Let $\mathbb{N}$ denote the nonnegative integers and $\mathbb{P}$ the positive integers. Fix $m \in \mathbb{P}$ and let $\mathcal{A} = \{x_{i,j} \mid i \in [r],\ j \in \mathbb{P}\}$. Everything that follows takes place in the ring of formal power series $R = \Bbbk[[\mathcal{A}]]$, where $\Bbbk$ is a field (assumed to be of characteristic 0 if necessary). There is an $\mathbb{N}^r$-grading on R; the homogeneous component of degree $\mathbf{u} = (u_1, \dots, u_r) \in \mathbb{N}^r$ is the vector space span of monomials having total degree u_i in the variables $\{x_{i,j} \mid j \in \mathbb{P}\}$.

The group $\mathfrak{S}$ of permutations of $\mathbb{P}$ acts diagonally on $\mathcal{A}$ by $\sigma(x_{i,j}) = x_{i,\sigma(j)}$; this action extends to R. The invariants of the action are the **MacMahon (symmetric) functions**. They form a graded subring $\mathfrak{M} \subseteq R$, with $\mathfrak{M}_{\mathbf{u}} = \mathfrak{M} \cap R_{\mathbf{u}}$.

Following Rosas [Ros01], the standard combinatorial bases for $\mathfrak{M}$ are as follows. For $\mathbf{u} \in \mathbb{N}^r$, each basis for $\mathfrak{M}_{\mathbf{u}}$ is indexed by the **vector partitions** $\lambda \vdash \mathbf{u}$. Such a thing is a multiset of nonzero vectors $\lambda^{(1)}, \dots, \lambda^{(k)} \in \mathbb{N}^r$ that sum to $\mathbf{u}$.

Let $\lambda = \{\lambda^{(1)}, \dots, \lambda^{(k)}\}$ be a vector partition, where $\lambda^{(j)} = (\ell_{1,j}, \dots, \ell_{r,j})$. The **monomial MacMahon function** m_λ is the sum of all monomials in the $\mathfrak{S}$-orbit of the monomial

$$\prod_{j=1}^{k} \prod_{i=1}^{r} x_{i,j}^{\ell_{i,j}}.$$

For $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{N}^r$, the **power-sum MacMahon function** is

$$p_{\mathbf{a}} = \sum_{i=1}^{\infty} x_{1,i}^{a_1} x_{2,i}^{a_2} \cdots x_{r,i}^{a_r};$$

the **elementary MacMahon function** is

$$e_{\mathbf{a}} = [t_1^{a_1} \cdots t_r^{a_r}] \prod_{i=1}^{\infty} 1 + t_1 x_{1,i} + t_2 x_{2,i} + \cdots + t_r x_{r,i}$$

and the **complete homogeneous MacMahon function** is

$$h_{\mathbf{a}} = [t_1^{a_1} \cdots t_r^{a_r}] \prod_{i=1}^{\infty} \frac{1}{1 - t_1 x_{1,i} - t_2 x_{2,i} - \cdots - t_r x_{r,i}}.$$

These are all multiplicative bases: for a vector partition $\lambda = \{\lambda^{(1)}, \dots, \lambda^{(k)}\}$ we set

$$p_\lambda = \prod_{j=1}^{k} p_{\lambda^{(j)}}, \qquad e_\lambda = \prod_{j=1}^{k} e_{\lambda^{(j)}}, \qquad h_\lambda = \prod_{j=1}^{k} h_{\lambda^{(j)}}.$$

Finally, there is an involutive automorphism $\omega \colon \mathfrak{M} \to \mathfrak{M}$ defined by $\omega(e_\lambda) = h_\lambda$. The **forgotten MacMahon functions** are defined by $f_\lambda = \operatorname{sign}(\lambda) m_\lambda$.

1 Transitions between bases

First, a standard linear algebra fact. Let V be a vector space of finite dimension n over ground field $\Bbbk$ (it doesn't matter what it is).

