

1 Transitions between bases

First, a standard linear algebra fact. Let V be a vector space of finite dimension n over ground field \mathbb{k} (it doesn't matter what it is).

Proposition 1. Suppose that $B = \{v_1, \dots, v_n\}$ and $B^* = \{v_1^*, \dots, v_n^*\}$ are bases for V that are orthogonal, i.e.,

$$\langle v_i, v_j^* \rangle = \delta_{ij} k_i$$

for some nonzero constants k_1, \dots, k_n . Then the unique expansion of u with respect to B is

$$u = \sum_{i=1}^n \frac{\langle v_i^*, u \rangle}{k_i} v_i. \quad (1)$$

Proof. For every $j \in [n]$ we have

$$\left\langle v_j^*, \sum_{i=1}^n \frac{\langle v_i^*, u \rangle}{k_i} v_i \right\rangle = \sum_{i=1}^n \frac{\langle v_i^*, u \rangle}{k_i} \langle v_j^*, v_i \rangle = \sum_{i=1}^n \langle v_i^*, u \rangle \delta_{ij} = \langle v_j^*, u \rangle.$$

That is, every element of B^* has the same inner product with the left and right-hand sides of (1). This is necessary and sufficient for equality. \square

This proposition reduces the problem of expanding a vector in a particular basis B to computing inner products with an orthogonal basis B^* . (The best possible case is that $k_i = 1$ for all i , i.e., B^* is the dual basis to B . However, we're not always that lucky.)

Notation:

- \mathbb{N} = nonnegative integers;
- For $u \in \mathbb{N}^k$, a **vector partition** $\lambda \vdash u$ is an unordered sequence of vectors (**parts**) summing to u ; zero vectors can be ignored.
- Weight of a vector partition = sum of all entries in all vectors in it (ex.: $\text{wt}(\{120, 100, 100, 013\}) = 9$)
- \mathfrak{M}_u = MacMahon symmetric functions of homogeneous multidegree u
- $\text{sign}(\lambda) = (-1)^{\text{number of parts with even sum}}$
- $m_\lambda, e_\lambda, h_\lambda, p_\lambda, f_\lambda$: monomial, elementary, homogeneous, power-sum, and forgotten MacMahon symmetric functions [Ros01, pp.327–328]
- For a vector partition λ in which each part v_i occurs with multiplicity m_i , define

$$|\lambda| = \prod_i m_i! \quad \text{and} \quad \lambda! = \prod_i \prod_{x \in v_i} (x!)^{m_i}.$$

A vector partition is **unitary** if it is a partition of the all-ones vector $\mathbf{1} \in \mathbb{N}^k$. There is an obvious bijection between unitary vector partitions and set partitions of $[k]$. Unitary MacMahon symmetric functions are the focus of [Dou72]. These are the graded pieces $\mathfrak{M}_{(1)^k}$.

The **type** of a set partition $\pi = B_1 | \dots | B_\ell$, with respect to a vector $u = (u_1, \dots, u_r)$ of weight n , is the vector partition $\text{type}_u(\pi) = \lambda = \lambda_1 \cdots \lambda_\ell$ where λ_k is the vector in \mathbb{N}^r whose i th coordinate is

$$\#\{j \in B_k \mid u_1 + \dots + u_{i-1} < j \leq i_1 + \dots + u_i\}.$$

the number of elements of B_k such that in the i th equivalence class.

For a vector \mathbf{u} of weight n and a vector partition λ of weight n , the number of set partitions π of type $\text{type}_{\mathbf{u}}(\pi)$ is

$$\binom{\mathbf{u}}{\lambda} := \frac{\mathbf{u}!}{\lambda! |\lambda|}.$$

Define a scalar product on \mathfrak{M} :

$$\boxed{\langle h_{\lambda}, m_{\mu} \rangle = \delta_{\lambda\mu}.} \quad (2)$$

The idea is to use this for basis transitions, along the lines discussed earlier. Specifically, formula (1) says that for any $F \in \mathfrak{M}_{\mathbf{u}}$, we have

$$F = \sum_{\lambda \vdash \mathbf{u}} \langle h_{\lambda}, F \rangle m_{\lambda} \quad \text{and} \quad F = \sum_{\lambda \vdash \mathbf{u}} \langle m_{\lambda}, F \rangle h_{\lambda}. \quad (3)$$

Thus we have to know how to compute inner products.

