CIS 4930.006S20/CIS 6930.013S20: Computational Methods for Imaging and Vision

Spring 2021
Homework #1

The University of South Florida
Department of Computer Science and Engineering
Tampa, FL

Assigned: January 27, 2021
Due: February 8, 2021

1 Gaussian Elimination

Our aim is to solve the system of linear equations $\mathbf{Ax} = \mathbf{y}$. (General conditions for the existence of a solution are given in **FSP Appendix 2.B.1**.) Comment on whether a solution to each of the following systems of equations exists, and, if it does, find it.

(a)

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 \\ 4 & 5 & 2 \\ -1 & -1 & 2 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 10 \\ 20 \\ 3 \end{bmatrix}.$$

(b)

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 4 & 5 & 8 \\ -1 & -1 & -2 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 7 \\ 38 \\ -9 \end{bmatrix}.$$

(c)

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 4 & 5 & 8 \\ -1 & -1 & -2 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

2 Eigenvalues and Eigenvectors (Linear algebra refresher)

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}.$$

(a) Find the eigenvalues and unit-norm eigenvectors of $\mathbf{A}$. Are the eigenvectors orthogonal? Check your answer with a using Matlab.

(b) Compute the determinant of $\mathbf{A}$, i.e. $\det \mathbf{A}$. Is $\mathbf{A}$ invertible? If it is, give its inverse; if not, say why.

(c) Find eigenvalues and unit-norm eigenvectors of $\mathbf{B}$. For $\alpha \in \{0, 1, 2, 3\}$ and $\beta \in [-3, 3]$, plot the eigenvalues of $\mathbf{B}$ (using Matlab). (This will be four pairs of curves that are functions of one variable.)

(d) Compute the determinant of $\mathbf{B}$. When is $\mathbf{B}$ invertible? For $(\alpha, \beta) \in [0, 5]^2$, plot $\det \mathbf{B}$ (with a computer, using Matlab). (This will be a surface plot of a function of two variables.)

3 Multiplication by an orthogonal matrix

Consider the vector space $\mathbb{R}^n$ with standard norm and standard inner product. Prove that

(a) multiplication by an orthogonal matrix $\mathbf{U}$ preserves lengths, that is,

$$\|\mathbf{U}\mathbf{x}\| = \|\mathbf{x}\|,$$

for any $\mathbf{x}$.

(b) multiplication by an orthogonal matrix $\mathbf{U}$ preserves angles, that is,

$$\langle \mathbf{U}\mathbf{x}, \mathbf{U}\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{y}\rangle,$$

for any $\mathbf{x}$ and $\mathbf{y}$.

4 Bases and frames of $\mathbb{R}^2$

Given the following sets of vectors:

$$\Phi_1 = \{\varphi_{1,0},\, \varphi_{1,1}\} = \left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \tag{1}$$

$$\Phi_2 = \{\varphi_{2,0},\, \varphi_{2,1},\, \varphi_{2,2},\, \varphi_{2,3}\} = \left\{ \begin{bmatrix} \frac{1}{2\sqrt{2}} \\ \frac{\sqrt{3}}{2\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{3}}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\} \tag{2}$$

$$\Phi_3 = \{\varphi_{3,0}, \varphi_{3,1}\} = \left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} \right\} \tag{3}$$

$$\Phi_4 = \{\varphi_{4,0}, \varphi_{4,1}, \varphi_{4,2}\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \tag{4}$$

For each of the sets of vectors, Φ_1 and Φ_3, do the following:

(a) Write the matrix representation for the set, that is, the synthesis operator associated with the set.

(b) Find the dual basis. Sketch (in other words, draw the arrows that represent each vector of) the original sets and their duals.

(c) Specify whether it is an orthonormal basis.

(d) For $\mathbf{x} = [2,\, 0]^\mathsf{T}$, write down the projection coefficients, $\alpha_{i,k} = \langle x, \widetilde{\varphi}_{i,k}\rangle$.

(e) For the same $\mathbf{x}$, verify the expansion formula $\Phi\widetilde{\Phi}^\mathsf{T} = \mathbf{I}$.

(f) Specify whether the expansion preserves the norm, that is, whether it is true that $\|\mathbf{x}\| = \sum_k |\alpha_{i,k}|^2$.

For each of the sets of vectors, Φ_2 and Φ_4, write the matrix representation for the set, that is, the synthesis operator associated with the set.

5 Inner product

True or False, two vectors, say $f(t)$ and $g(t)$, are **orthogonal** if their inner product is zero. Using your response to the above prove that $f(t) = \sin(\pi n t)$ and $g(t) = \sin(\pi m t)$ are orthogonal in the Hilbert space $\mathcal{L}^2[-1,\, +1]$, for any integers $n \neq m$ (i.e., when $n, m \in \mathbb{Z}$ and $n \neq m$).

6 Inner product computation by expansion sequences

Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be sequences in $\ell^2(\mathbb{N})$. Then, the functions

$$f(t) = \alpha_0 + \sum_{k=1}^{\infty} \alpha_k \sqrt{2} \cos(2\pi k t),$$
$$g(t) = \beta_0 + \sum_{k=1}^{\infty} \beta_k \sqrt{2} \cos(2\pi k t),$$

are in $\mathcal{L}^2\left(\left[-\frac{1}{2}, +\frac{1}{2}\right]\right)$. Demonstrate that the standard inner product between the functions, $f(t)$ and $g(t)$ can be written as the standard inner product between the sequences $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. That is, show that $\langle f(t), g(t) \rangle = \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle$.
Given the above results, write down (or derive, if you want) the norms of $f(t)$ and $g(t)$, i.e. write down $\|f(t)\|$ and $\|g(t)\|$.

7 Linear Independence (Optional, for extra credit.)

Find the values of the parameter $a \in \mathbb{C}$ such that the following set is linearly independent:

$$U = \left\{ \begin{bmatrix} 0 & a^2 \\ 0 & \mathrm{j} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & a-1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \mathrm{j}a & 1 \end{bmatrix} \right\}.$$

For $a = \mathrm{j}$, express the matrix

$$\begin{bmatrix} 0 & 5 \\ 2 & \mathrm{j}-2 \end{bmatrix}$$

as a linear combination of the elements of U. [*Note that* j *denotes the imaginary unit, i.e.* $\mathrm{j} = \sqrt{-1}$, *so that* $c\mathrm{j} \times d\mathrm{j} = c \times d \times \mathrm{j}^2 = c \times d \times -1 = -cd$.]

8 Vector space $\mathbb{C}^n$ (Optional, for extra credit.)

Prove that $\mathbb{C}^n$ is a vector space. Note: the symbol $\mathbb{C}$ means we are dealing with complex numbers. Thus, a vector $\mathbf{x} \in \mathbb{C}^\mathbf{n}$ means that $\mathbf{x}$ is a vector with n entries and each of its entry is a complex number.