Proposition 1. *Suppose that $B = \{v_1, \dots, v_n\}$ and $B^* = \{v_1^*, \dots, v_n^*\}$ are bases for V that are orthogonal, i.e.,*

$$\langle v_i, v_j^* \rangle = \delta_{ij} k_i$$

for some nonzero constants $k_1, \ldots, k_n$. Then the unique expansion of u with respect to B is

$$u = \sum_{i=1}^{n} \frac{\langle v_i^*,\, u\rangle}{k_i} v_i. \tag{1}$$

Proof. For every $j \in [n]$ we have

$$\left\langle v_j^*,\, \sum_{i=1}^{n} \frac{\langle v_i^*,\, u\rangle}{k_i} v_i \right\rangle = \sum_{i=1}^{n} \frac{\langle v_i^*,\, u\rangle}{k_i} \langle v_j^*,\, v_i\rangle = \sum_{i=1}^{n} \langle v_i^*,\, u\rangle \delta_{ij} = \langle v_j^*,\, u\rangle.$$

That is, every element of B^* has the same inner product with the left and right-hand sides of (1). This is necessary and sufficient for equality. □

This proposition reduces the problem of expanding a vector in a particular basis B to computing inner products with an orthogonal basis B^*. (The best possible case is that $k_i = 1$ for all i, i.e., B^* is the dual basis to B. However, we're not always that lucky.)

Notation:

- $\mathbb{N}$ = nonnegative integers
- For $\mathbf{u} \in \mathbb{N}^k$, a **vector partition** $\lambda \vdash \mathbf{u}$ is a multiset of nonzero vectors (the **parts** of λ) that sum to u.
- Weight of a vector partition = sum of all entries in all vectors in it (ex.: $\mathrm{wt}(\{120, 100, 100, 013\}) = 9$)
- $\mathfrak{M}_{\mathbf{u}}$ = MacMahon symmetric functions of homogeneous multidegree u
- $\mathrm{sign}(\lambda) = (-1)^{\text{number of parts with even sum}}$
- $m_\lambda, e_\lambda, h_\lambda, p_\lambda, f_\lambda$: monomial, elementary, homogeneous, power-sum, and forgotten MacMahon symmetric functions [Ros01, pp.327–328]
- For a vector partition λ in which each part v_i occurs with multiplicity m_i, define

$$|\lambda| = \prod_i m_i! \qquad \text{and} \qquad \lambda! = \prod_i \prod_{x \in v_i} (x!)^{m_i}.$$

A vector partition is **unitary** if it is a partition of the all-ones vector $\mathbf{1} \in \mathbb{N}^k$. There is an obvious bijection between unitary vector partitions and set partitions of $[k]$. Unitary MacMahon symmetric functions are the focus of [Dou72]. These are the graded pieces $\mathfrak{M}_{(1)^k}$.

The **type** of a set partition $\pi = B_1 | \cdots | B_\ell$, with respect to a vector $\mathbf{u} = (u_1, \ldots, u_r)$ of weight n, is the vector partition $\mathrm{type}_{\mathbf{u}}(\pi) = \lambda = \lambda_1 \cdots \lambda_\ell$ where λ_k is the vector in $\mathbb{N}^r$ whose ith coordinate is

$$\#\{j \in B_k \mid u_1 + \cdots + u_{i-1} < j \leq i_1 + \cdots + u_i\}.$$

the number of elements of B_k such that in the ith equivalence class.

For a vector $\mathbf{u}$ of weight n and a vector partition λ of weight n, the number of set partitions π of type $\mathrm{type}_{\mathbf{u}}(\pi)$ is

$$\binom{\mathbf{u}}{\lambda} := \frac{\mathbf{u}!}{\lambda!\,|\lambda|}.$$

Define a scalar product on $\mathfrak{M}$:

$$\boxed{\langle h_\lambda,\, m_\mu\rangle = \delta_{\lambda\mu}.} \tag{2}$$

The idea is to use this for basis transitions, along the lines discussed earlier. Specifically, formula (1) says that for any $F \in \mathfrak{M}_{\mathbf{u}}$, we have

$$F = \sum_{\lambda \vdash u} \langle h_\lambda,\, F \rangle m_\lambda \qquad \text{and} \qquad F = \sum_{\lambda \vdash u} \langle m_\lambda,\, F \rangle h_\lambda. \tag{3}$$

Thus we have to know how to compute inner products.

