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A COMBINATORIAL OVERVIEW OF THE THEORY OF MACMAHON SYMMETRIC
FUNCTIONS AND A STUDY OF THE KRONECKER PRODUCT OF SCHUR
FUNCTIONS.

A Dissertation


Presented to

The Faculty of the Graduate School of Arts ans Sciences

Brandeis University

Department of Mathematics

Professor Ira Gessel, Advisor

In partial Fulfillment
of the requirements for the Degree
Doctor of Philosophy

by

Mercedes H. Rosas

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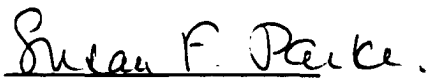
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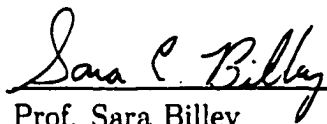
Dissertation Committe

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Prof. Sara Billey

*A mi Mamabiba, quien no pudo ver la culminacion de este projecto*
*y a mi Mamamia.*

*Este trabajo esta dedicado a mi Papa, por su optimismo y sus ganas de vivir,*

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ABSTRACT. The aim of this thesis is to study two problems closely related to the theory of symmetric functions.

First, we study the theory of MacMahon symmetric functions. A MacMahon symmetric function is a formal power series in a finite number of alphabets that is invariant under the diagonal action of the symmetric group. We show that the MacMahon symmetric functions are the generating functions for orbits of sets of functions (indexed by partitions) under the action of a Young subgroup of a symmetric group. This interpretation allows us to compute the transition matrices as well as the scalar product and the Kronecker product of the different bases of the ring of MacMahon symmetric functions. Moreover, we give some applications of the combinatorial theory of MacMahon symmetric functions developed in the first part of the thesis to various problems of combinatorics.

The second problem studied in this thesis is a classic problem of the theory of symmetric functions. How can we find explicit formulas for the Kronecker coefficients? We derive an explicit formula for the Kronecker coefficients corresponding to partitions of two two-row shapes, two hook shapes, and a hook shape and a two-row shape. We found a way to express the Kronecker coefficients in terms of regions and paths in $\mathbf{N}^2$.

# Contents

# CHAPTER 1

# Introduction

The aim of this thesis is to study two problems closely related to the theory of symmetric functions.

First, we look at the theory of MacMahon symmetric functions from a combinatorial viewpoint. A MacMahon symmetric function is a formal power series in a finite number of alphabets that is invariant under the diagonal action of the symmetric group.

In Chapter 1, we show that the MacMahon symmetric functions are the generating function for orbits of sets of functions (indexed by partitions) under the action of a Young subgroup of a symmetric group. This interpretation allows us to compute the transition matrices as well as the scalar product and the Kronecker product of the different bases of the ring of MacMahon symmetric functions.

In Chapter 2, we give some applications of the combinatorial theory of the MacMahon symmetric functions developed in Chapter 1 to various problems of combinatorics. More specifically, we reinterpret the chromatic symmetric function of Stanley [35], and show how to generalize it to include more information about the graph. We compute some specializations of the MacMahon symmetric functions and show how they are related to the Stirling numbers of

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both kinds. We prove a special case of a conjecture of Gessel [10] concerning

the signs in the matrix of change of basis from the one-dimensional MacMahon

symmetric functions to the homogeneous MacMahon symmetric functions.

The second problem studied in this thesis is how to derive an explicit

formula for the Kronecker coefficients corresponding to partitions of certain

shapes. The Kronecker coefficients, $\gamma^\lambda_{\mu\nu}$, arise when expressing a Kronecker

product (also called inner or internal product), $s_\mu * s_\nu$, of Schur functions in

the Schur basis,

$$s_\mu * s_\nu = \sum_{\mu,\nu} \gamma^\lambda_{\mu\nu} s_\lambda.$$

Remmel [23], [24] and Remmel and Whitehead [25] have studied the Kro-

necker product of Schur functions corresponding to two two-row shapes, two

hook shapes, and a hook shape and a two-row shape. We use the comultipli-

cation expansion for the Kronecker coefficients, and a formula for expanding

a Schur function of a difference of two alphabets due to Sergeev [3] to obtain

similar results in a simpler way. We believe that the formulas obtained using

this approach are elegant and reflect the symmetry of the Kronecker product.

In the three cases we found a way to express the Kronecker coefficients in terms

of regions and paths in $\mathbf{N}^2$.

# CHAPTER 2

# MacMahon symmetric functions, the partition lattice,

# and Young subgroups

## 1. Introduction

A MacMahon symmetric function is a formal power series in a finite number of alphabets that is invariant under the diagonal action of the symmetric group.

We study the relationship between the ring of MacMahon symmetric functions, the partition lattice, and Young subgroups of the symmetric group. We provide a combinatorial interpretation for MacMahon symmetric functions in terms of orbits of sets of functions (indexed by partitions) under the action of a Young subgroup of a symmetric group. We show how this interpretation allows us to compute the transition matrices as well as the scalar product and the Kronecker product of the different bases of the ring of MacMahon symmetric functions.

MacMahon showed the connection between MacMahon symmetric functions and combinatorics in [18], Vol. II, section XI, pp. 281-332. There, MacMahon applied the MacMahon symmetric functions to the problem of placing balls into boxes and to the theory of Latin squares.

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In [9] Ira Gessel used MacMahon symmetric functions to derive explicit

formulas for the number of $2 \times n$ Latin squares, of 0-1 matrices with trace 0,

and of words in a partially commuting monoid. Moreover, Gessel extended the

concept of $P$-recursiveness to MacMahon symmetric functions and showed that

for fixed $k$, the number of $k \times n$ latin squares is $P$-recursive as a function of

$n$. Similarly, Gessel showed that for fixed $k$, the number of $k \times n$ 0-1 matrices

with zeros on the diagonal and every row and column sum $k$ is $P$-recursive as

a function of $n$.

MacMahon symmetric functions appear naturally in the context of the the-

ory of resultants. Suppose that $F$ is a Cauchy form of degree $n$, that is, $F$

factors as a product of $n$ linear forms:

$$F(x, y, \cdots, z) = \prod_{i=1}^{n} (\alpha_i x + \beta_i y + \cdots + \gamma_j z).$$

Then the coefficient of $x^a y^b \cdots z^c$ in $F(x, y, \cdots, z)$ can be expressed in terms of

the coefficients of the linear factors using the elementary MacMahon symmetric

functions. This approach have been developed by Gian-Carlo Rota and Joel

Stein in connection with the theory of resultants [33].

MacMahon symmetric functions appear in other areas of mathematics.

They were introduced by people working in invariant theory. More recently,

Gelfand and Dikki [7], and Olver and Shakiban [21] used them in connections

with the theory of partial differential equations. Adem, Maginis and Milgram

[1] used them in the study of the cohomology of the symmetric group.

MacMahon symmetric functions have been called symmetric functions in several systems of parameters by MacMahon [18], multisymmetric functions by Mattuck [19], and vector symmetric functions and Gessel functions by Rota and Stein [32]. The name of MacMahon symmetric functions was introduced by Gessel in [9].

This work is inspired by Doubilet's study of the symmetric functions and their relation with the partition lattice [5]. There, Doubilet defined sets $\mathcal{M}_\pi$, $\mathcal{E}_\pi$, $\mathcal{P}_\pi$, $\mathcal{H}_\pi$, and $\mathcal{F}_\pi$ as done in Definition 9. He showed that their generating functions, $m_\pi$, $e_\pi$, $p_\pi$, $h_\pi$, and $f_\pi$, are scalar multiples of $m_\lambda$, $e_\lambda$, $p_\lambda$, $h_\lambda$, and $f_\lambda$, respectively, where $\lambda$ is a partition of $n$ determined by $\pi$. See Corollary 11 for a detailed statement.

Moreover, Doubilet realized that for most of his results he did not use the fact that $m_\pi$, $e_\pi$, $p_\pi$, $h_\pi$, and $f_\pi$ are symmetric functions. This motivated him to define a vector space freely generated by the symbols $\{\bar{m}_\pi\}_{\pi\in\Pi_n}$, called the space of Doubilet symmetric functions. He introduced other basis in this vector space by the equations $\bar{e}_\pi = \sum_{\sigma:\sigma\wedge\pi=\hat{0}} \bar{m}_\sigma$, $\bar{p}_\pi = \sum_{\sigma\geq\pi} \bar{m}_\sigma$, and so on. (See Appendix 6.2 for the other relations.) Afterwards, he computed the matrices of change of basis as well as the scalar and Kronecker product of the Doubilet symmetric functions using Möbius inversion.

The aim of this chapter is to reinterpret Doubilet's work by identifying Doubilet symmetric functions with the unitary symmetric functions, that is, with those MacMahon symmetric functions that are indexed by a partition

$(1, 1, \cdots, 1)$. This make sense for two different reasons. On the one hand, the unitary MacMahon symmetric functions and the Doubilet symmetric functions have the same matrices of change of basis, as well as the same scalar and Kronecker products. On the other hand, we define a projection map $\rho_n$ in a way that the image of a Doubilet symmetric function is a scalar factor times a symmetric function, as described in Corollary 11.

## 2. Basic definitions.

Let $u$ be a vector in $\mathbf{N}^k$. A vector partition of $u$ is an unordered sequence of vectors $\lambda = (b_1, r_1, \ldots, w_1)(b_2, r_2, \ldots, w_2) \ldots$ summing to $u$. We choose the letters $b$, $r$, and $w$ because sometimes it is useful to think of these letters as refering to colors. Then, $b$ is for blue, $r$ is for red, and $w$ is for white.

We consider two such sequences equal if they differ by a string of zero vectors. We write $\lambda \vdash u$ to indicate that $\lambda$ is a vector partition of $u$. The nonzero vectors $(b_i, r_i, \ldots, w_i)$ are called the parts of $\lambda$. The number of parts of $\lambda$ is defined to be the length of $\lambda$ and denoted by $l(\lambda)$.

Sometimes, we write $\lambda$ using block notation. That is, if $\lambda$ has $m(b_i, r_i, \cdots, w_i)$ copies of part $(b_i, r_i, \cdots, w_i)$ then we write $\lambda = \cdots (b_i, r_i, \ldots, w_i)^{m(b_i, r_i, \ldots, w_i)} \cdots$. To any vector partition $\lambda$ we associate two integer numbers:

$$\lambda! = b_1! \, r_1! \cdots w_1! \, b_2! \, r_2! \cdots w_2! \cdots$$

$$|\lambda| = \prod_i m(b_i, r_i, \ldots, w_i)!.$$

The weight of a vector $u$, written as weight($u$), is the sum of the coordinates of $u$. Let $X = \{x_1, x_2, \cdots\}$, $Y = \{y_1, y_2, \cdots\}$, $\cdots$, and $Z = \{z_1, z_2, \cdots\}$ be $k$ infinite alphabets. Given any formal power series in $\mathbf{Q}[[X, Y, \cdots, Z]]$, there is a natural action of $\pi \in S_\infty$ (where $S_\infty = \cup_{i \geq 1} S_i$ and $S_i$ is the symmetric group on $i$ letters) on $f$ called the diagonal action and defined by

$$\pi f(x_1, y_1, \cdots, z_1, x_2, y_2, \cdots, z_2, \cdots) =$$

$$f(x_{\pi 1}, y_{\pi 1}, \cdots, z_{\pi 1}, x_{\pi 2}, y_{\pi 2}, \cdots, z_{\pi 2}, \cdots).$$

A formal power series $f$ in $\mathbf{Q}[[X, Y, \cdots, Z]]$ is called a MacMahon symmetric functions in $k$ systems of indeterminates if it is invariant under the diagonal action of each $\pi \in S_\infty$ and if the multidegree of $f$ is bounded. We denote by $\mathfrak{M}^{(k)}$ the set of MacMahon symmetric functions on $k$ systems of indeterminates. In particular, $\mathfrak{M}^{(1)}$ is the ring of symmetric functions.

Let $f$ and $g$ be MacMahon symmetric functions. Since $f + g$ and $fg$ are in $\mathfrak{M}^{(k)}$ when both $f$ and $g$ are, it follows that $\mathfrak{M}^{(k)}$ has a ring structure. Moreover, it has a graded ring structure. We write

$$\mathfrak{M}^{(i)} = \bigcup_{\substack{u \in \mathbf{N}^k \\ \text{weight}(u)=i}} \mathfrak{M}_u.$$

where $\mathfrak{M}_u$ is the vector space of MacMahon symmetric functions of multihomogeneous degree $u$. Akin to the homogeneous pieces of the ring of symmetric functions, the vector spaces $\mathfrak{M}_u$ have bases indexed by vector partitions of

$u$, called the monomial, elementary, power sum, complete homogeneous, or forgotten MacMahon symmetric functions.

The first four bases are defined following MacMahon [18], the forgotten MacMahon symmetric functions are defined following Doubilet [5]. Finally, the one-dimensional MacMahon symmetric functions were introduced by Gessel [10]. We follow the notation of Macdonald [9, 17].

*The Monomial MacMahon Symmetric Functions.* Any vector partition

$$\lambda = (b_1, r_1, \ldots, w_1)(b_2, r_2, \ldots, w_2) \ldots$$

determines a monomial

$$\mathbf{x}^\lambda = x_1^{b_1} y_1^{r_1} \cdots z_1^{w_1} x_2^{b_2} y_2^{r_2} \cdots z_2^{w_2} \cdots x_l^{b_l} y_l^{r_l} \cdots z_l^{w_l}.$$

The monomial MacMahon symmetric function indexed by $\lambda$ is the sum of all distinct monomials that can be obtained from $\mathbf{x}^\lambda$ by a permutation $\pi$ in $S_\infty$, where the action of $\pi$ in $\mathbf{x}^\lambda$ is the diagonal action. That is,

$$m_\lambda = \sum x_{i_1}^{b_1} y_{i_1}^{r_1} \cdots z_{i_1}^{w_1} x_{i_2}^{b_2} y_{i_2}^{r_2} \cdots z_{i_2}^{w_2} \cdots.$$

where the sum is taken over all different monomials having exponents $(b_1, r_1, \ldots, w_1)$ $(b_2, r_2, \ldots, w_2) \ldots$, and where $\pi = (i_1, i_2, \ldots)$.

*The Elementary MacMahon Symmetric Functions.* Let $(b, r, \cdots, w)$ be a vector. Then $e_{(b,r,\cdots,w)}$ is defined by the generating function:

$$\sum_{b,r,\cdots,w} e_{(b,r,\cdots,w)} s^b t^r \cdots u^w = \prod_i (1 + x_i s + y_i t + \cdots + z_i u).$$

Let $\lambda$ be a vector partition. We set $e_\lambda = e_{(b_1,r_1,\cdots,w_1)}e_{(b_2,r_2,\cdots,w_2)}\cdots$.

*The Complete Homogeneous MacMahon Symmetric Functions.* Let $(b, r, \cdots, w)$ be a vector. Then $h_{(b,r,\cdots,w)}$ is defined by the generating function:

$$\sum_{b,r,\cdots,w} h_{(b,r,\cdots,w)} s^b t^r \cdots u^w = \prod_i \frac{1}{1 - x_i s - y_i t - \cdots - z_i u}.$$

Let $\lambda$ be a vector partition. We set $h_\lambda = h_{(b_1,r_1,\cdots,w_1)}h_{(b_2,r_2,\cdots,w_2)}\cdots$.

REMARK. MacMahon introduced two versions of the complete homogeneous MacMahon symmetric functions. The first version was defined using the generating function

$$\sum_{a,b} \hat{h}_{(a,b)} s^a t^b = \prod_i \frac{1}{(1 - x_i s)(1 - y_i t)}.$$

Note that $\hat{h}_{(a,b)} = \hat{h}_{(a,0)}\hat{h}_{(0,b)}$. We are following MacMahon's second definition. See [18], pp. 283–284.

*The Power Sum MacMahon Symmetric Functions.* Let $(b, r, \cdots, w)$ be a vector. Then $p_{(b,r,\cdots,w)}$ is defined by

$$p_{(b,r,\cdots,w)} = \sum_i x_i^b y_i^r \cdots z_i^w = m_{(b,r,\cdots,w)}.$$

Let $\lambda$ be a vector partition. We set $p_\lambda = p_{(b_1,r_1,\cdots,w_1)}p_{(b_2,r_2,\cdots,w_2)}\cdots$.

*The Forgotten MacMahon Symmetric Functions.* In the ring of MacMahon Symmetric Functions there is an involution defined by $\omega(e_\lambda) = h_\lambda$. (See [8] for more information on involution $\omega$.) We define the forgotten MacMahon

symmetric functions as the signed image of the monomial MacMahon symmetric functions functions under the involution $\omega$. More precisely, the forgotten MacMahon symmetric functions are defined by

$$\omega(m_\lambda) = (\text{sign } \lambda) \, f_\lambda$$

where the sign of $\lambda$ is defined as follows. To the vector partition $\lambda$ we associate the partition of $u$ defined by $(b_1 + \cdots + w_1, b_2 + \cdots + w_2, \cdots) = (1^{n_1} 2^{n_2} \cdots)$. Then. we define the sign of $\lambda$ as $(-1)^{n_2 + 2n_3 + 3n_4 + \cdots}$.

REMARK. We are following Doubilet's definition of the forgotten MacMahon symmetric functions [5] instead of the one in Macdonald [17].

DEFINITION 1. A vector partition is unitary if it is a partition of $(1)^n = (1, 1, \cdots, 1)$ in $\mathbf{N}^k$. Similarly, a MacMahon symmetric function is unitary if it is indexed by a unitary vector partition.

We assume the reader to be familiar with the notions of posets. lattices, and Möbius inversion. (See [29, 35] for more information.)

A partition $\pi$ of $[n]$ is a family of subsets of $E$, called the blocks of the partition $\pi$. with the following properties:

1. Every block of $\pi$ is a nonempty subset of $E$.

2. Two distinct blocks of $\pi$ are disjoint.

3. Every element of $E$ belongs to a block of $\pi$.

We denote by $\Pi_n$ the lattice of all partitions of $[n]$ ordered by the refinement relation. That is, for $\pi$ and $\sigma$ partitions in $\Pi_n$ we say that $\pi$ is a refinement of $\sigma$, written as $\pi \leq \sigma$, if every block of $\pi$ is a subset of some block of $\sigma$.

It is well-known, see for instance [29, 35], that the Möbius function of the partition lattice $\Pi_n$ is given by $\mu_n = (-1)^{n-1}(n-1)!$.

To compute the Möbius function of an interval in the partition lattice, we observe that if $\pi = \{B_1, B_2, \cdots, B_l\}$ and $B_i$ is partitioned into $\lambda_i$ blocks in the partition $\sigma$, then

$$[\sigma, \pi] \simeq \Pi_{\lambda_1} \times \Pi_{\lambda_2} \times \cdots \times \Pi_{\lambda_l}.$$

where we are assuming that $\lambda_1 \leq \lambda_2 \cdots \leq \lambda_l$. We define the type of $[\sigma, \pi]$, denoted as $\text{type}(\sigma, \pi)$, by

$$\text{type}(\sigma, \pi) = (\lambda_1, \lambda_2, \cdots, \lambda_l) = (1^{r_1} 2^{r_2} \cdots).$$

Then, we have that $\mu(\sigma, \pi) = \mu_{\lambda_1} \mu_{\lambda_2} \cdots \mu_{\lambda_l}$. We define the sign of an interval $[\sigma, \pi]$ of type $\lambda$ by $\text{sign}(\sigma, \pi) = \text{sign}\,\lambda = (-1)^{r_2 + 2r_3 + \cdots}$. Similarly, we define the sign of a partition $\pi$ by $\text{sign}\,\pi = \text{sign}(\hat{0}, \pi)$. (This definition is consistent with the previous definition of sign $\lambda$.) We obtain that

$$\mu(\sigma, \pi) = \text{sign}(\sigma, \pi)|\mu(\sigma, \pi)|.$$

NOTATION. We denote partitions in $\Pi_n$ by Greek letters like $\pi$, $\sigma$, or $\tau$. We denote vector partitions by Greek letters like $\lambda$, $\mu$ or $\nu$.

To any partition $\pi = \{B_1, B_2, \cdots, B_l\}$ we associate the unitary vector partition $\lambda = (\lambda_1, \lambda_2, \cdots)$, where the part $B_j$ has $i$th coordinate 1 if $i$ is in $B_j$ and 0 otherwise.

