, and use the same permutation on each alphabet, the function remains unchanged. We write M^n for the set of MacMahon symmetric functions with n alphabets. M^n has several bases analogous to the known bases of the symmetric functions, of which the most important for our purpose is the power-sum basis, which is indexed by vector partitions.

Definition 3.1.5. A **vector partition**, $\Lambda = \lambda^{(1)} \cdots \lambda^{(\ell)}$ of some vector $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{N}_{>0}^m$, is a set of vectors, $\lambda^{(i)} = (\lambda_{i,1}, \lambda_{i,2}, \ldots, \lambda_{i,m})$, none of which contain only zeros, each with m coordinates, such that $\lambda^{(1)} + \cdots + \lambda^{(\ell)} = \mathbf{u}$. Here $\lambda_{i,k}$ is a nonnegative integer which corresponds to the k^{th} coordinate of the i^{th} vector of Λ. There is no requirement that the vectors $\lambda^{(i)}$ be integer partitions. In other words, their coordinates need not be in decreasing order. Additionally, the order of the vectors is not significant. We index them only for notational convenience. To specify that λ is a vector partition of $\mathbf{u}$ we use the notation $\lambda \vdash \mathbf{u}$.

Definition 3.1.6. The **power sum** for MacMahon symmetric functions is defined for each $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ as

$$p_\lambda = \sum_{j=1}^{\infty} \prod_{k=1}^{m} (x_{j,k})^{\lambda_k}$$

where $(x_{j,k})^{\lambda_k}$ is the j^{th} variable of the k^{th} alphabet raised to the power of λ_k. The power sum is then defined for $\Lambda = \lambda^{(1)} \cdots \lambda^{(\ell)}$ to be

$$p_\Lambda = \prod_{i=1}^{\ell} \sum_{j=1}^{\infty} \prod_{k=1}^{m} (x_{j,k})^{\lambda_{i,k}}.$$

The set of all p_Λ generates M^n as a vector space, and is thus called the **power sum basis** Rosas (2001).

Example 3.1.7.

$$\begin{aligned} p_{(1,0,2)(2,2,3)} &= \left(\sum_{j=1}^{\infty} (x_{j,1})^1 (x_{j,2})^0 (x_{j,3})^2 \right) \left(\sum_{j=1}^{\infty} (x_{j,1})^2 (x_{j,2})^2 (x_{j,3})^3 \right) \\ &= (x_{1,1}x_{1,3}^2 + x_{2,1}x_{2,3}^2 + \cdots)(x_{1,1}^2 x_{1,2}^2 x_{1,3}^3 + x_{2,1}^2 x_{2,2}^2 x_{2,3}^3 + \cdots) \end{aligned}$$

Definition 3.1.8. The **bitype** of a weighted graph $G = (C_1, \ldots, C_{c(G)}, w)$, $\mathrm{btp}(G) = (n_1, w_1)\ (n_2, w_2) \cdots (n_{c(G)}, w_{c(G)})$ is the vector partition of $\mathbf{u} = (n(G), w(G))$. If $G = (V, E, w)$ is a weighted graph and $S \subseteq E$, we set $\mathrm{btp}(S) = \mathrm{btp}((V, S, w))$. Similarly, if $A \subseteq V$, we set $\mathrm{btp}(A) = \mathrm{btp}((A, E(A), w|_A))$, where $E(A)$ is the set of edges internal to A.

Definition 3.1.9. Given a weighted graph $G = (V, E, w)$, the **chromatic MacMahon function**, or **CMF**, of G is defined to be

$$\tilde{\mathbf{X}}_G = \sum_{\substack{k \in K(G) \\ \text{proper}}} \prod_{v \in V} x_{k(v)} y_{k(v)}^{w(v)}.$$

Observe that for a given weighted graph G, $\tilde{\mathbf{X}}_G$ determines $\mathbf{X}_{G,w}$ by setting $y_i = 1$ for every i. Additionally, $\tilde{\mathbf{X}}_G$ determines $\mathbf{X}_G$ by setting $x_i = 1$ for every i. However, the inverse is not true, the ordered pair of $\mathbf{X}_{G,w}$ and $\mathbf{X}_G$ does not determine $\tilde{\mathbf{X}}_G$.

Examine $\tilde{\mathbf{X}}_G$ on T_1 and T_2 from figures 2.2.1 and 2.2.2 respectively. Examine the terms acquired from using only two colors, color 1 and color 2. The function $\tilde{\mathbf{X}}_{T_1}$ gives the term $x_1^3 y_1^5 x_2^2 y_2^4 + x_2^3 y_2^5 x_1^2 y_1^4$ and $\tilde{\mathbf{X}}_{T_2}$ gives the term $x_1^3 y_1^4 x_2^2 y_2^5 + x_2^3 y_2^4 x_1^2 y_1^5$. Therefore the CMF distinguishes the two trees. However, as previously discussed, the CSF and wCSF don't distinguish them. Therefore the CSF and wCSF when taken together cannot distinguish the two trees, and therefore the CMF cannot be determined from the ordered pair of the CSF and wCSF.