There is a "lifting map" $\hat{\rho}$ [Ros01, Defn. 6] sending MacMahon functions to unitary (Doubilet) symmetric functions. Specifically:

$$\binom{\mathbf{u}}{\lambda} |\lambda|\, m_\lambda \stackrel{\hat{\rho}}{\longmapsto} \sum_{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda} m_\pi \tag{4a}$$

$$\binom{\mathbf{u}}{\lambda} \lambda!\, h_\lambda \stackrel{\hat{\rho}}{\longmapsto} \sum_{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda} h_\pi \tag{4b}$$

$$\binom{\mathbf{u}}{\lambda} \lambda!\, e_\lambda \stackrel{\hat{\rho}}{\longmapsto} \sum_{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda} e_\pi \tag{4c}$$

$$\binom{\mathbf{u}}{\lambda} p_\lambda \stackrel{\hat{\rho}}{\longmapsto} \sum_{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda} p_\pi \tag{4d}$$

$$\binom{\mathbf{u}}{\lambda} |\lambda|\, f_\lambda \stackrel{\hat{\rho}}{\longmapsto} \sum_{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda} f_\pi \tag{4e}$$

Moreover, for all $f, g \in \mathfrak{M}_{\mathbf{u}}$ we have [Ros01, Prop. 7]

$$\langle f,\, g \rangle = \mathbf{u}!\, \langle \hat{\rho}(f),\, \hat{\rho}(g) \rangle \tag{5}$$

Prop. 1 together with the scalar product (2) says in particular that for $F \in \mathfrak{M}_{\mathbf{u}}$ we have

$$F = \sum_{\lambda \vdash \mathbf{u}} \langle m_\lambda,\, F \rangle\, h_\lambda, \qquad\qquad F = \sum_{\lambda \vdash \mathbf{u}} \langle h_\lambda,\, F \rangle\, m_\lambda. \tag{6}$$

Doubilet [Dou72, Appendix 2] calculated the scalar products for all five families of unitary symmetric functions (which requires things like lattice operations and Möbius and zeta functions in the set partition lattice Π_n). Therefore, we should be able to compute scalar products, and thus basis expansions, for all MacMahon functions. Warning: Doubilet's notation for the standard bases is not the usual modern notation (he used k, a, h, s, f for what we call m, e, h, p, f — perhaps this was not standardized in 1970?), and each of his formulas has an extra $n!$ that needs to be deleted (he was working in an analogous ring where the $n!$ makes sense. Rosas reproduced Doubilet's calculations (in modern notation and without the $n!$'s that are extraneous in our context).

2 Expansions in the m-basis

$$\begin{aligned}
p_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat\rho(h_\lambda),\, \hat\rho(p_\mu) \rangle\, m_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{\binom{\mathbf{u}}{\lambda}\lambda!} \sum_{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda} h_\pi,\ \frac{1}{\binom{\mathbf{u}}{\mu}} \sum_{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu} p_\sigma \right\rangle m_\lambda && \text{(by (4b) and (4d))} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}\lambda!} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}} \langle h_\pi,\, p_\sigma \rangle\, m_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}\lambda!} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}} \zeta(\sigma,\pi) m_\lambda && \text{(by Doubilet formula \#3)} \\
p_\mu &= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}} \sum_{\lambda \vdash \mathbf{u}} \frac{\#\{(\pi,\sigma) \mid \mathrm{type}_{\mathbf{u}}(\pi)=\lambda,\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu,\ \sigma \le \pi\}}{\binom{\mathbf{u}}{\lambda}\lambda!} m_\lambda
\end{aligned}$$

$$\begin{aligned}
h_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat\rho(h_\lambda),\, \hat\rho(h_\mu) \rangle\, m_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{\binom{\mathbf{u}}{\lambda}\lambda!} \sum_{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda} h_\pi,\ \frac{1}{\binom{\mathbf{u}}{\mu}\mu!} \sum_{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu} h_\sigma \right\rangle m_\lambda && \text{(by (4b) twice)} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}\mu!} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}\lambda!} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}} \langle h_\pi,\, h_\sigma \rangle\, m_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}\mu!} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}\lambda!} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}} \lambda(\sigma \wedge \pi)!\, m_\lambda && \text{(by Doubilet formula \#9)}
\end{aligned}$$