For instance, to the partition 1236|457, we associate the unitary vector partition $(1, 1, 1, 0, 0, 1, 0)(0, 0, 0, 1, 1, 0, 1)$.

Note that we can recover $\pi$ from $\lambda$. Hence, sometimes we think of unitary vector partitions as partitions and we denote them by Greek letters like $\pi$, $\sigma$, or $\tau$.

Let $A$ be a noncommuting alphabet. A word over the alphabet $A$ is a finite sequence of elements of $A$. We include the empty sequence and call it the empty word. Let $\omega = a_1 a_2 \cdots a_n$ be a word with letters in $A$. The parameter $n$ is called the length of $\omega$ and is denoted by $|\omega|$. For each letter $a$ in $A$ we denote by $|\omega|_a$ the number of ocurrences of the letter $a$ in the word $\omega$.

The set of all words on the alphabet $A$ forms a monoid, denoted by $A^*$. The set of all nonempty words is denoted by $A^+$. We extend the total order of $A$ to a total order of $A^*$ using the lexicographic order.

The evaluation of a word $\omega$ in $A^*$, denoted by $\mathrm{ev}(\omega)$, is the monomial $x^b y^r \cdots z^w$ in $\mathbf{Q}[x, y, \cdots, z]$ where $b = |\omega|_X$, $r = |\omega|_Y$, and $w = |\omega|_Z$.

A sequence of words in $A^*$ is a totally ordered multiset of words. We denote the set of all sequences of words in $A^*$ by $A^S$.

The evaluation of a sequence of words $S = \omega_1 \omega_2 \cdots \omega_l$, denoted by $\mathrm{ev}(S)$, is the monomial $\mathrm{ev}(\omega_1)\mathrm{ev}(\omega_2)\cdots\mathrm{ev}(\omega_l)$ where $\mathrm{ev}(\omega_i) = x_i^{b_i} y_i^{r_i} \cdots z_i^{w_i}$, and $b_i =$

$|w_i|_X$, $r_i = |w_i|_Y$, and $w_i = |w_i|_Z$. Note that each word $w_i$ is evaluated in a different alphabet.

If $L$ is a set of words or of sequence of words, then the generating function of $L$ is the sum of the evaluations of the elements of $L$.

PROPOSITION 2. *The complete homogeneous MacMahon symmetric function $h_{(b,r,\cdots,w)}$ is the generating function for sequences of words $S$ in $A^S$ such that $|S|_X = b$, $|S|_Y = r, \cdots$, and $|S|_Z = w$.*

PROOF. The generating function for words in $A^*$ with evaluation $x_m^i y_m^j \cdots w_m^l$ is

$$\binom{i + j + \cdots + l}{i, j, \cdots, l} x_m^i y_m^j \cdots w_m^l.$$

Therefore, the generating function for sequences of words is given by

$$\prod_m \left( \sum_{i,j,\cdots,l} \binom{i + j + \cdots + l}{i, j, \cdots, l} x_m^i y_m^j \cdots w_m^l s^i t^j \cdots u^l \right)$$

$$= \prod_m \left( \sum_n (x_m s + y_m t + \cdots + w_m u)^n \right) = \sum_{b,r,\cdots,w} h_{(b,r,\cdots,w)} s^b t^r \cdots u^w.$$

$\square$

## 3. Basic constructions.

We associate to any vector in $\mathbf{N}^k$ of weight $n$ a Young subgroup of $S_n$ as follows. Let $u = (b, r, \cdots, w)$ be a vector in $\mathbf{N}^k$ of weight $n$. Let $S_u$ be the Young subgroup of $S_n$ defined by $S_u = S_{\{1,2,\cdots,b\}} \times S_{\{b+1,b+2,\cdots,b+r\}} \times \cdots \times S_{\{n-w+1,n-w+2,\cdots,n\}}$.

We have that $S_u$ acts on $[n]$ by restricting the canonical action of $S_n$ on $[n]$ to permutations in $S_u$. Moreover, this action partitions $[n]$ into equivalence classes. Two elements $n_1$ and $n_2$ belong to the same equivalence class if and only if there is a permutation $\sigma$ in $S_u$ such that $\sigma n_1 = n_2$. We order the equivalence classes using the smallest element in each of them.

Let $[n]_u$ be the set of ordered pairs $(i,j)$ where the element $i$ of $[n]$ belongs to the $j$th equivalence class of $[n]$ under the action of $S_u$. For simplicity, sometimes we denote the pair $(i,j)$ by putting $j$ dots over $i$. For instance, the element $(3,2)$ is sometimes denoted by $\ddot{3}$.

We may think of $[n]_u$ as a set of colored balls. If we interpret $(i,k)$ as saying that ball $i$ has been colored $k$, then in $[n]_u$ there are $b$ balls colored $1$, $r$ balls colored $2$, and so on.

Another way of thinking of $[n]_u$ is as follows. Before the action of the Young subgroup $S_u$ on $[n]$ we had balls $1, 2, \cdots, n$ and we knew how to distinguish between them. Then, the action of $S_u$ made us forget some information. In particular, if $u = (1)^n$ then we know how to distinguish between all the ball, that is, we *remember everything*. On the other hand, if $u = (n)$ then we are not able to distinguish between any of the balls, that is, we *forget everything*.

Let $(\Pi_n)_u$ be the lattice of partitions of $[n]_u$. Given a partition $\pi$ in $\Pi_n$, we associate a partition $\pi_u$ in $(\Pi)_u$ by replacing element $i$ in $\pi$ by the ordered pair $(i,k)$. The elements of $(\Pi_n)_u$ are called colored partitions.

Let $\pi_u$ be a colored partition. The type of $\pi_u$, written type($\pi_u$), is the vector partition of $u$ defined by saying that type($\pi_u$) has part $(b_i, r_i, \cdots, w_i)$ with multiplicity $m_i$ if and only if there are exactly $m_i$ blocks of $\pi_u$ with $b_i$ elements in the first equivalence class, $r_i$ elements in the second equivalence class, and so on until the last equivalence class that has $w_i$ elements. The notation $\pi \in \lambda$ means that type($\pi$) $= \lambda$.

EXAMPLE 3 ($S_{(1,2)}$ acts on $\Pi_3$). We explore the action of $S_{(1,2)}$ on $\Pi_3$.

First, we have that $[3]$ is partitioned into two equivalence classes: $\{1\}$ and $\{2,3\}$ by the action of $S_{(1,2)}$.

1. The set partition $\pi = 1|2|3$ has type $(1,0,0)(0,1,0)(0,0,1)$.

   Under the action of $S_{(1,2)}$, $\pi$ is colored as $\pi_{(1,2)} = \dot{1}|\ddot{2}|\dddot{3}$.

   Moreover, type($\pi_{(1,2)}$) $= (1,0)(0,1)(0,1) \vdash (1,2)$.

2. The set partition $\pi = 12|3$ has type $(1,1,0)(0,0,1)$.

   Under the action of $S_{(1,2)}$, $\pi$ is colored as $\pi_{(1,2)} = \dot{1}\ddot{2}|\dddot{3}$.

   Moreover, type($\pi_{(1,2)}$) $= (1,1)(0,1) \vdash (1,2)$.

DEFINITION 4. Let $g$ be a function from $[n]_u$ to $\mathbf{P}$. We weight $g$ by

$$\gamma(g) = \prod_{d \in [n]} c(d)_{g(d)}$$

where $c(d)$ denotes the color of ball $d$ and we use variables $x, y, \cdots, z$ to denote the colors of the balls. To any set of functions $T$ we associate the generating

function:

$$\gamma(T) = \sum_{f \in T} \gamma(f).$$

EXAMPLE 5. Suppose that $g : [5]_{(2,2,1)} \to \mathbf{P}$ is defined by $\dot{1} \mapsto 3$, $\dot{2} \mapsto 5$, $\ddot{3} \mapsto 3$, $\ddot{4} \mapsto 3$, and $\ddot{5} \mapsto 5$. Then, $\gamma(g) = x_3 y_3^2 x_5 z_5$.

DEFINITION 6. Let $F_n$ be the set of all functions from $[n]$ to $\mathbf{P}$. That is,

$$F_n = \{f : [n] \to \mathbf{P}\}.$$

We say that $f$ places ball $i$ in box $j$ if $f(i) = j$. Given any $f \in F_n$, we define a partition of $[n]$, denoted ker $f$, by saying that $n_1$ and $n_2$ are in the same block of ker $f$ if and only if $f(n_1) = f(n_2)$.

DEFINITION 7. We define a disposition to be an arrangement of the balls (that is, the elements of $[n]$) into the boxes (that is, the positive numbers, $\mathbf{P}$), in which the balls in each box may be placed in some configuration. Note that any function is a disposition where the balls are in no special configuration. (Doubilet use the word placing for what we are calling disposition. As pointed out in [4], dispositions are closely related to reluctant functions [20] and are reminiscent of MacMahon's distribution into groups [18]. We are following G.-C. Rota in calling them dispositions.)

We define the underlying function of a disposition $p$ to be the function obtained from $p$ if we forget about the extra data given by the configuration of the balls. The weight of a disposition is defined as the weight of its underlying

function. Similarly, the kernel of a disposition $p$, written as $\ker p$, is defined to be the kernel of its underlying function.

DEFINITION 8 (The projection map). Let $S_u$ be a Young subgroup of $S_n$ acting on $\Pi_n$. Let $p : [n] \to \mathbf{P}$ be a disposition. We define $p_u : [n]_u \to \mathbf{P}$ by

$$p_u((i, k)) = p(i).$$

The map sending $p \mapsto p_u$ is called the projection map and denoted by $\rho_u$.

## 4. Combinatorial interpretation.

In this section we define three classes of sets of functions and two classes of sets of dispositions. Each of these classes is indexed by partitions in $\Pi_n$. We show how their corresponding generating functions are related to the homonymous basis of the space of unitary MacMahon symmetric functions. Moreover, we show that any monomial, elementary, power sum, complete homogeneous, or forgotten MacMahon symmetric functions can be obtained as the generating function of the image of one these sets under the action of a Young subgroup of the symmetric group.

DEFINITION 9 (Doubilet). Let $\pi$ be a set partition of $[n]$.

1. Let $\mathcal{M}_\pi$ be the subset of $F_n$ defined by $\mathcal{M}_\pi = \{f : f \in F_n, \ker f = \pi\}$, and let $m_\pi$ be its generating function, $m_\pi = \gamma(\mathcal{M}_\pi)$.

2. Let $\mathcal{P}_\pi$ be the subset of $F_n$ defined by $\mathcal{P}_\pi = \{f : f \in F_n, \ker f \geq \pi\}$, and let $p_\pi$ be its generating function, $p_\pi = \gamma(\mathcal{P}_\pi)$.

3. Let $\mathcal{E}_\pi$ be the subset of $F_n$ defined by $\mathcal{E}_\pi = \{f : f \in F_n, \ker f \wedge \pi = \hat{0}\}$, and let $e_\pi$ be its generating function $e_\pi = \gamma(\mathcal{E}_\pi)$.

4. Let $\mathcal{H}_\pi$ be the set of dispositions $p$ such that within each box the balls from the same block of $\pi$ are linearly ordered. Let $h_\pi = \gamma(\mathcal{H}_\pi)$ be its generating function.

5. Let $\mathcal{F}_\pi$ be the set of dispositions such that balls from the same block of $\pi$ go into the same box, and within each box the blocks appearing are linearly ordered. Let $f_\pi = \gamma(\mathcal{F}_\pi)$ be its generating function.

THEOREM 10. *Let $S_u$ be a Young subgroup of $S_n$. Let $\pi$ be a partition of $[n]$ and let $\lambda$ be the type of $\pi_u$. We have that $\rho_u : \mathfrak{M}_{(1)^n} \to \mathfrak{M}_u$. Moreover,*

$$m_\pi \mapsto |\lambda| \, m_\lambda$$

$$p_\pi \mapsto p_\lambda$$

$$e_\pi \mapsto \lambda! \, e_\lambda$$

$$h_\pi \mapsto \lambda! \, h_\lambda$$

$$f_\pi \mapsto |\lambda| \, f_\lambda.$$

*where $\lambda! = b_1! \, r_1! \cdots w_1! \, b_2! \, r_2! \cdots w_2! \cdots$ and $|\lambda| = \prod_i m(b_i, r_i, \ldots, w_i)!$.*

PROOF. We observe that $m_\pi, p_\pi, e_\pi, h_\pi$, and $f_\pi$ belong to $\mathfrak{M}_{(1)^n}$ and that their image under $\rho_u$ belong to $\mathfrak{M}_u$.

1. Let $f$ be a function in $\mathcal{M}_\pi$. Suppose that the type of the image of $\pi$ under $S_u$ is $\lambda = (b_1, r_1, \ldots, w_1)(b_2, r_2, \ldots, w_2) \ldots (b_l, r_l, \ldots, w_l)$. Then,

$$\gamma(f) = x_{i_1}^{b_1} y_{i_1}^{r_1} \cdots z_{i_1}^{w_1} \, x_{i_2}^{b_2} y_{i_2}^{r_2} \cdots z_{i_2}^{w_2} \cdots x_{i_l}^{b_l} y_{i_l}^{r_l} \cdots z_{i_l}^{w_l}.$$

Therefore,

$$\gamma(\mathcal{M}_\pi) = \sum x_1^{b_1} y_1^{r_1} \cdots z_1^{w_1} \cdot x_2^{b_2} y_2^{r_2} \cdots z_2^{w_2} \cdots x_l^{b_l} y_l^{r_l} \cdots z_l^{w_l}.$$

where the sum is taken over all distinct indexes $i_1, i_2, \cdots i_l$. Any such monomial arises in $m(b_{i_1}, r_{i_1}, \ldots)! \, m(b_{i_2}, r_{i_2}, \ldots)! \ldots$ different ways. We have obtained that $\gamma(\mathcal{M}_\pi) = |\lambda| m_\lambda$.

2. We can rewrite $\mathcal{P}_\pi$ as

$$\mathcal{P}_\pi = \{f : f \in F \text{ and } f \text{ is constant in blocks } B_1, B_2, \ldots \text{ of } \pi\},$$

Therefore,

$$\gamma(\mathcal{P}_\pi) = \sum_{f \in P_\pi} \gamma(f) = \sum_{f \in P_\pi} \gamma(f|B_1)\gamma(f|B_2) \ldots$$

$$= \sum_{\substack{f \in P_\pi \\ f|B_i \text{ constant}}} \gamma(f|B_1)\gamma(f|B_2) \ldots = \prod_i \sum_{\substack{f:B_i \to X \\ f \text{ constant}}} \gamma(f)$$

$$= \prod_i (x_1^{b_i} y_1^{r_i} \cdots z_1^{w_i} + x_2^{b_i} y_2^{r_i} \cdots z_2^{w_i} + \ldots)$$

$$= \sum_{i_1, i_2, \ldots} x_{i_1}^{b_{i_1}} y_{i_1}^{r_{i_1}} \cdots z_{i_1}^{w_{i_1}} x_{i_2}^{b_{i_2}} y_{i_2}^{r_{i_2}} \cdots z_{i_2}^{w_{i_2}} \ldots = p_\lambda.$$

3. By definition $\mathcal{E}_\pi$ is the set of functions from $[n]$ to $\mathbf{P}$ that are injective on the blocks of $\pi$. Hence,

$$\gamma(\mathcal{E}_\pi) = \sum_{f \in E_\pi} \gamma(f) = \sum_{\substack{f \in F \\ f|B_i \text{ injective}}} \gamma(f|B_1)\gamma(f|B_2)\cdots$$

$$= \prod_i \left( \sum_{\substack{f:B_i \to X \\ f \text{ injective}}} \gamma(f) \right).$$

Note that $\gamma(f)$ is a monomial without repeated factor. Moreover, each such monomial arises from $b_i!\, r_i! \cdots w_i!$ different functions $f : B_i \to X$. Therefore,

$$\sum_{\substack{f:B_i \to X \\ f \text{ injective}}} \gamma(f) = b_i!\, r_i! \cdots w_i!\, e_{\lambda_i}$$

and $\gamma(\mathcal{E}_\pi) = b_1!\, r_1! \cdots w_1!\, b_2!\, r_2! \cdots w_2! \cdots e_{\lambda_1 \lambda_2 \ldots} = \lambda!\, e_\lambda$.

4. To construct a disposition $p$ in $H_\pi$, take one of the blocks of $\pi$ and send each of its balls to one of the boxes. Linearly order the balls inside each box. Repeat this process for all blocks of $\pi$. It follows that

$$h_\pi = \prod_{\text{Blocks } B} \gamma(B).$$

Therefore, it is enough to study $H_{\{[n]\}}$.

Suppose that in $[n]$ there are $b$ blue balls, $r$ red balls, and so on. In each box we have a word made out of these letters. Suppose we have defined the underlying function $f$ sending balls to boxes. The boxes are linearly ordered, Therefore, there are $n!$ different ways of linearly

ordering the balls of $[n]$, and we can make $\binom{n}{b,r,\cdots,w}$ sequences of words (with the same underlying function) out of these letters. Therefore there are $b!r!\cdots w!$ repetitions.

Proposition 2 says that $h_{(b,r,\cdots,w)}$ is the generating function for sequences of words $S$ in $A^S$ such that $|S|_X = b, |S|_Y = r, \cdots, |S|_Z = w$. Hence, $\gamma(H_{\{[n]\}}) = b!\,r!\cdots w!\,h_n$, so

$$\gamma(\mathcal{M}_\pi) = b_1!\,r_1!\cdots w_1!\,b_2!\,r_2!\cdots w_2!\cdots h_{\lambda_1\lambda_2\cdots} = \lambda!\,h_\lambda.$$

5. In [5] Doubilet showed that $\omega(m_\pi) = \text{sign}(\pi)f_\pi$ when $\pi$ is a unitary vector partition. Therefore, we have that

$$(\text{sign }\pi)f_\pi = \omega(m_\pi) \longmapsto \omega(|\lambda|m_\lambda)$$

$$= |\lambda|\omega(m_\lambda)$$

$$= |\lambda|(\text{sign }\lambda)f_\lambda.$$

Since $\text{sign }\lambda = \text{sign }\pi$ for any $\pi$ of type $\lambda$, we have that $f_\pi \longmapsto |\lambda|f_\lambda$.

$\square$

COROLLARY 11 (Doubilet). *Let $S_n$ be the symmetric group. Let $\pi$ be a partition in $\Pi_n$ and let $\lambda$ be the type of $\pi_n$. (Note that $\lambda$ is a partition of $n$). We have that*

$$\rho_n : \mathfrak{M}_{(1)^n} \to \mathfrak{M}_n$$

*where $\mathfrak{M}_n$ is the vector space of symmetric functions of degree $n$. Moreover, we obtain that $m_\lambda$, $p_\lambda$, $e_\lambda$, $h_\lambda$, and $f_\lambda$ are the usual symmetric functions.*

We interpret Corollary 11 as saying that the symmetric functions are the generating function for the set of functions given in Definition 9 if we forget how to distinguish between the elements of $[n]$. Similarly, the unitary MacMahon symmetric functions are the generating function for the set of functions given in Definition 9 if we know how to distinguish between all the elements of $[n]$. All other MacMahon symmetric functions fall between them.

The content of Theorem 10 is better described through an example.

EXAMPLE 12 ($S_{(4,2)}$ acts on $\Pi_6$). Let $\pi = 12|35|46$ be a partition in $\Pi_6$. The type of $\pi_{(4,2)}$ is given by $\lambda = (2,0)(1,1)^2$.

1. If $f \in \mathcal{M}_\pi$, then $\ker f = \pi$. It follows that $\dot{1}, \dot{2} \mapsto i$, $\dot{3}, \ddot{5} \mapsto j$, and $\dot{4}, \ddot{6} \mapsto k$, and $i, j$, and $k$ are different indices.

