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A Combinatorial Overview of the Hopf Algebra of MacMahon Symmetric Functions

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Abstract. A MacMahon symmetric function is a formal power series in a finite number of alphabets that is invariant under the diagonal action of the symmetric group. In this article, we give a combinatorial overview of the Hopf algebra structure of the MacMahon symmetric functions relying on the construction of a Hopf algebra from any alphabet of neutral letters obtained in [18, 19].

Keywords: MacMahon symmetric function, vector symmetric function, multi symmetric function, Gessel map

1. Introduction

In his seminal article [11] MacMahon initiated the systematic study of a new class of symmetric functions that he called symmetric functions of several systems of quantities. This class of functions had been previously considered by Cayley [2] and by Schläfli [22] in their investigations on polynomials. Following Ira Gessel [5, 6], and in honor of Major Percy MacMahon, we call this class of symmetric functions MacMahon symmetric functions.

One motivation for the study of MacMahon symmetric functions comes from the following analogy with symmetric functions. On the one hand, symmetric functions appear when expressing a monic polynomial in terms of its roots. On the other hand, suppose that we can express the coefficients of a polynomial in two variables as a product of linear factors. Then $e_{(0,0)}+\cdots+e_{(1,1)}xy+\cdots+e_{(n,n)}x^ny^n$ can be written as $(1+\alpha_1x+\beta_1y)\cdots(1+\alpha_nx+\beta_ny)$. Expanding the previous product, we obtain symmetric functions like $e_{(0,0)}=1$, $e_{(1,0)}=\alpha_1+\alpha_2+\cdots+\alpha_n$, and $e_{(0,1)}=\beta_1+\beta_2+\cdots+\beta_n$. But, we also get some things that are different, like $e_{(1,1)}=\alpha_1\beta_2+\alpha_2\beta_1+\cdots+\alpha_{n-1}\beta_n$, and $e_{(2,1)}=\alpha_1\alpha_2\beta_3+\alpha_1\alpha_3\beta_2+\cdots+\alpha_{n-2}\alpha_{n-1}\beta_n$.

The relevant fact about this new class of symmetric functions is that they are invariant under the diagonal action of the symmetric group, but not under its full action. The diagonal action of a permutation π in S_n on a monomial $\alpha_{i_1}\cdots\alpha_{i_k}\beta_{j_1}\cdots\beta_{j_l}$ is defined by $\alpha_{\pi(i_1)}\cdots\alpha_{\pi(i_k)}\beta_{\pi(j_1)}\cdots\beta_{\pi(j_l)}$. The class of functions that we just have found are the elementary MacMahon symmetric functions in two finite alphabets of size n.

As noted by MacMahon [10], in order to avoid syzygies between the elementary MacMahon symmetric functions, it is necessary to take the number of linear factors, or equivalently, the size of alphabets $X = \alpha_1 + \alpha_2 + \cdots$ and $Y = \beta_1 + \beta_2 + \cdots$, to be infinite. That observation leads to the following definition. A MacMahon symmetric function is a formal power series of bounded degree, in a finite number of infinite alphabets, that is invariant under the diagonal action of the symmetric group.

Probably, it is the fact that not all polynomials in several variables are a product of linear forms, even over an algebraically closed field, which makes the MacMahon symmetric functions less ubiquitous than their symmetric relatives.

In this article we look at the Hopf algebra structure of the MacMahon symmetric functions from a combinatorial point of view. Barnabei, Brini, Joni, and Rota [3, 8, 17] have suggested a combinatorial interpretation of the product and the coproduct of a bialgebra. They have proposed that the product corresponds to the process of putting things together, and that the coproduct corresponds to the process of splitting them apart. We use their ideas as a starting point towards a combinatorial interpretation of some instances of the theory of plethystic Hopf algebras developed in [7, 18–21]. In particular, we obtain a combinatorial interpretation for the Hopf algebra of MacMahon symmetric functions that extends the one developed by the first author [14–16] for their vector space structure.

We assume that the reader is familiar with the basic notions of algebra, coalgebra, bialgebra, Hopf algebra, and with Sweedler's notation. A fine exposition of these topics is given in [1].

2. The Hopf Algebra Gessel(A)

2.1. The Hopf Algebra Super$[A]$

The ordinary algebra of polynomials in a set of variables A is generalized in [7] to include the case where the variables can be of three different kinds: positively signed, neutral, and negatively signed. This new structure is called the supersymmetric algebra associated with the signed alphabet A, and denoted by Super$[A]$.

Let A^+ be a set of positive letters. The set of divided powers of A^+, denoted by Div(A^+), was originally constructed [7] as follows. Define Div(A^+) to be the quotient of $A^+ \times \mathbf{N}$ by the equivalence relations obtained by prescribing that $a^{(n)}$, the equivalence class of (a, n), behaves algebraically as $a^n/n!$. The map that sends the pair (a, n) to its equivalence class $a^{(n)}$ is called the divided powers operator.