There is a “lifting map” $\hat{\rho}$ [Ros01, Defn. 6] sending MacMahon functions to unitary (Doubilet) symmetric functions. Specifically:

$$\binom{\mathbf{u}}{\lambda} |\lambda| m_{\lambda} \xrightarrow{\hat{\rho}} \sum_{\pi: \text{type}_{\mathbf{u}}(\pi) = \lambda} m_{\pi} \quad (4a)$$

$$\binom{\mathbf{u}}{\lambda} \lambda! h_{\lambda} \xrightarrow{\hat{\rho}} \sum_{\pi: \text{type}_{\mathbf{u}}(\pi) = \lambda} h_{\pi} \quad (4b)$$

$$\binom{\mathbf{u}}{\lambda} \lambda! e_{\lambda} \xrightarrow{\hat{\rho}} \sum_{\pi: \text{type}_{\mathbf{u}}(\pi) = \lambda} e_{\pi} \quad (4c)$$

$$\binom{\mathbf{u}}{\lambda} p_{\lambda} \xrightarrow{\hat{\rho}} \sum_{\pi: \text{type}_{\mathbf{u}}(\pi) = \lambda} p_{\pi} \quad (4d)$$

$$\binom{\mathbf{u}}{\lambda} |\lambda| f_{\lambda} \xrightarrow{\hat{\rho}} \sum_{\pi: \text{type}_{\mathbf{u}}(\pi) = \lambda} f_{\pi} \quad (4e)$$

Moreover, for all $f, g \in \mathfrak{M}_{\mathbf{u}}$ we have [Ros01, Prop. 7]

$$\langle f, g \rangle = \mathbf{u}! \langle \hat{\rho}(f), \hat{\rho}(g) \rangle \quad (5)$$

Prop. 1 together with the scalar product (2) says in particular that for $F \in \mathfrak{M}_{\mathbf{u}}$ we have

$$F = \sum_{\lambda \vdash \mathbf{u}} \langle m_{\lambda}, F \rangle h_{\lambda}, \quad F = \sum_{\lambda \vdash \mathbf{u}} \langle h_{\lambda}, F \rangle m_{\lambda}. \quad (6)$$

Doubilet [Dou72, Appendix 2] calculated the scalar products for all five families of unitary symmetric functions (which requires things like lattice operations and Möbius and zeta functions in the set partition lattice Π_n). Therefore, we should be able to compute scalar products, and thus basis expansions, for all MacMahon functions. **Warning:** Doubilet’s notation for the standard bases is not the usual modern notation (he used k, a, h, s, f for what we call m, e, h, p, f — perhaps this was not standardized in 1970?), and each of his formulas has an extra $n!$ that needs to be deleted (he was working in an analogous ring where the $n!$ makes sense. Rosas reproduced Doubilet’s calculations (in modern notation and without the $n!$ ’s that are extraneous in our context).

2 Expansions in the m -basis

$$\begin{aligned}
p_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat{\rho}(h_\lambda), \hat{\rho}(p_\mu) \rangle m_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{(\frac{\mathbf{u}}{\lambda})\lambda!} \sum_{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda} h_\pi, \frac{1}{(\frac{\mathbf{u}}{\mu})} \sum_{\sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu} p_\sigma \right\rangle m_\lambda && \text{(by (4b) and (4d))} \\
&= \frac{\mathbf{u}!}{(\frac{\mathbf{u}}{\mu})} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\frac{\mathbf{u}}{\lambda})\lambda!} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \langle h_\pi, p_\sigma \rangle m_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{(\frac{\mathbf{u}}{\mu})} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\frac{\mathbf{u}}{\lambda})\lambda!} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \zeta(\sigma, \pi) m_\lambda && \text{(by Doubilet formula #3)} \\
p_\mu &= \frac{\mathbf{u}!}{(\frac{\mathbf{u}}{\mu})} \sum_{\lambda \vdash \mathbf{u}} \frac{\#\{(\pi, \sigma) \mid \text{type}_{\mathbf{u}}(\pi) = \lambda, \text{type}_{\mathbf{u}}(\sigma) = \mu, \sigma \leq \pi\}}{(\frac{\mathbf{u}}{\lambda})\lambda!} m_\lambda
\end{aligned}$$