Therefore $\gamma(\rho_{(4,2)}(f)) = x_i^2 x_j y_j x_k y_k$. Moreover,

$$\gamma(\rho_{(4,2)}(\mathcal{M}_\pi)) = \sum_{\substack{i,j,k \\ \text{diff.}}} x_i^2 x_j y_j x_k y_k = 2 \sum_{\substack{i,j<k \\ \text{diff.}}} x_i^2 x_j y_j x_k y_k = 2m_{(2,0)(1,1)^2}.$$

2. If $f \in \mathcal{P}_\pi$, then $\ker f \geq \pi$. It follows that $\dot{1}, \dot{2} \mapsto i$, $\dot{3}, \ddot{5} \mapsto j$, and $\dot{4}, \ddot{6} \mapsto k$, where $i, j, k$ are not necessarily different. Therefore, $\gamma(\rho_{(4,2)}(f)) = x_i^2 x_j y_j x_k y_k$. Moreover,

$$\gamma(\rho_{(4,2)}(\mathcal{P}_\pi)) = \sum_i x_i^2 \sum_j x_i y_j \sum_k x_k y_k = p_{(2,0)(1,1)^2}.$$

3. If $f \in \mathcal{E}_\pi$ then $\ker f \wedge \pi = \hat{0}$. It follows that $\dot{1}$ and $\dot{2}$ should be in different blocks of $\ker f$, and that $\dot{3}$ and $\ddot{5}$ (and $\dot{4}$ and $\ddot{6}$) also should be in different blocks of $\ker f$. We have that,

$$\gamma(\rho_{(4,2)}(\mathcal{E}_\pi)) = \sum_{i \neq j} x_i x_j \sum_{i \neq j} x_i y_j \sum_{i \neq j} x_i y_j$$

$$= 2 \sum_{i < j} x_i x_j \sum_{i \neq j} x_i y_j \sum_{i \neq j} x_i y_j = 2e_{(2,0)(1,1)^2}.$$

4. If $f \in \mathcal{H}_\pi$, then the balls coming from the same block of $\pi$ are linearly ordered. We can analyze each block separately.

$$\gamma(\rho_{(2)}(\mathcal{H}_{\{1,2\}})) = \sum_i x_i^2 + \sum_{i<j} x_i x_j$$

$$\gamma(\rho_{(1,1)}(\mathcal{H}_{\{1,2\}})) = \sum_i x_i y_i + \sum_{i \neq j} x_i y_j$$

Therefore,

$$\gamma(\rho_{(4,2)}(\mathcal{H}_\pi)) = 2(\sum_i x_i^2 + \sum_{i<j} x_i x_j)(2 \sum_i x_i y_i + \sum_{i \neq j} x_i y_j)^2 = 2h_{(2,0)}h_{(1,1)^2}.$$

5. Let $\pi$ be $12|34|56$. To compute $\gamma(\mathcal{F}_\pi)$ we proceed as follows:

The generating function for the set of $f \in \Pi_6$ that send all blocks of $\pi$ to the same block and linearly order them is $3! \, m_{123456}$.

The generating function for the set of $f \in \Pi_6$ that send two blocks to one box and the other one to another box, and then linearly order the two blocks is $2! \, m_{12|3456} + 2! \, m_{35|1256} + 2! \, m_{56|1234}$.

The generating function for the set of $f \in \Pi_6$ that send all blocks to different boxes is $m_{12|34|56}$. Therefore,

$$f_{12|34|56} = 3!\, m_{123456} + 2!\, m_{12|3456} + 2!\, m_{35||1256} + 2!\, m_{56||1234} + m_{12|34|56}.$$

Then, we apply the projection map to each of the unitary monomial MacMahon symmetric functions in the previous sum, as previously described.

## 5. The scalar and the Kronecker product. The transition matrix.

Following Gessel [9], we define a scalar product on $\mathfrak{M}^k$ by $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu}$. The relevance of the scalar product in enumerative combinatorics comes from the fact that for any MacMahon symmetric functions $f$ and for any vector partition $\lambda$, the scalar product $\langle h_\lambda, f \rangle$ gives the coefficient of $x_1^{b_1} \cdots z_1^{w_1} x_2^{b_2} \cdots z_2^{w_2} \cdots$ in $f$.

We define the Kronecker product in the ring of MacMahon symmetric functions by $p_\lambda * p_\mu = \langle p_\lambda, p_\mu \rangle p_\lambda$ and extend by linearity. Note that $\langle p_\lambda, p_\mu \rangle$ is the scalar product of MacMahon symmetric functions just defined.

The relevance of the Kronecker product of symmetric functions comes from the fact that it allows us to obtain the multiplicity of the irreducible representations of the symmetric group (or the general linear group) in the tensor product of their irreducible representations. It is not know whether there is an analogous interpretation for the Kronecker product for the MacMahon symmetric functions.

In this section we show how to compute the scalar and Kronecker products of MacMahon symmetric functions as well as the transition matrices. We need the following proposition to define a lifting map $\hat{\rho}_u : \mathfrak{M}_u \to \mathfrak{M}_{(1)^n}$ that will allow us to use Doubilet's calculations.

PROPOSITION 13. *Let $u$ be a vector in $\mathbf{N}^k$. The number of partitions $\pi \in \Pi_n$ such that $\pi_u$ has type $\lambda$ is given by*

$$\binom{u}{\lambda} = \frac{u!}{\lambda! |\lambda|}.$$

PROOF. Let $u = (b, r, \cdots, w)$. Order the blocks of $\lambda$ using the reverse lexicographic order. There are $u!$ permutations of $[n]_u$ that produce a partition $\pi_u$ satisfying the following two conditions.

1. If we place a dividing line at positions $b_i + c_i + \cdots + w_i$, for $1 \leq i \leq l(\lambda)$, then we obtain a colored partition of type $\lambda$.

2. The order of the boxes of $\pi_u$ extends the order of the parts of $\lambda$.

A partition $\pi_u$ may appear more than once for two different reasons. On the one hand, if we permute the balls of the same color within one box then $\pi_u$ appears $\lambda! = b_1! r_1! \cdots w_1! b_2! r_2! \cdots w_2! \cdots$ times. On the other hand, if we permute the boxes of the same size and kind then $\pi_u$ appears $|\lambda| = \prod_i m(b_i, r_i, \cdots, w_i)!$ times.

$\square$

DEFINITION 14 (The lifting map). Let $\lambda$ be a vector partition of $u$. Let

$M_\lambda = \binom{u}{\lambda}|\lambda|m_\lambda$. Then define the lifting map $\hat{\rho}_u : \mathfrak{M}_u \to \mathfrak{M}_{(1)^n}$ by

$$\hat{\rho}_u(M_\lambda) = \sum_{\pi \in \lambda} m_\pi.$$

where $\pi \in \lambda$ means that $\lambda$ is the type of $\pi_u$.

Similarly, we define $E_\lambda = \binom{u}{\lambda}\lambda! e_\lambda, P_\lambda = \binom{u}{\lambda}p_\lambda, H_\lambda = \binom{u}{\lambda}\lambda! h_\lambda$, and $F_\lambda = \binom{u}{\lambda}|\lambda| f_\lambda$.

It is easy to show that $E_\lambda \mapsto \sum_{\pi \in \lambda} e_\pi$, $P_\lambda \mapsto \sum_{\pi \in \lambda} p_\pi$, $H_\lambda \mapsto \sum_{\pi \in \lambda} h_\pi$, and $F_\lambda \mapsto \sum_{\pi \in \lambda} f_\pi$.

PROPOSITION 15. *The lifting map $\hat{\rho}_u$ has the property that $\rho_u \hat{\rho}_u = 1$. Moreover, for all $f, g \in \mathfrak{M}_u$,*

$$\langle f, g \rangle = u! \langle \hat{\rho}_u(f), \hat{\rho}_u(g) \rangle.$$

PROOF.     1. It is enough to show that $\rho_u \hat{\rho}_u(M_\lambda) = M_\lambda$, because the set $\{M_\lambda\}_\lambda$ is a basis for $\mathfrak{M}_u$. We have that

$$\rho_u \hat{\rho}_u(M_\lambda) = \rho_u(\sum_{\pi \in \lambda} m_\pi) = |\lambda| \sum_{\pi \in \lambda} m_\lambda = |\lambda| \binom{u}{\lambda} m_\lambda = M_\lambda.$$

2. It is enough to show that

$$\langle H_\lambda, M_\lambda \rangle = u! \langle \hat{\rho}_u(H_\lambda), \hat{\rho}_u(M_\lambda) \rangle$$

since both sets, $\{H_\lambda\}_\lambda$ and $\{M_\lambda\}_\lambda$, are basis for $\mathfrak{M}_u$. On the one hand, we have

$$\langle \hat{\rho}_u(H_\lambda), \hat{\rho}_u(M_\mu) \rangle = \left\langle \sum_{\pi \in \lambda} h_\pi, \sum_{\sigma \in \mu} m_\sigma \right\rangle$$

$$= \sum_{\pi \in \lambda} \sum_{\sigma \in \mu} \langle h_\pi, m_\sigma \rangle = \sum_{\pi \in \lambda} \delta_{\lambda,\mu} = \delta_{\lambda,\mu} \binom{u}{\lambda}.$$

On the other hand, we have that

$$\langle H_\lambda, M_\mu \rangle = \left\langle \binom{u}{\lambda} \lambda! h_\lambda, \binom{u}{\mu} |\mu| m_\mu \right\rangle = \binom{u}{\lambda}^2 \lambda! |\lambda| \delta_{\lambda,\mu} = u! \delta_{\lambda,\mu} \binom{u}{\lambda}.$$

$\square$

Proposition 15, together with Theorem 10 and [5], allows us to compute the transition matrices, as well as the scalar product of any two functions in $\mathfrak{M}_u$.

EXAMPLE 16. To compute the scalar product of two power sum MacMahon symmetric functions we proceed as follows. First, we assume that $\lambda = \mu$. (Otherwise their scalar product is 0.) Then,

$$\langle P_\lambda, P_\lambda \rangle = u! \langle \hat{\rho}_u(P_\lambda), \hat{\rho}_u(P_\lambda) \rangle = u! \left\langle \sum_{\pi \in \lambda} p_\pi, \sum_{\sigma \in \lambda} p_\sigma \right\rangle$$

$$= u! \sum_{\pi \in \lambda} \langle p_\pi, p_\pi \rangle = u! \binom{u}{\lambda} \frac{1}{|\mu(\hat{0}, \pi)|}.$$

because $\langle p_\pi, p_\pi \rangle = 1/\mu(\hat{0}, \pi)$ as stated in Appendix 6.2.

Therefore,

$$\langle p_\lambda, p_\lambda \rangle = \frac{|\lambda| \lambda!}{|\mu(\hat{0}, \lambda)|}.$$

where $\mu(\hat{0}, \lambda)$ is defined to be $\mu(\hat{0}, \pi)$ for any partition $\pi$ of type $\lambda$.

EXAMPLE 17. To compute the transition matrices between two basis of $\mathfrak{M}_u$, we write the corresponding transition basis for the unitary MacMahon symmetric functions according to Doubilet's calculation included in Appendix 6.1. Then, we apply the projection map to both sides of the corresponding identity. For example, suppose that $\pi \in \lambda$ and that $\sigma \in \eta$.

Then, since $\sigma \geq \pi$ if and only if $\nu \geq \lambda$, we have that

$$p_\pi = \sum_{\sigma \geq \pi} m_\sigma \longmapsto p_\lambda = \sum_{\eta \geq \lambda} \binom{u}{\eta} m_\eta.$$

PROPOSITION 18. *Let $f$ and $g$ be functions in $\mathfrak{M}_u$. Then*

1. *The map $\hat{\rho}_u$ satisfies*

$$\hat{\rho}_u(f * g) = u! \hat{\rho}_u(f) * \hat{\rho}_u(g)$$

2. *The Kronecker product on $\mathfrak{M}_{(1)^n}$ and $\mathfrak{M}_u$ are related by*

$$f * g = u! \rho_u(\hat{\rho}_u(f) * \hat{\rho}_u(g))$$

3. *The homomorphism $\omega$ is an algebra homomorphism. That is,*

$$\omega(f) * \omega(g) = f * g$$

*for all $f, g \in \mathfrak{M}_u$.*

PROOF. 1. The set $\{P_\lambda\}_\lambda$ is a basis for $\mathfrak{M}_u$. Therefore, it is enough to show

$$\hat{\rho}_u(P_\lambda * P_\mu) = u! \, \hat{\rho}_u(P_\lambda) * \hat{\rho}_u(P_\mu).$$

On the one hand, we have that

$$\hat{\rho}_u(P_\lambda * P_\mu) = \hat{\rho}_u\left(\binom{u}{\lambda}\binom{u}{\mu}\langle p_\lambda, p_\mu \rangle p_\lambda\right)$$

$$= \binom{u}{\lambda}\frac{|\lambda|\lambda!}{|\mu(\hat{0}, \pi)|}\delta_{\lambda,\mu} \, \hat{\rho}_u(P_\lambda) = \frac{u!\delta_{\lambda,\mu}}{|\mu(\hat{0}, \pi)|}\left(\sum_{\pi \in \lambda} p_\lambda\right).$$

On the other hand, we have that

$$\hat{\rho}_u(P_\lambda) * \hat{\rho}_u(P_\mu) = \sum_{\pi \in \lambda} p_\pi * \sum_{\sigma \in \mu} p_\sigma = \sum_{\substack{\pi \in \lambda \\ \sigma \in \mu}} p_\pi * p_\sigma$$

$$= \sum_{\substack{\pi \in \lambda \\ \sigma \in \mu}} \langle p_\pi, p_\sigma \rangle p_\pi = \frac{\delta_{\lambda,\mu}}{|\mu(\hat{0}, \pi)|}\left(\sum_{\pi \in \lambda} p_\lambda\right).$$

2. This follows from 1.

$$f * g = \rho_u(\hat{\rho}_u(f * g)) = \rho_u(u!\hat{\rho}_u(f) * \hat{\rho}_u(g)) = u! \, \rho_u(\hat{\rho}_u(f) * \hat{\rho}_u(g)).$$

since $\rho_u\hat{\rho}_u$ is the identity map.

3. It is enought to show that $\omega(p_\pi) * \omega(p_\sigma) = p_\pi * p_\sigma$. We have that

$$\omega(p_\pi) * \omega(p_\sigma) = \text{sign}(\pi)p_\pi * \text{sign}(\sigma)p_\sigma$$

$$= \text{sign}(\pi\sigma) \, p_\pi * p_\sigma = p_\pi * p_\sigma$$

since $p_\pi * p_\sigma = 0$ if $\pi \neq \sigma$.

$\square$

Proposition 18 together with Theorem 10 and [5] allows us to compute the Kronecker product of any two functions in $\mathfrak{M}^{(k)}$.

EXAMPLE 19. To compute the Kronecker product of two complete homogeneous MacMahon symmetric functions we proceed as follows:

$$
h_\lambda * h_\mu = \frac{1}{\lambda!\binom{u}{\lambda}\mu!\binom{u}{\mu}} H_\lambda * H_\mu = \frac{1}{\lambda!\binom{u}{\lambda}\mu!\binom{u}{\mu}} \rho_u\left(\hat{\rho}_u(H_\lambda * H_\mu)\right)
$$

$$
= \frac{|\lambda||\mu|}{u!} \rho_u\left(\sum_{\pi\in\lambda} h_\pi * \sum_{\sigma\in\mu} h_\sigma\right) = \frac{|\lambda||\mu|}{u!} \sum_{\substack{\pi\in\lambda\\\sigma\in\mu}} \rho_u(h_{\pi\wedge\sigma})
$$

$$
= \frac{|\lambda||\mu|}{u!} \sum_{\substack{\pi\in\lambda\\\sigma\in\mu}} \text{type}(\pi \wedge \sigma)! h_{\text{type}(\pi\wedge\sigma)}
$$

because $h_\pi * h_\sigma = h_{\pi\wedge\sigma}$, as stated in Appendix 6.3.

## 6. Appendices.

We reproduce the appendices in [5] making the appropriate changes in the notation to make it consistent with our notation.

Let $\pi, \sigma$, and $\tau$ be in $\Pi_n$. Then the following identities between MacMahon unitary symmetric functions hold. Note that the definition of the scalar product used differs from the one in [5].

### 6.1. The matrices of change of basis.

$$
p_\pi = \sum_{\sigma\geq\pi} m_\sigma, \qquad\qquad m_\pi = \sum_{\sigma\geq\pi} \mu(\pi,\sigma)p_\sigma,
$$

$$
e_\pi = \sum_{\sigma:\sigma\wedge\pi=\hat{0}} m_\sigma. \qquad\qquad m_\pi = \sum_{\tau\geq\pi} \frac{\mu(\pi,\tau)}{\mu(\hat{0},\tau)} \sum_{\sigma\leq\tau} \mu(\sigma,\tau)e_\sigma,
$$

$$e_\pi = \sum_{\sigma \le \pi} \mu(\hat{0}, \sigma) p_\sigma, \qquad\qquad p_\pi = \frac{1}{\mu(\hat{0}, \pi)} \sum_{\sigma \le \pi} \mu(\sigma, \pi) e_\sigma,$$

$$p_\pi = \sum_{\sigma \ge \pi} \text{sign}(\pi, \sigma) f_\sigma, \qquad\qquad f_\pi = \sum_{\sigma \ge \pi} |\mu(\pi, \sigma)| p_\sigma,$$

$$h_\pi = \sum_{\sigma : \sigma \wedge \pi = \hat{0}} \text{sign}(\sigma) f_\sigma, \qquad\qquad f_\pi = \sum_{\tau \ge \pi} \frac{|\mu(\pi, \tau)|}{|\mu(\hat{0}, \tau)|} \sum_{\sigma \le \tau} \mu(\sigma, \tau) h_\sigma,$$

$$h_\pi = \sum_{\sigma \le \pi} |\mu(\hat{0}, \sigma)| p_\sigma, \qquad\qquad p_\pi = \frac{1}{|\mu(\hat{0}, \pi)|} \sum_{\sigma \le \pi} \mu(\sigma, \pi) h_\sigma,$$

$$h_\pi = \sum_{\sigma} \text{type}(\pi \wedge \sigma)! \, m_\sigma, \qquad\qquad m_\pi = \sum_{\tau \ge \pi} \frac{\mu(\pi, \tau)}{|\mu(\hat{0}, \tau)|} \sum_{\sigma \le \pi} \mu(\sigma, \tau) h_\sigma,$$

$$e_\pi = \sum_{\sigma} \text{sign}(\sigma) \, \text{type}(\sigma \wedge \pi)! f_\sigma, \qquad\qquad f_\pi = \sum_{\tau \ge \pi} \frac{|\mu(\pi, \tau)|}{\mu(\hat{0}, \tau)} \sum_{\sigma \le \pi} \mu(\sigma, \tau) e_\sigma,$$

$$m_\pi = \sum_{\sigma \ge \pi} \text{sign}(\pi, \sigma) \, \text{type}(\pi, \sigma)! \, f_\sigma, \qquad\qquad f_\pi = \sum_{\sigma \ge \pi} \text{type}(\pi, \sigma)! \, m_\sigma,$$

$$e_\pi = \sum_{\sigma \le \pi} \text{sign}(\pi) \, \text{type}(\sigma, \pi)! \, h_\sigma, \qquad\qquad h_\pi = \sum_{\sigma \le \pi} \text{sign}(\sigma) \, \text{type}(\sigma, \pi)! \, e_\sigma.$$

## 6.2. The scalar product.

$$\langle h_\pi, m_\sigma \rangle = \delta_{\pi, \sigma}, \qquad\qquad \langle p_\pi, p_\sigma \rangle = \frac{\delta_{\pi, \sigma}}{|\mu(\hat{0}, \pi)|},$$

$$\langle h_\pi, p_\sigma \rangle = \zeta(\sigma, \pi), \qquad\qquad \langle e_\pi, p_\sigma \rangle = \text{sign}(\sigma) \, \zeta(\sigma, \pi),$$

$$\langle e_\pi, e_\sigma \rangle = \text{type}(\sigma \wedge \pi)!, \qquad\qquad \langle e_\pi, h_\sigma \rangle = \delta_{\pi \wedge \sigma, \hat{0}},$$

$$\langle e_\pi, m_\sigma \rangle = \text{sign}(\sigma) \, \text{type}(\sigma, \pi)! \zeta(\sigma, \pi), \qquad\qquad \langle h_\pi, h_\sigma \rangle = \text{type}(\sigma \wedge \pi)!,$$

$$\langle m_\pi, p_\sigma \rangle = \frac{\mu(\pi, \sigma)}{|\mu(\hat{0}, \sigma)|} \zeta(\pi, \sigma), \qquad\qquad \langle m_\pi, m_\sigma \rangle = \sum_{\tau \ge \pi \vee \sigma} \frac{\mu(\pi, \tau) \mu(\sigma, \tau)}{|\mu(\hat{0}, \tau)|},$$

$$\langle f_\pi, m_\sigma \rangle = \sum_{\tau \geq \pi \vee \sigma} \frac{|\mu(\pi, \tau)| \mu(\sigma, \tau)|}{|\mu(\hat{0}, \tau)|}, \qquad\qquad \langle f_\pi, e_\sigma \rangle = \text{sign}(\pi)\, \delta_{\pi, \sigma},$$

$$\langle f_\pi, h_\sigma \rangle = \text{type}(\pi, \sigma)!\, \zeta(\pi, \sigma), \qquad\qquad \langle f_\pi, p_\sigma \rangle = \text{sign}(\pi\sigma)\, \frac{\mu(\pi, \sigma)}{|\mu(\hat{0}, \sigma)|} \zeta(\pi, \sigma),$$

$$\langle f_\pi, f_\sigma \rangle = \text{sign}(\pi\sigma) \sum_{\tau \geq \pi \vee \sigma} \frac{\mu(\pi, \tau)\mu(\sigma, \tau)}{|\mu(\hat{0}, \tau)|}.$$

where $\zeta$ is the zeta function and $\delta_{\pi, \sigma}$ is the Kronecker symbol.

