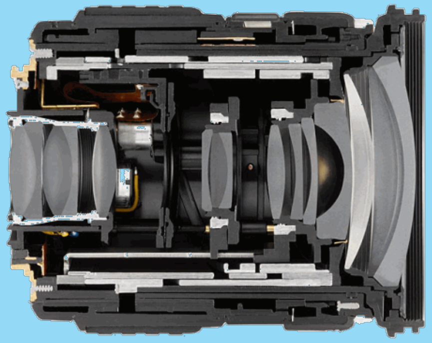
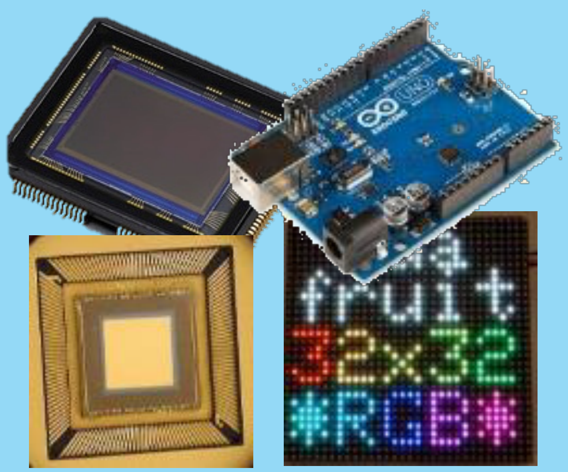


Optics

+



Sensors
&
devices

+

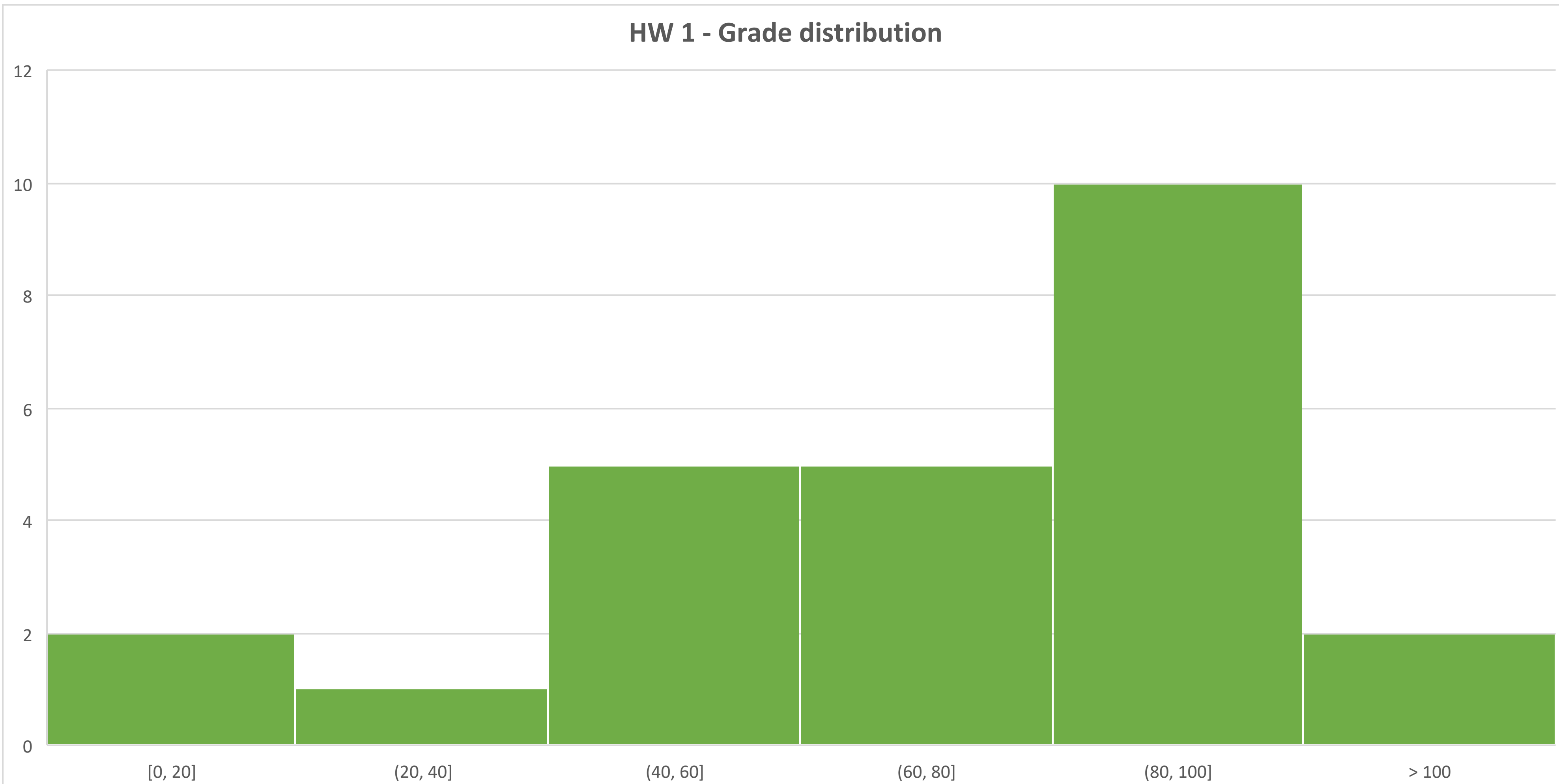
Signal
processing
&
algorithms

COMPUTATIONAL METHODS FOR IMAGING (AND VISION)

LECTURE 12: LSI IMAGING SYSTEMS

PROF. JOHN MURRAY-BRUCE

HW1 GRADES



Mean score: 71%

Median score: 77%

WHERE ARE WE

WE ARE HERE!