In this article, we introduce Div(A^+) in an equivalent way that has the advantage of showing its combinatorial nature. We represent a set of n distinguishable balls of weight a by the pair (a, n) in $A^+ \times \mathbf{N}$. Then, we define the divided powers operator acting on $A^+ \times \mathbf{N}$ as the operator that makes us forget how to distinguish between objects of the same weight. In consequence, the image of (a,n) under the divided power operator, denoted by $a^{(n)}$, represents a set of n indistinguishable balls of weight a.

A monomial element in $\mathrm{Div}(A^+)$ has the form $a^{(i)}b^{(j)}\cdots c^{(k)}$ and represents a set consisting of i indistinguishable objects of weight a, j indistinguishable objects of weight b, and k indistinguishable objects of weight c. As usual, the sum of monomial elements of $\mathrm{Div}(A^+)$ is interpreted as a disjoint union. In consequence, an arbitrary element of $\mathrm{Div}(A^+)$ is a disjoint union of sets of indistinguishable objects.

Sometimes, we think of an object of weight a as an object that has been colored a. In this framework, each positively signed letter corresponds to a color. Moreover, (a,n) represents a set of n distinguishable balls that have been colored a, and $a^{(i)}$ corresponds to a set of i indistinguishable objects colored a. In this framework, the effect of the divided power operator is that we forget how to distinguish between objects of the same color.

Following Barnabei, Brini, Joni, and Rota [3, 8, 17], we interpret the product of element of $\mathrm{Div}(A^+)$ as the process of putting objects together. From this combinatorial interpretation, we get the following algebraic rule for the product of elements of $\mathrm{Div}(A^+)$:

$$a^{(i)}a^{(j)} = \binom{i+j}{i} a^{(i+j)}.$$

Since having a set consisting of i indistinguishable objects, together with another set that consists of j indistinguishable objects, is the same as having a set with $i+j$ indistinguishable objects, together with a distinguished subset of i elements.

Similarly, the operation of exponentiation is defined by

$$(a^{(i)})^{(j)} = \frac{(ij)!}{j!(i!)^j} a^{(ij)}.$$

Since a set whose j elements are sets consisting of i objects is the same as a set with ij objects partitioned into j disjoint subsets, each of them consisting of i objects.

Finally, the following analog of Newton's identity for $\mathrm{Div}(A^+)$ relates the sum with the product: $(a+b)^{(i)} = \sum_{j+k=i} a^{(j)}b^{(k)}$. A set with i objects, where j of them are of weight a and k of them are of weight b is equivalent to a set of k indistinguishable objects of weight a, together with another set of j indistinguishable objects of weight b.

The unit of this product is given by the weight of the empty set, and denoted by 1. Multiplying an element W of $\mathrm{Div}(A^+)$ by 1 corresponds to adding nothing to the objects in W.

So far, we have described an algebra structure on Super$[A]$. To introduce its coalgebra structure, we follow Barnabei, Brini, Joni, and Rota [3, 8, 17], and interpret the coproduct of $\mathrm{Div}(A^+)$ as the process of splitting objects apart. In consequence, the coproduct of a monomial element W, denoted by ΔW, describes all different ways of splitting the objects being weight by W into two different boxes, where the boxes are expressed using the tensor product. For instance, $a \otimes a^{(2)}b$ indicates that in the first box we have an object of weight a, and that in the second box we have two indistinguishable objects of weight a and an object of weight b. The following identity describes all different ways of splitting i indistinguishable objects into two distinguishable boxes:

$$\Delta a^{(i)} = \sum_{j+k=i} a^{(j)} \otimes a^{(k)}.$$

Moreover, the order in which we place balls of different weights do not affect the result. Hence, the coproduct should be multiplicative: $\Delta(WW') = \Delta W \Delta W'$. For example,

$$\begin{aligned}\Delta a^{(2)}b &= (a^{(2)} \otimes 1 + a \otimes a + 1 \otimes a^{(2)})(b \otimes 1 + 1 \otimes b) \\ &= a^{(2)}b \otimes 1 + ab \otimes a + b \otimes a^{(2)} + a^{(2)} \otimes b + a \otimes ab + 1 \otimes a^{(2)}b.\end{aligned}$$

Let ε be the counit of our coalgebra. Using Sdweeler's notation, the counitary property says that $W = \sum W_{(1)}\varepsilon(W_2) = \sum \varepsilon(W_1)W_{(2)}$. Henceforth, from the unicity of the counit, we obtain that $\varepsilon(W)$ equals 1 if $W = 0$, and $\varepsilon(W)$ equals 0 if $W \neq 0$.

The study of neutral variables, A^0, is the algebraic analog of the study of weighted distinguishable objects. They behave as ordinary commuting variables. In consequence, the multiplication and the exponentiation are defined by $a^i a^j = a^{i+j}$ and $(a^i)^j = a^{ij}$, respectively. Moreover, Newton's identity relates the sum of neutral letters with their product.

