Homework 3

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Worked with Hunter Morera and Caio Da Silva On exercise/question 2.

Exercise 1

(a) Compute the continuous convolution of $g(x)$ and $h(x)$ for following three cases:

(i) $g(x) = h(x) = u(x) - u(x-4)$, where u(x) is the unit step function.

This must be done as a piecewise function from bounds $[0, x]$ and $[x, 8]$.

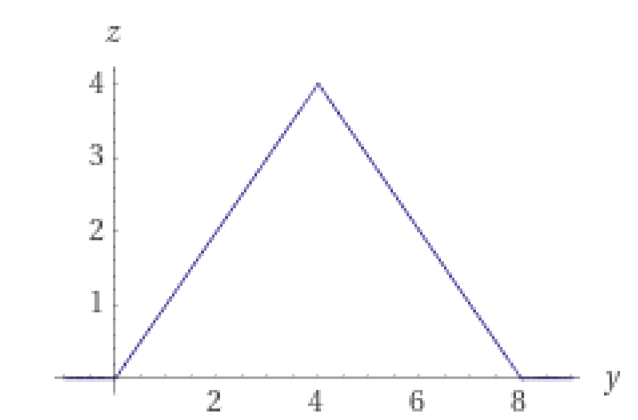
$\int_0^x g(t)h(x-t)dt$ and $\int_x^8 g(t)h(x-t)dt$ and since they are equal to 1 in this range from 0 to 8 we can integrate these as so:

$f(x)_0 = \int_0^x g(t)h(x-t)dt = \int_0^x 1dt = t]_0^x = x - 0 = x$

$f(x)_1 = \int_x^8 g(t)h(x-t)dt = \int_0^x 1dt = t]_x^8 = 8 - x$

The intersection of the lines $f(x)_0 = x$ and $f(x)_1 = 8 - x$ is $x = 4$.

$$\therefore f(x) = \begin{cases} x : & \text{if } 0 \le x \le 4 \\ 8 - x : & \text{if } 4 \le x \le 8 \\ 0 : & \text{otherwise} \end{cases}$$



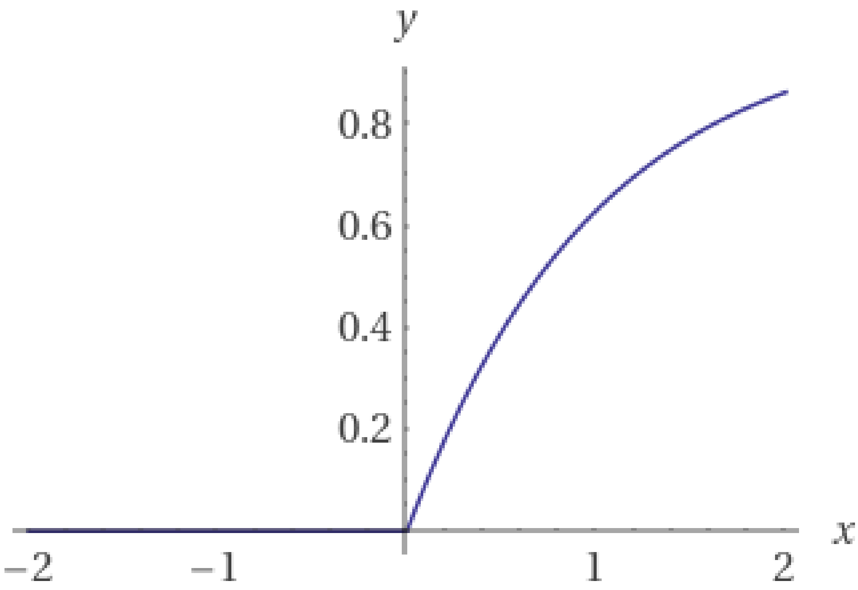
(ii) $g(x) = e^{-(x-1)}u(x-1)$ and $h(x) = u(x+1)$ where $u(x)$ is the unit step function.