$$\begin{aligned}
h_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat{\rho}(h_\lambda), \hat{\rho}(h_\mu) \rangle m_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{(\frac{\mathbf{u}}{\lambda})\lambda!} \sum_{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda} h_\pi, \frac{1}{(\frac{\mathbf{u}}{\mu})\mu!} \sum_{\sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu} h_\sigma \right\rangle m_\lambda && \text{(by (4b) twice)} \\
&= \frac{\mathbf{u}!}{(\frac{\mathbf{u}}{\mu})\mu!} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\frac{\mathbf{u}}{\lambda})\lambda!} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \langle h_\pi, h_\sigma \rangle m_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{(\frac{\mathbf{u}}{\mu})\mu!} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\frac{\mathbf{u}}{\lambda})\lambda!} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \lambda(\sigma \wedge \pi)! m_\lambda && \text{(by Doubilet formula #9)}
\end{aligned}$$

$$\begin{aligned}
e_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat{\rho}(h_\lambda), \hat{\rho}(e_\mu) \rangle m_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{(\frac{\mathbf{u}}{\lambda})\lambda!} \sum_{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda} h_\pi, \frac{1}{(\frac{\mathbf{u}}{\mu})\mu!} \sum_{\sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu} e_\sigma \right\rangle m_\lambda && \text{(by (4b) and (4c))} \\
&= \frac{\mathbf{u}!}{(\frac{\mathbf{u}}{\mu})\mu!} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\frac{\mathbf{u}}{\lambda})\lambda!} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \langle h_\pi, e_\sigma \rangle m_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{(\frac{\mathbf{u}}{\mu})\mu!} \sum_{\lambda \vdash \mathbf{u}} \frac{\#\{(\pi, \sigma) \mid \text{type}_{\mathbf{u}}(\pi) = \lambda, \text{type}_{\mathbf{u}}(\sigma) = \mu, \pi \wedge \sigma = \hat{\mathbf{0}}\}}{(\frac{\mathbf{u}}{\lambda})\lambda!} m_\lambda && \text{(by Doubilet formula #6)}
\end{aligned}$$

$$\begin{aligned}
f_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat{\rho}(h_\lambda), \hat{\rho}(f_\mu) \rangle m_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{(\mathbf{u})|\lambda|} \sum_{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda} h_\pi, \frac{1}{(\mathbf{u})|\mu|} \sum_{\sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu} f_\sigma \right\rangle m_\lambda && \text{(by (4b) and (4d))} \\
&= \frac{\mathbf{u}!}{(\mathbf{u})|\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\mathbf{u})|\lambda|} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \langle h_\pi, f_\sigma \rangle m_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{(\mathbf{u})|\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\mathbf{u})|\lambda|} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu \\ \sigma \leq \pi}} \lambda(\sigma, \pi)! m_\lambda && \text{(by Doubilet formula \#13)}
\end{aligned}$$

3 Expansions in the h -basis

$$\begin{aligned}
m_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat{\rho}(m_\lambda), \hat{\rho}(m_\mu) \rangle h_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{(\mathbf{u})|\lambda|} \sum_{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda} m_\pi, \frac{1}{(\mathbf{u})|\mu|} \sum_{\sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu} m_\sigma \right\rangle h_\lambda && \text{(by (4a) twice)} \\
&= \frac{\mathbf{u}!}{(\mathbf{u})|\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\mathbf{u})|\lambda|} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \langle m_\pi, m_\sigma \rangle h_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{(\mathbf{u})|\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\mathbf{u})|\lambda|} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \sum_{\tau \geq \pi \vee \sigma} \frac{\mu(\pi, \tau) \mu(\sigma, \tau)}{\mu(\hat{\mathbf{0}}, \tau)} h_\lambda && \text{(by Doubilet formula \#7)}
\end{aligned}$$

$$\begin{aligned}
p_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat{\rho}(m_\lambda), \hat{\rho}(p_\mu) \rangle h_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{(\mathbf{u})|\lambda|} \sum_{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda} m_\pi, \frac{1}{(\mathbf{u})} \sum_{\sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu} p_\sigma \right\rangle h_\lambda && \text{(by (4a) and (4d))} \\
&= \frac{\mathbf{u}!}{(\mathbf{u})} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\mathbf{u})|\lambda|} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \langle m_\pi, p_\sigma \rangle h_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{(\mathbf{u})} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\mathbf{u})|\lambda|} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu \\ \pi \leq \sigma}} \frac{\mu(\pi, \sigma)}{|\mu(\hat{\mathbf{0}}, \sigma)|} h_\lambda && \text{(by Doubilet formula \#10)}
\end{aligned}$$

