

# Einstein Summation Convention

This is a method to write equation involving several summations in a uncluttered form

Example:

$$\vec{A} \cdot \vec{B} = A_i \delta_{ij} B_j \quad \text{where } \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\text{or } = A_i B_i$$

- Summation runs over 1 to 3 since we are 3 dimension
- No indices appear more than two times in the equation
- Indices which is summed over is called **dummy indices** appear only in one side of equation
- Indices which appear on both sides of the equation is **free indices**.

# Vector or Cross or Outer Product

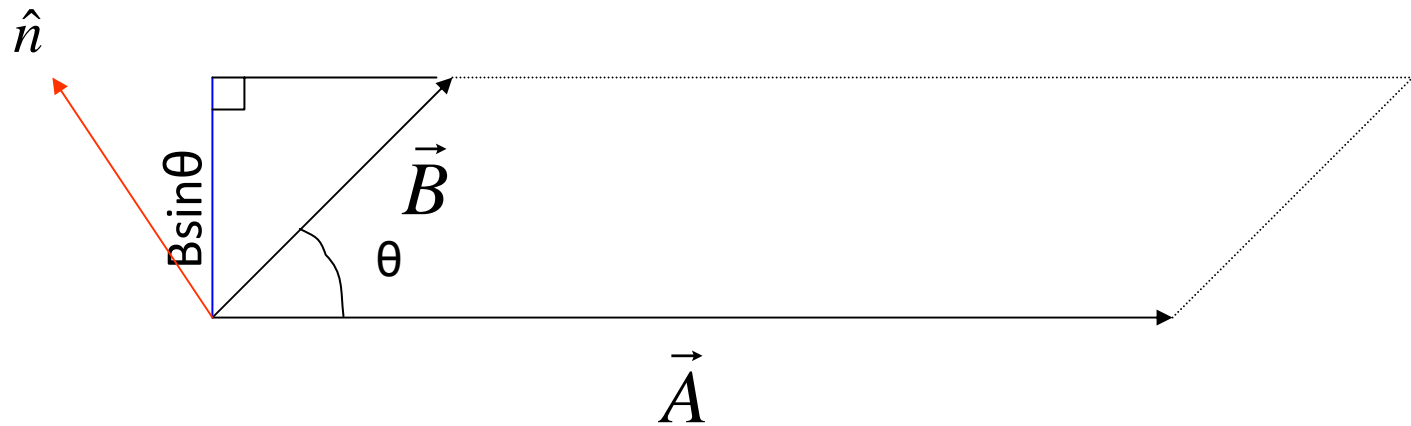
$$\vec{A} \times \vec{B} = AB \sin \theta \hat{n} = -\vec{B} \times \vec{A} \quad (\text{Not Commutative})$$

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \quad (\text{Distributive})$$

$$\left\{ \begin{array}{l} m(\vec{A} \times \vec{B}) = (m\vec{A}) \times \vec{B} = \vec{A} \times (m\vec{B}) = mAB \sin \theta \hat{n} \\ \vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C} \end{array} \right. \quad (\text{Not Associative})$$

- Product varies under change of basis, i.e. coordinate system
- Direction of the product is given by right hand screw rule
- Product gives the area of the parallelogram consisting the two vectors as its arms

# Cross Product: Graphical Representation



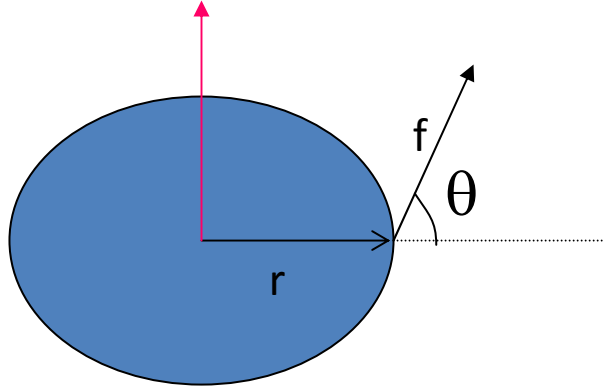
Examples:

Magnetic force on a moving charge

$$\vec{F}_{mag.} = q\vec{v} \times \vec{B}$$

Torque on a body

$$\vec{\tau} = \vec{r} \times \vec{f}$$



# In the component form

$$\vec{A} \times \vec{B} = \begin{bmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix}$$

$$= \vec{e}_x (A_y B_z - A_z B_y) + \vec{e}_y (A_z B_x - A_x B_z) + \vec{e}_z (A_x B_y - A_y B_x)$$