$f(x) = g(x) * h(x) = \int_{-\infty}^{\infty} g(t)h(x+1-t)dt$

From that, we can assume we must integrate from $[1, x+1]$ since $g(x)$ is 0 if $x < 1$ and $h(x+1-t)$ is 0 if $t \geq x+1$. We can rewrite the integral as follows:

$\int_1^{x+1} e^{1-t} dt$ and setting $u = 1-t$, $du = -dt$ and for upper bounds we get $u = 1-(x+1) = -x$ and for lower bounds we get $u = 1-(1) = 0$ which gives us $\int_0^{-x} -e^u du = -e^u]_0^{-x} = -e^{-x} - -e^0 = 1-e^{-x}$ and since bounds start at x=0, the final solution is $(1-e^{-x})u(x)$ where $u(x)$ is the unit step function. It can also be expressed in terms of hyperbolic Euler trig identities, $(sinh(x) - cosh(x) + 1)u(x)$.

$$\therefore f(x) = g(x) * h(x) = \begin{cases} 1 - e^{-x} : & \text{if } x \geq 0 \\ 0 : & \text{otherwise} \end{cases}$$



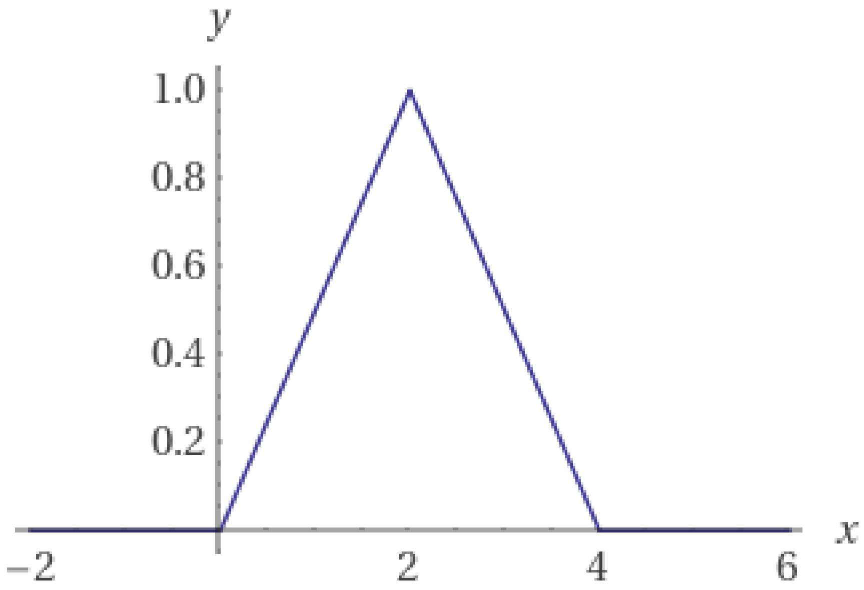
(iii)

$$g(x) = \Lambda(\tfrac{x}{2} - \tfrac{1}{2}) = \begin{cases} \frac{x}{2} + \frac{1}{2} : & \text{if } \; -1 \leq x \leq 1 \\ \frac{-x}{2} + \frac{3}{2} : & \text{if } \; 1 \leq x \leq 3 \\ 0 : & \text{otherwise} \end{cases}$$

$h(x) = \delta$ at $x = 2$

This basically shifts the function to center it at $x = 2$ which requires changing the y-intercepts by $\frac{1}{2}$ and the x-intercepts by 1.

$$\therefore g(x) * h(x) = \Lambda(\tfrac{x}{2} - 1) = \begin{cases} \frac{x}{2} : & \text{if } \; 0 \leq x \leq 2 \\ \frac{-x}{2} + 2 : & \text{if } \; 2 \leq x \leq 4 \\ 0 : & \text{otherwise} \end{cases}$$



(b) Let $g(x) = sinc(ax)$ and $h(x) = sinc(bx)$ where a,b are real scalars with $a \le b$. Using the convolution property of the Fourier transform, compute the convolution between the g(x) and h(x).

