COMPUTER SCIENCE 3P03 (FALL 2020)

Algorithms

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Assignment #1

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1. $t(n) = t(n-1) + O(1)$

$$t(n) = t(n-1) + 1$$

$$t(n-1) < t(n-1-1)+1 = t(n-2)+1$$

$$t(n) = t(n-2)+1+1 = t(n-2) + 2$$

$$t(n-2) = t(n-2-1)+1 = t(n-3)+1$$

$$t(n) = t(n-3)+1+2 = t(n-3)+3$$

$$t(n) = t(n-k)+k$$

$$n-k = 1$$

$$n = 1+k$$

$$n-1 = k$$

$$t(n) = t(1) + k$$

$$= 1 + n - 1$$

$$= n$$

$$\boxed{\therefore\ t(n) = O(n)}$$

2.

$$\left.\begin{array}{l}\text{for } (n) \\ \quad f(n^2)\end{array}\right\} \quad n \cdot n^2 = O(n^3)$$

$$\left.\begin{array}{l}\text{for } (n) \\ \quad \text{for } (n)\end{array}\right\} \quad n \cdot n = O(n^2)$$

$$\left.\begin{array}{l}\text{for } (n) \\ \quad \text{while } (\log n)\end{array}\right\} \quad n \cdot \log n = O(n \log n)$$

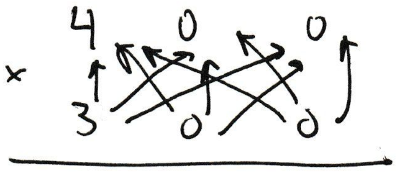
3. a) Adding two fixed digits $= O(1)$

Adding two n digits $= O(n)$

b) Multiplying two n digit numbers $= O(n^2)$

ex. $\begin{array}{r} 400 \\ \times\ 300 \\ \hline \end{array} = (0 \cdot 0)\ \&\ (0 \cdot 0)\ \&\ (0 \cdot 4) + (0 \cdot 0)\ \&\ (0 \cdot 0)\ \&\ (0 \cdot 4) + (3 \cdot 0)\ \&\ (3 \cdot 0)\ \&\ (3 \cdot 4)$

Each digit is multiplied n times



4.

A	B	O	o	Ω	Θ
$\log^k n$	$\log n^k$	yes	no	yes	yes
n^k	K^n	yes	yes	no	no
2^n	2^{cn}	yes	yes	no	no
2^n	2^{n+1}	yes	no	yes	yes
$n^{\log c}$	$c^{\log n}$	yes	no	yes	yes
$\log_3 5n$	$\log_{100} 500n$	no	no	yes	no
$\log(n!)$	$\log(n^n)$	yes	no	yes	yes
$100n + \log n$	$n + (\log n)^2$	yes	no	yes	yes
$\log n$	$\log(n^2)$	yes	no	yes	yes
$n^2/\log n$	$n \log^2 n$	yes	no	yes	yes
$n^{1/2}$	$(\log n)^5$	yes	no	yes	yes