Week	Date	Main Topic	Lecture	Readings	Homework Out	Homework Due
1	11-Jan-21		Introduction to computational imaging - Forward and Inverse problems - Common computational imaging problems			
	13-Jan-21		Vectors - Preliminaries			
2	18-Jan-21		**Dr. Martin Luther King, Jr. Holiday (no class)**			
	20-Jan-21	Mathematical preliminaries	Vectors and Vector Spaces - Subspaces, Finite dimensional spaces	IIP Appendix A; FSP 2.1 - 2.2		
3	25-Jan-21		Vector Spaces - Hilbert spaces	IIP Appendix B; FSP 2.3		
	27-Jan-21		Bases and Frames I - Orthonormal and Reisz Bases	IIP Appendix C; FSP 2.4 and 2.B	**HW 1**	
4	1-Feb-21		Bases and Frames II - Orthogonal Bases - Linear operators	IIP Appendix C; FSP 2.5 and 2.B		
	3-Feb-21		Fourier Analysis I - FT (1D and 2D) - FT properties	IIP 2.1, Appendix D; FSP 4.4		
5	8-Feb-21		Sampling and Interpolation - BL functions - Sampling	IIP 2.2, 2.3; FSP 5.4, 5.5		**HW 1**
	10-Feb-21		Fourier Analysis II (DFT)	IIP 2.4; FSP 3.6	**HW 2**	
6	15-Feb-21	Forward Modeling	LSI imaging: Forward problem I - Convolution	IIP 2.5 - 2.6, 3		
	17-Feb-21		LSI imaging: Forward problem I - Transfer functions	IIP 2.6		
7	22-Feb-21		LSI imaging: Forward problem I - Linear operators	IIP 3		
	24-Feb-21		LSI imaging: Forward problem I - Linear operators, Adjoints, and Inverses		**HW 3**	**HW 2**
8	1-Mar-21		**Mid-term Exams**			
	3-Mar-21		LSI imaging: Forward problem II - Sampling and Discretization: Matrix-vector form	IIP 2.7, 4		
9	8-Mar-21		LSI imaging: Forward problem II - Convolution matrix			
	10-Mar-21		LSI imaging: Forward problem II - Sampling and Discretization: Matrix-vector form - PSF, and Transfer functions			**HW3**

OUTLINE

- Linear Space Invariant (LSI) Imaging Systems
- Properties of LSI (imaging) systems
- Convolution - definition & properties

LEARNING GOALS

- To be able to identify LSI systems
- Identify the convolution integral
- Compute 1D and 2D convolutions

READING

- IIP 2.5 - 2.6
- IIP 3.1 - 3.3

DEFINITION

CONVOLUTION OPERATOR

CONVOLUTION & LSI IMAGING SYSTEMS

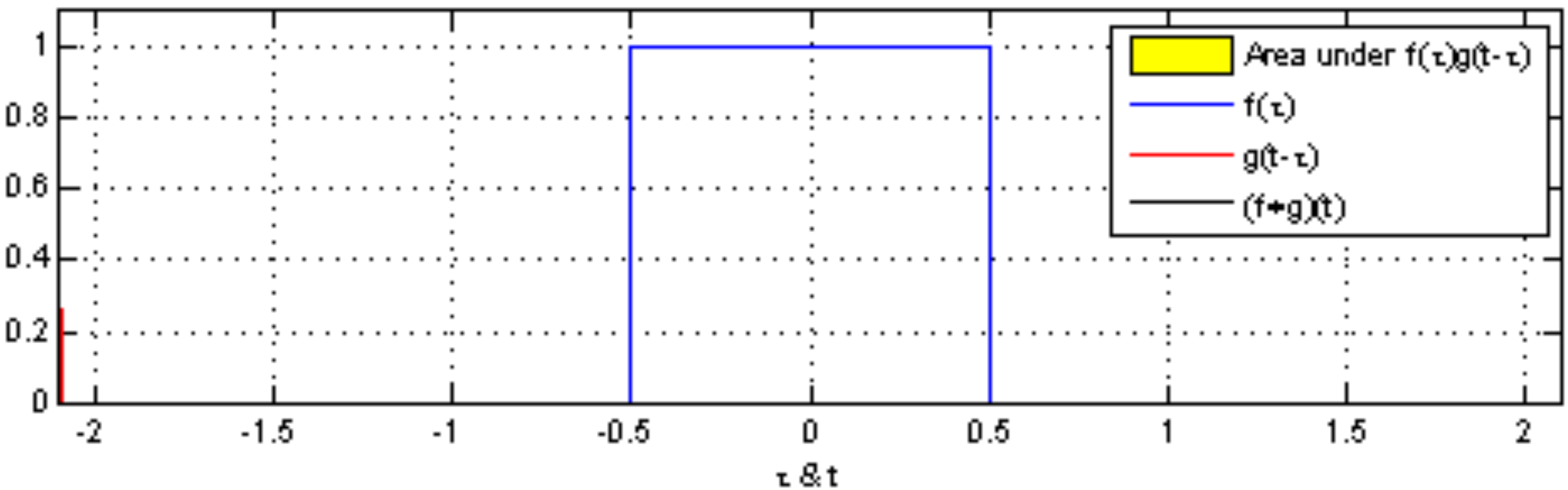
- **Linear shift-invariant systems:** the convolution integral relates the output $g_{\text{out}}(x)$ of the system given any input $g_{\text{in}}(x)$ to that system

$$g_{\text{out}}(x) = g_{\text{in}}(x) \star h(x) = \int_{-\infty}^{\infty} g_{\text{in}}(x')h(x - x')dx'$$

- $h(x)$ is called the system's impulse response
- The FT of the convolution of two functions in space is the product of their respective FTs, i.e. $g(x) \star h(x) \longleftrightarrow G(\omega)H(\omega)$

CONVOLUTION

EXAMPLE 1: CONVOLUTION OF BOX-CAR FUNCTION WITH ITSELF (GRAPHICAL)



Animated

 Wikipedia

OPERATOR FORM

SPECTRAL FORM

RANGE SPACE

NULL SPACE

ADJOINT

INVERSE

PROPERTIES OF THE CONVOLUTION OPERATOR

OPERATOR FORM

SPECTRAL FORM

RANGE SPACE

NULL SPACE

ADJOINT

INVERSE

PROPERTIES OF THE CONVOLUTION OPERATOR

CONVOLUTION
OPERATOR FORM

- **Convolution operator** A, for some LSI system with impulse response $h(x, y)$ is:

$$(Ag)(x, y) = h(x, y) \star g(x, y)$$
$$= \int_{-\infty}^{\infty} h(x - x', y - y')g(x', y')\, \mathrm{d}x'$$

