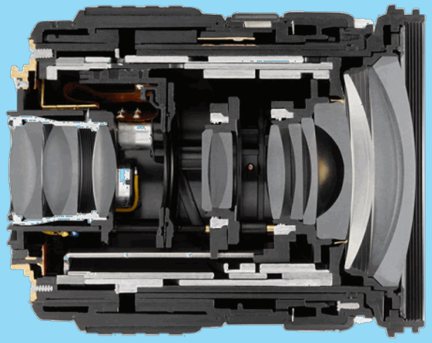

Optics

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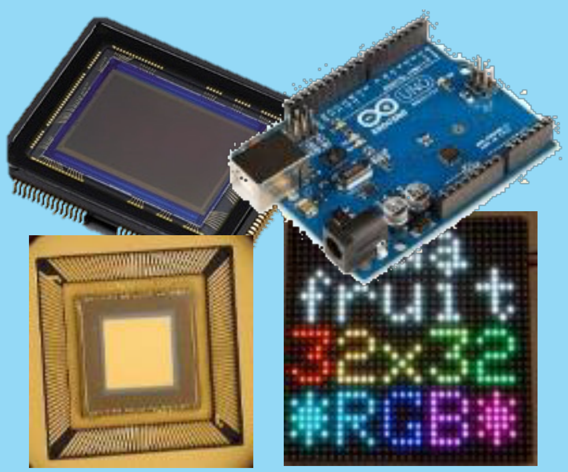


Sensors
&
devices

+

Signal
processing
&
algorithms

COMPUTATIONAL METHODS FOR IMAGING (AND VISION)

LECTURE 4: HILBERT SPACE & LINEAR OPERATORS

PROF. JOHN MURRAY-BRUCE

(EVEN FASTER) REVIEW

VECTORS

VECTORS AND VECTOR SPACES

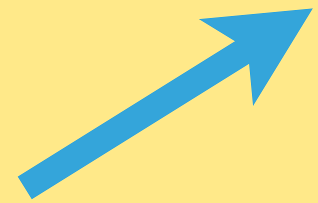
Span: is the set of all finite linear combinations of vectors of a vector space.

Magnitude (norm): For a vector $\mathbf{u} \in \mathbb{R}^n, \|\mathbf{u}\| = \sqrt{\sum_{i=1}^{n} u_i^2}$

Direction (inner product): For two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, the standard inner product is $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^{n} u_i v_i$

Linear independence: if any vector in the set can be written as a linear combination of the remaining vectors of the set, then the set is said to be *linearly dependent*.

Intuition: think of a vector as this little arrow.



Orthogonality: two vectors $\mathbf{u}, \mathbf{v}$ are orthogonal if their inner product is zero, i.e.: $\langle \mathbf{u}, \mathbf{v} \rangle = 0$

Subspace: a subset of a vector that is *closed* under vector addition and scalar multiplication.

Vector space: Collection of vectors (arrows) along with vector addition and scalar multiplication.

Unit Vector: is a vector with magnitude equal to one. Any vector can be normalized to get a unit length: $\mathbf{v}' = \frac{\mathbf{v}}{\|\mathbf{v}\|}$

OUTLINE

- Infinite dimensional vectors
  - Norms & inner products
  - Hilbert spaces
- Matrices (review)
- Linear operators

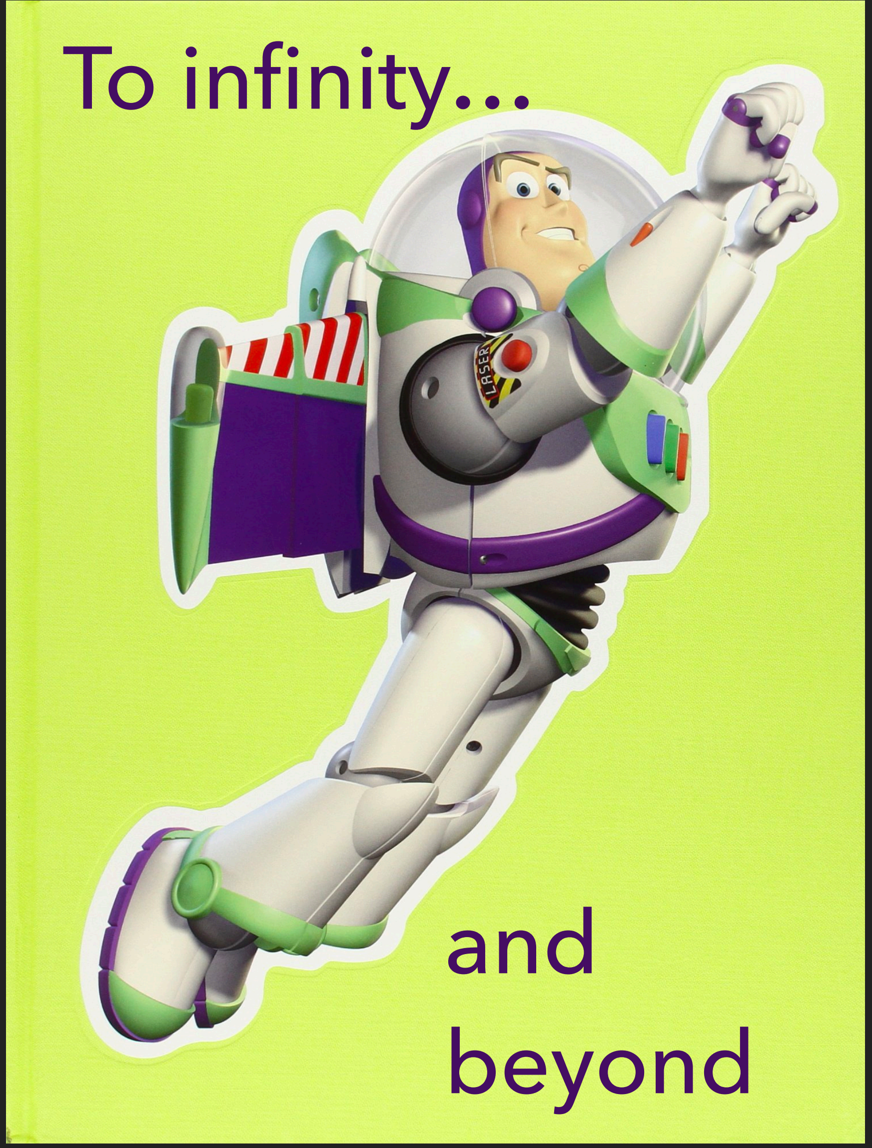
LEARNING GOALS

- Understand the basic Hilbert spaces terminology
- Understand importance of norms
- Understand linear operators as generalization of matrices

READING

- IIP Appendix B
- FSP 2.3 - 2.4

Finite dimensional vectors: such as $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$



$$\mathbb{R}^{\mathbb{N}}$$

BEYOND FINITE DIMENSIONS

INFINITE DIMENSIONAL VECTOR SPACES

BEYOND FINITE DIMENSIONAL VECTOR SPACES

INFINITE DIMENSIONAL VECTORS

- **Vector space of sequences over $\mathbb{Z}$:**

$$\mathbb{R}^{\mathbb{Z}} = \left\{ \mathbf{u} = \begin{bmatrix} \vdots \\ u_{-1} \\ u_0 \\ u_1 \\ \vdots \end{bmatrix} : u_n \in \mathbb{R}, n \in \mathbb{Z} \right\}$$

- As usual, addition and scalar multiplication are component-wise.
- Example your beloved Fibonacci sequence (every textbook example on recursive functions)

BEYOND FINITE DIMENSIONAL VECTOR SPACES
INFINITE DIMENSIONAL VECTORS

- **Standard inner product:** for the vector space of sequences over $\mathbb{Z}$
  - For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{\mathbb{Z}}$:
$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=-\infty}^{+\infty} u_i v_i$$
  - A **valid inner product** on a vector space $\mathscr{V}$ must return a unique, finite number for every pair of vectors in $\mathscr{V}$.
  - Remember what happens when we put $\mathbf{u}$ in place of $\mathbf{v}$?

We get: $\langle \mathbf{u}, \mathbf{u} \rangle = \sum_{i=-\infty}^{+\infty} u_i u_i = \sum_{i=-\infty}^{+\infty} u_i^2$

$$\langle \mathbf{u}, \mathbf{u} \rangle = \|\mathbf{u}\|^2$$

BEYOND FINITE DIMENSIONAL VECTOR SPACES

INFINITE DIMENSIONAL VECTORS

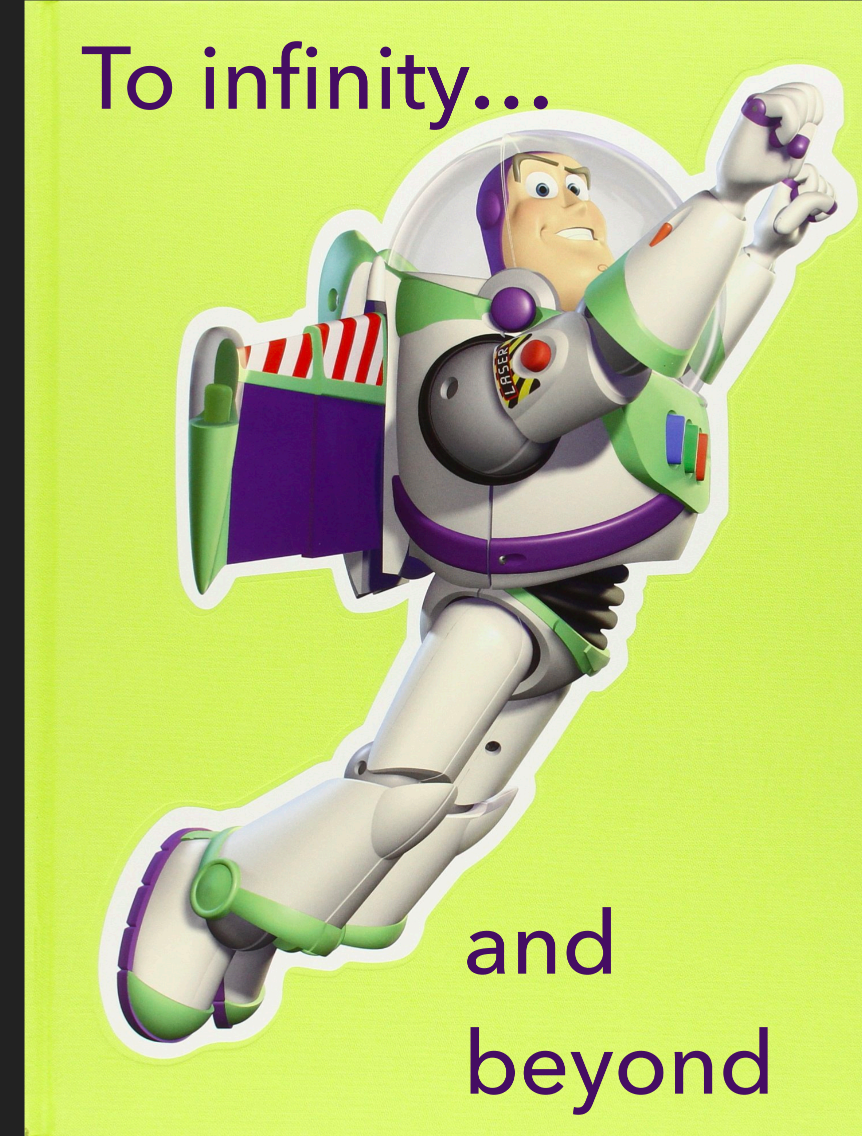
- **Standard norm:** for the vector space of sequences over $\mathbb{Z}$
  - For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{\mathbb{Z}}$:

$$\|\mathbf{u}\| = \sqrt{\sum_{i=-\infty}^{+\infty} u_i^2}$$

  - This norm is induced by the standard inner product.

