HW 2 SOLUTIONS

(1) If $g(x) \longleftrightarrow G(\omega)$, show that $g^*(x) \longleftrightarrow G^*(-\omega)$.

SOLUTION: One could consider the FT of $g^*(x)$ or the inverse FT of $G^*(\omega)$. Let's do the latter:
- From the definition of the FT,

$$\cancel{G(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)\, e^{j\omega x}\, dx}$$

$$G(\omega) = \int_{-\infty}^{\infty} g(x)\, e^{-j\omega x}\, dx$$

Set $\omega \to -\omega$ $\Rightarrow$ $G(-\omega) = \int_{-\infty}^{\infty} g(x)\, e^{-j(-\omega)x}\, dx$

$$= \int_{-\infty}^{\infty} g(x)\, e^{+j\omega x}\, dx$$

Take conjugate: $G^*(-\omega) = \left(\int_{-\infty}^{\infty} g(x)\, e^{+j\omega x}\, dx\right)^*$

$$= \int_{-\infty}^{\infty} \left(g(x)\, e^{+j\omega x}\right)^* dx$$

$$= \int_{-\infty}^{\infty} g^*(x)\, e^{-j\omega x}\, dx$$

$$= \mathcal{F}\{g^*(x)\}$$

Thus, $G^*(-\omega) = \mathcal{F}\{g^*(x)\}$

$$\Rightarrow g^*(x) \longleftrightarrow G^*(-\omega)$$

(2) $u(x) = \begin{cases} 1, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

(a) $f(x) = e^{-ax} u(x)$, $F(\omega) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} e^{-ax} u(x) e^{-j\omega x} dx$

$$= \int_0^{\infty} e^{-ax} e^{-j\omega x} dx = \int_0^{\infty} e^{-(a+j\omega)x} dx$$

$$= -\frac{1}{a+j\omega} \left[e^{-(a+j\omega)x} \right]_0^{\infty} = -\frac{1}{a+j\omega}(0-1)$$

$$= \frac{1}{a+j\omega}$$

(b) $f(x) = \text{sinc}(x) = \frac{\sin(x)}{x}$,

$f(x) \longleftrightarrow F(\omega)$

$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-j\omega x} dx$, by the definition of FT.

- From the lectures (see Lecture 7, slide 31 for example), we know that: $\text{rect}(x) \longleftrightarrow \text{sinc}(\omega/2)$ ——— (I)

- We want the Fourier transform of sinc(x), however; to do this we should exploit equation (I) above.
  - First, let's find the IFT of sinc(ω) using (I) and the fact that if $g(x) \longleftrightarrow G(\omega)$ then

$$G(\tfrac{\omega}{a}) = \int_{-\infty}^{\infty} g(x) e^{-j\frac{\omega}{a}x} dx$$

let $x' = x/a$

$\Rightarrow dx' = 1/a \, dx$

$$G(\omega/a) = \int_{-\infty}^{\infty} g(ax') e^{-j\omega x'} a\,dx'$$

$$= a \int_{-\infty}^{\infty} g(ax') e^{-j\omega x'} dx' = a \mathcal{F}\{g(ax')\}$$

Thus, $g(ax') \longleftrightarrow \frac{1}{a} G(\frac{\omega}{a})$ ~~and letting a=1/2~~

$$\Rightarrow \text{rect}(ax) \longleftrightarrow \frac{1}{a} \text{sinc}(\frac{\omega}{2a})$$

By letting $a = 1/2$, $\text{rect}(x/2) \longleftrightarrow 2\,\text{sinc}(\omega)$

$$\Rightarrow \mathcal{F}^{-1}\{\text{sinc}(\omega)\} = 1/2\, \text{rect}(x/2) \quad \text{——— (II)}$$

- At this stage, we know that $\mathcal{F}^{-1}\{F(\omega)\} = f(x)$ but want instead $\mathcal{F}\{F(x)\}$;

By definition of FT, $\mathcal{F}\{F(x)\} = \int_{-\infty}^{\infty} F(x) e^{-j\omega x} dx$

$$= \int_{-\infty}^{\infty} F(x) e^{j(-\omega)x} dx$$

$$= 2\pi \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} F(x) e^{j(-\omega)x} dx}_{f(-\omega)}.$$

