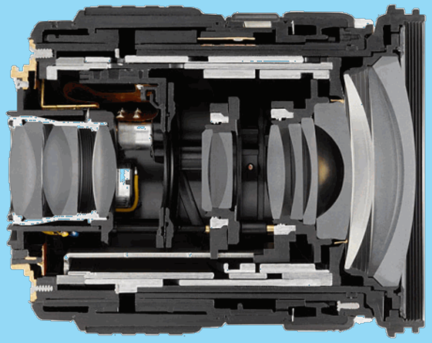

Optics

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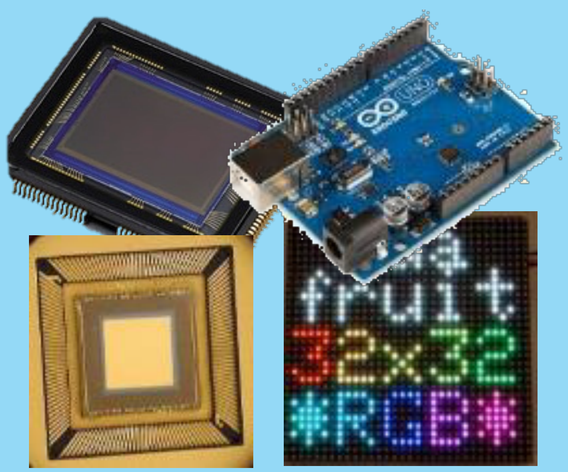


Sensors & devices

+

Signal processing & algorithms

COMPUTATIONAL METHODS FOR IMAGING (AND VISION)

LECTURE 7: BASES

PROF. JOHN MURRAY-BRUCE

WHERE ARE WE

WE ARE HERE!

Week	Date	Main Topic	Lecture	Readings	Homework Out	Homework Due
1	11-Jan-21		Introduction to computational imaging - Forward and Inverse problems - Common computational imaging problems			
	13-Jan-21		Vectors - Preliminaries			
2	18-Jan-21		Dr. Martin Luther King, Jr. Holiday (no class)			
	20-Jan-21	Mathematical preliminaries	Vectors and Vector Spaces - Subspaces, Finite dimensional spaces	IIP Appendix A; FSP 2.1 - 2.2		
3	25-Jan-21		Vector Spaces - Hilbert spaces	IIP Appendix B; FSP 2.3		
	27-Jan-21		Bases and Frames I - Orthonormal and Reisz Bases	IIP Appendix C; FSP 2.4 and 2.B	**HW 1**	
4	1-Feb-21		Bases and Frames II - Orthogonal Bases - Linear operators	IIP Appendix C; FSP 2.5 and 2.B		
	3-Feb-21		Fourier Analysis I - FT (1D and 2D) - FT properties	IIP 2.1, Appendix D; FSP 4.4		
5	8-Feb-21		Sampling and Interpolation - BL functions - Sampling	IIP 2.2, 2.3; FSP 5.4, 5.5		**HW 1**
	10-Feb-21		Fourier Analysis II (DFT)	IIP 2.4; FSP 3.6	**HW 2**	
6	15-Feb-21	Forward Modeling	LSI imaging: Forward problem I - Convolution	IIP 2.5 - 2.6, 3		
	17-Feb-21		LSI imaging: Forward problem I - Transfer functions	IIP 2.6		
7	22-Feb-21		LSI imaging: Forward problem I - Linear operators	IIP 3		
	24-Feb-21		LSI imaging: Forward problem I - Linear operators, Adjoints, and Inverses		**HW 3**	**HW 2**
8	1-Mar-21		**Mid-term Exams**			
	3-Mar-21		LSI imaging: Forward problem II - Sampling and Discretization: Matrix-vector form	IIP 2.7, 4		
9	8-Mar-21		LSI imaging: Forward problem II - Convolution matrix			
	10-Mar-21		LSI imaging: Forward problem II - Sampling and Discretization: Matrix-vector form - PSF, and Transfer functions			**HW3**

OUTLINE

- Bases (continued)
  - Biorthogonal bases
  - Matrix representation of linear operators

LEARNING GOALS

- Understand bases and related terminologies
- Linear operators and their matrix representations

