COT 6405 Introduction to Theory of Algorithms

Topic 15. Minimum Spanning Tree

Minimum Spanning Tree

- Problem:
  – given a <u>connected</u>, <u>undirected</u>, <u>weighted</u> graph
    G = (V, E)
  – find a spanning tree using edges that connects all nodes with a minimal total weight w(T)= SUM(w[u,v])
    - w[u,v] is the weight of edge (u,v)
- Objectives: we will learn
  – Generic MST
  – Kruskal's algorithm
  – Prim's algorithm

Motivation Example

- Problem definition
  - A town has a set of houses and a set of roads
  - Each road connects 2 and only 2 houses
  - A road connecting houses *u* and *v* has a repair cost *w(u, v)*
- Goal*:* Repair enough (and no more) roads such that
  - everyone stays connected: can reach every house from all other houses, and
  - The total repair cost is minimum

Model as a graph

- The problem can be modeled as a graph
  – Undirected weighted graph *G = (V, E)*.
  – Weight *w(u, v)* on each edge *(u, v)* ∈ *E*.
- Find $T \subseteq E$, such that
  – *T* connects all vertices (*T* is a spanning tree)
  – $w(T) = \sum_{(u,v) \in T} w(u, v)$ is minimized.
- A spanning tree whose weight is minimum over all spanning trees is called a minimum spanning tree, *MST*.

Growing a minimum spanning tree

- Building up the solution
  - We will build a set *A* of edges
  - Initially, *A* has no edges.
  - As we add edges to *A*, maintain a loop invariant
- Loop invariant: *A* is a subset of some MST
  - Add only edges that maintain the invariant
  - Definition: If *A* is a subset of some MST, an edge *(u, v)* is safe for *A*, if and only if $A \cup \{(u, v)\}$ is also a subset of some MST
  - So we will add only safe edges

Generic MST algorithm

GENERIC-MST(G, w)

$A = \emptyset$

while A is not a spanning tree

  find an edge (u, v) that is safe for A

  $A = A \cup \{(u, v)\}$

return A

Correctness

- Use the loop invariant to show that this generic algorithm works.
  - **Initialization:** The empty set trivially satisfies the loop invariant.
  - **Maintenance:** Since we add only safe edges, A remains a subset of some MST.
  - **Termination:** All edges added to A are in an MST, so A is a spanning tree that is also an MST, when we stop

Definitions

- Let $S \subset V$ (vertex set); $A \subseteq E$ (edge set).
- A cut $(S, V - S)$ is a partition of vertices into two disjoint sets: S and V-S
- Edge $(u, v) \in E$ **crosses** the cut $(S, V-S)$ if one endpoint is in S and the other is in $V-S$.
- A cut **respects** edge set A, if and only if no edge in A crosses the cut.
- An edge is a **light edge** crossing a cut, if and only if its weight is minimum over all edges crossing the cut.
  - For a given cut, there can be > 1 light edge crossing it.

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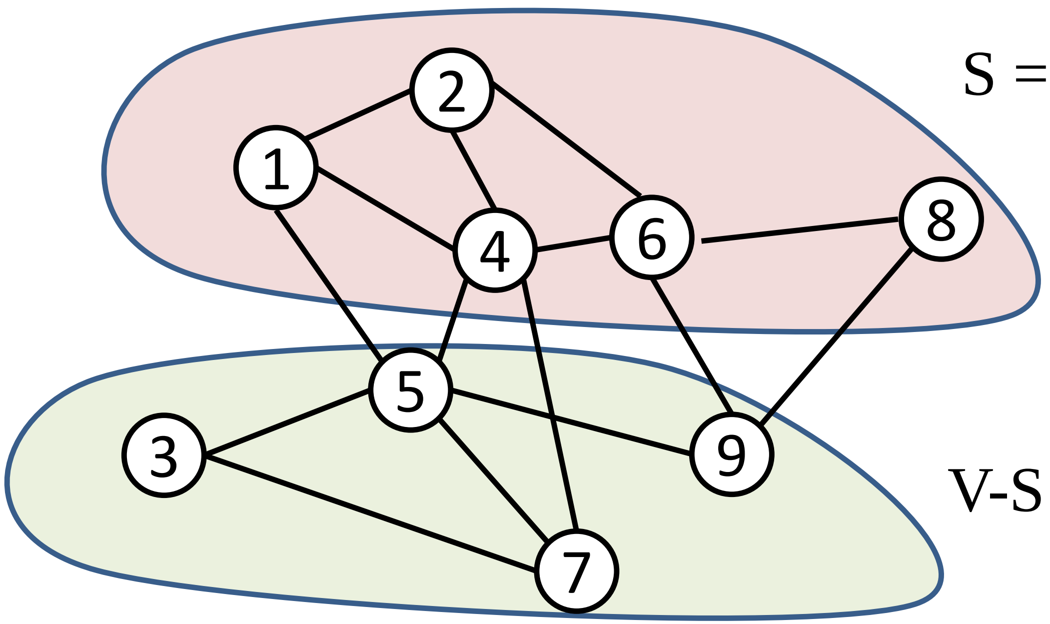


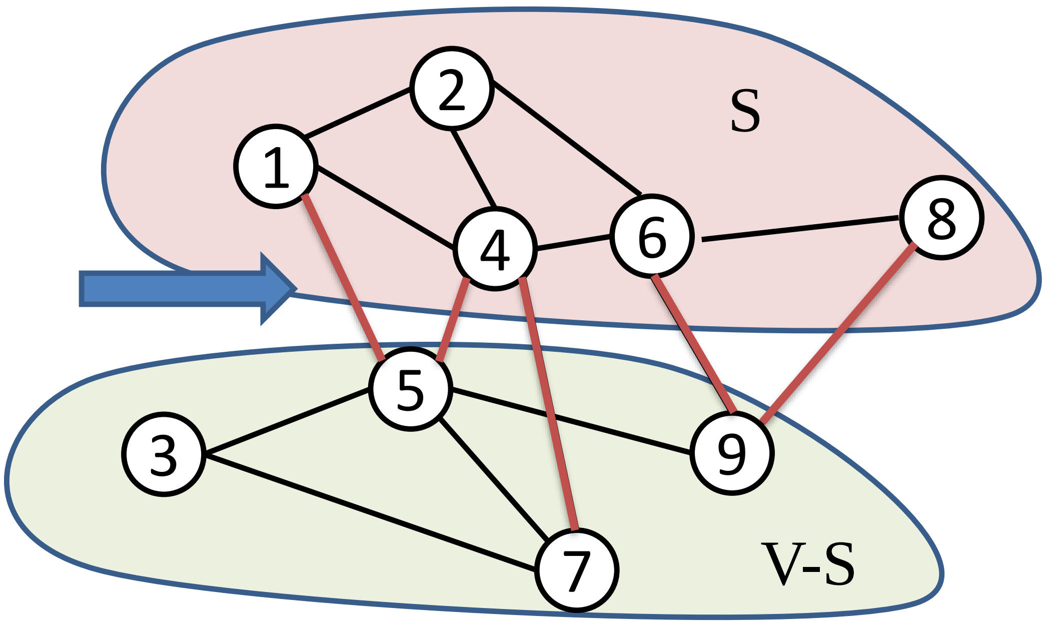
S = {1, 2, 4, 6, 8}

V-S = {3, 5, 7, 9}

Definitions

- Edge (u, v) ∈ E **crosses** the cut (S, V−S) if one endpoint is in S and the other is in V−S.

Edge(1, 5) crosses the cut (S, V-S)



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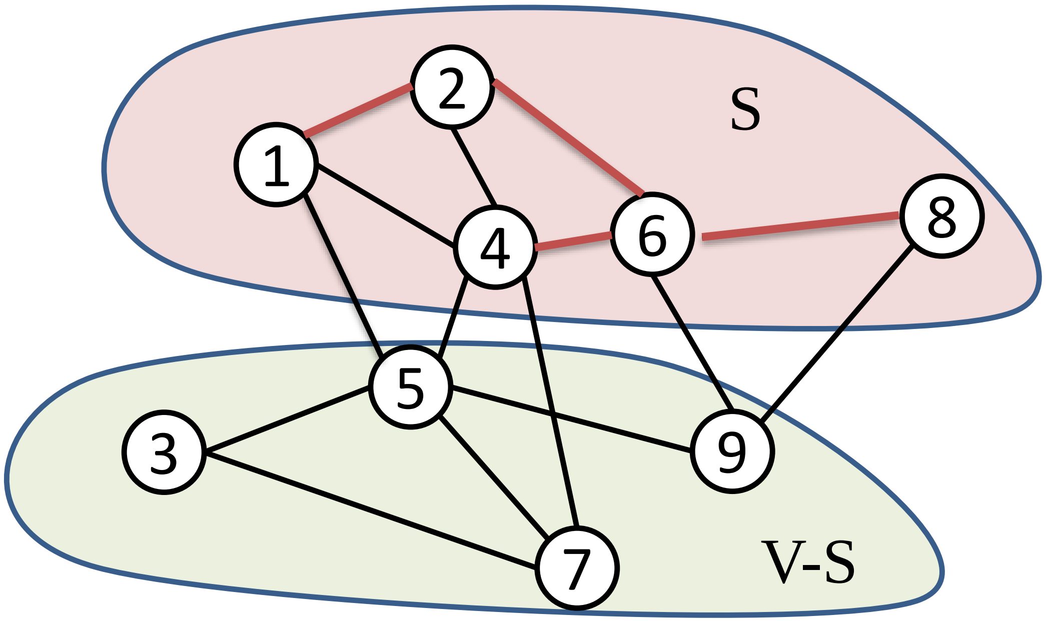

For example, A = {(1, 2),(2,6),(4,6),(6,8)}
cut(S, V-S) respects the edge set A

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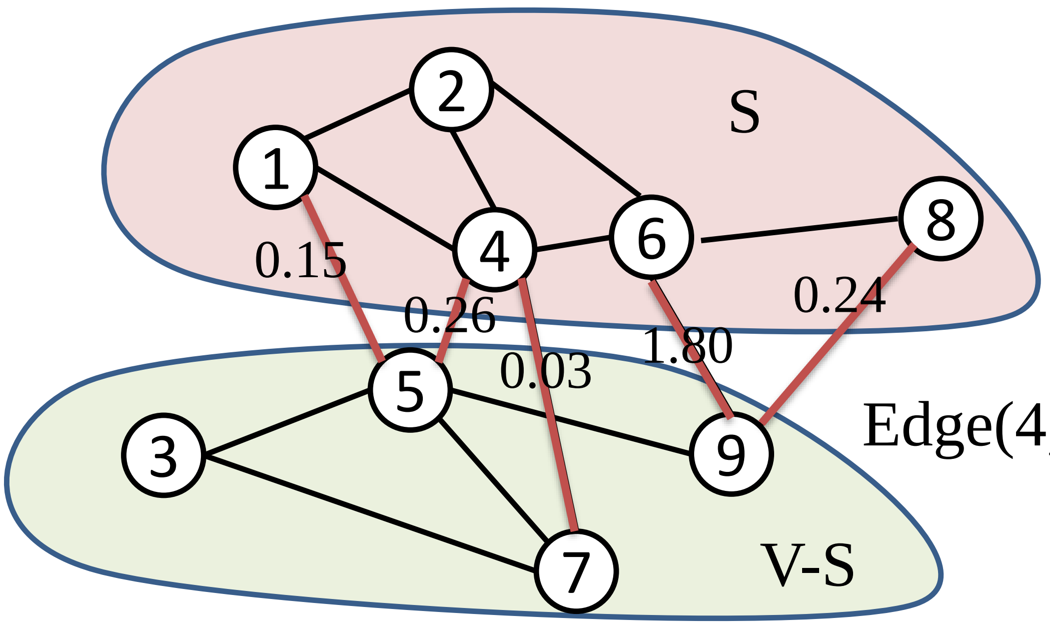
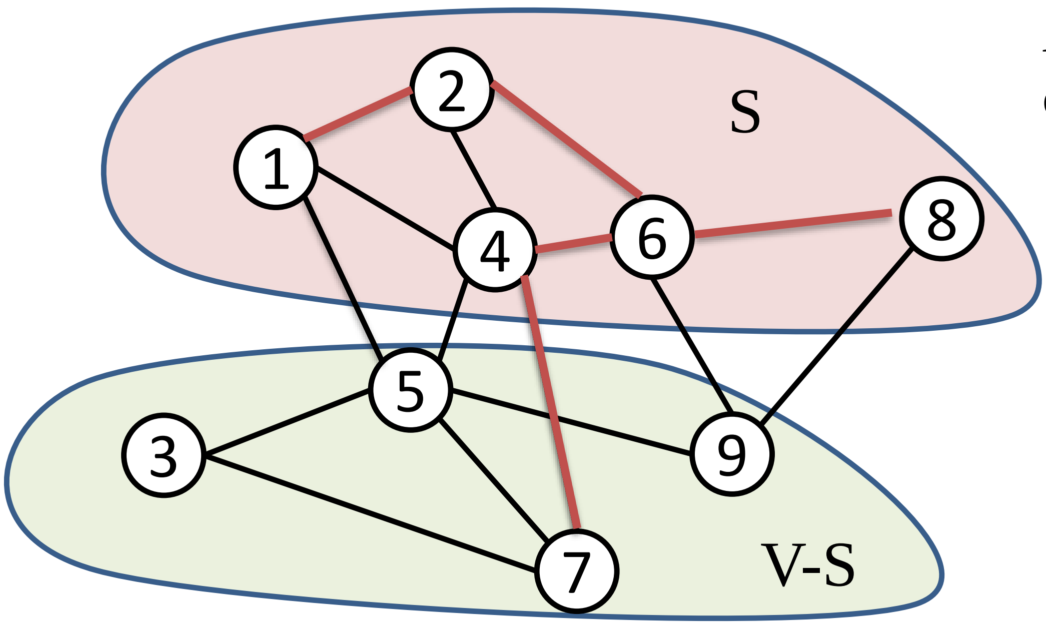


Edge(4,7) is the light edge

Theorem

- Let edge set *A* be a subset of some MST
- *(S, V −S)* be a cut that respects edge set *A*
  - No edges in A crosses the cut
- *(u, v)* be a light edge crossing cut *(S, V −S)*.
- Then, *(u, v)* is safe for *A*.