(i)

$F[g(x) * h(x)] = G(j\omega)H(j\omega)$

$\therefore F^{-1}[G(j\omega)H(j\omega)] = g(x) * h(x)$

(ii)

$g(x) = sinc(ax)$ and $h(x) = sinc(bx)$ where $a \le b$

$F[g(x)] = G(j\omega) = \frac{\pi rect(\frac{\omega}{2a})}{|a|}$

$F[h(x)] = H(j\omega) = \frac{\pi rect(\frac{\omega}{2b})}{|b|}$

(iii)

$G(j\omega)H(j\omega) = \frac{\pi^2 rect(\frac{\omega}{2a})rect(\frac{\omega}{2b})}{|ab|}$ and since $a \le b$ we get $G(j\omega)H(j\omega) = \frac{\pi^2 rect(\frac{\omega}{2a})}{|ab|}$

(iv)

$F^{-1}[\frac{\pi^2}{|ab|} rect(\frac{\omega}{2a})] = \frac{\pi^2}{|ab|} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} rect(\frac{\omega}{2a}) e^{ix\omega} d\omega = \frac{\pi}{2|ab|} \int_{-a}^{a} e^{ix\omega} d\omega = \frac{\pi}{2|ab|} \cdot \frac{-ie^{i\omega x}}{x}]_{-a}^{a}$

$\frac{\pi}{2|ab|} \cdot \frac{-ie^{i\omega x}}{x}]_{-a}^{a} = \frac{\pi}{2|ab|} \cdot [\frac{-ie^{iax}}{x} - \frac{-ie^{-iax}}{x} = \frac{-i}{x}(e^{iax} - e^{-iax}) \cdot \frac{2i}{2i} = \frac{-2i^2}{2ix}(e^{iax} - e^{-iax})] =$

$\frac{\pi}{2|ab|} \cdot [\frac{2}{x} \cdot \frac{(e^{iax} - e^{-iax})}{2i}] = \frac{\pi}{|ab|x} \cdot \frac{(e^{iax} - e^{-iax})}{2i} = \frac{\pi}{|ab|x} \cdot sin(ax) = \frac{\pi}{|b|} \cdot \frac{sin(ax)}{|a|x} = \frac{\pi}{|b|} \cdot sinc(ax) = \frac{\pi sinc(ax)}{|b|}$

$\therefore g(x) * h(x) = \frac{\pi sinc(ax)}{|b|}$

(c)

Let $g(x) = \frac{1}{\sqrt{2\pi}a} e^{\frac{-x^2}{2a^2}}$ and $h(x) = \frac{1}{\sqrt{2\pi}b} e^{\frac{-x^2}{2b^2}}$

(i) Compute the convolution between the functions $g(x)$ and $h(x)$, directly, that is by using the definition of the 1D convolution integral. $\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \sqrt{2\pi}\sigma$, where μ and σ are scalars.

$g(x) * h(x) = \int_{-\infty}^{\infty} g(t)h(x-t)dt = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}a} e^{\frac{-t^2}{2a^2}} \frac{1}{\sqrt{2\pi}b} e^{\frac{-(x-t)^2}{2b^2}} dt = \frac{1}{2\pi ab} \int_{-\infty}^{\infty} e^{\frac{-t^2}{2a^2}} e^{\frac{-(x-t)^2}{2b^2}} dt = \frac{1}{2\pi ab} \cdot \frac{\sqrt{2\pi}}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}} e^{\frac{-x^2}{2(a^2+b^2)}} = \frac{e^{\frac{-x^2}{2(a^2+b^2)}}}{\sqrt{2\pi}ab\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}$

(ii) Compute the convolution between the functions $g(x)$ and $h(x)$ by applying the convolution property of the Fourier transform.

