

1 Expanding a power-sum MacMahon symmetric function in the monomial basis

Fix a positive integer r . Throughout, we will work with MacMahon symmetric functions in r alphabets. The complete set of variables is

$$\{x_{i,j} \mid i \in \mathbb{P}, j \in [r]\}.$$

Fix vector partitions $\lambda = (\lambda_1, \dots, \lambda_\ell)$ and $\mu = (\mu_1, \dots, \mu_m)$ of the same vector, where

$$\begin{aligned}\lambda_i &= (\lambda_{i,1}, \dots, \lambda_{i,r}) \in \mathbb{P}^r \setminus \{\vec{0}\}, \\ \mu_i &= (\mu_{i,1}, \dots, \mu_{i,r}) \in \mathbb{P}^r \setminus \{\vec{0}\}.\end{aligned}$$

The monomial MacMahon symmetric function indexed by μ is the sum of all monomials in the diagonal \mathfrak{S}_n -orbit of the monomial

$$x_\mu = \prod_{i=1}^m \prod_{j=1}^r x_{i,j}^{\mu_{i,j}}.$$

The power-sum MacMahon symmetric function indexed by λ is

$$\begin{aligned}P_\lambda &= \prod_{i=1}^\ell P_{\lambda_i} \\ &= \prod_{i=1}^\ell \sum_{k \in \mathbb{P}} x_{k,1}^{\lambda_{i,1}} \cdots x_{k,r}^{\lambda_{i,r}} \\ &= \sum_{(k_1, \dots, k_\ell) \in \mathbb{P}^\ell} \prod_{i=1}^\ell x_{k_i,1}^{\lambda_{i,1}} \cdots x_{k_i,r}^{\lambda_{i,r}}.\end{aligned}$$

Thus the coefficient of x_μ in P_λ is

$$\# \left\{ (k_1, \dots, k_\ell) \in \mathbb{P}^\ell \mid \sum_{i: k_i=h} \lambda_i = \mu_h \quad \forall h \in [m] \right\}.$$

Note that we can replace \mathbb{P}^ℓ by m^ℓ .

For example, fix $r = 2$. Let $\mu = (1, 1)(1, 2)$, so that

$$x_\mu = x_{1,1}^1 x_{1,2}^1 x_{2,1}^1 x_{2,2}^2$$

and let $\lambda = (1, 1)(1, 0)(0, 1)(0, 1)$ so that $\ell = 4$ and

$$\begin{aligned}P_\lambda &= \sum_{(k_1, \dots, k_4)} \prod_{i=1}^4 x_{k_i,1}^{\lambda_{i,1}} x_{k_i,2}^{\lambda_{i,2}} \\ &= \sum_{(k_1, \dots, k_4)} x_{k_1,1}^{\lambda_{1,1}} x_{k_1,2}^{\lambda_{1,2}} x_{k_2,2}^{\lambda_{2,1}} x_{k_2,2}^{\lambda_{2,2}} x_{k_3,1}^{\lambda_{3,1}} x_{k_3,2}^{\lambda_{3,2}} x_{k_4,1}^{\lambda_{4,1}} x_{k_4,2}^{\lambda_{4,2}} \\ &= \sum_{(k_1, \dots, k_4)} x_{k_1,1}^1 x_{k_1,2}^1 x_{k_2,2}^1 x_{k_2,2}^0 x_{k_3,1}^0 x_{k_3,2}^1 x_{k_4,1}^0 x_{k_4,2}^1\end{aligned}$$

so in order to make this monomial equal to x_μ , we need

$$\sum_{i: k_i=1} \lambda_i = \mu_1 = (1, 1), \quad \sum_{i: k_i=2} \lambda_i = \mu_2 = (1, 2).$$