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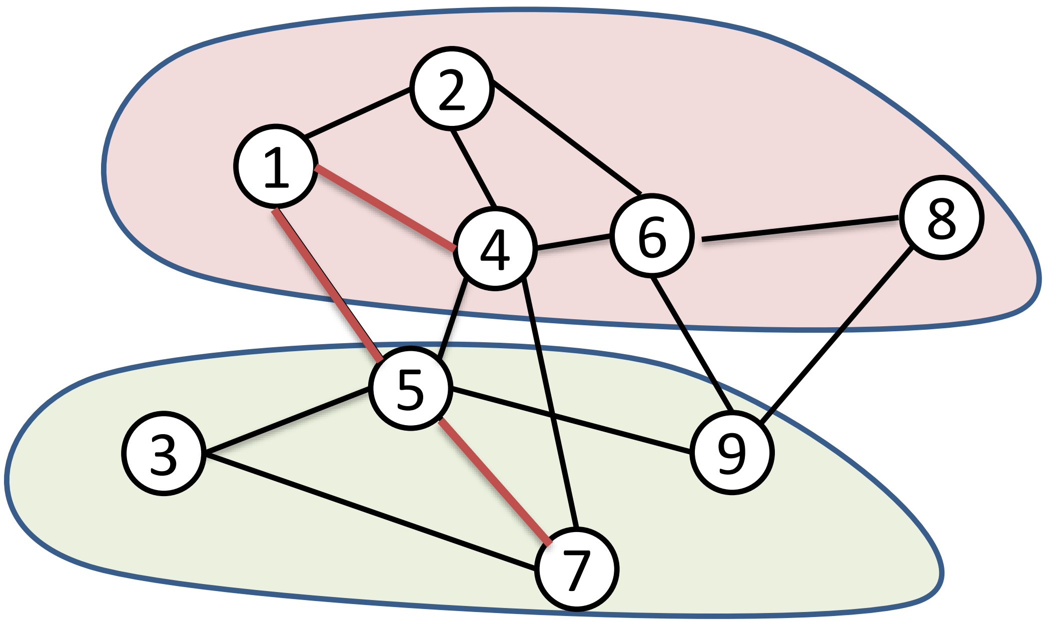
- Then, *(u, v)* is safe for *A*.
- **Proof**
  - Let tree T be an MST that includes edge set *A*
  - If *T* contains edge *(u, v)*, done.
  - So, now assume that *T* does not contain edge *(u, v)*
  - We'll construct a different MST T' that includes *A* ∪ {*(u, v)*}.

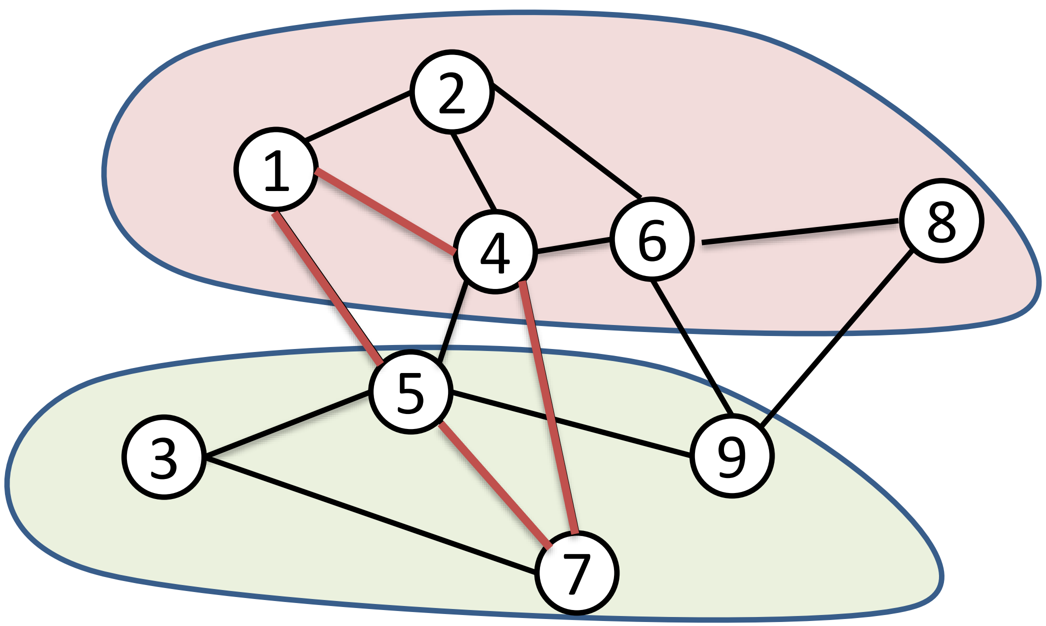
Proof

- Recall: a tree has a unique path between each pair of vertices (why?).
  - Since *T* is an MST, it contains a unique path *p* between *u* and *v*.
  - Path *p* must cross the cut *(S, V–S)* once
  - Let *(x, y)* be an edge of *p* that crosses the cut



Proof

- As *(u,v)* is a light edge, we have w(u,v) ≤ w(x,y)
- Since the cut respects *A*, edge *(x, y)* is not in *A*
- We can build tree T ' from *T*
  - Remove *(x, y)*: Breaks *T* into two components.
  - Reconnects them with edge *(u,v)* → *T'*



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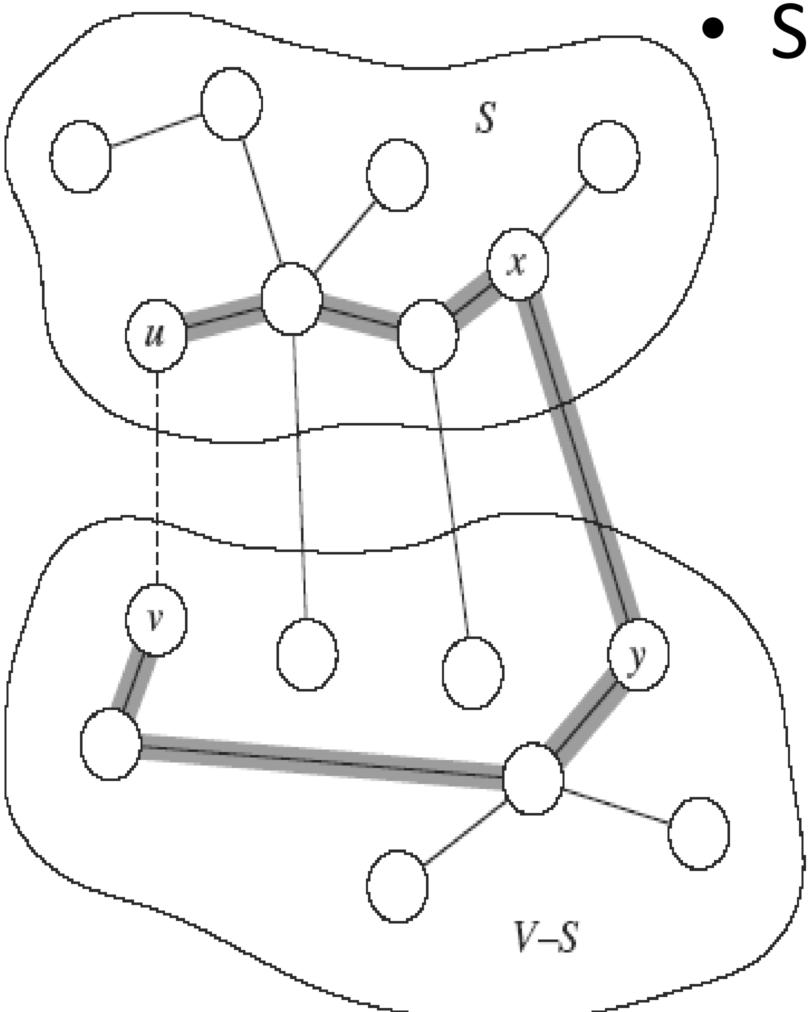


- Except for the dashed edge *(u, v)* , all edges shown are in *T*
- Shaded edges are the path *p*

Proof

- So $T' = T - \{(x, y)\} \cup \{(u, v)\}$.
- $\rightarrow$ T' is another spanning tree
- $w(T') = w(T) - w(x, y) + w(u, v) \leq w(T)$
  - since $w(u, v) \leq w(x, y)$
  - Since (1) T' is a spanning tree, (2) $w(T') \leq w(T)$, and (3) T is an MST $\rightarrow$ T' must be an MST
- Need to show that $A \cup \{(u, v)\} \subset T'$
  - $A \subseteq T$ and $(x, y) \notin A$ $\Rightarrow A \subseteq T - \{(x, y)\}$
  - $A \cup \{(u, v)\} \subseteq T - \{(x, y)\} \cup \{(u, v)\} = T'$
  - Since T' is an MST, edge (u, v) is safe for A.

MST: optimal substructure

- MSTs satisfy the optimal substructure property: an optimal tree is composed of optimal subtrees
  - Let T be an MST of G with an edge (u,v) in the middle
  - Removing (u,v) partitions T into two trees T_1 and T_2
  - Claim: T_1 is an MST of $G_1 = (V_1,E_1)$, and T_2 is an MST of $G_2 = (V_2,E_2)$
- Proof: $w(T) = w(u, v) + w(T_1) + w(T_2)$
  (There can't be a better tree than T_1 or T_2, or T would be suboptimal)

Kruskal's algorithm

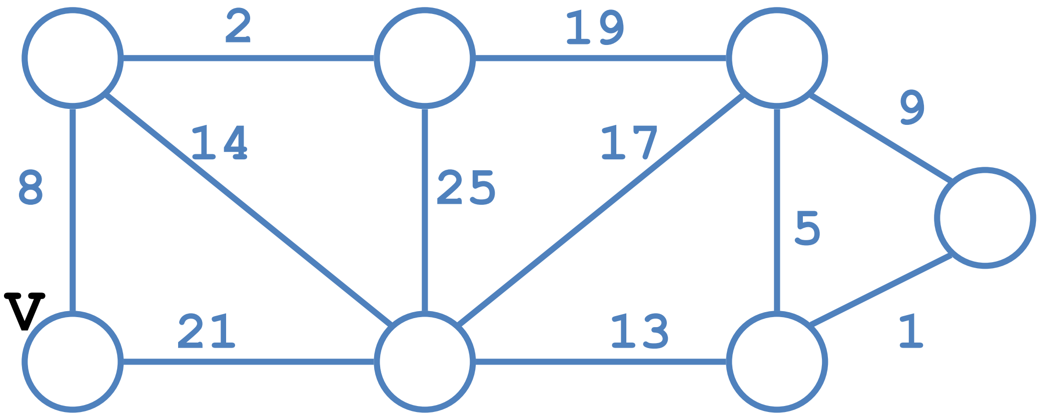
- Starts with each vertex being its own component
- Repeatedly merges two components into one by choosing the light edge that connects them
- Scans the set of edges in monotonically increasing order by weight
- Uses a disjoint-set data structure to determine whether an edge connects vertices in different components.