The coproduct of neutral letters is defined in terms of placing distinguishable balls into distinguishable boxes. Hence, $\Delta a = a \otimes 1 + 1 \otimes a$. Moreover, the order in which we place distinguishable objects into the boxes does not affect the result, so the coproduct is multiplicative. For example, $\Delta a^2 b = a^2 b \otimes 1 + 2ab \otimes a + b \otimes a^2 + a^2 \otimes b + 2a \otimes ab + 1 \otimes a^2 b$. The occurrence of the factor 2 in terms $ab \otimes a$ and $a \otimes ab$ comes from the fact that objects of weight a are distinguishable. Finally, the counitary property implies that the counit, $\varepsilon(W)$, equals 1 if $W = 0$, or equals 0 if $W \neq 0$.

Let A be an alphabet consisting of neutral and positively signed letters. (Negative letters do not appear in our study of the Hopf algebra structure of the MacMahon symmetric functions.) The superalgebra $\mathrm{Super}(A)$ is the algebra spanned by monomials in $\mathrm{Div}(A^+) \cup A^0$, where all letters commute and cocommute. So far, we have shown that $\mathrm{Super}(A)$ has structure of a bialgebra. But it has a richer structure; it is a graded, $\mathbf{Z}/2$ graded, commutative (negative letters anticommute), cocommutative Hopf algebra, i.e., it is a supersymmetric algebra.

Theorem 2.1. *Let A be a signed alphabet, then* $\mathrm{Super}(A)$ *has the structure of a supersymmetric algebra.*

Given a monomial element W in $\mathrm{Super}(A)$, the degree W is defined as the number of objects that W is weighting, and denoted by $|W|$. There are no negatively signed letters, so it is automatically $\mathbf{Z}/2$ graded. The antipode sends W to $(-1)^{\deg(W)}W$.

2.2. The Hopf Algebra Gessel(A)

In this section we give a combinatorial overview of the construction of the plethystic Hopf algebra Gessel(A) from Super$[A]$ introduced in [18]. Since the objective of this paper is to study the Hopf algebra of MacMahon symmetric functions, we assume that alphabet A is composed of neutral letters. Positively signed letters will appear as the result of the construction of Gessel(A).

There are two operators that we can apply to an alphabet. On the one hand, we can construct $\mathrm{Div}(A)$, making distinguishable objects indistinguishable. On the other hand, from any alphabet A' we can construct $\mathrm{Super}(A')$, forming packages of those objects being weighted by A'.

Let A be an alphabet of neutral letters. First, we construct $\mathrm{Super}[A]$. Then, we consider the monomials in $\mathrm{Super}[A]$ to be the positive letters of a new alphabet and construct the supersymmetric algebra Σ defined as $\mathrm{Super}(\mathrm{Div}(\mathrm{Super}[A]))$.

A monomial element in Σ has the form $(\omega)^{(i)}(\omega')^{(j)}\cdots(\omega'')^{(k)}$ where ω, ω', and ω'' are different monomial elements in $\mathrm{Super}[A]$ and is interpreted as the weight of a set of indistinguishable packages made out of distinguishable balls.

We set $(1)=1$. From the construction, we obtain that Σ is graded by saying that the degree of W is the number of packages of balls that W is weighting. We denote the degree of W by $|W|$.

2.3. The Laplace Pairing

We introduce a Laplace pairing on Σ sending (W,W') in $\Sigma\times\Sigma$ to $(W|W')$ in Σ according to the following rules. First, the Laplace pairing finds all possible bijections between the set of packages of W and the set of packages of W'. Then, for each such bijection, if package (ω) corresponds to package (ω'), the Laplace pairing puts all balls in ω and ω' together in the same package of $(W|W')$.

Theorem 2.2. *Let $U=(u_1)(u_2)\cdots(u_n)$ and $V=(v_1)(v_2)\cdots(v_n)$ be monomial elements of Σ. Let M be the square matrix obtained from U and V by making (u_iv_j) be its ij entry. Then,*

$$((u_1)(u_2)\cdots(u_n)|(v_1)(v_2)\cdots(v_n)) = \mathrm{Per}(M).$$

Proof. The symmetric group S_n is the set of bijections of $[n]$ onto itself. Therefore,

$$((u_1)(u_2)\cdots(u_n)|(v_1)(v_2)\cdots(v_n)) = \sum_{\sigma\in S_n}(u_1v_{\sigma 1})(u_2v_{\sigma 2})\cdots(u_nv_{\sigma n}).$$

By definition, this is the permanent of matrix M, see [12]. ∎

From the combinatorial definition of the Laplace pairing, we can deduce a recursive definition [18]:

(1) Set $(1|1)=1$.

If we pair the empty package with itself, we obtain the empty package.

(2) If $W=(\omega)^{(i)}$ and if $W'=(\omega')^{(j)}$, with $|\omega|>0$ and $|\omega'|>0$, then $(W|W')=(\omega\omega')^{(i)}$ if $i=j$ and $(W|W')=0$ otherwise.

Let $(\omega)^{(i)}$ and $(\omega')^{(j)}$ be sets of indistinguishable packages. If i equals j, there is only one bijection between then. On the other hand, if i is different from j, there are not bijections between them.

