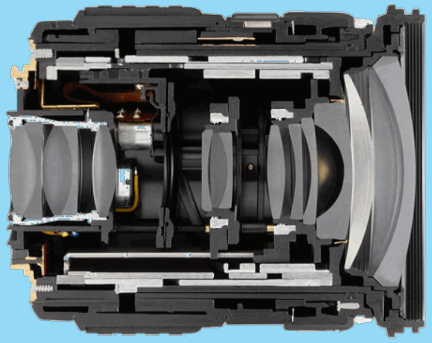
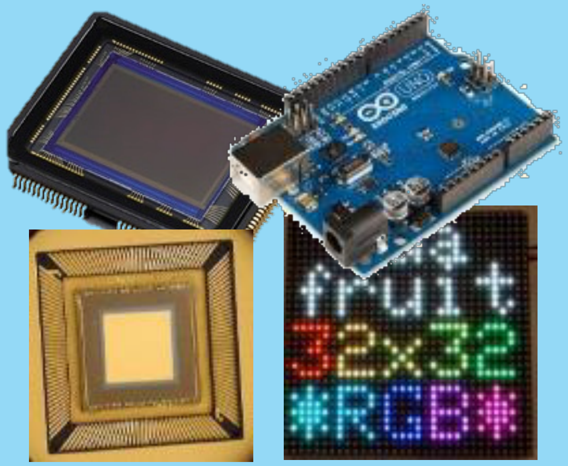


Optics

+



Sensors
&
devices

+

Signal
processing
&
algorithms

COMPUTATIONAL METHODS FOR IMAGING (AND VISION)

LECTURE ~~13~~ 14:
LSI IMAGING SYSTEMS (MATRIX-VECTOR FORMS)

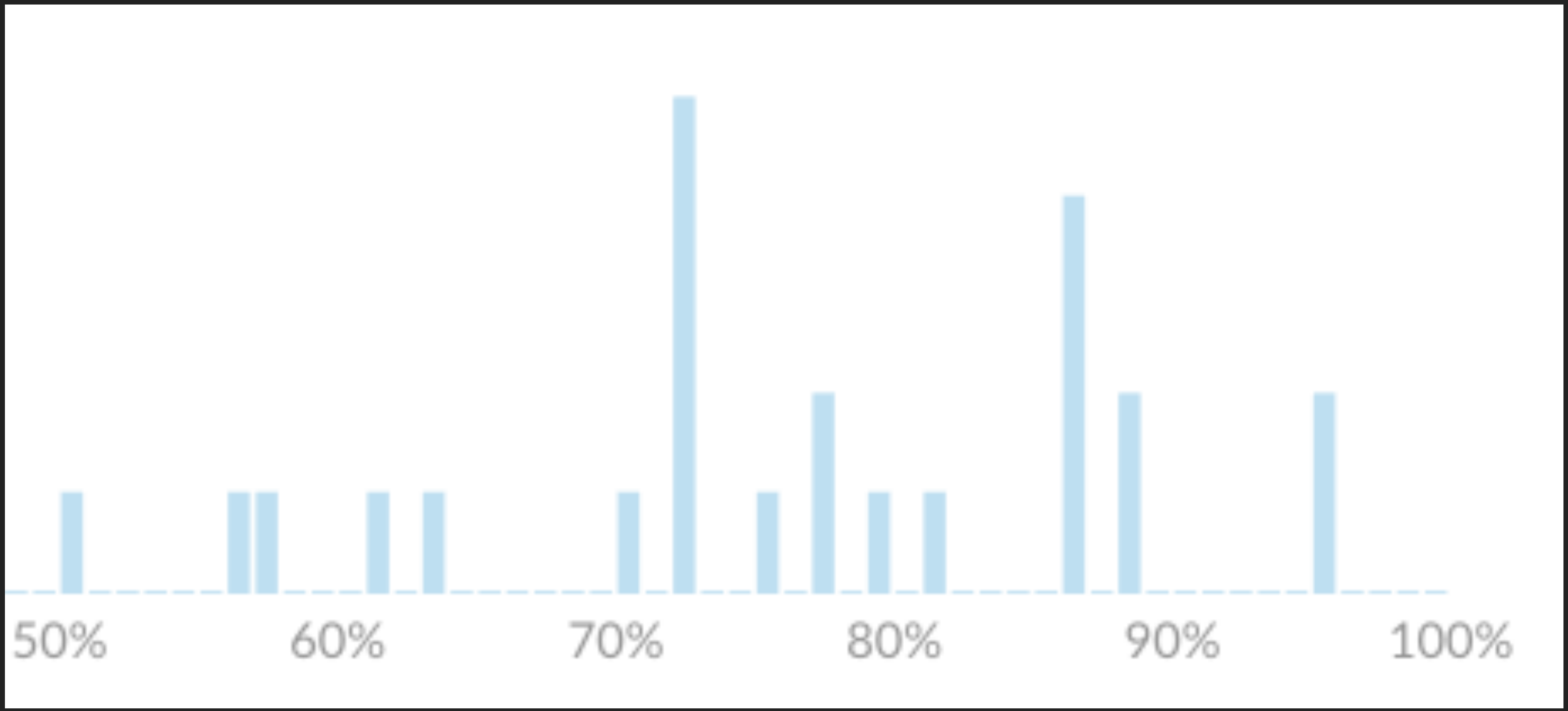
PROF. JOHN MURRAY-BRUCE

CIS 4930.006S20/6930.013S2

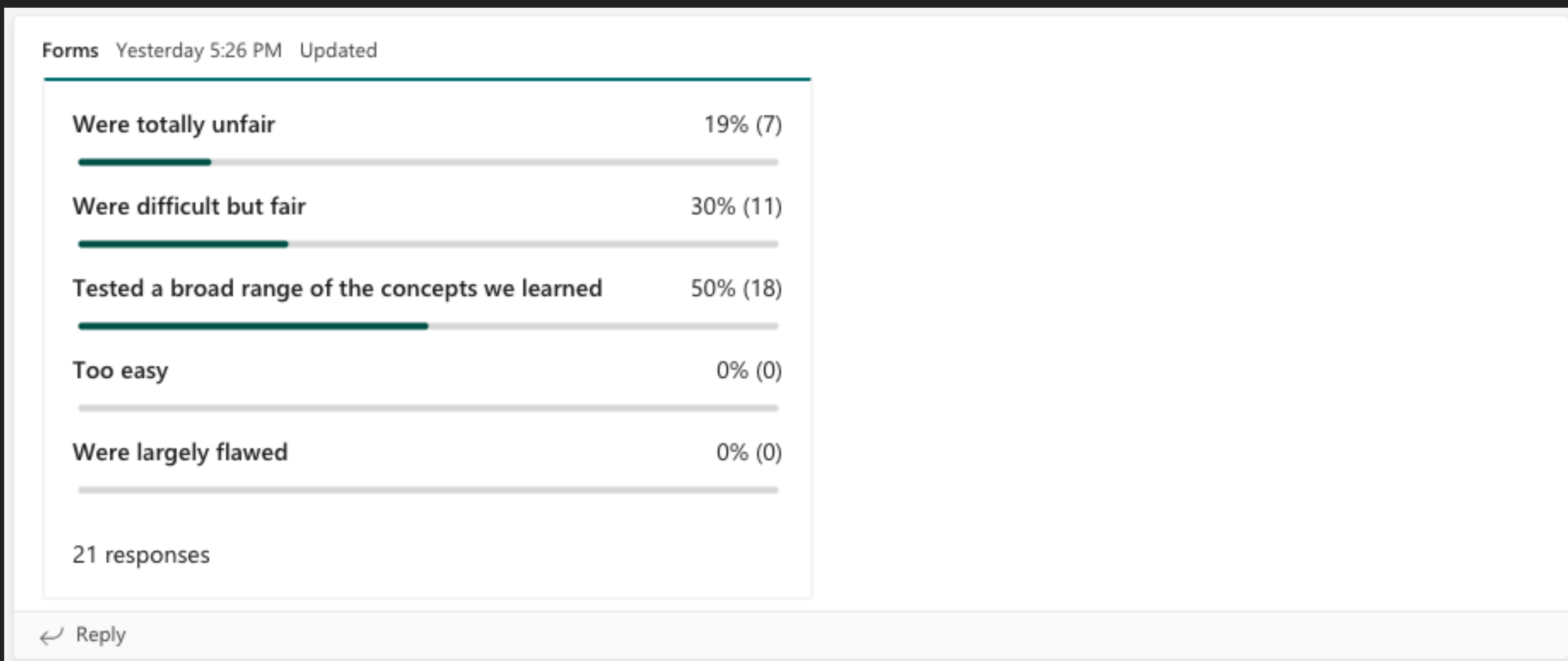
MIDTERM EXAM

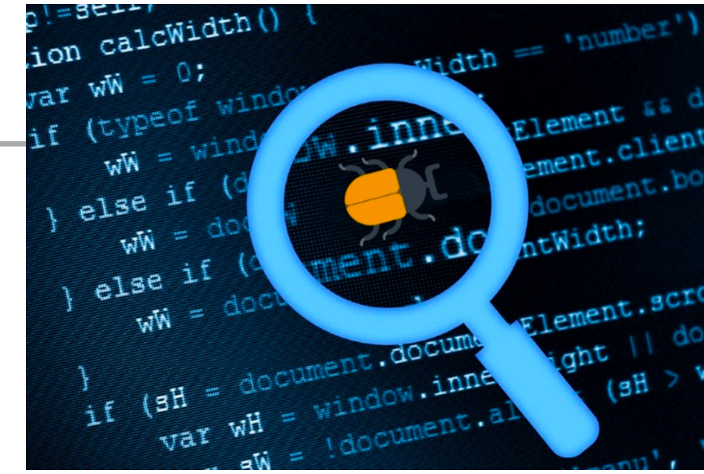
QUESTIONS REVIEW

- **Overall**
  - **Mean score: 77 % (UG 76, G 79)**
  - **Median score: 78 %**



THE QUESTIONS IN THE MID-TERM QUIZ (SELECT THE TWO MOST RELEVANT):





WHAT DIDN'T GO SO WELL?

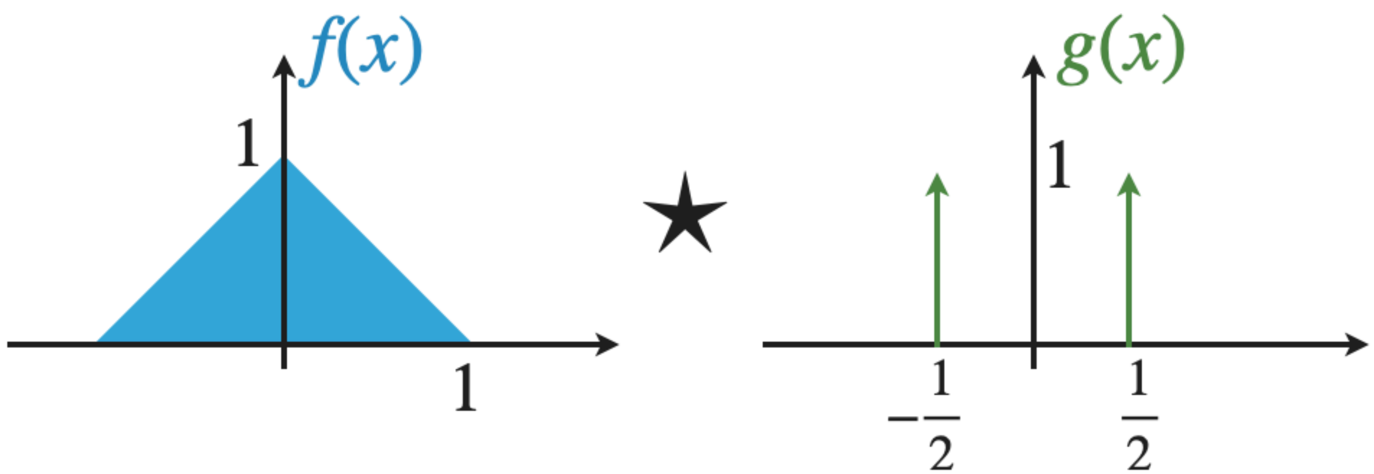
PESKY BUGS

Attempts: 24 out of 24

+0.12

Discrimination Index

Consider the sketches of the functions $f(x)$ and $g(x)$ shown below:



Which of the following options correctly shows the convolution between $f(x)$ and $g(x)$?