Disjoint Sets Data Structure

- A disjoint-set is a collection $C = \{S_1, S_2, \ldots, S_k\}$ of distinct dynamic sets
- Each set is identified by a member of the set, called representative.
- Disjoint set operations:
  - MAKE-SET(x): create a new set with only x
    - assume x is not already in some other set.
  - UNION(x,y): combine the two sets containing x and y into one new set.
    - A new representative is selected.
  - FIND-SET(x): return the representative of the set containing x.

Kruskal's Algorithm

Run the algorithm:

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{
    A = ∅;
    for each v ∈ G.V
        Make-Set(v);
    sort G.E by non-decreasing order by weight w
    for each (u,v) ∈ G.E (in sorted order)
        if FindSet(u) ≠ FindSet(v)
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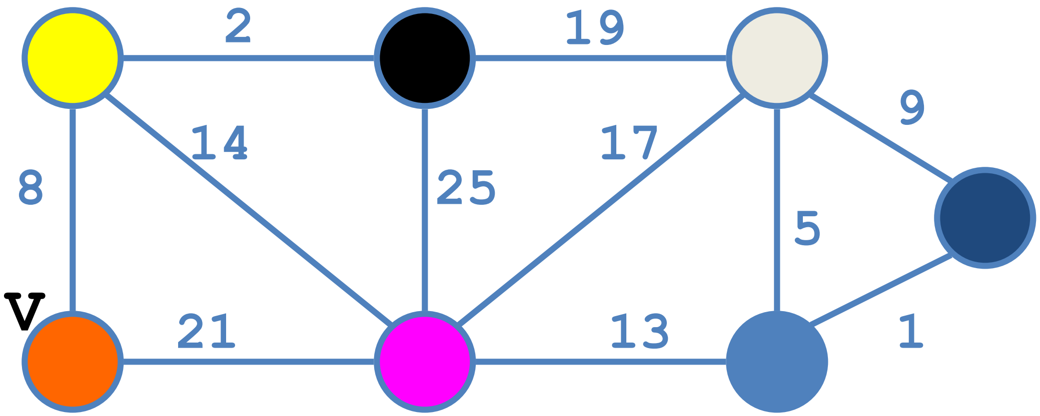


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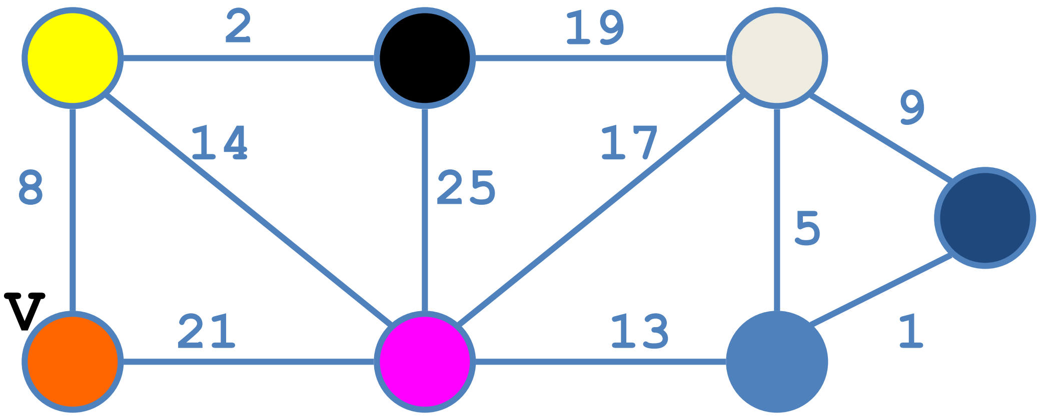


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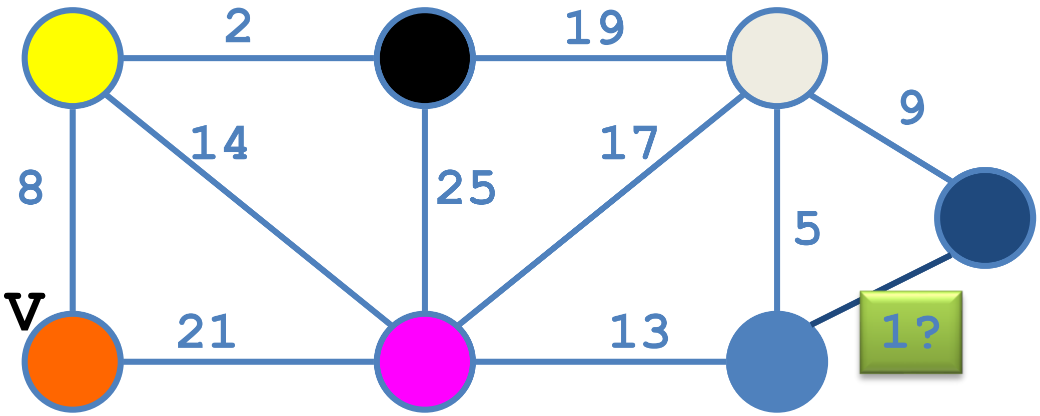


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        if FindSet(u) ≠ FindSet(v) // same tree?
            A = A U {{u,v}};
            Union(u, v);
}
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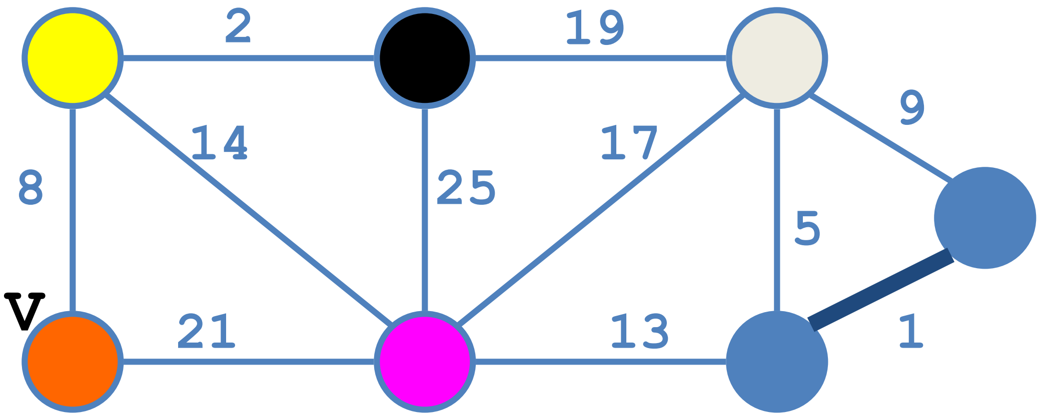


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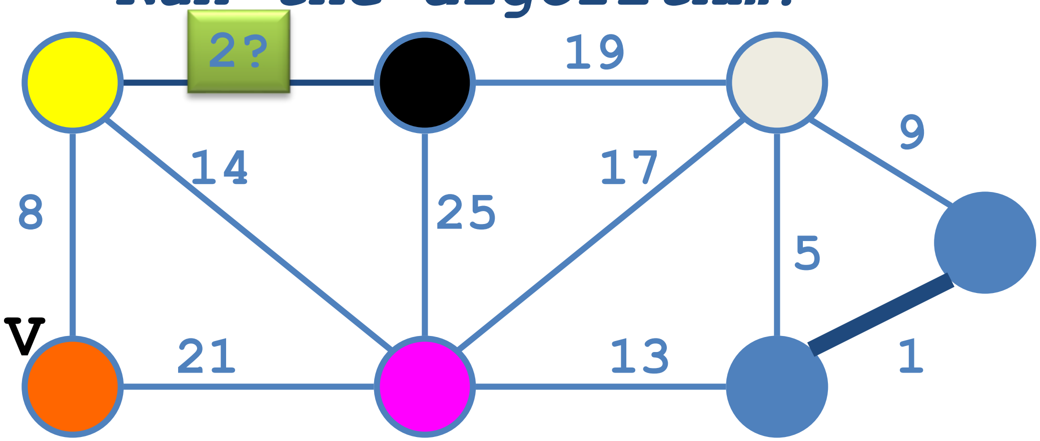

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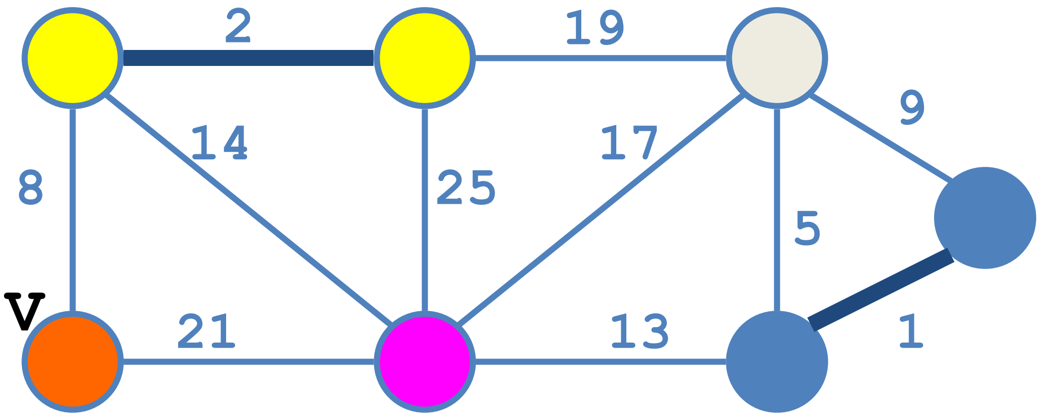


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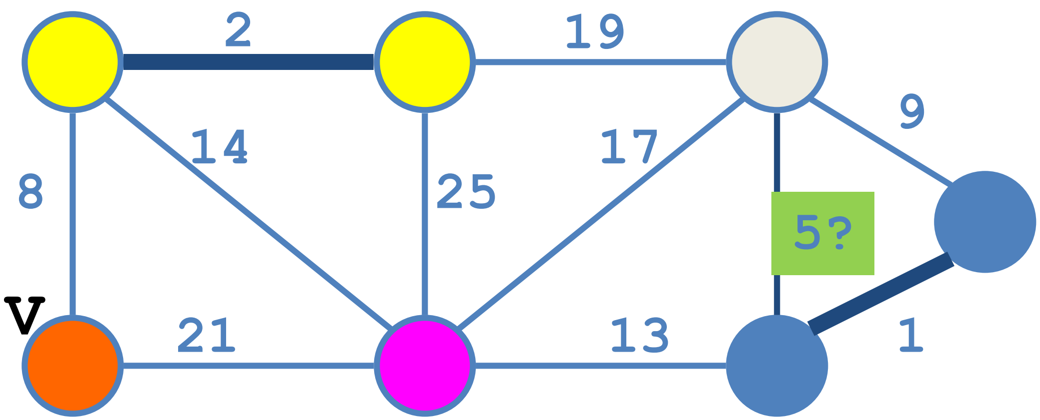


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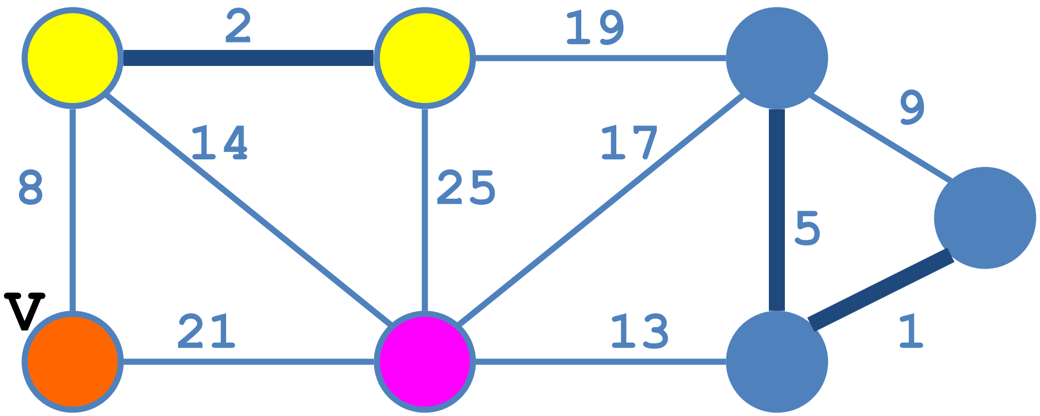