(3) If W has packages of different weights, then $(W|W')$ can be defined recursively by the Laplace identity:

$$(UV|W) = \sum (U|W_{(1)})(V|W_{(2)}),$$

where $\Delta W = \sum W_{(1)} \otimes W_{(2)}$.

Suppose that W has more than one class of packages. Let UV be an arbitrary partition of W into two non-empty parts. The Laplace pairing splits the packages weighted by W' in all possible ways by taking its coproduct. Then, it proceeds recursively.

Nonzero terms can only occur when the degree of U equals the degree of $W_{(1)}$ and where the degree of V equals the degree of $W_{(2)}$. In particular, $(W|W')$ is equal to zero if the degree of W is different than the degree of W'.

Similarly, if W' has packages of different weights, $(W|W')$ can be defined recursively by the dual Laplace identities:

$$(U|VW) = \sum (U_{(1)}|V)(U_{(2)}|W),$$

where $\Delta U = \sum U_{(1)} \otimes U_{(2)}$.

The Laplace pairing allows us to define the circle product between elements of Σ by

$$U \circ V = \sum U_{(1)}(U_{(2)}|V_{(1)})V_{(2)}. \tag{2.1}$$

For any signed alphabet A, the pair $(\Sigma, \circ)$ is called the Cliffordization of $\mathrm{Super}[A]$, and is denoted by $\mathrm{Pleth}(\mathrm{Super}[A])$.

The construction of Cliffordization of a supersymmetric algebra is studied in a more general setting in [18], where the following result was obtained. When A is an alphabet of neutral letters, we denote $\mathrm{Pleth}(\mathrm{Super}[A])$ by $\mathrm{Gessel}(A)$.

Theorem 2.3. *Let A be any signed alphabet. The Cliffordization of* $\mathrm{Super}[A]$, *denoted by* $\mathrm{Pleth}(\mathrm{Super}[A])$ *is an associative Hopf algebra.*

The antipode is given by the Schmitt's formula. Let $W_{<i>} = (1-\varepsilon)W_{(i)}$. Then,

$$s_0 W = \sum_{r \geq 1} (-1)^r W_{<1>} \circ W_{<2>} \circ \cdots \circ W_{<r>}. \tag{2.2}$$

For each k the antipode looks for all possible ways of splitting the packages of balls weighted by W into k different boxes, so that packages of balls weighted by $W_{<k>}$ are in box k, and such that no box remains empty. Then, it takes the signed circle product of the weight of the packages obtained in this way.

There are two different products on $\mathrm{Gessel}(A)$. One comes from the algebra structure of $\mathrm{Super}[A]$, and is called the juxtaposition product, and another one comes from the Hopf algebra structure, and is called the circle product. Later, we introduce a third product on $\mathrm{Gessel}(A)$, the square product, and study some connections between the three of them.

3. The Hopf Algebra of MacMahon Symmetric Functions

Let A be an alphabet consisting of n neutral letters. There is an isomorphism between the Hopf algebra $\mathrm{Gessel}(A)$ and the MacMahon symmetric functions in n alphabets, denoted by $\mathfrak{M}^n$, the Gessel map [19]. In the particular case where A consists of one neutral letter, the Gessel map defines an isomorphism between $\mathrm{Gessel}(A)$ and the Hopf algebra of symmetric functions.

Moreover, we can associate a Hopf algebra $\mathrm{Pleth}(H)$ to any supersymmetric algebra H [18] obtaining a generalization of the Hopf algebra of MacMahon symmetric functions. This is an intriguing object of study that we do not pursue here. A particularly striking case appears when A is an alphabet of negative letters. Then, $\mathrm{Pleth}(\mathrm{Super}[A])$ is isomorphic under the Gessel map to the skew-symmetric MacMahon symmetric functions. In this case, the role of the permanent is played by the determinant function.

3.1. The Gessel Map

We define the Gessel map as the generating function for a process of placing balls into boxes according to certain rules [14, 15]. Suppose that we have an infinite set of boxes labeled by the natural numbers. Let a be a letter in A. We write $(a|i)$ to indicate that we have placed a ball of weight a in box i. Sometimes, we denote $(a|i)$ by x_i, $(b|i)$ by y_i, $(c|i)$ by z_i, and so on.

Definition 3.1. *We define the Gessel map $G\colon\ \mathrm{Gessel}(A) \to \mathfrak{M}^n$ as the linear map that sends the monomial W in* $\mathrm{Gessel}(A)$ *to the generating function for the process of placing those balls being weighted by W into distinguishable boxes labeled by the natural numbers, according to the following rules*:

- *Balls that belong to different packages are placed into different boxes.*
- *Balls that belong to the same package are placed into the same box.*

A vector partition is a decomposition of a vector as a sum of vector with positive integer coordinates, with the assumption that the order of the summands is not relevant. For example, $(1,1)(1,0)$ is a vector partition of $(2,1)$. There is a bijection between monomial elements in $\mathrm{Gessel}(A)$ and vector partitions. The bijection sends monomial $W = (\omega_1)^{(i_1)} \cdots (\omega_l)^{(i_l)}$ of $\mathrm{Gessel}(A)$, to vector partition $\lambda = (a_1, b_1, \ldots, c_1)^{i_1} \cdots (a_l, b_l, \ldots, c_l)^{i_l}$, where a_j is the number of elements of weight a in ω_j, b_j is the number of elements of weight b in ω_j, and so on. In this case, we say that λ is the vector partition associated to the monomial element W.