READING

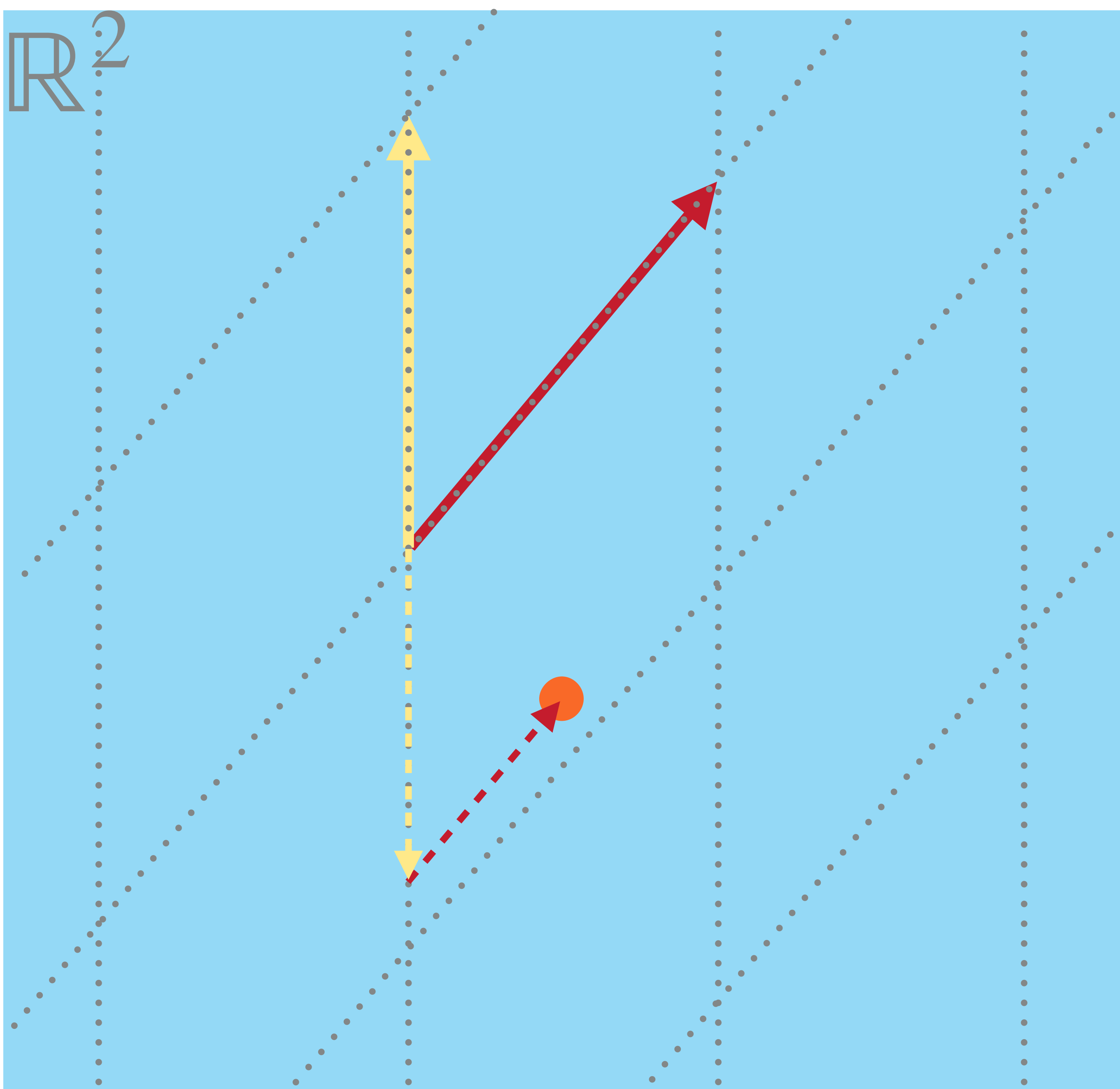
- IIP Appendix C
- FSP 2.5

BIORTHOGONAL BASES

BASIS

https://centonomy.com/you-cant-skip-the-basics/

BIORTHOGONAL BASES



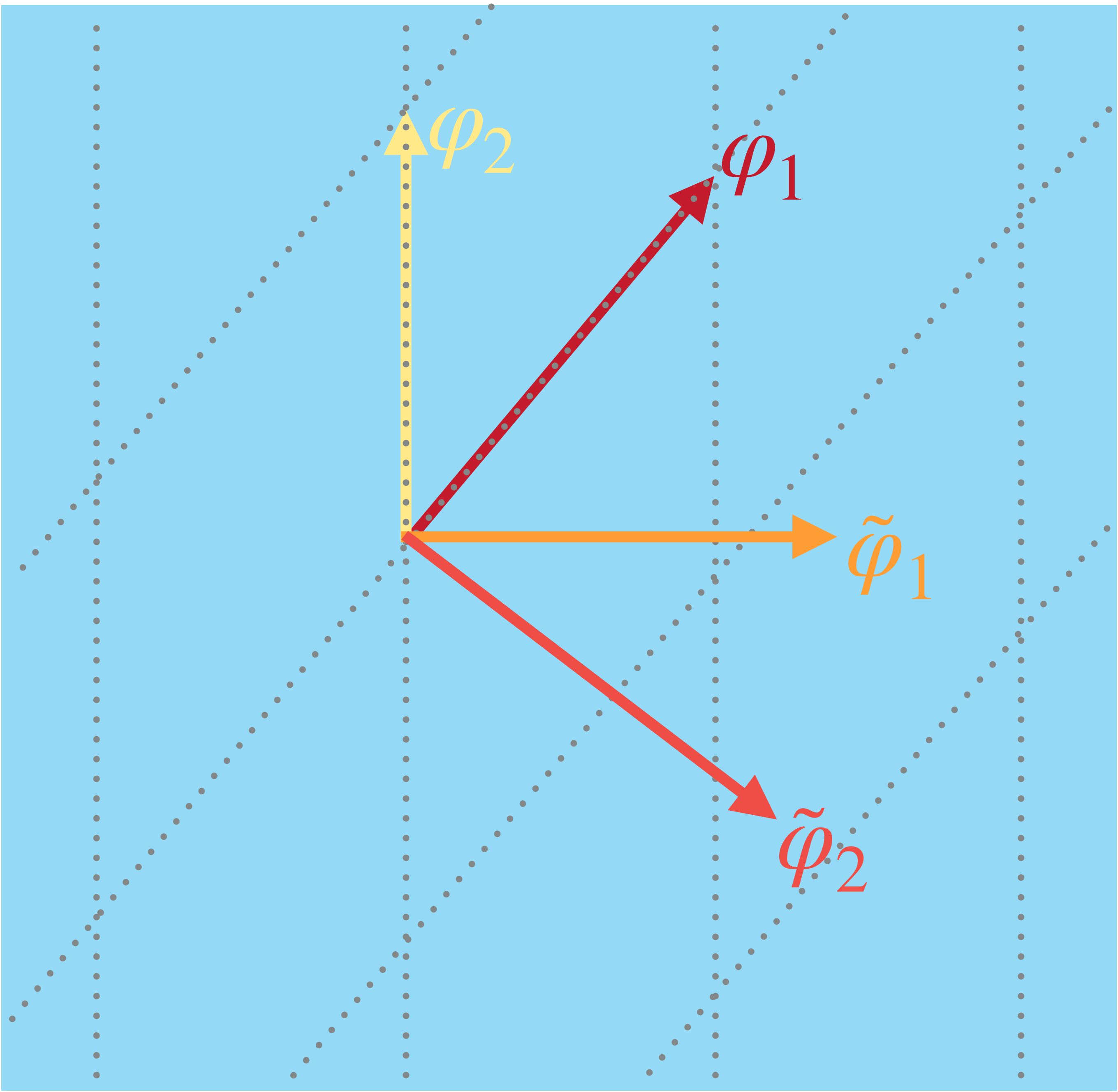
$+$

$\times$

$\langle u, v \rangle$

BIORTHOGONAL BASES

Basis $\{\varphi_1, \varphi_2\}$ and its **dual** $\{\tilde{\varphi}_1, \tilde{\varphi}_2\}$.



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$\times$

$\langle u, v \rangle$

BIORTHOGONAL BASES

- **Definition (Biorthogonal Bases):** $\Phi = \{\varphi_k\}_{k\in\mathscr{K}} \subset H$ and $\widetilde{\Phi} = \{\tilde{\varphi}_k\}_{k\in\mathscr{K}} \subset H$ is a biorthogonal pair of bases when
  - Φ and $\widetilde{\Phi}$ are both **bases for** H
  - Φ and $\widetilde{\Phi}$ are **biorthogonal**, i.e.:
  $$\langle \varphi_j, \tilde{\varphi}_k \rangle = \delta_{j-k} \text{ for all } j, k \in \mathscr{K}$$
- Notice that the roles of Φ and $\tilde{\Phi}$ are interchangeable

- $\langle \varphi_1, \tilde{\varphi}_1 \rangle = \int \varphi_1 \tilde{\varphi}_1 dt = 1$
- $\langle \varphi_1, \tilde{\varphi}_2 \rangle = 0$

BIORTHOGONAL BASES

EXAMPLE

$$\varphi_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \varphi_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \varphi_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \Rightarrow \Phi = [\varphi_1 \quad \varphi_2 \quad \varphi_3] = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\tilde{\varphi}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \tilde{\varphi}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \tilde{\varphi}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad \Rightarrow \widetilde{\Phi} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