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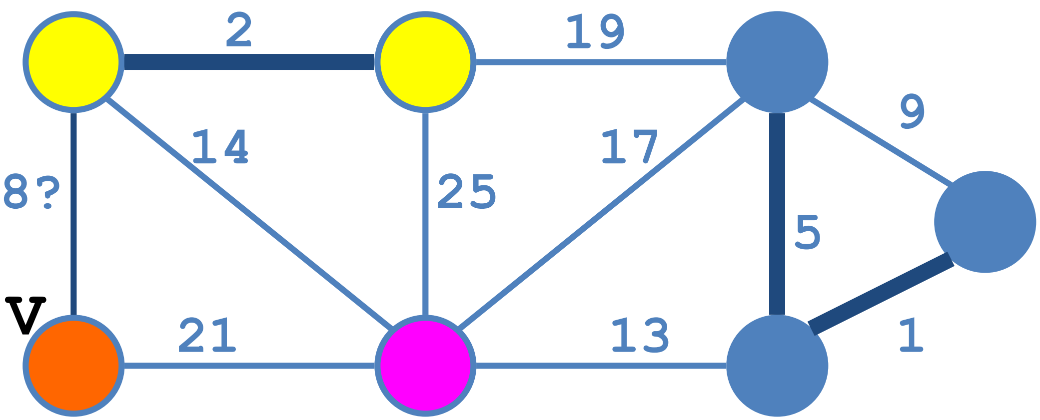


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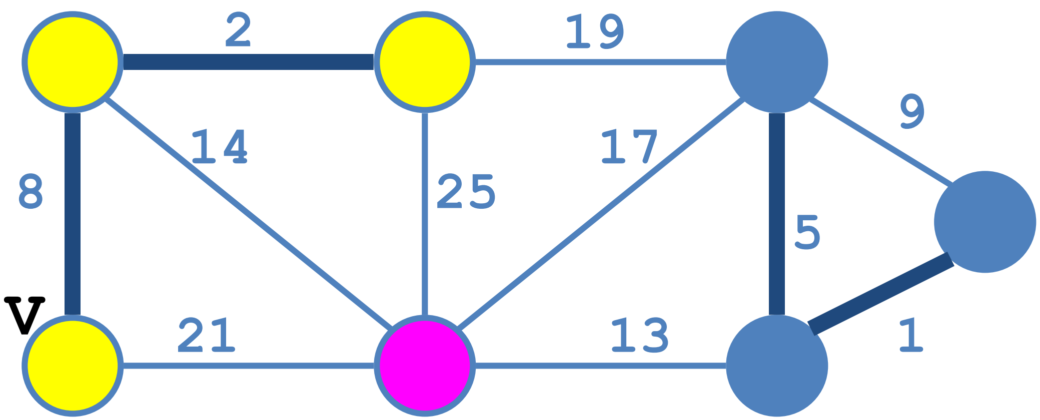


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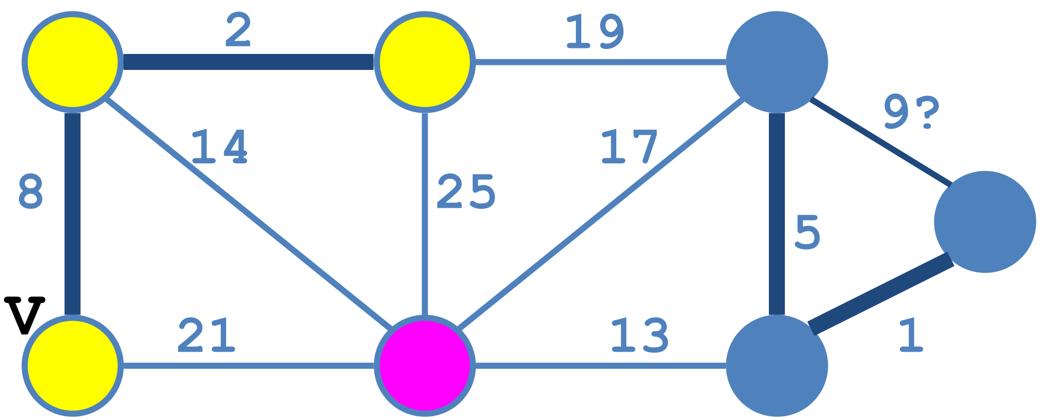


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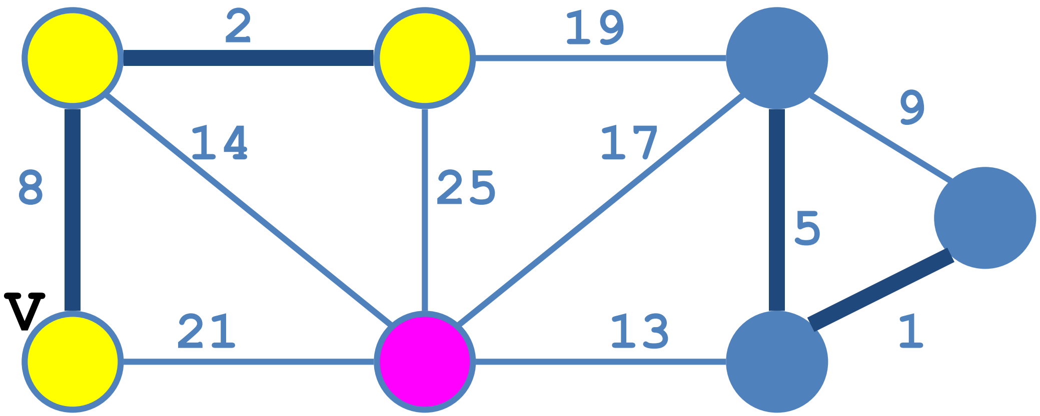


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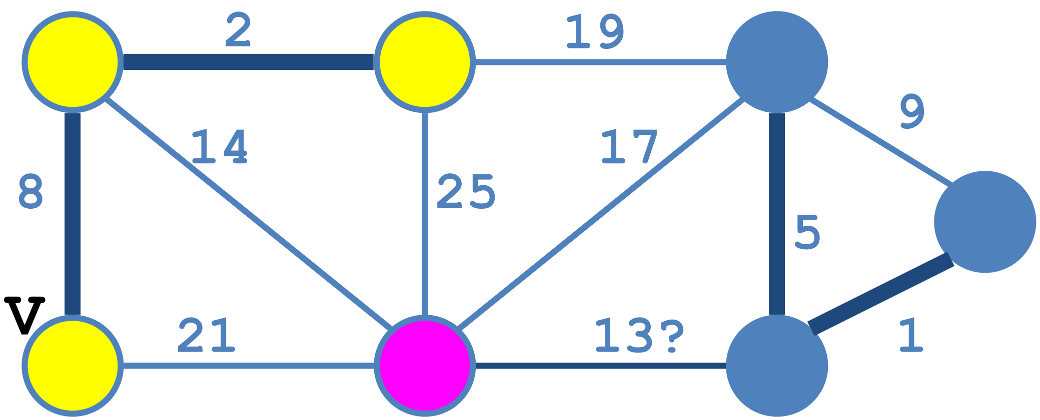


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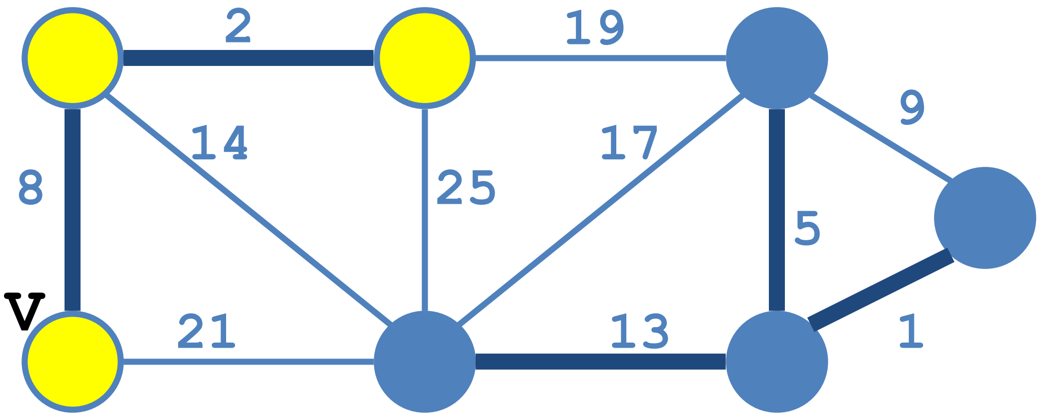


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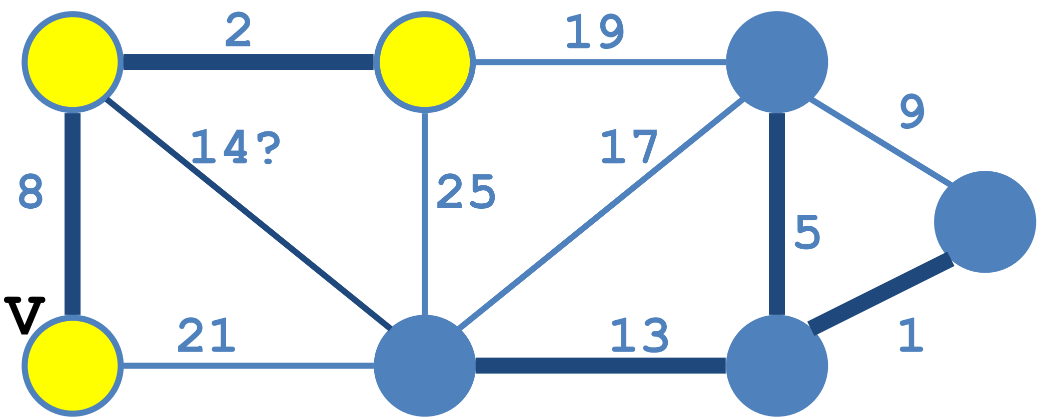


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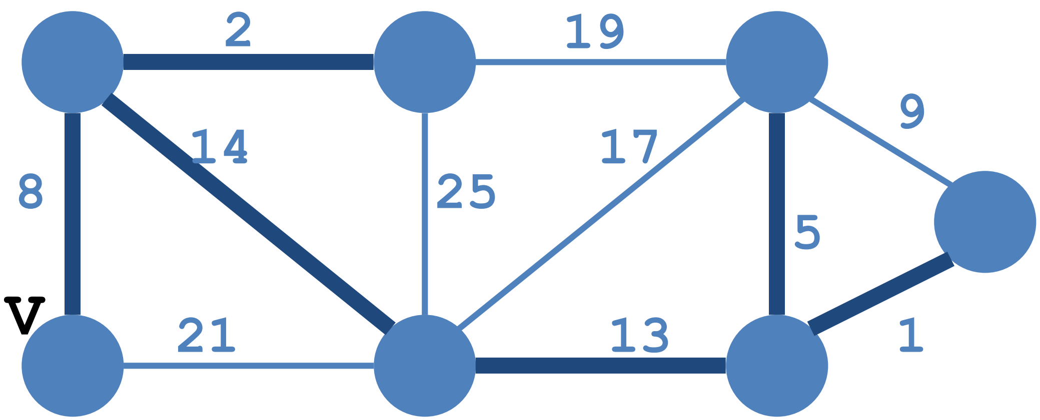


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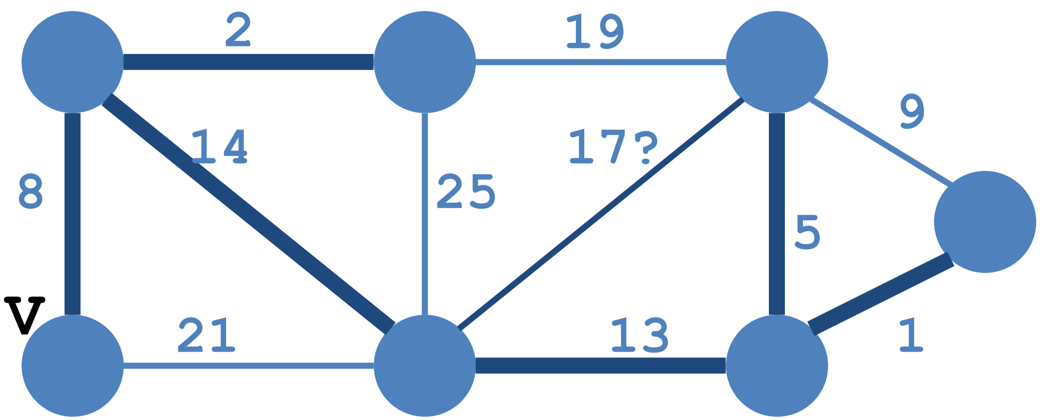