Infinite dimensional vectors:
infinite sequences $\mathbb{R}^{\mathbb{Z}}$

$$\mathbb{R}^{\mathbb{R}^{\mathbb{Z}}}$$



BEYOND FINITE DIMENSIONS

INFINITE DIMENSIONAL VECTOR SPACES

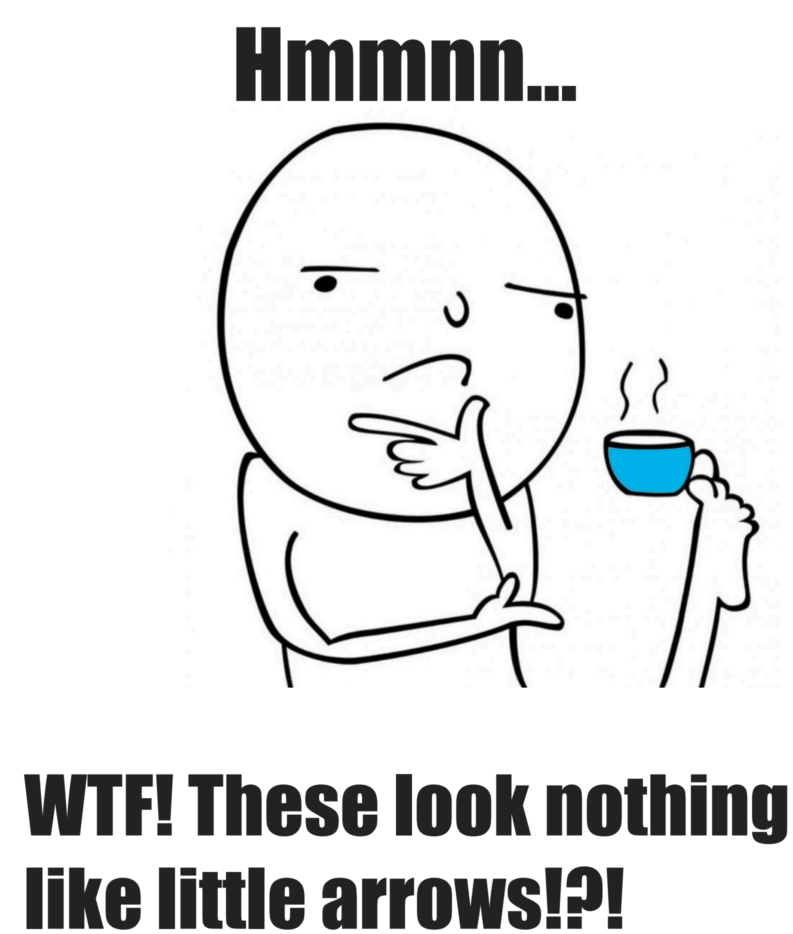
BEYOND FINITE DIMENSIONAL VECTOR SPACES
FUNCTIONS

- **Vector space of functions over $\mathbb{R}$:**

$$\mathbb{R}^{\mathbb{R}} = \left\{ f : f(t) \in \mathbb{R}, t \in \mathbb{R} \right\}$$

- **Examples:**

1. Sinusoids $f(t) = 3.5\sin(2\pi t)$,
2. Linear functions $f(t) = 2t + 3$,
3. General polynomials $f(t) = \sum_{k=0}^{K} \alpha_k t^k$.



BEYOND FINITE DIMENSIONAL VECTOR SPACES
FUNCTIONS

- **Vector space of functions over $\mathbb{R}$:**

$$\mathbb{R}^{\mathbb{R}} = \left\{ f : f(t) \in \mathbb{R}, t \in \mathbb{R} \right\}$$

- Addition and scalar multiplication defined naturally:
  - Addition: $(f + g)(t) = f(t) + g(t)$
  - Scalar multiplication: $(\alpha f)(t) = \alpha f(t)$
- They can be given **magnitude (norm, area)** and **direction (inner product)**

STANDARD INNER PRODUCT
VECTOR SPACE OF FUNCTIONS

- **Vector space of functions over $\mathbb{R}$:**

$$\mathbb{R}^{\mathbb{R}} = \left\{ f : f(t) \in \mathbb{R}, t \in \mathbb{R} \right\}$$

- For $f(t), g(t) \in \mathbb{R}^{\mathbb{R}}$:

$$\langle f(t), g(t) \rangle = \int_{-\infty}^{+\infty} f(t) g(t) \, \mathrm{d}t$$

- Again, for validity the integral must exist and be finite!
- **Question: this inner product induces a norm. What is it?**
  - Hint: substitute $g(t) = f(t)$

STANDARD NORM
VECTOR SPACE OF FUNCTIONS

- **Vector space of functions over $\mathbb{R}$:**

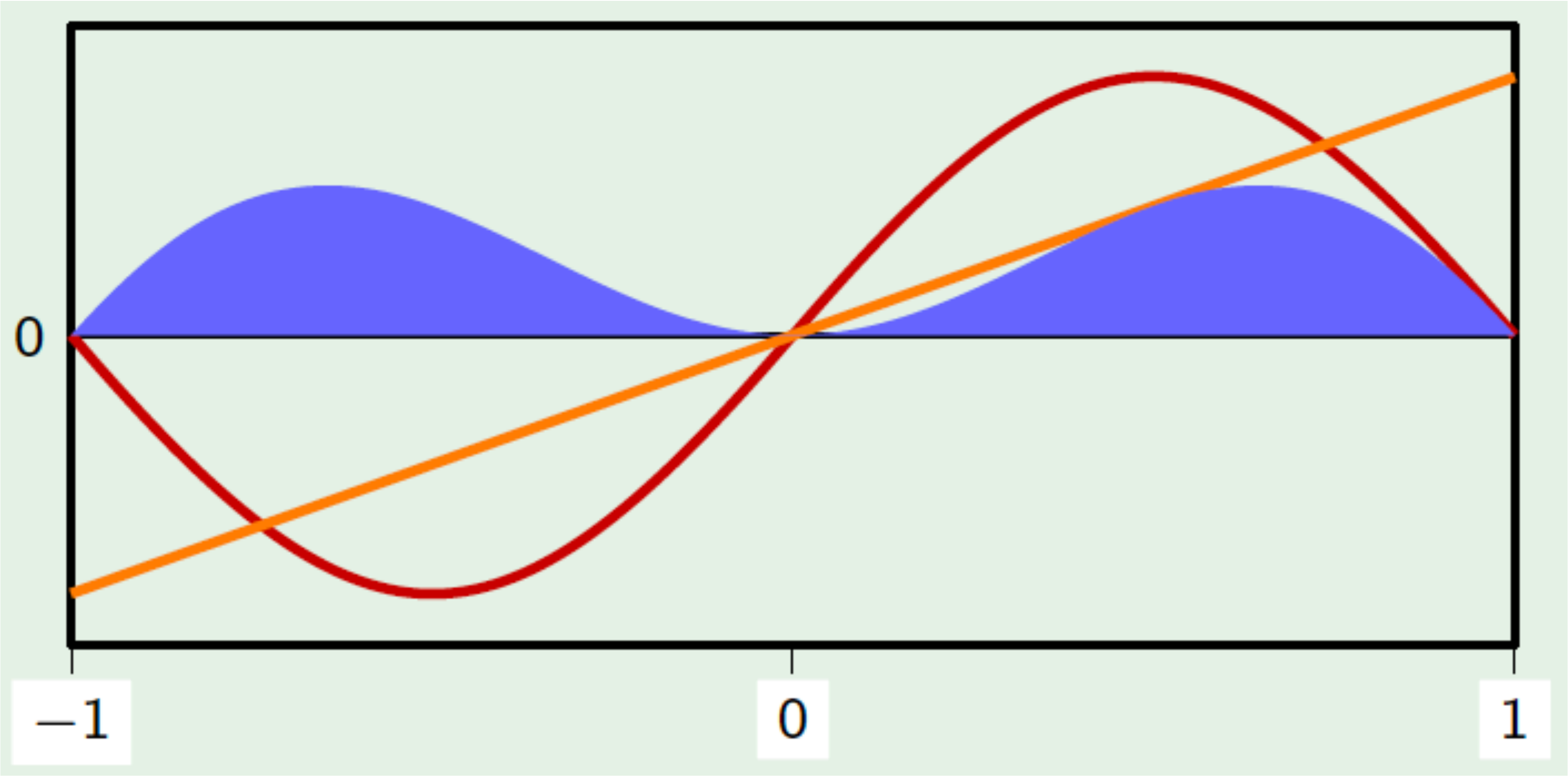
$$\mathbb{R}^{\mathbb{R}} = \left\{ f : f(t) \in \mathbb{R}, t \in \mathbb{R} \right\}$$

- For $f(t) \in \mathbb{R}^{\mathbb{R}}$:

$$\|f(t)\| = \sqrt{\int_{-\infty}^{+\infty} |f(t)|^2 \, \mathrm{d}t}$$

EXAMPLE 1
INNER PRODUCT IN $\mathscr{L}^2[-1,\ 1]$

Compute the inner product between $x(t) = t$ and $y(t) = \sin(\pi t)$, over the vector space $\mathscr{L}^2[-1,\ 1]$.