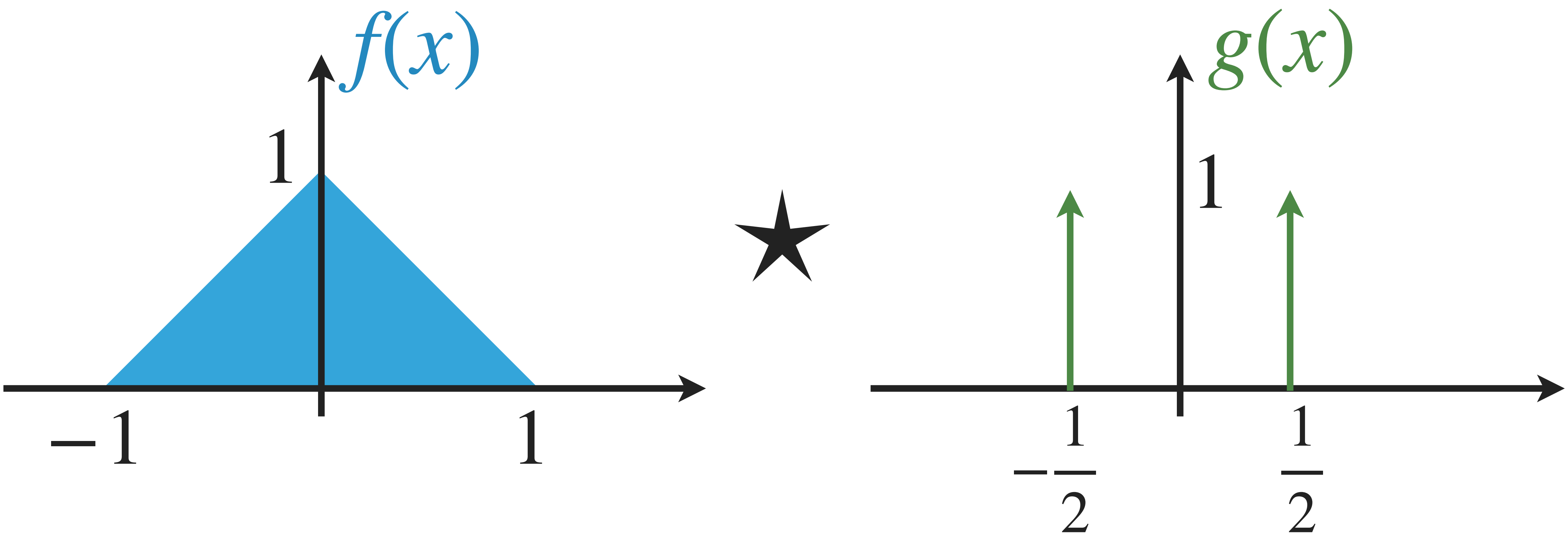
No answer text provided.		0 %
No answer text provided.	2 respondents	8 %
No answer text provided.	21 respondents	88 %
No answer text provided.	1 respondent	4 %

88% answered correctly

WHAT DIDN'T GO SO WELL?

PESKY BUGS

- **Convolution question:**



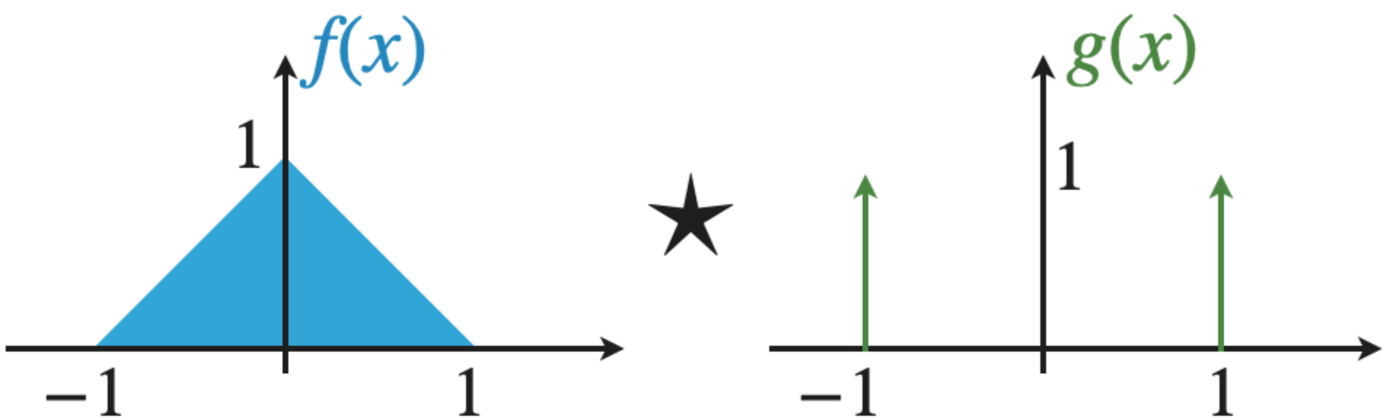
WHAT DIDN'T GO SO WELL?
PESKY BUGS

Attempts: 24 out of 24

+0.22

Discrimination Index

Consider the sketches of the functions $f(x)$ and $g(x)$ shown below:

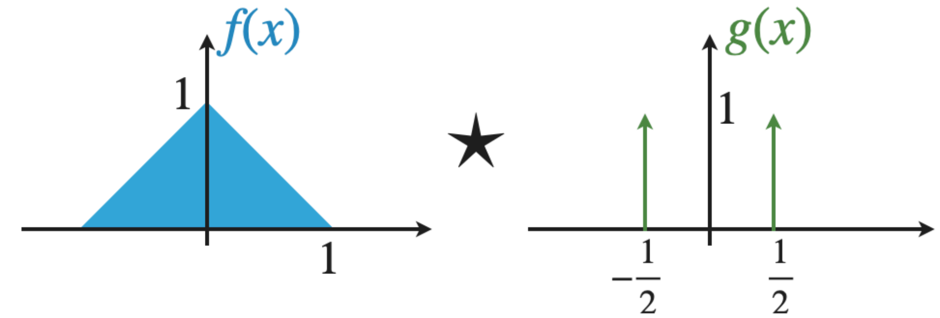


Which of the following options correctly shows the convolution between $f(x)$ and $g(x)$?

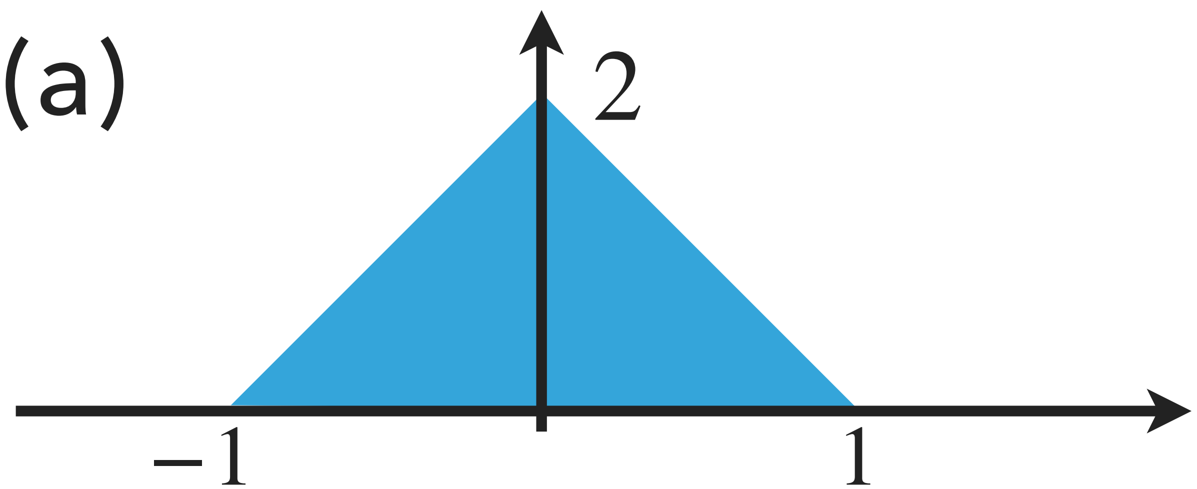
	20 respondents	83 %	83% answered correctly
No answer text provided.		0 %	
No answer text provided.	1 respondent	4 %	
No answer text provided.	3 respondents	13 %	

CONVOLUTION QUESTION 2

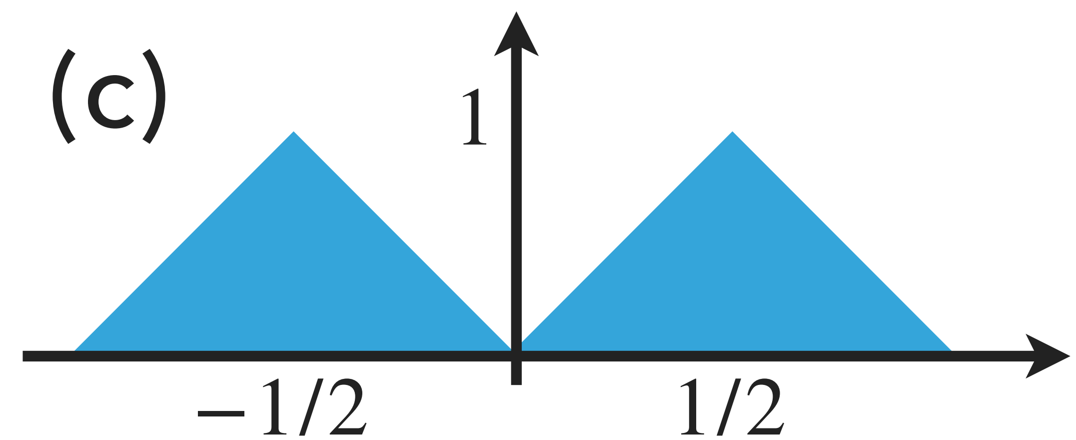
Consider the sketches of the functions $f(x)$ and $g(x)$ shown below:



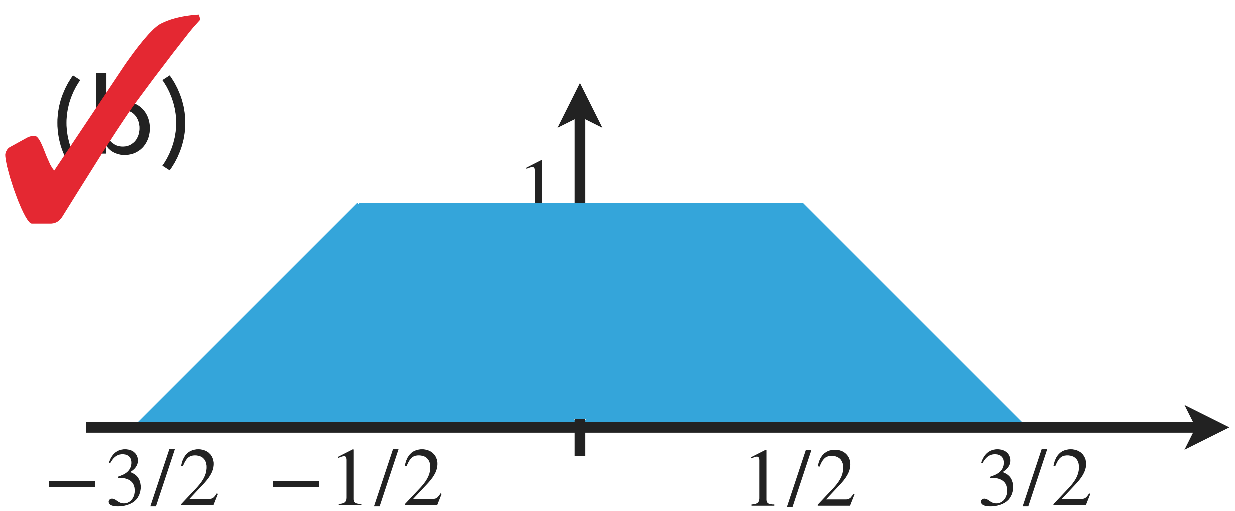
(a)



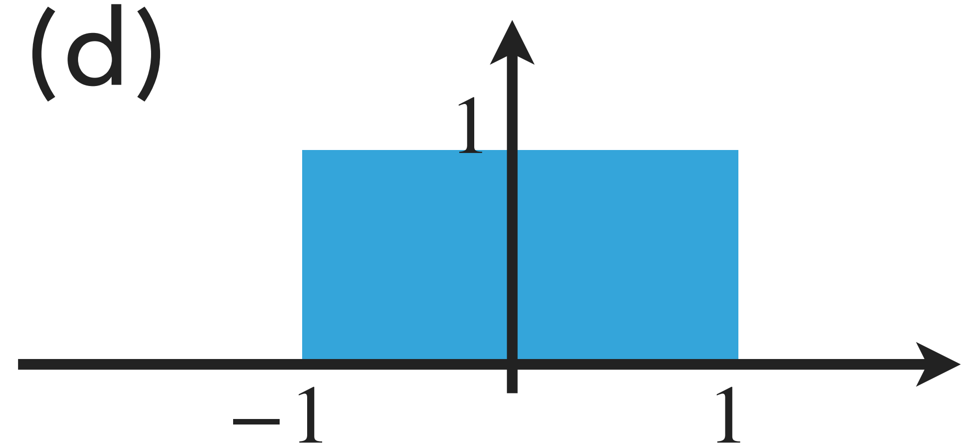
(c)



(b)



(d)



WHAT DIDN'T GO SO WELL?