## 6.3. The Kronecker product.

$$p_\pi * p_\sigma = \frac{\delta_{\pi, \sigma}}{|\mu(\hat{0}, \pi)|} p_\pi, \qquad\qquad h_\pi * p_\sigma = \zeta(\sigma, \pi) p_\sigma,$$

$$e_\pi * p_\sigma = \text{sign}(\sigma)\, \zeta(\sigma, \pi) p_\sigma. \qquad\qquad f_\pi * p_\sigma = \frac{|\mu(\pi, \sigma)|}{|\mu(\hat{0}, \sigma)|} \zeta(\pi, \sigma) p_\sigma.$$

$$m_\pi * p_\sigma = \frac{\mu(\pi, \sigma)}{|\mu(\hat{0}, \sigma)|} \zeta(\pi, \sigma) p_\sigma, \qquad\qquad h_\pi * h_\sigma = h_{\pi \wedge \sigma}.$$

$$e_\pi * e_\sigma = h_{\pi \wedge \sigma}, \qquad\qquad e_\pi * h_\sigma = e_{\pi \wedge \sigma}.$$

$$m_\pi * h_\sigma = \sum_{\tau \in [\pi, \sigma]} \mu(\pi, \tau) p_\tau. \qquad\qquad f_\pi * h_\sigma = \sum_{\tau \in [\pi, \sigma]} |\mu(\pi, \tau)| p_\tau.$$

$$m_\pi * e_\sigma = \text{sign}(\pi) \sum_{\tau \in [\pi, \sigma]} |\mu(\pi, \tau)| p_\tau. \qquad m_\pi * m_\sigma = \sum_{\tau \geq \pi \vee \sigma} \frac{\mu(\pi, \tau)\mu(\sigma, \tau)}{|\mu(\hat{0}, \tau)|} p_\tau.$$

$$f_\pi * f_\sigma = \sum_{\tau \geq \pi \vee \sigma} \frac{|\mu(\pi, \tau)||\mu(\sigma, \tau)|}{|\mu(\hat{0}, \tau)|} p_\tau, \qquad m_\pi * f_\sigma = \sum_{\tau \geq \pi \vee \sigma} \frac{\mu(\pi, \tau)|\mu(\sigma, \tau)|}{|\mu(\hat{0}, \tau)|} p_\tau,$$

$$f_\pi * e_\sigma = \text{sign}(\pi) \sum_{\tau \in [\pi, \sigma]} \mu(\pi, \tau) p_\tau.$$

# CHAPTER 3

# Applications

In this chapter we study some applications of the combinatorial construction of the MacMahon symmetric function developed in the previous chapter.

As our first application we define the chromatic MacMahon symmetric function of a graph $G$ in a way that allows us to keep more information about the graph $G$ than the chromatic symmetric function defined by Stanley [37]. For instance, suppose that we are interested in the degrees of the vertices of $G$. Then we define the chromatic MacMahon symmetric functions in a way that determines the degree sequence of $G$.

The chromatic MacMahon symmetric function of a graph $G$ specializes to the chromatic symmetric function and to the chromatic polynomial.

Moreover, we proof a special case of a conjecture of Gessel [10] concerning the signs in the matrix of change of basis from the one-dimensional MacMahon symmetric functions to the complete homogeneous MacMahon symmetric functions.

Finally, we define the principal specialization of a MacMahon symmetric function. Then we proceed to study the effect of the principal specialization when applied to the different basis of the ring of MacMahon symmetric functions.

We explain how to use the combinatorial interpretation of the MacMahon symmetric functions obtained in the previous chapter to obtain the connection constants between the power sequence, the lower, and the upper factorials basis of the ring of polynomials in a beatiful way.

# 1. The chromatic symmetric function.

In this section we show that the chromatic symmetric functions of a graph can be defined in terms of the set $\mathcal{M}_\pi$. We show that this definition can be extended to the case of MacMahon symmetric functions, giving us the oportunity to keep relevant information about the graph.

We assume the reader to be familiar with the notion of graph and of directed graph.

A simple graph is a graph without loops or multiple edges. Let $G$ be a simple graph with vertex set $[n]$ and edge set $E$. A function $\kappa : [n] \to \mathbf{P}$ is called a proper coloring of $G$ if $\kappa(u) \neq \kappa(v)$ whenever $u$ and $v$ are vertices of an edge of $G$.

A stable partition $\pi$ of $G$ is a partition of $[n]$ such that each block of $\pi$ is totally disconnected (i.e., each block is a stable (or independent) set of vertices). Let $S(G)$ be the set of all stable partitions of $G$.

DEFINITION 20 (Stanley [37]). Let $G$ be a simple graph with vertex set $[n]$

and edge set $E$. The chromatic symmetric function is defined as

$$X_G = \sum_\kappa x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)}.$$

where the sum ranges over all proper colorings $\kappa : [n] \to \mathbf{P}$.

We define the set $\mathcal{F}_n^G$ in terms of the $\mathcal{M}_\pi$ and show that the generating

function of this set is the chromatic symmetric function.

DEFINITION 21. Let $G$ be a simple graph with vertex set $[n]$ and edge set

$E$. Let $\mathcal{F}_n^G$ be the subset of $F_n$ defined by

$$\mathcal{F}_n^G = \{ f : [n] \to \mathbf{P} : \text{for any } \{n_1, n_2\} \in E, f(n_1) \neq f(n_2) \}$$

$$= \bigcup_{\lambda \vdash n} \bigcup_{\pi \in S(G)} \mathcal{M}_\pi.$$

Stanley showed in [35] the following proposition.

PROPOSITION 22.

$$\gamma(\rho_n(\mathcal{F}_n^G)) = X_G.$$

We obtain a MacMahon symmetric function if instead of $S_n$ we have one of

its Young subgroup acting on $\mathcal{M}_\pi$. This may be desirable if we have a graph

(or directed graph) $G$ and some invariant set of vertices that we want to keep

track of. Then, we may distinguish between the different kinds of vertices by

using different kinds variables to weight them. For instance, suppose that we

are interested in the degree sequence of graph $G$. We define the MacMahon chromatic symmetric function as follows.

DEFINITION 23. Let $G$ be a simple graph with vertex set $[n]$ and edge set $E$. The chromatic MacMahon symmetric function (in alphabets $X_1, X_2, \cdots$) is defined by

$$\bar{X}_G = \sum_{\kappa} x_{d(1),\kappa_1} x_{d(2),\kappa_2} \cdots x_{d(n),\kappa_n}.$$

where $d(i)$ denoted the degree of vertex $i$.

PROPOSITION 24. Let $G$ be a graph with vertex set $[n]$ and degree sequence $(1^{u_1} 2^{u_2} \cdots n^{u_n})$. Then $\bar{X}_G \in \mathfrak{M}_u$. Moreover, we can recover the chromatic symmetric function $X_G$ from $\bar{X}_G$ through the projection map:

$$\rho_u(\bar{X}_G) = X_G$$

where we define $\rho_u(\bar{X}_G)$ to be the image under $\rho_n$, where $n$ is the weight of $\lambda$, of any preimage of $\bar{X}_G$ under $\rho_n$.

PROOF. Note that the multiset $\{d(1), \cdots, d(n)\}$ does not depend on the coloring of graph $G$, but only on its degree sequence. Moreover, it determines its degree sequence.

The second part of our proposition follows from Proposition 22. By obtaining any preimage for $X_G$ under $\rho_u$ we are forgetting the degree sequence of the graph. Then, by applying $\rho_n$ we recover Stanley's chromatic symmetric function. □

EXAMPLE 25. In [35] Stanley showed that the chromatic symmetric function for graphs $G$ and $H$ (See Figure 1) does not distinguish between them.



Graph G                Graph H
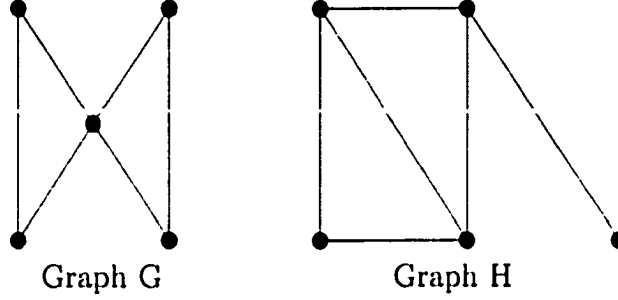
FIGURE 1.

On the other hand. the chromatic MacMahon symmetric function of $G$ and $H$ are different.

$$\bar{X}_G = 2m_{(0,2,0,0)^2(0,0,0,1)} + 4m_{(0,2,0,0)(0,1,0,0)^2(0,0,0,1)} + m_{(0,1,0,0)^4(0,0,0,1)}.$$

$$\bar{X}_H = 2m_{(0,1,1)(1,0,1)(0,0,1)} + 2m_{(1,0,1)(0,0,1)^2(0,1,0)}$$

$$+ m_{(0,1,1)(0,0,1)^2(1,0,0)} + m_{(1,1,0)(0,0,1)^3} + m_{(0,0,1)^3(0,1,0)(1,0,0)}.$$

We have that $\bar{X}_G \in \mathfrak{M}_{(0,4,0,1)}$. Therefore, the degree sequence of $G$ is $(2,2,2,2,4)$. Similarly, we have that $X_H \in \mathfrak{M}_{(1,1,3)}$. Therefore, the degree sequence of $H$ is $(1,2,3,3,3)$. On the other hand. $\rho_{(0,4,0,1)}(\bar{X}_G) = \rho_{(1,1,3)}(\bar{X}_H) = 2 \cdot 2! \, m_{221} + 4 \cdot 3! \, m_{2111} + 5! \, m_{1111}$ is the chromatic symmetric function of both graph $G$ and graph $H$.

EXAMPLE 26. The MacMahon chromatic symmetric function encloses more information than what we can obtain by having both the chromatic symmetric function and the degree sequence of a graph.



Graph L          Graph M
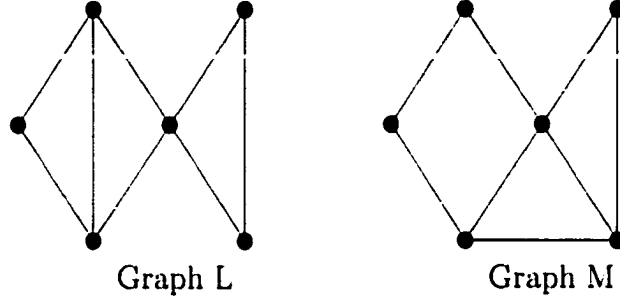
FIGURE 2.

The MacMahon chromatic symmetric function can distinguish between some graphs having the same degree sequence and chromatic symmetric function.

The chromatic symmetric function of graphs $L$ and $M$ is $3!m_{222} + 7 \cdot 2!2!m_{2211} + 7 \cdot 4!m_{21111} + 5!m_{11111}$.

The degree sequence of graphs $L$ and $M$ is $(222334)$.

$$\bar{X}_L = 2!\, m_{(0,1,0,1)(0,1,1,0)^2} + 2!\, m_{(0,1,1,0)^2\cdots} + 4m_{(0,1,0,1)(0,1,1,0)^2\cdots}$$

$$+ 2m_{(0,2,0,0)(0,1,1,0))\cdots} + m_{(0,1,0,1)\cdots} + 2m_{(0,2,0,0)} + 4m_{(0,1,1,0)} + m\cdots$$

$$\bar{X}_M = 2!\, m_{(0,1,0,1)(0,1,1,0)^2} + 3 \cdot 2!\, m_{(0,1,1,0)\cdots} + m_{(0,0,2,0)(0,1,0,1)\cdots}$$

$$+ 3m_{(0,2,0,0)(0,1,0,1)\cdots} + 3m_{(0,1,1,0)\cdots} + 3m_{(0,2,0,0)\cdots} + m_{(0,1,0,1)\cdots} + m\cdots$$

where having a MacMahon symmetric function indexed by a vector partition

where a subindex "···" appear indicates that we should complete that vector

partition with parts of weight 1 until it becomes a vector partition of $(0,3,2,1)$.

REMARK. I don't know of two graphs with the same MacMahon chromatic

symmetric function.

There are several other ways to define the chromatic MacMahon symmetric

functions depending on which information we want to remember about the

graph. For instance, if we have a directed graph, we may want to remember

their sources, or sinks, or both, or we may want to remember their outdegree,

etc. If we have a multipartite graph, we may want to remember to which part

each vertex belongs, etc.

## 2. The one-dimensional MacMahon symmetric functions.

In this section we prove a special case of a conjecture of Gessel concerning

the signs in the matrix of change of basis from the one-dimensional MacMa-

hon symmetric functions to the complete homogeneous MacMahon symmetric

functions. Before stating Gessel's conjecture we introduce some definitions.

DEFINITION 27. Let $A^*$ be the monoid formed by all words with letters in

the alphabet $A$, $A^+$ be the subset of $A^*$ of nonempty word, and $A^\sharp$ be the set

of all sequences of words in $A^*$.

A word $\omega$ in $A^*$ is primitive if it is not possible to find any word $u$ in $A^*$ such that $\omega = u^n$, with $n > 1$. A word $l$ is a Lyndon word if it is a primitive word and if it is minimal in its conjugancy class. That is, if for all words $u$ and $v$ in $A^+$, $l = uv$ implies $l < vu$. We denote by $L(A)$ the set of Lyndon words on $A$.

THEOREM 28 (Lyndon). *Any word $w \in A^+$ may be written uniquely as a nonincreasing product of Lyndon words:*

$$w = l_1 l_2 \cdots l_n.$$

*where $l_i \in L$ and $l_1 \geq l_2 \geq \cdots \geq l_n$.*

For a proof of Lyndon's theorem and further information on words and Lyndon words see [16, 26].

*The One-dimensional MacMahon Symmetric Functions.* Let $(b, r, \cdots, w)$ be a vector. Then $G_{(b,r,\cdots,w)}$ is defined by the generating function

$$1 + \sum_{b,r,\cdots,w} h_{(b,r,\cdots,w)} s^b t^r \cdots u^w = \prod_{\gcd(b,r,\cdots,w)=1} \left( 1 + \sum_{i=1}^{\infty} g_{(b_i,r_i,\cdots,w_i)} s^{b_i} t^{r_i} \cdots u^{w_i} \right).$$

Let $\lambda$ be a vector partition. We set $g_\lambda = g_{(b_1,r_1,\cdots,w_1)} g_{(b_2,r_2,\cdots,w_2)} \cdots$.

Let $L^S$ be the subset of $A^S$ consisting of sequences of Lyndon words $s$. We have the following proposition.

PROPOSITION 29 (Gessel). *Let $b, r, \cdots, w$ be coprime numbers. Let $G_{(nb,nr,\cdots,nw)}$ be the subset of $M \in L^S$ consisting of sequences of Lyndon words $M$ such that*

$$|M|_X = nb, |M|_Y = nr, \cdots, |M|_Z = nw,$$

*and such that for each $l \in M$ there is some $k \in \mathbb{N}$ such that*

$$|l|_X = kb, |l|_Y = kr, \cdots |l|_Z = kw,$$

*Then, $\gamma(G_{(nb,nr,\cdots,nw)}) = g_{(nb,nr,\cdots,nw)}$.*

PROOF. Define the function $h^i_{(b,r,\cdots,w)}$ by the generating function

$$\frac{1}{1 - x_i s - y_i t - \cdots - z_i t} = 1 + \sum_{b,r,\cdots,w} h^i_{(b,r,\cdots,w)} s^b t^r \cdots u^w.$$

Hence,

$$1 + \sum_{b,r,\cdots,w} h_{(b,r,\cdots,w)} s^b t^r \cdots u^w = \prod_i \left( 1 + \sum_{b,r,\cdots,w} h^i_{(b,r,\cdots,w)} s^b t^r \cdots u^w \right)$$

Therefore, $h_{(b,r,\cdots,w)} = \sum h^{i_1}_{(b_1,r_1,\cdots,w_1)} h^{i_2}_{(b_2,r_2,\cdots,w_2)} \cdots$, where the sum is taken over all sequences $i_1 < i_2 < \cdots$, such that $(b_1, r_1, \cdots, w_1)(b_2, r_2, \cdots, w_2) \cdots \vdash (b, r, \cdots, w)$.

We proceed similarly to obtain that

$$g_{(b,r,\cdots,w)} = \sum g^{i_1}_{(b_1,r_1,\cdots,w_1)} g^{i_2}_{(b_2,r_2,\cdots,w_2)} \cdots,$$

where the sum is taken over all sequences $i_1 < i_2 < \cdots$, such that $(b_1, r_1, \cdots, w_1)$ $(b_2, r_2, \cdots, w_2) \cdots \vdash (b, r, \cdots, w)$.

Therefore, we have that it is enough to show that $g^i_{(b,r,\cdots,w)}$ is the generating function for sequences of words in position $i$. For the sake of simplicity, we omit the subindex $i$ in the rest of the demonstration.

Let $L$ be the set of Lyndon words on the alphabet $A$. Theorem 28 implies that

$$\sum_{\omega \in A^*} \omega = \sum_n \sum_{l_1 \geq l_2 \geq \cdots \geq l_n} l_1 l_2 \cdots l_n$$

Let $\Theta : A^* \to \mathbf{Q}[\mathbf{x}, \mathbf{y}, \cdots, \mathbf{z}]$ be the homomorphism that sends $X$ to $x$, $Y$ to $y$, and so on, where $x, y, \cdots, z$ are commuting variables. Then, we have that

$$\frac{1}{1 - x - y - \cdots - z} = \prod_{l \in G_{(b,r,\cdots,w)}} \frac{1}{1 - \Theta(l)}$$

$$= \prod_{\substack{b,r,\cdots,w \\ \gcd(b,r,\cdots,w)=1}} \prod_{l \in G_{(b,r,\cdots,w)}} \frac{1}{1 - \Theta(l)}$$

where $G_{(b,r,\cdots,w)} = \bigcup_{n \geq 1} G_{(nb,nr,\cdots,nw)}$.

On the other hand, let

$$f_{(b,r,\cdots,w)} = \prod_{l \in L_{b,r,\cdots,w}} \frac{1}{1 - \Theta(l)}$$

$$= 1 + \sum_{m \geq 1} c_{m,b,r,\cdots,w} (x^b y^r \cdots z^w)^m$$

$$= F(x^b y^r \cdots z^w)$$

for some coefficients $c_{m,b,r,\cdots,w}$.

For example, if $(b, r) = (1, 1)$, then a list of the possible Lyndon words is given by $xy$, $xxyy$, $xxxyyy$, $xxyxyy$, $xxyyxy$, $\cdots$. Therefore,

$$f_{(1,1)} = \frac{1}{1 - xy} \frac{1}{1 - x^2 y^2} \frac{1}{(1 - x^3 y^3)^3} \cdots$$

Then, we have that

$$(1) \qquad \frac{1}{1-x-y\cdots-z} = \prod_{\substack{b,r,\cdots,w \\ \gcd(b,r,\cdots,w)=1}} F_{(b,r,\cdots,w)}(x^b y^r \cdots z^w)$$

From proposition 2 we have that

$$\frac{1}{1-x-y\cdots-z} = \prod_{\substack{b,r,\cdots,w \\ \gcd(b,r,\cdots,w)=1}} \left( 1 + \sum_n g_{(nb,nr,\cdots,nw)}(x,y,\cdots,z) \right)$$

$$(2) \qquad\qquad\qquad = \prod_{\substack{b,r,\cdots,w \\ \gcd(b,r,\cdots,w)=1}} G_{(b,r,\cdots,w)}(x^b y^r \cdots z^w)$$

Therefore, if take logarithms on both sides of the equation obtained from (1) and (2) then, we obtain that $F_{(b,r,\cdots,w)}(x^b y^r \cdots z^w)$ equals $G_{(b,r,\cdots,w)}(x^b y^r \cdots z^w)$

□

We proceed to prove a special case of a conjecture of Gessel.