EXAMPLE 1: WORKED SOLUTION

$$
\begin{aligned}
\langle x(t), y(t)\rangle &= \int_{-1}^{+1} x(t)y(t)\,\mathrm{d}t, && \text{from the definition of the inner product}\\
&= \int_{-1}^{+1} t\sin(\pi t)\,\mathrm{d}t, && \text{substituting } x(t) = t \text{ and } y(t) = \sin(\pi t)\\
&= \left[-\frac{t}{\pi}\cos(\pi t)\right]_{-1}^{+1} - \left(-\int_{-1}^{+1}\frac{1}{\pi}\cos(\pi t)\,\mathrm{d}t\right), && \text{integration by parts}\\
&= \left[-\frac{t}{\pi}\cos(\pi t)\right]_{-1}^{+1} + \left[\frac{1}{\pi^2}\sin(\pi t)\right]_{-1}^{+1}\\
&= \left(-\frac{1}{\pi}\cos(\pi) - \frac{1}{\pi}\cos(-\pi)\right) + \frac{1}{\pi^2}\left(\sin(\pi) - \sin(-\pi)\right)\\
&= \frac{2}{\pi}
\end{aligned}
$$

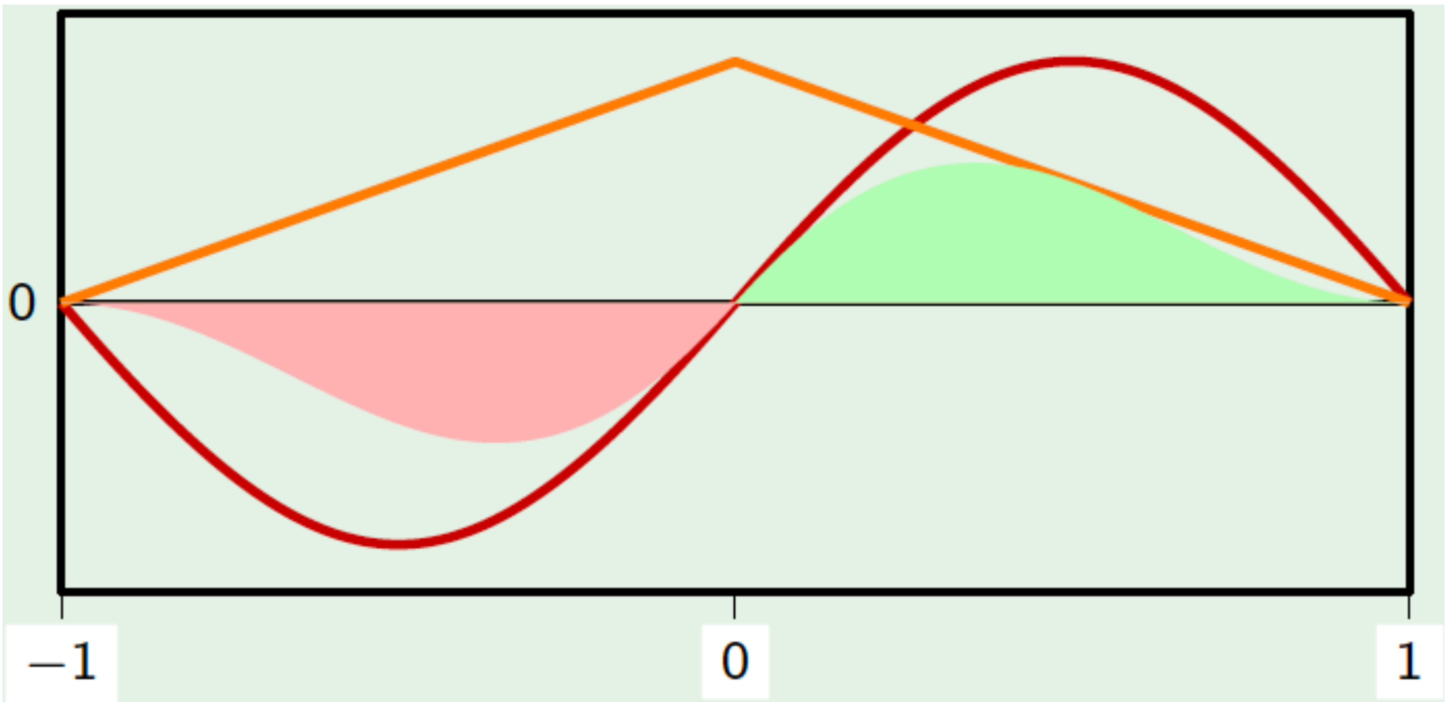
EXAMPLE 2

ORTHOGONAL SUBSPACES OF $\mathscr{L}^2[-1,\ 1]$

Show that $f(t) = 1 - |\,t\,|$ and $g(t) = \sin(\pi t)$ form orthogonal subspaces of $\mathscr{L}^2[-1,\ 1]$.

Solution: For orthogonality $\langle f(t), g(t) \rangle = \int_{-\infty}^{+\infty} f(t) g(t)\, \mathrm{d}t = 0$

$$\langle f(t), g(t) \rangle = \int_{-1}^{+1} (1 - |\,t\,|) \sin(\pi t)\, \mathrm{d}t$$
$$= 0$$

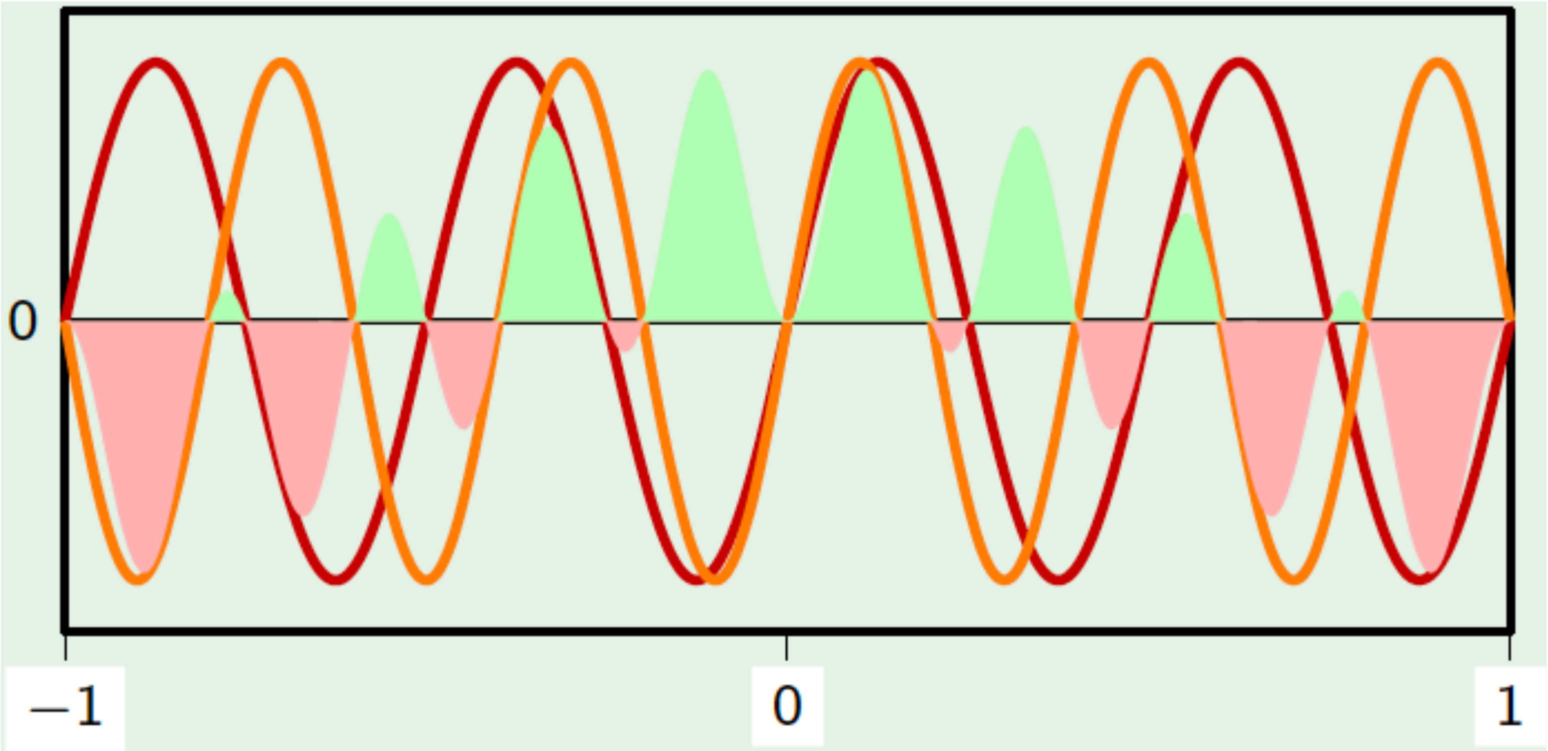


EXAMPLE 3
ORTHOGONAL SUBSPACES OF $\mathscr{L}^2[-1,\ 1]$

Show that $f(t) = \sin(4\pi t)$ and $g(t) = \sin(5\pi t)$ form orthogonal subspaces of $\mathscr{L}^2[-1,\ 1]$. {Hint: $\sin(a)\sin(b) = 1/2[\cos(a-b) - \cos(a+b)]$}

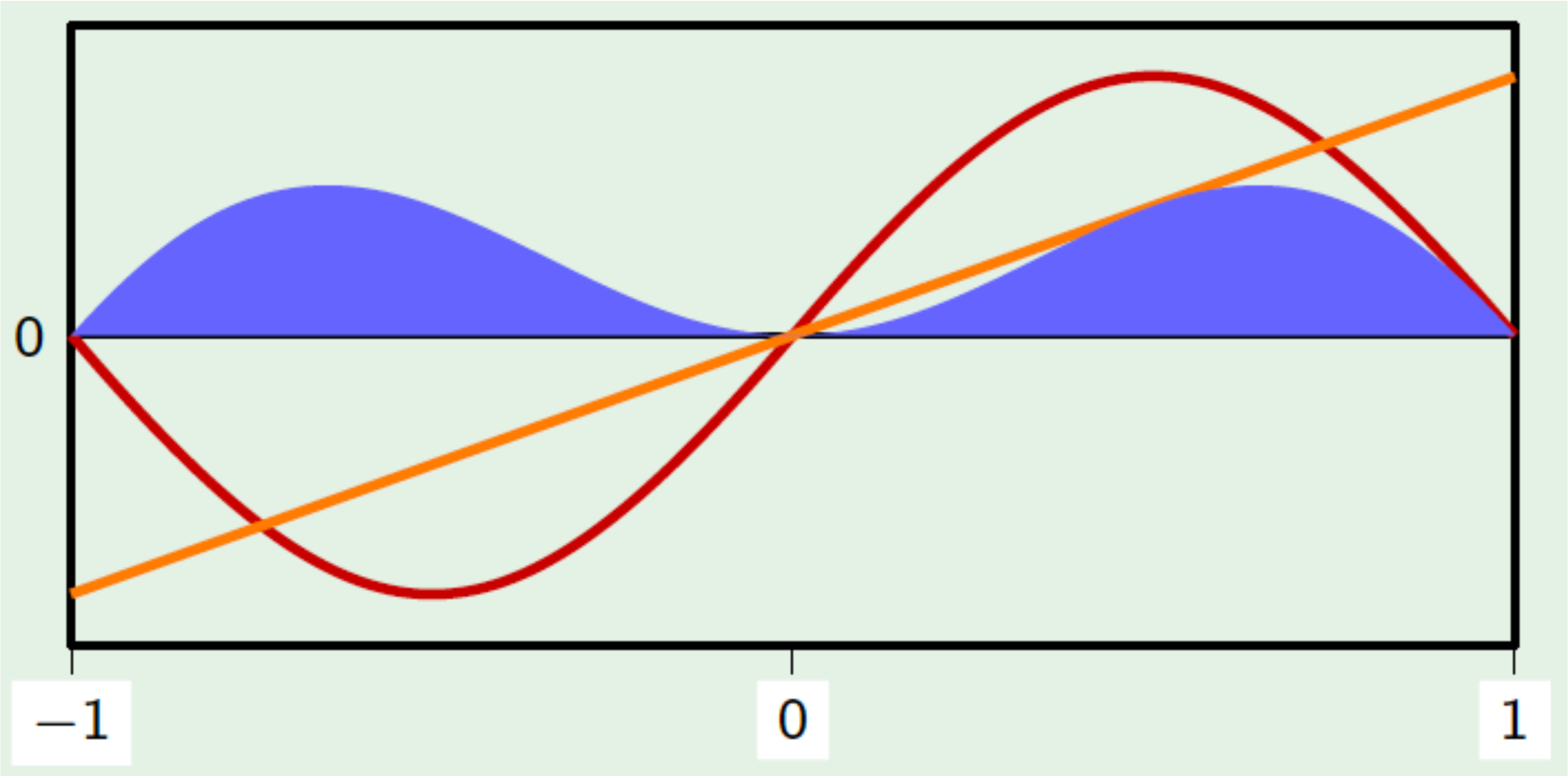
Solution: For orthogonality $\langle f(t), g(t)\rangle = \int_{-\infty}^{+\infty} f(t)g(t)\,\mathrm{d}t = 0$

$$\langle f(t), g(t)\rangle = \int_{-1}^{+1} \sin(4\pi t)\sin(5\pi t)\,\mathrm{d}t$$
$$= 0$$