$$\begin{aligned}
e_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat{\rho}(m_\lambda), \hat{\rho}(e_\mu) \rangle h_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{(\mathbf{u})|\lambda|} \sum_{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda} m_\pi, \frac{1}{(\mathbf{u})\mu!} \sum_{\sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu} e_\sigma \right\rangle h_\lambda && \text{(by (4a) and (4c))} \\
&= \frac{\mathbf{u}!}{(\mathbf{u})\mu!} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\mathbf{u})|\lambda|} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \langle m_\pi, e_\sigma \rangle h_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{(\mathbf{u})\mu!} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\mathbf{u})|\lambda|} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu \\ \pi \leq \sigma}} \text{sign}(\pi) \lambda(\pi, \sigma)! h_\lambda && \text{(by Doubilet formula #8)}
\end{aligned}$$

$$\begin{aligned}
f_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat{\rho}(m_\lambda), \hat{\rho}(f_\mu) \rangle h_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{(\mathbf{u})|\lambda|} \sum_{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda} m_\pi, \frac{1}{(\mathbf{u})|\mu|} \sum_{\sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu} f_\sigma \right\rangle h_\lambda && \text{(by (4a) and (4e))} \\
&= \frac{\mathbf{u}!}{(\mathbf{u})|\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\mathbf{u})|\lambda|} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \langle m_\pi, f_\sigma \rangle h_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{(\mathbf{u})|\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\mathbf{u})|\lambda|} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \sum_{\tau \geq \sigma \vee \pi} \frac{|\mu(\sigma, \tau)| \cdot \mu(\pi, \tau)}{\mu(\hat{\mathbf{0}}, \tau)} h_\lambda && \text{(by Doubilet formula #11)}
\end{aligned}$$

4 Expansions in the p -basis

The power-sum basis is orthogonal (though not orthonormal). I would like to verify this, since it is not evident from going through the lifting map. A remark in Rosas [Ros01, p.333] says that for $\lambda \vdash \mathbf{u}$,

$$\langle p_\lambda, p_\mu \rangle = \frac{\delta_{\lambda, \mu} |\lambda| \lambda!}{|\mu(\hat{\mathbf{0}}, \lambda)|} \quad (7)$$

where $\mu(\hat{\mathbf{0}}, \lambda)$ means $\mu(\hat{\mathbf{0}}, \pi)$ for any set partition π such that $\text{type}_{\mathbf{u}}(\pi) = \lambda$.

Therefore, Proposition 1 says that for every $F \in \mathfrak{M}$,

$$F = \sum_{\lambda} \frac{\langle p_\lambda, F \rangle \mu(\hat{\mathbf{0}}, \lambda)}{|\lambda| \lambda!} p_\lambda = \sum_{\lambda} \frac{\langle p_\lambda, F \rangle \mu(\hat{\mathbf{0}}, \lambda) (\mathbf{u})}{\mathbf{u}!} p_\lambda \quad (8)$$

So, as before:

$$\begin{aligned}
m_\mu &= \sum_\lambda \left\langle \binom{\mathbf{u}}{\lambda} \hat{\rho}(p_\lambda), \hat{\rho}(m_\mu) \right\rangle \mu(\hat{\mathbf{0}}, \lambda) p_\lambda && \text{(by (8) and (5))} \\
&= \frac{1}{\binom{\mathbf{u}}{\mu} |\mu|} \sum_\lambda \mu(\hat{\mathbf{0}}, \lambda) \sum_{\substack{\sigma: \text{type}_{\mathbf{u}}(\sigma)=\lambda \\ \pi: \text{type}_{\mathbf{u}}(\pi)=\mu}} \langle p_\sigma, m_\pi \rangle p_\lambda && \text{(by (4d) and (4a) and bilinearity)} \\
&= \frac{1}{\binom{\mathbf{u}}{\mu} |\mu|} \sum_\lambda \left[\mu(\hat{\mathbf{0}}, \lambda) \sum_{\substack{\sigma: \text{type}_{\mathbf{u}}(\sigma)=\lambda \\ \pi: \text{type}_{\mathbf{u}}(\pi)=\mu \\ \pi \leq \sigma}} \frac{\mu(\pi, \sigma)}{|\mu(\hat{\mathbf{0}}, \sigma)|} \right] p_\lambda && \text{(by Doubilet formula \#10)}
\end{aligned}$$