CONJECTURE 30 (Gessel). *Let $\lambda$ be a vector partition of $u$. The sign of the coefficient of $h_\lambda$ in $g_u$ is $(-1)^{l(\lambda)-1}$.*

We obtain that the following proposition holds.

PROPOSITION 31.      1. *Let $\pi$ be a partition of $(1)^n$. Then,*

$$[h_\pi]g_{(1)^n} = (-1)^{(l(\pi)-1)}(l(\pi) - 1)!.$$

2. *Let $\lambda$ be a vector partition of $u$, where $u$ has at least one coordinate equal to one. Then*

$$[h_\pi]g_u = \frac{(-1)^{(l(\lambda)-1)}(l(\lambda) - 1)!}{|\lambda|}.$$

*where $\lambda$ is the type of $\pi$ under $\rho_u$.*

PROOF.    1. Suppose that $h_\pi$ and $g_\sigma$ are unitary MacMahon symmetric functions. From Proposition 2 and Proposition 29 we obtain the identity

$$h_{\hat{1}} = \sum_\pi g_\pi$$

Therefore, from Möbius inversion we obtain that

$$(3) \qquad g_{\hat{1}} = \sum_\pi \mu(\pi, \hat{1}) h_\pi = \sum_\pi (-1)^{l(\pi)-1} (l(\pi) - 1)! h_\pi.$$

because $[\pi, \hat{1}]$ is isomorphic to the lattice $\Pi_n$ for $n = l(\pi) - 1$.

2. In general. the projection map behaves poorly when applied to the basis of one-dimensional MacMahon symmetric functions. The reason is that the image of a Lyndon word is no longer a Lyndon word. If $u$ has some coordinate equal to 1. by symmetry we can assume that its first coordinate is 1. In this case. the projection map behaves nicely.

A Lyndon word belongs to $g_{(1)^n}$ if and only if it starts with the first letter of the alphabet. Similarly, a Lyndon word belongs to $g_{(1,\cdots)}$ if and only if it starts with the first letter of the alphabet. Therefore, if $u$ has one coordinate equal to 1 then

$$\rho_u(g_{(1)^n}) = u! g_u.$$

Applying the projection map to both sides of equation (1) we obtain that

$$u!g_u = \sum_\lambda (-1)^{l(\lambda)-1}(l(\lambda) - 1)!\lambda!\binom{u}{\lambda}h_\lambda.$$

where $\lambda$ is the image of $\pi$. Therefore,

$$g_u = \sum_\lambda \frac{(-1)^{l(\lambda)-1}(l(\lambda) - 1)!h_\lambda}{|\lambda|}.$$

$\square$

## 3. The principal specialization.

In [12] Joni, Rota, and Sagan studied three sequences of binomial type and their connection constants: the power sequence, lower, and upper factorials. To each pair of sequences, they associated a partially ordered set. Then, the connection constants where obtained by summing over the associated poset, and by Möbius inversion.

They mentioned that the most difficult part of their work consisted in guessing the correct poset. In this section we show how the posets appear naturally when studing specializations of the MacMahon symmetric functions.

Let $f$ be a MacMahon symmetric function in alphabets $X^i = \{x^i_1, x^i_2, \cdots\}$, with $i = 1, 2, \cdots, k$. Following Stanley [36] we define the principal specialization $ps^1_k$ by setting $x^j_i = 1$ if $i \leq k$ and $x^j_i = 0$ otherwise.

Recall that a disposition is a function $f$ from $[n]$ to $[k]$ together with a linear order of the elements coming from the same block of $\ker f$.

NOTATION. Let $\lambda$ be a partition of $n$ of length $l$. We define

$$(k)_n = k(k - 1) \cdots (k - n + 1)$$

$$(k)^n = k(k + 1) \cdots (k + n - 1)$$

$$(k)_\lambda = (k)_{\lambda_1}(k)_{\lambda_2} \cdots (k)_{\lambda_l}$$

$$(k)^\lambda = (k)^{\lambda_1}(k)^{\lambda_2} \cdots (k)^{\lambda_l}$$

THEOREM 32. *Let $\pi$ be a partition in $\Pi_n$ with type $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{l(\pi)})$,
where $l(\pi)$ is the number of blocks of $\pi$. Then*

(4)                           $$\mathrm{ps}_k^1(m_\pi) = (k)_{l(\pi)}$$

(5)                           $$\mathrm{ps}_k^1(h_\pi) = (k)^\lambda$$

(6)                           $$\mathrm{ps}_k^1(p_\pi) = k^{l(\pi)}$$

(7)                           $$\mathrm{ps}_k^1(e_\pi) = (k)_\lambda$$

(8)                           $$\mathrm{ps}_k^1(f_\pi) = (k)^{l(\pi)}$$

PROOF. Let $T$ be any subset of $F_n$ and let $t$ be its generating function. Then $\mathrm{ps}_k^1(t)$ is equal to the number of functions in $T$ such that their image is contained in $[k]$. From Definition 9 we obtain the following results.

1. There are $(k)_{l(\pi)}$ functions from $[n]$ to $[k]$ with kernel $\pi$.

2. Look at each block of $\lambda$ separately. There are $(k)^n$ dispositions from $[n]$ to $[k]$. Hence, there are $(k)^\lambda$ dispositions from $[n]$ to $[k]$.

3. There are $k^{l(\pi)}$ functions from $[n]$ to $[k]$ that are constant on the blocks of $\pi$.

4. Look at each block of $\lambda$ separately. There are $(k)_n$ functions from $[n]$ to $[k]$ that are injective on the blocks of $\pi$. Hence, there are $(k)_\lambda$ dispositions from $[n]$ to $[k]$.

5. There are $(k)^\lambda$ dispositions from the set of blocks of $\ker \pi$ to $k$.

To count the number of disposition from $[n]$ to $[k]$ we proceed as follows. The image of 1 can be choosen in $k$ different ways, the image of 2 can be choosen in $k + 1$ ways. (There are $k$ possible images for 2, but in the case where $f(1) = f(2)$ we must also choose the order of 1 and 2). Using induction, we see that the number of dispositions os $(k)^n = k(k + 1) \cdots (k + n - 1)$.

$\square$

Now, we are able to compute the transition matrices between the different basis of the space of polynomials of degree $n$.

NOTATION. Let $s(n, i)$ denote the Stirling number of the first kind and $S(n, i)$ denote the Stirling number of the second kind.

The number of partitions of a set of $n$ elements into $k$ blocks is counted by $S(n, k)$. Similarly, the number of permutations $\pi$ in $S_n$ with exactly $k$ cycles is counted by $|s(n, k)|$.

COROLLARY 33.  *The connection constant formulae for* $\{k^n\}$, $\{(k)_n\}$, *and* $\{(k)^n\}$ *are*

$$k^n = \sum_{i \geq 0} S(n, i)(k)_i$$

$$(k)_n = \sum_{i \geq 0} s(n, i)k^i$$

$$(k)^n = \sum_{i \geq 0} \frac{n!}{i!}\binom{n-1}{i-1}(k)_i$$

$$(k)_n = \sum_{i \geq 0}(-1)^{n-i}\frac{n!}{i!}\binom{n-1}{i-1}(k)^i$$

$$(k)^n = \sum_{i \geq 0}|s(n, i)|k^i$$

$$k^n = \sum_{i \geq 0}(-1)^{n-i}S(n, i)(k)^i$$

PROOF.     1. Appplying the homomorphism $\text{ps}_k^1$ to the generating functions of $F_n = \bigcup_{\pi \in \Pi_n} \mathcal{M}_\pi$, we obtain that

(9) $$k^n = \sum_{\pi \in \Pi_n} (k)_{l(\pi)}$$

$$= \sum_{i \geq 0} S(n, i)(k)_i,$$

If we apply Möbius inversion to Equation (9) we obtain

$$(k)_n = \sum_{\pi \in \Pi_n} \mu(\pi) k^{l(\pi)}$$

$$= \sum_i k^i \sum_{\substack{\pi \in \Pi_n \\ l(\pi)=i}} \mu(\pi)$$

$$= \sum_{i \geq 0} s(n, i) k^i.$$

2. Define a linear partition to be a partition of $n$, together with a total order on each block. Let $\mathcal{L}_n$ be the poset of linear partition of $[n]$.

To any disposition $p : [n] \to \mathbf{P}$ we associate a linear partition defined by the $\ker p$ together with the ordering of the balls. Hence, we have the following equation:

$$(10) \qquad h_{[n]} = \sum_{\sigma \in \mathcal{L}_n} m_\sigma.$$

Apply homomorphism $\mathrm{ps}_k^1$ to both sides of equation (10) to obtain

$$(11) \qquad (k)^n = \sum_{\sigma \in \mathcal{L}_n} (k)_{l(\sigma)}.$$

$$= \sum_{i \geq 0} \frac{n!}{i!} \binom{n-1}{i-1} (k)_i.$$

because the number of linear partitions of $[n]$ into $i$ blocks is $\frac{n!}{i!}\binom{n-1}{i-1}$. These numbers are known as the Lah numbers [29].

If we apply Möbius inversion to equation (11), then we obtain $(k)_n = \sum_{\lambda \in \mathcal{L}_n} \mu(\lambda)(k)^{l(\lambda)}$.

Moreover, for all $\lambda$ in $\mathcal{L}_n$ we have that $\mu(\lambda) = (-1)^{n-l(\lambda)}$, because $[0, \lambda]$ is a Boolean lattice. Therefore,

$$(k)_n = \sum_{\sigma \in \mathcal{L}_n} (-1)^{n-l(\lambda)} (k)^{l(\lambda)}$$

$$= \sum_{i \geq 0} (-1)^{n-i} \frac{n!}{i!} \binom{n-1}{i-1} (k)^i.$$

3. To each partition $\sigma = B_1 | B_2 | \cdots | B_l$, we can associate in a canonical way $\mu(\hat{0}, \sigma) = (B_1 - 1)!(B_2 - 1)! \cdots (B_l - 1)!$ sets of Lyndon words. Given one of such sets of Lyndon words, for each $\sigma' \geq \sigma$ we obtain one different linear partition. Hence,

$$h_{[n]} = \sum_{\sigma \in \Pi_n} |\mu(\hat{0}, \sigma)| p_\sigma.$$

Apply the homomorphism $ps_k^1$ to both sides of the previous equation to obtain

$$(12) \qquad\qquad h_{[n]} = \sum_{\sigma \in \Pi_n} k^{c(\pi)}$$

$$(13) \qquad\qquad = \sum_{i \geq 0} |s(n, i)| k^i.$$

where $c(\sigma)$ is the number of cycles of $\sigma$ and where $|s(n, i)|$ is the signless Stirling number of the second kind which counts the number of permutations in $S_n$ with $i$ cycles.

We give $S_n$ a poset structure induced by the refinement order on partitions. We say that $\sigma \leq \tau$ if each cycle of $\sigma$, written with the

smallest element first, is composed of a string of consecutive integers

from the cycles of $\tau$.

If we apply Möbius inversion to the previous equation we obtain

$$k^n = \sum_{\sigma \in S_n} \mu(\sigma)(x)^{l(\sigma)}$$

$$= \sum_{i \geq 0} (-1)^{n-i} S(n, i)(k)^i.$$

because $\mu(\sigma)$ is zero unless each cycle of $\sigma$ increases from left to right,

in which case $\mu(\sigma) = (-1)^{n-l(\sigma)}$.

All three Möbius inversion arguments are fully explained in [12].     □

COROLLARY 34. *Let* $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ *be a partition of* $n$. *Let* $\pi = 1\,2 \cdots \lambda_1 | \cdots | n - \lambda_l + 1 \cdots n$.

*We define* $\mathrm{disj}(\pi, i)$ *to be the number of partitions* $\sigma$ *of length* $i$ *such that* $\sigma \wedge \pi = \dot{0}$. *Then*

(14)  $$(k)_{\lambda_1}(k)_{\lambda_2} \cdots (k)_{\lambda_l} = \sum_{i \geq 0} \mathrm{disj}(\pi, i)(k)_i.$$

(15)  $$(k)^{\lambda_1}(k)^{\lambda_2} \cdots (k)^{\lambda_l} = \sum_{i \geq 0} (-1)^{n-i} \mathrm{disj}(\pi, i)(k)^i.$$

PROOF. From Appendix 6.1 we have that

(16)  $$e_\pi = \sum_{\sigma: \sigma \wedge \pi = \dot{0}} m_\sigma.$$

(17)  $$h_\pi = \sum_{\sigma: \sigma \wedge \pi = \dot{0}} \mathrm{sign}(\sigma) f_\sigma.$$

Apply the homomorphism $ps^1_k$ to both sides of 16 to obtain 14. Similarly, apply the homomorphism $ps^1_k$ to both sides of 17 to obtain 15. $\qquad\square$

THEOREM 35. *Let $\lambda$ be a vector partition.*

$$ps^1_k(m_\lambda) = \frac{(k)_{l(\lambda)}}{|\lambda|}$$

$$ps^1_k(h_\lambda) = \frac{(k)^\lambda}{\lambda!}$$

$$ps^1_k(p_\lambda) = k^{l(\lambda)}$$

$$ps^1_k(e_\lambda) = \frac{(k)_\lambda}{\lambda!}$$

$$ps^1_k(f_\lambda) = \frac{(k)_{l(\lambda)}}{|\lambda|}$$

PROOF. It follows from Theorem 10 and Theorem 32. $\qquad\square$

# The Kronecker Product of Schur Functions indexed by

# Two-Row Shapes or Hook Shapes.

## 1. Introduction

The aim of this paper is to derive an explicit formula for the Kronecker coefficients corresponding to partitions of certain shapes. The Kronecker coefficients, $\gamma_{\mu\nu}^{\lambda}$, arise when expressing a Kronecker product (also called inner or internal product), $s_\mu * s_\nu$, of Schur functions in the Schur basis,

$$(18) \qquad s_\mu * s_\nu = \sum_{\mu,\nu} \gamma_{\mu\nu}^{\lambda} s_\lambda.$$

These coefficients can also be defined as the multiplicities of the irreducible representations in the tensor product of two irreducible representations of the symmetric group. A third way to define them is by the comultiplication expansion. Given two alphabets $X = \{x_1, x_2, \cdots\}$ and $Y = \{y_1, y_2, \cdots\}$

$$(19) \qquad s_\lambda[XY] = \sum_{\mu,\nu} \gamma_{\mu\nu}^{\lambda} s_\mu[X] s_\nu[Y],$$

where $s_\lambda[XY]$ means $s_\lambda(x_1 y_1, x_1 y_2, \cdots, x_i y_j, \cdots)$. Remmel [23, 24] and Remmel and Whitehead [25] have studied the Kronecker product of Schur functions corresponding to two two-row shapes, two hook shapes, and a hook shape and

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a two-row shape. We will use the comultiplication expansion (19) for the Kro-
necker coefficients, and a formula for expanding a Schur function of a difference
of two alphabets due to Sergeev [3] to obtain similar results in a simpler way.
We believe that the formulas obtained using this approach are elegant and re-
flect the symmetry of the Kronecker product. In the three cases we found a
way to express the Kronecker coefficients in terms of regions and paths in $N^2$.

## 2. Basic definitions

A partition $\lambda$ of a positive integer $n$, written as $\lambda \vdash n$, is an unordered
sequence of natural numbers adding to $n$. We write $\lambda$ as $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$,
where $\lambda_1 \geq \lambda_2 \geq \cdots$, and consider two such strings equal if they differ by a
string of zeroes. The nonzero numbers $\lambda_i$ are called the parts of $\lambda$, and the
number of parts is called the length of $\lambda$, denoted by $l(\lambda)$. In some cases, it is
convenient to write $\lambda = (1^{d_1} 2^{d_2} \cdots n^{d_n})$ for the partition of $n$ that has $d_i$ copies
of $i$. Using this notation, we define the integer $z_\lambda$ to be $1^{d_1} d_1! \, 2^{d_2} d_2! \cdots n^{d_n} d_n!$.

We identify $\lambda$ with the set of points $(i, j)$ in $N^2$ defined by $1 \leq j \leq \lambda_i$, and
refer to them as the Young diagram of $\lambda$. The Young diagram of a partition $\lambda$
is thought of as a collection of boxes arranged using matrix coordinates. For
instance, the Young diagram corrresponding to $\lambda = (4, 3, 1)$ is
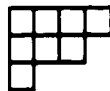
To any partition $\lambda$ we associate the partition $\lambda'$. its conjugate partition, defined by $\lambda_i' = |\{j : \lambda_j \geq i\}|$. Geometrically, $\lambda'$ can be obtained from $\lambda$ by flipping the Young diagram of $\lambda$ around its main diagonal. For instance, the conjugate partition of $\lambda$ is $\lambda' = (3, 2, 2, 1)$, and the corresponding Young diagram is
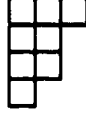
We recall some facts about the theory of representations of the symmetric group, and about symmetric functions. See [17] or [34] for proofs and details.

Let $R(S_n)$ be the space of class function in $S_n$, the symmetric group on $n$ letters. and let $\Lambda^n$ be the space of homogeneous symmetric functions of degree $n$. A basis for $R(S_n)$ is given by the characters of the irreducible representations of $S_n$. Let $\chi^\mu$ be the irreducible character of $S_n$ corresponding to the partition $\mu$. There is a scalar product $\langle \, . \, \rangle_{S_n}$ on $R(S_n)$ defined by

$$\langle \chi^\mu, \chi^\nu \rangle_{S_n} = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^\mu(\sigma) \chi^\nu(\sigma),$$

and extended by linearity.

A basis for the space of symmetric functions is given by the Schur functions. There exists a scalar product $\langle \, , \, \rangle_{\Lambda^n}$ on $\Lambda^n$ defined by

$$\langle s_\lambda, s_\mu \rangle_{\Lambda^n} = \delta_{\lambda\mu},$$

where $\delta_{\lambda\mu}$ is the Kronecker delta, and extended by linearity.

Let $p_\mu$ be the power sum symmetric function corresponding to $\mu$, where

$\mu$ is a partition of $n$. There is an isometry $\mathrm{ch}^n : R(S_n) \longmapsto \Lambda^n$, given by the

characteristic map,

$$\mathrm{ch}^n(\chi) = \sum_{\mu \vdash n} z_\mu^{-1} \chi(\mu) p_\mu.$$

This map has the remarkable property that if $\chi^\lambda$ is the irreducible character

of $S_n$ indexed by $\lambda$, then $\mathrm{ch}^n(\chi^\lambda) = s_\lambda$, the Schur function corresponding to $\lambda$.

In particular, we obtain that $s_\lambda = \sum_{\mu \vdash n} z_\mu^{-1} \chi^\lambda(\mu) p_\mu$. Hence.

(20)                                   $\chi^\lambda(\mu) = \langle s_\lambda, p_\mu \rangle.$

Let $\lambda$, $\mu$, and $\nu$ be partitions of $n$. The Kronecker coefficients $\gamma_{\mu\nu}^\lambda$ are defined

by

(21)            $\gamma_{\mu\nu}^\lambda = \langle \chi^\lambda, \chi^\mu \chi^\nu \rangle_{S_n} = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^\lambda(\sigma) \chi^\mu(\sigma) \chi^\nu(\sigma).$

Equation (21) shows that the Kronecker coefficients $\gamma_{\mu\nu}^\lambda$ are symmetric in $\lambda$,

$\mu$, and $\nu$. The relevance of the Kronecker coefficients comes from the following

fact: Let $X^\mu$ be the representation of the symmetric group corresponding to

the character $\chi^\mu$. Then $\chi^\mu \chi^\nu$ is the character of $X^\mu \otimes X^\nu$, the representation

obtained by taking the tensor product of $X^\mu$ and $X^\nu$. Moreover, $\gamma_{\mu\nu}^\lambda$ is the

multiplicity of $X^\lambda$ in $X^\mu \otimes X^\nu$.

