P. Doubilet, Symmetric Functions through the Theory of Distribution and Occupancy, Stud. Appl. Math., Vol. 51 (1972), pp. 377–396.



9. $k_\pi = \sum_{\sigma \geq \pi} \operatorname{sign}(\pi, \sigma)\lambda(\pi, \sigma)!\, f_\sigma \qquad f_\pi = \sum_{\sigma \geq \pi} \lambda(\pi, \sigma)!\, k_\sigma$

10. $a_\pi = \sum_{\sigma \leq \pi} (\operatorname{sign} \sigma)\lambda(\sigma, \pi)!\, h_\sigma \qquad h_\pi = \sum_{\sigma \leq \pi} (\operatorname{sign} \sigma)\lambda(\sigma, \pi)!\, a_\sigma$

## Appendix 2: The inner product on $\tilde{\mathscr{S}}$

1. $(\tilde{h}_\pi, \tilde{k}_\sigma) = n!\,\delta_{\pi\sigma}$
2. $(\tilde{s}_\pi, \tilde{s}_\sigma) = \delta_{\pi\sigma} \dfrac{n!}{|\mu(0, \pi)|}$
3. $(\tilde{h}_\pi, \tilde{s}_\sigma) = n!\,\zeta(\sigma, \pi)$, where $\zeta(\sigma, \pi) = \begin{cases} 1 & \text{if } \sigma \leq \pi \\ 0 & \text{if not} \end{cases}$
4. $(\tilde{a}_\pi, \tilde{s}_\sigma) = n!(\operatorname{sign} \sigma)\zeta(\sigma, \pi)$
5. $(\tilde{a}_\pi, \tilde{a}_\sigma) = n!\,\lambda(\sigma \wedge \pi)!$
6. $(\tilde{a}_\pi, \tilde{h}_\sigma) = \begin{cases} n! & \text{if } \pi \wedge \sigma = 0 \\ 0 & \text{if } \pi \wedge \sigma \neq 0 \end{cases}$
7. $(\tilde{k}_\pi, \tilde{k}_\sigma) = n! \sum_{\tau \geq \pi \vee \sigma} \dfrac{\mu(\pi, \tau)\mu(\sigma, \tau)}{|\mu(0, \tau)|}$
8. $(\tilde{a}_\pi, \tilde{k}_\sigma) = n!(\operatorname{sign} \sigma)\lambda(\sigma, \pi)!\,\zeta(\sigma, \pi)$
9. $(\tilde{h}_\pi, \tilde{h}_\sigma) = n!\,\lambda(\sigma \wedge \pi)!$
10. $(\tilde{k}_\pi, \tilde{s}_\sigma) = n! \dfrac{\mu(\pi, \sigma)}{|\mu(0, \sigma)|}\zeta(\pi, \sigma)$
11. $(\tilde{f}_\pi, \tilde{k}_\sigma) = n! \sum_{\tau \geq \pi \vee \sigma} \dfrac{|\mu(\pi, \tau)|\mu(\sigma, \tau)}{|\mu(0, \tau)|}$
12. $(\tilde{f}_\pi, \tilde{f}_\sigma) = (\operatorname{sign} \pi)(\operatorname{sign} \sigma) n! \sum_{\tau \geq \pi \vee \sigma} \dfrac{\mu(\pi, \tau)\mu(\sigma, \tau)}{|\mu(0, \tau)|}$
13. $(\tilde{f}_\pi, \tilde{h}_\sigma) = n!\,\lambda(\pi, \sigma)!\,\zeta(\pi, \sigma)$
14. $(\tilde{f}_\pi, \tilde{s}_\sigma) = (\operatorname{sign} \pi)(\operatorname{sign} \sigma) n! \dfrac{\mu(\pi, \sigma)}{|\mu(0, \sigma)|}\zeta(\pi, \sigma)$
15. $(\tilde{f}_\pi, \tilde{a}_\sigma) = n!(\operatorname{sign} \pi)\delta_{\pi\sigma}$

\pi and \sigma are set partitions of [n].
\mu is the Mobius function in ∏_n

| Doubilet's notation | Standard notation |
|---|---|
| k | m |
| a | e |
| h | h |
| s | p |
| f | f |

sign(pi) = (-1)^(number of even blocks of pi)
= (-1)^(rank of pi in partition lattice)

## Appendix 3: The Kronecker inner product on $\tilde{\mathscr{S}}$

1. $[\tilde{s}_\pi, \tilde{s}_\sigma] = \dfrac{n!}{|\mu(0, \pi)|}\delta_{\pi\sigma}\tilde{s}_\pi$
2. $[\tilde{h}_\pi, \tilde{s}_\sigma] = n!\,\zeta(\sigma, \pi)\tilde{s}_\sigma$
3. $[\tilde{a}_\pi, \tilde{s}_\sigma] = n!(\operatorname{sign} \sigma)\zeta(\sigma, \pi)\tilde{s}_\sigma$