Definition 3.1.10. Given a weighted graph $G = (V, E, w)$, the **extended generalized degree polynomial**, or **EGDP**, of G is defined to be

$$\tilde{\mathbf{G}}_G = \sum_{A \subseteq V} w^{|A|} x^{w(A)} y^{d(A)} z^{e(A)}.$$

We sometimes also write the EGDP as

$$\tilde{\mathbf{G}}_G(w,x,y,z) = \sum_{a,b,c,d} g_T(a,b,c,d) w^d x^a y^b z^c.$$

Here $g_T(a,b,c,d) = |\{A \subseteq V : |A| = d, w(A) = a, d(A) = b, e(A) = c\}|$ is the coefficient of $w^d x^a y^b z^c$ in $\tilde{\mathbf{G}}_G$.

3.2 Rewriting the CMF in the power sum basis

Proposition 3.2.1. *Let $G = (V, E, w)$ be a weighted graph with $n = |G|$ and $w = w(G)$. Then*

$$\tilde{\mathbf{X}}_G = \sum_{S \subseteq E(G)} (-1)^{|S|} p_{\mathrm{btp}(S)}(x, y).$$

Proof. Let $S \subseteq E$ such that the weighted graph (V, S, w) has connected components $C_1, \ldots, C_\ell$, of sizes $n_1, \ldots, n_\ell$ and weights $w_1, \ldots, w_\ell$, so that $\operatorname{btp}(S) = (n_1, w_1) \cdots (n_\ell, w_\ell)$ and

$$\begin{aligned}
p_{\operatorname{btp}(S)}(x,y) &= \prod_{i=1}^{\ell} p_{(n_i,w_i)}(x,y) \\
&= \sum_{(k_1,\ldots,k_\ell)\in\mathbb{N}^\ell} x_{k_1}^{n_1} y_{k_1}^{w_1} \cdots x_{k_\ell}^{n_\ell} y_{k_\ell}^{w_\ell} \\
&= \sum_{(k_1,\ldots,k_\ell)\in\mathbb{N}^\ell} x_{k_1}^{|C_1|} y_{k_1}^{\sum_{v\in C_1} w(v)} \cdots x_{k_\ell}^{|C_\ell|} y_{k_\ell}^{\sum_{v\in C_\ell} w(v)} \\
&= \sum_{k\in K_S(G)} \prod_{v\in V} x_{k(v)} y_{k(v)}^{w(v)}
\end{aligned}$$

where $K_S(G)$ is the set of all colorings $V \to \mathbb{N}$ that are monochromatic on the components of S (assigning color k_i to all vertices of C_i).

Multiplying by $(-1)^{|S|}$ and summing over all S, we obtain

$$\begin{aligned}
\sum_{S\subseteq E(G)} (-1)^{|S|} p_{\operatorname{btp}(S)}(x,y) &= \sum_{S\subseteq E(G)} (-1)^{|S|} \sum_{k\in K_S(G)} \prod_{v\in V} x_{k(v)} y_{k(v)}^{w(v)} \\
&= \sum_{k\in K(G)} \prod_{v\in V} x_{k(v)} y_{k(v)}^{w(v)} \sum_{S\in E_k(G)} (-1)^{|S|}
\end{aligned}$$

where $E_k(G)$ is the set of all k-monochromatic edges $e \in E(G)$. The second equality above is true, as the restriction that all edges of S are k-monochromatic is exactly equivalent to the restriction that k must be monochromatic on connected components of S.

And by a simple inclusion-exclusion argument, $\sum_{S\subseteq E_k(G)} (-1)^{|S|} = 0$ whenever k is not a proper coloring. When k is a proper coloring, $E_k(G)$ is empty, in which case $\sum_{S\subseteq E_k(G)} (-1)^{|S|} = 1$. Therefore

$$\tilde{\mathbf{X}}_G = \sum_{\substack{k\in K(G)\\ \text{proper}}} \prod_{v\in V} x_{k(v)} y_{k(v)}^{w(v)} = \sum_{S\subseteq E(G)} (-1)^{|S|} p_{\operatorname{btp}(S)}(x,y).$$

□

Example 3.2.2. We calculate the CMF for the tree T from example 2.1.7. The chromatic MacMahon function on T is therefore equal to

$$\tilde{\mathbf{X}}_T = p_{(1,2)(1,1)(1,3)} - p_{(2,3)(1,3)} - p_{(1,2)(2,4)} + p_{(3,6)}.$$

Corollary 3.2.3. *When F is a weighted forest, the chromatic MacMahon function of F can be expanded in the power sum basis as*

$$\tilde{\mathbf{X}}_F = \sum_{\Lambda \vdash (n,w)} b_\Lambda(F)(-1)^{n-\ell(\Lambda)} p_\Lambda(x,y)$$

where $b_\Lambda(F) = |\{S \subseteq E(F) \colon \operatorname{btp}(S) = \Lambda\}|$.