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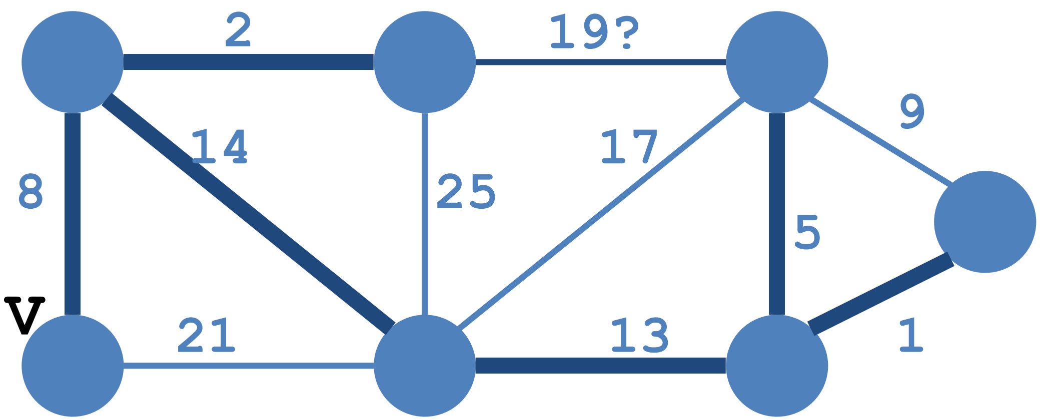


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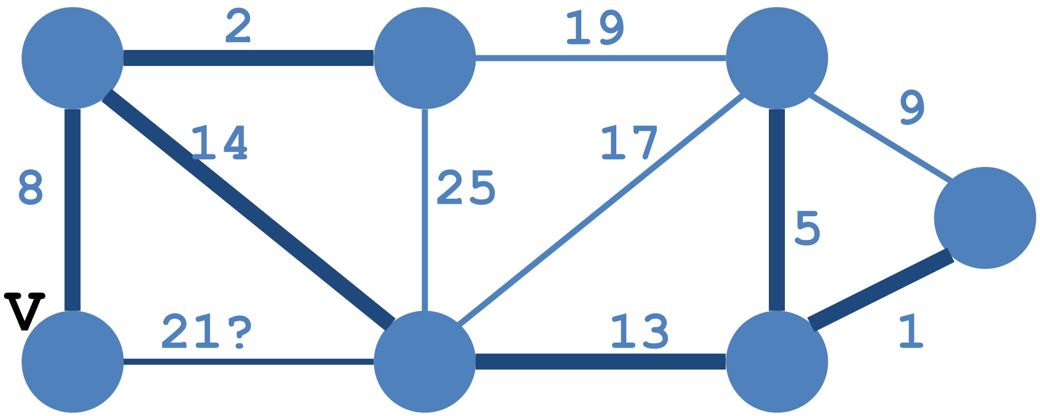


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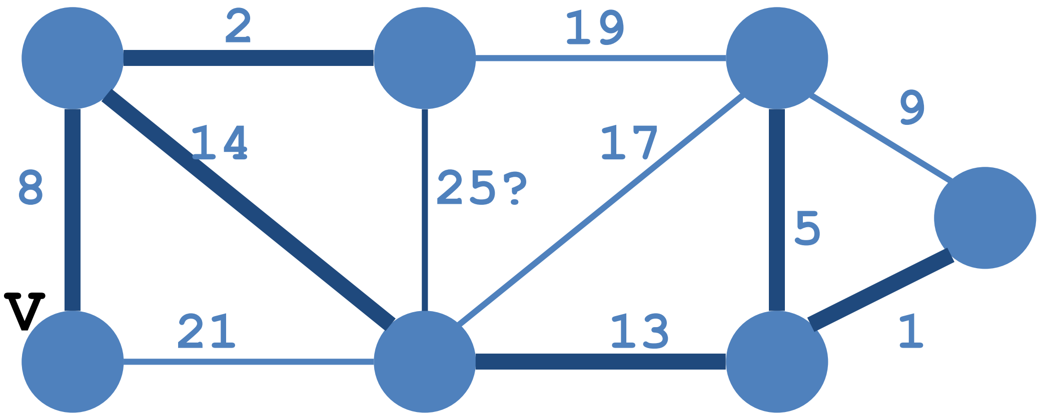
    for each v ∈ G.V
        Make-Set(v);
    sort G.E by non-decreasing order by weight w
    for each (u,v) ∈ G.E (in sorted order)
        if FindSet(u) ≠ FindSet(v)
            A = A U {{u,v}};
            Union(u, v);
}
```

Kruskal's Algorithm

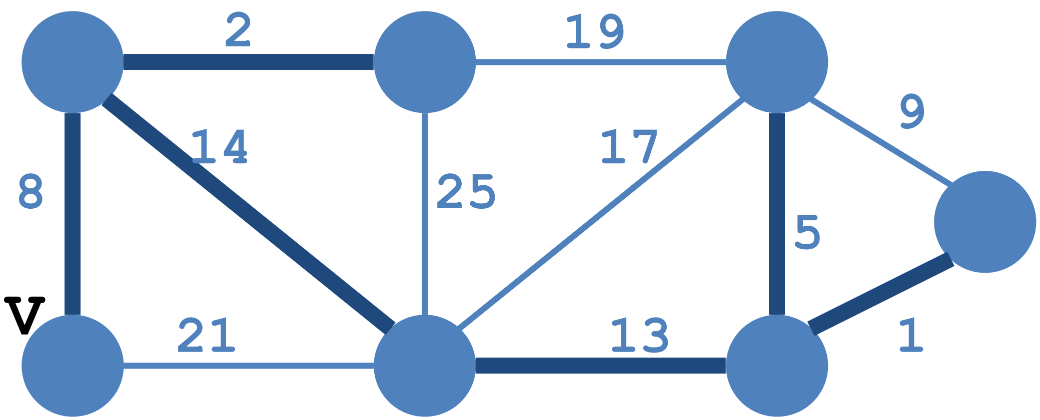
Run the algorithm:




```
Kruskal(G, w)
{
    A = ∅;
    for each v ∈ G.V
        Make-Set(v);
    sort G.E by non-decreasing order by weight w
    for each (u,v) ∈ G.E (in sorted order)
        if FindSet(u) ≠ FindSet(v)
            A = A U {{u,v}};
            Union(u, v);
}
```

Kruskal's Algorithm: Done

Run the algorithm:

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        if FindSet(u) ≠ FindSet(v)
            A = A U {{u,v}};
            Union(u, v);
}
```

Correctness Of Kruskal’s Algorithm

- Sketch of a proof: this algorithm produces an MST of *T*
  - Assume algorithm is wrong: result is not an MST
  - Then, algorithm adds a wrong edge at some point
  - If it adds a wrong edge, there must be another lower weight edge
  - But algorithm chooses lowest weight edge at each step. Contradiction

Kruskal's Algorithm

```
Kruskal(G, w)
{
    A = ∅;
    for each v ∈ G.V
        Make-Set(v);
    sort G.E by non-decreasing order by weight w
    for each (u,v) ∈ G.E (in sorted order)
        if FindSet(u) ≠ FindSet(v)
            A = A U {{u,v}};
            Union(u, v);
}
```

What will affect the running time?

Initialize A	**O(1)**
1st FOR loop	**|V| MakeSet() calls**
Sort	**O(E lgE)**
FINDSET()/Union()	**O(E) calls**

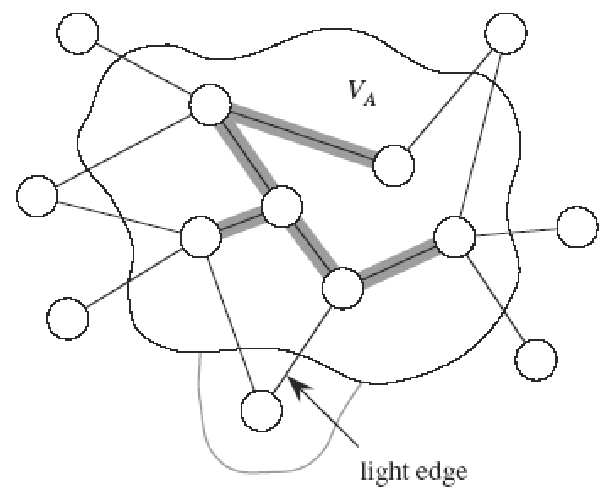
Kruskal's Algorithm: Running Time

- Initialize A: O(1)
- First for loop: |V| MAKE-SETs
- Sort E: O(E lg E)
- Second for loop: O(E) FIND-SETs and UNIONs
- **O(V) +O (E α(V)) + O(E lg E)**
  - Since G is connected, $|E| \geq |V|-1 \Rightarrow O(E\ \alpha(V)) + O(E \lg E)$
  - $\alpha(|V|) = O(\lg V) = O(\lg E)$
  - Therefore, the total time is O(E lg E)
  - $|E| \leq |V|^2 \Rightarrow \lg |E| = O(2 \lg V) = O(\lg V)$
  - Therefore, O(E lg V) time

Prim's algorithm

- Build a tree *A*
  - Starts from an arbitrary "root" *r*.
  - At each step, find a light edge crossing the cut $(V_A, V - V_A)$, where V_A = vertices that *A* is incident on.
  - Add this light edge to *A*.
- GREEDY CHOICE*:*

  add min weight to A



[Edges of A are shaded.]

How to find the light edge quickly?

- Use a priority queue Q
  - Each object is a vertex in $V{-}V_A$
  - Key of v is the minimum weight of any edge (u, v), where $u \in V_A$
  - the vertex returned by EXTRACT-MIN is v
    - such that there exists $u \in V_A$, and edge (u, v) is a light edge crossing $(V_A, V{-}V_A)$
- Key of v is ∞, if v is not adjacent to any vertices in V_A

How to find the light edge quickly?

- The edges of A form a rooted tree with root r
  - r is given as an input to the algorithm, but it can be any vertex
  - Each vertex knows its parent in the tree by the attribute v.π = parent of v
  - $\pi[v]$ = NIL, if $v = r$ or v has no parent.
  - As the algorithm progresses, $A = \{(v, v.\pi) : v \in V - \{r\} - Q\}$

Prim's Algorithm

```
MST-Prim(G, w, r)
    for each u ∈ G.V
        u.key = ∞
         u.π = NIL
    r.key = 0
    Q = G.V
    while (Q not empty)
        u = ExtractMin(Q)
        for each v ∈ G.Adj[u]
            if (v ∈ Q and w(u,v) < v.key )
                 v.π = u
                v.key = w(u,v)
```

Prim's Algorithm

```
MST-Prim(G, w, r)
    for each u ∈ G.V
        u.key = ∞
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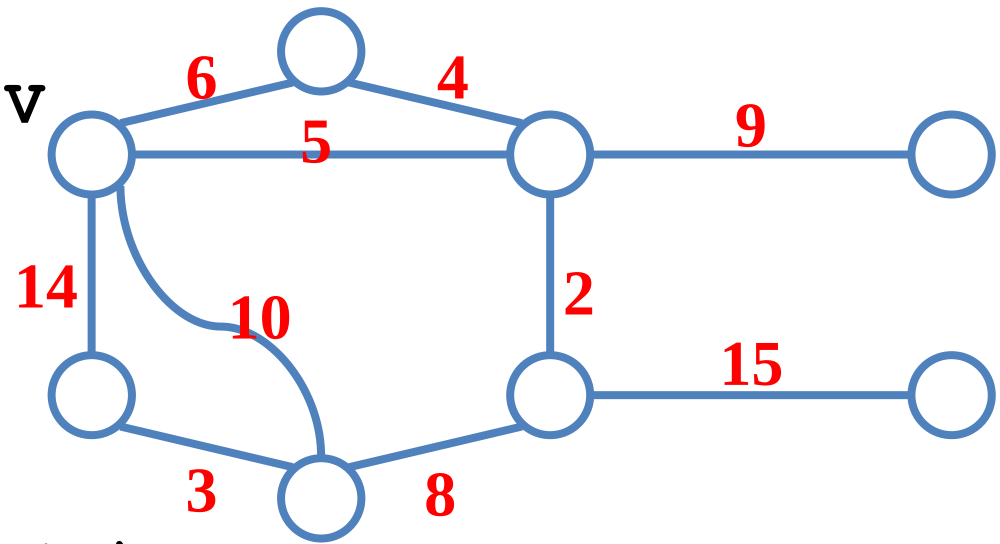
```