A vector partition $\lambda = (a_1, b_1, \ldots, c_1)(a_2, b_2, \ldots, c_2) \cdots (a_l, b_l, \ldots, c_l)$ determines a monomial $\mathbf{x}^\lambda = x_1^{a_1} y_1^{b_1} \cdots z_1^{c_1} x_2^{a_2} y_2^{b_2} \cdots z_2^{c_2} \cdots x_l^{a_l} y_l^{b_l} \cdots z_l^{c_l}$. The monomial MacMahon symmetric function indexed by λ is the sum of all distinct monomials that can be obtained from $\mathbf{x}^\lambda$ by a permutation π in S_∞, where the action of π in $\mathbf{x}^\lambda$ is the diagonal action. That is,

$$m_\lambda = \sum_{\substack{\text{different} \\ \text{monomials}}} x_{i_1}^{a_1} y_{i_1}^{b_1} \cdots z_{i_1}^{c_1} x_{i_2}^{a_2} y_{i_2}^{b_2} \cdots z_{i_2}^{c_2} \cdots x_{i_l}^{a_l} y_{i_l}^{b_l} \cdots z_{i_l}^{c_l}.$$

The monomial MacMahon symmetric functions are a basis for the space of MacMahon symmetric functions.

Theorem 3.2. *Let W be a monomial element in* $\mathrm{Gessel}(A)$, *and let λ be the associated vector partition. The image under the Gessel map of W is the monomial MacMahon symmetric function m_λ. Moreover, if λ is a partition of a number, then its image is the monomial symmetric function m_λ.*

Proof. Let W be a monomial element in Gessel(A), and let $(a_1b_1\cdots c_1)^{(i_1)}\cdots(a_lb_l\cdots c_l)^{(i_l)}$ be the corresponding vector partition. If f is one of the placing described by the Gessel map, then the weight of f is $x_{i_1}^{a_1}y_{i_1}^{b_1}\cdots z_{i_1}^{c_1}\cdots x_{i_l}^{a_l}y_{i_l}^{b_l}\cdots z_{i_l}^{c_l}$, where the subindices are all different.

The weight of f uniquely determines f, because packages of the same kind are indistinguishable. Henceforth, the image of W under G is the monomial MacMahon symmetric function m_λ. ∎

Theorem 3.2 shows that the Gessel map defines a vector space isomorphism between Gessel(A) and the MacMahon symmetric functions. Moreover, it allows us to define the monomial MacMahon symmetric functions as the image under the Gessel map of monomial elements of Gessel(A).

For instance, let $A=\{a,b\}$ be an alphabet with two neutral letters. Then, the monomial MacMahon symmetric function $m_{(2,1)(0,1)}$ is $G(a^2b)(b)$, that is, it equals $\sum_{i\neq j}(a|i)^{(2)}(b|i)(b|j)=\sum_{i\neq j}x_i^2y_iy_j$. Similarly, the monomial symmetric function $m_{(2)(1)}$ is $G(a^2)(a)$, i.e.,

$$\sum_{i\neq j}(a|i)^{(2)}(a|j)=\sum_{i\neq j}x_i^2x_j.$$

Definition 3.3. *A monomial element in* Gessel(A) *is called elementary if it corresponds to the weight of a set of one-ball packages. Elementary monomials have the form* $(a)^{(i)}(b)^{(j)}\cdots(c)^{(k)}$, *with* i, j, k *greater than or equal to zero, and where* $a,b,\ldots,$ *and* c *are letters in* A. *A monomial element* W *in* Gessel(A) *is called primitive if it is the weight of exactly one package of balls. Primitive elements have the form* $W=(\omega)$, *for some monomial element* ω *in* Super$[A]$.

Elementary and primitive monomials correspond to the two extreme ways of distributing packages of balls into boxes. We either place them into different boxes or we put them in the same box.