In the Einstein summation notation

$$(\vec{A} \times \vec{B})_i = \varepsilon_{ijk} A_j B_k$$

Where  $\varepsilon_{ijk}$  is a Levi-Civita Tensor

# The tensor operator $\varepsilon_{ijk}$ and $\varepsilon^{ijk}$

- The tensor  $\varepsilon_{ijk}$  is defined for  $i,j,k=1,\dots,3$  as

$$\varepsilon_{ijk} = \begin{cases} 0 & \text{unless } i,j,\text{and } k \text{ are distinct} \\ +1, & \text{if } rst \text{ is an even permutation of } 123 \\ -1, & \text{if } rst \text{ is an odd permutation of } 123 \end{cases}$$

odd	even
3 2 1	1 2 3
1 3 2	2 3 1
2 1 3	3 1 2

# **What is a Tensor?**

**A Tensor is a method to represent the Physical Properties in an anisotropic system**

**For example:**

**You apply a force in one direction and look for the affect in other direction  
(Piezo-electricity)**

<b>Elasticity</b>	<b>: Elastic Tensor</b>
<b>Dielectric constant</b>	<b>: Dielectric Tensor</b>
<b>Conductivity</b>	<b>: Conductivity Tensor</b>

This generalized notation allows an easy writing of equations of the continuum mechanics, such as the generalized Hook's law :

## 2<sup>nd</sup> rank tensors

- Stress, strain
- Conductivity (  $J_i = \sigma_{ij} E_j$  )
- susceptibility
- Kroneker Delta  $\delta_{ij}$

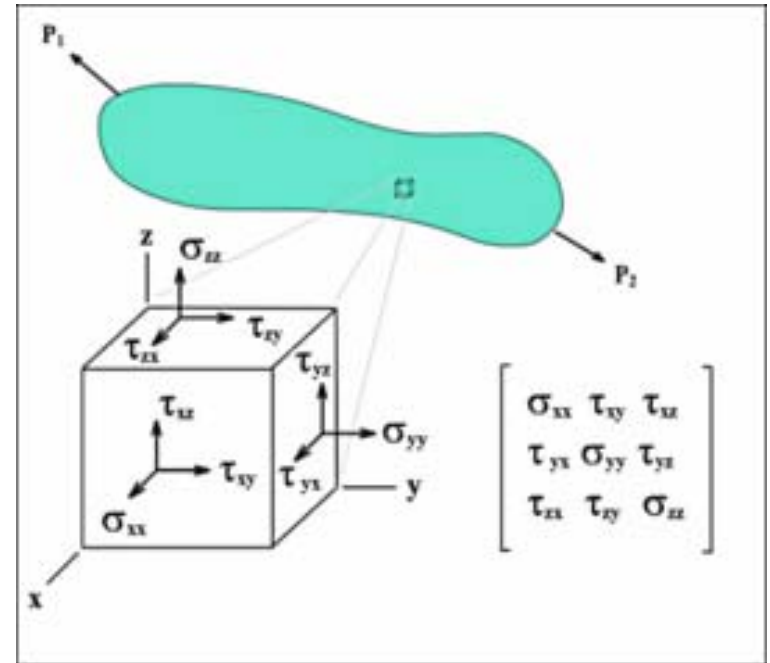
## 3<sup>rd</sup> rank tensors

- Piezoelectricity
- Levi-Civita

## 4<sup>th</sup> rank tensors

- Elastic moduli

**n<sup>th</sup> rank tensor has 3<sup>n</sup> components in 3 - dimensional space**



$$\sigma_{ij} = \sum_{kl} C_{ijkl} \cdot \varepsilon_{kl}$$

$\sigma_{ij} \rightarrow$  Stress on  $i^{th}$  plane  
in  $j^{th}$  direction



**Moment of Inertia tensor:** When axis of rotation is not given, then we can generalize moment of inertia into a tensor of rank 2.