5. $t(n) = 2t(n/2) + 1$

Iteration: $t(n/2) = 2t(n/2^2) + 1 \rightarrow t(n) = 2(2t(n/2^2)+1)+1$

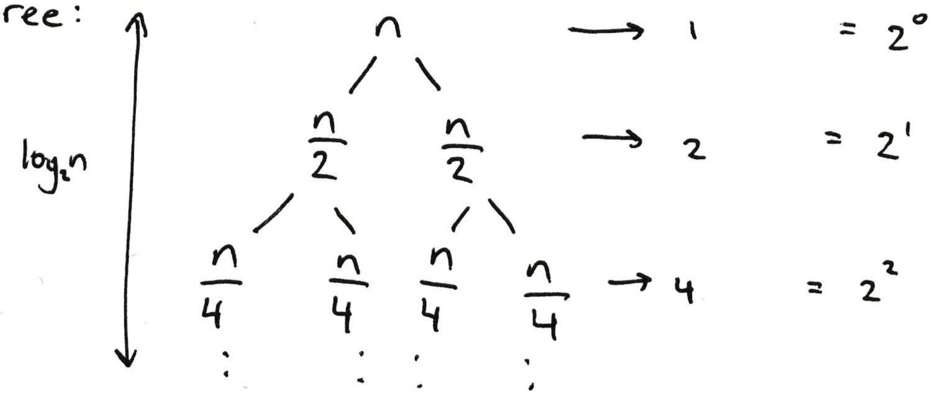
$$= 4t(n/2^2) + 2$$

$$t(n) = 2^k t(n/2^k) + k$$

$$1 = \frac{n}{2^k}, \quad n = 2^k, \quad k = \log n$$

$$t(n) = n\,t(1) + \log n = n + \log n = O(n)$$

Tree:



$$= 2^{\log n} = n^{\log_2 2} = O(n)$$

Master: $a = 2,\ b = 2,\ k = 0,\ p = 0$

$$a > b^k, \quad 2 > 2^0, \quad 2 > 1$$

$$\therefore t(n) = O(n^{\log_b a}) = O(n^{\log_2 2}) = O(n)$$

6. a) $t(1) = 1$, $t(n) = 3t(n/2) + n$

$$t(n) = 3t(n/2) + n$$

$$t(n/2) = 3t(n/2^2) + n/2$$

$$t(n) = 3\left(3t(n/2^2) + \frac{n}{2}\right) + n$$

$$t(n) = 3^2 t(n/2^2) + \frac{3n}{2} + n$$

$$t(n/2^2) = 3t(n/2^3) + n/2^2$$

$$t(n) = 3^2\left(3t(n/2^3) + \frac{n}{2^2}\right) + \frac{3n}{2} + n$$

$$t(n) = 3^3 t(n/2^3) + \frac{3^2 n}{2^2} + \frac{3n}{2} + n$$

$$t(n) = 3^k t(n/2^k) + \left(\frac{3}{2}\right)^{k-1} n + \left(\frac{3}{2}\right)^{k-2} n + \dots + \left(\frac{3}{2}\right)^{k-k} n$$

$$t(n) = 3^k t(n/2^k) + \sum_{i=0}^{L} \left(\frac{3}{2}\right)^i n$$

$$t(n) = 3^{\log_2 n} t(1)$$

$$t(n) = n^{\log_2 3}$$

$$t(n) = O(n^{\log_2 3})$$

6. b) $t(n) = 2t(n/2) + n\log n$

Guess: $t(n) \leq cn\log^2 n$

$$t(n) \leq 2c\frac{n}{2}\log^2(n/2) + n\log n$$

$$= cn\log^2(n/2) + n\log n$$

$$= cn\log^2 n - cn\log(2) + n\log n$$

$$\leq cn\log^2 n$$

$$\therefore t(n) = O(n\log^2 n)$$

7. $t(n) = 4t(n/2) + n$ $\quad a=4,\ b=2,\ k=1,\ p=0$

$a > b^k = 4 > 2^1$

$$\boxed{\therefore\ t(n) = \Theta(n^{\log_2 4}) = \Theta(n^2)}$$

$t(n) = 4t(n/2) + n^2$ $\quad a=4,\ b=2,\ k=2,\ p=0$

$a = b^k = 4 = 2^2$

$p > -1 = 0 > -1$

$$\boxed{\therefore\ t(n) = \Theta(n^{\log_b^a} \log^{p+1} n) = \Theta(n^{\log_2 4} \log^{0+1} n) = \Theta(n^2 \log n)}$$

$t(n) = 4t(n/2) + n^3$ $\quad a=4,\ b=2,\ k=3,\ p=0$

$a < b^k = 4 < 8$

$p \geq 0$

$$\boxed{\therefore\ t(n) = \Theta(n^k \log^p n) = \Theta(n^3 \log^0 n) = \Theta(n^3)}$$

$t(n) = 2t(n/2) + n/\log n$

Here $f(n) = n/\log n$ is not a polynomial, so we cannot use the master method.