Run on example graph

Prim's Algorithm

```
MST-Prim(G, w, r)
    for each u ∈ G.V
        u.key = ∞
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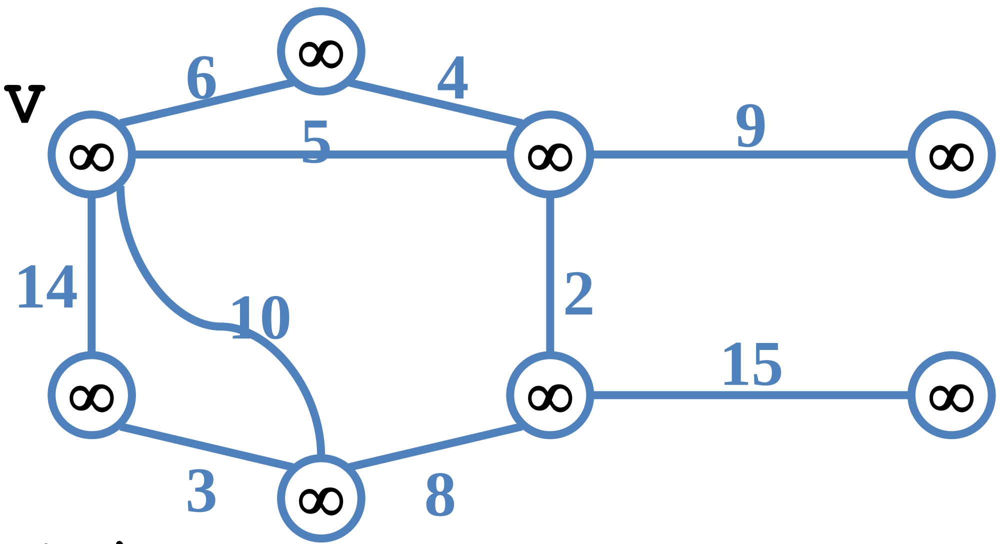
```



Run on example graph

Prim's Algorithm

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    for each u ∈ G.V
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        for each v ∈ G.Adj[u]
            if (v ∈ Q and w(u,v) < v.key)
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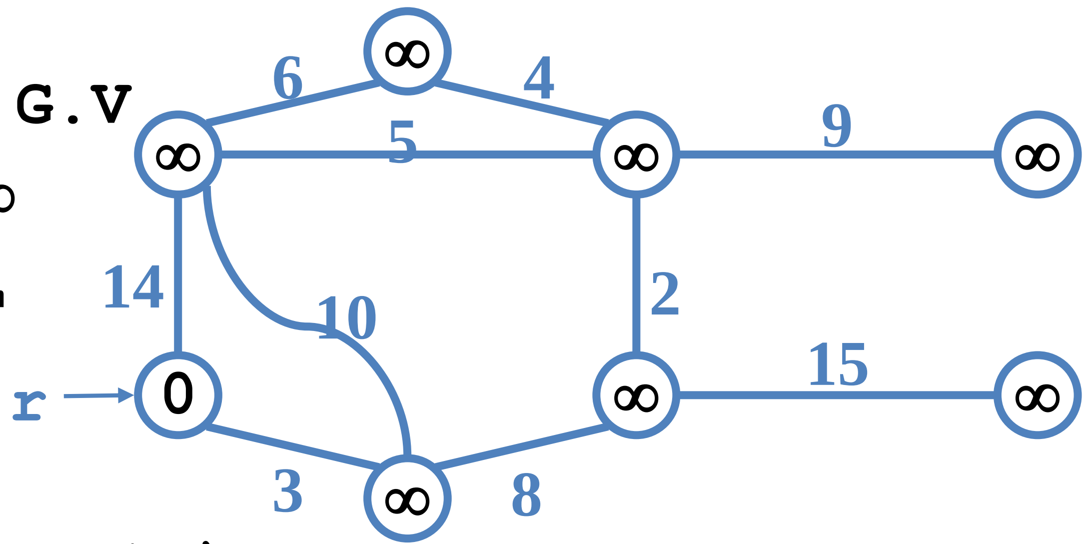
```



Pick a start vertex r

Prim's Algorithm

```
MST-Prim(G, w, r)
    for each u ∈ G.V
        u.key = ∞
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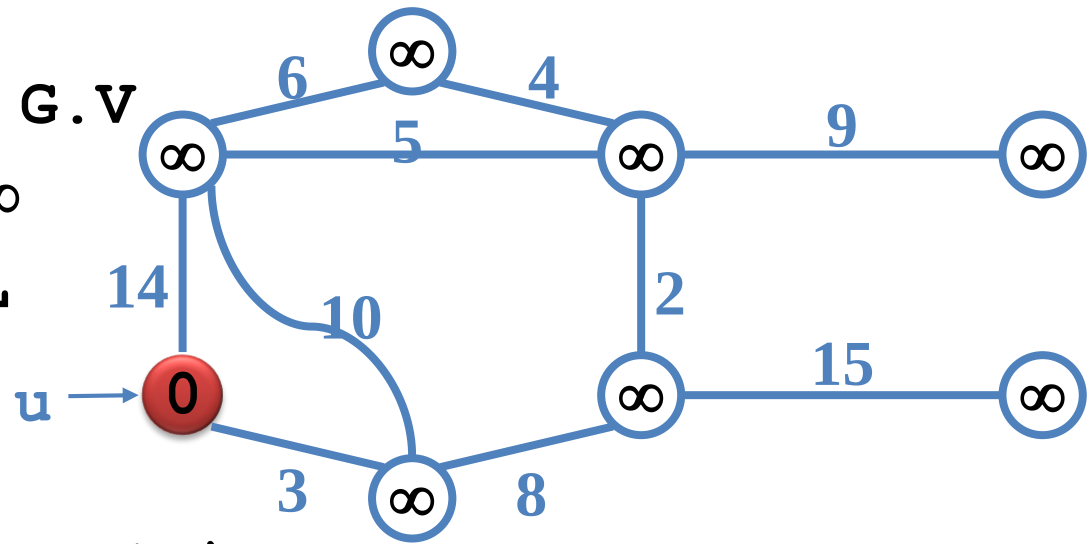
    r.key = 0
    Q = G.V
    while (Q not empty)
        u = ExtractMin(Q)
        for each v ∈ G.Adj[u]
            if (v ∈ Q and w(u,v) < v.key)
                v.π = u
                v.key = w(u,v)
```



Red vertices have been removed from Q

Prim's Algorithm

```
MST-Prim(G, w, r)
    for each u ∈ G.V
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            if (v ∈ Q and w(u,v) < v.key)
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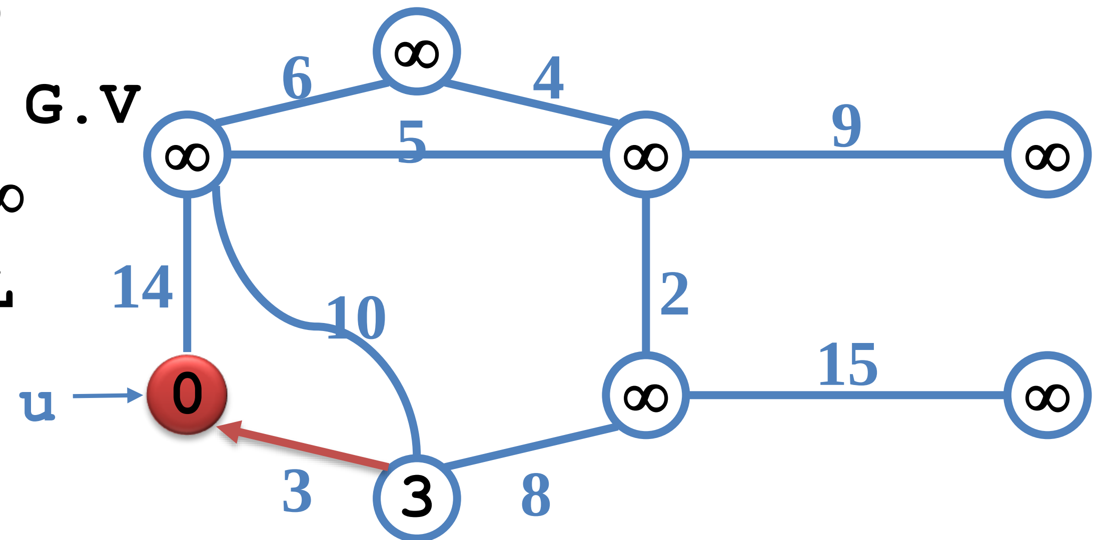