- Verify that

  1. Φ and $\widetilde{\Phi}$ are both bases for H (what is H?)
  2. Φ and $\widetilde{\Phi}$ are **biorthogonal**, i.e. $\langle \varphi_j, \tilde{\varphi}_k \rangle = \delta_{j-k}$ for all $j, k \in \mathscr{K}$

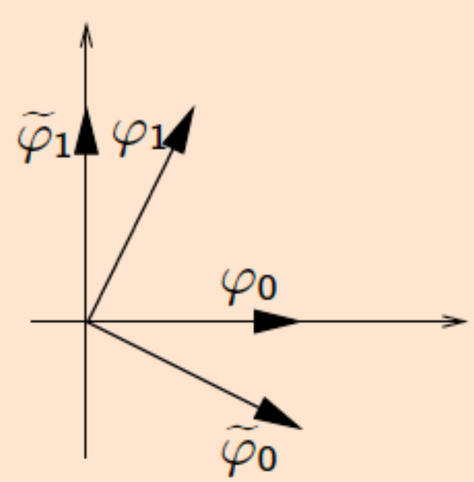
PLEASE DO TRY IT!!!

BIORTHOGONAL BASES

EXPANSION

Theorem

- $\Phi = \{\varphi_k\}_{k \in \mathcal{K}}, \widetilde{\Phi} = \{\widetilde{\varphi}_k\}_{k \in \mathcal{K}}$ *biorthogonal pair of bases for* H



BIORTHOGONAL BASES

EXPANSION

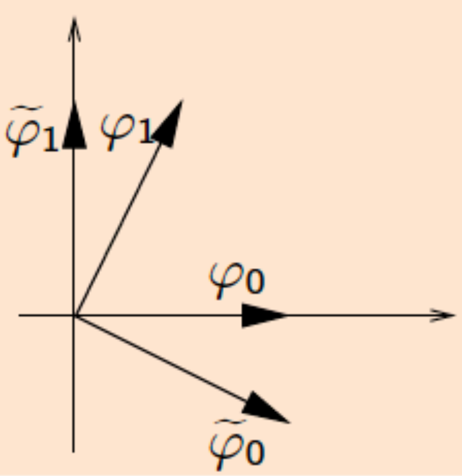
Theorem

- $\Phi = \{\varphi_k\}_{k \in \mathcal{K}}, \widetilde{\Phi} = \{\widetilde{\varphi}_k\}_{k \in \mathcal{K}}$ *biorthogonal pair of bases for* H
- *Any* $x \in H$ *has expansion coefficients*

$$\alpha_k = \langle x, \widetilde{\varphi}_k \rangle, \ \ k \in \mathcal{K}, \ \ \text{or } \alpha = \widetilde{\Phi}^* x$$

- *Synthesis:*
$$\begin{aligned} x &= \sum_{k \in \mathcal{K}} \langle x, \widetilde{\varphi}_k \rangle \varphi_k \\ &= \Phi\alpha = \Phi\widetilde{\Phi}^* x \end{aligned}$$

- *Also* $x = \sum_{k \in \mathcal{K}} \langle x, \varphi_k \rangle \widetilde{\varphi}_k$



DUAL BASIS

HOW DO WE COMPUTE THE DUAL Φ^*

Theorem (Dual basis)

DUAL BASIS

HOW DO WE COMPUTE THE DUAL $\widetilde{\Phi}$

Theorem (Dual basis)

- *Given a Riesz basis* $\Phi = \{\varphi_k\}_{k\in\mathcal{K}}$ *for Hilbert space* H*, the set* $\widetilde{\Phi} = \{\widetilde{\varphi}_k\}_{k\in\mathcal{K}}$ *defined via*

$$\widetilde{\varphi}_k = \sum_{\ell\in\mathcal{K}} a_{\ell,k}\varphi_\ell, \qquad \text{for each } k \in \mathcal{K},$$

$$\boxed{\widetilde{\Phi} = \Phi A = \Phi(\Phi^*\Phi)^{-1},}$$

is a basis for H*, called the dual basis, and the sets* Φ *and* $\widetilde{\Phi}$ *are a biorthogonal pair of bases*