EXAMPLE 4
NORM IN $\mathscr{L}^2[-1,\ 1]$

Compute norm of $f(t) = t$ and $g(t) = \sin(\pi t)$, over the vector space $\mathscr{L}^2[-1,\ 1]$.



TRY IT !!!

PLOTTING 1D FUNCTIONS

MATLAB PRACTICE 2

PLOTTING (1D) FUNCTIONS IN MATLAB

- A function can be “represented” by a dense set of numbers (i.e. vector)
  - One way to get these numbers is to evaluate the function at certain points
  - Plot the values points and then join them up (like high school times)
- **Example:** Let’s plot $x(t) = t$ and $y(t) = \cos(\pi t)$ between [-1, 1]


Wikipedia: David Hilbert

HILBERT SPACE

INFINITE DIMENSIONAL VECTOR SPACES

NORMED VECTOR SPACES

THE DEFAULTS

Finite dimension ($\mathbb{R}^n$):

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^{n} u_i v_i, \text{ and } \|\mathbf{u}\| = \left(\sum_{i=1}^{n} |u_i|^2 \right)^{1/2}$$

$\ell^2(\mathbb{Z})$: Square-summable sequences

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=-\infty}^{+\infty} u_i v_i, \text{ and } \|\mathbf{u}\| = \left(\sum_{i=-\infty}^{+\infty} |u_i|^2 \right)^{1/2}$$

$\mathscr{L}^2(\mathbb{R})$: Square-integrable functions

$$\langle f(t), g(t) \rangle = \int_{-\infty}^{+\infty} f(t) g(t)\, \mathrm{d}t, \text{ and } \|f(t)\| = \left(\int_{-\infty}^{+\infty} f^2(t)\, \mathrm{d}t \right)^{1/2}$$

HILBERT SPACE

- **Definition (Hilbert space):** is a complete inner product space.
  - Inner product space: a vector space, equipped with a valid inner product.
  - Completeness is just a technical requirement: i.e. "any sequence of vectors whose elements become eventually arbitrarily close."
- **Examples:**
  - $\mathbb{R}^n$ equipped with the 2-norm.
  - $\ell^2(\mathbb{Z}), \mathscr{L}^2(\mathbb{R})$.

HILBERT SPACE

DISTANCES, NORMS & INNER PRODUCTS

DISTANCES, NORMS AND INNER PRODUCTS

- Norm induces a metric (or distance)

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

- An inner product induces a norm

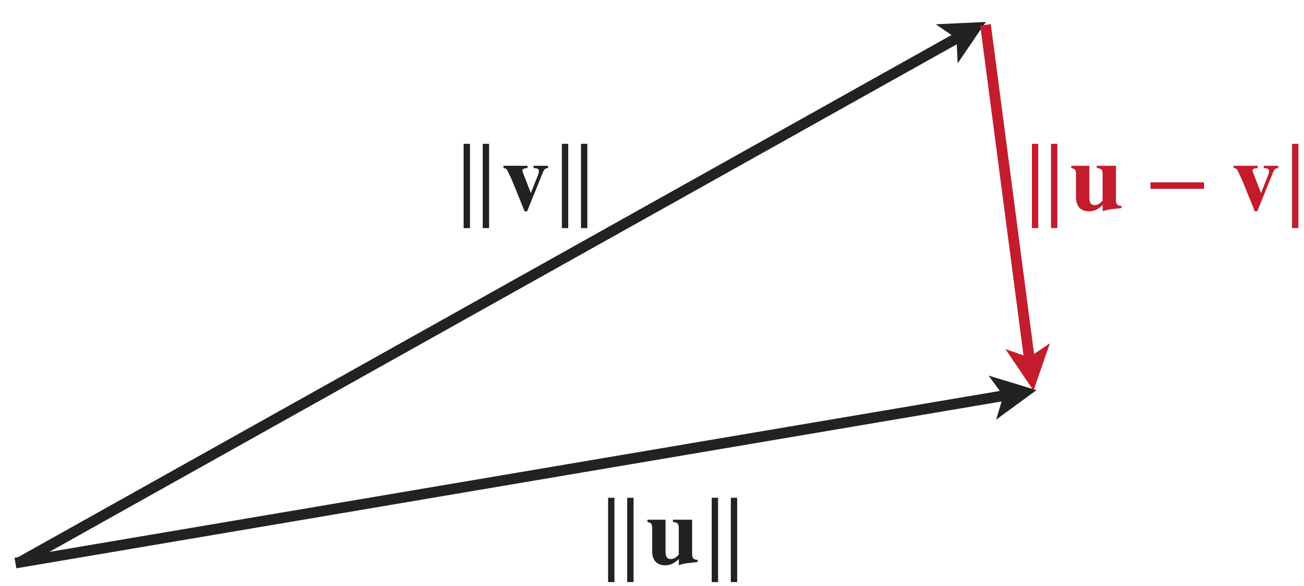
$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$$

- Note:
  - Not all norms are induced by an inner product
  - Not all metrics are induced by a norm

DISTANCES, NORMS AND INNER PRODUCTS

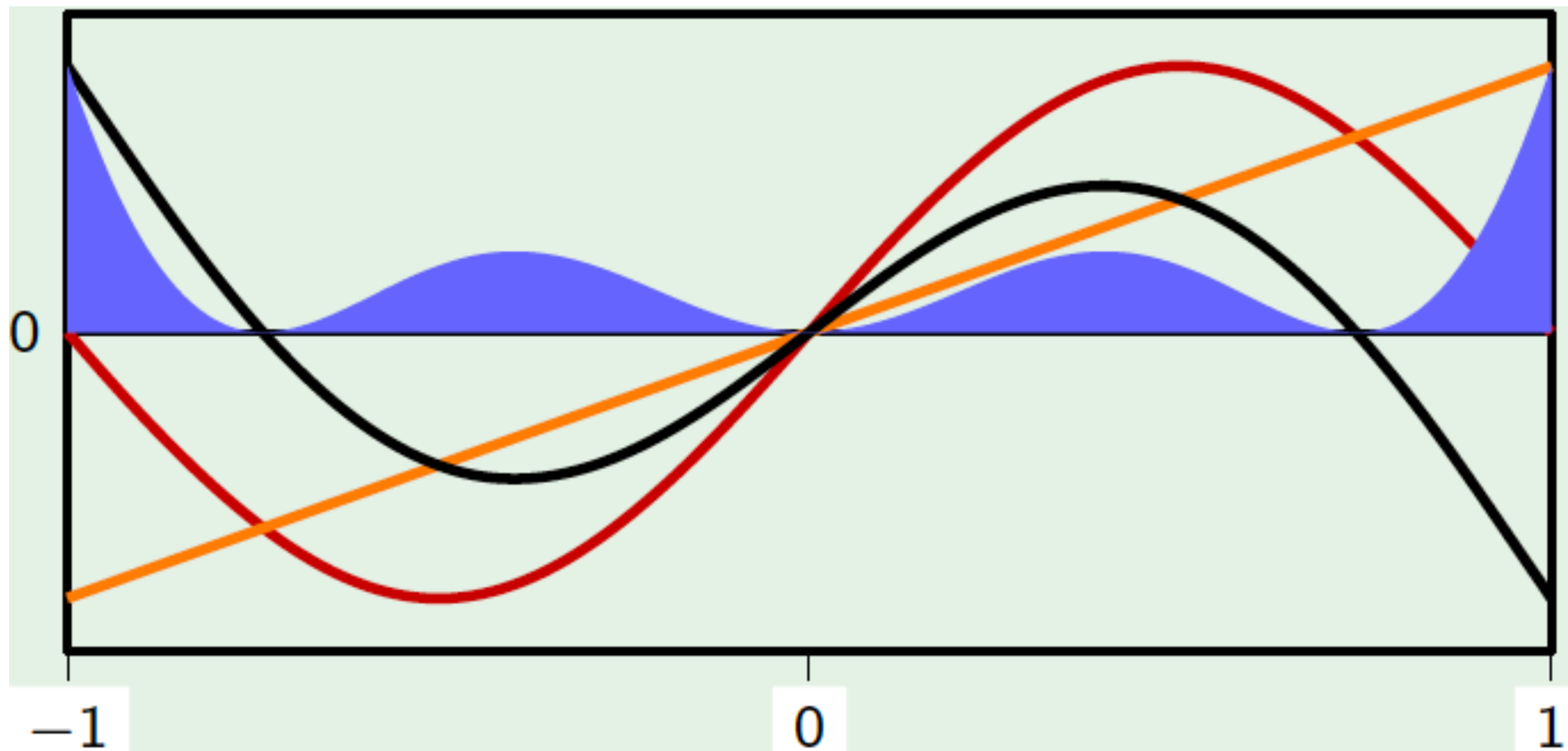
EXAMPLE: NORM AND DISTANCE IN $\mathbb{R}^2$

- $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{u_1^2 + u_2^2}$
- $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{v_1^2 + v_2^2}$
- $\|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle} = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$



DISTANCES, NORMS AND INNER PRODUCTS
EXAMPLE METRIC: MEAN SQUARED ERROR

$f(t) = t, \quad g(t) = \sin(\pi t), \quad g(t) - f(t),$ and
$\|g(t) - f(t)\| = \sqrt{5/3 - 4/\pi} \approx 0.6272.$



A GEOMETRIC VIEW

WHY ARE NORMS USEFUL?