PESKY BUGS

- **Discrimination indices** are quite similar
  - No possible way to get the other answers, so some ways this was actually a great question
  - It's an **inverse problem**
  - Nonetheless, everyone will get the points! Well the three respondents that were "tripped-up" by the absence of that '-1')
- **All responses to Q37** were lost, for some unknown reason. Everyone gets the point for this.

12 respondents	50 %	✓
1 respondent	4 %	
1 respondent	4 %	
10 respondents	42 %	

Question 8

Consider the n-dimensional vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, which one of the following is the $\ell^p(\mathbb{Z}^n)$ -norm of the vector $\mathbf{u}$

- $\|\mathbf{u}\|_p = \sum_{i=1}^{n} |u_i|^p$
- $\|\mathbf{u}\|_p = \left(\sum_{i=1}^{n} |u_i|^{2p}\right)^{1/2}$
- $\|\mathbf{u}\|_p = \left(\sum_{i=1}^{n} |u_i|^{2}\right)^{1/p}$
- ✓ $\|\mathbf{u}\|_p = \left(\sum_{i=1}^{n} |u_i|^{p}\right)^{1/p}$

NORMED VECTOR SPACES

P-NORMS

- Finite dimensions, $\mathbb{R}^n$:

$$\|\mathbf{u}\|_p = \left(\sum_{i=1}^{n} |u_i|^p\right)^{1/p}, p \in [1,\infty)$$

- Square-summable sequences, $\ell^p(\mathbb{Z})$:

$$\|\mathbf{u}\|_p = \left(\sum_{i=-\infty}^{+\infty} |u_i|^p\right)^{1/p}, p \in [1,\infty)$$

- Square-integrable function, $\mathscr{L}^p(\mathbb{R})$:

$$\|f(t)\|_p = \left(\int_{-\infty}^{+\infty} |f(t)|^p \,\mathrm{d}t\right)^{1/p}, p \in [1,\infty)$$

- Vectors in $\ell^p(\mathbb{Z})$ different to those in $\ell^q(\mathbb{Z})$ when $p \neq q$. Similarly for $\mathscr{L}^p(\mathbb{R})$ and $\mathscr{L}^q(\mathbb{R})$.
- $p = 2$ is the only $\ell^p(\mathbb{Z})$, or $\mathscr{L}^p(\mathbb{R})$, norm induced by an inner product.

What is the inner product between $f(t) = t$ and $g(t) = \sin(\pi t)$, over the vector space $\mathcal{L}^2([-1, 1])$.

None of the other options are valid	29 %
$-\frac{1}{\pi} \approx -0.3183$	58 %
✓ $\frac{2}{\pi} \approx 0.6367$	0 %
$\frac{1}{\pi} \approx 0.3183$	13 %

EXAMPLE 1: WORKED SOLUTION

$$
\begin{aligned}
\langle x(t), y(t)\rangle &= \int_{-1}^{+1} x(t)y(t)\,\mathrm{d}t, && \text{from the definition of the inner product}\\
&= \int_{-1}^{+1} t\sin(\pi t)\,\mathrm{d}t, && \text{substituting } x(t) = t \text{ and } y(t) = \sin(\pi t)\\
&= \left[-\frac{t}{\pi}\cos(\pi t)\right]_{-1}^{+1} - \left(-\int_{-1}^{+1}\frac{1}{\pi}\cos(\pi t)\,\mathrm{d}t\right), && \text{integration by parts}\\
&= \left[-\frac{t}{\pi}\cos(\pi t)\right]_{-1}^{+1} + \left[\frac{1}{\pi^2}\sin(\pi t)\right]_{-1}^{+1}\\
&= \left(-\frac{1}{\pi}\cos(\pi) - \frac{1}{\pi}\cos(-\pi)\right) + \frac{1}{\pi^2}\left(\sin(\pi) - \sin(-\pi)\right)\\
&= \frac{2}{\pi}
\end{aligned}
$$

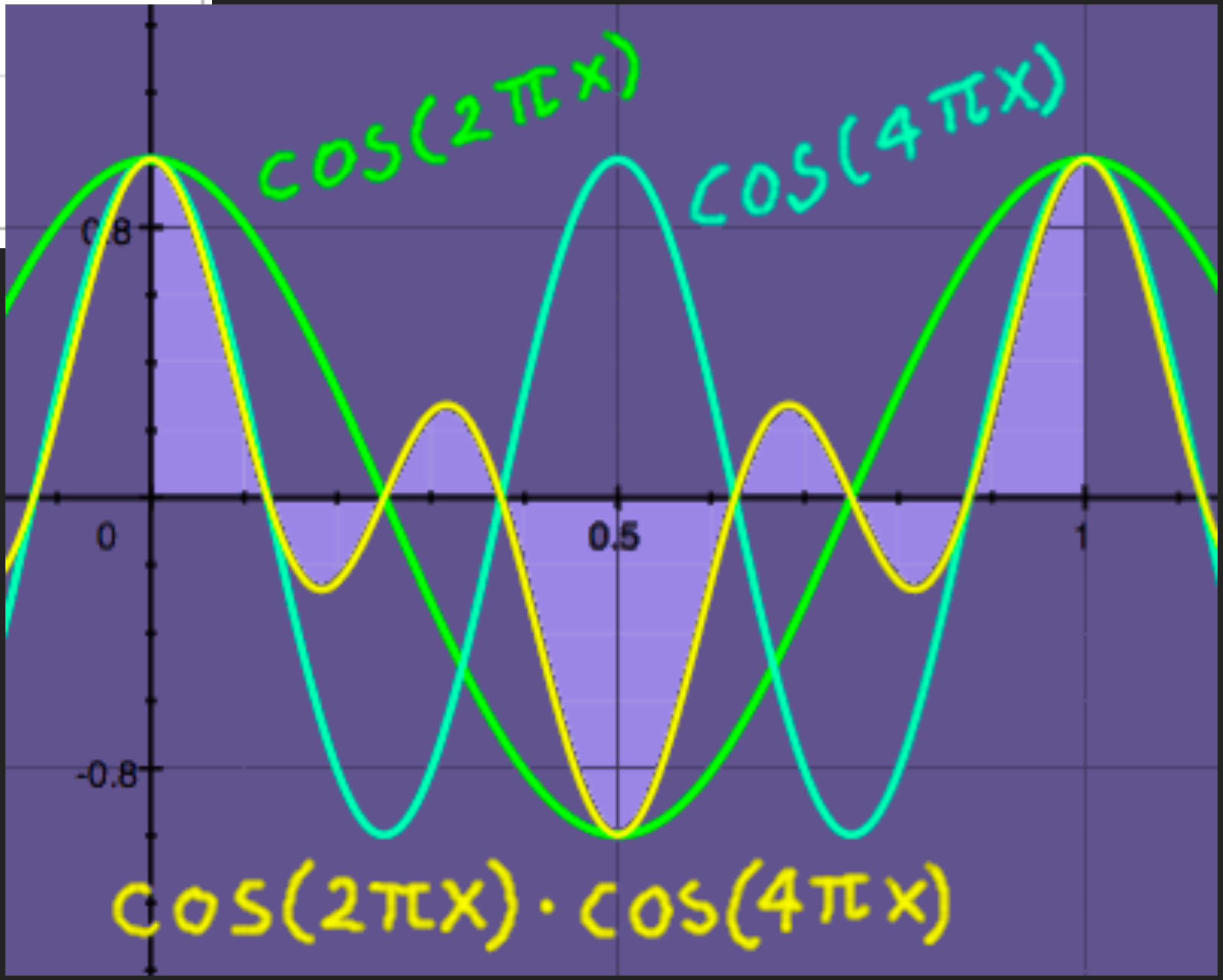
TEXT

Question 22 1 pts

Consider the functions $f(x) = \cos(2\pi x)$, and $g(x) = \cos(8\pi x)$. Do the functions form orthogonal subspaces of the vector space $\mathcal{L}^2([-1,1])$?
[Hint: you may find the identity
$\cos(x)\cos(y) = 1/2[\cos(x+y) + \cos(x-y)]$ to be useful.]

- Yes, because their inner product is not zero
- No, they do not because their inner product is zero
- None of the other options, because this has nothing to do with their inner products
- ✓ Yes, because their inner product is zero

	0 %	
1 respondent	4 %	
21 respondents	88 %	✓
2 respondents	8 %	



See also solutions to HW1, Question 5.

TEXT

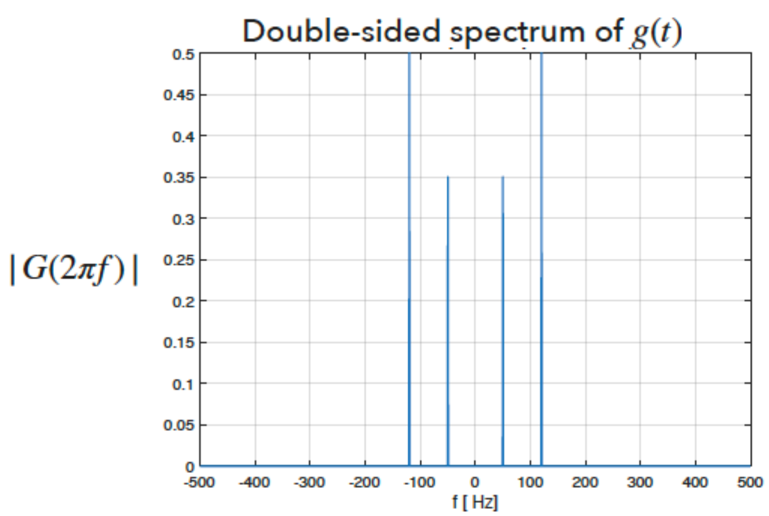
What is the 1D inverse Fourier transform of $F(\omega) = \delta(\omega - 2) + \delta(\omega + 2)$?

- $\frac{1}{2\pi}\cos(2x)$
- $-2\,\mathrm{sinc}(2x)$
- $\frac{1}{2\pi}(\cos(2x) + \sin(2x))$
- $\frac{1}{2\pi}\sin(2x)$

MATLAB EXAMPLE

MATLAB

- Notice the peaks are at 50 Hz and 120 Hz, corresponding to the frequencies of the sinusoids in the original signal $g(t)$

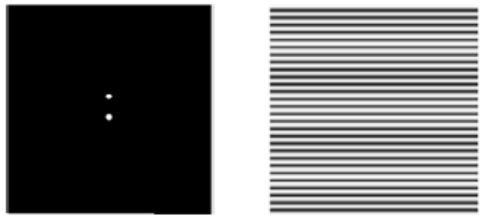


Note: for a complex number z, $|z|^2 = zz^* = z^*z$

SOME IMPORTANT FOURIER TRANSFORM PAIRS

MATLAB

- Two delta functions
  $f(x, y) = 1/2\left[\delta(x, y - y_0) + \delta(x, y + y_0)\right]$



$$F(\omega_x, \omega_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, e^{-j(\omega_x x + \omega_y y)}\, dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 1/2\left[\delta(x, y - y_0) + \delta(x, y + y_0)\right] e^{-j(\omega_x x + \omega_y y)}\, dxdy$$

$$= 1/2\left(e^{j\omega_y y_0} + e^{-j\omega_y y_0}\right)$$

$$= \cos(\omega_y y_0)$$

TEXT

Attempts: 22 out of 24

This question is about bases. Please read carefully and ensure you answer all parts of this question.

A. Describe the key differences between an **orthogonal basis** and a **biorthogonal basis.**

B. You are given a biorthogonal basis **only**, describe clearly the steps you would take to compute the expansion coefficients for the vector $\boldsymbol{x}$ in terms of the basis. (Assume that $\boldsymbol{x}$ is in the space spanned by the basis set.)