Proof.

$$\begin{aligned}
\tilde{\mathbf{X}}_F &= \sum_{S \subseteq E(F)} (-1)^{|S|} p_{\operatorname{btp}(S)}(x,y) \\
&= \sum_{S \subseteq E(F)} (-1)^{n-\ell(\operatorname{btp}(S))} p_{\operatorname{btp}(S)}(x,y) \\
&= \sum_{\Lambda \vdash (n,w)} \sum_{\substack{S \subseteq E(F) \\ \operatorname{btp}(S)=\Lambda}} (-1)^{n-\ell(\Lambda)} p_\Lambda(x,y) \\
&= \sum_{\Lambda \vdash (n,w)} b_\Lambda(F)(-1)^{n-\ell(\Lambda)} p_\Lambda(x,y).
\end{aligned}$$ □

3.3 The Hopf algebra coproduct on MacMahon symmetric functions

Here we find the Hopf algebra structure on the ring of MacMahon symmetric functions. There is a very clear associative multiplication map, so all that must be shown is that there is a comultiplication map that is coassociative and compatible with multiplication. We then show that this bialgebra is graded and connected.

Lemma 3.3.1. *The coproduct defined on the MacMahon power sum basis by*

$$\Delta(p_\Lambda) = \sum_{K \subseteq [\ell(\Lambda)]} p_{\Lambda|_K} \otimes p_{\Lambda|_{\overline{K}}} \tag{3.3.1}$$

is coassociative and compatible with multiplication.

Proof. To prove coassociativity, we show diagram 3.1.1 commutes.

$$
\begin{aligned}
I \otimes \Delta(\Delta(p_\Lambda)) &= I \otimes \Delta\left(\sum_{K\subseteq[\ell(\Lambda)]} p_{\Lambda|_K} \otimes p_{\Lambda|_{\overline{K}}}\right) \\
&= \sum_{K\subseteq[\ell(\Lambda)]} p_{\Lambda|_K} \otimes \Delta(p_{\Lambda|_{\overline{K}}}) \\
&= \sum_{K\subseteq[\ell(\Lambda)]} p_{\Lambda|_K} \otimes \left(\sum_{K_1\sqcup K_2=\overline{K}} p_{\Lambda|_{K_1}} \otimes p_{\Lambda|_{K_2}}\right) \\
&= \sum_{K_1\sqcup K_2\sqcup K_3=[\ell(\Lambda)]} p_{\Lambda|_{K_1}} \otimes p_{\Lambda|_{K_2}} \otimes p_{\Lambda|_{K_3}} \\
&= \sum_{K\subseteq[\ell(\Lambda)]} \left(\sum_{K_1\sqcup K_2=K} p_{\Lambda|_{K_1}} \otimes p_{\Lambda|_{K_2}}\right) \otimes p_{\Lambda|_{\overline{K}}} \\
&= \sum_{K\subseteq[\ell(\Lambda)]} \Delta(p_{\Lambda|_K}) \otimes p_{\Lambda|_{\overline{K}}} \\
\Delta \otimes I(\Delta(p_\Lambda)) &= \Delta \otimes I\left(\sum_{K\subseteq[\ell(\Lambda)]} p_{\Lambda|_K} \otimes p_{\Lambda|_{\overline{K}}}\right)
\end{aligned}
$$

To prove coassociativity, we show diagram 3.1.2 commutes.

$$
\begin{aligned}
m_{1,3}\otimes m_{2,4}\left(\Delta\otimes\Delta\left(p_\Lambda\otimes p_\Omega\right)\right) &= m_{1,3}\otimes m_{2,4}(\Delta(p_\Lambda)\otimes\Delta(p_\Omega)) \\
&= m_{1,3}\otimes m_{2,4}\left(\left(\sum_{K\subseteq[\ell(\Lambda)]} p_{\Lambda|_K}\otimes p_{\Lambda|_{\overline{K}}}\right)\otimes\left(\sum_{K'\subseteq[\ell(\Omega)]} p_{\Omega|_{K'}}\otimes p_{\Omega|_{\overline{K'}}}\right)\right) \\
&= m_{1,3}\otimes m_{2,4}\left(\sum_{\substack{K\subseteq[\ell(\Lambda)]\\ K'\subseteq[\ell(\Omega)]}} p_{\Lambda|_K}\otimes p_{\Lambda|_{\overline{K}}}\otimes p_{\Omega|_{K'}}\otimes p_{\Omega|_{\overline{K'}}}\right) \\
&= \sum_{\substack{K\subseteq[\ell(\Lambda)]\\ K'\subseteq[\ell(\Omega)]}} (p_{\Lambda|_K}\cdot p_{\Omega|_{K'}})\otimes(p_{\Lambda|_{\overline{K}}}\cdot p_{\Omega|_{\overline{K'}}}) \\
&= \sum_{J\subseteq[\ell(\Lambda\Omega)]} p_{\Lambda\Omega|_J}\otimes p_{\Lambda\Omega|_{\overline{J}}}
\end{aligned}
$$

where $J = K\cup K'$, and $\Lambda\Omega$ is the vector partition obtained by concatenating Ω with Λ

$$
\begin{aligned}
&= \Delta(p_{\Lambda\Omega}) \\
&= \Delta(p_\Lambda\cdot p_\Omega) \\
&= \Delta\left(m\left(p_\Lambda\otimes p_\Omega\right)\right).
\end{aligned}
$$

Therefore Δ satisfies the requirements for our coproduct. □

Lemma 3.3.2. *The vector space of MacMahon symmetric functions, M^n, is a graded and connected bialgebra, and therefore a Hopf algebra.*

Proof. It has already been shown that the function Δ defined in equation 3.3.1 is coassociative and compatible. All that needs to be seen now is that M^n is graded and connected.