$$\begin{aligned}
h_\mu &= \sum_\lambda \langle \hat{\rho}(p_\lambda), \hat{\rho}(h_\mu) \rangle \mu(\hat{\mathbf{0}}, \lambda) \binom{\mathbf{u}}{\lambda} p_\lambda && \text{(by (8) and (5))} \\
&= \frac{1}{\binom{\mathbf{u}}{\mu} \mu!} \sum_\lambda \mu(\hat{\mathbf{0}}, \lambda) \sum_{\substack{\sigma: \text{type}_{\mathbf{u}}(\sigma)=\lambda \\ \pi: \text{type}_{\mathbf{u}}(\pi)=\mu}} \langle p_\sigma, h_\pi \rangle p_\lambda && \text{(by (4d) and (4b) and bilinearity)} \\
&= \frac{1}{\binom{\mathbf{u}}{\mu} \mu!} \sum_\lambda \left[\mu(\hat{\mathbf{0}}, \lambda) \sum_{\substack{\sigma: \text{type}_{\mathbf{u}}(\sigma)=\lambda \\ \pi: \text{type}_{\mathbf{u}}(\pi)=\mu \\ \sigma \leq \pi}} \right] p_\lambda && \text{(by Doubilet formula \#3)}
\end{aligned}$$

$$\begin{aligned}
e_\mu &= \sum_\lambda \langle \hat{\rho}(p_\lambda), \hat{\rho}(e_\mu) \rangle \mu(\hat{\mathbf{0}}, \lambda) \binom{\mathbf{u}}{\lambda} p_\lambda && \text{(by (8) and (5))} \\
&= \frac{1}{\binom{\mathbf{u}}{\mu} \mu!} \sum_\lambda \mu(\hat{\mathbf{0}}, \lambda) \sum_{\substack{\sigma: \text{type}_{\mathbf{u}}(\sigma)=\lambda \\ \pi: \text{type}_{\mathbf{u}}(\pi)=\mu}} \langle p_\sigma, e_\pi \rangle p_\lambda && \text{(by (4d) and (4c) and bilinearity)} \\
&= \frac{1}{\binom{\mathbf{u}}{\mu} \mu!} \sum_\lambda \left[\mu(\hat{\mathbf{0}}, \lambda) \sum_{\substack{\sigma: \text{type}_{\mathbf{u}}(\sigma)=\lambda \\ \pi: \text{type}_{\mathbf{u}}(\pi)=\mu \\ \sigma \leq \pi}} \text{sign}(\sigma) \right] p_\lambda && \text{(by Doubilet formula \#4)}
\end{aligned}$$

$$\begin{aligned}
f_\mu &= \sum_{\lambda} \langle \hat{\rho}(p_\lambda), \hat{\rho}(f_\mu) \rangle \mu(\hat{\mathbf{0}}, \lambda) \binom{\mathbf{u}}{\lambda} p_\lambda && \text{(by (8) and (5))} \\
&= \frac{1}{\binom{\mathbf{u}}{\mu} |\mu|} \sum_{\lambda} \mu(\hat{\mathbf{0}}, \lambda) \sum_{\substack{\sigma: \text{type}_{\mathbf{u}}(\sigma)=\lambda \\ \pi: \text{type}_{\mathbf{u}}(\pi)=\mu}} \langle p_\sigma, f_\pi \rangle p_\lambda && \text{(by (4d) and (4e) and bilinearity)} \\
&= \frac{1}{\binom{\mathbf{u}}{\mu} |\mu|} \sum_{\lambda} \left[\mu(\hat{\mathbf{0}}, \lambda) \sum_{\substack{\sigma: \text{type}_{\mathbf{u}}(\sigma)=\lambda \\ \pi: \text{type}_{\mathbf{u}}(\pi)=\mu \\ \pi \leq \sigma}} \text{sign}(\pi) \text{sign}(\sigma) \frac{\mu(\pi, \sigma)}{|\mu(\hat{\mathbf{0}}, \sigma)|} \right] p_\lambda && \text{(by Doubilet formula #14)}
\end{aligned}$$

