

$$9. k_\pi = \sum_{\sigma \geq \pi} \text{sign}(\pi, \sigma) \lambda(\pi, \sigma)! f_\sigma \quad f_\pi = \sum_{\sigma \geq \pi} \lambda(\pi, \sigma)! k_\sigma$$

$$10. a_\pi = \sum_{\sigma \leq \pi} (\text{sign } \sigma) \lambda(\sigma, \pi)! h_\sigma \quad h_\pi = \sum_{\sigma \leq \pi} (\text{sign } \sigma) \lambda(\sigma, \pi)! a_\sigma$$

### Appendix 2: The inner product on $\tilde{\mathcal{P}}$

- | Doubilet's notation  | Standard notation   |
|--|---|
| $(\tilde{h}_\pi, \tilde{k}_\sigma) = n! \delta_{\pi\sigma}$  | $k$ $m$   |
| $(\tilde{s}_\pi, \tilde{s}_\sigma) = \delta_{\pi\sigma} \frac{n!}{ \mu(0, \pi) }$  | $a$ $e$   |
| $(\tilde{h}_\pi, \tilde{s}_\sigma) = n! \zeta(\sigma, \pi)$ , where $\zeta(\sigma, \pi) = \begin{cases} 1 & \text{if } \sigma \leq \pi \\ 0 & \text{if not} \end{cases}$   | $h$ $h$   |
| $(\tilde{a}_\pi, \tilde{s}_\sigma) = n! (\text{sign } \sigma) \zeta(\sigma, \pi)$  | $s$ $p$   |
| $(\tilde{a}_\pi, \tilde{a}_\sigma) = n! \lambda(\sigma \wedge \pi)!$   | $f$ $f$   |
| $(\tilde{a}_\pi, \tilde{h}_\sigma) = \begin{cases} n! & \text{if } \pi \wedge \sigma = 0 \\ 0 & \text{if } \pi \wedge \sigma \neq 0 \end{cases}$                           |   |
| $(\tilde{k}_\pi, \tilde{k}_\sigma) = n! \sum_{\tau \geq \pi \vee \sigma} \frac{\mu(\pi, \tau) \mu(\sigma, \tau)}{ \mu(0, \tau) }$  |   |
| $(\tilde{a}_\pi, \tilde{k}_\sigma) = n! (\text{sign } \sigma) \lambda(\sigma, \pi) \zeta(\sigma, \pi)$   |   |
| $(\tilde{h}_\pi, \tilde{h}_\sigma) = n! \lambda(\sigma \wedge \pi)!$   |   |
| $(\tilde{k}_\pi, \tilde{s}_\sigma) = n! \frac{\mu(\pi, \sigma)}{ \mu(0, \sigma) } \zeta(\pi, \sigma)$  | sign(pi) = $(-1)^{\text{number of even blocks of pi}}$<br>= $(-1)^{\text{rank of pi in partition lattice}}$ |
| $(\tilde{f}_\pi, \tilde{k}_\sigma) = n! \sum_{\tau \geq \pi \vee \sigma} \frac{ \mu(\pi, \tau)   \mu(\sigma, \tau) }{ \mu(0, \tau) }$                                      |   |
| $(\tilde{f}_\pi, \tilde{f}_\sigma) = (\text{sign } \pi) (\text{sign } \sigma) n! \sum_{\tau \geq \pi \vee \sigma} \frac{\mu(\pi, \tau) \mu(\sigma, \tau)}{ \mu(0, \tau) }$ |   |
| $(\tilde{f}_\pi, \tilde{h}_\sigma) = n! \lambda(\pi, \sigma) \zeta(\pi, \sigma)$   |   |
| $(\tilde{f}_\pi, \tilde{s}_\sigma) = (\text{sign } \pi) (\text{sign } \sigma) n! \frac{\mu(\pi, \sigma)}{ \mu(0, \sigma) } \zeta(\pi, \sigma)$                             |   |
| $(\tilde{f}_\pi, \tilde{a}_\sigma) = n! (\text{sign } \pi) \delta_{\pi\sigma}$   |   |

### Appendix 3: The Kronecker inner product on $\tilde{\mathcal{P}}$

1.  $[\tilde{s}_\pi, \tilde{s}_\sigma] = \frac{n!}{|\mu(0, \pi)|} \delta_{\pi\sigma} \tilde{s}_\pi$
2.  $[\tilde{h}_\pi, \tilde{s}_\sigma] = n! \zeta(\sigma, \pi) \tilde{s}_\sigma$
3.  $[\tilde{a}_\pi, \tilde{s}_\sigma] = n! (\text{sign } \sigma) \zeta(\sigma, \pi) \tilde{s}_\sigma$