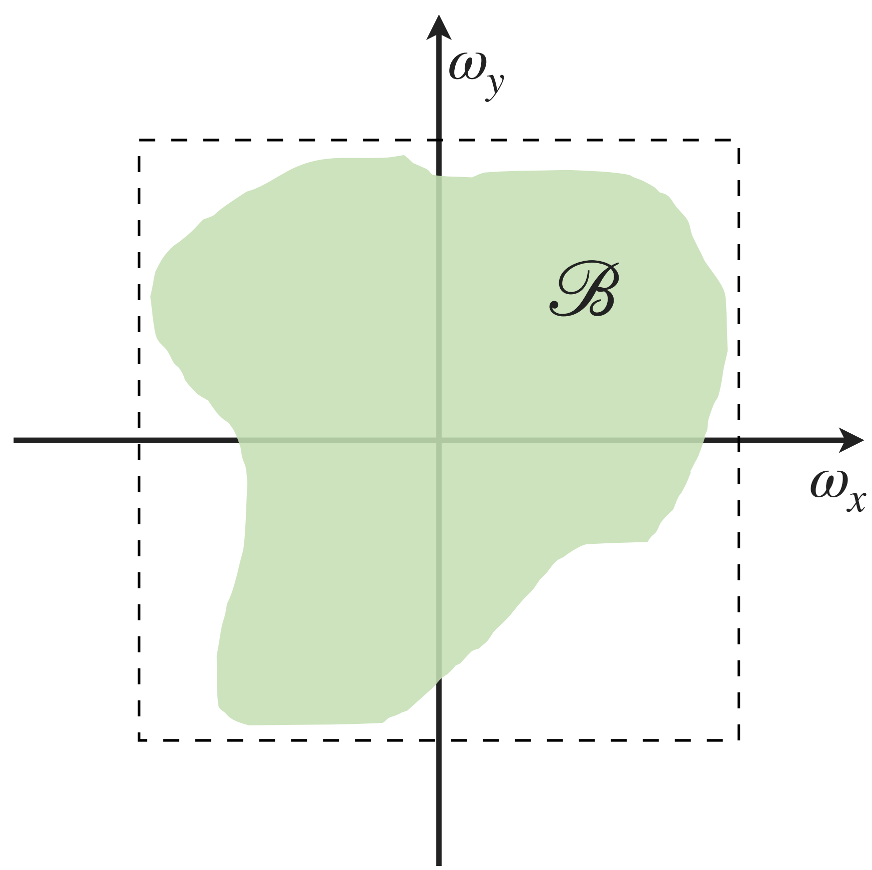
- **Spectral representation:** or the frequency domain representation

$$(Ag)(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(\omega_x, \omega_y)G(\omega_x, \omega_y)e^{\mathrm{j}(\omega_x x + \omega_y y)}\, \mathrm{d}\omega_x \mathrm{d}\omega_y$$

  - From the convolution property of the FT
- The operator A is both **linear** and **bounded. [WHY?]**

BANDLIMITED SYSTEM AND ITS CONVOLUTION OPERATOR

- Consider a system with frequency support (Bandwidth) $\mathscr{B}$
  - The system is said to be **bandlimited**
  - Meaning that the system's transfer function $H(\omega_x, \omega_y) = 0$, for $(\omega_x, \omega_y) \notin \mathscr{B}$



The shaded area shows the region $\mathscr{B}$, i.e. the region where $H(\omega_x, \omega_y) \neq 0$.

OPERATOR FORM

SPECTRAL FORM

RANGE SPACE

NULL SPACE

ADJOINT

INVERSE

PROPERTIES OF THE CONVOLUTION OPERATOR

PROPERTIES OF CONVOLUTION OPERATOR
RANGE SPACE

- The **range** of the convolution operator A, denoted $\mathrm{Range}(A)$
  - $\mathrm{Range}(A)$ contains all bandlimited functions with a bandwidth $\mathscr{B}_0$ contained in $\mathscr{B}$ (i.e. $\mathscr{B}_0 \subseteq \mathscr{B}$)
  - $\mathrm{Range}(A) \subset \left\{ g \in \mathscr{L}^2(\mathbb{R}) : G(\omega_x, \omega_y) = 0, (\omega_x, \omega_y) \notin \mathscr{B} \right\}$