Answers which scored in the top 27%	9 respondents	38 %
Answers which scored in the middle 46%	12 respondents	50 %
Answers which scored in the bottom 27%	3 respondents	13 %

View in SpeedGrader

- **A:**
  - Orthogonal - vectors are orthogonal (pairwise inner products are zero). [+1]
  - Biorthogonal - basis vectors are not orthogonal, and have dual basis. Pair of bases (not necessarily orthogonal). Pairwise inner products of basis and duals are zero (apart from one). [+1]
- **B:**
  - [+1]: Find the dual
  - [+1]: State how this can be done
  - [+1]: Take inner product between dual and x

FLATCAM: THIN, LENSLESS CAMERAS USING CODED APERTURE AND COMPUTATION

CONFOCAL NON-LINE-OF-SIGHT IMAGING BASED ON THE LIGHT-CONE TRANSFORM

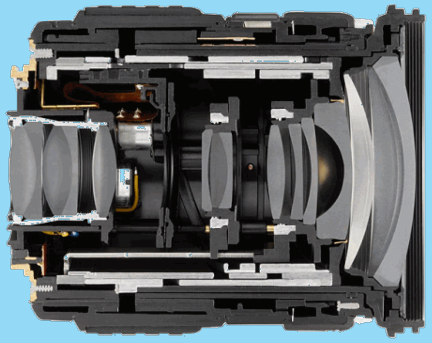
DIFFUSERCAM: LENSLESS SINGLE-EXPOSURE 3D IMAGING

COMPUTATIONAL PERISCOPY WITH AN ORDINARY DIGITAL CAMERA

REMOVING REFLECTION FROM A SINGLE IMAGE WITH GHOSTING EFFECT

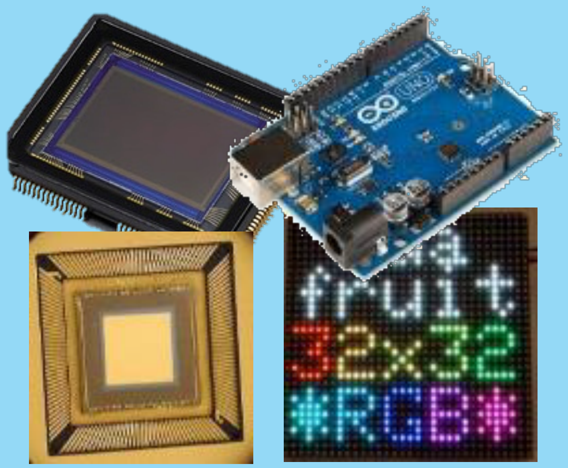
GROUP PROJECTS

PLEASE SIGN UP TO A TEAM BY 5 PM ON FRIDAY, MARCH 5, 2021



Optics

+



Sensors & devices

+

Signal processing & algorithms

COMPUTATIONAL METHODS FOR IMAGING (AND VISION)

LECTURE 14: LSI IMAGING SYSTEMS (MATRIX-VECTOR FORMS)

PROF. JOHN MURRAY-BRUCE

WHERE ARE WE

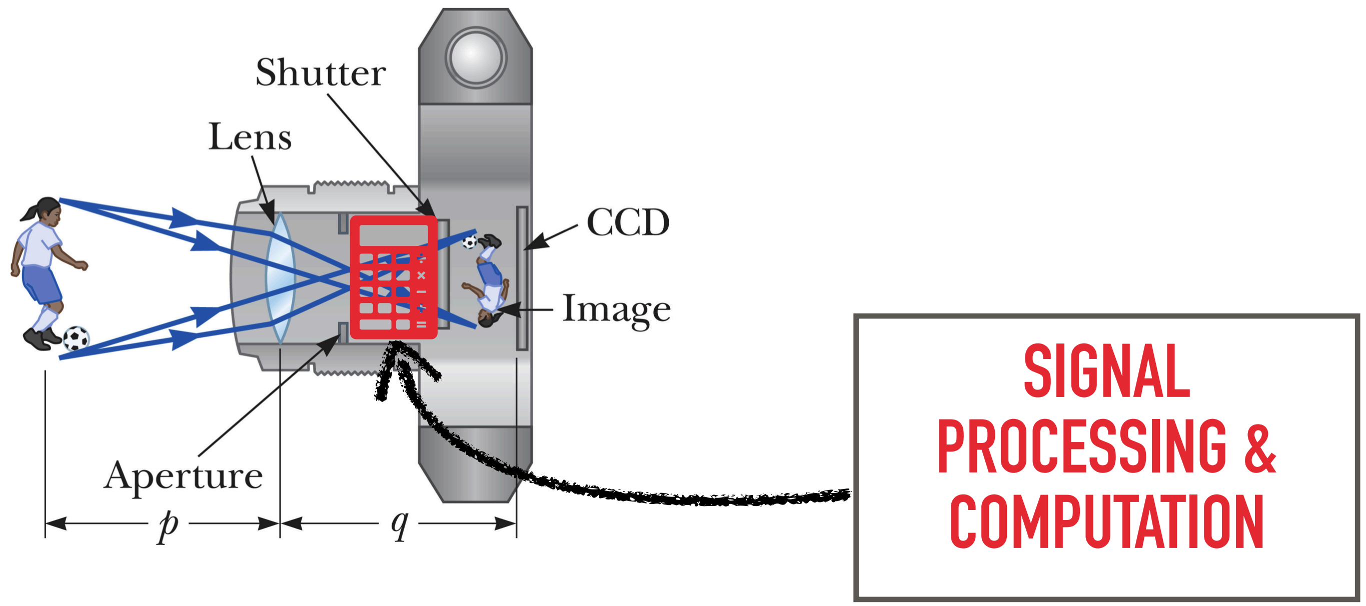
WE ARE HERE!

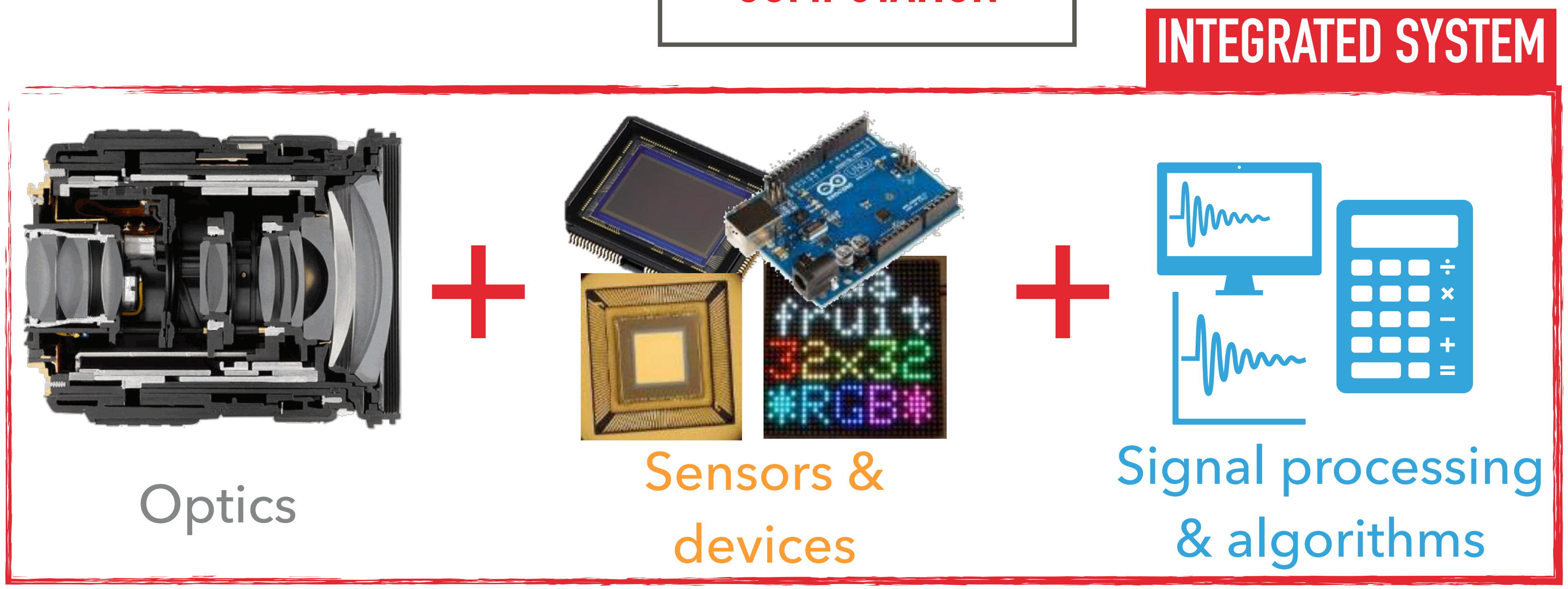
Week	Date	Main Topic	Lecture	Readings	Homework Out	Homework Due
1	11-Jan-21		Introduction to computational imaging - Forward and Inverse problems - Common computational imaging problems			
	13-Jan-21		Vectors - Preliminaries			
2	18-Jan-21		**Dr. Martin Luther King, Jr. Holiday (no class)**			
	20-Jan-21	Mathematical preliminaries	Vectors and Vector Spaces - Subspaces, Finite dimensional spaces	IIP Appendix A; FSP 2.1 - 2.2		
3	25-Jan-21		Vector Spaces - Hilbert spaces	IIP Appendix B; FSP 2.3		
	27-Jan-21		Bases and Frames I - Orthonormal and Reisz Bases	IIP Appendix C; FSP 2.4 and 2.B	**HW 1**	
4	1-Feb-21		Bases and Frames II - Orthogonal Bases - Linear operators	IIP Appendix C; FSP 2.5 and 2.B		
	3-Feb-21		Fourier Analysis I - FT (1D and 2D) - FT properties	IIP 2.1, Appendix D; FSP 4.4		
5	8-Feb-21		Sampling and Interpolation - BL functions - Sampling	IIP 2.2, 2.3; FSP 5.4, 5.5		**HW 1**
	10-Feb-21		Fourier Analysis II (DFT)	IIP 2.4; FSP 3.6	**HW 2**	
6	15-Feb-21	Forward Modeling	LSI imaging: Forward problem I - Convolution	IIP 2.5 - 2.6, 3		
	17-Feb-21		LSI imaging: Forward problem I - Transfer functions	IIP 2.6		
7	22-Feb-21		LSI imaging: Forward problem I - Linear operators	IIP 3		
	24-Feb-21		LSI imaging: Forward problem I - Linear operators, Adjoints, and Inverses		**HW 3**	**HW 2**
8	1-Mar-21		**Mid-term Exams**			
	3-Mar-21		LSI imaging: Forward problem II - Sampling and Discretization: Matrix-vector form	IIP 2.7, 4		
9	8-Mar-21		LSI imaging: Forward problem II - Convolution matrix			
	10-Mar-21		LSI imaging: Forward problem II - Sampling and Discretization: Matrix-vector form - PSF, and Transfer functions			**HW3**

LSI IMAGING
ANALOG

- Analyzed **Continuous LSI** imaging systems from a **function spaces perspective**.
- **Input-Output** relationship:
  - **Convolution integral (continuous spatial domain)**
    - Output is convolution between input function/signal and system impulse response
  - **Transfer functions (frequency domain)**
    - Output is product between $G(\omega_x, \omega_y)$, the FT of input, and the system's transfer function $H(\omega_x, \omega_y)$
- Generally speaking, traditional **optics perform analog processing.**

TOWARD COMPUTATIONAL IMAGING





DIGITAL VS ANALOG IMAGING
THE TRADEOFF

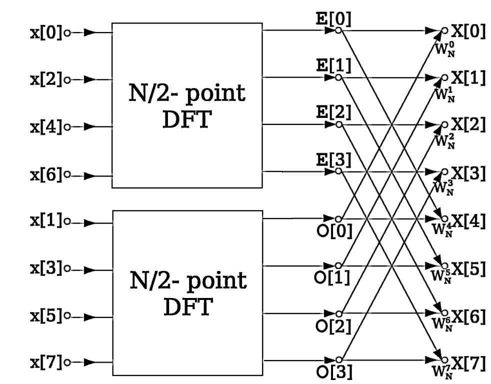
- **Optics:** Fast, but limited by
  - Physical laws
  - Material constraints
  - Manufacturing capabilities
  - Cost, size, and …

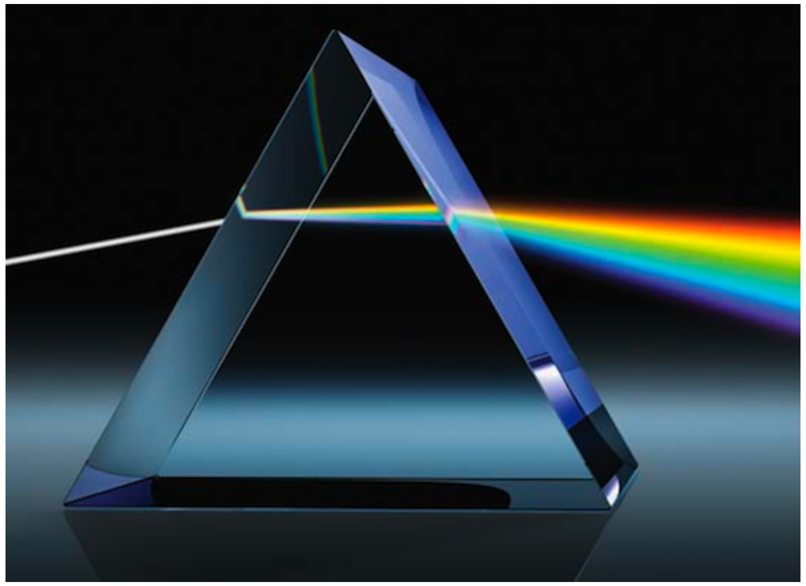
- **Digital signal processing:** Fundamentally slower, but “easier” to work with
  - Flexibility, compactness, etc

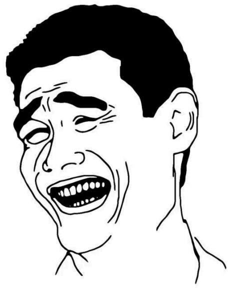
Optimization:

Best of analog optics and digital computation

Fast Fourier transform, you say?