$$\begin{aligned}
e_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat\rho(h_\lambda),\, \hat\rho(e_\mu) \rangle\, m_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{\binom{\mathbf{u}}{\lambda}\lambda!} \sum_{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda} h_\pi,\ \frac{1}{\binom{\mathbf{u}}{\mu}\mu!} \sum_{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu} e_\sigma \right\rangle m_\lambda && \text{(by (4b) and (4c))} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}\mu!} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}\lambda!} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}} \langle h_\pi,\, e_\sigma \rangle\, m_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}\mu!} \sum_{\lambda \vdash \mathbf{u}} \frac{\#\{(\pi,\sigma) \mid \mathrm{type}_{\mathbf{u}}(\pi)=\lambda,\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu,\ \pi \wedge \sigma = \hat{\mathbf{0}}\}}{\binom{\mathbf{u}}{\lambda}\lambda!} m_\lambda && \text{(by Doubilet formula \#6)}
\end{aligned}$$

$$\begin{aligned}
f_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat{\rho}(h_\lambda),\, \hat{\rho}(f_\mu) \rangle \, m_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{\binom{\mathbf{u}}{\lambda} \lambda!} \sum_{\pi\colon \operatorname{type}_{\mathbf{u}}(\pi)=\lambda} h_\pi,\ \frac{1}{\binom{\mathbf{u}}{\mu} |\mu|} \sum_{\sigma\colon \operatorname{type}_{\mathbf{u}}(\sigma)=\mu} f_\sigma \right\rangle m_\lambda && \text{(by (4b) and (4d))} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu} |\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda} \lambda!} \sum_{\substack{\pi\colon \operatorname{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma\colon \operatorname{type}_{\mathbf{u}}(\sigma)=\mu}} \langle h_\pi,\, f_\sigma \rangle \, m_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu} |\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda} \lambda!} \sum_{\substack{\pi\colon \operatorname{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma\colon \operatorname{type}_{\mathbf{u}}(\sigma)=\mu \\ \sigma \le \pi}} \lambda(\sigma,\pi)! m_\lambda && \text{(by Doubilet formula \#13)}
\end{aligned}$$

3 Expansions in the h-basis

$$\begin{aligned}
m_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat{\rho}(m_\lambda),\, \hat{\rho}(m_\mu) \rangle \, h_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{\binom{\mathbf{u}}{\lambda} |\lambda|} \sum_{\pi\colon \operatorname{type}_{\mathbf{u}}(\pi)=\lambda} m_\pi,\ \frac{1}{\binom{\mathbf{u}}{\mu} |\mu|} \sum_{\sigma\colon \operatorname{type}_{\mathbf{u}}(\sigma)=\mu} m_\sigma \right\rangle h_\lambda && \text{(by (4a) twice)} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu} |\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda} |\lambda|} \sum_{\substack{\pi\colon \operatorname{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma\colon \operatorname{type}_{\mathbf{u}}(\sigma)=\mu}} \langle m_\pi,\, m_\sigma \rangle \, h_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu} |\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda} |\lambda|} \sum_{\substack{\pi\colon \operatorname{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma\colon \operatorname{type}_{\mathbf{u}}(\sigma)=\mu}} \sum_{\tau \ge \pi \vee \sigma} \frac{\mu(\pi,\tau)\mu(\sigma,\tau)}{\mu(\hat{\mathbf{0}},\tau)} \, h_\lambda && \text{(by Doubilet formula \#7)}
\end{aligned}$$