## Angular momentum

$$L_i = I_{ij} \omega_j$$

### For continuous mass distribution

$$I_{ij} = \iiint_V \rho(r) (r_l^2 \delta_{ij} - r_i r_j) dV$$

For axis of rotation about  $\hat{n}$  the scalar form can be calculated as

$$I = \hat{n}^\top \mathbf{I} \hat{n} = \sum_{j=1}^3 \sum_{k=1}^3 n_j I_{jk} n_k$$

### For discrete particles

$$\mathbf{I} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}$$

$$I_{11} = I_{xx} \stackrel{\text{def}}{=} \sum_{k=1}^N m_k (y_k^2 + z_k^2),$$

$$I_{22} = I_{yy} \stackrel{\text{def}}{=} \sum_{k=1}^N m_k (x_k^2 + z_k^2),$$

$$I_{33} = I_{zz} \stackrel{\text{def}}{=} \sum_{k=1}^N m_k (x_k^2 + y_k^2),$$

$$I_{12} = I_{xy} \stackrel{\text{def}}{=} - \sum_{k=1}^N m_k x_k y_k,$$

$$I_{13} = I_{xz} \stackrel{\text{def}}{=} - \sum_{k=1}^N m_k x_k z_k,$$

$$I_{23} = I_{yz} \stackrel{\text{def}}{=} - \sum_{k=1}^N m_k y_k z_k$$

## Rank of a Tensor

Rank = 0 : Scalar Only One component

Rank = 1 : Vector                      Three components

Rank = 2      Nine Components

Rank = 3      Twenty Seven Components

Rank = 4      Eighty One Components

Symmetry plays a very important role in evaluating these components

# Tensor notation

- In tensor notation a superscript stands for a column vector
- a subscript for a row vector (useful to specify lines)

$$P^i = (p^i)_{i=1,2,3} = \begin{pmatrix} p^1 \\ p^2 \\ p^3 \end{pmatrix}$$

$$L_i = (l_1, l_2, l_3)$$

- A matrix is written as

**You know about Matrix Methods**

$$\begin{aligned} M_i^j &= \begin{pmatrix} m_1^1 & m_1^2 & m_1^3 \\ m_2^1 & m_2^2 & m_2^3 \\ m_3^1 & m_3^2 & m_3^3 \end{pmatrix} = \begin{pmatrix} M_i^1 \\ M_i^2 \\ M_i^3 \end{pmatrix} \\ &= (M_1^j, M_2^j, M_3^j) \end{aligned}$$

# Tensor notation

- Tensor summation convention:
  - an index repeated as sub and superscript in a product represents summation over the range of the index.
- Example:

$$L_i P^i = l_1 p^1 + l_2 p^2 + l_3 p^3$$

# Tensor notation

- Scalar product can be written as

$$L_i P^i = l_1 p^1 + l_2 p^2 + l_3 p^3$$

- where the subscript has the same index as the superscript. This implicitly computes the sum.
- This is commutative

$$L_i P^i = P^i L_i$$

- Multiplication of a matrix and a vector

$$P^j = M_i^j P^i$$

- This means a change of P from the coordinate system i to the coordinate system j (transformation).

# Line equation

- In classical methods, a line is defined by the equation  $ax + by + c = 0$
- In homogenous coordinates we can write this as

$$\vec{L}^T \cdot \vec{P} = (a, b, c) \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

- In tensor notation we can write this as

$$L_i P^i = 0$$

# Determinant in tensor notation

$$M_i^j = (m_1^j, m_2^j, m_3^j)$$

$$\det(M_i^j) = \varepsilon_{ijk} m_1^i m_2^j m_3^k$$

# Cross product in tensor notation

$$\vec{c} = \vec{a} \times \vec{b}$$

$$c_i = (a \times b)_i = \varepsilon_{ijk} a^j b^k$$



# Example

- Intersection of two lines
- L:  $l_1x + l_2y + l_3 = 0$ , M:  $m_1x + m_2y + m_3 = 0$
- Intersection:

$$x = \frac{l_2m_3 - l_3m_2}{l_1m_2 - l_2m_1}, y = \frac{l_3m_1 - l_1m_3}{l_1m_2 - l_2m_1}$$

- Tensor:  $P^i = E^{ijk} L_j M_k$
- Result:  
 $p^1 = l_2m_3 - l_3m_2$   
 $p^2 = l_3m_1 - l_1m_3$   
 $p^3 = l_1m_2 - l_2m_1$

