# COT 6405 Introduction to Theory of Algorithms

Topic 5. Master Theorem

# Solving the recurrences

- Substitution method
- Recursion tree
- Master method

# The Master Theorem

- Given: a *divide-and-conquer* algorithm
  - An algorithm that divides the problem of size *n* into *a* subproblems, each of input size *n/b*
  - Let the cost of each stage (i.e., the work to divide the problem + combine solved subproblems) be described by the function *f*(n)
- Then, the Master Theorem gives us a cookbook for the algorithm's running time

# The Master Theorem (Cont'd)

- If T($n$) = $a$T($n$/$b$) + f($n$) then

$\boxed{f(n) < n^{\log_b a}}$

$$
T(n) = \begin{cases}
\Theta\left(n^{\log_b a}\right) & f(n) = O\left(n^{\log_b a - \varepsilon}\right) \\
\Theta\left(n^{\log_b a} \lg n\right) & f(n) = \Theta\left(n^{\log_b a}\right) \\
\Theta\left(f(n)\right) & f(n) = \Omega\left(n^{\log_b a + \varepsilon}\right) \text{AND} \\
 & af(n/b) < cf(n) \text{ for large } n
\end{cases}
\left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\end{matrix}}\right\} \begin{matrix}\varepsilon > 0 \\ c < 1\end{matrix}
$$

$\boxed{f(n) > n^{\log_b a}}$

# The Master Theorem (Cont'd)

$$T(n) = aT(n/b) + f(n), \text{ where } a \geq 1,\ b > 1$$

Compare $n^{\log_b a}$ vs. $f(n)$:

**Case 1:** $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$.

($f(n)$ is polynomially smaller than $n^{\log_b a}$.)

***Solution:*** $T(n) = \Theta(n^{\log_b a})$.

(Intuitively: cost is dominated by leaves.)

# The Master Theorem (Cont'd)

**Case 2:** $f(n) = \Theta(n^{\log_b a} \lg^k n)$, where $k \geq 0$.

*[This formulation of Case 2 is more general than in Theorem 4.1, and it is given in Exercise* 4.6-2]

($f(n)$ is within a polylog factor of $n^{\log_b a}$, but not smaller.)

***Solution:*** $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.

(Intuitively: cost is $n^{\log_b a} \lg^k n$ at each level, and there are $\Theta(\lg n)$ levels.)

***Simple case:*** $k = 0 \Rightarrow f(n) = \Theta(n^{\log_b a}) \Rightarrow T(n) = \Theta(n^{\log_b a} \lg n)$.

# The Master Theorem (Cont'd)

**Case 3:** $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$ and $f(n)$ satisfies the regularity condition $af(n/b) \le cf(n)$ for some constant $c < 1$ and all sufficiently large $n$.
($f(n)$ is polynomially greater than $n^{\log_b a}$.)
***Solution:*** $T(n) = \Theta(f(n))$.
(Intuitively: cost is dominated by root.)

# The Master Theorem (Cont'd)

- If T($n$) = $a$T($n$/$b$) + f($n$) then

$$f(n) < n^{\log_b a}$$

$$\boldsymbol{T(n)} = \begin{cases} \Theta\left(n^{\log_b a}\right) & f(n) = O\left(n^{\log_b a - \varepsilon}\right) \\ \Theta\left(n^{\log_b a} \lg^{k+1} n\right) & f(n) = \Theta\left(n^{\log_b a} \lg^k n\right) \\ \Theta\left(f(n)\right) & f(n) = \Omega\left(n^{\log_b a + \varepsilon}\right) \text{AND} \\ & af(n/b) < cf(n) \text{ for large } n \end{cases} \Bigg\} \begin{matrix} \varepsilon > 0 \\ c < 1 \end{matrix}$$

$$f(n) > n^{\log_b a}$$

# Using the Master Theorem, Case 1

- Solve $T(n) = 9T(n/3) + n$
  - $a=9$, $b=3$, $f(n) = n$
  - $n^{\log_b a} = n^{\log_3 9} = n^2$
  - Since $f(n) = O(n^{2 - \varepsilon})$, where $\varepsilon=1$, case 1 applies:

$$T(n) = \Theta\left(n^{\log_b a}\right) \text{when } f(n) = O\left(n^{\log_b a - \varepsilon}\right)$$