$F(\omega) = F[g(x)] = \frac{1}{\sqrt{2\pi}a} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2a^2}} \cdot e^{-i\omega x} dx = e^{-\frac{a^2\omega^2}{2}}$

$F(\omega) = F[h(x)] = \frac{1}{\sqrt{2\pi}b} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2b^2}} \cdot e^{-i\omega x} dx = e^{-\frac{b^2\omega^2}{2}}$

$F^{-1}[G(\omega)H(\omega)] = g(x) * h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{a^2\omega^2+b^2\omega^2}{2}} e^{j\omega x}\, d\omega = \frac{1}{2\pi} \cdot \frac{\sqrt{2}\sqrt{\pi}e^{\frac{j^2x^2}{2b^2+2a^2}}}{\sqrt{b^2+a^2}} = \frac{\sqrt{2}\sqrt{\pi}e^{\frac{-x^2}{2b^2+2a^2}}}{2\pi\sqrt{b^2+a^2}} = \frac{\sqrt{2\pi}e^{\frac{-x^2}{2b^2+2a^2}}}{\sqrt{2\pi}\sqrt{2\pi}\sqrt{b^2+a^2}} = \frac{e^{\frac{-x^2}{2b^2+2a^2}}}{\sqrt{2\pi}\sqrt{b^2+a^2}}$

Exercise 2

(a)

Write down the mathematical expression for the operator that represents the imaging system, in both standard form and spectral form.

Standard:

$g_{out}(x, y; z) = \frac{1}{j\lambda z} \iint g_{in}(x', y') e^{jk\frac{(x-x')^2+(y-y')^2}{2z}}\, dx'\, dy'$

Spectral:

$g_{out}(x, y; z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{in}(\omega_x, \omega_y) e^{-j\pi\lambda z(\omega_x^2+\omega_y^2)} e^{j(\omega_x x+\omega_y y)}\, d\omega_x\, d\omega_y$

(b) Find the range, null space, adjoint, and inverse of the system operator.

Range Space:

$R(A) \subset \{g_{out} \in L^2(\mathbb{R}) : G_{out}(\omega_x, \omega_y) = 0, (\omega_x, \omega_y) \notin B\}$

Null Space:

$N(A) \supset \{g_{in} \in L^2(\mathbb{R}) : G_{in}(\omega_x, \omega_y; z) = 0, (\omega_x, \omega_y) \in B\}$

Adjoint:
I am showing it in spectral form, we simply take the complex conjugate of $H(\omega_x, \omega_y) =$

$H^*(\omega_x, \omega_y)$. It simply flips the sign of the imaginary part of the function $H(\omega_x, \omega_y)$.

$\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\omega_x, \omega_y) e^{j\pi\lambda z(\omega_x^2+\omega_y^2)} e^{j(\omega_x x+\omega_y y)}\ d\omega_x\ d\omega_y$

Inverse:
The inverse is shown in spectral form.

$\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{G(\omega_x,\omega_y)}{e^{-j\pi\lambda z(\omega_x^2+\omega_y^2)}} e^{j(\omega_x x+\omega_y y)}\ d\omega_x\ d\omega_y$

(c) Construct the forward model in matrix form of the imaging system. Assume square pixels with size Δ and number of pixels in each spatial dimension is N. Discretization of the object space is performed on the standard basis with square pixel of size δ, and number of pixels in each spatial dimension M is chosen such that $M\delta = N\Delta$.

$A_{m_1,m_2,n_1,n_2} = \Delta^2 \frac{1}{j\lambda z} e^{jk\frac{(m_1\delta-n_1\Delta)^2+(m_2\delta-n_2\Delta)^2}{2z}}$

(d) Find the range, null space, adjoint, and inverse of the discretized system.

Range Space:
The range space is eigenvectors of A with non-zero eigen-values. The range space for A is:

$R(A) = \{g_{out}(x, y)\ :\ Ag_{in}(x, y; z) = g_{out}(x, y)\}$

Null Space:
The null space is eigenvectors of A with zero eigen-values. The null space for A is:

$N(A) = \{g_{out}(x, y)\ :\ Ag_{in}(x, y; z) = 0\}$

Adjoint:
The adjoint of A can be described as taking the complex conjugate of A and then transposing the matrix. The adjoint of A is:

$Adjoint(A) = [A^\star]^T$

Inverse:
The inverse of A is such that the inverse multiplied by A is an identity matrix. The inverse of A is.