Define the degree of the power sum p_λ for some vector λ to be the sum of the coordinates of λ. The degree of the power sum p_Λ for some vector partition $\Lambda = \lambda^{(1)}\cdots\lambda^{(\ell)}$ is defined to be the sum of the degrees of $p_{\lambda^{(i)}}$ for all $i\in[\ell]$.

Property 3.1.3 is equivalent to saying this notion of degree is well defined. This is immediate from the above definition of degree and lemma 3.3.1.

Property 3.1.4 is immediate as any power series of degree k multiplied by a power series of degree $n-k$ will have degree n.

Property 3.1.5 is immediate from the definition of the coproduct. For any element in M_n^ℓ, its coproduct will be a sum of elements with smaller degree tensor each other such that the two degrees add up to n.

Therefore M^n is graded.

For any $c \in M_0^n$, $c = c_\varnothing p_\varnothing = c_\varnothing$, in other words c is a scalar, and thus M_0^n has the same dimension as the base field as a vector space, that is to say it has dimension 1. Therefore M^n is connected. $\square$

Proposition 3.3.3. *Given two -linear maps $f,g : M^n \to R$, where R is a commutative ring, the convolution $f * g$ satisfies*

$$(f*g)(\tilde{\mathbf{X}}_F) = \sum_{A \subseteq V(F)} f(\tilde{\mathbf{X}}_{F|_A}) \cdot g(\tilde{\mathbf{X}}_{F|_{\overline{A}}}).$$

Proof. In a Hopf algebra, the convolution must satisfy,

$$\Delta(B) = \sum B_1 \otimes B_2 \implies (f*g) = \sum f(B_1) \cdot g(B_2),$$

where $B, B_1, B_2 \in M^n$. Here we are using the Sweedler notation for the coproduct.

Therefore to find the convolution on $\tilde{\mathbf{X}}_F$ we must take the coproduct on $\tilde{\mathbf{X}}_F$. By Prop. 3.2.1,

$$\begin{aligned}
\Delta(\tilde{\mathbf{X}}_F) &= \sum_{S \subseteq E(F)} (-1)^{|S|} \Delta(p_{\mathrm{btp}(S)}) \\
&= \sum_{S \subseteq E(F)} (-1)^{|S|} \sum_{K \subseteq [\ell(\mathrm{btp}(S))]} p_{\mathrm{btp}(S|_K)} \otimes p_{\mathrm{btp}(S|_{\overline{K}})},
\end{aligned}$$

where $S|_K$ is defined to be the edge set induced on the components of S indexed by K. Therefore,

$$\begin{aligned}
\Delta(\tilde{\mathbf{X}}_F) &= \sum_{S\subseteq E(F)} \sum_{K\subseteq[\ell(\mathrm{btp}(S))]} (-1)^{|S|_K|} p_{\mathrm{btp}(S|_K)} \otimes (-1)^{\left|S|_{\overline{K}}\right|} p_{\mathrm{btp}(S|_{\overline{K}})} \\
&= \sum_{A\subseteq V(F)} \sum_{S_1\subseteq E(A)} \sum_{S_2\subseteq E(\overline{A})} (-1)^{|S_1|} p_{\mathrm{btp}(S_1)} \otimes (-1)^{|S_2|} p_{\mathrm{btp}(S_2)} \\
&= \sum_{A\subseteq V(F)} \left(\sum_{S_1\subseteq E(A)} (-1)^{|S_1|} p_{\mathrm{btp}(S_1)} \right) \otimes \left(\sum_{S_2\subseteq E(\overline{A})} (-1)^{|S_2|} p_{\mathrm{btp}(S_2)} \right) \\
&= \sum_{A\subseteq V(F)} \tilde{\mathbf{X}}_{F|_A} \otimes \tilde{\mathbf{X}}_{F|_{\overline{A}}}.
\end{aligned}$$

This gives us the convolution on $\tilde{\mathbf{X}}_F$ of f and g to be,

$$(f*g)(\tilde{\mathbf{X}}_F) = \sum_{A\subseteq V(F)} f(\tilde{\mathbf{X}}_{F|_A}) \cdot g(\tilde{\mathbf{X}}_{F|_{\overline{A}}}) \qquad \square$$

3.4 Hopf-theoretic proof of MacMahon analogue of Crew's conjecture

Liu and Tang proved Crew's conjecture in (Liu & Tang, 2024, Prop. 2.4). Here we reproduce Liu and Tang's method in the setting of MacMahon symmetric functions.

Lemma 3.4.1. *There exists a $\mathbb{Q}$-linear map $\varphi_{t,u,v} : M^2 \to \mathbb{Q}[t,u,v]$ such that for every forest F we have*

$$\varphi_{t,u,v}(\tilde{\mathbf{X}}_F) = t^{|F|} u^{e(F)} v^{w(F)}.$$

Proof. Define $\varphi_{t,u,v}$ on the power sum basis $\{p_\Lambda(x,y)\}$ where Λ is a vector partition of $\mathbf{u} = (n,w)$ by

$$\varphi_{t,u,v}(p_\Lambda(x,y)) = t^n (1-u)^{n-\ell(\Lambda)} v^w.$$

Note that when $\Lambda = \mathrm{btp}(F)$ for some forest F, $n = |F|$ and $w = w(F)$.

