Fourier Analysis Notes (Stein-Shakarchi)

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Theorem 1. *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be twice differentiable in both variables, and define a change of variables via polar coordinates from $(x, y) \in \mathbb{R}^2$ to $(r, \theta) \in \mathbb{R}_+ \times [0, 2\pi)$:*

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

Then, we can write:

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2}\frac{1}{r^2} + \frac{\partial u}{\partial r}\frac{1}{r}$$

Proof: First note:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ \implies \frac{\partial r}{\partial x} &= \frac{x}{r} = \cos\theta \end{aligned}$$

and similarly:

$$\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$$

for θ we have:

$$\begin{aligned} \theta &= \arctan\frac{y}{x} \\ \implies \frac{\partial \theta}{\partial x} &= \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot -\frac{y}{x^2} = -\frac{y}{x^2 + y^2} = -\frac{\sin\theta}{r} \\ \frac{\partial \theta}{\partial y} &= \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2} = \frac{\cos\theta}{r} \end{aligned}$$

Thus for any function f of $(x, y) \in \mathbb{R}^2$, we have:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial f}{\partial r}\cos\theta - \frac{\partial f}{\partial \theta}\frac{\sin\theta}{r} \\ \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial f}{\partial r}\sin\theta + \frac{\partial f}{\partial \theta}\frac{\cos\theta}{r} \end{aligned}$$

Hence we can evaluate:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}$$

so:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial u_x}{\partial r}\cos\theta - \frac{\partial u_x}{\partial \theta}\frac{\sin\theta}{r} \\ &= \left(\frac{\partial^2 u}{\partial r^2}\cos\theta - \frac{\partial^2 u}{\partial r \partial \theta}\frac{\sin\theta}{r} + \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r^2}\right)\cos\theta \\ &\quad - \left(\frac{\partial^2 u}{\partial r \partial \theta}\cos\theta - \frac{\partial u}{\partial r}\sin\theta - \frac{\partial^2 u}{\partial \theta^2}\frac{\sin\theta}{r} - \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\right)\frac{\sin\theta}{r} \\ &= \frac{\partial^2 u}{\partial r^2}\cos^2\theta + \frac{\partial^2 u}{\partial \theta^2}\frac{\sin^2\theta}{r^2} - 2\frac{\partial^2 u}{\partial r \partial \theta}\frac{\sin\theta\cos\theta}{r} + \frac{\partial u}{\partial r}\frac{\sin^2\theta}{r} + 2\frac{\partial u}{\partial \theta}\frac{\sin\theta\cos\theta}{r^2} \end{aligned}$$

and similarly:

$$\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r} \\
\implies \frac{\partial^2 u}{\partial y^2} &= \frac{\partial u_y}{\partial r}\sin\theta + \frac{\partial u_y}{\partial \theta}\frac{\cos\theta}{r} \\
&= \left(\frac{\partial^2 u}{\partial r^2}\sin\theta + \frac{\partial^2 u}{\partial r \partial \theta}\frac{\cos\theta}{r} - \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r^2}\right)\sin\theta \\
&\quad + \left(\frac{\partial^2 u}{\partial \theta \partial r}\sin\theta + \frac{\partial u}{\partial r}\cos\theta + \frac{\partial^2 u}{\partial \theta^2}\frac{\cos\theta}{r} - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)\frac{\cos\theta}{r} \\
&= \frac{\partial^2 u}{\partial r^2}\sin^2\theta + \frac{\partial^2 u}{\partial \theta^2}\frac{\cos^2\theta}{r^2} + 2\frac{\partial^2 u}{\partial r \partial \theta}\frac{\cos\theta\sin\theta}{r} + \frac{\partial u}{\partial r}\frac{\cos^2\theta}{r} - 2\frac{\partial u}{\partial \theta}\frac{\sin\theta\cos\theta}{r^2}
\end{aligned}$$

hence:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2}\frac{1}{r^2} + \frac{\partial u}{\partial r}\frac{1}{r}$$

□

Theorem 2 (Stein-Shakarchi Theorem 2.1)**.** *Suppose f is an integrable function on the circle with all Fourier coefficients equal to zero. Then $f(\theta_0) = 0$ whenever f is continuous at θ_0.*

Proof. First let us show that without loss of generality we can assume $\theta_0 = 0$. To see this, define $g(x) = f(x + \theta_0)$ where f is considered to be periodic on $\mathbb{R}$. Then for all $n \in \mathbb{Z}$:

$$\left|\int_{-\pi}^{\pi} g(x)e^{inx}dx\right| = \left|e^{-in\theta_0}\int_{-\pi+\theta_0}^{\pi+\theta_0} f(x)e^{inx}dx\right| = \left|\int_{-\pi}^{\pi} f(x)e^{inx}dx\right|$$

Thus the Fourier coefficients of g are equal to the Fourier coefficients of f in modulus, meaning they are all 0 if and only if the coefficients of f are all 0 (which is an assumption of the theorem). If we can prove $g(0) = 0$ when the coefficients of g are all 0, then we know $f(\theta_0) = g(0) = 0$ too.