Red arrows indicate parent pointers

Prim's Algorithm

```
MST-Prim(G, w, r)
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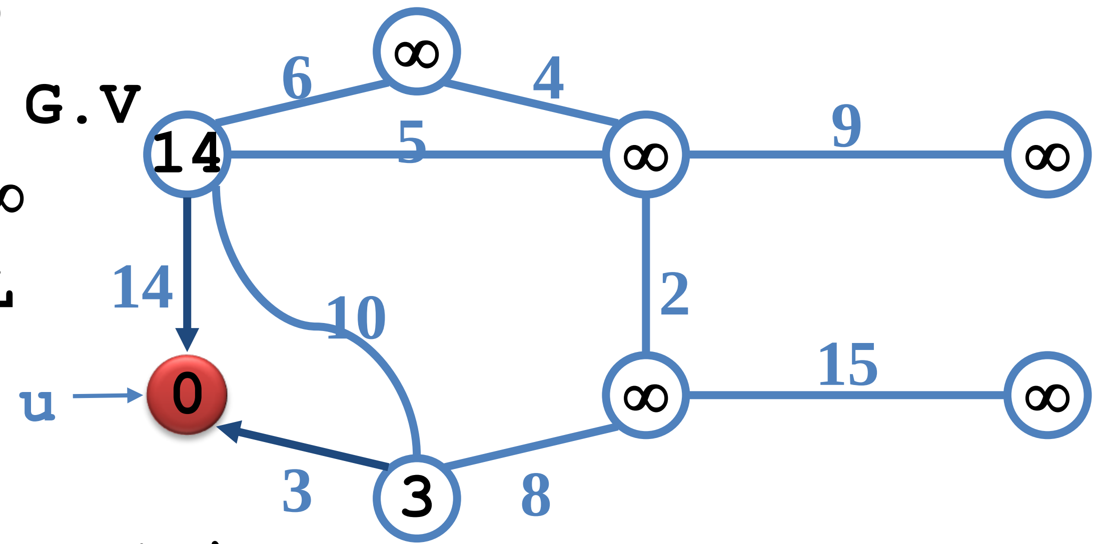

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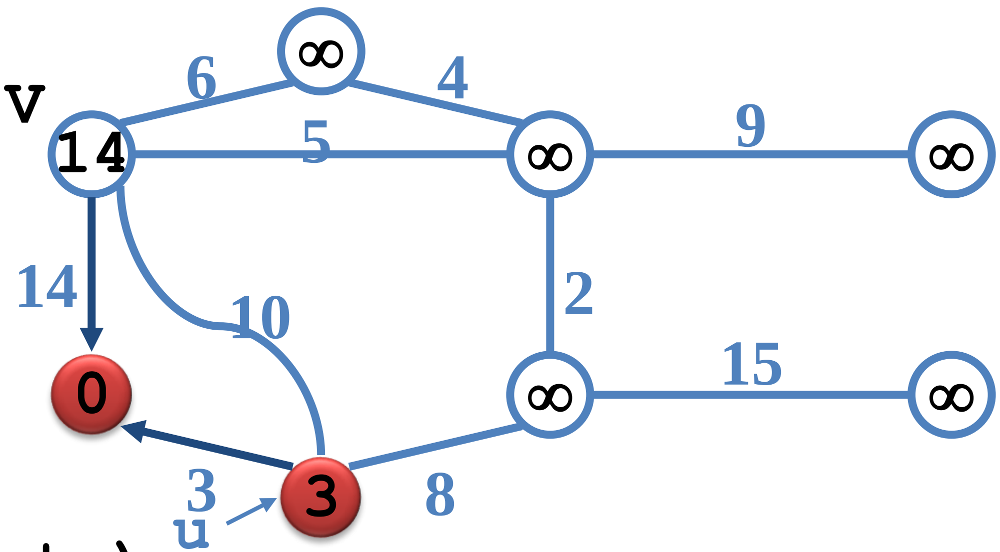

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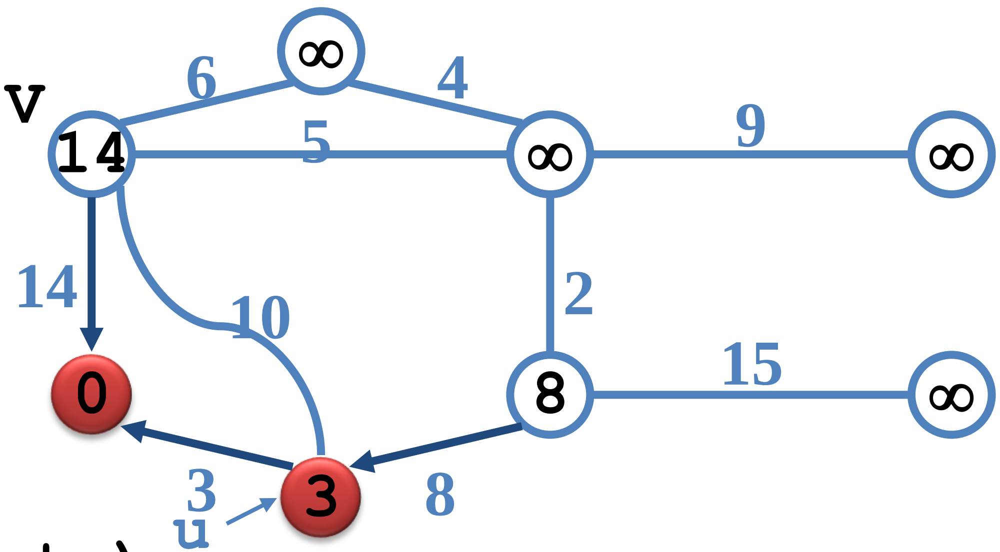

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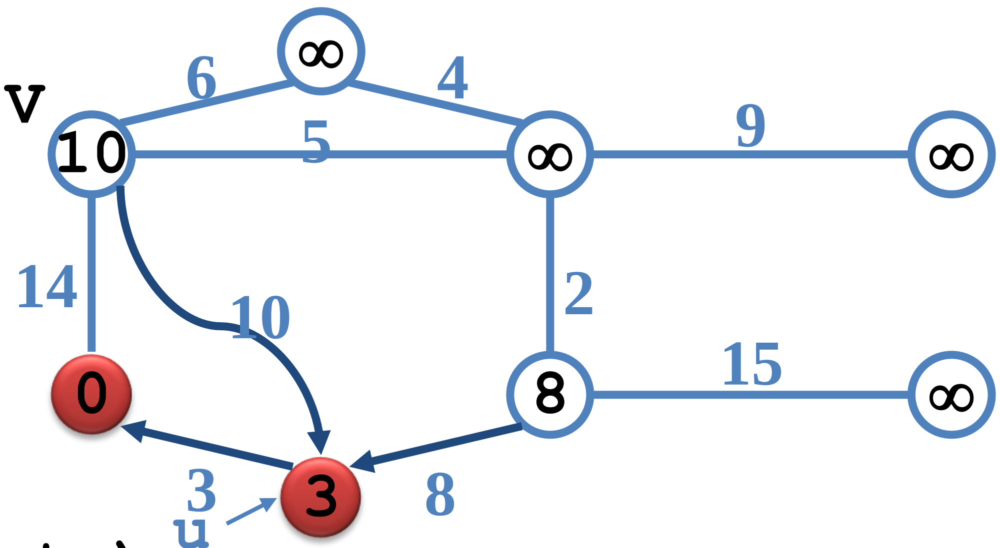

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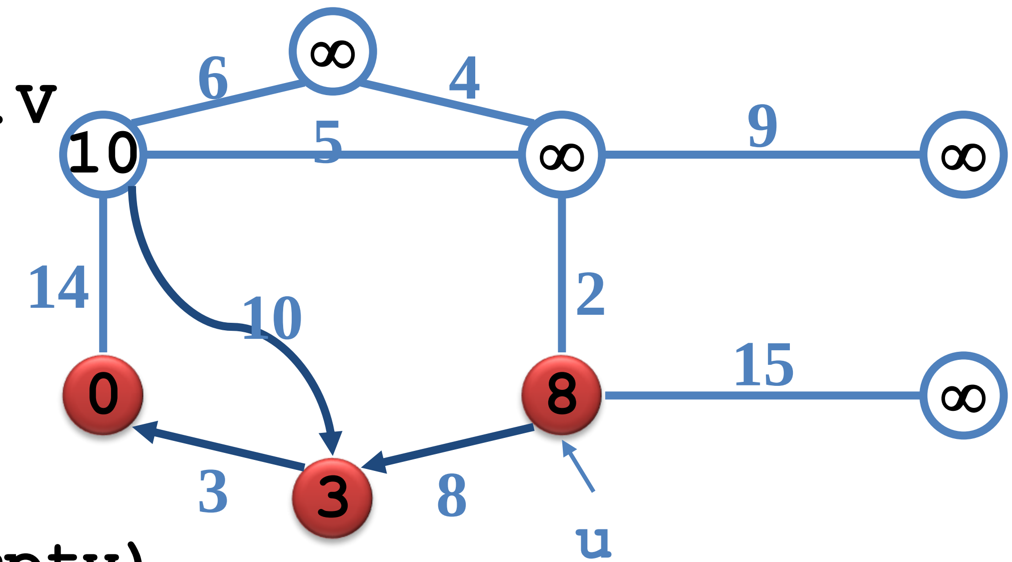

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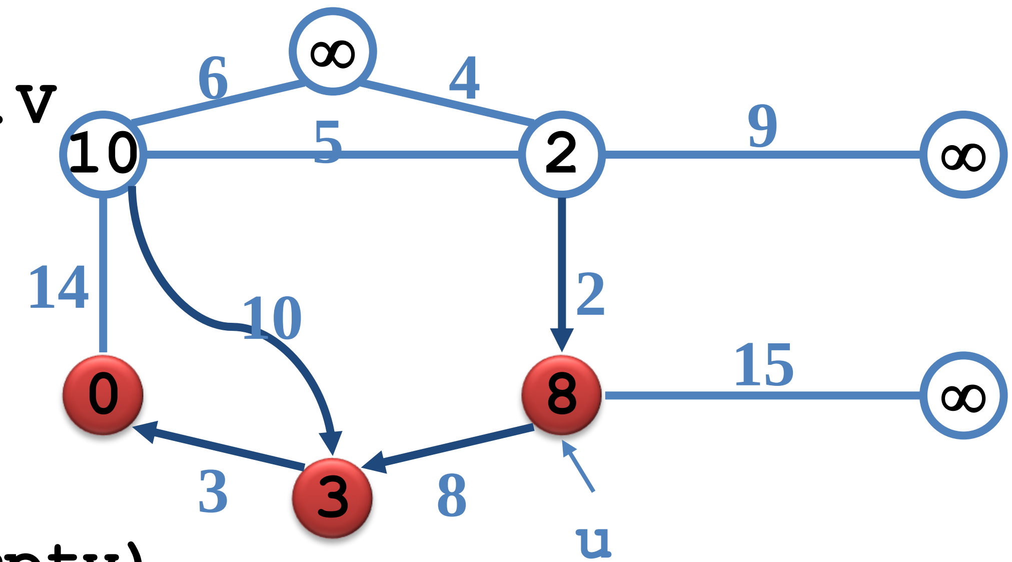

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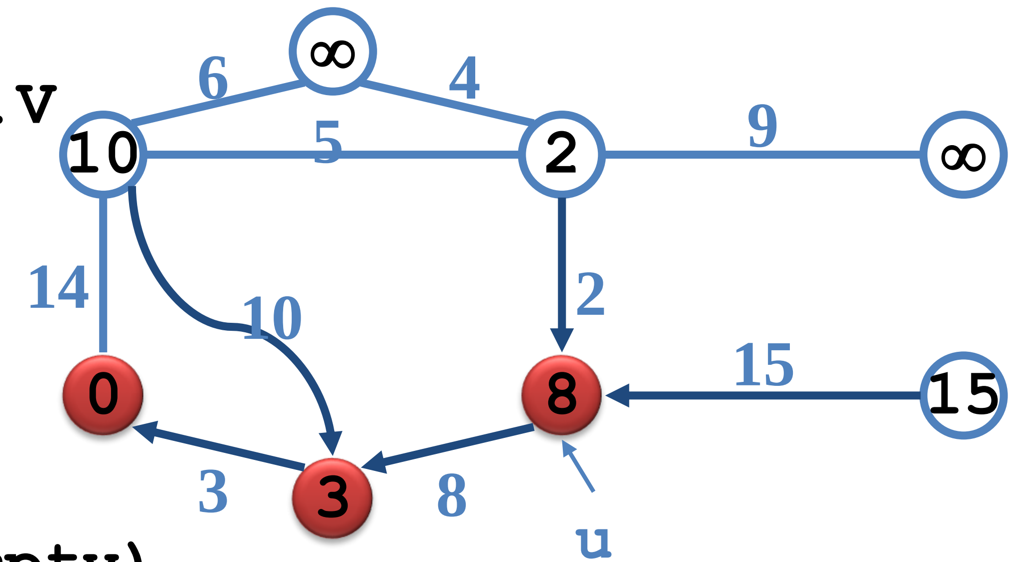

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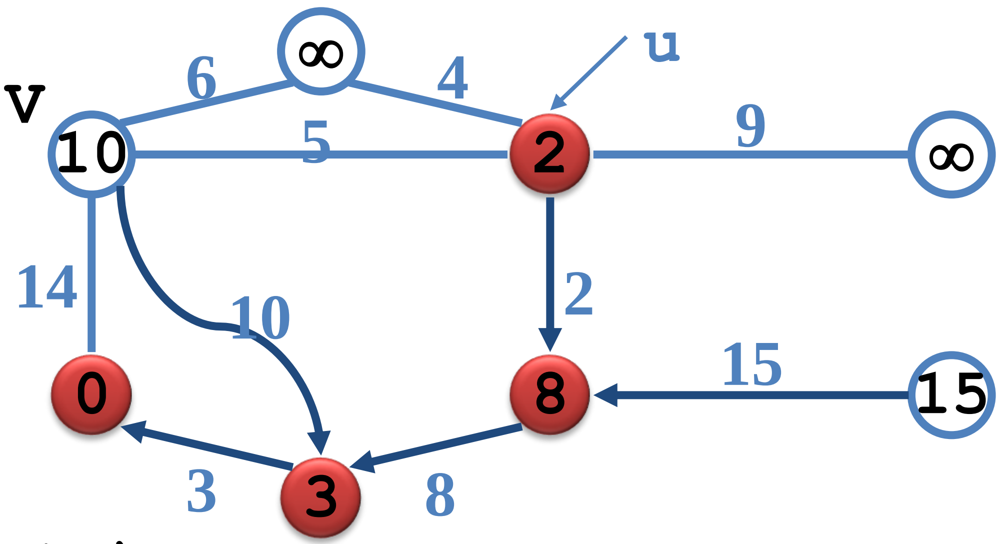

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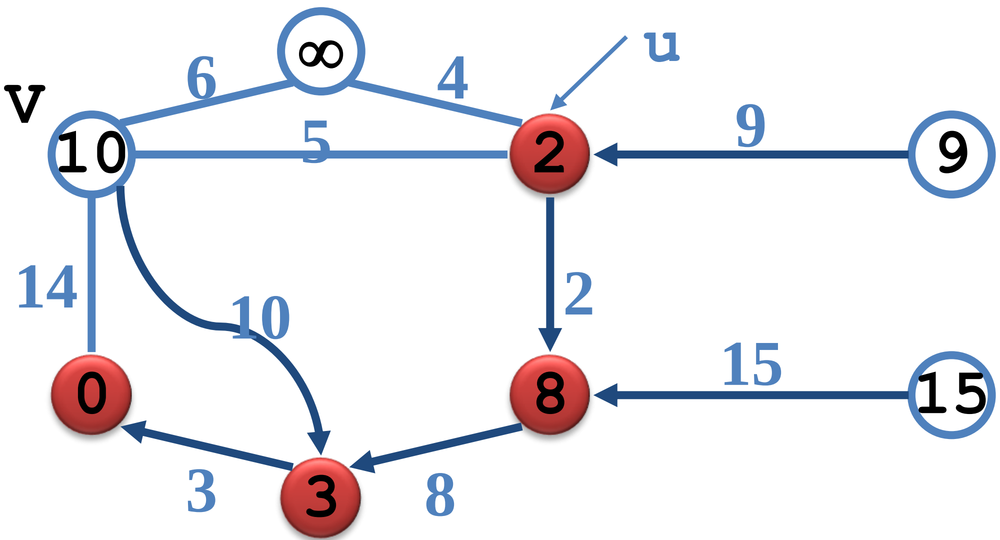

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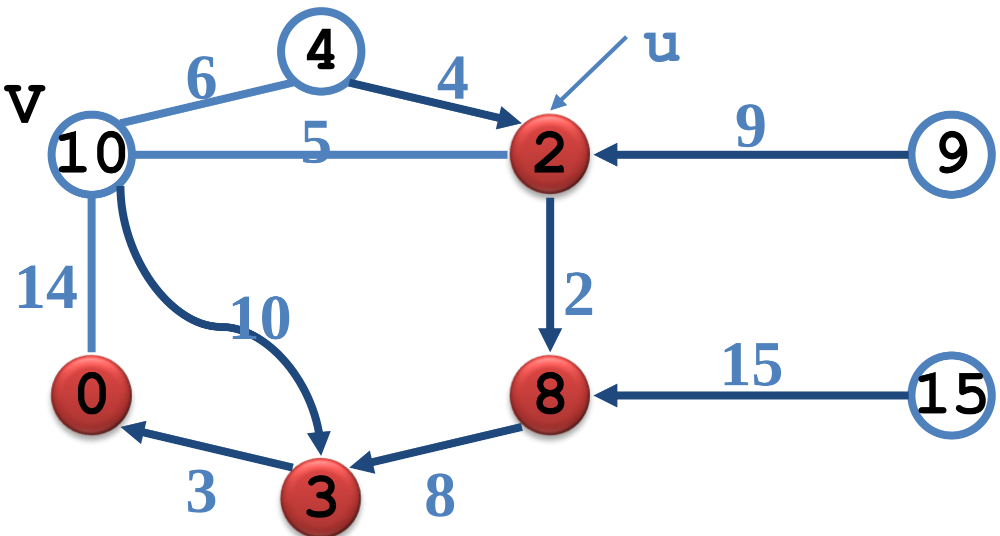

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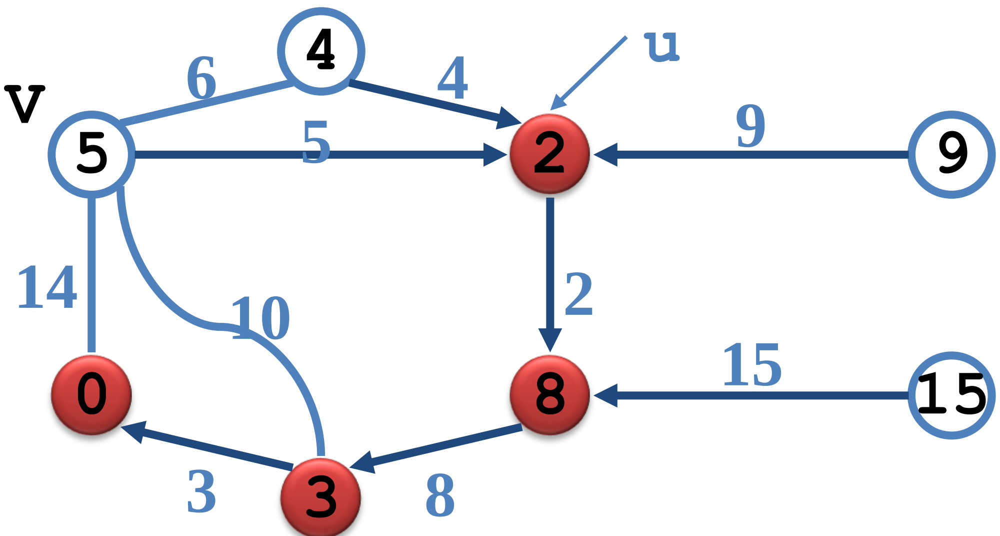

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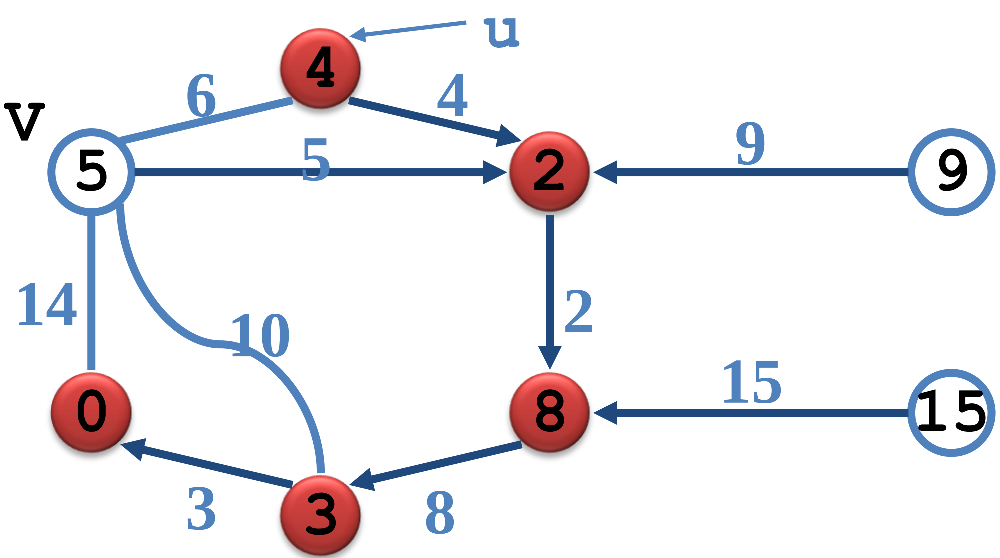

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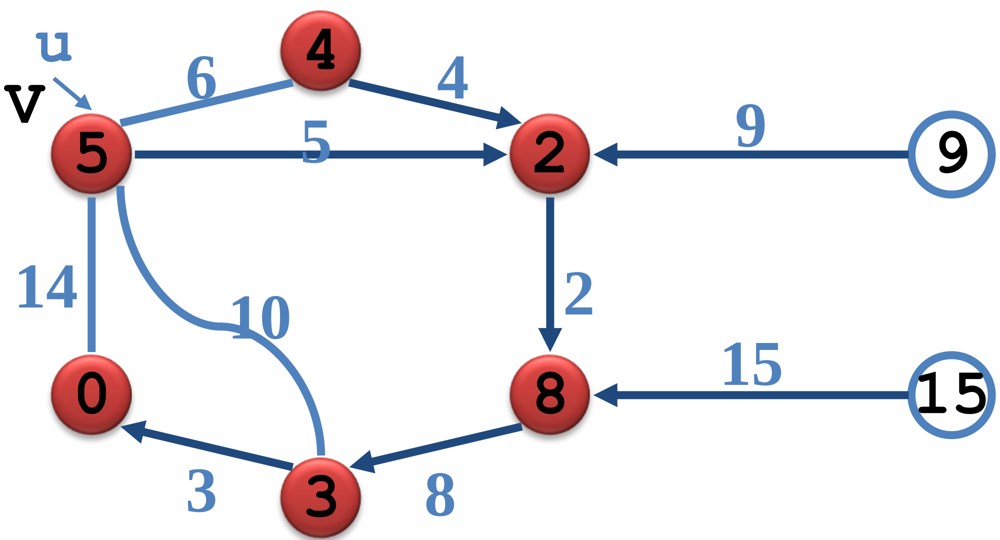

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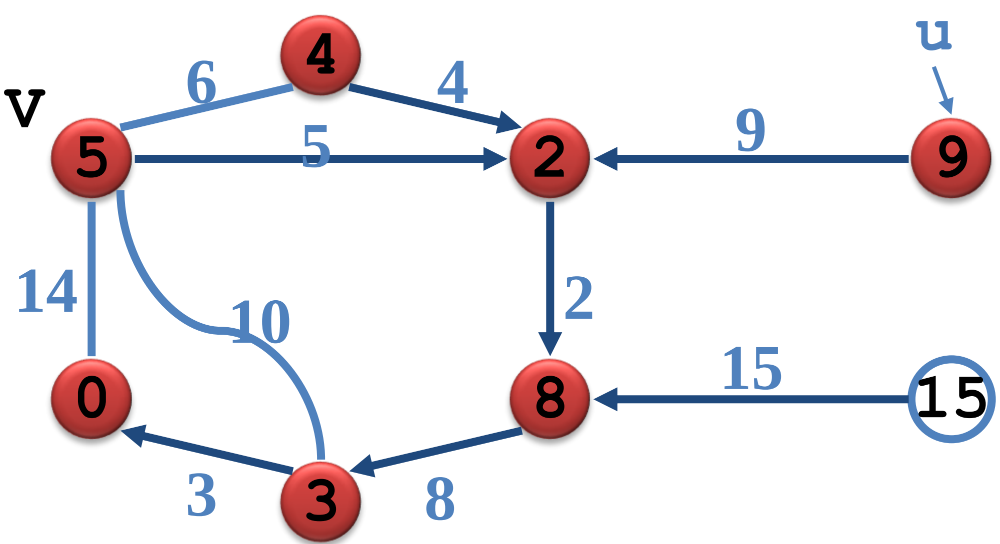

Prim's Algorithm

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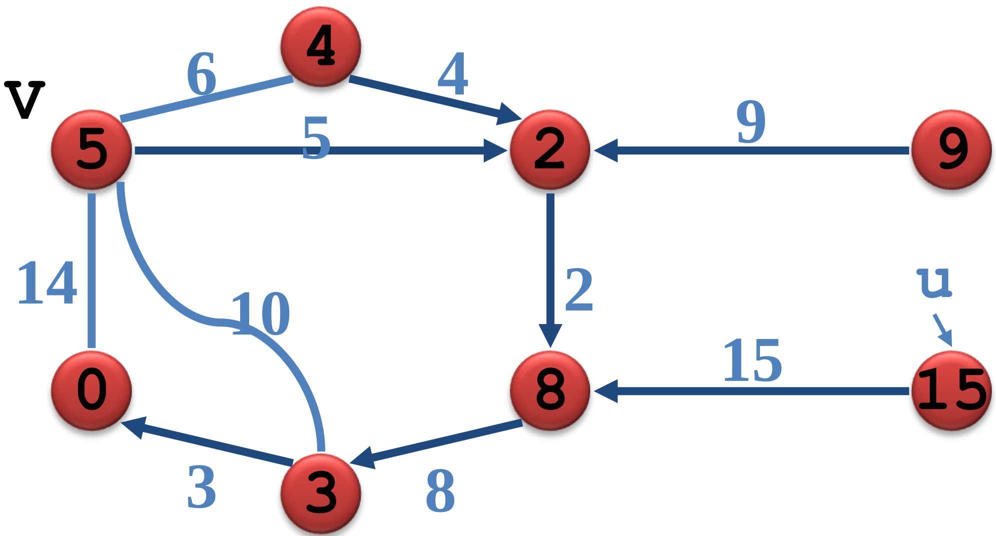

Review: Prim's Algorithm