Now, due to Cor. 3.2.3 we know $\tilde{\mathbf{X}}_F = \sum_{\Lambda \vdash (n,w)} b_\Lambda(F)(-1)^{n-\ell(\Lambda)} p_\Lambda(x,y)$, where $n = |F|$, $w = w(F)$, and $b_\Lambda(F) = |\{S \subseteq E(F) \colon \mathrm{btp}(S) = \Lambda\}|$. Therefore,

$$\begin{aligned}
\varphi_{t,u,v}(\tilde{\mathbf{X}}_F) &= \sum_{\Lambda \vdash (n,w)} b_\Lambda(F)(-1)^{n-\ell(\Lambda)} \cdot \varphi_{t,u,v}(p_\Lambda(x,y)) \\
&= \sum_{\Lambda \vdash (n,w)} b_\Lambda(F)(-1)^{n-\ell(\Lambda)} \cdot t^n (1-u)^{n-\ell(\Lambda)} v^w \\
&= t^n v^w \sum_{\Lambda \vdash (n,w)} b_\Lambda(F)(u-1)^{n-\ell(\Lambda)}.
\end{aligned}$$

For any choice of $n-\ell$ edges from $E(F)$, there is a specific edge set with bitype Λ for some vector partition Λ where $\ell(\Lambda) = \ell$. Different choices of $n-\ell$ edges may produce different bitypes, but each bitype derived from this process will have the same length. Therefore $\sum_{\ell(\Lambda)=\ell} b_\Lambda = \binom{e(F)}{n-\ell}$. Therefore,

$$\begin{aligned}
\varphi_{t,u,v}(\tilde{\mathbf{X}}_F) &= t^n v^w \sum_{\ell} \sum_{\substack{\Lambda \vdash (n,w) \\ \ell(\Lambda)=\ell}} b_\Lambda(F)(u-1)^{n-\ell} \\
&= t^n v^w \sum_{\ell} \binom{e(F)}{n-\ell} (u-1)^{n-\ell} \\
&= t^n u^{e(F)} v^w. \qquad \square
\end{aligned}$$

All this lemma is stating is that there is a linear function which determines the weights, edges, and vertices of a forest from its CMF.

Proposition 3.4.2. *There exists a $\mathbb{Q}$-linear map $\gamma : M^2 \to \mathbb{Q}(w,x,y,z)$ such that for every forest F we have*

$$\gamma(\tilde{\mathbf{X}}_F) = y^{c(F)} \tilde{\mathbf{G}}_F(w,x,y,z).$$

Proof. We will define $\gamma = \varphi_{xy,y^{-1}z,w} * \varphi_{y,y^{-1},1}$. Then by Prop. 3.3.3,

$$\begin{aligned}
(\varphi_{xy,y^{-1}z,w} * \varphi_{y,y^{-1},1})(\tilde{X}_F) &= \sum_{A \subseteq V(F)} \varphi_{xy,y^{-1}z,w}(\tilde{X}_{F|_A}) \cdot \varphi_{y,y^{-1},1}(\tilde{X}_{F|_{\overline{A}}}) \\
&= \sum_{A \subseteq V(F)} (xy)^{|A|}(y^{-1}z)^{e(A)}(w)^{w(A)}(y)^{|\overline{A}|}(y^{-1})^{e(\overline{A})} \\
&= \sum_{A \subseteq V(F)} w^{w(A)} x^{|A|} y^{|F|-e(A)-e(\overline{A})} z^{e(A)} \\
&= \sum_{A \subseteq V(F)} w^{w(K)} x^{|A|} y^{d(A)+c(F)} z^{e(A)} \\
&= y^{c(F)} \mathbf{G}_F(w,x,y,z).
\end{aligned}$$

□

Therefore the CMF determines the CSF by setting every $y_i = 1$, the wCSF by setting every $x_i = 1$, the EGDP by the above map γ, the wGDP by γ where $w = 1$, and the GDP by γ where $x = 1$.

Appendix A

Finding the coproduct for MacMahon symmetric functions

In this section, we explain how the coproduct was found. We will use the coproduct for noncommutative symmetric functions from (Lauve & Mastnak, 2011, p.538), and use the projection map from NCSym to MacMahon symmetric functions from Rosas (2001) to transform this into a coproduct for MacMahon symmetric functions.

Definition A.0.1. A **Noncommutative symmetric function**, or **NCSym** for short, is a symmetric function where the underlying variables do not commute. Noncommutative symmetric functions are better understood through their power sum basis, which is indexed by set partitions

Definition A.0.2. A **set partition** $\pi = A_1|\cdots|A_\ell$ is a partition of the set $[n]$ into ℓ blocks such that $A_1 \sqcup \cdots \sqcup A_\ell = [n]$. We will choose to always write blocks in order from least to greatest by least element.

An **interval partition** $\pi = A_1|\cdots|A_\ell$ is a set partition where every block is an interval. We define a correspondence between positive integer vectors and interval partitions in the following way: given some interval partition, $\pi = A_1|\cdots|A_\ell$, define the vector $\pi = (\pi_1, \dots, \pi_\ell)$ where $|A_i| = \pi_i$. This correspondence goes both ways. Given some interval partition π, we say π is the vector determined by π. Given some positive integer vector μ, we say u is the interval partition determined by μ.