$Inverse(A) = A^{-1}$

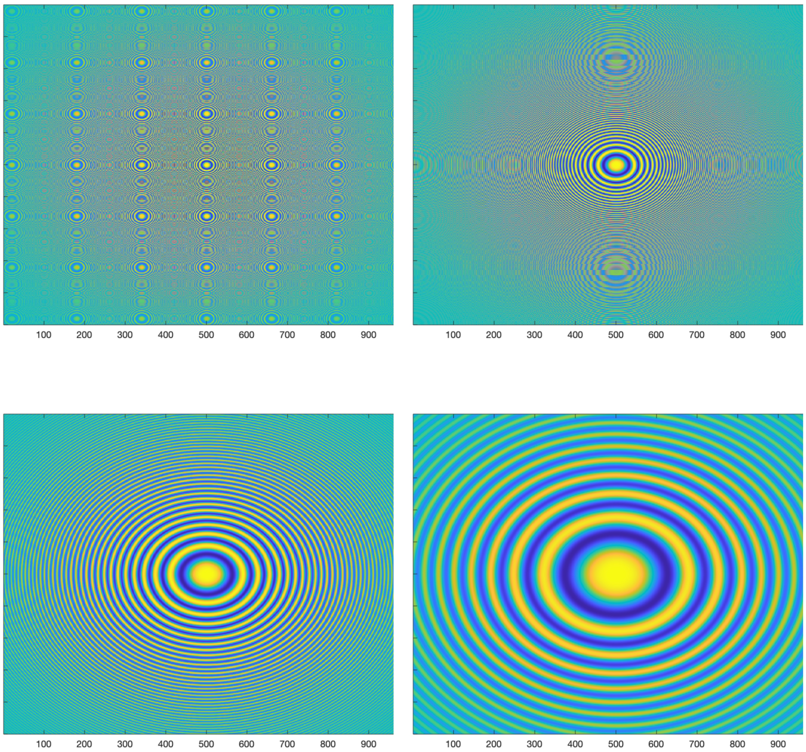
(e) Write a Matlab/Python script for the forward model in (c).

```
function H = H_system(D, d, zed)
M = 1000;
N = 1000;
lambda = 0.5*10^-6;
z = zed*10^-3;

Delta = D*10^-6;
delta = d*10^-6;

dm = 1/(M*delta);
dn = 1/(N*Delta);

m = [-M/2:1:(M/2)-1].*dm;
n = [-N/2:1:(N/2)-1].*dn;

[mm, nn] = meshgrid(m, n);

H = exp( (-1 * sqrt(-1) * pi * lambda * z) .* (mm.^2 + nn.^2));

% Assuming square pixels
P = sinc(Delta*nn).*sinc(Delta*nn); %N and Delta

% Assuming square object discretization
Q = sinc(delta*mm).*sinc(delta*mm); %M and delta

H_sys = (H.*P.*Q);

figure;
imagesc(real(H_sys));

figure;
surf(abs(real(H_sys)), 'edgecolor', 'none');
end
```

(f) Plot the system transfer functions with the following parameters: $\lambda = 0.5\mu m$, N = 1000.