OPERATOR FORM

SPECTRAL FORM

RANGE SPACE

NULL SPACE

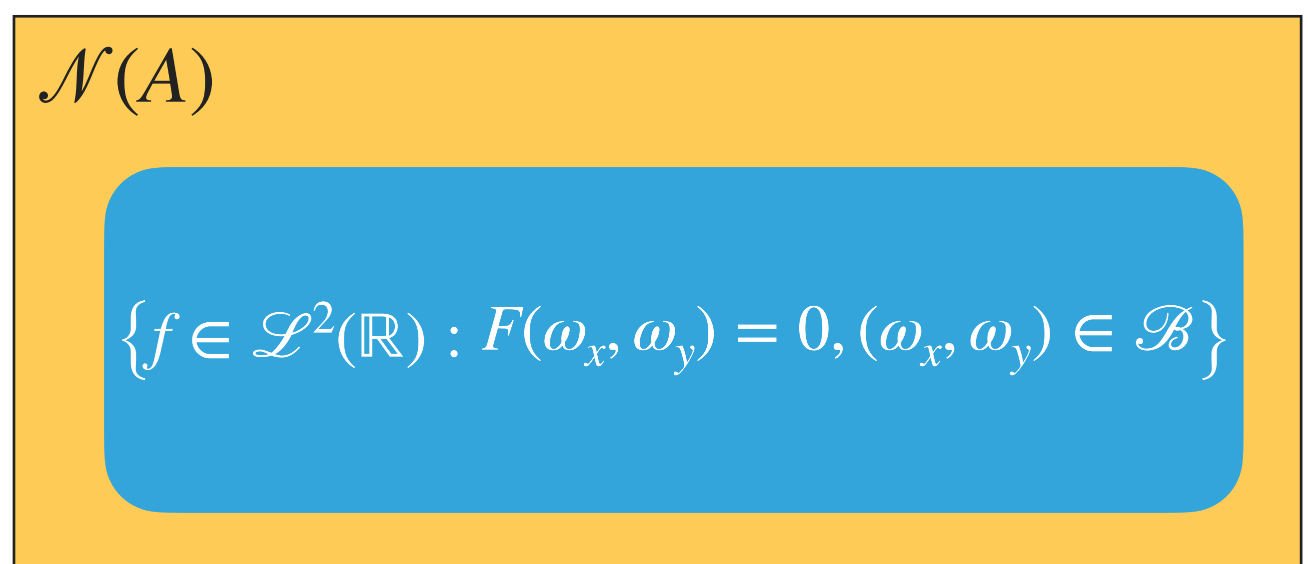
ADJOINT

INVERSE

PROPERTIES OF THE CONVOLUTION OPERATOR

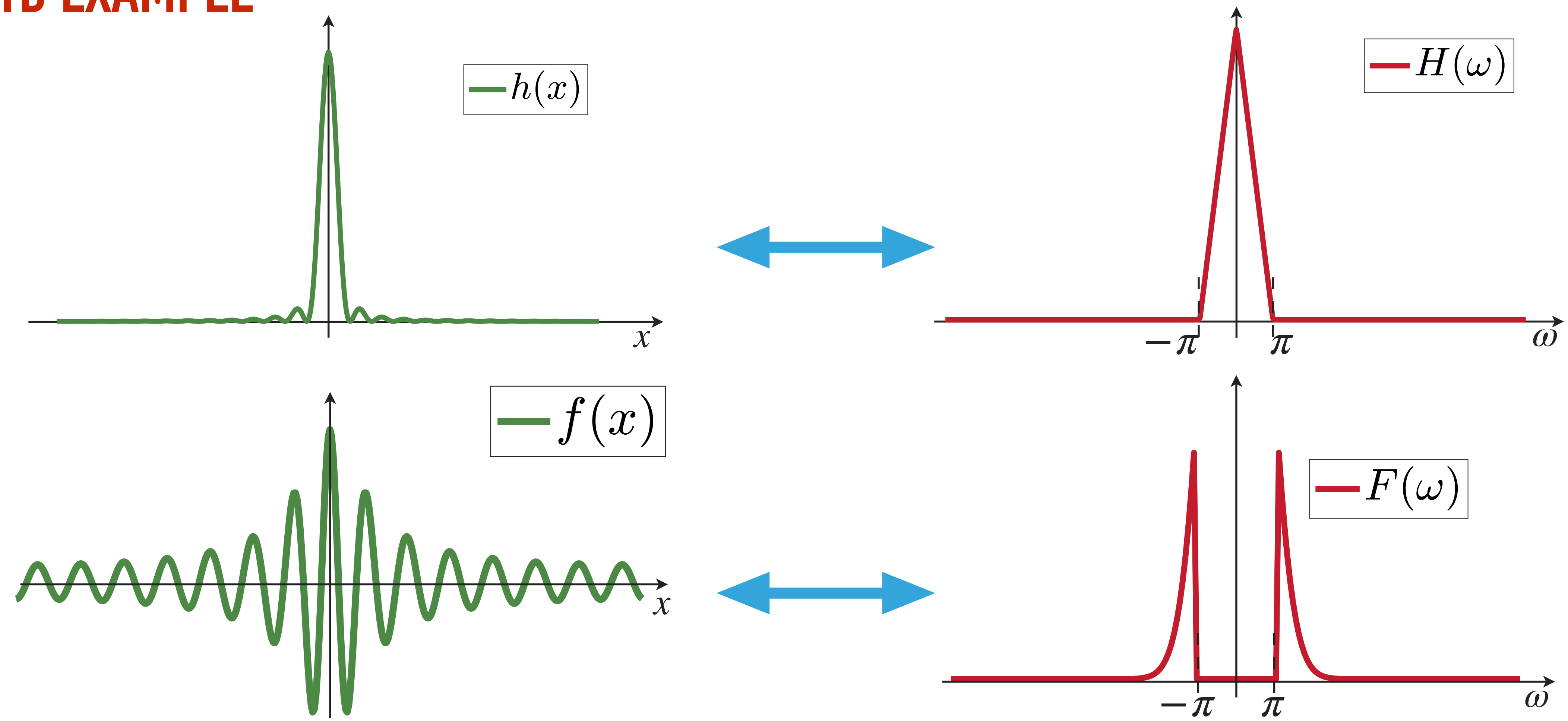
PROPERTIES OF CONVOLUTION OPERATOR
NULL SPACE

- **Null space** of the convolution operator A, denoted $\mathcal{N}(A)$
  - $\mathcal{N}(A)$ contains all bandlimited functions with a bandwidth $\mathcal{B}_0$ contained in $\mathcal{B}$ (i.e. $\mathcal{B}_0 \subseteq \mathcal{B}$)
  - $\mathcal{N}(A) \supset \left\{ f \in \mathcal{L}^2(\mathbb{R}) : F(\omega_x, \omega_y) = 0, (\omega_x, \omega_y) \in \mathcal{B} \right\}$
  - **Null space:** tell us about the so-called "invisible objects"



NULL SPACE OF A CONVOLUTION OPERATOR

1D EXAMPLE



- $f(x)$ is in the null space of the system (invisible) because when it is the input of the system $h(x)$, the system's output is zero.
- If $f_1 \in \mathscr{R}(A)$ and $f_2 \in \mathscr{N}(A)$, then $f_1 \perp f_2$ (i.e. $\langle f_1, f_2 \rangle = 0$).

OPERATOR FORM

SPECTRAL FORM

RANGE SPACE

NULL SPACE

ADJOINT

INVERSE

PROPERTIES OF THE CONVOLUTION OPERATOR

PROPERTIES OF CONVOLUTION OPERATOR
THE ADJOINT

- The **adjoint** operator A^* of the convolution operator A is such that

$$(A^*g)(x) = h^*(-x, -y) \star g(x, y)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^*(x'-x, y'-y)g(x', y')\, \mathrm{d}x'\mathrm{d}y'$$

- **Spectral representation**

$$(A^*g)(x) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H^*(\omega_x, \omega_y)G(\omega_x, \omega_y)e^{\mathrm{j}(\omega_x x+\omega_y y)}\, \mathrm{d}\omega_x\mathrm{d}\omega_y$$

OPERATOR FORM

SPECTRAL FORM

RANGE SPACE

NULL SPACE

ADJOINT

INVERSE

PROPERTIES OF THE CONVOLUTION OPERATOR

PROPERTIES OF CONVOLUTION OPERATOR

INVERSE OPERATOR

- If $H(\omega_x, \omega_y)$ is not bandlimited
  - It is non-zero everywhere in frequency space (apart from a few isolated points)
  - That means it is not finitely supported in frequency space
- The **inverse operator** is given by

$$(A^{-1}g)(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{G(\omega_x, \omega_y)}{H(\omega_x, \omega_y)} e^{\mathrm{j}(\omega_x x + \omega_y y)} \, \mathrm{d}\omega_x \mathrm{d}\omega_y$$

  - The operator is unbounded if $H(\omega_x, \omega_y)$ contains zeros.