  - Thus the solution is $T(n) = \Theta(n^2)$

# Using the Master Theorem, Case 2

- $T(n) = T(2n/3) + 1$
  - a=1,  b=3/2, f(n) = 1
  - $n^{\log_b a} = n^{\log_{3/2} 1} = n^0 = 1$ → compare with f(n)=1
  - Since $f(n) = \Theta(n^{\log_b a}) = \Theta(1)$, the simple form of case 2 applies:

$$T(n) = \Theta\left(n^{\log_b a} \lg n\right) \text{when } f(n) = \Theta\left(n^{\log_b a}\right)$$

  - Thus the solution is $T(n) = \Theta(\lg n)$

# Using the Master Theorem, Case 3

- T(n) = 3T(n/4) + n lg n
  - a=3, b=4, f(n) = nlgn
  - $n^{\log_b a}$= $n^{\log_4 3}$ = n $^{0.793}$ → compare with f(n)=n lg n
  - Since f(n) = $\Omega(n^{0.793+\varepsilon})$, where ε=0.207
    - Also for c=3/4 < 1 , a*f(n/b) <= c*f(n)
      →3(n/4)*lg(n/4) <= (3/4)n lg n
  - case 3 applies:

$$T(n) = \Theta\big(f(n)\big) \text{ when } f(n) = \Omega\left(n^{\log_b a+\varepsilon}\right)$$

  - Thus the solution is T(n) = Θ(n lg n)

# Exercises

- T(n) =  5T(n/2) +$\Theta(n^2)$
- T(n) = 27 T(n/3) +$\Theta(n^3 lgn)$
- T(n) = 5T(n/2) +$\Theta(n^3)$

$\boxed{f(n) < n^{\log_b a}}$

$$T(n) = \begin{cases} \Theta\left(n^{\log_b a}\right) & f(n) = O\left(n^{\log_b a - \varepsilon}\right) \\ \Theta\left(n^{\log_b a} \lg^{k+1} n\right) & f(n) = \Theta\left(n^{\log_b a} \lg^k n\right) \\ \Theta\left(f(n)\right) & f(n) = \Omega\left(n^{\log_b a + \varepsilon}\right) \text{AND} \\ & af(n/b) < cf(n) \ \text{ for large } n \end{cases} \quad \begin{matrix} \varepsilon > 0 \\ c < 1 \end{matrix}$$

$\boxed{f(n) > n^{\log_b a}}$

# Exercises (cont'd)

- T(n) =  5T(n/2) +$\Theta(n^2)$
- a = 5, b = 2, f(n) = $\Theta(n^2)$
- $n^2 \ \in O\left( n^{\log_2 5 - \varepsilon}\right)$
- Case 1, T(*n*) = $\Theta( n^{\log_2 5})$

# Exercises (cont'd)

- T(n) = 27 T(n/3) +$\Theta(n^3 lgn)$
- a = 27, b = 3, f(n) = $\Theta(n^3 lgn)$
- $n^{\log_3 27}$ = $n^3$
- Case 2: k = 1, and f(n) = $\Theta$ ($n^{\log_3 27}$lg$n$)
- T($n$) = $\Theta$ ($n^3 lg^2 n$)

# Exercises (cont'd)

- T(n) = 5T(n/2) +$\Theta(n^3)$
- a = 5, b = 2, f(n) = $\Theta(n^3)$
- $n^3 \in \Omega\left( n^{\log_2 5+\varepsilon}\right)$
- Case 3, check the regularity condition
  - *a* f(*n*/*b*) = 5$(\frac{n}{2})^3$ = (5/8) $n^3 \leq \mathrm{c}n^3$ for c = 5/8 <1
- T(*n*) = $\Theta(n^3)$