For example, if $\pi = 12|3|456|7$ then $\pi = (2,1,3,1)$, and if $\mu = (1,3,3)$ then $u = 1|234|567$.

Definition A.0.3. The **Power Sum Basis** for NCSym is indexed by set partitions. For each $\pi = A_1|\cdots|A_\ell$ with $A_1 \sqcup \cdots \sqcup A_\ell = [n]$,

$$p_\pi = \sum_{(i_1,\ldots,i_n)} x_{i_1}\cdots x_{i_n}$$

where $i_j = i_k$ if j and k are in the same block of π.

For example:

$$p_{14|23} = x_1x_2^2x_1 + x_2x_1^2x_2 + x_1x_3^2x_1 + x_3x_1^2x_3 + x_2x_3^2x_2 + x_3x_2^2x_3 + \cdots + x_1^4 + x_2^4 + \cdots$$

Definition A.0.4. Fixing some set partition $\mu = B_1|B_2|\cdots|B_m$, the **type** of a set partition $\pi = A_1|A_2|\cdots|A_\ell$ under μ, denoted $\Lambda = \text{type}_\mu(\pi)$, is the vector partition $\Lambda = \lambda^{(1)}\cdots\lambda^{(\ell)}$ where $\lambda^{(k)}$ is the vector whose i^{th} coordinate is the intersection $A_k \cap B_i$. For example, say $\pi = 12|3|456|7$ and $\mu = 1|234|567$. Then the following table presents the cardinality of the intersections of each block of π and μ.

	π			
μ	12	3	456	7
1	1	0	0	0
234	1	1	1	0
567	0	0	2	1

(A.0.1)

Therefore $\text{btp}_\mu(\pi) = (1,1,0)(0,1,0)(0,1,2)(0,0,1)$, the columns of the table.

Definition A.0.5. Let π be a set partition with ℓ blocks, and let $K \subseteq [\ell]$. Then π_K is the set partition containing only the parts indexed by K. Furthermore, $(\pi_K)^\downarrow$ is the set partition with the same pattern as π_K with an underlying set being an interval of the form $[1,r]$ for some r. Here by pattern we mean the blocks have the same relative order, but each is shifted as to leave no gaps on the underlying interval.

As an example of the above, let $\pi = 12|3|456|789$ and $K = \{2,4\}$. Then $\pi_K = 3|789$ and $\pi_{\overline{K}} = 12|456$. But $(\pi_K)^{\downarrow} = 1|234$ and $(\pi_{\overline{K}})^{\downarrow} = 12|345$. This notation now allows us to define the coproduct for NCSym. Lauve and Mastnak (Lauve & Mastnak, 2011, p.538) show that given π, a set partition with ℓ blocks, the coproduct of p_π for NCSym is

$$\Delta(p_\pi) = \sum_{K \subseteq [\ell]} p_{(A_K)^{\downarrow}} \otimes p_{(A_{\overline{K}})^{\downarrow}}.$$

For each positive integer vector μ, there is a projection map ρ_μ defined, as in (Rosas, 2001, Def. 1), by

$$\begin{aligned} \rho_\mu : \mathrm{NCSym} &\to M^n \\ p_\pi &\mapsto p_{\mathrm{btp}_\mu(\pi)}. \end{aligned}$$

Therefore, in order to find the coproduct on MacMahon symmetric functions, we must take an arbitrary MacMahon power sum symmetric function indexed by Λ. The vector μ we use for the projection map we use will be explicitly defined as the vector that is the sum of the vectors of Λ. We then use the projection map backwards to get to the noncommutative symmetric function indexed by π for which $\Lambda = \mathrm{btp}_\mu(\pi)$. We then use the coproduct in NCSym on p_π. Finally we use the projection map forwards to give us the coproduct for MacMahon symmetric functions. The diagram below illustrates this.

$$\begin{array}{ccc} \mathrm{NCSym} & \xrightarrow{\rho_\mu} & M^n \\ \Delta\big\downarrow & & \big\downarrow ? \\ \mathrm{NCSym} \otimes \mathrm{NCSym} & \xrightarrow{\rho_\mu \otimes \rho_\mu} & M^n \otimes M^n \end{array}$$

We now describe the process by which this coproduct was found. Begin with the element of the MacMahon power sum basis p_Λ given by the vector partition $\Lambda = \lambda^{(1)}\lambda^{(2)} \cdots \lambda^{(\ell)}$ where each $\lambda^{(k)}$ is a vector and $\lambda^{(1)} + \cdots + \lambda^{(\ell)} = \mu$. The vector μ is fixed by this process since the coordinate μ_k of μ is the sum of the k^{th} coordinate of each vector in Λ. The vector μ determines an interval partition

$\mu = B_1|\cdots|B_m$ where B_i is an interval with size μ_i. Then going backwards along the projection map given by μ gives you the element of the noncommutative power sum basis, p_π, where π is a set partition that makes the statement $\Lambda = \text{type}_\mu(\pi)$ true. There are multiple set partitions that satisfy this requirement since the projection map is not injective. We choose the one that is an interval partition given by the vector $\pi = (\pi_1, \dots, \pi_\ell)$ where $\pi_i = |\lambda^{(i)}|$. Notice that in Table A.0.1, the sum of the i^{th} column equals π_i, the size of the corresponding block of π, and the sum of the k^{th} row equals μ_k, the size of the corresponding block of μ.