Bit*h Please! I can Fourier transform at the speed of light!

LSI IMAGING

DISCRETE SYSTEMS: FROM FILMS TO DIGITAL SENSORS

- **Digital (r-)evolution:** processing is done digitally, leading to **discrete LSI systems**
  - **Digital sensors**
    - Sampling: how do imaging sensors work?
    - Digitization: digital representation of the object and the imaging system
  - **"Transfer function" of discrete LSI system:** are the **Eigenvalues** of the imaging matrix
- We will consider these three things
  - Sampling, digitization and transfer functions of discrete LSI systems

OUTLINE

- Discrete LSI Imaging Systems
- Matrix-Vector models
  - Sampling, discretization and “transfer functions” of discrete LSI systems

LEARNING GOALS

- Understand simple discretization schemes for object and image spaces
- Discretize image space
- Describe effect of such discretization

READING

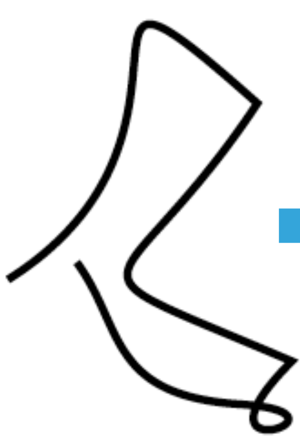
- IIP 2.5 - 2.7
- IIP 3.4

SAMPLING
DISCRETIZATION
TRANSFER FUNCTIONS

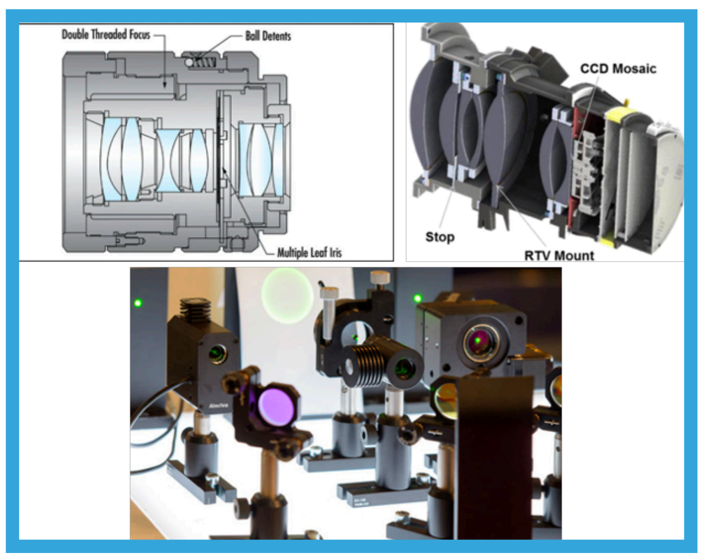
HOW DO IMAGING SENSORS WORK?

SAMPLING
HOW DO IMAGING SENSORS/DETECTORS TYPICALLY WORK?

- The operator is the same at any point in space (LSI)



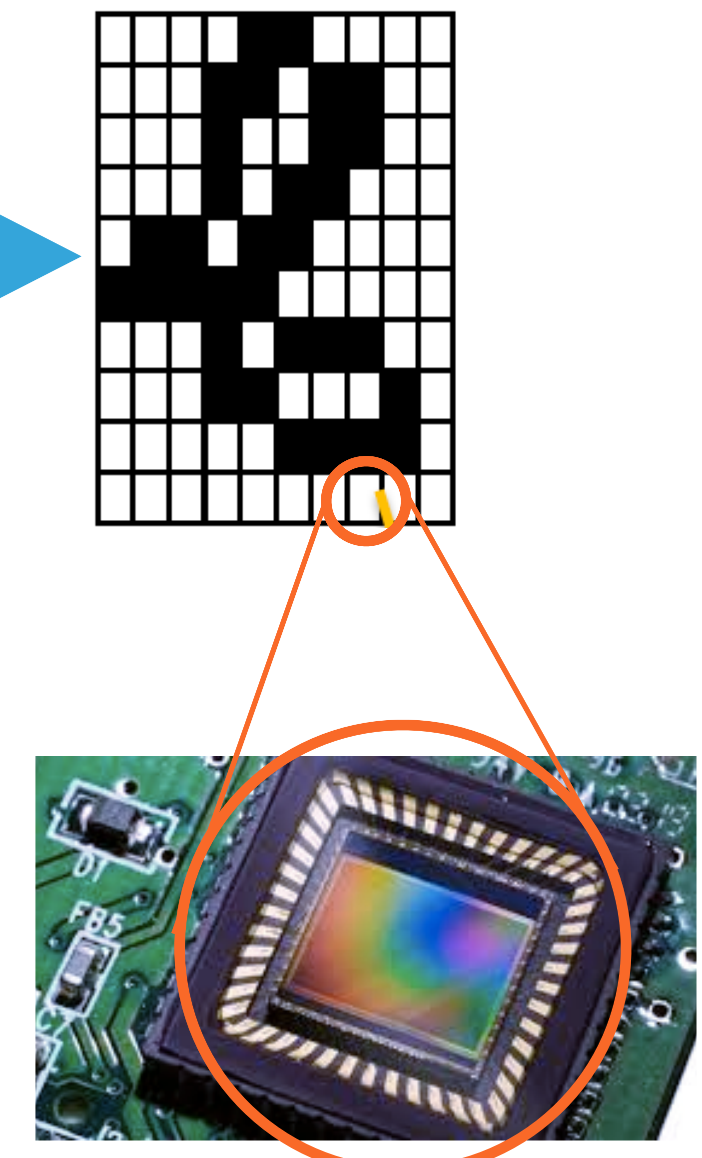
$g(x, y)$



Optics $h(x, y)$

$g(x, y) \star h(x, y)$

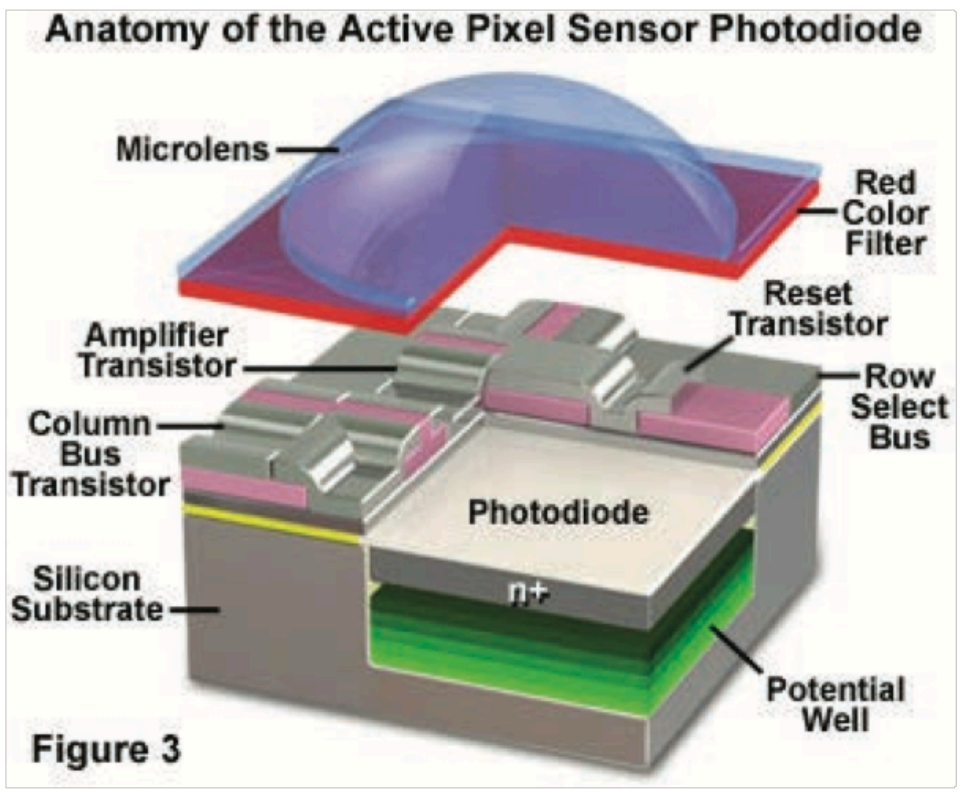
Sampling $\{g_{mn}\}$



HOW DO IMAGING DETECTORS WORK?

THE PHYSICS

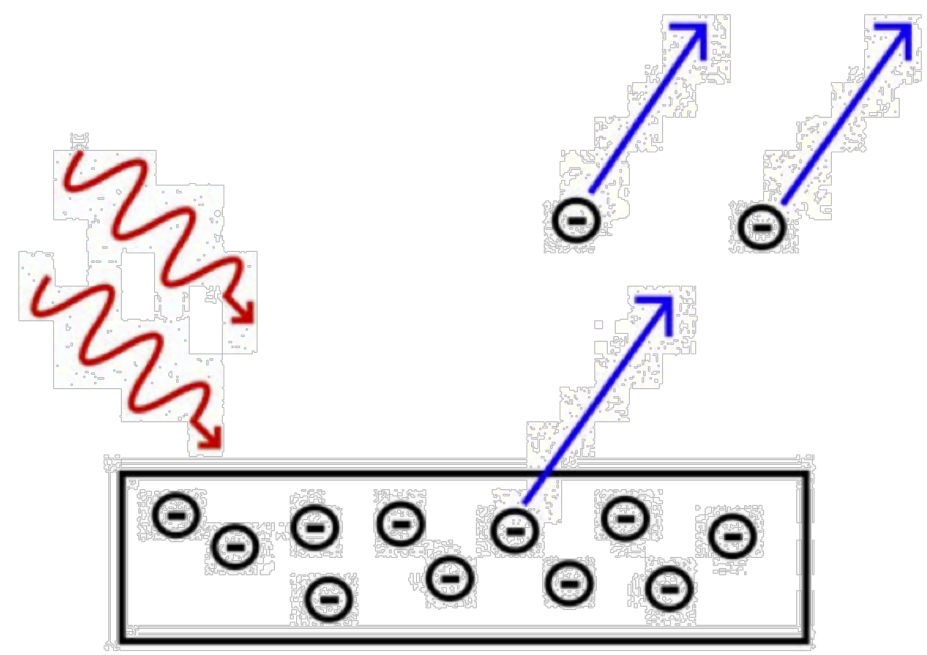
DETECTOR ELEMENT
FROM PHOTONS TO ELECTRONS



source: Molecular Expressions

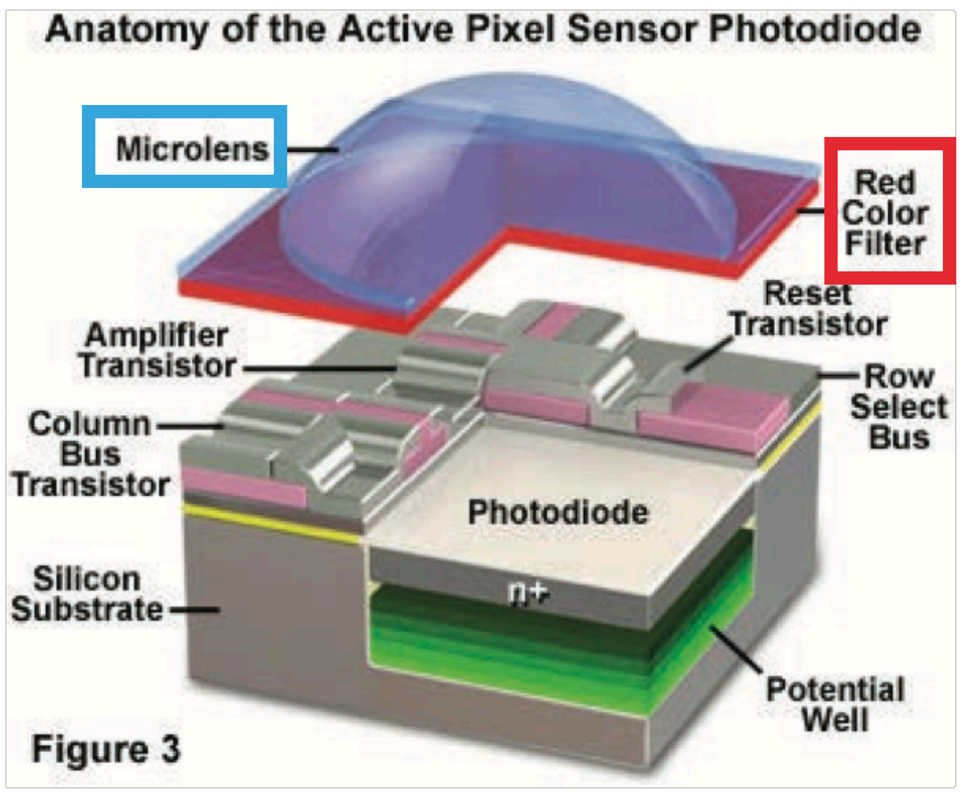
photon to electron converter

→ photoelectric effect!