The solutions for (k_1, k_2, k_3, k_4) and the corresponding expressions for x_μ are

$$\begin{array}{ll} (1, 2, 2, 2) & x_{1,1}^1 x_{1,2}^1 x_{2,1}^1 x_{2,2}^2 = (x_{1,1}^1 x_{1,2}^1) (x_{2,2}^1 x_{2,2}^0) (x_{2,1}^0 x_{2,2}^1) (x_{2,1}^0 x_{2,2}^1) \\ (2, 1, 1, 2) & x_{1,1}^1 x_{1,2}^1 x_{2,1}^1 x_{2,2}^2 = (x_{2,1}^1 x_{2,2}^1) (x_{1,2}^1 x_{1,2}^0) (x_{1,1}^0 x_{1,2}^1) (x_{2,1}^0 x_{2,2}^1) \\ (2, 1, 2, 1) & x_{1,1}^1 x_{1,2}^1 x_{2,1}^1 x_{2,2}^2 = (x_{2,1}^1 x_{2,2}^1) (x_{1,2}^1 x_{1,2}^0) (x_{2,1}^0 x_{2,2}^1) (x_{1,1}^0 x_{1,2}^1) \end{array}$$

So the coefficient is 3. This could certainly be calculated algorithmically.

2 Transitions between bases, from Rosas 2001

First, a standard linear algebra fact. Let V be a vector space of finite dimension n over ground field \mathbb{k} (it doesn't matter what it is).

Proposition 1. Suppose that $B = \{v_1, \dots, v_n\}$ and $B^* = \{v_1^*, \dots, v_n^*\}$ are bases for V that are orthogonal, i.e.,

$$\langle v_i, v_j^* \rangle = \delta_{ij} k_i$$

for some nonzero constants k_1, \dots, k_n . Then the unique expansion of u with respect to B is

$$u = \sum_{i=1}^n \frac{\langle v_i^*, u \rangle}{k_i} v_i. \quad (1)$$

Proof. For every $j \in [n]$ we have

$$\left\langle v_j^*, \sum_{i=1}^n \frac{\langle v_i^*, u \rangle}{k_i} v_i \right\rangle = \sum_{i=1}^n \frac{\langle v_i^*, u \rangle}{k_i} \langle v_j^*, v_i \rangle = \sum_{i=1}^n \langle v_i^*, u \rangle \delta_{ij} = \langle v_j^*, u \rangle.$$

That is, every element of B^* has the same inner product with the left and right-hand sides of (1). This is necessary and sufficient for equality. \square

This proposition reduces the problem of expanding a vector in a particular basis B to computing inner products with an orthogonal basis B^* . (The best possible case is that $k_i = 1$ for all i , i.e., B^* is the dual basis to B . However, we're not always that lucky.)

Notation:

- \mathbb{N} = nonnegative integers;
- For $u \in \mathbb{N}^k$, a **vector partition** $\lambda \vdash u$ is an unordered sequence of vectors (**parts**) summing to u ; zero vectors can be ignored.
- Weight of a vector partition = sum of all entries in all vectors in it (ex.: $\text{wt}(\{120, 100, 100, 013\}) = 9$)
- \mathfrak{M}_u = MacMahon symmetric functions of homogeneous multidegree u
- $\text{sign}(\lambda) = (-1)^{\text{number of parts with even sum}}$

- $m_\lambda, e_\lambda, h_\lambda, p_\lambda, f_\lambda$: monomial, elementary, homogeneous, power-sum, and forgotten MacMahon symmetric functions (Rosas 327–328)
- For a vector partition λ in which each part v_i occurs with multiplicity m_i , define

$$|\lambda| = \prod_i m_i! \quad \text{and} \quad \lambda! = \prod_i \prod_{x \in v_i} (x!)^{m_i}.$$

A vector partition is **unitary** if it is a partition of the all-ones vector $\mathbf{1} \in \mathbb{N}^k$. There is an obvious bijection between unitary vector partitions and set partitions of $[k]$. There is a whole theory of MacMahon symmetric functions of unitary partitions, due to Doubilet. These are the graded pieces $\mathfrak{M}_{(1)^k}$.