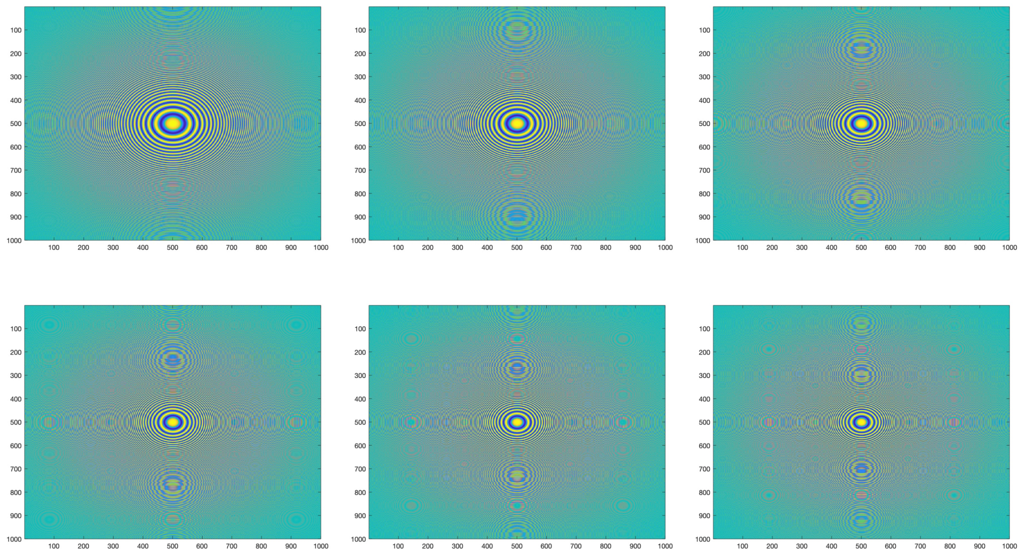
(i) Keep z = 50 mm fixed, $\delta = \Delta = 2\mu m, 5\mu m, 10\mu m, 20\mu m$. How does sampling affect the performance of digital holography?

In the first figure below, we can see the change in the imaging system based on the change in delta. The top left is the lowest delta at $2\mu m$ and the bottom right is the highest delta at $10\mu m$. If you look at the second figure you can see the surface plots for the lowest and highest delta from left to right respectively. We can see well from the surface plot that as the delta increases the imaging systems impulse response becomes more jagged and aliased. This makes since, if we have a smaller delta and thus more samples for an incoming signal, our reconstruction will be smother and more similar to the incoming signal. Whereas with a larger delta we are taking less samples of the incoming signal and the aliasing becomes very apparent. This example shows me that more sampling (smaller delta) for digital holography will produce a more accurate reconstruction with less aliasing compared to less sampling (larger delta).



(ii) Keep $\delta = \Delta = 5\mu m$ fixed, while z ranges from 30 mm to 70 mm (use a small step-size). How does object distance affect the performance of digital holography?

The figure below shows the output as the z ranges from 30mm to 70mm. The top leftmost image shows the smallest range of 30mm and the bottom rightmost image shows the largest range of 70mm. Each image in between is an increase of 10mm from left to right, top to bottom. We can see that as the z increases (object distance increases) the impulse response shrinks. This makes since, if we were imaging a single point of light, as the light moves farther away the response would become smaller as less intense light is interacting with the imaging system. Therefore from these images we can see that the object distance will reduce the response from the imaging system and provide less detail as one would expect. As the z increases the smoothing effect becomes more noticeable in that things would become blurry and less detailed.



(iii) Keep $\Delta = 5\mu m$ and z=50mm fixed,while δ ranges from $1\mu m$ to $10\mu m$ in $2\mu m$ step-size. How does discretization of the object affect the performance of digital holography?

In the first figure below you can see the output for the range of δ from $1 - 10\mu m$. The top leftmost images shows with the smallest value of δ and the bottom rightmost image shows the largest value of δ. In the second figure we can see the surface plots of the smallest and largest values of δ from left to right respectively. Note that the Δ for the imaging system is kept the same for all of these outputs. We can see that similar to (i) when the value of the delta is smaller and thus more samples are taken we get a smoother and less aliased camera response. When the value is then changed to its max for this problem the output shows much more aliasing and jagged edges. This is to be expected, because even if the camera pixels are fine grained, if the object itself is not discretized in a fine grain way then the imaging will obviously result in an image that is blocky and aliased. This example shows me that more sampling (smaller delta) in the discretization of the object will produce a more accurate reconstruction with less aliasing compared to less sampling (larger delta) of the object.