# Limitation of the Master Theorem

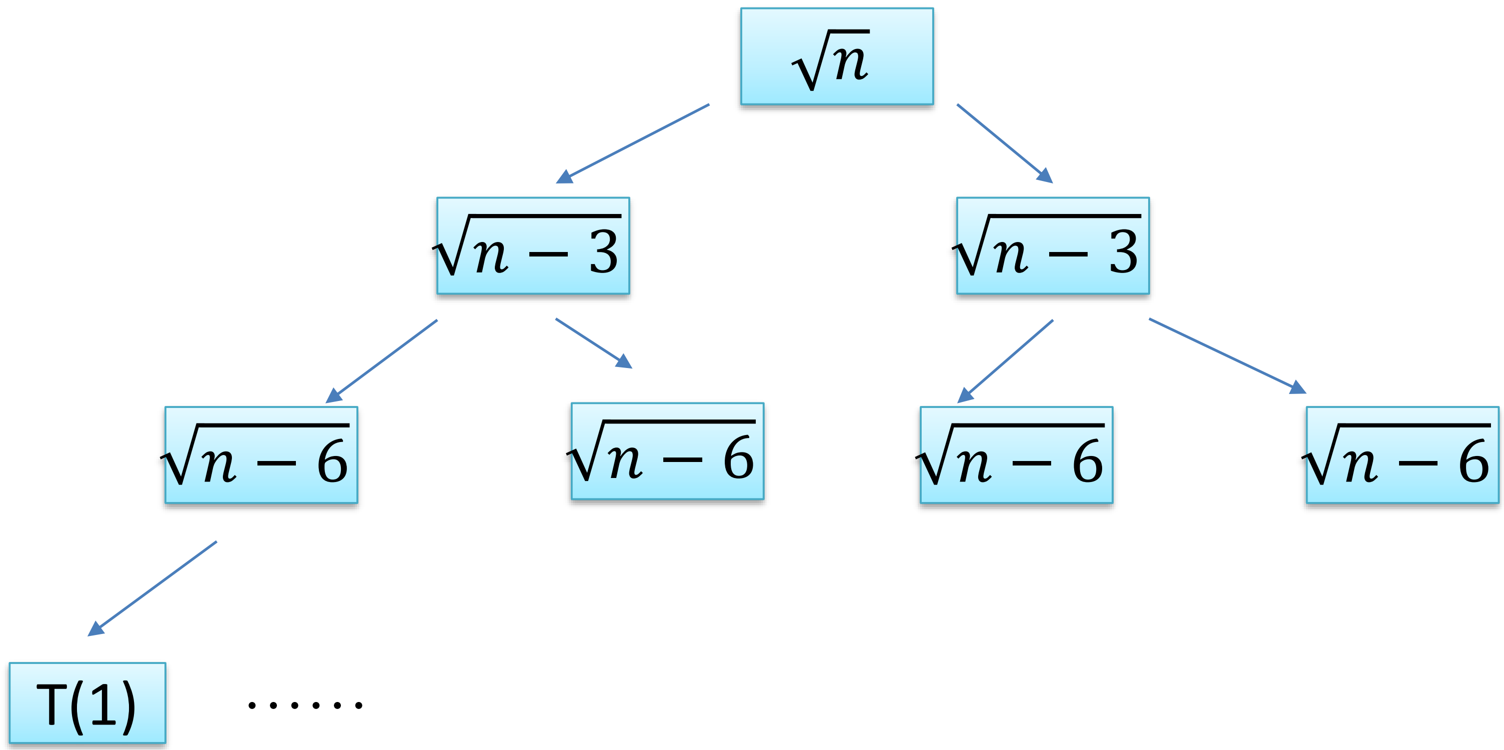
- Master Theorem does not apply to all f(n)!
  - The regularity condition in Case 3
  - Situations that don't look anything like that of the Master Theorem
    - T(n) = 2T(n-3) + $\sqrt{n}$