GRAM MATRIX

- $G = \Phi^*\Phi$ is called the **Gram matrix**
- $G_{i,k}$ entry i, k of the gram matrix, and is $G = \langle \varphi_k, \varphi_i \rangle$, for every $i, k \in \mathscr{K}$

$$G = \begin{bmatrix} \langle \varphi_1, \varphi_1 \rangle & \langle \varphi_2, \varphi_1 \rangle & \langle \varphi_3, \varphi_1 \rangle & \cdots & \langle \varphi_K, \varphi_1 \rangle \\ \langle \varphi_1, \varphi_2 \rangle & \langle \varphi_2, \varphi_2 \rangle & \langle \varphi_3, \varphi_2 \rangle & \cdots & \langle \varphi_K, \varphi_2 \rangle \\ \langle \varphi_1, \varphi_3 \rangle & \langle \varphi_2, \varphi_3 \rangle & \langle \varphi_3, \varphi_3 \rangle & \cdots & \langle \varphi_K, \varphi_3 \rangle \\ \vdots & \vdots & \cdots & \ddots & \cdots \\ \langle \varphi_1, \varphi_K \rangle & \langle \varphi_2, \varphi_K \rangle & \langle \varphi_3, \varphi_K \rangle & \cdots & \langle \varphi_K, \varphi_K \rangle \end{bmatrix}$$

- **Inner products:** Consider $x = \Phi\boldsymbol{\alpha}$ and $y = \Phi\boldsymbol{\beta}$, then:

$$\langle x, y \rangle = \langle \Phi\boldsymbol{\alpha}, \Phi\boldsymbol{\beta} \rangle = \langle \Phi^*\Phi\boldsymbol{\alpha}, \boldsymbol{\beta} \rangle = \langle G\boldsymbol{\alpha}, \boldsymbol{\beta} \rangle$$

We have a **non-standard inner product** unless $G = I$, which only happens if the basis is orthonormal

BIORTHOGONAL BASES
EXAMPLE: COMPUTING THE DUAL

SOLUTIONS

1. Write the **matrix representation** (i.e., the synthesis operator) associated with the set:

$$\varphi_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \varphi_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad \varphi_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

2. Is the set an **orthonormal basis**?

3. Find the **dual** $\widetilde{\Phi}$

$$\Phi = [\varphi_1 \quad \varphi_2 \quad \varphi_3] = \begin{bmatrix}1 & 0 & 1\\1 & 1 & 1\\0 & 1 & 1\end{bmatrix}$$

Check all pairwise inner products

$$\langle \varphi_1, \varphi_2 \rangle = (1 \times 0) + (1 \times 1) + (0 \times 1) = 1$$
$$\langle \varphi_1, \varphi_3 \rangle = (1 \times 1) + (1 \times 1) + (0 \times 1) = 2$$
$$\langle \varphi_2, \varphi_3 \rangle = (0 \times 1) + (1 \times 1) + (1 \times 1) = 2$$

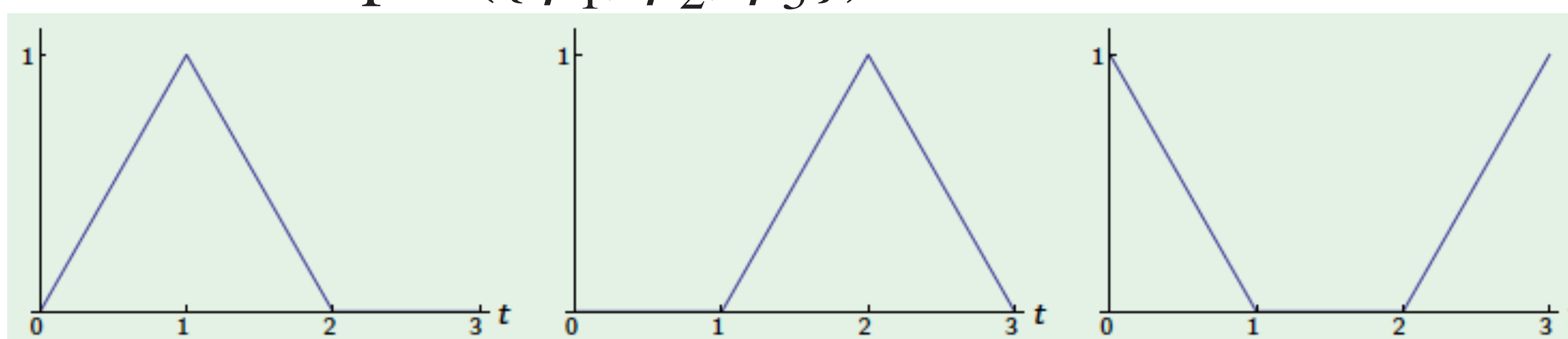
$$\widetilde{\Phi} = \Phi(\Phi^* \Phi)^{-1} = \begin{bmatrix}0 & -1 & 1\\1 & 1 & -1\\-1 & 0 & 1\end{bmatrix}$$