$$\begin{aligned}
p_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat{\rho}(m_\lambda),\, \hat{\rho}(p_\mu) \rangle \, h_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{\binom{\mathbf{u}}{\lambda} |\lambda|} \sum_{\pi\colon \operatorname{type}_{\mathbf{u}}(\pi)=\lambda} m_\pi,\ \frac{1}{\binom{\mathbf{u}}{\mu}} \sum_{\sigma\colon \operatorname{type}_{\mathbf{u}}(\sigma)=\mu} p_\sigma \right\rangle h_\lambda && \text{(by (4a) and (4d))} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda} |\lambda|} \sum_{\substack{\pi\colon \operatorname{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma\colon \operatorname{type}_{\mathbf{u}}(\sigma)=\mu}} \langle m_\pi,\, p_\sigma \rangle \, h_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda} |\lambda|} \sum_{\substack{\pi\colon \operatorname{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma\colon \operatorname{type}_{\mathbf{u}}(\sigma)=\mu \\ \pi \le \sigma}} \frac{\mu(\pi,\sigma)}{|\mu(\hat{\mathbf{0}},\sigma)|} \, h_\lambda && \text{(by Doubilet formula \#10)}
\end{aligned}$$

$$
\begin{aligned}
e_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat\rho(m_\lambda),\, \hat\rho(e_\mu) \rangle\, h_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda} m_\pi,\ \frac{1}{\binom{\mathbf{u}}{\mu}\mu!} \sum_{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu} e_\sigma \right\rangle h_\lambda && \text{(by (4a) and (4c))} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}\mu!} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}} \langle m_\pi,\, e_\sigma \rangle\, h_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}\mu!} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu \\ \pi \le \sigma}} \mathrm{sign}(\pi)\lambda(\pi,\sigma)!\, h_\lambda && \text{(by Doubilet formula \#8)}
\end{aligned}
$$

$$
\begin{aligned}
f_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat\rho(m_\lambda),\, \hat\rho(f_\mu) \rangle\, h_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda} m_\pi,\ \frac{1}{\binom{\mathbf{u}}{\mu}|\mu|} \sum_{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu} f_\sigma \right\rangle h_\lambda && \text{(by (4a) and (4e))} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}|\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}} \langle m_\pi,\, f_\sigma \rangle\, h_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}|\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}} \sum_{\tau \ge \sigma \vee \pi} \frac{|\mu(\sigma,\tau)| \cdot \mu(\pi,\tau)}{\mu(\hat{\mathbf{0}},\tau)}\, h_\lambda && \text{(by Doubilet formula \#11)}
\end{aligned}
$$

4 Expansions in the p-basis

Rosas [Ros01, p.333] states that for $\lambda \vdash \mathbf{u}$,

$$\langle p_\lambda,\, p_\mu \rangle = \frac{\delta_{\lambda,\mu}|\lambda|\,\lambda!}{|\mu(\hat{\mathbf{0}},\lambda)|} \tag{7}$$

where $\mu(\hat{\mathbf{0}},\lambda)$ means $\mu(\hat{\mathbf{0}},\pi)$ for any set partition π such that $\mathrm{type}_u(\pi) = \lambda$. This can be verified using the lifting maps and Doubilet's formula.

Therefore, Proposition 1 says that for every $F \in \mathfrak{M}$,

$$F = \sum_\lambda \frac{\langle p_\lambda,\, F\rangle |\mu(\hat{\mathbf{0}},\lambda)|}{|\lambda|\,\lambda!} p_\lambda = \sum_\lambda \frac{\langle p_\lambda,\, F\rangle |\mu(\hat{\mathbf{0}},\lambda)| \binom{\mathbf{u}}{\lambda}}{\mathbf{u}!} p_\lambda \tag{8}$$

So, as before:

$$
\begin{aligned}
m_\mu &= \sum_\lambda \left\langle \binom{\mathbf{u}}{\lambda} \hat{\rho}(p_\lambda),\ \hat{\rho}(m_\mu) \right\rangle |\mu(\hat{\mathbf{0}}, \lambda)| p_\lambda && \text{(by (8) and (5))} \\
&= \frac{1}{\binom{\mathbf{u}}{\mu}|\mu|} \sum_\lambda |\mu(\hat{\mathbf{0}}, \lambda)| \sum_{\substack{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\lambda \\ \pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\mu}} \langle p_\sigma,\ m_\pi \rangle\, p_\lambda && \text{(by (4d) and (4a) and bilinearity)} \\
&= \frac{1}{\binom{\mathbf{u}}{\mu}|\mu|} \sum_\lambda \left[|\mu(\hat{\mathbf{0}}, \lambda)| \sum_{\substack{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\lambda \\ \pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\mu \\ \pi \le \sigma}} \frac{\mu(\pi, \sigma)}{|\mu(\hat{\mathbf{0}}, \sigma)|} \right] p_\lambda && \text{(by Doubilet formula \#10)}
\end{aligned}
$$