wikipedia

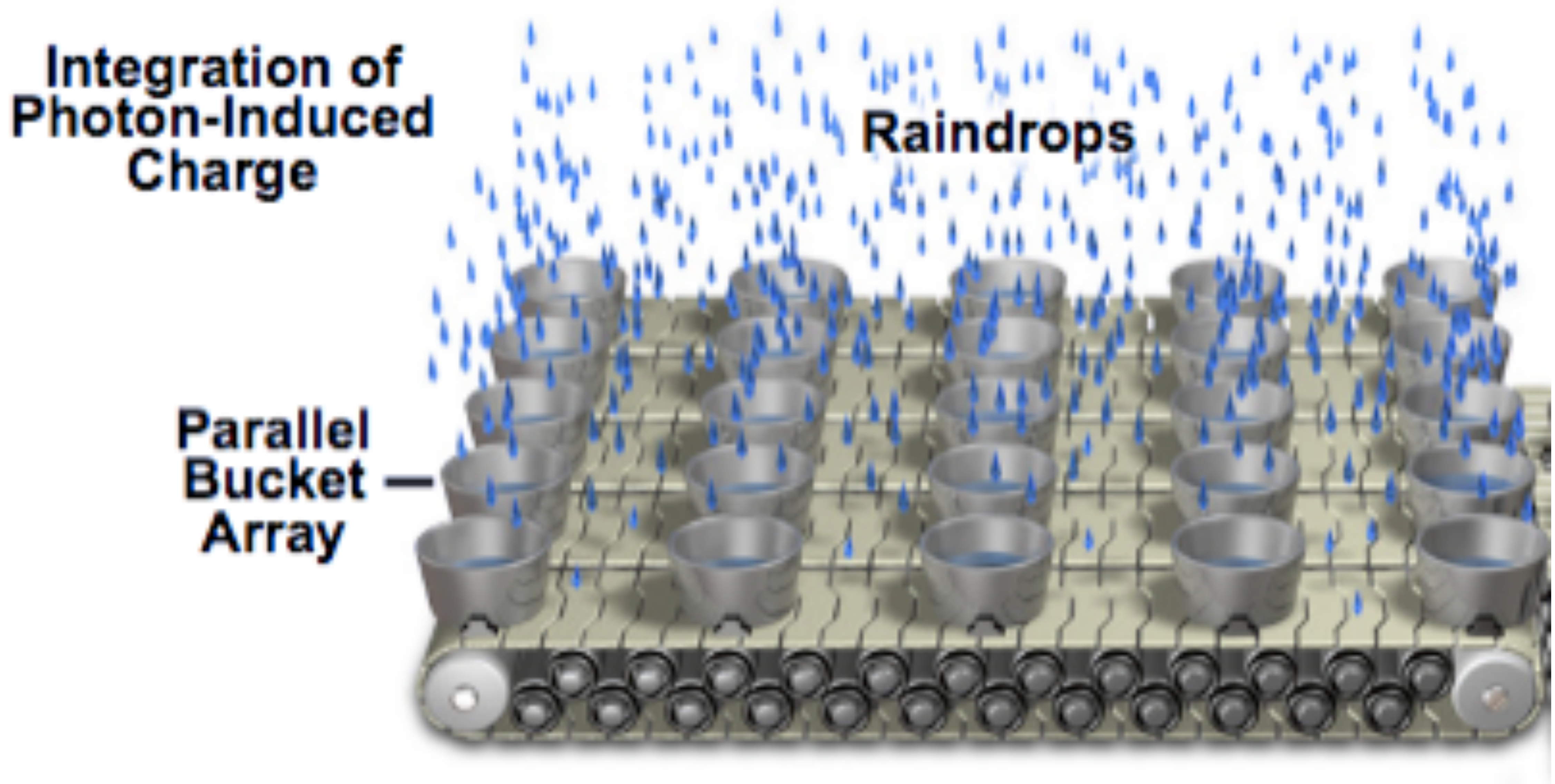
DETECTOR ELEMENT
FROM PHOTONS TO ELECTRONS



source: Molecular Expressions

- **Microlens:** focus light onto photodiode
- **Color filter:** Transmits the desired color channel
- **Quantum efficiency:** ratio of charge carries (electrons) generated to the number of incident photons. (~50%)
- **Fill factor:** fraction of surface area used for light gathering