Solved on next page →

$$t(n) = 2t(n/2) + \frac{n}{\log n}$$

$$t(n/2) = 2t(n/2^2) + \frac{(n/2)}{\log(n/2)}$$

$$t(n) = 2\left(2t(n/2^2) + \frac{(n/2)}{\log(n/2)}\right) + \frac{n}{\log n}$$

$$t(n) = 2^2 t(n/2^2) + \frac{n}{\log(n/2)} + \frac{n}{\log n}$$

$$t(n/2^2) = 2t(n/2^2) + \frac{(n/2^2)}{\log(n/2^2)}$$

$$t(n) = 2^2\left(2t(n/2^2) + \frac{(n/2^2)}{\log(n/2^2)}\right) + \frac{n}{\log(n/2)} + \frac{n}{\log n}$$

$$t(n) = 2^3 t(n/2^3) + \frac{n}{\log(n/2^2)} + \frac{n}{\log(n/2)} + \frac{n}{\log n}$$

$$t(n) = 2^k t(n/2^k) + n \cdot \sum_{i=0}^{k-1} \frac{1}{\log\left(\frac{n}{2^i}\right)} \quad , \quad \frac{n}{2^k} = 1 \; , \; n = 2^k \; , \; k = \log_2 n$$

$$t(n) = n\,t(1) + n \cdot \sum_{i=0}^{k-1} \frac{1}{\log n - i}$$

$$t(n) = n + n \cdot \log(k)$$

$$t(n) = n + n \cdot \log\log n \qquad \boxed{t(n) = O(n\log\log n)}$$

8. $2n^2 + 5n^3 = \Theta(n^3)$ $\quad f(n) = 2n^2 + 5n^3$, $g(n) = n^3$

① $f(n) = O(g(n))$ if $\exists$ c and K such that $0 \le f(n) \le c\,g(n)$ for all $n \ge K$.

$0 \le 2n^2 + 5n^3 \le cn^3$

try: $c = 6$, $K = 1$

$0 \le 2 + 5 \le 6$

$0 \le 7 \le 6$ ×

try: $c = 6$, $K = 2$

$0 \le 2^3 + 5 \cdot 2^3 \le 6 \cdot 2^3$

$0 \le 6 \cdot 2^3 \le 6 \cdot 2^3$ ✓

$\therefore f(n) = O(g(n))$ with constants $c = 6$ and $K = 2$.

② $f(n) = \Omega(g(n))$ if $\exists$ c and K such that $0 \le c\,g(n) \le f(n)$ for all $n \ge K$.

$0 \le cn^3 \le 2n^2 + 5n^3$

try: $c = 1$, $K = 2$

$0 \le 2^3 \le 2^3 + 5 \cdot 2^3$

$0 \le 2^3 \le 6 \cdot 2^3$ ✓

$\therefore f(n) = \Omega(g(n))$ with constants $c = 1$ and $k = 2$.

Since $f(n) = O(n^3)$ and $f(n) = \Omega(n^3)$ it is also $\Theta(n^3)$

$\square$

9. $t_1(n) = (n\log n^{100})/2 + f(n) = O(n\log n)$

$t_2(n) = \log^{1.5} n + \log(n^2) = O(\log^{1.5} n)$

$t_3(n) = 1 + 3 + 3^2 + \ldots + 3^n + 3^{n+1} + 3^{n+2} = O(3^n)$

$t_4(n) = \log 1 + \log 2 + \cdots + \log n = O(n\log n)$

$t_5(n) = 4n^{3/4} + 5n\log n + 2n\log\log n = O(n\log n)$

$t_6(n) = 1 + 2 + 3 + 4 + \ldots + 2020 = \frac{n(n+1)}{2} = O(n^2)$

$t_7(n) = \sum_{i=0}^{n} \binom{n}{i} = O(n^i)$

$\therefore t_2(n) < (t_5(n) < t_4(n) < t_1(n)) < (t_6(n) < t_7(n)) < t_3(n)$