The **type** of a set partition $\pi = B_1 | \cdots | B_\ell$, with respect to a vector $u = (u_1, \dots, u_r)$ of weight n , is the vector $\text{type}_u(\pi) = \lambda = \lambda_1 \cdots \lambda_\ell$ where λ_k is the vector in \mathbb{N}^r whose i th coordinate is

$$\#\{j \in B_k \mid u_1 + \cdots + u_{i-1} < j \leq i_1 + \cdots + u_i\}.$$

the number of elements of B_k such that in the i th equivalence class.

For a vector u of weight n and a vector partition λ of weight n , the number of set partitions π of type $\text{type}_u(\pi)$ is

$$\binom{u}{\lambda} := \frac{u!}{\lambda! |\lambda|}.$$

Define a scalar product on \mathfrak{M} :

$$\boxed{\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}.} \quad (2)$$

The idea is to use this for basis transitions, along the lines discussed earlier. Specifically, formula (1) says that for any $F \in \mathfrak{M}_u$, we have

$$F = \sum_{\lambda \vdash u} \langle h_\lambda, F \rangle m_\lambda \quad \text{and} \quad F = \sum_{\lambda \vdash u} \langle m_\lambda, F \rangle h_\lambda. \quad (3)$$

Thus we have to know how to compute inner products.

There is a “lifting map” $\hat{\rho}$ [Rosas 2001, Defn. 6] sending MacMahon functions to unitary (Doubilet) symmetric functions. Specifically:

$$\binom{\mathbf{u}}{\lambda} |\lambda| m_\lambda \xrightarrow{\hat{\rho}} \sum_{\pi: \text{type}_u(\pi)=\lambda} m_\pi \quad (4a)$$

$$\binom{\mathbf{u}}{\lambda} \lambda! h_\lambda \xrightarrow{\hat{\rho}} \sum_{\pi: \text{type}_u(\pi)=\lambda} h_\pi \quad (4b)$$

$$\binom{\mathbf{u}}{\lambda} \lambda! e_\lambda \xrightarrow{\hat{\rho}} \sum_{\pi: \text{type}_u(\pi)=\lambda} e_\pi \quad (4c)$$

$$\binom{\mathbf{u}}{\lambda} p_\lambda \xrightarrow{\hat{\rho}} \sum_{\pi: \text{type}_u(\pi)=\lambda} p_\pi \quad (4d)$$

$$\binom{\mathbf{u}}{\lambda} |\lambda| f_\lambda \xrightarrow{\hat{\rho}} \sum_{\pi: \text{type}_u(\pi)=\lambda} f_\pi \quad (4e)$$

Moreover, for all $f, g \in \mathfrak{M}_u$ we have

$$\langle f, g \rangle = \mathbf{u}! \langle \hat{\rho}(f), \hat{\rho}(g) \rangle \quad (5)$$

[Rosas, Prop. 7].

Prop. 1 together with the scalar product (2) says in particular that for $F \in \mathfrak{M}_{\mathbf{u}}$ we have

$$F = \sum_{\lambda \vdash \mathbf{u}} \langle m_{\lambda}, F \rangle h_{\lambda}, \quad F = \sum_{\lambda \vdash \mathbf{u}} \langle h_{\lambda}, F \rangle m_{\lambda}. \quad (6)$$

Doubilet (Appendix 2) calculated the scalar products for all five families of unitary symmetric functions (which requires things like lattice operations and Möbius and zeta functions in the set partition lattice Π_n). Therefore, we should be able to compute scalar products, and thus basis expansions, for all MacMahon functions.

3 Expansions in the m -basis

$$\begin{aligned}
p_{\mu} &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat{\rho}(h_{\lambda}), \hat{\rho}(p_{\mu}) \rangle m_{\lambda} && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{(\mathbf{u})_{\lambda}!} \sum_{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda} h_{\pi}, \frac{1}{(\mathbf{u})_{\mu}!} \sum_{\sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu} p_{\sigma} \right\rangle m_{\lambda} && \text{(by (4b) and (4d))} \\
&= \frac{\mathbf{u}!}{(\mathbf{u})_{\mu}!} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\mathbf{u})_{\lambda}!} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \langle h_{\pi}, p_{\sigma} \rangle m_{\lambda} && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{(\mathbf{u})_{\mu}!} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\mathbf{u})_{\lambda}!} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \zeta(\sigma, \pi) m_{\lambda} && \text{(by Doubilet formula #3)} \\
p_{\mu} &= \boxed{\frac{\mathbf{u}!}{(\mathbf{u})_{\mu}!} \sum_{\lambda \vdash \mathbf{u}} \frac{\#\{(\pi, \sigma) \mid \text{type}_{\mathbf{u}}(\pi) = \lambda, \text{type}_{\mathbf{u}}(\sigma) = \mu, \sigma \leq \pi\}}{(\mathbf{u})_{\lambda}!} m_{\lambda}}
\end{aligned}$$