$$
\begin{aligned}
h_\mu &= \sum_\lambda \langle \hat{\rho}(p_\lambda),\ \hat{\rho}(h_\mu) \rangle |\mu(\hat{\mathbf{0}}, \lambda)| \binom{\mathbf{u}}{\lambda} p_\lambda && \text{(by (8) and (5))} \\
&= \frac{1}{\binom{\mathbf{u}}{\mu}\mu!} \sum_\lambda |\mu(\hat{\mathbf{0}}, \lambda)| \sum_{\substack{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\lambda \\ \pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\mu}} \langle p_\sigma,\ h_\pi \rangle\, p_\lambda && \text{(by (4d) and (4b) and bilinearity)} \\
&= \frac{1}{\binom{\mathbf{u}}{\mu}\mu!} \sum_\lambda \left[|\mu(\hat{\mathbf{0}}, \lambda)| \sum_{\substack{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\lambda \\ \pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\mu \\ \sigma \le \pi}} \right] p_\lambda && \text{(by Doubilet formula \#3)}
\end{aligned}
$$

$$
\begin{aligned}
e_\mu &= \sum_\lambda \langle \hat{\rho}(p_\lambda),\ \hat{\rho}(e_\mu) \rangle |\mu(\hat{\mathbf{0}}, \lambda)| \binom{\mathbf{u}}{\lambda} p_\lambda && \text{(by (8) and (5))} \\
&= \frac{1}{\binom{\mathbf{u}}{\mu}\mu!} \sum_\lambda |\mu(\hat{\mathbf{0}}, \lambda)| \sum_{\substack{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\lambda \\ \pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\mu}} \langle p_\sigma,\ e_\pi \rangle\, p_\lambda && \text{(by (4d) and (4c) and bilinearity)} \\
&= \frac{1}{\binom{\mathbf{u}}{\mu}\mu!} \sum_\lambda \left[|\mu(\hat{\mathbf{0}}, \lambda)| \sum_{\substack{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\lambda \\ \pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\mu \\ \sigma \le \pi}} \mathrm{sign}(\sigma) \right] p_\lambda && \text{(by Doubilet formula \#4)}
\end{aligned}
$$

$$
\begin{aligned}
f_\mu &= \sum_\lambda \langle \hat\rho(p_\lambda),\ \hat\rho(f_\mu)\rangle |\mu(\hat{\mathbf{0}},\lambda)| \binom{\mathbf{u}}{\lambda} p_\lambda && \text{(by (8) and (5))}\\
&= \frac{1}{\binom{\mathbf{u}}{\mu}|\mu|} \sum_\lambda |\mu(\hat{\mathbf{0}},\lambda)| \sum_{\substack{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\lambda\\ \pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\mu}} \langle p_\sigma,\ f_\pi\rangle\, p_\lambda && \text{(by (4d) and (4e) and bilinearity)}\\
&= \frac{1}{\binom{\mathbf{u}}{\mu}|\mu|} \sum_\lambda \left[|\mu(\hat{\mathbf{0}},\lambda)| \sum_{\substack{\sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\lambda\\ \pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\mu\\ \pi\le\sigma}} \mathrm{sign}(\pi)\,\mathrm{sign}(\sigma) \frac{\mu(\pi,\sigma)}{|\mu(\hat{\mathbf{0}},\sigma)|} \right] p_\lambda && \text{(by Doubilet formula \#14)}
\end{aligned}
$$