NORMED VECTOR SPACES
THE DEFAULTS

Finite dimension ($\mathbb{R}^n$):

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^{n} u_i v_i, \text{ and } \|\mathbf{u}\| = \left(\sum_{i=1}^{n} |u_i|^2 \right)^{1/2}$$

$\ell^2(\mathbb{Z})$: Square-summable sequences

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=-\infty}^{+\infty} u_i v_i, \text{ and } \|\mathbf{u}\| = \left(\sum_{i=-\infty}^{+\infty} |u_i|^2 \right)^{1/2}$$

$\mathscr{L}^2(\mathbb{R})$: Square-integrable functions

$$\langle f(t), g(t) \rangle = \int_{-\infty}^{+\infty} f(t) g(t)\, \mathrm{d}t, \text{ and } \|f(t)\| = \left(\int_{-\infty}^{+\infty} f^2(t)\, \mathrm{d}t \right)^{1/2}$$

NORMED VECTOR SPACES
P-NORMS

- Finite dimensions, $\mathbb{R}^n$:

$$\|\mathbf{u}\|_p = \left(\sum_{i=1}^{n} | u_i |^p \right)^{1/p}, p \in [1,\infty)$$

- Square-summable sequences, $\ell^p(\mathbb{Z})$:

$$\|\mathbf{u}\|_p = \left(\sum_{i=-\infty}^{+\infty} | u_i |^p \right)^{1/p}, p \in [1,\infty)$$

- Square-integrable function, $\mathscr{L}^p(\mathbb{R})$:

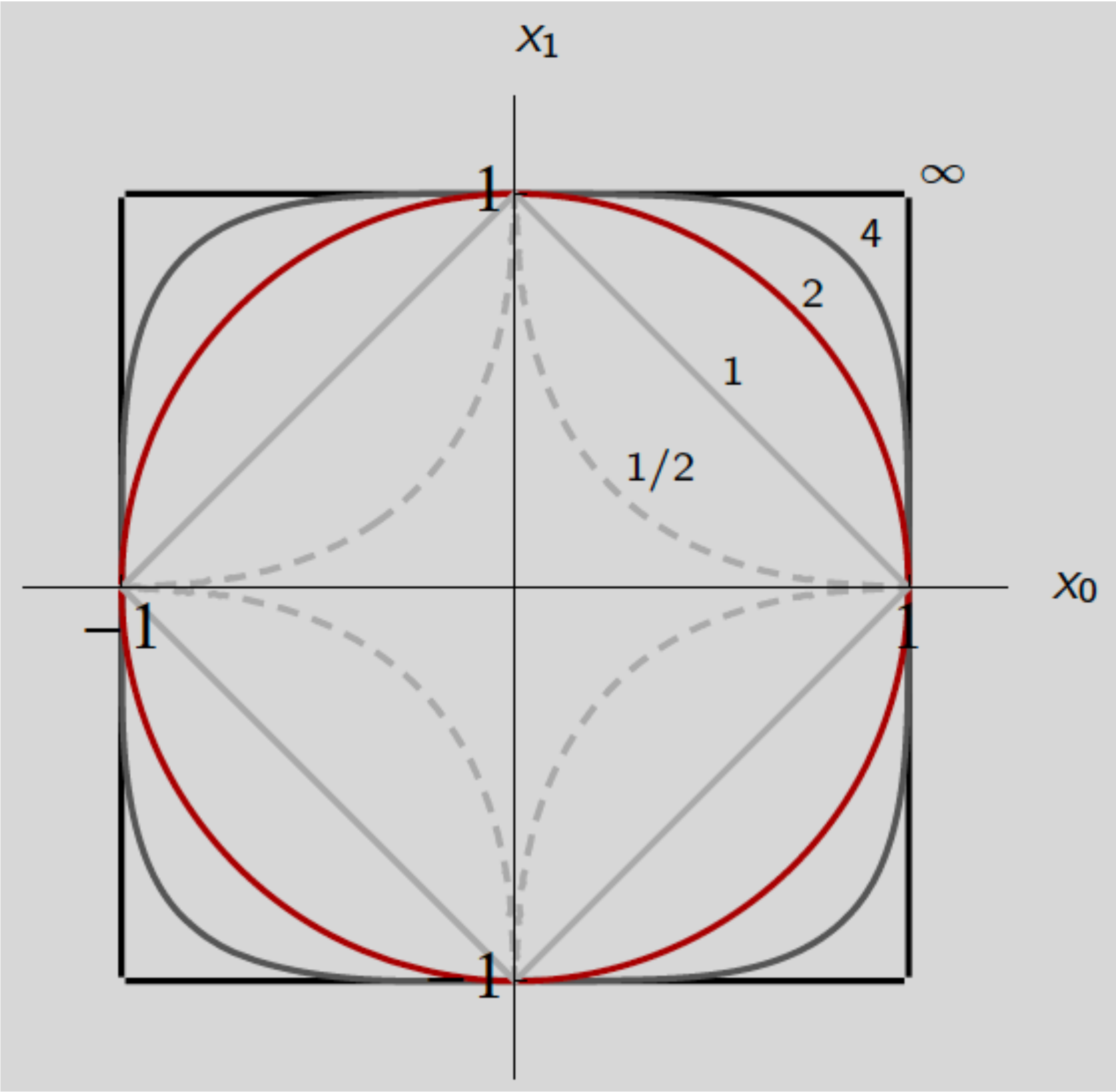
$$\|f(t)\|_p = \left(\int_{-\infty}^{+\infty} | f(t) |^p \, \mathrm{d}t \right)^{1/p}, p \in [1,\infty)$$

- Vectors in $\ell^p(\mathbb{Z})$ different to those in $\ell^q(\mathbb{Z})$ when $p \neq q$. Similarly for $\mathscr{L}^p(\mathbb{R})$ and $\mathscr{L}^q(\mathbb{R})$.

- $p = 2$ is the only $\ell^p(\mathbb{Z})$, or $\mathscr{L}^p(\mathbb{R})$, norm induced by an inner product.

WORLD LOOKS DIFFERENT UNDER DIFFERENT NORMS

- Unit balls in different norms: $\ell^{1/2}$ (semi-norm), ℓ^1, ℓ^2, ℓ^4, ℓ^∞



SOLUTIONS TO LINEAR SYSTEMS

USING DIFFERENT NORMS

- Consider a simple linear system (i.e. $y = Ax$)

$$\boxed{y = \frac{1}{5}\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}\alpha_0 \\ \alpha_1\end{bmatrix}}$$

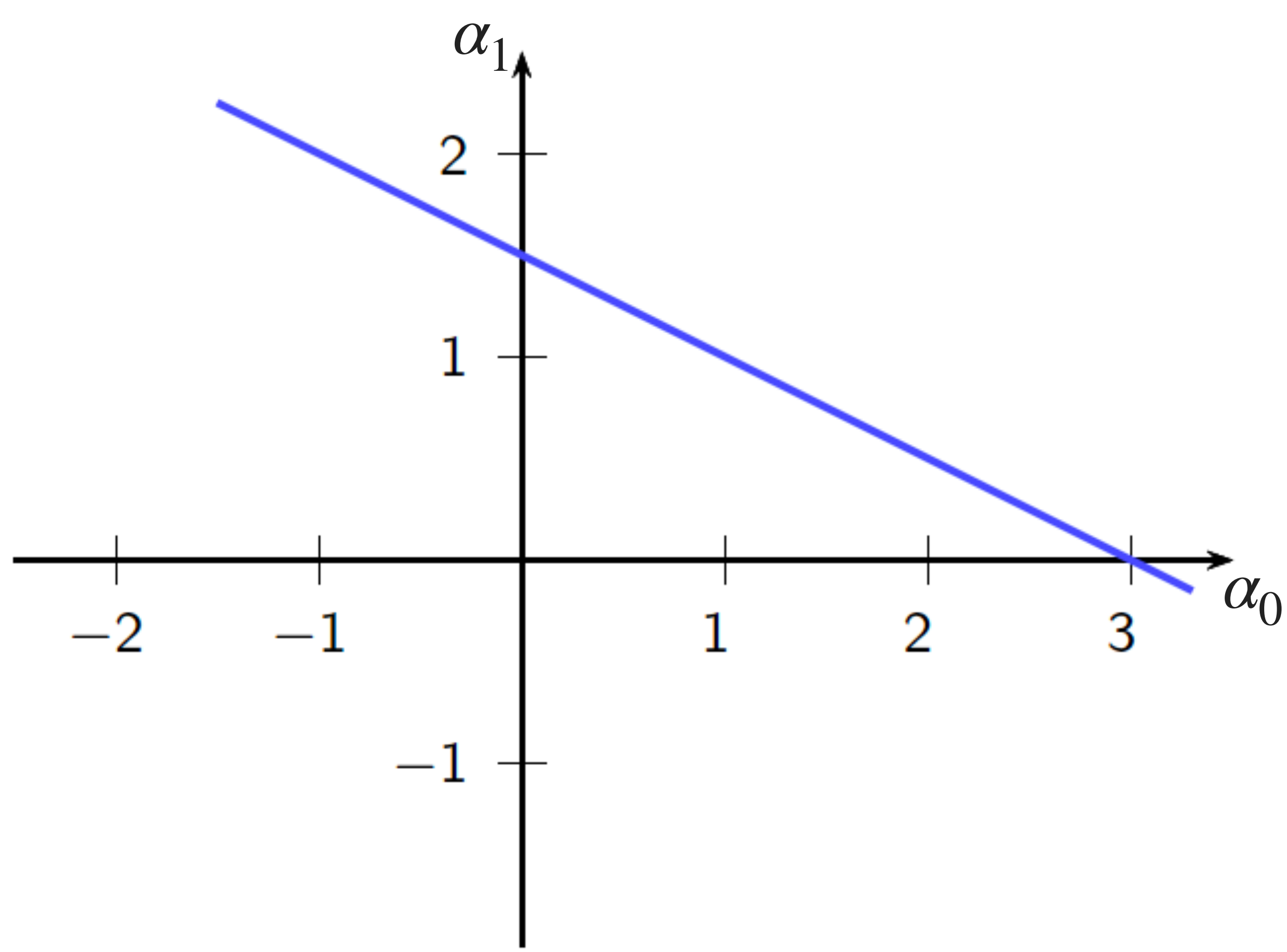
  - The solutions behave very differently under different norm constraints on x.