5 The omega involution, and expansions in the e - and f -bases

There is an automorphism $\omega: \mathfrak{M} \rightarrow \mathfrak{M}$ (called θ in [Dou72]) defined by

$$\omega(e_\lambda) = h_\lambda$$

This is defined in [Ros01, p.328] (and used to define the forgotten basis $f_\lambda = \text{sign}(\lambda)\omega(m_\lambda)$). I would like to know what ω does to p_λ . For symmetric functions, we have $\omega(p_\lambda) = \text{sign}(\lambda)p_\lambda$ (where λ is a partition), and for unitary MacMahon functions, we have $\omega(p_\pi) = \text{sign}(\pi)p_\pi$ (where π is a set partition). It would be natural to hope that $\omega(p_\lambda) = \text{sign}(\lambda)p_\lambda$ in general, though Rosas does not state that. Can we prove it? And can we prove that ω is an isometry?

Or should we just assume all of these things? If so, then applying ω to the conversions to the h -basis gives

$$f_\mu = \text{sign}(\mu) \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu} |\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda} |\lambda|} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \sum_{\tau \geq \pi \vee \sigma} \frac{\mu(\pi, \tau) \mu(\sigma, \tau)}{\mu(\hat{\mathbf{0}}, \tau)} e_\lambda$$

$$p_\mu = \text{sign}(\mu) \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda} |\lambda|} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu \\ \pi \leq \sigma}} \frac{\mu(\pi, \sigma)}{|\mu(\hat{\mathbf{0}}, \sigma)|} e_\lambda$$

$$h_\mu = \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu} \mu!} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda} |\lambda|} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu \\ \pi \leq \sigma}} \text{sign}(\pi) \lambda(\pi, \sigma)! e_\lambda$$

$$m_\mu = \text{sign}(\mu) \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu} |\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda} |\lambda|} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \sum_{\tau \geq \sigma \vee \pi} \frac{|\mu(\sigma, \tau)| \cdot \mu(\pi, \tau)}{\mu(\hat{\mathbf{0}}, \tau)} e_\lambda$$

and likewise applying ω to the conversions to the m -basis gives

$$p_\mu = \text{sign}(\mu) \frac{\mathbf{u}!}{(\mathbf{u})_\mu} \sum_{\lambda \vdash \mathbf{u}} \text{sign}(\lambda) \frac{\#\{(\pi, \sigma) \mid \text{type}_{\mathbf{u}}(\pi) = \lambda, \text{type}_{\mathbf{u}}(\sigma) = \mu, \sigma \leq \pi\}}{(\mathbf{u})_\lambda \lambda!} f_\lambda$$

$$e_\mu = \frac{\mathbf{u}!}{(\mathbf{u})_\mu \mu!} \sum_{\lambda \vdash \mathbf{u}} \text{sign}(\lambda) \frac{1}{(\mathbf{u})_\lambda \lambda!} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi) = \lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma) = \mu}} \lambda(\sigma \wedge \pi)! f_\lambda$$

$$h_\mu = \frac{\mathbf{u}!}{(\mathbf{u})_\mu \mu!} \sum_{\lambda \vdash \mathbf{u}} \text{sign}(\lambda) \frac{\#\{(\pi, \sigma) \mid \text{type}_{\mathbf{u}}(\pi) = \lambda, \text{type}_{\mathbf{u}}(\sigma) = \mu, \pi \wedge \sigma = \hat{\mathbf{0}}\}}{(\mathbf{u})_\lambda \lambda!} f_\lambda$$

$$m_\mu = \text{sign}(\mu) \frac{\mathbf{u}!}{(\mathbf{u})_\mu |\mu|} \sum_{\lambda \vdash \mathbf{u}} \text{sign}(\lambda) \frac{1}{(\mathbf{u})_\lambda \lambda!} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi) = \lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma) = \mu \\ \sigma \leq \pi}} \lambda(\sigma, \pi)! f_\lambda$$

References

- [Dou72] Peter Doubilet, *On the foundations of combinatorial theory. VII. Symmetric functions through the theory of distribution and occupancy*, Studies in Appl. Math. **51** (1972), 377–396. MR 429577 [1](#), [2](#), [7](#)
- [Ros01] Mercedes H. Rosas, *MacMahon symmetric functions, the partition lattice, and Young subgroups*, J. Combin. Theory Ser. A **96** (2001), no. 2, 326–340. MR 1864127 [1](#), [2](#), [5](#), [7](#)