We now apply the NCSym coproduct to π. From the blocks of π, index some by K. Then we take the pattern on the blocks in K and in $\overline{K}$, and apply the projection map $\rho_{(\mu_K)^\downarrow} \otimes \rho_{(\mu_{\overline{K}})^\downarrow}$ to $(\pi_K)^\downarrow \otimes (\pi_{\overline{K}})^\downarrow$. This is best understood through example.

Say that $\Lambda = (1,1,0)(0,1,0)(0,1,2)(0,0,1)$, the same as in table A.0.1. We therefore get that $\mu = (1,3,3)$ and $\pi = (2,1,3,1)$ by summing the rows and the columns respectively. This gives $\mu = 1|234|567$ and $\pi = 12|3|456|7$. We will now display half the possible choices for K and $\overline{K}$, with the unshown half being symmetric to the ones shown.

K	π_K	$\pi_K \otimes \pi_{\overline{K}}$	$(\pi_K)^\downarrow \otimes (\pi_{\overline{K}})^\downarrow$	$p_{(\pi_K)^\downarrow} \otimes p_{(\pi_{\overline{K}})^\downarrow}$
$\varnothing$	$\varnothing$	$\varnothing \otimes 12\|3\|456\|7$	$\varnothing \otimes 12\|3\|456\|7$	$p_\varnothing \otimes p_{12\|3\|456\|7}$
$\{1\}$	12	$12 \otimes 3\|456\|7$	$12 \otimes 1\|234\|5$	$p_{12} \otimes p_{1\|234\|5}$
$\{2\}$	3	$3 \otimes 12\|456\|7$	$1 \otimes 12\|345\|6$	$p_1 \otimes p_{12\|345\|6}$
$\{3\}$	456	$456 \otimes 12\|3\|7$	$123 \otimes 12\|3\|4$	$p_{123} \otimes p_{12\|3\|4}$
$\{1,2\}$	$12\|3$	$12\|3 \otimes 456\|7$	$12\|3 \otimes 123\|4$	$p_{12\|3} \otimes p_{123\|4}$
$\{1,3\}$	$12\|456$	$12\|456 \otimes 3\|7$	$12\|345 \otimes 1\|2$	$p_{12\|345} \otimes p_{1\|2}$
$\{2,3\}$	$3\|456$	$3\|456 \otimes 12\|7$	$1\|234 \otimes 12\|3$	$p_{1\|234} \otimes p_{12\|3}$
$\{1,2,3\}$	$12\|3\|456$	$12\|3\|456 \otimes 7$	$12\|3\|456 \otimes 1$	$p_{12\|3\|456} \otimes p_1$

Then we repeat the table above to calculate the type for each of these interval partitions. However, because we calculate the type of $(\pi_K)^\downarrow$ with respect to $(\mu_K)^\downarrow$, we can ignore the arrow on both and get the same answer. However, then what this comes down to, is for the type of any π_K, simply consider the table that originally described Λ as being the type of π, and only take the columns indexed by K. Therefore, we now copy the same table as above, adding in an extra column giving the vector partition after the projection map. We remove commas and some parentheses for clarity.

π_K	$\pi_K \otimes \pi_{\overline{K}}$	$\rho_\mu(p_{\pi_K}) \otimes \rho_\mu(p_{\pi_{\overline{K}}})$
$\varnothing$	$\varnothing \otimes 12\|3\|456\|7$	$p_\varnothing \otimes p_{(110)(010)(012)(001)}$
12	$12 \otimes 3\|456\|7$	$p_{(110)} \otimes p_{(010)(012)(001)}$
3	$3 \otimes 12\|456\|7$	$p_{(010)} \otimes p_{(110)(012)(001)}$
456	$456 \otimes 12\|3\|7$	$p_{(012)} \otimes p_{(110)(010)(001)}$
$12\|3$	$12\|3 \otimes 456\|7$	$p_{(110)(010)} \otimes p_{(012)(001)}$
$12\|456$	$12\|456 \otimes 3\|7$	$p_{(110)(012)} \otimes p_{(010)(001)}$
$3\|456$	$3\|456 \otimes 12\|7$	$p_{(010)(012)} \otimes p_{(110)(001)}$
$12\|3\|456$	$12\|3\|456 \otimes 7$	$p_{(110)(010)(012)} \otimes p_{(001)}$

At the end of this process, we find exactly what the coproduct is for MacMahon symmetric functions. For some vector partition $\Lambda = \lambda^{(1)} \cdots \lambda^{(\ell)}$,

$$\Delta(p_\Lambda) = \sum_{K \subseteq [\ell]} p_{\Lambda|_K} \otimes p_{\Lambda|_{\overline{K}}}$$

Appendix B

An alternate proof of MacMahon analogue of Crew's conjecture

In this section, we provide an alternate proof of the MacMahon analogue of Crew's conjecture. This proof is generalized from (Aliste-Prieto et al., 2024, Thm. 6).