SOLUTIONS TO LINEAR SYSTEMS
USING DIFFERENT NORMS

- Consider a simple linear system (i.e. $y = Ax$)

$$\boxed{y = \frac{1}{5}\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}\alpha_0 \\ \alpha_1\end{bmatrix}}$$

  - The solutions behave very differently under different norm constraints on x.
  - Let $y = \frac{3}{5}$



SOLUTIONS TO LINEAR SYSTEMS
USING DIFFERENT NORMS

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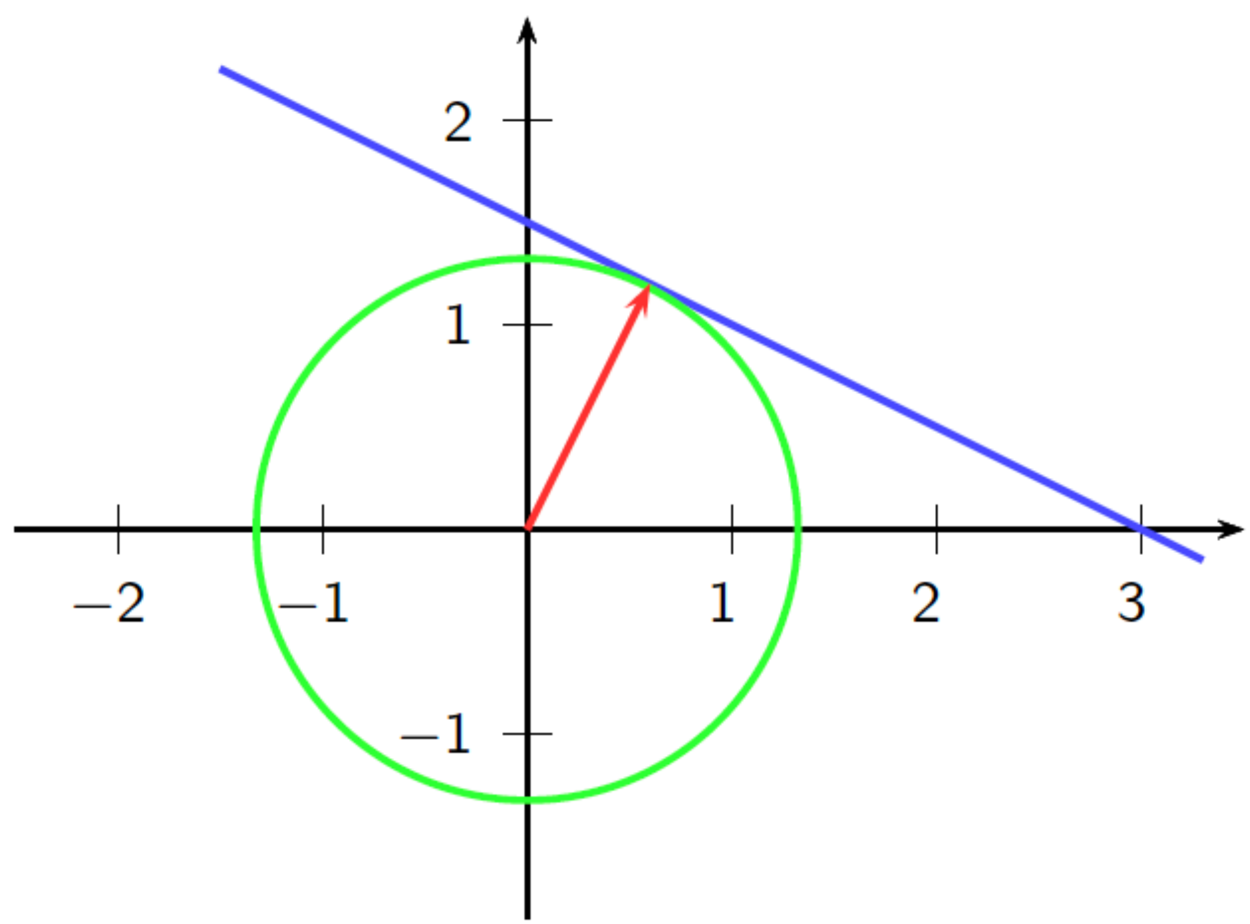

SOLUTIONS TO LINEAR SYSTEMS
USING DIFFERENT NORMS

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$$\boxed{y = \frac{1}{5}\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}\alpha_0 \\ \alpha_1\end{bmatrix}}$$

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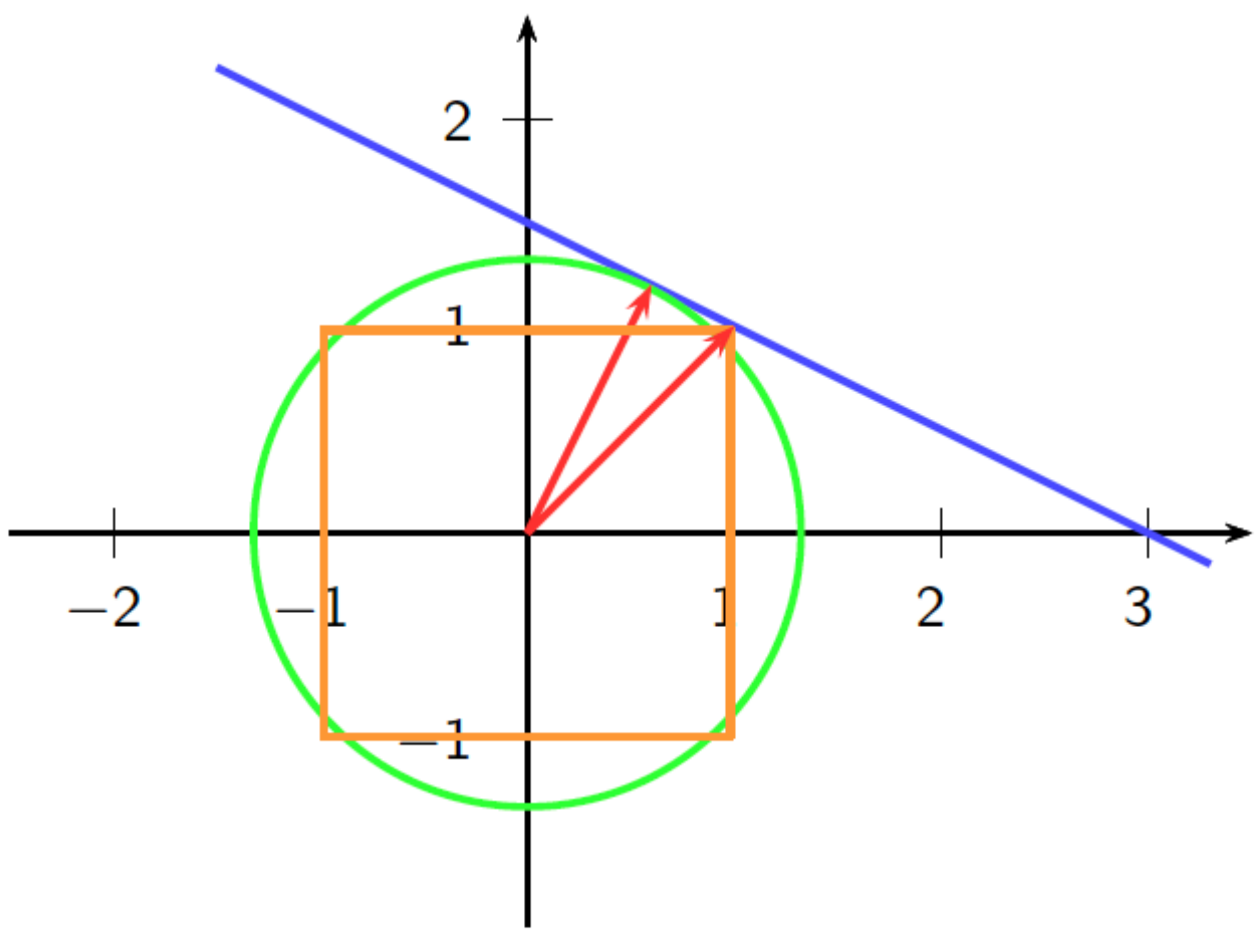
  - Let $y = \dfrac{3}{5}$

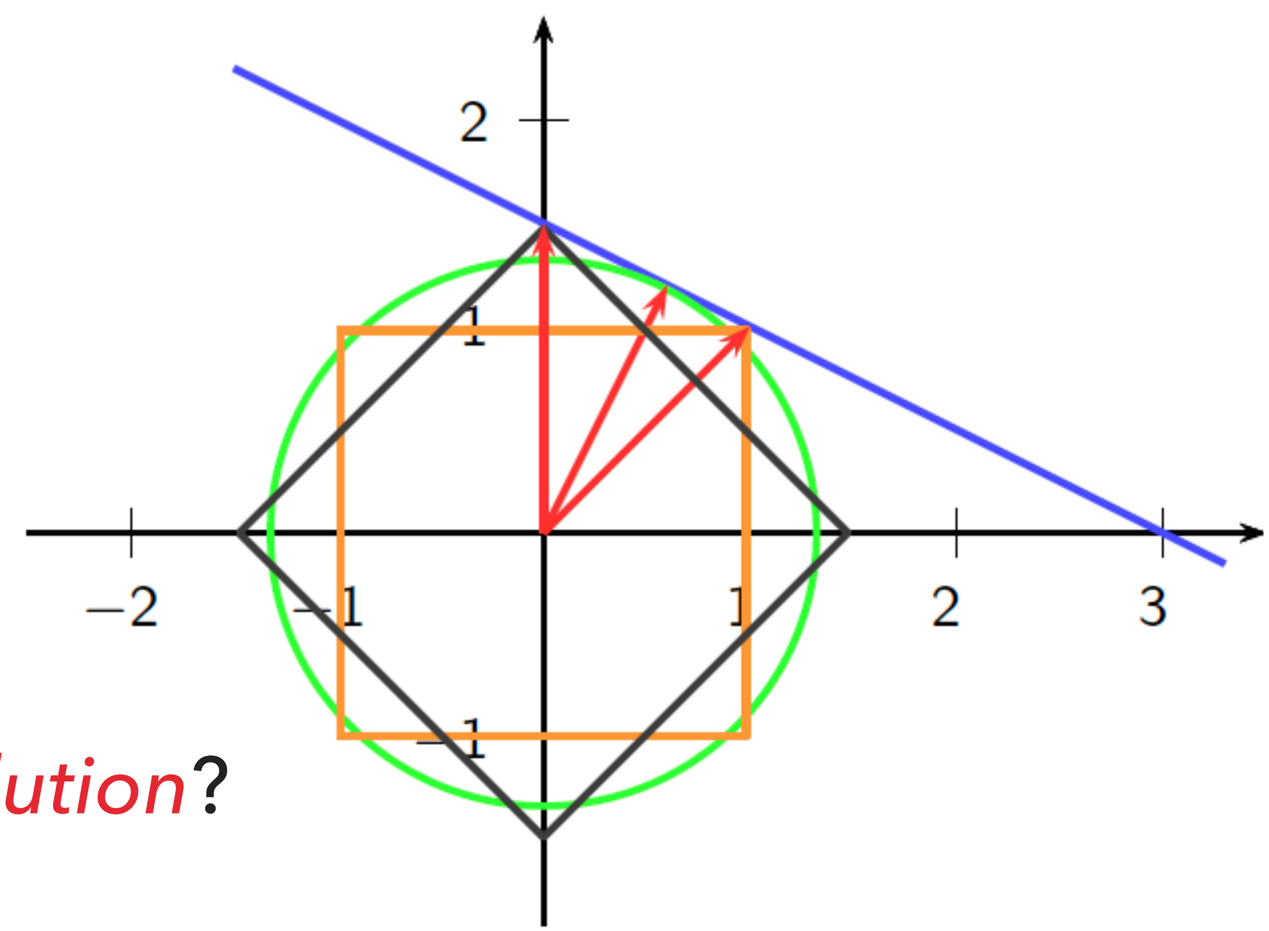


SOLUTIONS TO LINEAR SYSTEMS
USING DIFFERENT NORMS

- Consider a simple linear system (i.e. $y = Ax$)

$$\boxed{y = \frac{1}{5}\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}\alpha_0 \\ \alpha_1\end{bmatrix}}$$

  - The solutions behave very differently under different norm constraints on x.
  - Let $y = \frac{3}{5}$.