Let $f$ and $g$ be homogeneous symmetric functions of degree $n$. The Kronecker product, $f * g$, is defined by

$$\text{(22)} \qquad\qquad f * g = \text{ch}^n(uv),$$

where $\text{ch}^n u = f$, $\text{ch}^n v = g$, and $uv(\sigma) = u(\sigma)v(\sigma)$. To obtain (18) from this definition, we set $f = s_\mu$, $g = s_\nu$, $u = \chi^\mu$, and $v = \chi^\nu$ in (22).

The Kronecker product has the following symmetries:

$$s_\mu * s_\nu = s_\nu * s_\mu.$$

$$s_\mu * s_\nu = s_{\mu'} * s_{\nu'}.$$

Moreover, if $\lambda$ is a one-row shape

$$\gamma^\lambda_{\mu\nu} = \delta_{\mu,\nu}.$$

We introduce the operation of substitution or plethysm into a symmetric function. Let $f$ be a symmetric function, and let $X = \{x_1, x_2, \cdots\}$ be an alphabet. We write $X = x_1 + x_2 + \cdots$, and define $f[X]$ by,

$$f[X] = f(x_1, x_2, \cdots).$$

In general, if $u$ is any element of $\mathbf{Q}[[x_1, x_2, \cdots]]$, we write $u$ as $\sum_\alpha c_\alpha u_\alpha$ where $u_\alpha$ is a monomial with coefficient 1. Then $p_\lambda[u]$ is defined by setting

$$p_n[u] = \sum_\alpha c_\alpha u_\alpha^n$$

$$p_\lambda[u] = p_{\lambda_1}[u] \cdots p_{\lambda_n}[u]$$

for $\lambda = (\lambda_1, \cdots, \lambda_n)$. We define $f[u]$ for all symmetric functions $f$ by saying that $f[u]$ is linear in $f$.

The operation of substitution into a symmetric function has the following properties. For $\alpha$ and $\beta$ rational numbers, $(\alpha f + \beta g)[u] = \alpha f[u] + \beta g[u]$. Moreover, if $c_\alpha = 1$ for all $\alpha$, then $f[u] = f(\cdots, u_\alpha, \cdots)$.

Let $X = x_1 + x_2 + \cdots$ and $Y = y_1 + y_2 + \cdots$ be two alphabets. Define the sum of two alphabets by $X + Y = x_1 + x_2 + \cdots + y_1 + y_2 + \cdots$, and the product of two alphabets by $XY = x_1 y_1 + \cdots + x_i y_j + \cdots$. Then

$$p_n[X + Y] = p_n[X] + p_n[Y],$$

$$(23) \qquad\qquad p_n[XY] = p_n[X]p_n[Y].$$

The inner product of function in the space of symmetric functions in two infinite alphabets is defined by

$$\langle \cdot \rangle_{XY} = \langle \cdot \rangle_X \langle \cdot \rangle_Y.$$

where for any given alphabet $Z$, $\langle\ ,\ \rangle_Z$ denotes the inner product of the space of symmetric functions in $Z$.

For all partitions $\rho$, we have that $p_\rho[XY] = p_\rho[X]p_\rho[Y]$. If we rewrite (20) as $p_\rho = \sum_\lambda \chi^\lambda(\rho) s_\lambda$, then

$$(24) \qquad\qquad \sum_\lambda \chi^\lambda s_\lambda[XY] = \sum_{\mu,\nu} \chi^\mu \chi^\nu s_\mu[X] s_\nu[Y].$$

Taking the coefficient of $\chi^\lambda$ on both sides of the previous equation we obtain

$$s_\lambda[XY] = \sum \langle \chi^\lambda, \chi^\mu \chi^\nu \rangle s_\mu[X] s_\nu[Y].$$

Finally, using the definition of Kronecker coefficients (21) we obtain the comultiplication expansion (19).

NOTATION. Let $p$ be a point in $\mathbf{N}^2$. We say that $(i, j)$ can be reached from $p$, written $p \rightsquigarrow (i, j)$, if $(i, j)$ can be reached from $p$ by moving any number of steps south-west or north-west. We define the weight function $\omega$ by

$$\omega_p(i, j) = \begin{cases} x^i y^j, & \text{if } p \rightsquigarrow (i, j), \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $\sigma_{k,l}(h) = 0$ if $h < 0$.

NOTATION. We denote by $\lfloor x \rfloor$ the largest integer less than or equal to $x$ and by $\lceil x \rceil$ the smallest integer greater than or equal to $x$.

If $f$ is a formal power series, then $[x^\alpha] f$ denotes the coefficient of $x^\alpha$ in $f$.

Following Donald Knuth we denote the characteristic function applied to a proposition $P$ by enclosing $P$ with brackets,

$$(P) = \begin{cases} 1, & \text{if proposition } P \text{ is true,} \\ 0, & \text{otherwise.} \end{cases}$$

## 3. The case of two two-row shapes

The object of this section is to find a closed formula for the Kronecker coefficients when $\mu = (\mu_1, \mu_2)$ and $\nu = (\nu_1, \nu_2)$ are two-row shapes, and when we do not have any restriction on the partition $\lambda$. We describe the Kronecker coefficients $\gamma^\lambda_{\mu\nu}$ in terms of paths in $\mathbf{N}^2$. More precisely, we define two rectangular regions in $\mathbf{N}^2$ using the parts of $\lambda$. Then we count the number of points in $\mathbf{N}^2$ inside each of these rectangles that can be reached from $(\nu_2, \mu_2 + 1)$, if we are allowed to move any number of steps south-west or north-west. Finally, we subtract these two numbers.

We begin by introducing two lemmas that allow us to state Theorem 39 in a concise form.

NOTATION. We use the coordinate axes as if we were working with matrices with first entry $(0,0)$. That is, the point $(i, j)$ belongs to the $i$th row and the $j$th column.

LEMMA 36. *Let $k$ and $l$ be positive numbers. Let $R$ be the rectangle with width $k$, height $l$, and upper-left square $(0,0)$. Define*

$$\sigma_{k,l}(h) = \left| \{ (u, v) \in R \cap \mathbf{N}^2 : (0, h) \rightsquigarrow (u, v) \} \right|$$

*Then*

$$
\sigma_{k,l}(h) = \begin{cases}
0, & \text{if } h < 0 \\[2ex]
\lfloor (\tfrac{h}{2} + 1)^2 \rfloor, & \text{if } 0 \le h < \min(k,l) \\[2ex]
\sigma_{k,l}(s) + (\tfrac{h-s}{2})\min(k,l), & \text{if } \min(k,l) \le h < \max(k,l) \\[2ex]
\lceil \tfrac{kl}{2} \rceil - \sigma_{k,l}(k+l-h-4), & \text{if } h \text{ is even and } \max(k,l) \le h \\[2ex]
\lfloor \tfrac{kl}{2} \rfloor - \sigma_{k,l}(k+l-h-4), & \text{if } h \text{ is odd and } \max(k,l) \le h
\end{cases}
$$

*where $s$ is defined as follows: If $h - \min(k,l)$ is even, then $s = \min(k,l) - 2$; otherwise $s = \min(k,l) - 1$.*

PROOF. If $h$ is to the left of the 0th column, then we cannot reach any of the points in $\mathbf{N}^2$ inside $R$. Hence, $\sigma_{k,l}(h)$ should be equal to zero.

If $0 \le h \le \min(k,l)$, then we are counting the number of points in $\mathbf{N}^2$ that can be reached from $(0,h)$ inside the square $S$ of side $\min(k,l)$. We have to consider two cases. If $h$ is odd, then we are summing $2 + 4 + \cdots + (h + 1) = \lfloor (\tfrac{h}{2} + 1)^2 \rfloor$. On the other hand, if $h$ is even, then we are summing $1 + 3 + \cdots + (h + 1) = (\tfrac{h}{2} + 1)^2$.

If $\min(k,l) \le h < \max(k,l)$, then we subdivide our problem into two parts. First, we count the number of points in $\mathbf{N}^2$ that can be reached from $(0,h)$ inside the square $S$ by $\sigma_{k,l}(s)$. Then we count those points in $\mathbf{N}^2$ that are in $R$ but not in $S$. Since $h < \max(k,l)$ all diagonals have length $\min(k,l)$ and there are $\tfrac{h-s}{2}$ of them. See Table 1.

If $\max(k, l) \leq h$, then it is easier to count the total number of points in $\mathbf{N}^2$ that can be reached from $(0, h)$ inside $R$ by choosing another parameter $\hat{h}$ big enough and with the same parity as $h$. Then we subtract those points in $\mathbf{N}^2$ in $R$ that are not reachable from $(0, h)$ because $h$ is too close. If $h$ is even this number is $\lceil kl/2 \rceil$. If $h$ is odd this number is $\lfloor kl/2 \rfloor$.

Then we subtract those points that we should not have counted. we express this number in terms of the function $\sigma$. The line $y = -x + h + 2$ intersects the line $y = l - 1$ at $x = h - l + 3$. This is the $x$ coordinate of the first point on the last row that is not reachable from $(0, h)$. Then to obtain the number of points that can be reached from this point by moving south-west or north-west, we subtract $h - l + 3$ to $k - 1$. We have obtained that are $\sigma_{k,l}(k + l - h - 4)$ points that we should not have counted.                                                     $\square$

EXAMPLE 37. By definition $\sigma_{9,5}(4)$ counts the points in $\mathbf{N}^2$ in Table 1 marked with o. Then $\sigma_{9,5}(4) = 9$. Similarly, $\sigma_{9,5}(8)$ counts the points in $\mathbf{N}^2$ in Table 1 marked either with the symbol o or with the symbol •. Then $\sigma_{9,5}(8) = 19$.

LEMMA 38. *Let $a, b, c$, and $d$ be in $\mathbf{N}$. Let $R$ be the rectangle with vertices $(a, c)$, $(a + b, c)$, $(a, c + d)$, and $(a + b, c + d)$. We define*

$$\Gamma(a, b, c, d)(x, y) = \left| \{(u, v) \in R : (x, y) \rightsquigarrow (u, v)\} \right|.$$

TABLE 1

*Suppose that* $(x, y)$ *is such that* $x \geq y$. *Then*

$$\Gamma(a, b, c, d)(x, y) = \begin{cases} \sigma_{b+1,d+1}(x + y - a - c), & 0 \leq y \leq c \\[2mm] \sigma_{b+1,y-c+1}(x - a) + \sigma_{b+1,c+d-y+1}(x - a) - \delta, & c < y < c + d \\[2mm] \sigma_{b+1,d+1}(x - y + c + d - a), & c + d \leq y \end{cases}$$

*where* $\delta$ *is defined as follows If* $x < a$, *then* $\delta = 0$. *If* $a \leq x \leq a + b$, *then* $\delta = \left\lceil \frac{x-a+1}{2} \right\rceil$. *Finally, if* $x > a + b$ *then we consider two cases: If* $x - a - b$ *is even then* $\delta = \left\lceil \frac{b+1}{2} \right\rceil$; *otherwise,* $\delta = \left\lfloor \frac{b+1}{2} \right\rfloor$.

PROOF. We consider three cases. If $0 \leq y \leq c$ then the first position inside $R$ that we reach is $(x + y - a - c, c)$. Therefore, we assume that we are starting at this point. Similarly, if $y \geq c + d$, then the first position inside $R$ that we reach is $(x - y + c + d - a, c)$. Again, we can assume that we are starting at this point.

On the other hand, if $c < y < c + d$, then we subdivide the problem in two parts. The number of position to the north of us is counted by $\sigma_{b+1,y-c+1}(x-a)$.

The number of position to the south of us is counted by $\sigma_{b+1,c+d-y+1}(x-a)$.

We define $\delta$ to be the number of points in $\mathbf{N}^2$ that we counted twice during this process. Then it is easy to see that $\delta$ is given by the previous definition.    $\square$

To compute the coefficient $u_\nu$ in the expansion $f[X] = \sum_\eta u_\eta s_\eta[X]$ for $f \in \Lambda$. it is enough to expand $f[x_1 + \cdots + x_n] = \sum_\eta u_\eta s_\eta[x_1 + \cdots + x_n]$ for any $n \geq l(\nu)$. (See [7, section I.3], for proofs and details.) Therefore, in this section we work with symmetric functions in a finite number of variables.

Let $\mu$ and $\nu$ be two-row partitions. Set $X = 1 + x$ and $Y = 1 + y$ in the comultiplication expansion (19) to obtain

$$(25) \qquad s_\lambda[(1+y)(1+x)] = \sum_{\mu,\nu} \gamma_{\mu\nu}^\lambda s_\mu[1+y] s_\nu[1+x].$$

Note that the Kronecker coefficients are zero when $l(\lambda) > 4$.

Jacobi's definition of a Schur function on a finite alphabet $s_\lambda[X]$ as a quotient of alternants says that

$$(26) \qquad s_\lambda[X] = s_\lambda(x_1, \cdots, x_n) = \frac{\det(x_i^{\lambda_j + n - j})_{1 \leq i,j \leq n}}{\prod_{i<j}(x_i - x_j)}.$$

By the symmetry properties of the Kronecker product it is enough to compute the Kronecker coefficients $\gamma_{\mu\nu}^\lambda$ when $\nu_2 \leq \mu_2$.

THEOREM 39. *Let $\mu$, $\nu$, and $\lambda$ be partitions of $n$, where $\mu = (\mu_1, \mu_2)$ and $\nu = (\nu_1, \nu_2)$ are two two-row partitions and let $\lambda = (\lambda_1, \lambda_2, \lambda_4, \lambda_4)$ be a partition of length less than or equal to 4. Assume that $\nu_2 \leq \mu_2$. Then*

$$\gamma_{\mu\nu}^\lambda = \big(\Gamma(a,b,a+b+1,c) - \Gamma(a,b,a+b+c+d+2,c)\big)(\nu_2, \mu_2 + 1).$$

*where* $a = \lambda_3 + \lambda_4$, $b = \lambda_2 - \lambda_3$, $c = \min(\lambda_1 - \lambda_2, \lambda_3 - \lambda_4)$ *and* $d = |\lambda_1 + \lambda_4 - \lambda_2 - \lambda_3|$.

PROOF. We expand the polynomial $s_\lambda[(1+y)(1+x)] = s_\lambda(1, y, x, xy)$ in two different ways and obtain the Kronecker coefficients by equating both results. Let $\varphi$ be the polynomial defined by $\varphi = (1 - x)(1 - y)s_\lambda(1, y, x, xy)$. Using Jacobi's definition of a Schur function we obtain

$$(27) \qquad \varphi = \frac{\begin{vmatrix} 1 & 1 & 1 & 1 \\ y^{\lambda_1+3} & y^{\lambda_2+2} & y^{\lambda_3+1} & y^{\lambda_4} \\ x^{\lambda_1+3} & x^{\lambda_2+2} & x^{\lambda_3+1} & x^{\lambda_4} \\ (xy)^{\lambda_1+3} & (xy)^{\lambda_2+2} & (xy)^{\lambda_3+1} & (xy)^{\lambda_4} \end{vmatrix}}{xy(1 - xy)(y - x)(1 - x)(1 - y)}.$$

On the other hand, we may use Jacobi's definition to expand $s_\mu[1 + y]$ and $s_\nu[1 + x]$ as quotients of alternants. Substitute this into (25):

$$s_\lambda[(1 + y)(1 + x)] = \sum_{\substack{\mu=(\mu_1,\mu_2) \\ \nu=(\nu_1,\nu_2)}} \gamma^\lambda_{\mu\nu} \left( \frac{y^{\mu_2} - y^{\mu_1+1}}{1 - y} \right) \left( \frac{x^{\nu_2} - x^{\nu_1+1}}{1 - x} \right)$$

$$(28) \qquad = \sum_{\substack{\mu=(\mu_1,\mu_2) \\ \nu=(\nu_1,\nu_2)}} \gamma^\lambda_{\mu\nu} \frac{x^{\nu_2} y^{\mu_2} - x^{\nu_2} y^{\mu_1+1} - x^{\nu_1+1} y^{\mu_2} + x^{\nu_1+1} y^{\mu_1+1}}{(1 - x)(1 - y)}.$$

Since $\nu_1 + 1$ and $\mu_1 + 1$ are both greater than $\lfloor \frac{n}{2} \rfloor$, equation (28) implies that *the coefficient of* $x^{\nu_2} y^{\mu_2}$ *in* $\varphi$ *is* $\gamma^\lambda_{\mu\nu}$. It is convenient to define an auxiliary polynomial by

$$(29) \qquad \zeta = (1 - xy)(y - x)\varphi.$$

Let $\xi$ be the polynomial obtained by expanding the determinant appearing in
(27). Equations (27) and (29) imply

$$\zeta = \frac{\xi}{xy(1-x)(1-y)}.$$

Let $\xi_{i,j}$ be the coefficient of $x^i y^j$ in $\xi$. ( Then $\xi_{i,j}$ is zero if $i \leq 0$ or $j \leq 0$,
because $\xi$ is a polynomial divisible by $xy$.) Let $\zeta_{i,j}$ be the coefficient of $x^i y^j$ in
$\zeta$. Then

$$(30) \qquad \sum_{i,j \geq 0} \zeta_{i,j} x^i y^j = \frac{1}{xy(1-x)(1-y)} \sum_{i,j \geq 0} \xi_{i,j} x^i y^j = \sum_{i,j,k,l \geq 0} \xi_{i-k,j-l} x^{i-1} y^{j-1}.$$

Comparing the coefficient of $x^i y^j$ on both sides of equation (30) we obtain that

$$(31) \qquad \zeta_{i,j} = \sum_{k,l \geq 0} \xi_{i+1-k,j+1-l} = \sum_{k=0}^{i} \sum_{l=0}^{j} \xi_{k+1,l+1}$$

We compute $\zeta_{i,j}$ from (31) by expanding the determinant appearing on (27).
We consider two cases.

**Case 1.** Suppose that $\lambda_1 + \lambda_4 > \lambda_2 + \lambda_3$. Then

$$\lambda_1 + \lambda_2 + 4 > \lambda_1 + \lambda_3 + 3 > \lambda_1 + \lambda_4 + 2 \geq \lambda_2 + \lambda_3 + 2 > \lambda_2 + \lambda_4 + 1 > \lambda_3 + \lambda_4.$$

We record the values of $\xi_{j+1,i+1}$ in Table 2. We use the convention that $\xi_{i+1,j+1}$
is zero whenever the $(i,j)$ entry is not in Table 2.