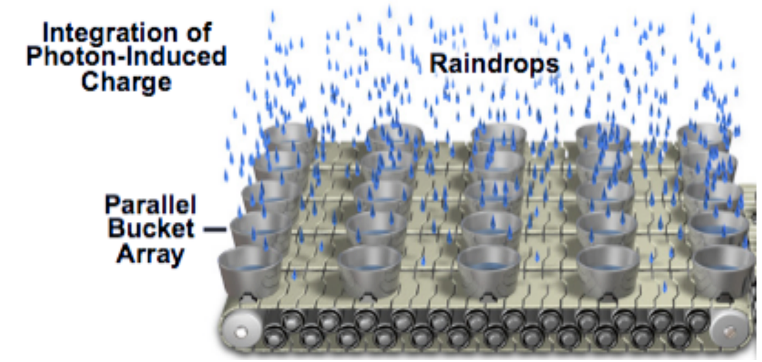
ARRAY OF DETECTOR ELEMENTS
INTUITION

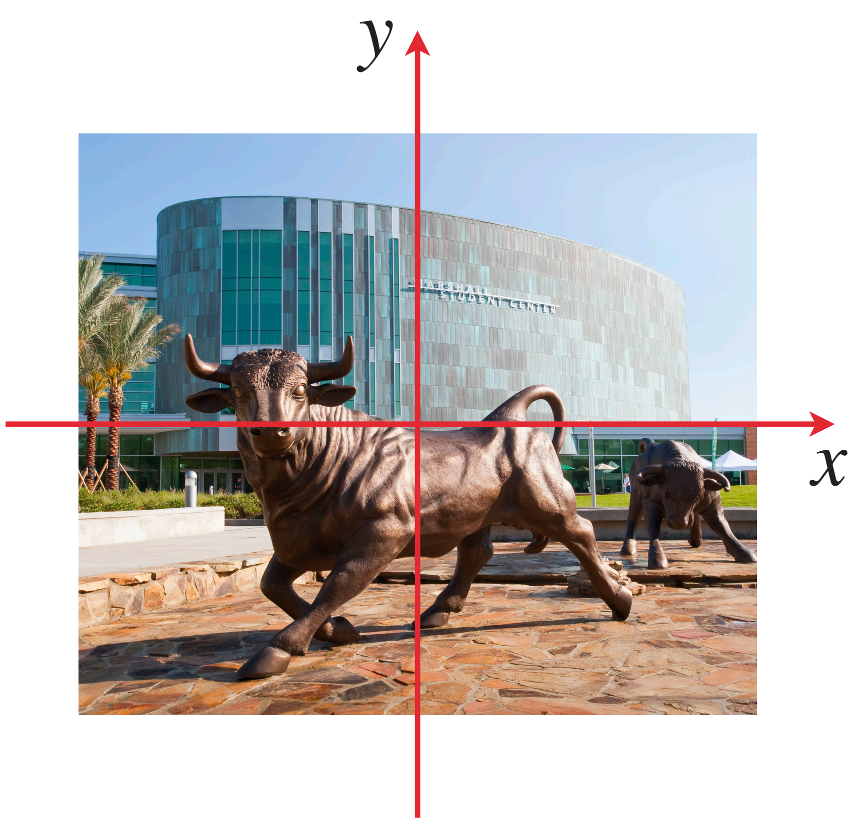


TURNING THE PHYSICS INTO MATH

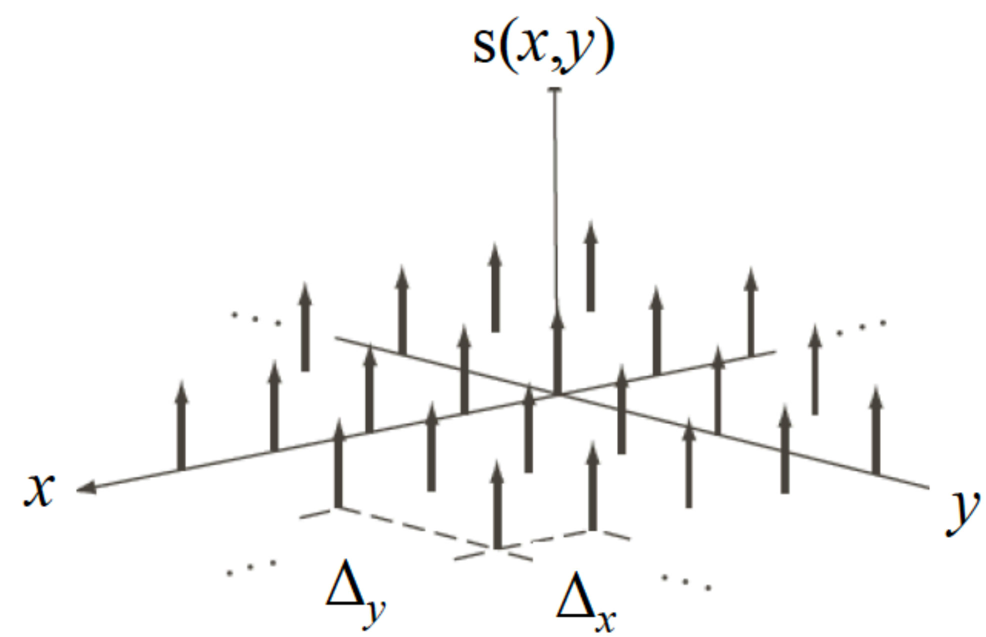
THE MATHEMATICAL ABSTRACTION

IDEALIZED IMPULSE SAMPLING



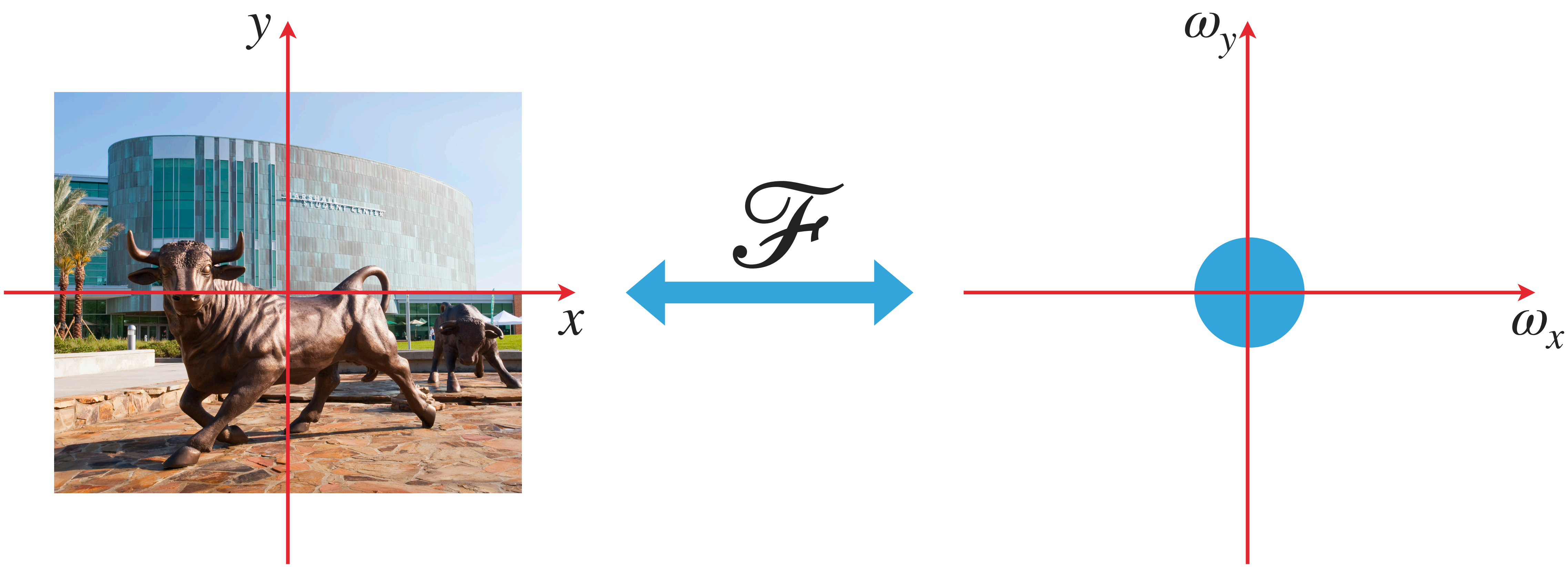


$\times$

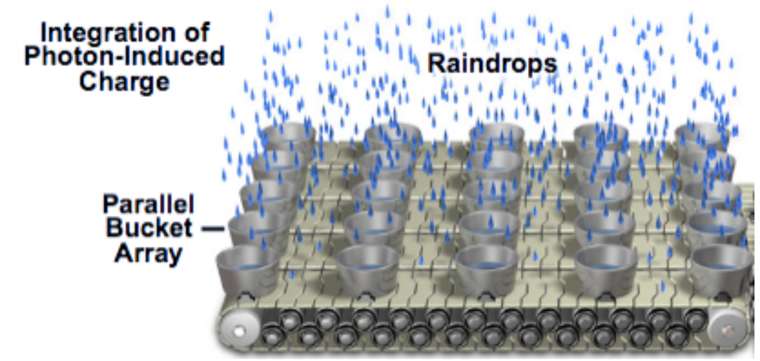


See lectures 8 & 9, slides 15 - 21.

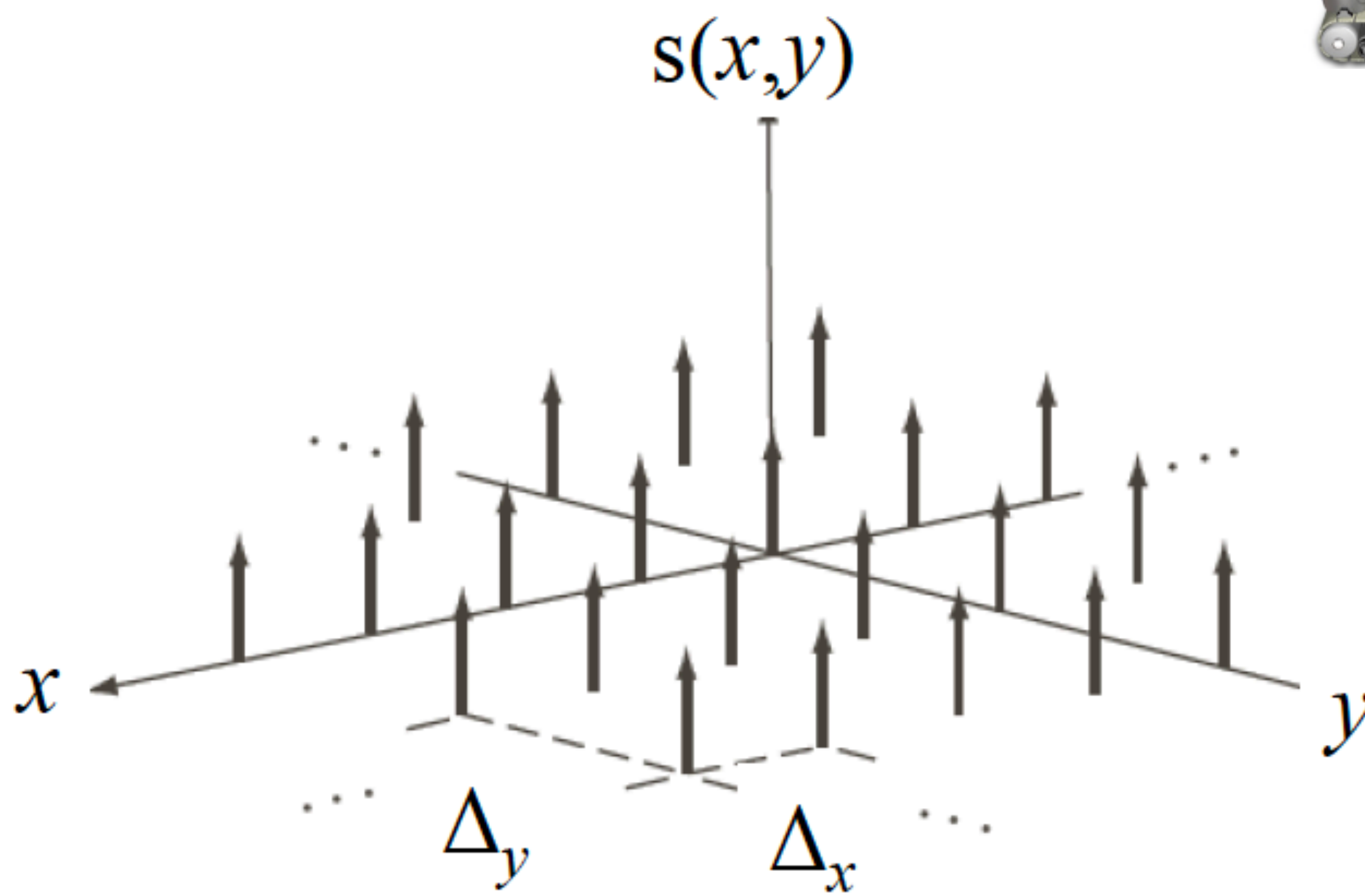
2D FOURIER TRANSFORM
FREQUENCY-DOMAIN REPRESENTATION OF THE OBJECT



See lectures 8 & 9, slides 15 - 21.



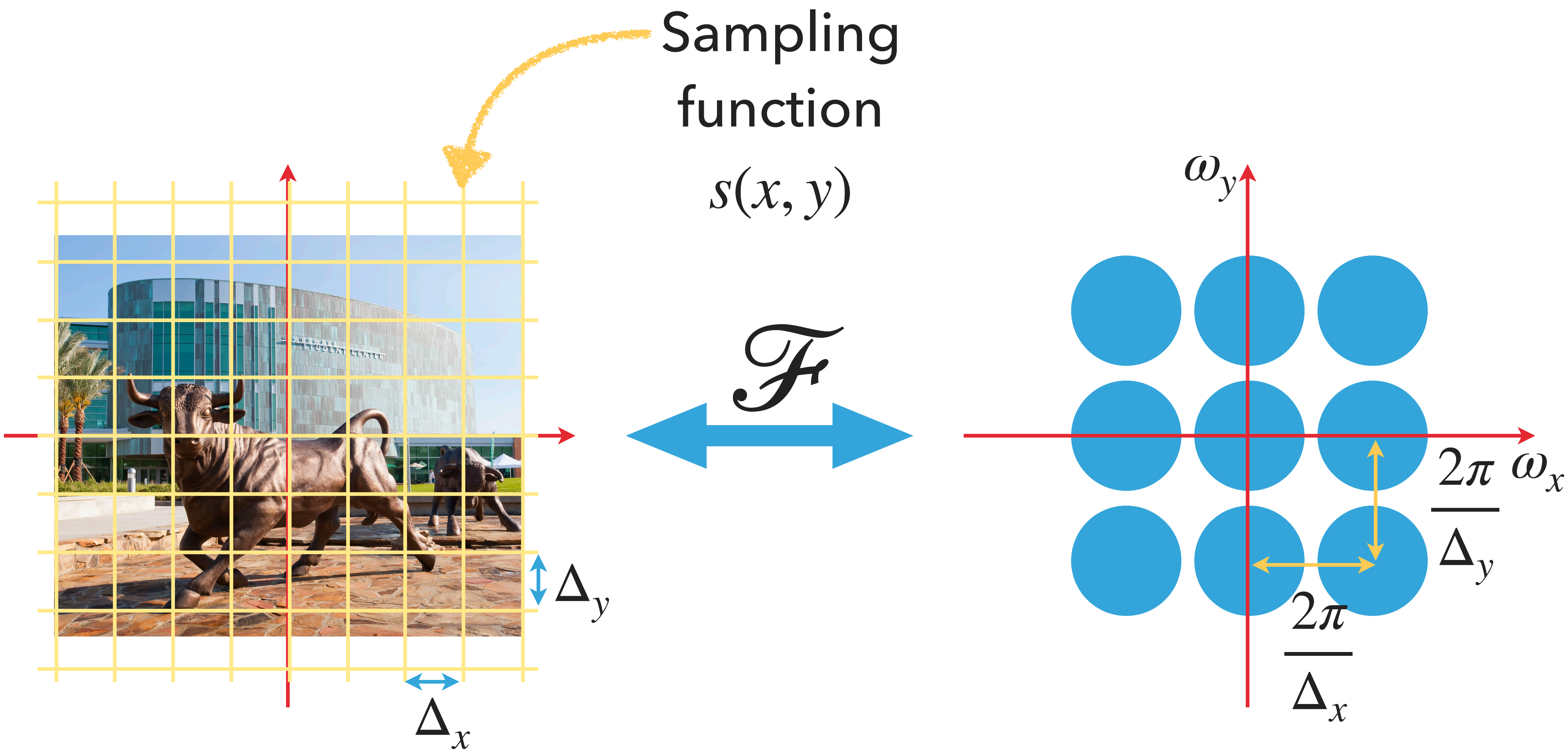
IDEALIZED IMPULSE SAMPLING
FREQUENCY-DOMAIN REPRESENTATION OF SAMPLING



Fourier transform of $s(x, y)$ is also a train of 2D Dirac delta functions, but with spacing $\left(m\frac{2\pi}{\Delta_x}, n\frac{2\pi}{\Delta_y}\right)$

See lectures 8 & 9, slides 15 - 21.

SAMPLING IN 2D



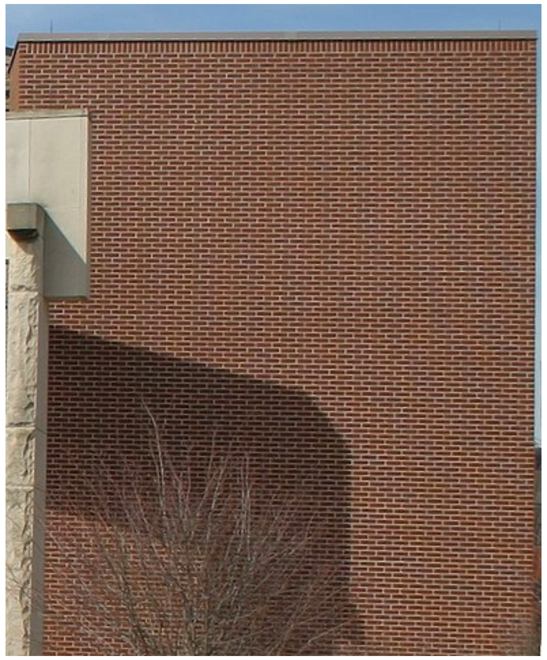
$$g_s(m\Delta_x, n\Delta_y) = g(x, y)s(x, y)$$

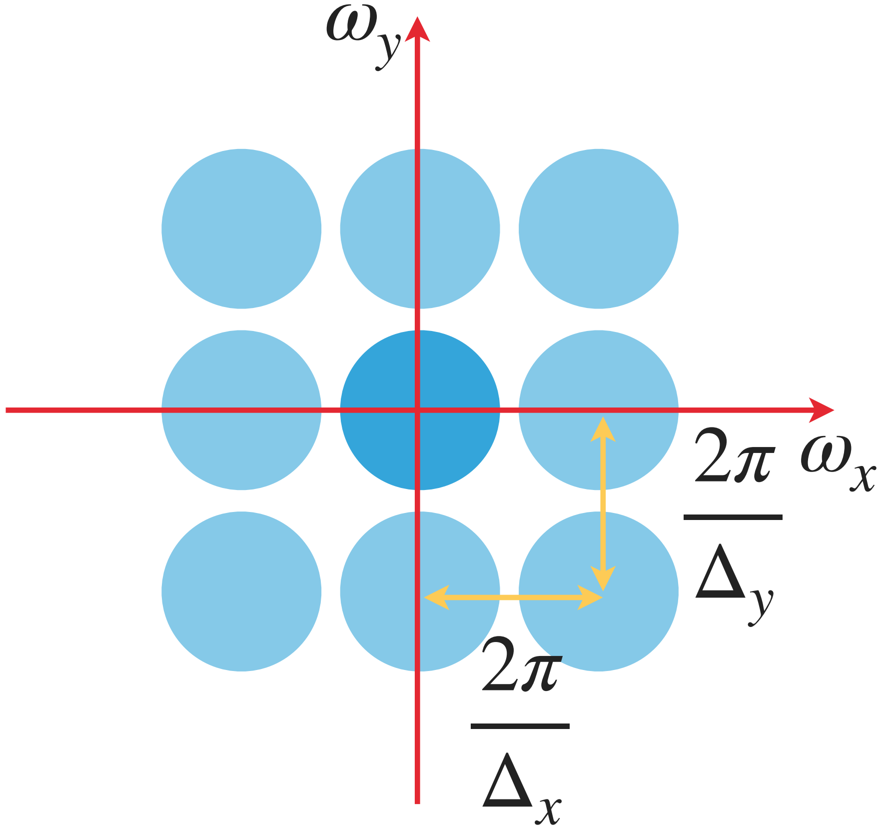
$$G_s(\omega_x, \omega_y) = \frac{1}{2\pi} G(\omega_x, \omega_y) \star S(\omega_x, \omega_y)$$

See lectures 8 & 9, slides 15 - 21.