Also, note that: $\Phi \widetilde{\Phi}^* = \mathbf{I} \quad \Rightarrow \widetilde{\Phi}^* = \Phi^{-1}$

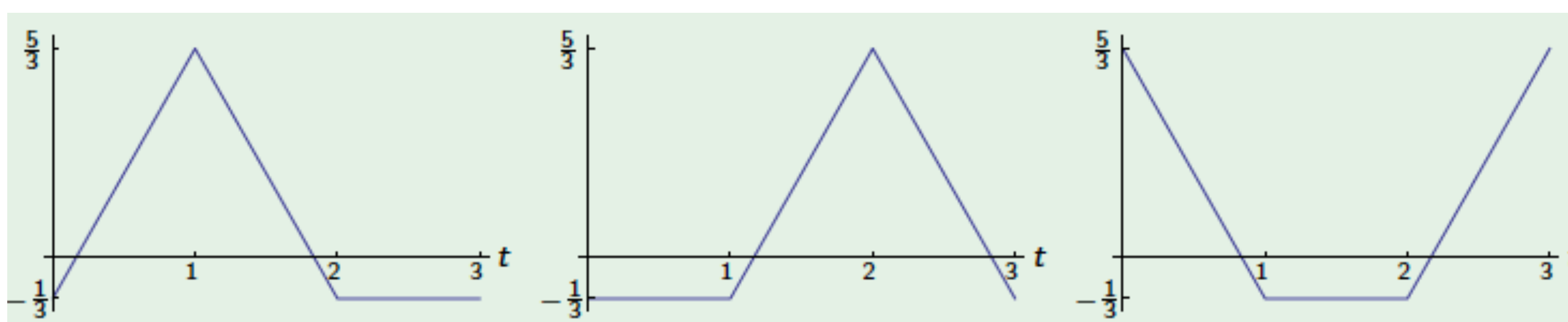
DUAL BASIS

EXAMPLE: HOW DO WE COMPUTE THE DUAL $\widetilde{\Phi}$

- Consider the function $\varphi_1 = \begin{cases} t & t \in [0,1); \\ 2-t, & t \in (1,2]; \\ 0, & t \in (2,3] \end{cases}$ in $\mathscr{L}^2([0,3])$ and its circular shifts by 1 and 2.
- $\Phi = \{\varphi_1, \varphi_2, \varphi_3\}$ is basis for $\mathrm{span}(\{\varphi_1, \varphi_2, \varphi_3\})$



$\varphi_1(t)$ $\varphi_2(t)$ $\varphi_3(t)$

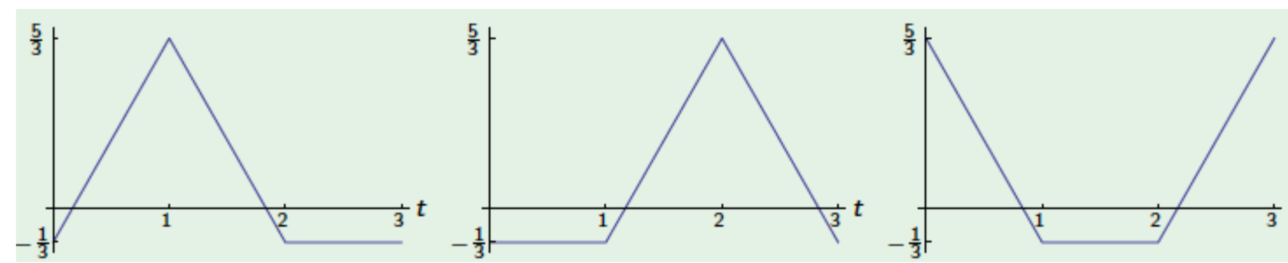


$\tilde{\varphi}_1(t)$ $\tilde{\varphi}_2(t)$ $\tilde{\varphi}_3(t)$

DUAL BASIS

EXAMPLE: HOW DO WE COMPUTE THE DUAL $\widetilde{\Phi}$

$\varphi_1(t)$ $\varphi_2(t)$ $\varphi_3(t)$



$\tilde{\varphi}_1(t)$ $\tilde{\varphi}_2(t)$ $\tilde{\varphi}_3(t)$

- $\operatorname{span}(\{\varphi_1, \varphi_2, \varphi_3\})$ is the subspace of functions $f(t)$ that are piecewise linear on $[0, 3]$ with breakpoints at 1 and 2, and satisfy $f(0) = f(3)$.

$$G = \begin{bmatrix} 2/3 & 1/6 & 1/6 \\ 1/6 & 2/3 & 1/6 \\ 1/6 & 1/6 & 2/3 \end{bmatrix}$$

CAN YOU DERIVE G?