# Limitations (cont’d)

- Situations that don’t look anything like that of the Master Theorem
- T(n) = 2T(n-3) + $\sqrt{n}$

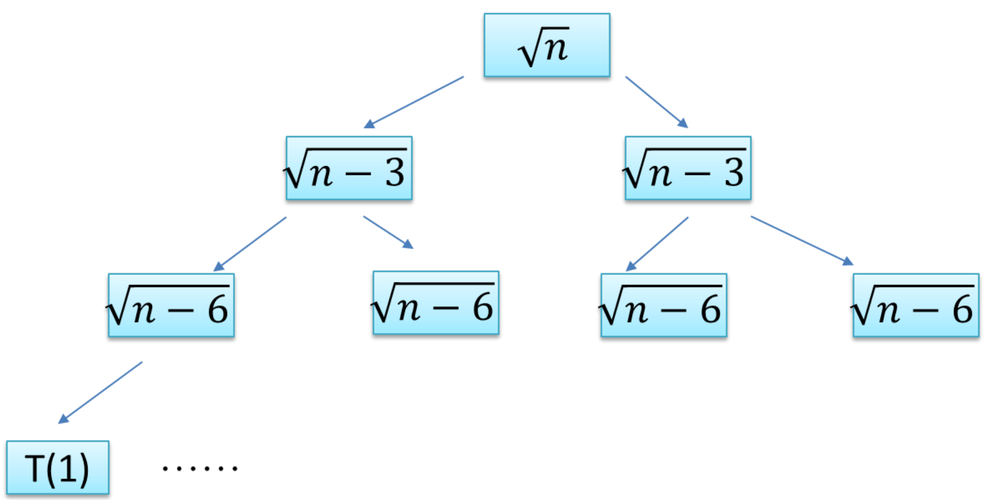
# What to do when it doesn't apply

- The recursion-tree method



# Cont'd

- The sub-problem size for a node at depth *i* is $n - 3i$
- The sub-problem size hits T(1), when $n - 3i$= 1, or $i = (n - 1)/3$
- Thus, tree has 1+ (n-1)/3 levels (*i* = 0,1,…, (n-1)/3 )



# Cont'd

- Each node at level *i* has a cost of $\sqrt{n-3i}$
- Each level has $2^i$ nodes
  - Level 0: 1, level 1: 2, level 2:4, level 3: 8….
- Thus, the total cost of level *i* is $2^i\sqrt{n-3i}$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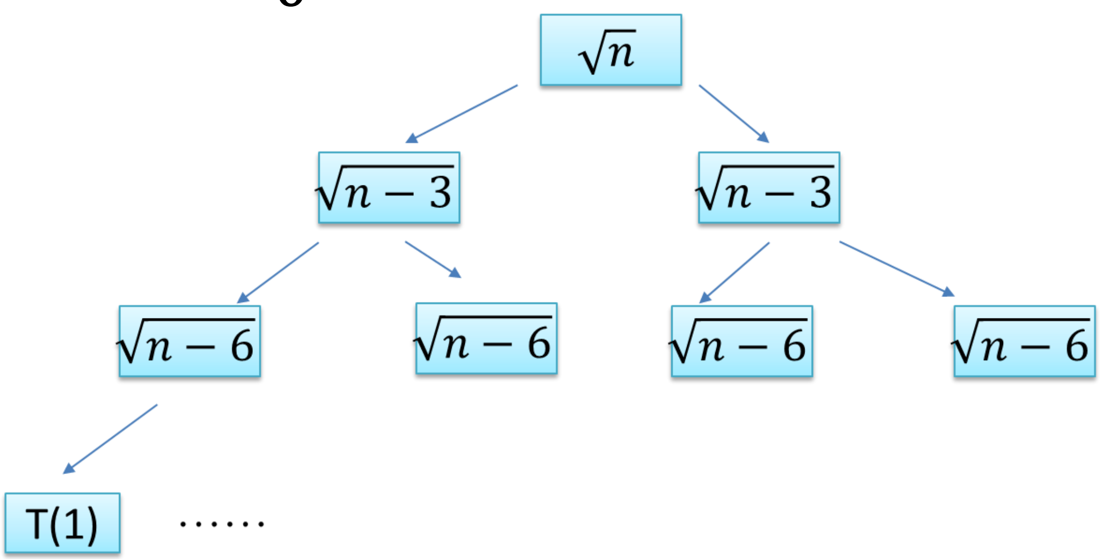
# Cont'd