It should also be obvious we can assume $f(0) > 0$ (if not we can just replace f with $-f$ in the following).

Now, choose $\delta \in (0, \pi/2]$ such that $f > f(0)/2$ on $(-\delta, \delta)$. This is possible because f is continuous so we can always find a ball around 0 where f is sufficiently close to $f(0)$.

Then we can define $p(\theta) := \varepsilon + \cos\theta$ where we can choose $\varepsilon > 0$ sufficiently small such that $|p(\theta)| < 1 - \varepsilon/2$ whenever $|\theta| \in [\delta, \pi]$. We can think of this like shifting up so instead of $p(\delta) \in (0, 1)$ and $p(\pi) = -1$ we increase p a bit so that $p(\delta) < 1$ still but $p(\pi) > -1$.

Lastly we can choose $\eta \in (0, \delta)$ so that $p(\theta) \geq 1 + \varepsilon/2$ for $|\theta| < \eta$. This works because we already shifted up cos by ε.

We can now let $p_k(\theta) = [p(\theta)]^k$ Since this is a trigonometric polynomial we see for all $k \in \mathbb{N}$:

$$\int_{-\pi}^{\pi} f(\theta)\, p_k(\theta)\, d\theta = 0$$

Also we can see:

$$\left|\int_{B_\delta(0)^C} f(\theta)\, p_k(\theta)\, d\theta\right| \leq 2\pi\,(1 - \varepsilon/2)^k \sup|f|$$

and:

$$\int_{B_\delta(0)\cap B_\eta(0)^C} f(\theta)\, p_k(\theta)\, d\theta \geq 0$$

since $f \geq f(0)/2 > 0$ and $p > 0$ on $B_\delta(0)$. Finally:

$$\int_{B_\eta(0)} f(\theta)\, p_k(\theta)\, d\theta \geq \eta\,(1 + \varepsilon/2)^k f(0)$$

Since this grows unboundedly in k, we can see:

$$\int_{-\pi}^{\pi} f(\theta)\, p_k(\theta)\, d\theta \xrightarrow[k\to\infty]{} \infty$$

which is a contradiction since those integrals must all be 0.

We can generalise the proof to complex valued f by writing $f = u + iv$. This lets us recover u and v from f and $\overline{f}$. Using the fact that the n-th Fourier coefficient of f is the $-n$-th coefficient of its conjugate, we can show if all the Fourier coefficients of f are zero then all the Fourier coefficients of u and v are zero, and we are done. $\square$

From this we can see if f is continuous on the circle and all of its Fourier coefficients are zero then f is zero identically.

Theorem 3 (Convergence of Cesaro sums)**.** *Suppose a sequence z_n converges to A. Then the Cesaro sums of z_n also converge to A*

Proof. Without loss of generality assume $A = 0$. If not define $y_n = z_n - A$, and notice that $y_n \to 0$ and

$$\frac{\sum_{i=1}^{N} y_i}{N} = \frac{\sum_{i=1}^{N} (z_i - A)}{N} = \frac{\sum_{i=1}^{N} z_i}{N} - A$$

meaning if we show the Cesaro sums of y_n converge to zero, we show the Cesaro sums of z_n converge to A. Let $\varepsilon > 0$ be given and choose $M > 0$ such that $|y_m| < \varepsilon/2$ for $m > M$. Also note that we know y_n are bounded since they are convergent.

For any $n > M$:

$$\begin{aligned}\left|\frac{1}{n}\sum_{i=1}^{n} y_i\right| &\leq \frac{1}{n}\sum_{i=1}^{M} |y_i| + \frac{1}{n}\sum_{i=M+1}^{n} |y_i| \\ &\leq \frac{M}{n}\sup_{i\in\mathbb{N}} |y_i| + \left(1 - \frac{M}{n}\right)\frac{\varepsilon}{2} \xrightarrow[n\to\infty]{} \frac{\varepsilon}{2}\end{aligned}$$

Hence there exists some $N > M$ such that for all $n > N$ we have:

$$\left|\frac{1}{n}\sum_{i=1}^{n} y_i\right| < \varepsilon$$

$\square$

By linearity of convolution we know the Cesaro means of the Fourier series are given by convolution with Fejer kernels:

$$F_n = \frac{1}{n}\sum_{i=0}^{n-1} D_i$$

We can define a notion of a good kernel:

Definition 1 (Good kernel)**.** *A family of kernels (K_n) is a family of good kernels if it satisfies:*

1. *For all $n \in \mathbb{N}$*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t)\, \mathrm{d}t = 1$$

2. *There exists $M > 0$ such that for all $n \in \mathbb{N}$*

$$\int_{-\pi}^{\pi} |K_n(t)|\, \mathrm{d}t \leq M$$

3. *For all* $\delta > 0$ *we have*

$$\lim_{n\to\infty} \int_{B_\delta(0)^C} |K_n(t)| \, \mathrm{d}t = 0$$

These are important because they approximate the identity in convolution.