- We find the dual $\widetilde{\Phi}$ by using $\widetilde{\Phi} = \Phi G^{-1}$, where

$$G^{-1} = \begin{bmatrix} 5/3 & -1/3 & -1/3 \\ -1/3 & 5/3 & -1/3 \\ -1/3 & -1/3 & 5/3 \end{bmatrix}$$

APPROXIMATIONS

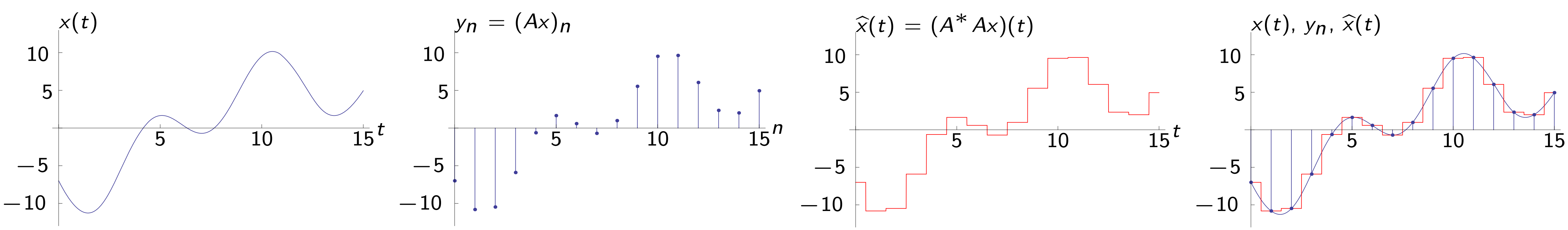
BASIS

https://centonomy.com/you-cant-skip-the-basics/

APPROXIMATION

ORTHOGONAL PROJECTION ON $\mathscr{L}^2(\mathbb{R})$

- Local averaging on $\mathscr{L}^2(\mathbb{R})$ $\quad A : \mathcal{L}^2(\mathbb{R}) \to \ell^2(\mathbb{Z}) \quad (Ax)_n = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} x(t)\, dt$



adjoint $A^* : \ell^2(\mathbb{Z}) \to \mathcal{L}^2(\mathbb{R})$ that produces staircase function

- The local averaging operator A, and its adjoint A^* are such that $AA^* = I$, so that $P = A^*A$ is an orthogonal projection

HANDLING MATRICES

MATLAB PRACTICE 4

MATRICES IN MATLAB

- Matrices
  - Matrix transpose (adjoint)
  - Matrix inverse
    - Left inverses
    - Right inverses
- Eigenvalue decomposition
  - Eigenvalues & Eigenvectors
- *Apply these concepts to obtain plot in Slide 15*
  - Homework question 2
- Plotting 2D functions (surface plots)

WHAT WE COVERED

- **Bases** (definition)
  - Reisz basis (stability)
  - Orthogonal & Orthonormal bases
  - Biorthogonal bases (Basis & Dual pair)
- **Analysis** and **Synthesis** with bases
  - Computing basis expansion/analysis coefficients of a vector
  - Synthesizing (making) the vector given basis and expansion/analysis coefficients
- The **Gram matrix**

SEE YOU NEXT TIME!

INTRODUCTION TO FOURIER ANALYSIS

TECHNICAL ASIDE
CLOSED SUBSPACE

Definition (Closed subspace)

A subspace S of a normed vector space V is called *closed* when it contains all limits of sequences of vectors in S

Problem and resolution:

- Subspaces can fail to be closed ("weird" because subspaces of finite-dimensional normed vector spaces are always closed)
- Subspaces often arise from span of a set of vectors
- Span defined with *finite* linear combinations
  Span not necessarily closed
- Very often work with closure of span (which is a closed subspace):

$$\overline{\text{span}}(\{\varphi_k\}_{k\in\mathcal{K}}) = \left\{ \sum_{k\in\mathcal{K}} \alpha_k \varphi_k \mid \alpha_k \in \mathbb{C} \text{ and the sum converges} \right\}$$