WHAT WE COVERED TODAY

- **Convolution (definition)**
  - Operator form and spectral form
- **Convolution operator properties**
  - Null space & Range space
  - Inverse
  - Adjoint

SOME DEFINITIONS

LSI IMAGING SYSTEMS

IMAGING SYSTEM

- **Point Spread Function:** The point spread function (PSF) of an imaging system is its impulse response:
  - Denoted by $\mathrm{PSF}(x, y)$ or $h(x, y)$
  - The PSF, $h(x, y)$, is the image of a point source (which is located at the origin, in this case)

BLURRING

TRANSFER FUNCTION DEFINITIONS

- **Optical Transfer Function:** Fourier transform of the point spread function

$$\mathrm{OTF}(\omega_x, \omega_y) = \mathscr{F}\left\{\mathrm{PSF}(x, y)\right\}$$

  - Because FT is generally complex-valued, OTF can be split into **magnitude** and **phase** contributions:

$$\mathrm{OTF}(\omega_x, \omega_y) = \mathrm{MTF} \times \mathrm{PTF}$$

Magnitude (contrast) Transfer Function

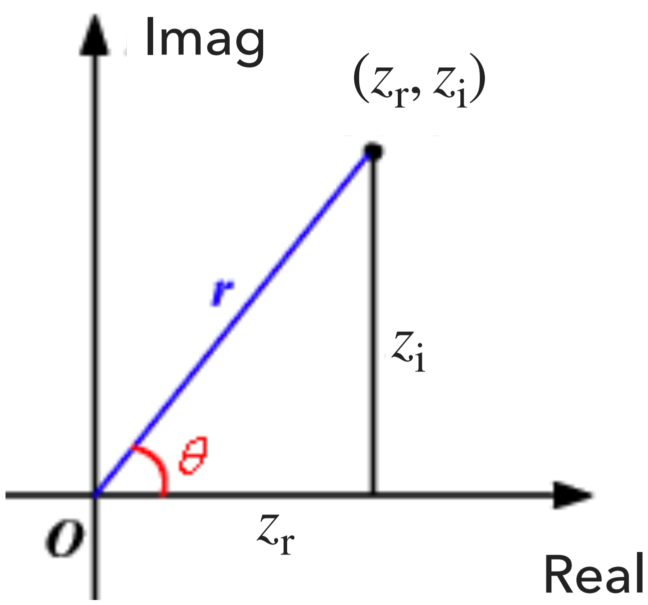
Phase Transfer Function

$$\mathrm{MTF} = |\,\mathrm{OTF}\,|$$

$$\mathrm{PTF} = \measuredangle \mathrm{OTF} = e^{-\mathrm{j}2\pi\Lambda(\omega_x,\omega_y)}$$

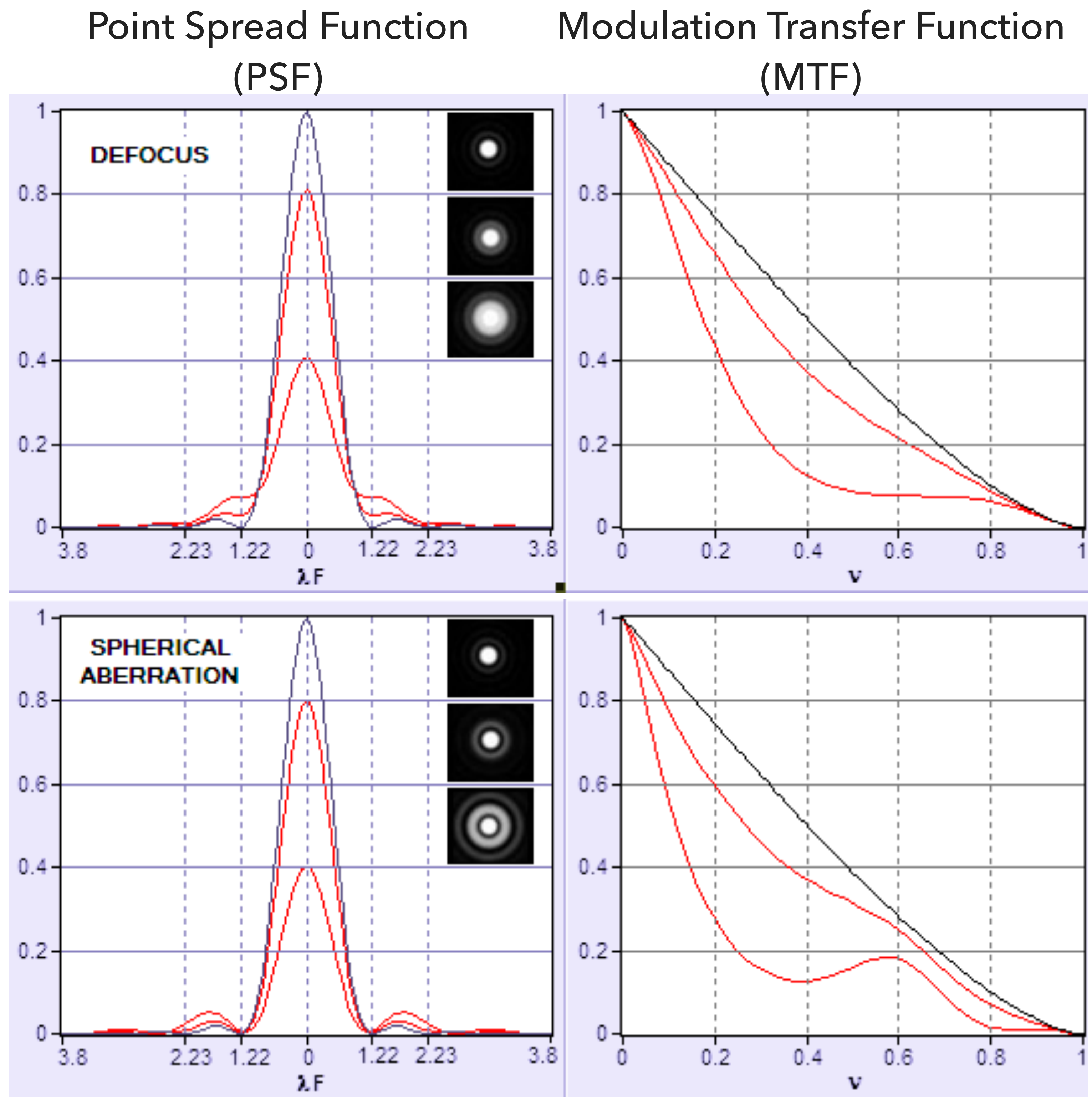
COMPLEX NUMBERS
REVIEW

- Let, $z \in \mathbb{C}$, such that $z = z_\mathrm{r} + \mathrm{j}z_\mathrm{i}$
- **Polar form:** $z = r\cos(\theta) + \mathrm{j}\,r\sin(\theta)$
  - **Euler form:** $z = r\,e^{\mathrm{j}\theta}$
  - Obtained using Euler's formula:
    - $e^{\mathrm{j}\theta} = \cos(\theta) + \mathrm{j}\sin(\theta)$