Theorem 4. *If* (K_n) *is a family of good kernels, and* f *is continuous at* x*, we have* $K_n * f(x) \to f(x)$ *as* $n \to \infty$*. Furthermore, if* f *is uniformly continuous, then* $K_n * f \to f$ *in the supremum norm.*

We can directly compute the Fejer kernels and show they are good kernels.

Theorem 5. *The Fejer kernels are good kernels.*

First we will to show a brief identity for our computation:

Lemma 1.

$$e^{i\theta} + e^{-i\theta} - 2 = -4 \sin^2 \frac{\theta}{2}$$

Proof:

$$\sin^2 \frac{\theta}{2} = \left(\frac{e^{i\theta/2} - e^{-i\theta/2}}{2i} \right)^2 = -\frac{e^{i\theta} + e^{-i\theta} - 2}{4}$$

□

With this we can continue with the proof.

Proof of theorem 5: First we will consider the Dirichlet kernels D_N, where convolution with these kernels gives the Fourier series.

$$D_N(\theta) = \sum_{n=-N}^{N} e^{in\theta}$$

We know for all $n \in \mathbb{Z}$ we have:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\theta} \mathrm{d}\theta = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

so we have:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(\theta) \, \mathrm{d}\theta = 1$$

We can compute D_N in closed form

$$\begin{aligned} D_N(\theta) &= \sum_{n=-N}^{N} e^{in\theta} \\ &= \sum_{n=-N}^{N} \cos n\theta + i \sum_{n=-N}^{N} \sin n\theta \end{aligned}$$

noting $\sin n\theta = -\sin(-n\theta)$:

$$= 1 + 2 \sum_{n=1}^{N} \cos n\theta$$

using that $2\cos u \sin v = \sin(u+v) - \sin(u-v)$:

$$
\begin{aligned}
&= 1 + \sum_{n=1}^{N} \frac{\sin\left(\left(n+\frac{1}{2}\right)\theta\right) - \sin\left(\left(n-\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{\theta}{2}\right)} \\
&= 1 + \frac{1}{\sin\left(\frac{\theta}{2}\right)}\left(\sin\left(\frac{3}{2}\theta\right) - \sin\left(\frac{1}{2}\theta\right) + \sin\left(\frac{5}{2}\theta\right) - \sin\left(\frac{3}{2}\theta\right) + \dots\right. \\
&\quad \left. + \sin\left(\left(N+\frac{1}{2}\right)\theta\right) - \sin\left(\left(N-\frac{1}{2}\right)\theta\right)\right) \\
&= 1 + \frac{\sin\left(\left(N+\frac{1}{2}\right)\theta\right) - \sin\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \\
&= \frac{\sin\left(\left(N+\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{\theta}{2}\right)}
\end{aligned}
$$

Now we can directly compute:

$$
\begin{aligned}
F_n(\theta) &= \frac{1}{n}\sum_{k=0}^{n-1} D_k(\theta) \\
&= \frac{1}{n}\sum_{k=0}^{n-1}\sum_{m=-k}^{k} e^{im\theta} \\
&= \frac{1}{n}\sum_{k=0}^{n-1} e^{-ik\theta}\sum_{m=-k}^{k} e^{i(m+k)\theta} \\
&= \frac{1}{n}\sum_{k=0}^{n-1} e^{-ik\theta}\sum_{m=0}^{2k} e^{im\theta} \\
&= \frac{1}{n}\sum_{k=0}^{n-1} \frac{e^{-ik\theta}\left(1-e^{i(2k+1)\theta}\right)}{1-e^{i\theta}} \\
&= \frac{1}{n\left(1-e^{i\theta}\right)}\sum_{k=0}^{n-1}\left(e^{-ik\theta} - e^{i(k+1)\theta}\right) \\
&= \frac{1}{n\left(1-e^{i\theta}\right)}\left(\frac{1-e^{-in\theta}}{1-e^{-i\theta}} - e^{i\theta}\cdot\frac{1-e^{in\theta}}{1-e^{i\theta}}\right) \\
&= \frac{1}{n\left(1-e^{i\theta}\right)}\left(\frac{1-e^{-in\theta}}{1-e^{-i\theta}} + \frac{1-e^{in\theta}}{1-e^{-i\theta}}\right) \\
&= \frac{2-e^{-in\theta}-e^{in\theta}}{n\left(1-e^{i\theta}\right)\left(1-e^{-i\theta}\right)} \\
&= \frac{2-e^{-in\theta}-e^{in\theta}}{n\left(2-e^{i\theta}-e^{-i\theta}\right)}
\end{aligned}
$$

by lemma 1:

$$
= \frac{\sin^2\frac{N\theta}{2}}{n\sin^2\frac{\theta}{2}}
$$

$\square$