Definition B.0.1. Define

$$\omega(\Lambda,a,b,c,d) := (-1)^{n-b-1} \sum_{\Omega \vdash (a,d)} \binom{d-\ell(\Omega)}{c} \binom{\Lambda}{\Omega} \binom{n-\ell(\Lambda)+\ell(\Omega)-d}{n-b-c-1}.$$

Definition B.0.2. Say that Λ and Ω are two vector partitions such that $\Lambda,\Omega \vdash (n,w)$. Define

$$\binom{\Lambda}{\Omega} := \prod_{i=1}^{a}\prod_{j=1}^{d} \binom{m_{i,j}(\Lambda)}{m_{i,j}(\Omega)}$$

where $m_{i,j}(\Lambda) :=$ the number of times the vector (i,j) appears in Λ. Observe that if $\Lambda = \mathrm{btp}(S)$ for some $S \subseteq E(G)$ then $\binom{\mathrm{btp}(S)}{\Omega} = |\{A \subseteq V(G) : \mathrm{btp}(A) = \Omega, S \subseteq E(A) \cup E(\overline{A})\}|$.

Theorem B.0.3. *When $T = (V,E,w)$ is a tree, the coefficients of $\tilde{\mathbf{G}}_T(w,x,y,z)$ can be recovered from the coefficients of $\tilde{\mathbf{X}}_T$.*

$$g_T(a,b,c,d) = \sum_{\Lambda \vdash (n,w)} b_\Lambda(T)(-1)^{n-\ell(\Lambda)}\omega(\Lambda,a,b,c,d)$$

Proof. Let RHS equal the right hand side of the above equation, and LHS equal the left hand side.

$$\mathrm{RHS} = \sum_{\Lambda \vdash (n,w)} (-1)^{n-\ell(\Lambda)} |\{S \subseteq E : \mathrm{btp}(S) = \Lambda\}| (-1)^{n-b-1} \omega(\Lambda,a,b,c,d)$$

$$= \sum_{S\subseteq E} (-1)^{n-\ell(\mathrm{btp}(S))}(-1)^{n-b-1}\omega(\mathrm{btp}(S),a,b,c,d)$$

$$= \sum_{S\subseteq E} (-1)^{n-\ell(\mathrm{btp}(S))}(-1)^{n-b-1} \sum_{\Omega\vdash(a,d)} \binom{d-\ell(\Omega)}{c}\binom{\mathrm{btp}(S)}{\Omega}\binom{n-\ell(\mathrm{btp}(S))+\ell(\Omega)-d}{n-b-c-1}$$

Simply by substituting in the definition of $\omega(\Lambda,a,b,c,d)$.

$$= \sum_{S\subseteq E} (-1)^{|S|+n-b-1} \sum_{\Omega\vdash(a,d)} \binom{d-\ell(\Omega)}{c}\binom{\mathrm{btp}(S)}{\Omega}\binom{|S|+\ell(\Omega)-d}{n-b-c-1}$$

Since S is a forest and thus has cardinality equal to $n-\ell(\mathrm{btp}(S))$.

$$= \sum_{S\subseteq E} (-1)^{|S|+n-b-1} \sum_{\Omega\vdash(a,d)} \sum_{\substack{A\subseteq V\\|A|=a\\w(A)=d\\\mathrm{btp}(A)=\Omega\\S\subseteq E(A)\cup E(\overline{A})}} \binom{d-\ell(\Omega)}{c}\binom{|S|+\ell(\Omega)-d}{n-b-c-1}$$

By the observation that $\binom{\mathrm{btp}(S)}{\Omega} = |A\subseteq V(G) : \mathrm{btp}(A)=\Omega, S\subseteq E(A)\cup E(\overline{A})|$.

$$= \sum_{S\subseteq E} (-1)^{|S|+n-b-1} \sum_{\substack{A\subseteq V\\|A|=a\\w(A)=d\\S\subseteq E(A)\cup E(\overline{A})}} \binom{d-\ell(\mathrm{btp}(S))}{c}\binom{|S|+\ell(\mathrm{btp}(S))-d}{n-b-c-1}$$

$$= \sum_{\substack{A\subseteq V\\|A|=a\\w(A)=d}} \sum_{S\subseteq E(A)\cup E(\bar{A})} (-1)^{|S|+n-b-1}\binom{d-\ell(\mathrm{btp}(S))}{c}\binom{|S|+\ell(\mathrm{btp}(S))-d}{n-b-c-1}$$

$$= \sum_{\substack{A\subseteq V\\|A|=a\\w(A)=d}} \sum_{S(A)\subseteq E(A)} \sum_{S(\bar{A})\subseteq E(\bar{A})} (-1)^{|S(A)|+|S(\bar{A})|+n-b-1}\binom{|S(A)|}{c}\binom{|S(\bar{A})|}{n-b-c-1}$$

$$= \sum_{\substack{A\subseteq V\\|A|=a\\w(A)=d}} \left(\sum_{S(A)\subseteq E(A)} (-1)^{|S(A)|+c}\binom{|S(A)|}{c}\right)\left(\sum_{S(\bar{A})\subseteq E(\bar{A})} (-1)^{|S(\bar{A})|+n-b-c-1}\binom{|S(\bar{A})|}{n-b-c-1}\right)$$

$$= |\{A \subseteq V : w(A) = d, e(A) = c, e(\bar{A}) = n - b - c - 1, |A| = a\}| = \text{LHS}$$

by (Aliste-Prieto et al., 2024, Lemma 5). $\square$

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