  - Which norm gives a *sparse solution*?

GENERALIZATION OF MATRICES

LINEAR OPERATORS

LINEAR SYSTEMS
A REVIEW OF MATRICES

- A **matrix** is a 2D array of numbers
  - Example:

$$\mathbf{A} = \begin{bmatrix} -1 & 3 & 0 \\ \frac{1}{2} & -8 & 17 \end{bmatrix}$$ is a 2×3 (read: 2-by-3) matrix

  - We will write $\mathbf{A} \in \mathbb{R}^{2 \times 3}$

1. **Question:** what is the dimension of the matrix $\mathbf{A} = \begin{bmatrix} 1 & -3 \\ -1 & 2.5 \\ 6 & 0 \\ 1 & 1 \end{bmatrix}$

  - **Answer:** $\mathbf{A} \in \mathbb{R}^{4 \times 2}$

LINEAR SYSTEMS
A REVIEW OF MATRICES

MATLAB

- A mathematical construct used to represent various phenomena
- Yeah okay! BUT – what is it good for?
  - In linear algebra, it was used to represent a system of linear (simultaneous) equations

Solve:

$$\begin{aligned} 2x_1 - 4x_2 + x_3 &= 7 \\ -4x_1 - 1x_2 + 7x_3 &= -3 \\ 5x_1 - 13x_2 - 7x_3 &= -13 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} 2 & -4 & 1 \\ -4 & -1 & 7 \\ 5 & -13 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ -13 \end{bmatrix}$$

Can store these on a computer and apply computer a host of computer algorithms to the matrix and vectors.

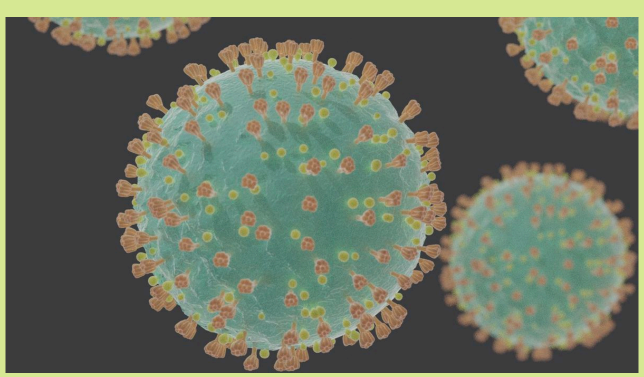
LINEAR SYSTEMS
A REVIEW OF MATRICES

- **Operations on a matrix**
  - Addition, Scalar Multiplication, Subtraction and Matrix Multiplication
  - Transpose
- **Properties of a matrix**
  - Rank
  - Determinant
  - **Range space:** span of its column vectors
  - **Null space:** orthogonal complement of range space
  - **Norm**
  - **Condition number**
- **Eigenvalue Decomposition and Singular value decomposition (SVD)**

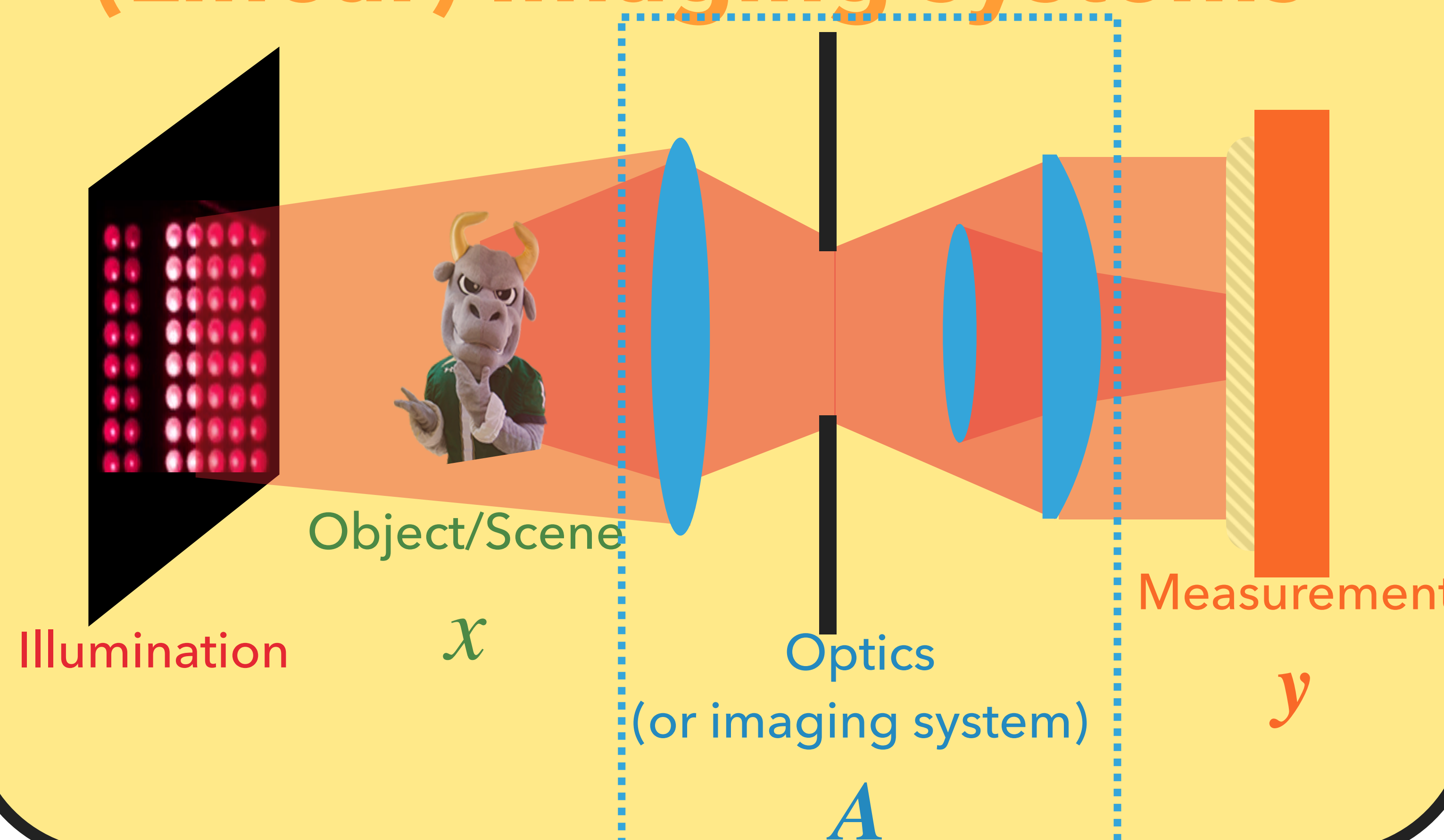
LINEAR SYSTEMS
MATRICES

- Here, we will use **matrices** to represent
  - Images
  - (Linear) Imaging systems

Images



(Linear) Imaging systems



https://commons.wikimedia.org/wiki/File:Coronavirus_SARS-CoV-2.jpg

UTILITY OF MATRICES (& LINEAR OPERATORS)

- **Imaging systems** can be described by **linear operators**
  - Satisfy addition and scalar multiplication
- Properties of linear operators can be studied to reason about the imaging system
  - Range space: **span** of column vectors
  - Null space: **orthogonal complement** of range space
  - Boundedness (norm)
  - Condition number (how well-behaved is the system)

LINEAR OPERATORS
GENERALIZATION OF MATRICES

- **Definition:** $A : H_0 \to H_1$ is a linear operator when, for all $x, y \in H_0$ and $\alpha \in \mathbb{R}$, the following hold:
  - *Additivity*: $A(x + y) = Ax + Ay$
  - *Scalability*: $A(\alpha x) = \alpha(Ax)$
- **Terminology:**
  - **Null space** (a subspace of H_0): $\mathcal{N}(A) = \{x \in H_0 \,|\, Ax = \mathbf{0}\}$
  - **Range space** (a subspace of H_1): $\mathcal{R}(A) = \{Ax \in H_1 \,|\, x = H_0\}$
  - **Boundedness**: A is bounded when $\|A\| < \infty$.
    - $\|A\|$ is a norm of A.

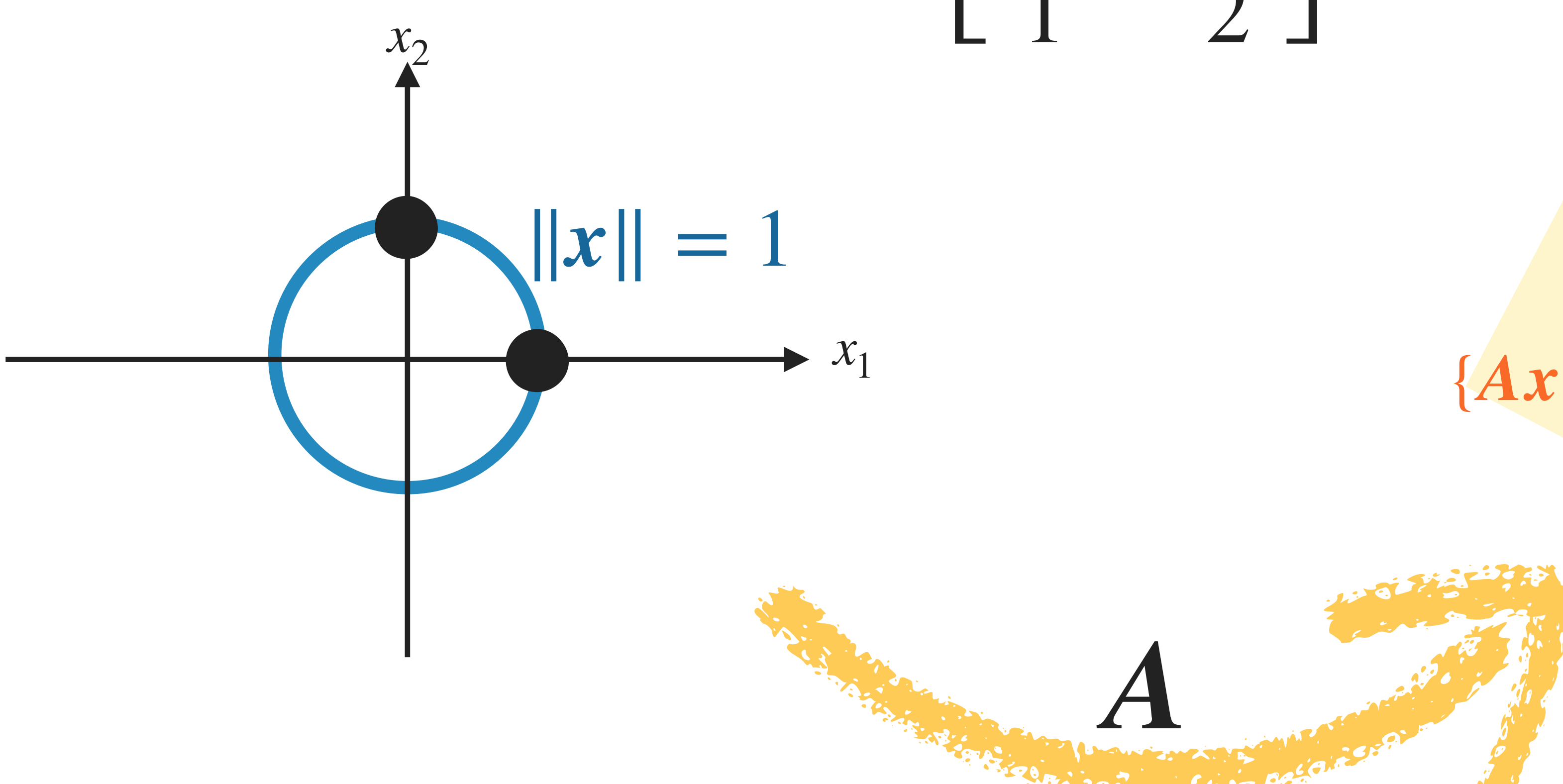
LINEAR OPERATORS
GENERALIZATION OF MATRICES