```
MST-Prim(G, w, r)
    for each u ∈ G.V
        u.key = ∞
        u.π = NIL
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    Q = G.V
    while (Q not empty)
        u = ExtractMin(Q)
        for each v ∈ G.Adj[u]
            if (v ∈ Q and w(u,v) < v.key)
                v.π = u
                v.key = w(u,v)
```

What is the hidden cost in this code?

Review: Prim's Algorithm

```
MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈ Q
        key[u] = ∞;
    key[r] = 0;
    p[r] = NULL;
    while (Q not empty)
        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
                DecreaseKey(v, w(u,v));
```

Prim's Algorithm: running time

- We can use the BUILD-MIN-HEAP procedure to perform the initialization in lines 1–5 in $O(V)$ time
- EXTRACT-MIN operation is called $|V|$ times, and each call takes $O(\lg V)$ time, the total time for all calls to EXTRACT-MIN is $O(V \lg V)$

Running time (cont'd)

- The for loop in lines 8–11 is executed O(E) times altogether, since the sum of the lengths of all adjacency lists is 2 |E|.
  - Lines 9 -10 take constant time
  - line 11 involves an implicit DECREASE-KEY operation on the min-heap, which takes $O(\lg V)$ time
- Thus, the total time for Prim's algorithm is $O(V) + O(V \lg V) + O(E \lg V) = O(E \lg V)$
  - The same as Kruskal's algorithm

Summary

- We learned
  - Generic MST
  - Kruskal’s and Prim’s algorithm
- Common mistakes: Don’t mix Kruskal’s algorithm with Prim’s algorithm