# Translation

- Classic
- Homogenous coordinates

$$x_2 = x_1 + t_x$$

$$y_2 = y_1 + t_y$$

$$\begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix}$$

- Tensor notation

$$P^B = T_A^B P^A \quad \text{with } A, B = 1, 2, 3$$

- T is a transformation from the system A to B

# Rotation

- Classic
- Homogenous coordinates

$$x_2 = \cos(a)x_1 - \sin(a)y_1$$

$$y_2 = \sin(a)x_1 + \cos(a)y_1$$

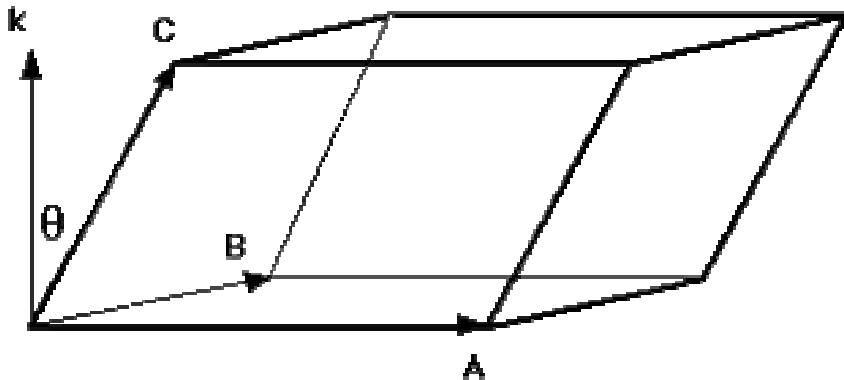
$$\begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(a) & -\sin(a) & 0 \\ \sin(a) & \cos(a) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix}$$

- Tensor notation

$$P^B = R_A^B P^A \quad \text{with } A, B = 1, 2, 3$$

## Scalar Triple Product

$$\vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$



Can we take these vectors in any other sequence?

$|\vec{A} \times \vec{B} \cdot \vec{C}|$  = Volume of the parallelepiped

Area of Base Parallelogram =  $|\vec{A} \times \vec{B}|$

$\vec{A} \times \vec{B} \cdot \vec{C} = |\vec{A} \times \vec{B}| |\vec{C}| \cos \theta = \text{Base Area} \cdot \text{height} = \text{Volume}$

## vector triple product

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

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Exercise: Prove it: Hint: use  $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$

This vector is in the plane spanned by the vectors  $\vec{b}$  and  $\vec{c}$  (when these are not parallel).

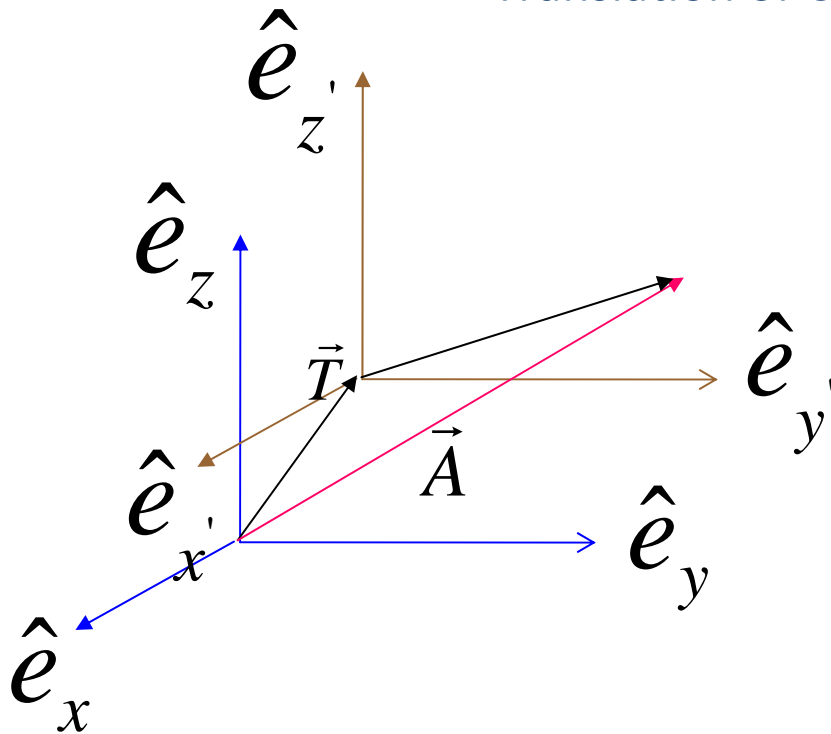
**Note that the use of parentheses in the triple cross products is necessary, since the cross product operation is not associative, i.e., generally we have**

$$(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$$

# Coordinate Transformations: translation

In engineering it is often necessary to express vectors in different coordinate frames. This requires the rotation and translation matrixes, which relates coordinates, i.e. basis (unit) vectors in one frame to those in another frame.

## Translation of Coordinate systems



Position coordinate

$$\vec{A}' = \vec{A} - \vec{T}$$