- **Complex conjugate:** $z^* \in \mathbb{C}$, is such that $z^* = z_\mathrm{r} - \mathrm{j}z_\mathrm{i}$
  - Note also that: $z = r\,e^{\mathrm{j}\theta} \Rightarrow z^* = r\,e^{-\mathrm{j}\theta}$

PSF AND MTF: CROSS-SECTIONS



Aberration is
due to diffraction

LINEAR OPERATORS

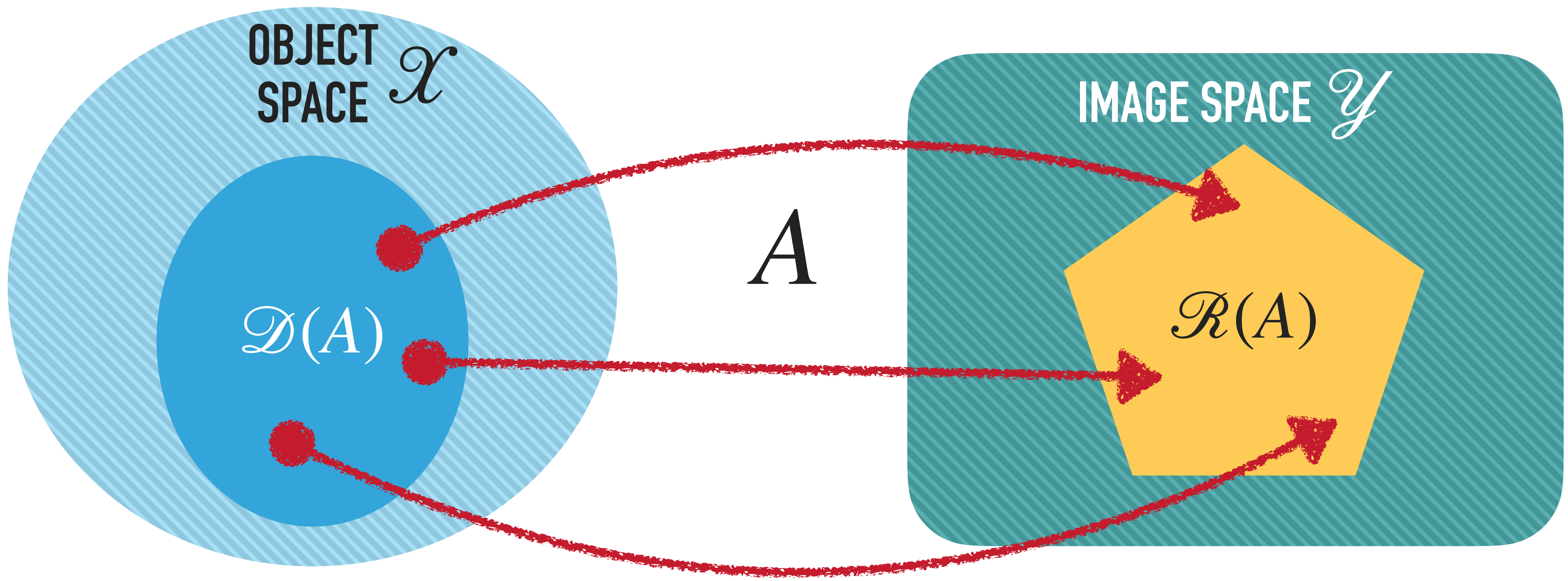
- RANGE SPACE
- NULL SPACE
- ADJOINT
- INVERSE

FUNCTION SPACES TREATMENT

KEY SPACES FOR IMAGING SYSTEMS

- Imaging system described by operator $A : \mathscr{X} \mapsto \mathscr{Y}$
  - Maps $f \in \mathrm{dom}(A)$ to $g \in \mathrm{Range}(A)$
  - Notation: $A(f) = g$
- Key spaces:
  - Object space
  - Image space
  - Domain (e.g. square integrable functions, the $\mathscr{L}^2-$space)
  - Range space
  - Null space

KEY SPACES FOR IMAGING SYSTEMS



- Consider pair of Hilbert spaces
  - $\mathscr{X}$ constitutes the **object space**, $\mathscr{Y}$ forms the **image space**
- Operator A maps an element of a subset of $\mathscr{X}$, i.e. $\mathrm{dom}(A)$, to an element of $\mathscr{Y}$, i.e. $\mathrm{range}(A)$
  - **Domain:** $\mathscr{D}(A) = \mathrm{dom}(A)$
  - **Range:** $\mathscr{R}(A) = \mathrm{range}(A)$

NULL SPACE

- The **null space** of a linear operator A (denoted as $\mathcal{N}(A)$)
    - The set of all elements f, such that $Af = 0$,

$$\mathcal{N}(A) = \left\{f \in \mathcal{X} : Af = 0\right\}$$

    - Important for understanding inverse problems
    - The set of invisible objects

ADJOINT OPERATOR

- The **adjoint** operator, denoted A^*, of a linear bounded operator A
  - Is the unique operator such that