Equation (31) shows that the value of $\zeta_{i,j}$ can be obtained by adding the
entries northwest of the point $(i,j)$. In Table 3 we record the values of $\zeta_{i,j}$.
where

$$I_1 = [0, \lambda_3 + \lambda_4), \qquad\qquad I_2 = [\lambda_3 + \lambda_4, \lambda_2 + \lambda_4],$$

| $i \backslash j$ | $\lambda_3 + \lambda_4$ | $\lambda_2 + \lambda_4 + 1$ | $\lambda_2 + \lambda_3 + 2$ | $\lambda_1 + \lambda_4 + 2$ | $\lambda_1 + \lambda_3 + 3$ | $\lambda_1 + \lambda_2 + 4$ |
|---|---|---|---|---|---|---|
| $\lambda_3 + \lambda_4$ | 0 | $-1$ | $+1$ | $+1$ | $-1$ | 0 |
| $\lambda_2 + \lambda_4 + 1$ | $+1$ | 0 | $-1$ | $-1$ | 0 | $+1$ |
| $\lambda_2 + \lambda_3 + 2$ | $-1$ | $+1$ | 0 | 0 | $+1$ | $-1$ |
| $\lambda_1 + \lambda_4 + 2$ | $-1$ | $+1$ | 0 | 0 | $+1$ | $-1$ |
| $\lambda_1 + \lambda_3 + 3$ | $+1$ | 0 | $-1$ | $-1$ | 0 | $+1$ |
| $\lambda_1 + \lambda_2 + 4$ | 0 | $-1$ | $+1$ | $+1$ | $-1$ | 0 |

TABLE 2. The values of $\xi_{j+1,i+1}$ when $\lambda_1 + \lambda_4 \geq \lambda_2 + \lambda_3$

| $i \backslash j$ | $I_1$ | $I_2$ | $I_3$ | $I_4$ | $I_5$ | $I_6$ | $I_7$ |
|---|---|---|---|---|---|---|---|
| $I_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $I_2$ | 0 | 0 | $-1$ | 0 | $+1$ | 0 | 0 |
| $I_3$ | 0 | $+1$ | 0 | 0 | 0 | $-1$ | 0 |
| $I_4$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $I_5$ | 0 | $-1$ | 0 | 0 | 0 | $+1$ | 0 |
| $I_6$ | 0 | 0 | $+1$ | 0 | $-1$ | 0 | 0 |
| $I_7$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

TABLE 3. The values of $\zeta_{i,j}$ when $\lambda_1 + \lambda_4 \geq \lambda_2 + \lambda_3$

$$I_3 = [\lambda_2 + \lambda_4 + 1, \lambda_2 + \lambda_3 + 1], \qquad I_4 = [\lambda_2 + \lambda_3 + 2, \lambda_1 + \lambda_4 + 1],$$

$$I_5 = [\lambda_1 + \lambda_4 + 2, \lambda_1 + \lambda_3 + 2], \qquad I_6 = [\lambda_1 + \lambda_3 + 3, \lambda_1 + \lambda_2 + 3],$$

$$I_7 = [\lambda_1 + \lambda_2 + 4, \infty).$$

**Case 2.** Suppose that $\lambda_1 + \lambda_4 \leq \lambda_2 + \lambda_3$. Then

$$\lambda_1 + \lambda_2 + 4 > \lambda_1 + \lambda_3 + 3 > \lambda_2 + \lambda_3 + 2 > \lambda_1 + \lambda_4 + 2 > \lambda_2 + \lambda_4 + 1 > \lambda_3 + \lambda_4.$$

Note that in Table 3, the rows and columns corresponding to $\lambda_1 + \lambda_4 + 2$ and $\lambda_2 + \lambda_3 + 2$ are the same. Therefore, the values of $\xi_{i,j}$ for $\lambda_1 + \lambda_4 \leq \lambda_2 + \lambda_3$ are recorded in Table 3, if we set

$$I_3 = [\lambda_2 + \lambda_4 + 1, \lambda_1 + \lambda_4 + 1]$$

$$I_4 = [\lambda_2 + \lambda_4 + 2, \lambda_2 + \lambda_3 + 1]$$

$$I_5 = [\lambda_2 + \lambda_3 + 2, \lambda_1 + \lambda_3 + 2],$$

and define the other intervals as before.

In both cases, let $\varphi_{i,j}$ be the coefficient of $x^i y^j$ in $\varphi$. Using (29) we obtain that

$$\varphi = \frac{1}{(1 - xy)(y - x)} \sum_{i,j \geq 0} \zeta_{i,j} x^i y^j$$

(32)
$$= \frac{1}{y - x} \sum_{i,j,l \geq 0} \zeta_{i-l,j-l} x^i y^j$$

$$= \sum_{i,j,k,l \geq 0} \zeta_{i-k-l,j+k-l+1} x^i y^j.$$

(Note: We can divide by $y - x$ because $\varphi = 0$ when $x = y$.) Comparing the coefficients of $x^i y^j$ on both sides of equation (32), we obtain $\varphi_{i,j} =$

| | | | | |
|---|---|---|---|---|
| | | | | |
| o | | | | |
| | o | | | |
| o | | o | | |
| | o | | | |
| o | | | | |

TABLE 4

$\sum_{k,l \geq 0} \zeta_{i-k-l,j+k-l+1}$. Therefore,

$$(33) \qquad \varphi_{\nu_2,\mu_2} = \sum_{i,j=0}^{\nu_2} \zeta_{\nu_2-i-j,\mu_2+i-j+1}.$$

We have shown that $\gamma_{\mu\nu}^{\lambda} = \varphi_{\nu_2,\mu_2}$ can be obtained by adding the entries in Table 3 in all points in $\mathbf{N}^2$ that can be reached from $(\nu_2, \mu_2 + 1)$. See Table 4.

The right-most point in Table 4 has coordinates $(3,2)$.

By hypothesis $\nu_2 \leq \mu_2 \leq \lfloor n/2 \rfloor$. Then, if we start at $(\nu_2, \mu_2 + 1)$ and move as previously described, the only points in $\mathbf{N}^2$ that we can possibly reach and that are nonzero in Table 3 are those in $I_2 \times I_3$ or $I_2 \times I_5$. Hence, we have that $\varphi_{\nu_2,\mu_2}$ is the number of points in $\mathbf{N}^2$ inside $I_2 \times I_3$ that can be reached from $(\nu_2, \mu_2 + 1)$ minus the ones that can be reached in $I_2 \times I_5$.

**Case 1** The inequality $\lambda_1 + \lambda_4 > \lambda_2 + \lambda_3$ implies that $\lambda_1 + \lambda_4 + 1 > \lfloor \frac{n}{2} \rfloor$. Moreover, $\mu_2 \geq \nu_2$ implies that we are only considering the region of $\mathbf{N}^2$ given by $0 \leq i \leq j \leq \lfloor \frac{n}{2} \rfloor$. The number of points in $\mathbf{N}^2$ that can be reached from

$(\nu_2, \mu_2 + 1)$ inside $I_2 \times I_3$ is given by $\Gamma(\lambda_3 + \lambda_4, \lambda_2 - \lambda_3, \lambda_2 + \lambda_4 + 1, \lambda_3 - \lambda_4)$.

Similarly, the number of points in $\mathbf{N}^2$ that can be reached from $(\nu_2, \mu_2 + 1)$ inside $I_2 \times I_5$ is given by $\Gamma(\lambda_3 + \lambda_4, \lambda_2 - \lambda_3, \lambda_1 + \lambda_4 + 2, \lambda_3 - \lambda_4)$.

**Case 2** The inequality $\lambda_2 + \lambda_3 \geq \lambda_1 + \lambda_4$, implies that $\lambda_1 + \lambda_4 + 1 > \lfloor \frac{n}{2} \rfloor$.

Moreover, $\mu_2 \geq \nu_2$ implies that we are only considering the region of $\mathbf{N}^2$ given by $0 \leq i \leq j \leq \lfloor \frac{n}{2} \rfloor$. The number of points in $\mathbf{N}^2$ that can be reached from $(\nu_2, \mu_2 + 1)$ inside $I_2 \times I_3$ is given by $\Gamma(\lambda_3 + \lambda_4, \lambda_2 - \lambda_3, \lambda_2 + \lambda_4 + 1, \lambda_1 - \lambda_2)$.

Similarly, the number of points in $\mathbf{N}^2$ that can be reached from $(\nu_2, \mu_2 + 1)$ inside $I_2 \times I_5$ is given by $\Gamma(\lambda_3 + \lambda_4, \lambda_2 - \lambda_3, \lambda_2 + \lambda_3 + 2, \lambda_1 - \lambda_2)$. $\qquad\square$

COROLLARY 40. *Let $\mu = (\mu_1, \mu_2)$, $\nu = (\nu_1, \nu_2)$, and $\lambda = (\lambda_1, \lambda_2)$ be partitions of $n$. Assume that $\nu_2 \leq \mu_2 \leq \lambda_2$. Then*

$$\gamma_{\mu\nu}^{\lambda} = (y - x)(y \geq x),$$

*where $x = \max\left(0, \left\lceil \frac{\mu_2 + \nu_2 + \lambda_2 - n}{2} \right\rceil\right)$ and $y = \left\lceil \frac{\mu_2 + \nu_2 - \lambda_2 + 1}{2} \right\rceil$.*

PROOF. Set $\lambda_3 = \lambda_4 = 0$ in Theorem 39. Then we notice that the second possibility in the definition of $\Gamma$, that is, when $c < y < c + d$, never occurs. Note that $\nu_2 + \mu_2 - \lambda_2 \geq \nu_2 + \mu_2 - \lambda_1 - 1$ for all partitions $\mu, \nu$, and $\lambda$. Therefore,

$$\gamma_{\mu\nu}^{\lambda} = \sigma_{\lambda_2+1,1}(\nu_2 + \mu_2 - \lambda_2) - \sigma_{\lambda_2+1,1}(\nu_2 + \mu_2 - \lambda_1 - 1).$$

Suppose that $\nu_2 + \mu_2 - \lambda_2 < 0$. From the definition of $\sigma_{\lambda_2+1,1}$ we obtain that $\gamma_{\mu\nu}^{\lambda} = 0$. Therefore, in order to have $\gamma_{\mu\nu}^{\lambda}$ not equal to zero, we should assume that $\nu_2 + \mu_2 - \lambda_2 \geq 0$.

If $0 \le \nu_2 + \mu_2 - \lambda_2 < \lambda_2 + 1$, then

$$\sigma_{\lambda_2+1,1}(\nu_2 + \mu_2 - \lambda_2) = \left\lceil \frac{\nu_2 + \mu_2 - \lambda_2 + 1}{2} \right\rceil$$

Similarly, if $0 \le \nu_2 + \mu_2 - \lambda_2 < \lambda_2 + 1$, then

$$\sigma_{\lambda_2+1,1}(\nu_2 + \mu_2 - \lambda_1 - 1) = \left\lceil \frac{\nu_2 + \mu_2 + \lambda_2 - n}{2} \right\rceil$$

It is easy to see that all other cases on the definition of $\sigma_{k,l}$ can not occur. Therefore, defining $x$ and $y$ as above, we obtain the desired result.          $\square$

EXAMPLE 41. If $\mu = \nu = \lambda = (l,l)$ or $\mu = \nu = (2l,2l)$ and $\lambda = (3l,l)$, then from the previous corollary, we obtain that

$$\gamma_{\mu\nu}^{\lambda} = \left\lceil \frac{l+1}{2} \right\rceil - \left\lceil \frac{l}{2} \right\rceil = (l \text{ is even})$$

Note that to apply Corollary 40 to the second family of shapes, we should first use the symmetries of the Kronecker product to set $\nu = \lambda = (2l,2l)$ and $\mu = (3l,l)$.

COROLLARY 42. *The Kronecker coefficients $\gamma_{\mu\nu}^{\lambda}$, where $\mu$ and $\nu$ are two-row partitions, are unbounded.*

PROOF. It is enough to construct an unbounded family of Kronecker coefficients. Assume that $\mu = \nu = \lambda = (3l,l)$. Then from the previous corollary we obtain that

$$\gamma_{\mu\nu}^{\lambda} = \left\lceil \frac{l+1}{2} \right\rceil$$

□

## 4. Sergeev's formula

In this section we state Sergeev's formula for a Schur function of a difference of two alphabets. See [3] or [7, section I.3] for proofs and comments.

DEFINITION 43. Let $X_m = x_1 + \cdots + x_m$ be a finite alphabet, and let $\delta_m = (m - 1, m - 2, \cdots, 1, 0)$. We define $X_m^{\delta_m}$ by $X_m^{\delta_m} = x_1^{m-1} \cdots x_{m-1}$.

DEFINITION 44. Let $i(\alpha)$ denote the number of inversions of the permutation $\alpha$. We define the alternant to be

$$A_m^x P = \sum_{\alpha \in S_m} (-1)^{i(\alpha)} P(x_{\alpha(1)}, \cdots, x_{\alpha(m)}),$$

for any polynomial $P(x_1, \cdots, x_n)$.

DEFINITION 45. Let $\Delta$ be the operation of taking the Vandermonde determinant of an alphabet, i.e..

$$\Delta(X_m) = \det(x_i^{m-j})_{i,j=1}^m.$$

THEOREM 46 (Sergeev's Formula). *Let* $X_m = x_1 + \cdots + x_m$, *and* $Y_n = y_1 + \cdots + y_n$ *be two alphabets. Then*

$$s_\lambda[X_m - Y_n] = \frac{1}{\Delta(X_m)\Delta(Y_n)} A_m^x A_n^y X_m^{\delta_m} Y_n^{\delta_n} \prod_{(i,j)\in\lambda} (x_i - y_j)$$

The notation $(i,j) \in \lambda$ means that the point $(i,j)$ belongs to the diagram of $\lambda$. We set $x_i = 0$ for $i > m$ and $y_j = 0$ for $j > n$.

We use Sergeev's formula as a tool for making some calculations we need for the next two sections.

1. Let $\mu = (1^{e_1}m_2)$ be a hook. (We are assuming that $e_1 \geq 1$ and $m_2 \geq 2$.) Let $X^1 = \{x_1\}$ and $X^2 = \{x_2\}$.

$$(34) \qquad s_\mu[x_1 - x_2] = (-1)^{e_1} x_1^{m_2-1} x_2^{e_1}(x_1 - x_2).$$

2. Let $\nu = (\nu_1, \nu_2)$ be a two-row partition. Let $Y = \{y_1, y_2\}$. Then

$$(35) \qquad s_\nu[y_1 + y_2] = \frac{(y_1 y_2)^{\nu_2}(y_1^{\nu_1-\nu_2+1} - y_2^{\nu_1-\nu_2+1})}{y_1 - y_2}.$$

3. We say that a partition $\lambda$ is a double hook if $(2,2) \in \lambda$ and it has the form $\lambda = (1^{d_1}2^{d_2}n_3 n_4)$. In particular any two-row shape is considered to be a double hook.

Let $\lambda$ be a double hook. Let $U = \{u_1, u_2\}$ and $V = \{v_1, v_2\}$. Then if $n_4 \neq 0$ then $s_\lambda[u_1 + u_2 - v_1 - v_2]$ equals

$$(36) \qquad \frac{(u_1 - v_1)(u_2 - v_1)(u_1 - v_2)(u_2 - v_2)}{(u_1 - u_2)(v_1 - v_2)}(-1)^{d_1}(u_1 u_2)^{n_3-2}(v_1 v_2)^{d_2}$$

$$\times \left(u_2^{n_4-n_3+1} - u_1^{n_4-n_3+1}\right)\left(v_2^{d_1+1} - v_1^{d_1+1}\right).$$

On the other hand, if $n_4 = 0$ then to compute $s_\lambda[u_1 + u_2 - v_1 - v_2]$ we

should write $\lambda$ as $(1^{d_1}2^{d_2-1}2\,n_3)$. That is, we set $d_1 := d_1$, $d_2 := d_2 - 1$,

$n_3 := 2$, and $n_4 = n_3$ in (36).

4. Let $\lambda$ be a hook shape, $\lambda = (1^{d_1}n_2)$. (We are assuming that $d_1 \geq 1$ and

$n_2 \geq 2$.) Let $U = \{u_1, u_2\}$ and $V = \{v_1, v_2\}$. Then $s_\lambda[u_1 + u_2 - v_1 - v_2]$

equals

$$(37) \quad (-1)^{d_1-1}\frac{1}{(u_1 - u_2)}\frac{1}{(v_1 - v_2)} \times$$

$$\Big\{ u_1 v_1(u_1 - v_1)(u_1 - v_2)(u_2 - v_1)u_1^{n_2-2}v_1^{d_1-1}$$

$$- u_1 v_2(u_1 - v_2)(u_1 - v_1)(u_2 - v_2)u_1^{n_2-2}v_2^{d_1-1}$$

$$- u_2 v_1(u_2 - v_1)(u_2 - v_2)(u_1 - v_1)u_2^{n_2-2}v_1^{d_1-1}$$

$$+ u_2 v_2(u_2 - v_2)(u_2 - v_1)(u_1 - v_2)u_2^{n_2-2}v_2^{d_1-1}. \Big\}$$

## 5. The case of two hook shapes

In this section we derive an explicit formula for the Kronecker coefficients

$\gamma_{\mu\nu}^\lambda$ in the case in which $\mu = (1^e u)$, and $\nu = (1^f v)$ are both hook shapes. Given

a partition $\lambda$ the Kronecker coefficient $\gamma_{\mu\nu}^\lambda$ tells us whether point $(u, v)$ belongs

to some regions in $\mathbf{N}^2$ determined by $\mu$, $\nu$ and $\lambda$.

Recall that we denote the characteristic function by enclosing a proposition

$P$ with brackets, $(P)$.

LEMMA 47. *Let* $(u, v) \in \mathbb{N}^2$ *and let* $R$ *be the rectangle with vertices* $(a, b)$,

$(b, a)$, $(c, d)$, *and* $(d, c)$, *with* $a \geq b$, $c \geq d$, $c \geq a$ *and* $d \geq b$. *(Sometimes, when*

$c = d = e$, *we denote this rectangle as* $(a, b; e)$.*)*

*Then* $(u, v) \in R$ *if and only if* $|v - u| \leq a - b$ *and* $a + b \leq u + v \leq c + d$

PROOF. Let $L_1$ be the line of slope 1 passing through $(u, v)$, and let $L_2$ be

the line of slope $-1$ passing through $(u, v)$. Then we have that

$$L_1 : y = x + v - u$$

$$L_2 : y = -x + u + v$$

The point $(u, v)$ is in $R$ if and only if $L_1$ is between the lines of slope 1 passing

throught $(a, b)$ and $(b, a)$. That is, $a - b \leq v - u \leq b - a$ and $L_2$ is between the

lines of slope $-1$ passing throught $(a, b)$ and $(c, d)$. That is, $a + b \leq u + v \leq$

$c + d$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

THEOREM 48. *Let* $\lambda$, $\mu$ *and* $\nu$ *be partitions of* $n$, *where* $\mu = (1^e u)$ *and*

$\nu = (1^f v)$ *are hook shapes. Then the Kronecker coefficients* $\gamma_{\mu\nu}^\lambda$ *are given by*

*the following:*

1. *If* $\lambda$ *is a one-row shape, then* $\gamma_{\mu\nu}^\lambda = \delta_{\mu,\nu}$.

2. *If* $\lambda$ *is not contained in a double hook shape, then* $\gamma_{\mu\nu}^\lambda = 0$.

3. *Let* $\lambda = (1^{d_1}2^{d_2}n_3n_4)$ *be a double hook. Let* $x = 2d_2 + d_1$. *Then*

$$\gamma_{\mu\nu}^\lambda = (n_3 - 1 \leq \frac{e + f - x}{2} \leq n_4)(|f - e| \leq d_1)$$

$$+ (n_3 \leq \frac{e + f - x + 1}{2} \leq n_4)(|f - e| \leq d_1 + 1).$$

*Note that if* $n_4 = 0$, *then we shall rewrite* $\lambda = (1^{d_1}2^{d_2-1}2\,n_3)$ *before using the previous formula.*

4. *Let* $\lambda = (1^d w)$ *be a hook shape. Suppose that* $e \leq u$, $f \leq v$, *and* $d \leq w$. *Then*

$$\gamma_{\mu\nu}^\lambda = (e \leq d + f)(d \leq e + f)(f \leq e + d).$$

PROOF. Set $X = \{1, x\}$ and $Y = \{1, y\}$ in the comultiplication expansion (19) to obtain

$$(38) \qquad s_\lambda[(1 - x)(1 - y)] = \sum_{\mu,\nu} \gamma_{\mu\nu}^\lambda s_\mu[1 - x]s_\nu[1 - y].$$

We use equation (34) to replace $s_\mu$ and $s_\nu$ in the right hand side of (38). Then we divide the resulting equation by $(1 - x)(1 - y)$ to get

$$\frac{s_\lambda[1 - y - x + xy]}{(1 - x)(1 - y)} = \sum_{\mu,\nu} \gamma_{\mu\nu}^\lambda (-x)^e(-y)^f.$$

Therefore,

$$\gamma_{\mu\nu}^\lambda = \left[(-x)^e(-y)^f\right]\frac{s_\lambda[1 - y - x + xy]}{(1 - x)(1 - y)},$$

when $\mu$ and $\nu$ are hook shapes.

**Case 1.** If $\lambda$ is not contained in any double hook, then the point $(3, 3)$ is in $\lambda$, and by Sergeev's formula, $s_\lambda[1 - y - x + xy]$ equals zero.

**Case 2.** Let $\lambda = (1^{d_1} 2^{d_2} n_3\, n_4)$ be a double hook. Set $u_1 = 1$, $u_2 = xy$, $v_1 = x$, and $v_2 = y$ in (36). Then we divide by $(1 - x)(1 - y)$ on both sides of the resulting equation to obtain

$$(39) \quad \frac{s_\lambda[1 - y - x + xy]}{(1 - x)(1 - y)} = (-1)^{d_1} (xy)^{n_3 + d_2 - 1}$$

$$\times (1 - x)(1 - y) \left( \frac{1 - (xy)^{n_4 - n_3 + 1}}{1 - xy} \right) \left( \frac{x^{d_1 + 1} - y^{d_1 + 1}}{x - y} \right).$$

Note: If $n_4 = 0$ then we should write $\lambda = (1^{d_1} 2^{d_2 - 1} 2\, n_4)$ in order to use (36).