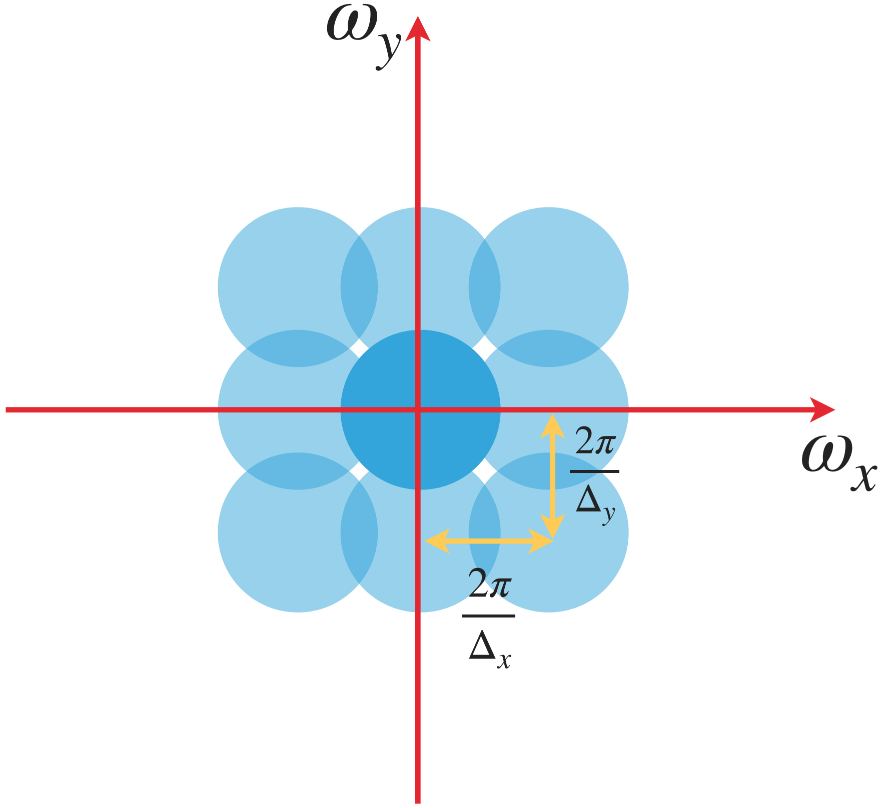
ALIASING
EXAMPLES (2D)

Properly sampled image





Poorly sampled image, leads to aliasing

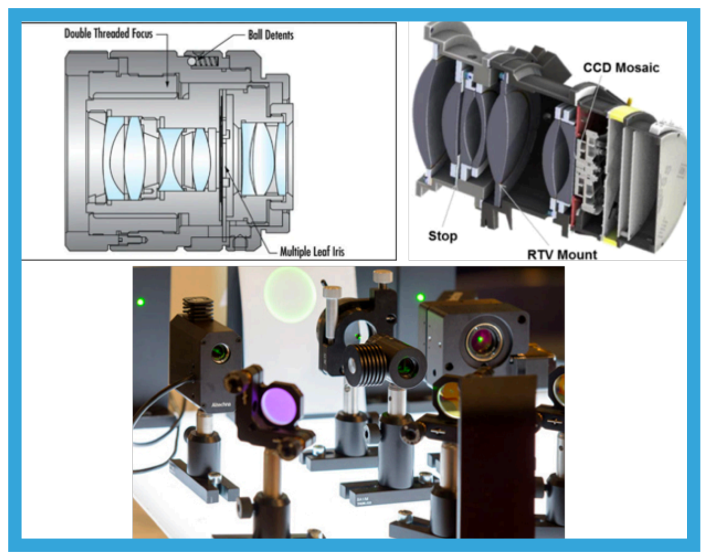


SAMPLING

HOW DO IMAGING SENSORS TYPICALLY WORK?

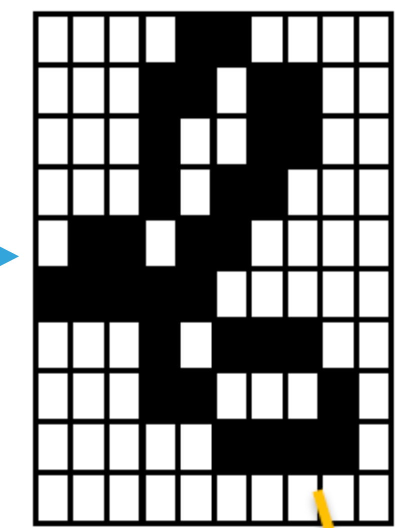
ANALOG PART

$g(x, y)$



Optics $h(x, y)$

$g(x, y) \star h(x, y)$



Sampling $\{g_{mn}\}$

Where $g(x, y) \star h(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x', y')h(x - x', y - y')\, \mathrm{d}y' \mathrm{d}x'$

$$g_{mn} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\frac{\Delta_x}{2}}^{\frac{\Delta_x}{2}} \int_{-\frac{\Delta_y}{2}}^{\frac{\Delta_y}{2}} g(x, y)h(x' - x, y' - y)\boxed{p(x' - m\Delta_x, y' - n\Delta_y)}\mathrm{d}y'\mathrm{d}x'\mathrm{d}y\mathrm{d}x$$

$p(x, y)$ is the pixel sampling function

How can we describe square pixels? **2D RECTANGULAR/BOX-CAR FUNCTIONS**

EFFECT OF PIXEL SAMPLING

PIXEL TRANSFER FUNCTION

- The sample at each pixel is given by

$$g_{mn} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\Delta_y/2}^{\Delta_y/2}\int_{-\Delta_x/2}^{\Delta_x/2} g(x',y')h'(x-x',y-y')\,\underbrace{\boxed{p(x-m\Delta_x, y-n\Delta_y)}}_{\text{PIXEL SAMPLING FUNCTION}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}x'\mathrm{d}y'$$

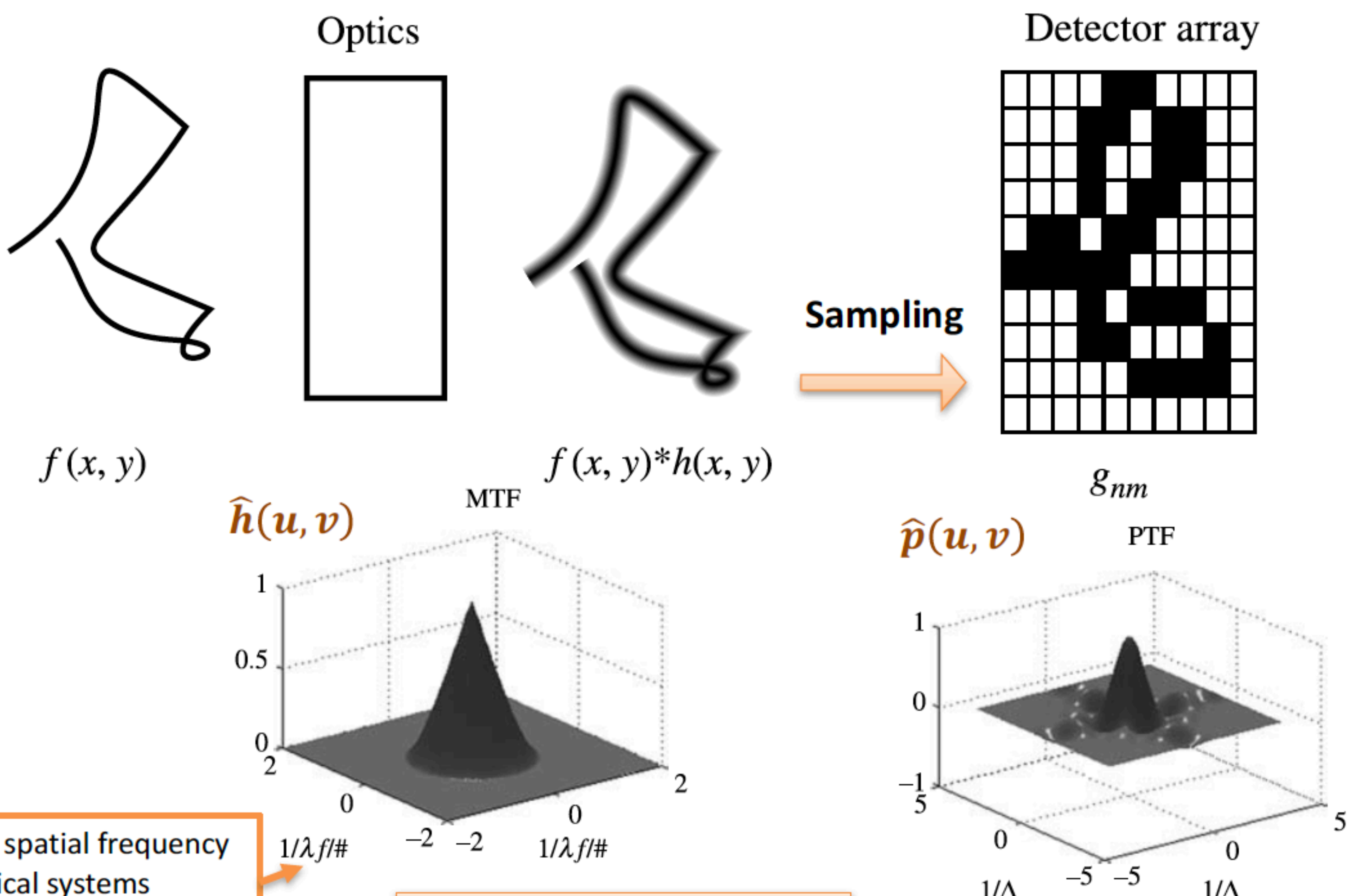
- First, take the **Fourier transform** $\mathscr{F}$ of the continuous version:

$$G_{\text{out}}(\omega_x,\omega_y) = G(\omega_x,\omega_y)H(\omega_x,\omega_y)\,\underbrace{\boxed{P(\omega_x,\omega_y)}}_{\text{PIXEL TRANSFER FUNCTION}}$$

- Take inverse **Fourier transform** $\mathscr{F}^{-1}$ and evaluate at $(m\Delta_x, n\Delta_y)$

$$g_{mn} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\omega_x,\omega_y)H(\omega_x,\omega_y)P(\omega_x,\omega_y)e^{\mathrm{j}(m\Delta_x\omega_x+n\Delta_y\omega_y)}\,\mathrm{d}\omega_x\mathrm{d}\omega_y$$

PIXEL TRANSFER FUNCTION
EXAMPLE



Normalized spatial frequency unit for optical systems
System Bandwidth = $1/\lambda f/\#$

Set the bandwidth of the signal before sampling

D. Brady, Optical Imaging and Spectroscopy, Chap 7.2.

SAMPLING
GENERAL DESCRIPTION

- Taking discrete samples from a continuous function $g(x)$ to produce a vector

$$\mathbf{g} = \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_{M-1} \end{bmatrix}$$

- Let the m-th pixel have the sampling function $p_m(x)$
- Then $g_m = \int g(x)p_m(x)\mathrm{d}x = \langle g(x), p_m(x) \rangle$
- Thus, $\mathbf{g} = \left[\langle g(x), p_0(x)\rangle, \langle g(x), p_1(x)\rangle, \ \cdots, \langle g(x), p_{M-1}(x)\rangle\right]^T$