Thus, when $f(x) \leftrightarrow F(\omega)$

~~then~~, $F(x) \longleftrightarrow 2\pi f(-\omega)$

$$\Rightarrow \mathcal{F}\{\operatorname{sinc}(x)\} = 2\pi\left(\tfrac{1}{2}\operatorname{rect}\left(\frac{-\omega}{2}\right)\right)$$

$$= \pi \operatorname{rect}(\omega/2).$$

(c) $f(x) = \operatorname{rect}(x)$

$$F(\omega) = \int_{-\infty}^{\infty} \operatorname{rect}(x) e^{-j\omega x} dx$$

$$= \int_{-1/2}^{1/2} e^{-j\omega x} dx = -\frac{1}{j\omega}\left[e^{-j\omega x}\right]_{-1/2}^{+1/2}$$

$$= 1/j\omega\left(e^{j\omega/2} - e^{-j\omega/2}\right) = \frac{2\sin(\omega/2)}{\omega}$$

$$= \frac{\sin(\omega/2)}{\omega/2} = \operatorname{sinc}(\omega/2).$$

(d) $f(x) = \Lambda(x)$; Notice that $f(x) = \operatorname{rect}(x) * \operatorname{rect}(x)$

- Recall from lecture 11, that $g(x) * h(x) \leftrightarrow G(\omega)H(\omega)$
- Thus, $\mathcal{F}\{\Lambda(x)\} = \mathcal{F}\{\operatorname{rect}(x)\} \cdot \mathcal{F}\{\operatorname{rect}(x)\}$

$$= \operatorname{sinc}^2(\omega/2).$$

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③ IFT :

ⓐ $F(\omega) = rect(\omega)$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} rect(\omega)\, e^{j\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1} e^{j\omega x}\, dx = \frac{1}{2\pi}\left[\frac{1}{jx} e^{j\omega x}\right]_{-1}^{1}$$

$$= 1/2\pi \left(\frac{e^{jx} - e^{-jx}}{jx}\right) = \frac{1}{2\pi}\cdot\frac{2\sin(x)}{x}$$

$$= \frac{1}{\pi}\,\frac{\sin(x)}{x} = \frac{1}{\pi}\, sinc(x)$$

ⓑ $F(\omega) = e^{-a|\omega|}$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-a|\omega|}\, e^{-j\omega x}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{0} e^{a\omega}\, e^{-j\omega x}\, d\omega$$

$$+ \frac{1}{2\pi}\int_{0}^{\infty} e^{-a\omega}\, e^{-j\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} e^{(a-jx)\omega}\, d\omega + \frac{1}{2\pi}\int_{0}^{\infty} e^{-(a+jx)\omega}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{1}{a-jx} e^{(a-jx)\omega}\right]_{-\infty}^{0} + \frac{1}{2\pi}\left[-\frac{1}{a+jx} e^{-(a+jx)\omega}\right]_{0}^{\infty}$$

$$= \frac{1}{2\pi}\cdot\frac{1}{a-jx}(1-0) + \frac{1}{2\pi}\cdot\frac{-1}{a+jx}(0-1)$$

$$= \frac{1}{2\pi}\left[\frac{1}{a-jx} + \frac{1}{a+jx}\right] = \frac{1}{2\pi}\cdot\frac{a+\cancel{jx}+a-\cancel{jx}}{a^2+x^2}$$

$$= \frac{1}{\pi}\left(\frac{a}{a^2+x^2}\right)$$

4 2D FT :