Let $W=(a)^{(i)}(b)^{(j)}\cdots(c)^{(k)}$ be an elementary monomial in Gessel(A). Then, the image of W under the Gessel map is the elementary MacMahon symmetric function $e_{(i,j,\ldots,k)}$. It corresponds to the generating function for all different ways of placing the balls being weighted by W into different boxes. That is,

$$e_{(i,j,\ldots,k)}=[s^it^j\cdots u^k]\prod_l(1+sx_l+ty_l+\cdots+uz_l).$$

For instance, the elementary symmetric function $e_{(n)}$ is $G((a)^{(n)})$. Similarly, the elementary MacMahon symmetric function $e_{(1,2)}$ is $G((a)(b)^{(2)})$. It corresponds to

$$\sum_{\substack{i_i<i_2\\ \text{all different}}} x_jy_{i_1}y_{i_2}.$$

Let $(a^ib^j\cdots d^k)$ be a primitive element in Gessel(A). Then, its image under the Gessel map is the power sum MacMahon symmetric function $p_{(i,j,\ldots,k)}$. It corresponds to the generating function for all different ways of placing the balls being weighted by (ω) into one box. That is,

$$p_{(i,j,\ldots,k)}=\sum_l x_l^iy_l^j\cdots z_l^k.$$

For instance, the power sum symmetric function $p_{(n)}$ is defined as $G(a^n) = \sum_i x_i^n$, and the power sum MacMahon symmetric functions $p_{(p,q)}$ is defined as $G(a^p b^q) = \sum_i x_i^p y_i^q$. They are often called polarized power sums.

3.2. The Algebra Structure

We define the circle product of elements of Gessel(A) so that it corresponds to the ordinary product of MacMahon symmetric functions.

Theorem 3.4. *The Gessel map is an algebra map*:

$$G(W \circ W') = G(W)G(W').$$

For example, let $W = (a)^{(2)}(ab)$, and $W' = (ac)$. On the one hand,

$$W \circ W' = (a^2c)(a)(ab) + (a)^{(2)}(a^2bc) + (a)^{(2)}(ab)(ac).$$

On the other hand, $G(W) = \sum_{i<j,\,\mathrm{diff.}} x_i x_j x_k y_k$, $G(W') = \sum_i x_i y_i$, and

$$G(W \circ W') = \sum_{\mathrm{diff.}} x_i^2 z_i x_j x_k y_k + \sum_{i<j,\,\mathrm{diff.}} \left(x_i x_j x_k^2 y_k z_k + x_i x_j x_k y_k x_l z_l\right).$$

Let $W_1, W_2, \ldots, W_l$ be elementary (primitive) monomials in Gessel(A), and let λ_1, $\lambda_2, \ldots, \lambda_l$ be the associated vectors. Then, we say that $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_l$ is the vector partition associated to $W_1 \circ W_2 \circ \cdots \circ W_l$. In particular, if W is an elementary (primitive) monomial, then the vector partition associated to W has exactly one part.

A vector partition is unitary if it is a partition of a vector such that all its coordinates are equal to one. Unitary vector partitions can be identified with partitions of a set.

Theorem 3.5. *Let $W_1, W_2, \ldots, W_l$ be elementary monomials such that the associated vector partition λ is unitary. Then, the expression for $W_1 \circ W_2 \circ \cdots \circ W_l$ in the monomial basis is the following*:

$$W_1 \circ W_2 \circ \cdots \circ W_l = \sum_{\pi \wedge \lambda = \hat{0}} \pi.$$

where the join is taken on the partition lattice. It corresponds to all different ways of placing the balls being weighted by $W_1 \circ W_2 \circ \cdots \circ W_l$ into boxes, with the condition that balls in different packages of π go into different boxes.

Proof. It follows from the combinatorial interpretation for the elementary MacMahon symmetric functions found in [14]. ∎

For example, $(a)(b) \circ (c) = (ac)(b) + (a)(bc) + (a)(b)(c)$.

Theorem 3.6. *Let $W_1, W_2, \ldots W_l$ be primitive monomials such that the associated vector partition λ is unitary. Then, the expression for $W_1 \circ W_2 \circ \cdots \circ W_l$ in the monomial basis is given as the sum of all monomials π bigger than or equal to the partition $W_1|W_2|\cdots|W_l$. It corresponds to all different ways of placing the balls being weighted by π into boxes, with the condition that balls in the same package of $W_1|W_2\cdots|W_l$ go into the same box.*

Proof. It follows from the combinatorial interpretation for the power sum MacMahon symmetric functions found in [14]. ∎

For example, $(ab) \circ (cd) \circ (e) = (abcde) + (abcd)(e) + (abe)(cd) + (ab)(cde) + (ab)(cd)(e)$.

What happen in Theorems 3.5 and 3.6, when the partition λ is not unitary? After we make two distinguishable balls indistinguishable, equation $p_{(1,0)(0,1)} = m_{(1,0)(0,1)} + m_{(1,1)}$ becomes $p_{(1)(1)} = 2m_{(1)(1)} + m_{(2)}$. There are two different situations in the term $m_{(1,0)(0,1)}$ that become the same one under the divided powers operator, explaining the occurrence of the factor of two in front of $m_{(1)(1)}$.

It is possible to compute the transition matrix between the different bases of the ring of MacMahon symmetric functions, as well as their scalar or inner product, by lifting those computations on the ring of MacMahon symmetric functions to the partition lattice, and using its Möbius function [14, 15]. This is done by the introduction of two operators on the space of MacMahon symmetric functions, the projection operator and the lifting operator, fitting nicely on this framework.