5 The omega involution, and expansions in the e- and f-bases

There is an automorphism $\omega\colon \mathfrak{M}\to\mathfrak{M}$ (called θ in [Dou72]) defined by

$$\omega(e_\lambda) = h_\lambda$$

This is defined in [Ros01, p.328] (and used to define the forgotten basis $f_\lambda = \mathrm{sign}(\lambda)\omega(m_\lambda)$). I would like to know what ω does to p_λ. For symmetric functions, we have $\omega(p_\lambda) = \mathrm{sign}(\lambda)p_\lambda$ (where λ is a partition), and for unitary MacMahon functions, we have $\omega(p_\pi) = \mathrm{sign}(\pi)p_\pi$ (where π is a set partition). It would be natural to hope that $\omega(p_\lambda) = \mathrm{sign}(\lambda)p_\lambda$ in general, though Rosas does not state that. Can we prove it? And can we prove that ω is an isometry?

Or should we just assume all of these things? If so, then applying ω to the conversions to the h-basis gives

$$f_\mu = \mathrm{sign}(\mu)\frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}|\mu|} \sum_{\lambda\vdash\mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda\\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}} \sum_{\tau\ge\pi\vee\sigma} \frac{\mu(\pi,\tau)\mu(\sigma,\tau)}{\mu(\hat{\mathbf{0}},\tau)}\, e_\lambda$$

$$p_\mu = \mathrm{sign}(\mu)\frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}} \sum_{\lambda\vdash\mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda\\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu\\ \pi\le\sigma}} \frac{\mu(\pi,\sigma)}{|\mu(\hat{\mathbf{0}},\sigma)|}\, e_\lambda$$

$$h_\mu = \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}\mu!} \sum_{\lambda\vdash\mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda\\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu\\ \pi\le\sigma}} \mathrm{sign}(\pi)\lambda(\pi,\sigma)!\, e_\lambda$$

$$m_\mu = \mathrm{sign}(\mu)\frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}|\mu|} \sum_{\lambda\vdash\mathbf{u}} \frac{1}{\binom{\mathbf{u}}{\lambda}|\lambda|} \sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda\\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}} \sum_{\tau\ge\sigma\vee\pi} \frac{|\mu(\sigma,\tau)|\cdot\mu(\pi,\tau)}{\mu(\hat{\mathbf{0}},\tau)}\, e_\lambda$$

and likewise applying ω to the conversions to the m-basis gives

$$p_\mu = \mathrm{sign}(\mu)\frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}}\sum_{\lambda\vdash\mathbf{u}}\mathrm{sign}(\lambda)\frac{\#\{(\pi,\sigma)\mid \mathrm{type}_{\mathbf{u}}(\pi)=\lambda,\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu,\ \sigma\le\pi\}}{\binom{\mathbf{u}}{\lambda}\lambda!}f_\lambda$$

$$e_\mu = \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}\mu!}\sum_{\lambda\vdash\mathbf{u}}\mathrm{sign}(\lambda)\frac{1}{\binom{\mathbf{u}}{\lambda}\lambda!}\sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda\\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu}}\lambda(\sigma\wedge\pi)!\,f_\lambda$$

$$h_\mu = \frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}\mu!}\sum_{\lambda\vdash\mathbf{u}}\mathrm{sign}(\lambda)\frac{\#\{(\pi,\sigma)\mid \mathrm{type}_{\mathbf{u}}(\pi)=\lambda,\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu,\ \pi\wedge\sigma=\hat{\mathbf{0}}\}}{\binom{\mathbf{u}}{\lambda}\lambda!}f_\lambda$$

$$m_\mu = \mathrm{sign}(\mu)\frac{\mathbf{u}!}{\binom{\mathbf{u}}{\mu}|\mu|}\sum_{\lambda\vdash\mathbf{u}}\mathrm{sign}(\lambda)\frac{1}{\binom{\mathbf{u}}{\lambda}\lambda!}\sum_{\substack{\pi:\ \mathrm{type}_{\mathbf{u}}(\pi)=\lambda\\ \sigma:\ \mathrm{type}_{\mathbf{u}}(\sigma)=\mu\\ \sigma\le\pi}}\lambda(\sigma,\pi)!f_\lambda$$

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