ⓐ $f(x,y) = \text{rect}(ax)\,\text{rect}(by)$

$$F(\omega_x, \omega_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{rect}(ax)\,\text{rect}(by)\, e^{-j(\omega_x x + \omega_y y)}\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \text{rect}(ax)\, e^{-j\omega_x x}\, dx \int_{-\infty}^{\infty} \text{rect}(by)\, e^{-j\omega_y y}\, dy$$

$$= \int_{-1/2a}^{1/2a} e^{-j\omega_x x}\, dx \int_{-1/2b}^{1/2b} e^{-j\omega_y y}\, dy$$

$$= -\frac{1}{j\omega_x}\left[e^{-j\omega_x x}\right]_{-1/2a}^{\frac{1}{2a}} \cdot -\frac{1}{j\omega_y}\left[e^{-j\omega_y y}\right]_{-\frac{1}{2b}}^{\frac{1}{2b}}$$

$$= -\frac{1}{j\omega_x}\left(e^{-j\frac{\omega_x}{2a}} - e^{+j\frac{\omega_x}{2a}}\right) \cdot -\frac{1}{j\omega_y}\left(e^{-j\frac{\omega_y}{2b}} - e^{+j\frac{\omega_y}{2b}}\right)$$

$$= \frac{1}{j\omega_x}\left(e^{j\frac{\omega_x}{2a}} - e^{-j\frac{\omega_x}{2a}}\right)\left(\frac{1}{j\omega_y}\right)\left(e^{j\frac{\omega_y}{2b}} - e^{-j\frac{\omega_y}{2b}}\right)$$

$$= \frac{2\sin\left(\frac{\omega_x}{2a}\right)}{\omega_x} \cdot \frac{2\sin\left(\frac{\omega_y}{2b}\right)}{\omega_y}$$

$$= \frac{1}{a}\frac{\sin\left(\frac{\omega_x}{2a}\right)}{\omega_x/2a} \cdot \frac{1}{b}\frac{\sin\left(\frac{\omega_y}{2b}\right)}{\omega_y/2b}$$

$$= \frac{1}{ab}\,\text{sinc}\left(\frac{\omega_x}{2a}\right)\text{sinc}\left(\frac{\omega_y}{2b}\right)$$

(6) $f(x,y) = \Lambda(x)\,\Lambda(y)$

$$F(\omega_x, \omega_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j(\omega_x x + \omega_y y)}\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \Lambda(x) e^{-j\omega_x x}\, dx \int_{-\infty}^{\infty} \Lambda(y) e^{-j\omega_y y}\, dy$$

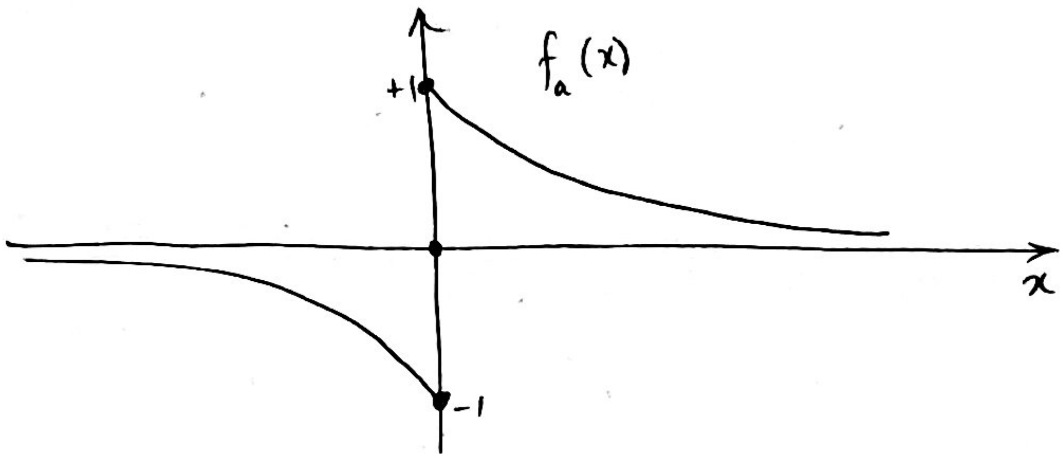
Using result of Q2(d), $\int_{-\infty}^{\infty} \Lambda(x) e^{-j\omega_x x}\, dx = \text{Sinc}^2(\omega_x/2)$

similarly, $\int_{-\infty}^{\infty} \Lambda(x) e^{-j\omega_y y}\, dy = \text{Sinc}^2\left(\frac{\omega_y}{2}\right)$

$$\therefore\ F(\omega_x, \omega_y) = \text{Sinc}^2\left(\frac{\omega_x}{2}\right) \text{Sinc}^2\left(\frac{\omega_y}{2}\right).$$