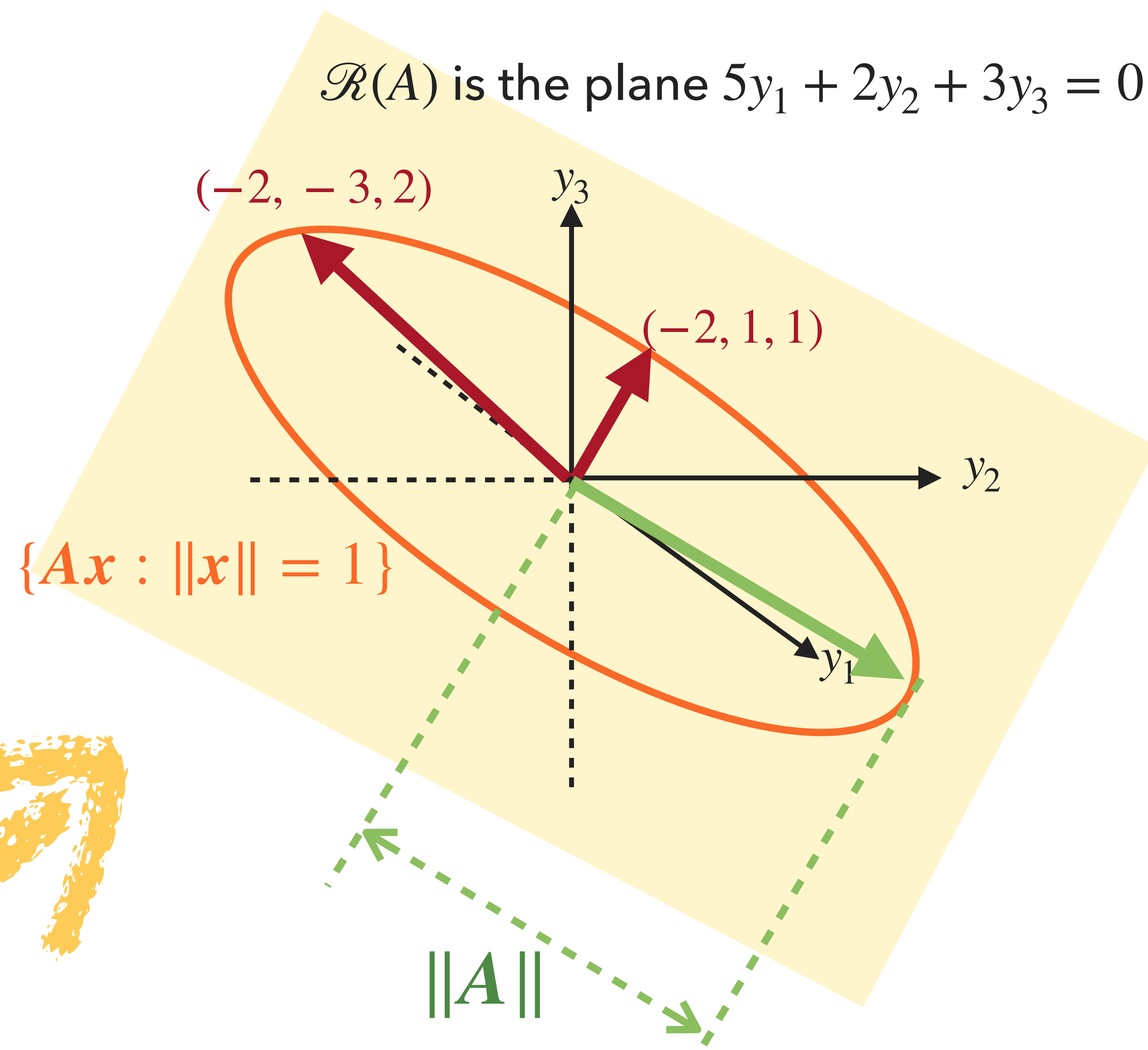
- **Operator norm:** induced by vector norms

$$\|A\| = \sup_{\|x\|=1} \|Ax\|$$

- Consider $\boldsymbol{Ax} = \boldsymbol{y}$

$$\boldsymbol{A} = \begin{bmatrix} -2 & -2 \\ 1 & -3 \\ 1 & 2 \end{bmatrix}$$





A REMINDER MATRIX-VECTOR MULTIPLICATION

$$\begin{bmatrix} -2 & -2 \\ 1 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = ?$$

$$\begin{bmatrix} -2 & -2 \\ 1 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = ?$$

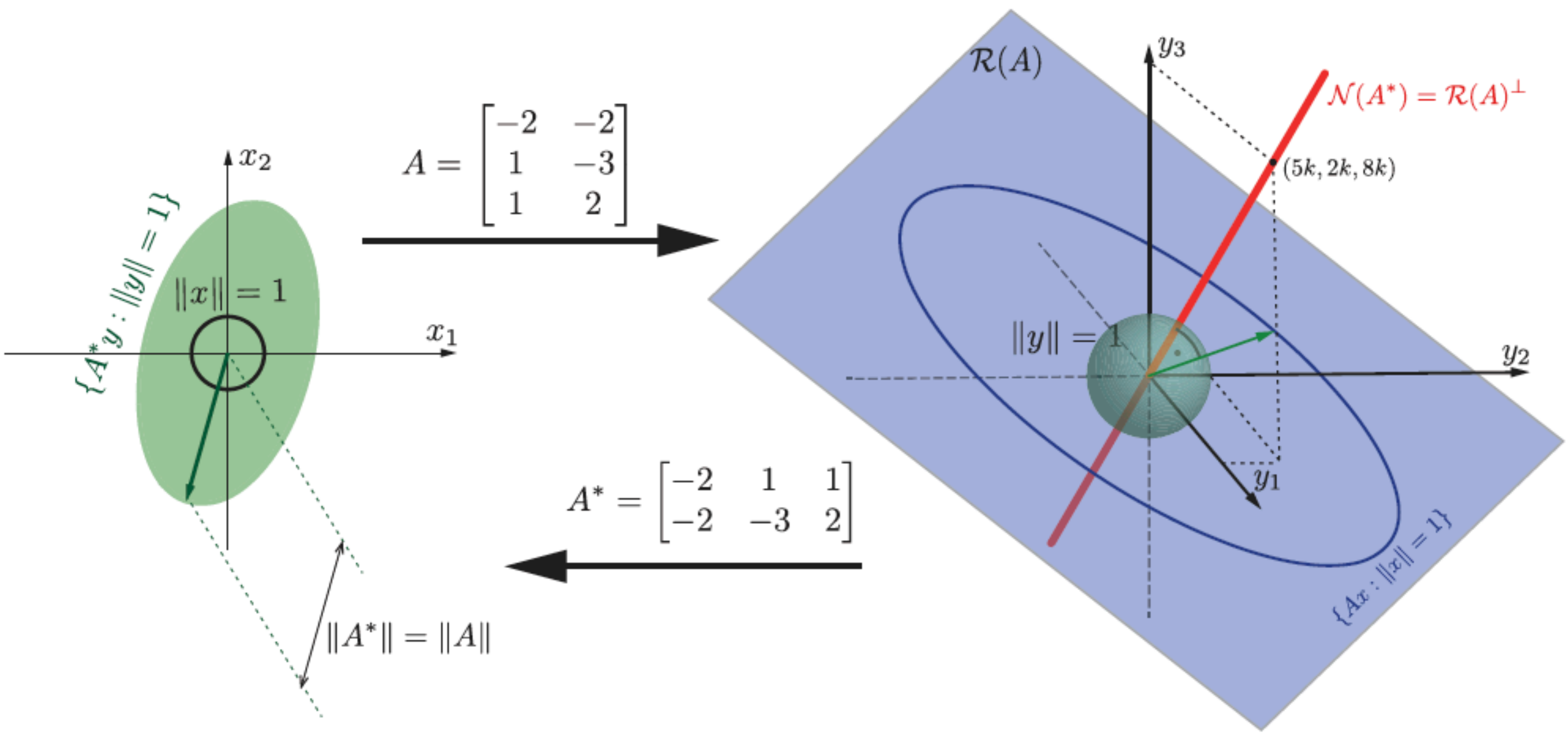
LINEAR OPERATORS
ADJOINT

- **Definition (Adjoint):** $A^* : H_1 \to H_0$ is the adjoint of linear operator $A : H_0 \to H_1$ when, $\langle Ax, y \rangle_{H_1} = \langle x, A^* y \rangle_{H_0}$ for every $x \in H_0$ and $y \in H_1$.

- **Self-adjoint**: If $A = A^*$, then A is called self-adjoint (or symmetric).

- Note that $\mathcal{N}(A^*) = \mathcal{R}(A)^\perp$.
- Generalization of the matrix transpose to operators.

LINEAR OPERATORS
ADJOINT



$\mathcal{N}(A^*)$ is the line $\frac{1}{5}y_1 = \frac{1}{2}y_2 = \frac{1}{3}y_3$

HANDLING MATRICES

MATLAB PRACTICE 3

MATRICES IN MATLAB

- Defining a matrix
- Adding/Subtracting matrices
- Multiplying matrices by
  - Scalars
  - Vectors
  - Matrices
- Rank and determinants
- Matrix Transpose

SEE YOU NEXT TIME!

BASES