$$\begin{aligned}
h_{\mu} &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat{\rho}(h_{\lambda}), \hat{\rho}(h_{\mu}) \rangle m_{\lambda} && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{(\mathbf{u})_{\lambda}!} \sum_{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda} h_{\pi}, \frac{1}{(\mathbf{u})_{\mu}!} \sum_{\sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu} h_{\sigma} \right\rangle m_{\lambda} && \text{(by (4b) twice)} \\
&= \frac{\mathbf{u}!}{(\mathbf{u})_{\mu}!} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\mathbf{u})_{\lambda}!} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \langle h_{\pi}, h_{\sigma} \rangle m_{\lambda} && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{(\mathbf{u})_{\mu}!} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\mathbf{u})_{\lambda}!} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \lambda(\sigma \wedge \pi)! m_{\lambda} && \text{(by Doubilet formula #9)}
\end{aligned}$$

$$\begin{aligned}
e_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat{\rho}(h_\lambda), \hat{\rho}(e_\mu) \rangle m_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{(\mathbf{u})_\lambda \lambda!} \sum_{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda} h_\pi, \frac{1}{(\mathbf{u})_\mu \mu!} \sum_{\sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu} e_\sigma \right\rangle m_\lambda && \text{(by (4b) and (4c))} \\
&= \frac{\mathbf{u}!}{(\mathbf{u})_\mu \mu!} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\mathbf{u})_\lambda \lambda!} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \langle h_\pi, e_\sigma \rangle m_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{(\mathbf{u})_\mu \mu!} \sum_{\lambda \vdash \mathbf{u}} \frac{\#\{(\pi, \sigma) \mid \text{type}_{\mathbf{u}}(\pi)=\lambda, \text{type}_{\mathbf{u}}(\sigma)=\mu, \pi \wedge \sigma = \hat{\mathbf{0}}\}}{(\mathbf{u})_\lambda \lambda!} m_\lambda && \text{(by Doubilet formula #6)}
\end{aligned}$$

$$\begin{aligned}
f_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat{\rho}(h_\lambda), \hat{\rho}(f_\mu) \rangle m_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{(\mathbf{u})_\lambda \lambda!} \sum_{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda} h_\pi, \frac{1}{(\mathbf{u})_\mu |\mu|} \sum_{\sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu} f_\sigma \right\rangle m_\lambda && \text{(by (4b) and (4d))} \\
&= \frac{\mathbf{u}!}{(\mathbf{u})_\mu |\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\mathbf{u})_\lambda \lambda!} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \langle h_\pi, f_\sigma \rangle m_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{(\mathbf{u})_\mu |\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\mathbf{u})_\lambda \lambda!} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu \\ \sigma \leq \pi}} \lambda(\sigma, \pi)! m_\lambda && \text{(by Doubilet formula #13)}
\end{aligned}$$