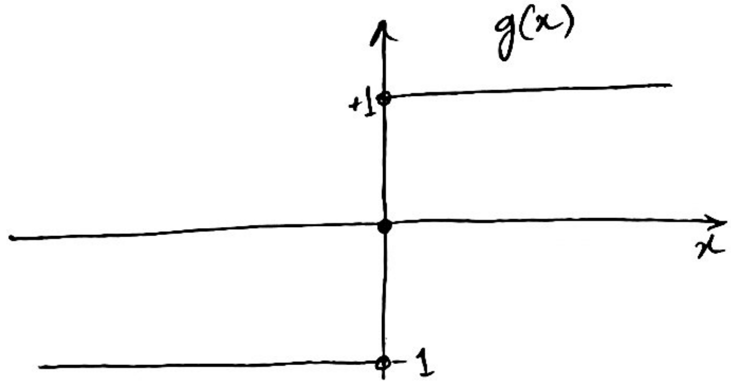
(5)

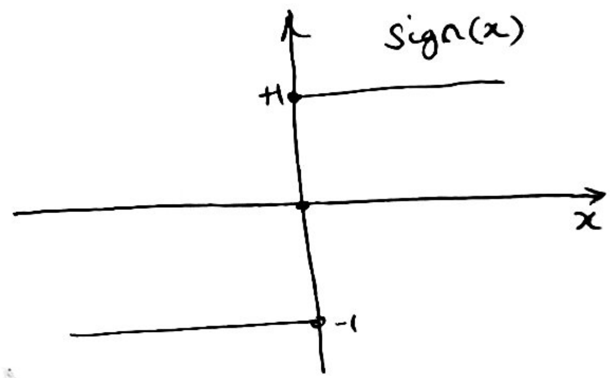
(a)



(b) Let $g(x) = \lim_{a\to\infty} f_a(x)$

Signum function





obvious from plot that $g(x) = \text{sign}(x)$.

(c) $$G(\omega) = \mathcal{F}\{g(x)\} = \int_{-\infty}^{\infty} g(x)\, e^{-j\omega x}\, dx$$

$$= \int_{-\infty}^{\infty} \lim_{a\to\infty} f_a(x)\, e^{-j\omega x}\, dx$$

$$= \lim_{a\to\infty} \int_{-\infty}^{\infty} f_a(x)\, e^{-j\omega x}\, dx = \lim_{a\to\infty} F_a(\omega)$$

But, $F_a(\omega) = \int_{-\infty}^{0} -e^{ax}\, e^{-j\omega x}\, dx + \int_{0}^{\infty} e^{-ax}\, e^{-j\omega x}\, dx$

$$= -\int_{-\infty}^{0} e^{(a-j\omega)x}\, dx + \int_{0}^{\infty} e^{-(a+j\omega)x}\, dx$$

$$= -\frac{1}{a-j\omega}\left[e^{(a-j\omega)x}\right]_{-\infty}^{0} + \left(-\frac{1}{a+j\omega}\right)\left[e^{-(a+j\omega)x}\right]_{0}^{\infty}$$

$$= -\frac{1}{a-j\omega}(1-0) - \frac{1}{a+j\omega}(0-1)$$

$$= \frac{1}{a+j\omega} - \frac{1}{a-j\omega}$$

$$= \frac{a-j\omega - a - j\omega}{a^2+\omega^2} = -\frac{2j\omega}{a^2+\omega^2}$$

Thus, $G(\omega) = \mathcal{F}\{g(x)\} = \lim_{a\to\infty} \frac{-2j\omega}{a^2+\omega^2}$

$$= -\frac{2j\omega}{\omega^2} = -\frac{2j}{\omega}$$

$\therefore\ \mathcal{F}\{\mathrm{sign}(x)\} = +\underline{\underline{\frac{2}{j\omega}}}$.