The operation induced by the Laplace pairing on the algebra of MacMahon symmetric functions gives it the structure of a plethystic Hopf algebra. The Laplace pairing corresponds to permanent function on the monomial basis.

Let $(\omega_1)(\omega_2)\cdots(\omega_k)$ and $(\omega'_1)(\omega'_2)\cdots(\omega'_l)$ be monomials in Gessel(A). Then

$$((\omega_1)(\omega_2)\cdots(\omega_k)|(\omega'_1)(\omega'_2)\cdots(\omega'_l))$$

equals zero if $k \neq l$. When $k = l$, it equals the permanent of the matrix whose $(i,\ j)$-entry is $(\omega_i\omega_j)$. For example, $((a^2)(a)|(a^2)(a)) = (a^4)(a^2) + (a^3)(a^3) = (a^4)(a^2) + 2(a^3)^{(2)}$. Then, $(m_{21}|m_{21})$ equals $m_{42} + 2m_{33}$.

The operation induced by the Laplace pairing does not correspond to the plethysm of symmetric function. In particular, the Laplace pairing is symmetric.

4. Involution ω and the Square Product

On the MacMahon symmetric functions there is a remarkable operation called involution ω [9, 23] corresponding, up to a sign, to the antipode s_0 of Gessel(A). We use the antipode to define two remarkable basis for the MacMahon symmetric functions: the homogeneous and the forgotten MacMahon symmetric functions. We describe involution ω in terms of the antipode of Super(A), and of the antipode of Gessel(A) as follows:

$$\omega((W_1)(W_2)\cdots(W_k)) = s_0((s(W_1))(s(W_2))\cdots(s(W_k))),$$

where s denotes the antipode of Super(A) that sends W to $(-1)^{\deg(W)}W$. Moreover, we define the square product between monomial elements in Gessel(A) by

$$W \square W' = \omega[\omega[W]\omega[W']].$$

For example, $(a)\square(a) = \omega[\omega[(a)]\omega[(a)]] = 2!s_0[(-1)^2(a)^{(2)}] = 2!((a^{(2)} + (a^2))$. The image under the Gessel map of $\frac{1}{2!}(a)\square(a)$ is the homogeneous symmetric function h_2. We define

$$(a)^{[n]} = \frac{(a)\square(a)\square\cdots\square(a)}{n!}.$$

An element of Gessel(A) of the form $(a)^{[i]}\Box(b)^{[j]}\cdots\Box(c)^{[k]}$ is called a Wronski element. The homogeneous MacMahon symmetric functions are the image under the antipode of the elementary MacMahon symmetric functions. The image of a Wronski element under the Gessel map is a homogeneous MacMahon symmetric function, as we can easily check by taking the antipode at both sides of the defining equation.

Theorem 4.1. *The image of $(a)^{[i]}\Box(b)^{[j]}\cdots\Box(c)^{[k]}$ under the Gessel map is the MacMahon symmetric function $h_{(i,j,\dots,k)}$.*

Moreover, $h_{(i,j,\dots,k)}$ is the generating function for the process of placing the balls being weighted by $(a)^{(i)}(b)^{(j)}\cdots(c)^{(k)}$ into boxes, with the condition that distinguishable balls within the same box are linearly ordered. That is,

$$\sum_{i,j,\dots,k} h_{(i,j,\dots,k)} s^i t^j \cdots u^k = \prod_i \frac{1}{1 - x_i s - y_i t - \cdots - z_i u}.$$

Proof. The combinatorial description follows from [14, 15]. ∎

The homogeneous MacMahon symmetric functions are a multiplicative basis for the space of MacMahon symmetric functions.

To introduce the forgotten MacMahon symmetric functions we need the following lemma.

Lemma 4.2. *Properties of the square product*:

(1) $s_0[W \circ W'] = s_0[W] \circ s_0[W']$,

(2) $s_0[WW'] = s_0[W]\Box s_0[W']$,

(3) $s_0[W\Box W'] = s_0[W]s_0[W']$.

Moreover, the associativity of the juxtaposition product implies the associativity of the square product.

The forgotten MacMahon symmetric functions are defined as the image under the antipode map of the monomial MacMahon symmetric functions [4]. We have seen that the monomial MacMahon symmetric functions correspond to the image under the Gessel map of the monomial elements of Gessel(A). So, the previous lemma implies that the forgotten MacMahon symmetric functions are the image of monomial elements when using the square product instead of the juxtaposition product.

Theorem 4.3. *Let $D = (\omega_1)^{[i]}\Box(\omega_2)^{[j]}\Box\cdots\Box(\omega_l)^{[k]}$ be a Doubilet element and let λ be the vector partition associated of its underlying monomial. The image of D under the Gessel map is the forgotten MacMahon symmetric function f_λ.*

The forgotten MacMahon symmetric function corresponding to the Doubilet element D is the generating function for the process of placing the balls being weighted by D into boxes, where balls coming from the same package go to the same box, and where we require that within each box, the packages appearing are linearly ordered.