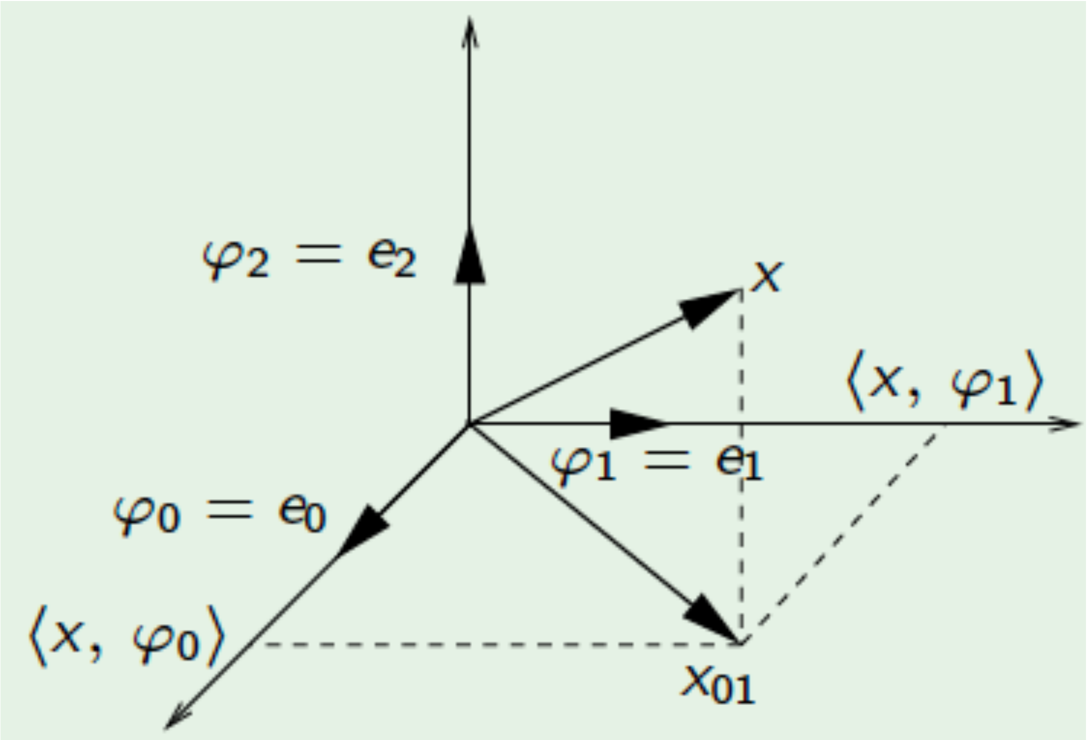
ORTHONORMAL BASIS
SYNTHESIS

- Let $\Phi = \{\varphi_k\}_{k\in\mathcal{K}}$ be an orthonormal basis for H, then for any $f \in H$ such that $\alpha_k = \langle f, \varphi_k\rangle$ for $k \in \mathcal{K}$ (or equivalently, $\boldsymbol{\alpha} = \Phi^* f$), the sequence $\boldsymbol{\alpha}$ is unique.
- **Synthesis**: how can we can get back the original f ?

$$f = \sum_k \langle f, \varphi_k\rangle\varphi_k$$

$$\Phi\boldsymbol{\alpha} = \Phi\Phi^* f$$

Notice that $\Phi\Phi^*$ is identity matrix (operator).



BEST APPROXIMATION AND NORMAL EQUATIONS

Theorem (Normal equations)

- *$x \in H$ and $\{\phi_k\}_{k\in\mathcal{I}}$ a Riesz basis for a closed subspace S*
- *The closest vector to x in S is*

$$\widehat{x} \; = \; \sum_{k\in\mathcal{I}} \beta_k \phi_k \; = \; \Phi\beta$$

where β is the unique solution to

$$\Phi^*\Phi\beta \; = \; \Phi^* x \qquad \text{or}$$

$$\sum_{k\in\mathcal{I}} \beta_k \langle \phi_k, \phi_i \rangle \; = \; \langle x, \phi_i \rangle \text{ for all } i \in \mathcal{I}$$

Normal equations

- $\widehat{x} \; = \; \Phi(\Phi^*\Phi)^{-1}\Phi^* x \; = \; Px$

P is an orthogonal projection

- We can verify $P^2 \; = \; \Phi\,\underbrace{(\Phi^*\Phi)^{-1}\Phi^*\Phi}_{I}(\Phi^*\Phi)^{-1}\Phi^* \; = \; P$ and $P^* = P$

A REMINDER MATRIX-VECTOR MULTIPLICATION

$$\begin{bmatrix} -2 & -2 \\ 1 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = ?$$

$$\begin{bmatrix} -2 & -2 \\ 1 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = ?$$

APPROXIMATION

ORTHOGONAL PROJECTION ON $\mathscr{L}^2(\mathbb{R})$

- Orthogonal projection gives “best” approximation
  - Best here means the norm of the error is minimized!
  - Minimum mean squared error