$$\langle Af, g \rangle_{\mathcal{Y}} = \langle f, A^* g \rangle_{\mathcal{X}}$$

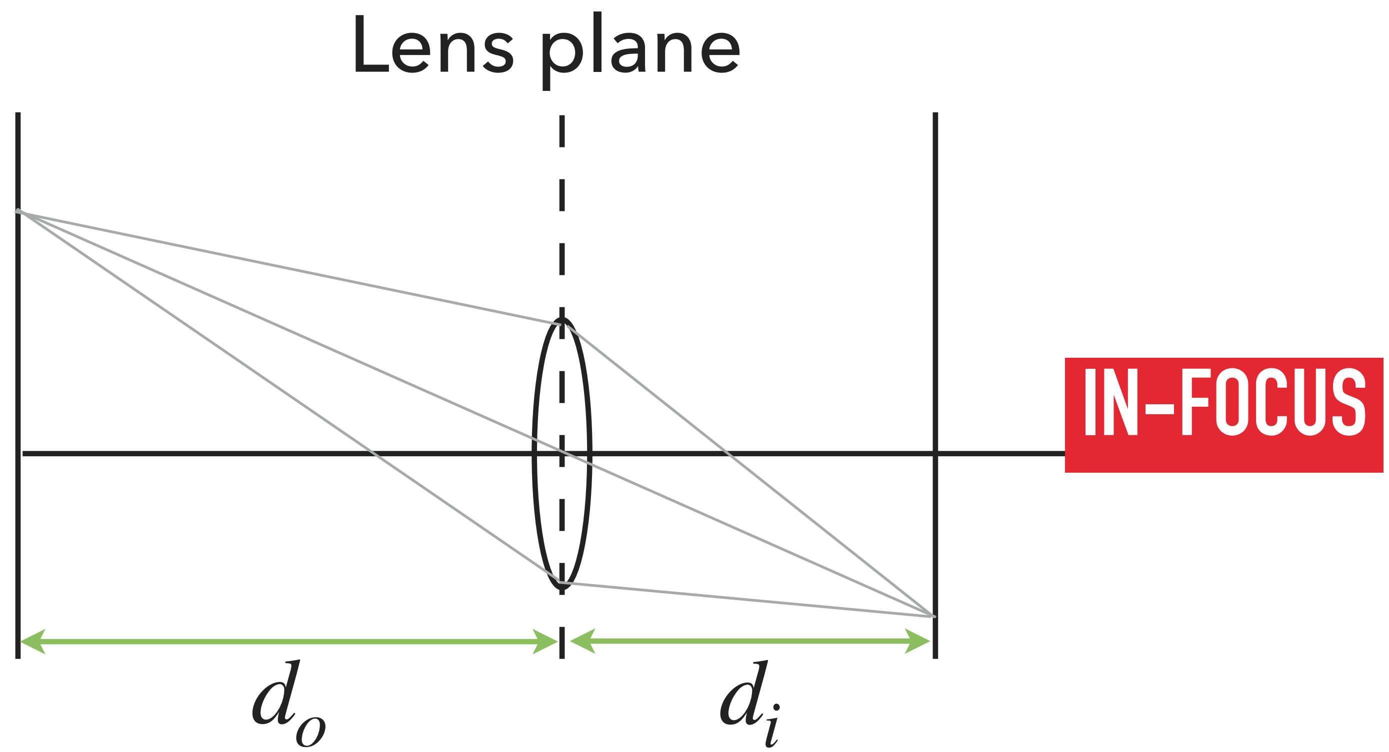
- **Self-adjoint:** $A = A^*$
- Recall that: the adjoint operator is a generalization of the **Hermitian** (complex conjugate, transpose) of a matrix