Let $p$ be a point in $\mathbf{N}^2$. We say that $(i, j)$ can be reached from $p$, written $p \leadsto (i, j)$, if $(i, j)$ can be reached from $p$ by moving any number of steps south-west or north-west. We have defined a weight function by

$$\omega_p(i, j) = \begin{cases} x^i y^j, & \text{if } p \leadsto (i, j); \\ \\ 0, & \text{otherwise.} \end{cases}$$

Let $\omega_p(T) = \sum_{(i,j) \in T} \omega_p(i, j)$ be the generating function of a region $T$ in $\mathbf{N}^2$. Let $R$ be the rectangle with vertices $(0, d_1)$, $(d_1, 0)$, $(d_1 + n_4 - n_3, n_4 - n_3)$ and $(n_4 - n_3, d_1 + n_4 - n_3)$. Then

$$\omega_{(d_1 + n_4 - n_3, n_4 - n_3)}(R) = \left( \frac{1 - (xy)^{n_4 - n_3 + 1}}{1 - xy} \right) \left( \frac{x^{d_1 + 1} - y^{d_1 + 1}}{x - y} \right)$$

$$= \sum_{k=0}^{n_4 - n_3} \sum_{i+j=d_1} (xy)^k x^i y^j.$$

See Table 5.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 1 | | | | |
| | | | 1 | | 1 | | | |
| | | 1 | | 1 | | 1 | | |
| | 1 | | 1 | | 1 | | 1 | |
| 1 | | 1 | | 1 | | 1 | | 1 |
| | 1 | | 1 | | 1 | | 1 | |
| | | 1 | | 1 | | 1 | | |
| | | | 1 | | 1 | | | |
| | | | | 1 | | | | |

TABLE 5. $d_1 = 4$ and $n_3 - n_4 = 4$

Recall that we are using matrix coordinates, and that the upper-left corner has coordinates $(0,0)$. The coordinates of the four vertices of $R$ in Table 5 are $(0,4)$, $(4,0)$, $(8,4)$, and $(4,8)$.

We interpret the right-hand side of (39) as the sum of four different generating functions. To be more precise, the right-hand side of (39) can be written as $\sum_{i=1}^{4} \omega_{p_i}(r_i)$ where $p_1 = (n_4 + d_2 - 1, n_4 + d_2 + d_1 - 1)$ and $R_1 = \{n_3 + d_2 + d_1 - 1, n_3 + d_2 - 1; n_4 - n_3\}$, $p_2 = (n_4 + d_2, n_3 4 + d_2 + d_1 - 1)$ and $R_2 = \{n_3 + d_2 + d_1, n_3 + d_2 - 1; n_4 - n_3\}$, $p_3 = (n_4 + d_2 - 1, n_4 + d_2 + d_1)$ and $R_3 = \{n_3 + d_2 + d_1 - 1, n_3 + d_2; n_4 - n_3\}$, and $p_4 = (n_4 + d_2 + d_1, n_4 + d_2 + d_1)$ and $R_4 = \{n_3 + d_2 + d_1, n_3 + d_2 + d_1; n_4 - n_3\}$.

We observe that $R_1 \cup R_2$ (and $R_3 \cup R_4$ ) are rectangles in $\mathbb{N}^2$. Moreover,

$$(40) \qquad \gamma_{\mu\nu}^\lambda = ((e, f) \in R_1 \cup R_2) + ((e, f) \in R_3 \cup R_4).$$

The vertices of rectangle $R_1 \cup R_4$ are given (using the notation of Lemma 47)
by

$$a = n_3 + d_2 + d_1 - 1 \qquad\qquad b = n_3 + d_2 - 1$$

$$c = n_4 + d_2 + d_1 \qquad\qquad d = n_4 + d_2$$

Similarly, the vertices of rectangle $R_2 \cup R_3$ are given by

$$a = n_3 + d_2 + d_1 \qquad\qquad b = n_3 + d_2 - 1$$

$$c = n_4 + d_2 + d_1 \qquad\qquad d = n_4 + d_2 - 1$$

Applying Lemma 47 to (40) we obtain

$$\gamma_{\mu\nu}^\lambda = (n_3 - 1 \le \frac{e + f - x}{2} \le n_4)(|f - e| \le d_1)$$

$$+ (n_3 \le \frac{e + f - x + 1}{2} \le n_4)(|f - e| \le d_1 + 1).$$

**Case 3. $\lambda$ is a hook.** Suppose that $\lambda$ is a hook, $\lambda = (1^d w)$. Set $u_1 = 1$,

$u_2 = xy$, $v_1 = x$, and $v_2 = y$ in (37). Then we divide by $(1 - x)(1 - y)$ on both

sides of the resulting equation to obtain

$$(41) \quad \frac{s_\lambda[1 - y - x + xy]}{(1-x)(1-y)} = (-1)^d \left( \frac{x^{d+1} - y^{d+1}}{x - y} \right) \left( \frac{1 - (xy)^w}{1 - xy} \right)$$

$$+ (-1)^{d-1} xy \left( \frac{x^d - y^d}{x - y} \right) \left( \frac{1 - (xy)^{w-1}}{1 - xy} \right).$$

We want to interpret this equation as a generating function for a region $T$ using the weight $\omega$. We proceed as follows:

Let $R_1$ be the rectangle with vertices $(d, 0)$, $(0, d)$, $(d + w - 1, w - 1)$, and $(w - 1, d + w - 1)$. Then

$$(42) \quad \omega_{(w-1,d+w-1)}(R_1) = \left( \frac{1 - (xy)^w}{1 - xy} \right) \left( \frac{x^{d+1} - y^{d+1}}{x - y} \right) = \sum_{k=0}^{w-1} \sum_{i+j=d} (xy)^k x^i y^j.$$

(See Table 5.) Similarly, let $R_2$ be the rectangle with vertices $(d, 1)$, $(1, d)$, $(d + w - 2, w - 1)$, and $(w - 1, d + w - 2)$. Then

$$(43)$$

$$\omega_{(w-1,d+w-2)}(R_2) = xy \left( \frac{1 - (xy)^{w-1}}{1 - xy} \right) \left( \frac{x^d - y^d}{x - y} \right) = xy \sum_{k=0}^{w-2} \sum_{i+j=d-1} (xy)^k x^i y^j.$$

Observe that the points in $\mathbf{N}^2$ that can be reached from $(0, d)$ in $R_1$ and the points in $\mathbf{N}^2$ that can be reached from $(1, d)$ in $R_2$ are disjoint. Moreover, they completely fill the rectangle $R_1 \cup R_2$. See Table 6.

Note that $R_2$ is contained in $R_1$. We obtain that

$$\omega_{(w-1,d+w-1)}(R_1) + \omega_{(w-1,d+w-2)}(R_2) = |(e, f) \in R_1|$$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | 1 | | | | | |
| | | | 1 | −1 | 1 | | | | |
| | | 1 | −1 | 1 | −1 | 1 | | | |
| | 1 | −1 | 1 | −1 | 1 | −1 | 1 | | |
| 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | |
| | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 |
| | | 1 | −1 | 1 | −1 | 1 | −1 | 1 | |
| | | | 1 | −1 | 1 | −1 | 1 | | |
| | | | | 1 | −1 | 1 | | | |
| | | | | | 1 | | | | |

TABLE 6. $d = 4, w = 6$.

We use apply Lemma 47 to the previous equation to obtain:

$$(|e - f| \leq d)(d \leq e + f \leq d + 2w - 2).$$

But, by hypothesis. $e \leq u$, $f \leq v$, and $d \leq w$. Therefore, this system is equivalent to $(d \leq e + f)(f \leq e + d)(e \leq d + f)$, as desired. $\square$

COROLLARY 49. *Let* $\lambda$, $\mu$, *and* $\nu$ *be partitions of* $n$, *where* $\mu = (1^e u)$ *and* $\nu = (1^f v)$ *are hook shapes and* $\lambda = (\lambda_1, \lambda_2)$ *is a two-row shape. Then the Kronecker coefficients* $\gamma^\lambda_{\mu\nu}$ *are given by*

$$\gamma^\lambda_{\mu\nu} = (\lambda_2 - 1 \leq e \leq \lambda_1)(e = f) + (\lambda_2 \leq \frac{e + f + 1}{2} \leq \lambda_1)(|e - f| \leq 1).$$

PROOF. In Theorem 48, set $d_1 = d_2 = 0$, $n_3 = \lambda_2$ and $n_4 = \lambda_1$. $\square$

COROLLARY 50. *Let $\lambda$, $\mu$ and $\nu$ be partitions of $n$, where $\mu$ and $\nu$ are hook shapes. Then the Kronecker coefficients are bounded. Moreover, the only possible values for the Kronecker coefficients are 0, 1 or 2.*

## 6. The case of a hook shape and a two-row shape

In this section we derive an explicit formula for the Kronecker coefficients in the case $\mu = (1^{e_1}m_2)$ is a hook and $\nu = (\nu_1, \nu_2)$ is a two-row shape. Given a partition $\lambda$, the Kronecker coefficients $\gamma^\lambda_{\mu\nu}$ tell us whether the point $(e_1, \nu_2)$ belongs to some regions in $\mathbf{N}^2$ determined by $\mu$, $\nu$ and $\lambda$.

Using the symmetry properties of the Kronecker product, we may assume that if $\lambda = (1^{d_1}2^{d_2}n_3n_4)$ then $n_4 - n_3 \leq d_1$. (If $n_4 = 0$ then we should rewrite $\lambda$ as $(1^{d_1}2^{d_2-1}2n_3)$. Moreover, our hypothesis becomes $n_3 - 2 \leq d_1$.)

Recall that we denote the value of the characteristic function at proposition $P$ by $(P)$.

THEOREM 51. *Let $\lambda$, $\mu$ and $\nu$ be partitions of $n$, where $\mu = (1^{e_1}m_2)$ is a hook and $\nu = (\nu_1, \nu_2)$ is a two-row shape. Then the Kronecker coefficients $\gamma^\lambda_{\mu\nu}$ are given by the following:*

1. *If $\lambda$ is a one-row shape, then $\gamma^\lambda_{\mu\nu} = \delta_{\mu,\nu}$.*

2. *If $\lambda$ is not contained in any double hook, then $\gamma^\lambda_{\mu\nu} = 0$.*

3. *Suppose* $\lambda = (1^{d_1}2^{d_2}n_3n_4)$ *is a double hook. Assume that* $n_4 - n_3 \le d_1$.

*(If* $n_4 = 0$, *then we should write* $\lambda = (1^{d_1}2^{d_2-1}2\,n_3)$*.) Then*

$$\gamma_{\mu\nu}^{\lambda} = (n_3 \le \nu_2 - d_2 - 1 \le n_4)(d_1 + 2d_2 < e_1 < d_1 + 2d_2 + 3)$$

$$+ (n_3 \le \nu_2 - d_2 \le n_4)(d_1 + 2d_2 \le e_1 \le d_1 + 2d_2 + 3)$$

$$+ (n_3 \le \nu_2 - d_2 + 1 \le n_4)(d_1 + 2d_2 < e_1 < d_1 + 2d_2 + 3)$$

$$- (n_3 + d_2 + d_1 = \nu_2)(d_1 + 2d_2 + 1 \le e_1 \le d_1 + 2d_2 + 2).$$

4. *If* $\lambda$ *is a hook, see Corollary 49.*

PROOF. Set $X = 1 + x$ and $Y = 1 + y$ in the comultiplication expansion (19) to obtain

$$(44) \qquad s_\lambda[(1 - x)(1 + y)] = \sum_{\mu,\nu} \gamma_{\mu\nu}^{\lambda} s_\mu[1 - x]s_\nu[1 + y].$$

Use (34) and (35) to replace $s_\mu$ and $s_\nu$ in the right-hand side of (44), and divide by $(1 - x)$ to obtain

$$(45) \qquad \frac{s_\lambda[(1 - x)(1 + y)]}{1 - x} = \sum_{\substack{\mu=(1^{e_1}m_2) \\ \nu=(\nu_1,\nu_2)}} \gamma_{\mu\nu}^{\lambda}(-x)^{e_1}y^{\nu_2}\left(\frac{1 - y^{\nu_1-\nu_2+1}}{1 - y}\right).$$

If $\lambda$ is not contained in any double hook, then the point $(3,3)$ is in $\lambda$, and by Sergeev's formula, $s_\lambda[(1 - x)(1 + y)]$ equals zero.

Since we already computed the Kronecker coefficients when $\lambda$ is contained in a hook, we can assume for the rest of this proof that $\lambda$ is a double hook. Let

$\lambda = (1^{d_1} 2^{d_2} n_3 \, n_4)$. (Note: If $n_4 = 0$ then we should write $\lambda = (1^{d_1} 2^{d_2-1} 2 \, n_4)$.)

Set $u_1 = 1$, $u_2 = y$, $v_1 = x$, and $v_2 = xy$ in (36), and multiply by $\frac{1-y}{1-x}$ on both sides of the resulting equation.

$$(46) \quad \sum_{\substack{\mu=(1^{e_1} m_2) \\ \nu=(\nu_1,\nu_2)}} \gamma_{\mu\nu}^{\lambda}(-x)^{e_1} y^{\nu_2}\left(1 - y^{\nu_1-\nu_2+1}\right) = (y-x)(1-xy)(1-x)$$

$$\times \, (-x)^{d_1+2d_2} y^{n_3+d_2-1} \left(\frac{(1-y^{n_4-n_3+1})(1-y^{d_1+1})}{1-y}\right).$$

We have that $(y-x)(1-xy)(1-x) = y - x(1+y+y^2) + x^2(1+y+y^2) - x^3 y$.

Therefore, looking at the coefficient of $x$ on both sides of the equation, we see that $\gamma_{\mu\nu}^{\lambda}$ is zero if $e_1$ is different from $d_1 + 2d_2, d_1 + 2d_2 + 1, d_1 + 2d_2 + 2$, or $d_1 + 2d_2 + 3$.

Let $e_1 = d_1 + 2d_2$ or $e_1 = d_1 + 2d_2 + 3$. Since $\nu_2 \le n/2$, we have that

$$\gamma_{\mu\nu}^{\lambda} = [y^{\nu_2}] \sum_{\substack{\mu=(1^{e_1} m_2) \\ \nu=(\nu_1,\nu_2)}} \gamma_{\mu\nu}^{\lambda} y^{\nu_2}$$

$$= [y^{\nu_2}] \sum_{\substack{\mu=(1^{e_1} m_2) \\ \nu=(\nu_1,\nu_2)}} \gamma_{\mu\nu}^{\lambda} y^{\nu_2}(1 - y^{\nu_1-\nu_2+1}) \qquad (\nu_1 + 1 > n/2)$$

$$= [y^{\nu_2}] y^{n_3+d_2}(1 - y^{d_1+1})\left(\frac{1-y^{n_4-n_3+1}}{1-y}\right) \qquad \text{(Eq. 46)}$$

$$= [y^{\nu_2}] y^{n_3+d_2}(1 - y^{d_1+1}) \sum_{k=0}^{n_4-n_3} y^k$$

$$= [y^{\nu_2}] y^{n_3+d_2} \sum_{k=0}^{n_4-n_3} y^k. \qquad\qquad (n_3 + d_2 + d_1 \geq n/2)$$

We have obtained that for $e_1 = d_1 + 2d_2$ or $e_1 = d_1 + 2d_2 + 3$

$$\gamma_{\mu\nu}^\lambda = (n_3 \leq \nu_2 - d_2 \leq n_4).$$

Let $e_1 = d_1 + 2d_2 + 1$ or $e_1 = d_1 + 2d_2 + 2$. Since $\nu_2 \leq \lfloor \frac{n}{2} \rfloor$ we have that

$$\gamma_{\mu\nu}^\lambda = [y^{\nu_2}] \sum_{\substack{\mu=(1^{e_1}m_2) \\ \nu=(\nu_1,\nu_2)}} \gamma_{\mu\nu}^\lambda y^{\nu_2}$$

$$= [y^{\nu_2}] \sum_{\substack{\mu=(1^{e_1}m_2) \\ \nu=(\nu_1,\nu_2)}} \gamma_{\mu\nu}^\lambda (1 - y^{\nu_1-\nu_2+1})$$

$$= [y^{\nu_2}] y^{n_3+d_2-1}(1 + y + y^2)(1 - y^{d_1+1}) \left( \frac{1 - y^{n_4-n_3+1}}{1-y} \right)$$

$$= \left( [y^{\nu_2}] y^{n_3+d_2-1}(1 + y + y^2) \left( \frac{1 - y^{n_4-n_3+1}}{1-y} \right) \right) - (n_3 + d_2 + d_1 = \nu_2)$$

$$= \left( [y^{\nu_2}] y^{n_3+d_2-1}(1 + y + y^2) \sum_{k=0}^{n_4-n_3} y^k \right) - (n_3 + d_2 + d_1 = \nu_2)$$

We have obtained that for $e_1 = d_1 + 2d_2 + 1$ or $e_1 = d_1 + 2d_2 + 2$

$$(47) \quad \gamma_{\mu\nu}^\lambda = (n_3 \leq \nu_2 - d_2 - 1 \leq n_4) + (n_3 \leq \nu_2 - d_2 \leq n_4)$$

$$+ (n_3 \leq \nu_2 - d_2 + 1 \leq n_4) - (n_3 + d_2 + d_1 = \nu_2).$$

$\square$

COROLLARY 52. *The Kronecker cofficients, $\gamma_{\mu\nu}^\lambda$, where $\mu$ is a hook and $\nu$ is a two-row shape are always 0, 1, 2 or 3.*

## 7. Final comments

The inner product of symmetric functions was discovered by J. H. Redfield [22] in 1927, together with the scalar product of symmetric functions. He called them cup and cap products, respectively. D.E. Littlewood [14, 15] reinvented the inner product in 1956.

More recently, I.M. Gessel [11] and A. Lascoux [13] obtained combinatorial interpretations for the Kronecker coefficients in some restricted cases: Lascoux in the case where $\mu$ and $\nu$ are hooks, and $\lambda$ a straight tableaux, and Gessel in the case that $\mu$ and $\nu$ are zigzag shapes and $\lambda$ is an arbitrary skew shape. A. Lascoux interpreted the Kronecker coefficients, when two of the shapes are hooks as counting clases of words under some equivalence relation. We refer to [13] or [6] for a complete statement of his results. The Corollary of Theorem 3 in this paper shows that each class of words, under Lascoux's equivalence, contains either 0, 1, or 2 different representatives. I. Gessel worked on a more general framework, contemplating the occurrence of skew tableaux. It was shown in [6] that in the case where two of the partitions are hook shapes, and the third one is an arbitrary straight shape, his result is equivalent to Lascoux's.

In [6], A.M. Garsia and J.B. Remmel founded a way to relate shuffles of permutations and Kronecker coefficients. From here they obtained a combinatorial interpretation for the Kronecker coefficients when $\lambda$ is a product of

homogeneous symmetric functions, and $\mu$ and $\nu$ are arbitrary skew shapes. They also showed how Gessel's and Lascoux's results are related.

J.B. Remmel [23], [24], and J.B. Remmel and T. Whitehead [25], obtained formulas for computing the Kronecker coefficients in the same cases considered in this paper. Their approach was mainly combinatorial:

First, they expanded the Kronecker product $s_\mu * s_\nu$ in terms of Schur functions using the Garsia-Remmel algorithm [6]. The problem of computing the Kronecker coefficients was reduced to computing signed sums of certain products of skew Schur functions.

Then they obtained a description of the coefficients that arise in the expansion of the resulting product of skew Schur functions in terms of counting *3-colored diagrams* in [23], and [24] or *4-colored diagrams* in [25]. At this point, they reduced the problem to computing a signed sum of *colored diagram*.

Finally, they defined involutions on these signed sums to cancel negative terms, and obtained the desired formulas by counting classes of restricted *colored diagram*.

In general, it is not obvious how to go from the determination of the Kronecker coefficients $\gamma^\lambda_{\mu,\nu}$ when $\mu$ and $\nu$ are two-row shapes found in this paper, and the one obtained by J.B. Remmel and T. Whitehead [25]. But, in some particular cases this is easy to see. For instance, when $\lambda$ is also a two-row shape, both formulas are exactly the same.

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