To summarize, we describe the action of the Gessel map on the three different products defined for the MacMahon symmetric functions.

(1) *The juxtaposition product.* The image of a monomial element in Gessel(A)

$$(w)^{(i)}(w')^{(j)}\cdots(w'')^{(k)}$$

under the Gessel map is the generating function for the process of placing balls into boxes, where balls in the same block go to the same box, and where balls in different blocks go into different boxes.

(2) *The square product.* The image of a Doubilet element in Gessel(A)

$$(w)^{[i]}\square(w')^{[j]}\square\cdots\square(w'')^{[k]}$$

under the Gessel map is the generating function for the process of placing balls into boxes with the condition that balls that come from the same package go into the same box, and that within each box the packages appearing are linearly ordered.

(3) *The circle product.* The image of a circle product of monomial elements in Gessel(A)

$$W_1\circ W_2\circ\cdots\circ W_k$$

under the Gessel map is the generating function for the process of placing balls into boxes, where for each i, balls in the same block of W_i go to the same box.

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References

1. E. Abe, Hopf Algebras, Cambridge Tracts in Mathematics, Cambridge University Press, 1997.
2. A. Cayley, On the symmetric functions of the roots of certain systems of two equations, Phil. Trans. **147** (1857).
3. M. Barnabei, A. Brini, and G.-C. Rota, Section coefficients and section sequences, reprinted in: Gian-Carlo Rota on Combinatorics, Birkhäuser, 1995.
4. P. Doubilet, On the foundations of combinatorial theory, VII: symmetric functions through the theory of distribution and occupancy, Stud. Appl. Math. Vol. LI No. 4 (1972) 377–396, reprinted in: Gian-Carlo Rota on Combinatorics, J.P. Kung Ed., Birkhäuser, 1995, pp. 403–422.
5. I.M. Gessel, Symmetric functions and P-recursiveness, J. Combin. Theory, Ser. A **53** (1988) 257–285.
6. I.M. Gessel, Enumerative Applications of Symmetric Functions, Actes 17^e Séminaire Lothaingien, Publ. I.R.M.A. Strasbourg, 348/5-17 (1988) 5–21.
7. F.D. Grosshans, G.-C. Rota, and J. Stein, Invariant Theory and Superalgebras, CBMS Regional Conference Series in Mathematics **69**, American Mathematical Society, 1987.
8. S.A. Joni and G.-C. Rota, Coalgebras and bialgebras in combinatorics, reprinted in: Gian-Carlo Rota on Combinatorics, Birkhäuser, 1995.
9. I.G. Macdonald, Symmetric Functions and Hall Polynomials, Second Edition, Oxford Science Publications, Oxford University Press, 1995.
10. P.A. MacMahon, Combinatorial Analysis, Vols. 1–2, Cambridge University Press 1915, 1916, reprinted by Chelsea, New York, 1960.

11. P.A. MacMahon, Memoir on symmetric functions of the roots of systems of equations, reprinted in: Percy Alexander MacMahon, Collected Papers, Vol. II, MIT Press, 1986.
12. H. Minc, Permanents, Encyclopedia of Mathematics and Its Applications, Vol. 6, Addison-Wesley, 1978.
13. P.J. Olver, Classical Invariant Theory, London Mathematical Society Student Texts **44**, Cambridge University Press, 1999.
14. M.H. Rosas, A combinatorial overview of the theory of MacMahon symmetric functions and a study of the Kronecker product of Schur functions, Ph.D. Thesis, Brandeis University, 2000.
15. M.H. Rosas, MacMahon symmetric functions, the partition lattice, and Young tableaux, J. Combin. Theory, Ser. A **96** (2) (2001) 326–340.
16. M.H. Rosas, Specializations of MacMahon symmetric functions and the Stirling numbers, Discrete Math. **246** (1–3) (2002) 285–293.
17. G.-C. Rota, Hopf algebra methods in combinatorics, reprinted in: Gian-Carlo Rota on Combinatorics, Birkhäuser, 1995.
18. G.-C. Rota and J. Stein, Plethystic Hopf algebras, Proc. Natl. Acad. Sci. USA **91** (1994) 13057–13061.
19. G.-C. Rota and J. Stein, Plethystic algebras and vector symmetric functions, Proc. Natl. Acad. Sci. USA **91** (1994) 13062–13066.
20. G.-C. Rota and J. Stein, A formal theory of resultants I, an algorithm in invariant theory, In: Algebraic Combinatorics and Computer Science, Springer Italia, Milan, 2001, pp. 267–314.
21. G.-C. Rota and J. Stein, A formal theory of resultants II, a constructive definition of the resultant, In: Algebraic Combinatorics and Computer Science, Springer Italia, Milan, 2001, pp. 315–342.
22. Schläfli, Ueber die Resultante Eines Systemes mehrerer algebraischen Gleichungen, Vienna Academy Denkschriften, Vol. 4, 1852.
23. R. Stanley, Enumerative Combinatorics, Vol. II, Cambridge Studies in Advanced Mathematics **62**, Cambridge University Press, 1999.