4 Expansions in the h -basis

$$\begin{aligned}
m_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat{\rho}(m_\lambda), \hat{\rho}(m_\mu) \rangle h_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{(\mathbf{u})_\lambda |\lambda|} \sum_{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda} m_\pi, \frac{1}{(\mathbf{u})_\mu |\mu|} \sum_{\sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu} m_\sigma \right\rangle h_\lambda && \text{(by (4a) twice)} \\
&= \frac{\mathbf{u}!}{(\mathbf{u})_\mu |\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\mathbf{u})_\lambda |\lambda|} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \langle m_\pi, m_\sigma \rangle h_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{(\mathbf{u})_\mu |\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\mathbf{u})_\lambda |\lambda|} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \sum_{\tau \geq \pi \vee \sigma} \frac{\mu(\pi, \tau) \mu(\sigma, \tau)}{\mu(\hat{\mathbf{0}}, \tau)} h_\lambda && \text{(by Doubilet formula #7)}
\end{aligned}$$

$$\begin{aligned}
p_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat{\rho}(m_\lambda), \hat{\rho}(p_\mu) \rangle h_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{(\frac{\mathbf{u}}{\lambda})|\lambda|} \sum_{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda} m_\pi, \frac{1}{(\frac{\mathbf{u}}{\mu})} \sum_{\sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu} p_\sigma \right\rangle h_\lambda && \text{(by (4a) and (4d))} \\
&= \frac{\mathbf{u}!}{(\frac{\mathbf{u}}{\mu})} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\frac{\mathbf{u}}{\lambda})|\lambda|} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \langle m_\pi, p_\sigma \rangle h_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{(\frac{\mathbf{u}}{\mu})} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\frac{\mathbf{u}}{\lambda})|\lambda|} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu \\ \pi \leq \sigma}} \frac{\mu(\pi, \sigma)}{|\mu(\hat{\mathbf{0}}, \sigma)|} h_\lambda && \text{(by Doubilet formula #10)}
\end{aligned}$$

$$\begin{aligned}
e_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat{\rho}(m_\lambda), \hat{\rho}(e_\mu) \rangle h_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{(\frac{\mathbf{u}}{\lambda})|\lambda|} \sum_{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda} m_\pi, \frac{1}{(\frac{\mathbf{u}}{\mu})\mu!} \sum_{\sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu} e_\sigma \right\rangle h_\lambda && \text{(by (4a) and (4c))} \\
&= \frac{\mathbf{u}!}{(\frac{\mathbf{u}}{\mu})\mu!} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\frac{\mathbf{u}}{\lambda})|\lambda|} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \langle m_\pi, e_\sigma \rangle h_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{(\frac{\mathbf{u}}{\mu})\mu!} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\frac{\mathbf{u}}{\lambda})|\lambda|} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu \\ \pi \leq \sigma}} \text{sign}(\pi) \lambda(\pi, \sigma)! h_\lambda && \text{(by Doubilet formula #8)}
\end{aligned}$$

$$\begin{aligned}
f_\mu &= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \langle \hat{\rho}(m_\lambda), \hat{\rho}(f_\mu) \rangle h_\lambda && \text{(by (3) and (5))} \\
&= \mathbf{u}! \sum_{\lambda \vdash \mathbf{u}} \left\langle \frac{1}{(\frac{\mathbf{u}}{\lambda})|\lambda|} \sum_{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda} m_\pi, \frac{1}{(\frac{\mathbf{u}}{\mu})|\mu|} \sum_{\sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu} f_\sigma \right\rangle h_\lambda && \text{(by (4a) and (4e))} \\
&= \frac{\mathbf{u}!}{(\frac{\mathbf{u}}{\mu})|\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\frac{\mathbf{u}}{\lambda})|\lambda|} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \langle m_\pi, f_\sigma \rangle h_\lambda && \text{(by bilinearity)} \\
&= \frac{\mathbf{u}!}{(\frac{\mathbf{u}}{\mu})|\mu|} \sum_{\lambda \vdash \mathbf{u}} \frac{1}{(\frac{\mathbf{u}}{\lambda})|\lambda|} \sum_{\substack{\pi: \text{type}_{\mathbf{u}}(\pi)=\lambda \\ \sigma: \text{type}_{\mathbf{u}}(\sigma)=\mu}} \sum_{\tau \geq \sigma \vee \pi} \frac{|\mu(\sigma, \tau)| \cdot \mu(\pi, \tau)}{\mu(\hat{\mathbf{0}}, \tau)} h_\lambda && \text{(by Doubilet formula #11)}
\end{aligned}$$