- The bottom level has $2^{(n-1)/3}$ nodes, each costing T(1)
- Assume T(1) = $c_0$. The total cost of the bottom level will be $c_0 2^{(n-1)/3}$



# Cont'd

- We add up the costs over all levels to determine the total cost for the entire tree:

$$T(n) = \sum_{i=0}^{\frac{n-1}{3}-1} 2^i\sqrt{n-3i} + c_0 2^{(n-1)/3}$$

$$\leq \sum_{i=0}^{\frac{n-1}{3}-1} 2^i\sqrt{n} + c_0 2^{(n-1)/3}$$

$$= \sqrt{n}(2^{(n-1)/3}\text{-}1) + c_0 2^{(n-1)/3}$$

$$= \sqrt{n}2^{(n-1)/3} + c_0 2^{(n-1)/3} - \sqrt{n}$$

$$= O(\sqrt{n}2^{n/3})$$

# Processing floors and ceilings

- T(n) = 2T($\lfloor n/2 \rfloor$) + n has the solution of T(n) = $\Theta(nlgn)$
- T(n ) = 2T($\lfloor n/2 \rfloor$) + n

$$\leq 2\mathrm{T}\left(\frac{n}{2}\right) + n$$ - > O(nlgn)

- T(n ) = 2T($\lfloor n/2 \rfloor$) + n

$$\geq 2\mathrm{T}\left(\frac{n}{2} - 1\right) + n$$

= $2\mathrm{T}\left(\frac{n-2}{2}\right) + (n$-2) +2 $\geq 2\mathrm{T}\left(\frac{n-2}{2}\right) + (n$-2)

= Ω((*n*-2)lg(*n*-2)) - > Ω(*n*lg*n*)

# Cont'd

- T(n) = 2T($\lfloor n/2 \rfloor$) + n
- We guess the solution is T(*n*) = $\Omega$(*n*lgn) -> we can find a constant *d* larger than 0 such that T(*n*) $\geq d\ n\lg n$
- Assume: T(*k*) $\geq d\ k\lg k$ for all $k \leq n$
- T(*n*) = 2T($\lfloor n/2 \rfloor$) + *n*

  $\geq$ 2d $\lfloor n/2 \rfloor$ lg$\lfloor n/2 \rfloor$+ *n*

  = *dn*lg $\lfloor n/2 \rfloor$ + *n*

  $\geq$*dn*(lg*n* – 2) + *n* (lg $\lfloor n/2 \rfloor \geq$ lg(*n*/2) - 1)

  = *dnlgn* – 2*dn* + *n* $\geq$ *dnlgn* when *d* < 0.5

# Processing floors and ceilings (cont'd)

- T(n) = 2T($\lceil n/2 \rceil$) + n has the solution of T(n) = $\Theta(nlgn)$
- T(n ) = 2T($\lceil n/2 \rceil$) + n

  $\leq 2\mathrm{T}\left(\frac{n}{2}+1\right)+n$

  = $2\mathrm{T}\left(\frac{n+2}{2}\right)+(n$ +2) -2 $\leq 2\mathrm{T}\left(\frac{n+2}{2}\right)+(n$ +2)

  -> $O$((n+2)lg(n+2)) -> O(nlgn)
- T(n ) = 2T($\lceil n/2 \rceil$) + n

  $\geq 2\mathrm{T}\left(\frac{n}{2}\right)+n \;\; -> \Omega(nlgn)$