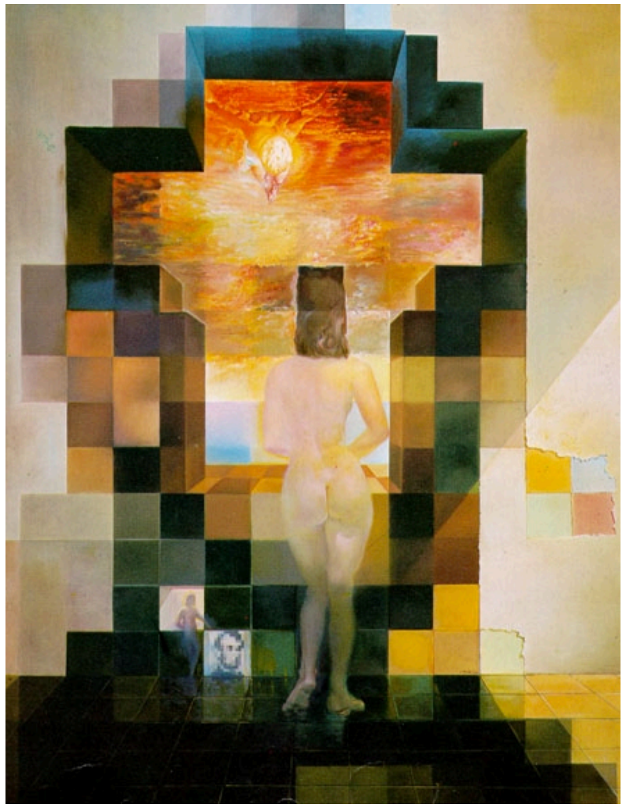
LSI IMAGING SYSTEMS

- CAMERA
- TOMOGRAPHY
- OPTICAL MICROSCOPY
- HOLOGRAPHY

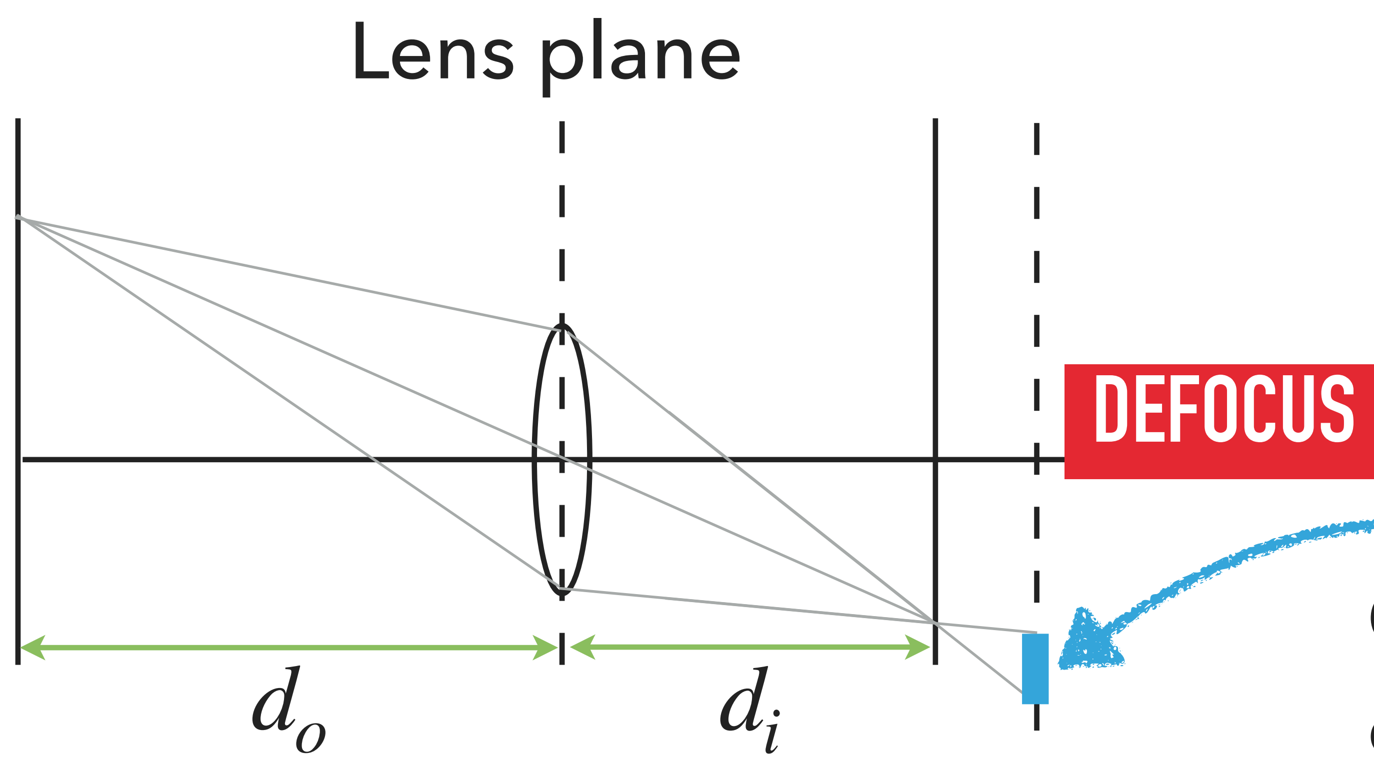
EXAMPLES

CAMERA



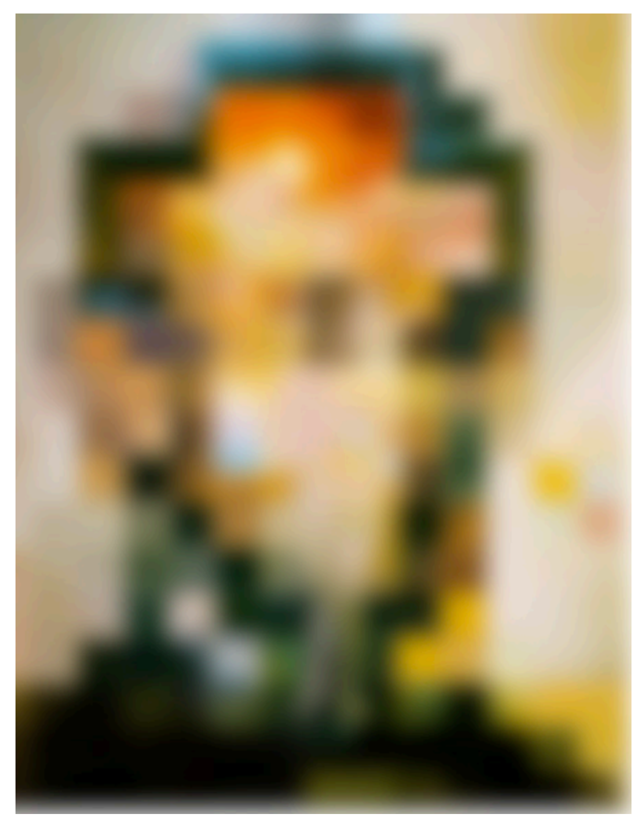


CAMERA



Circle of confusion

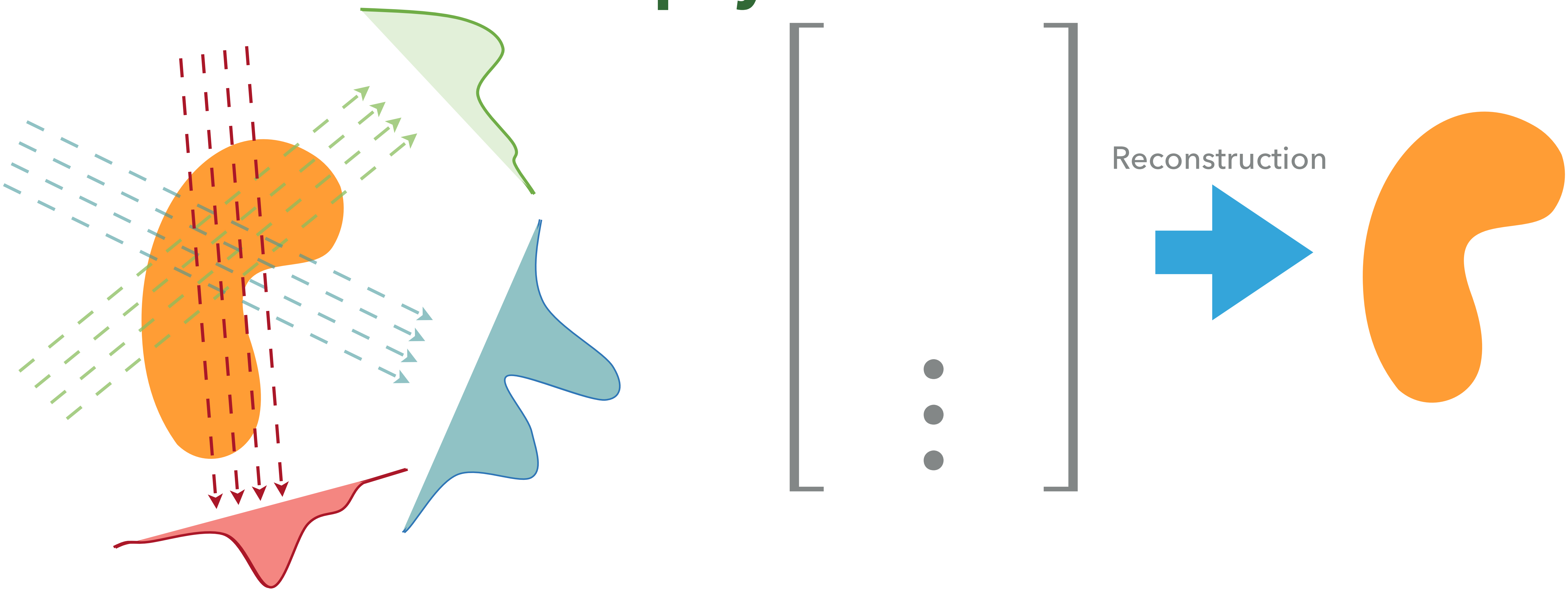
It is so-called because the light is no longer focussed a single point, but is instead spread over some larger circular region.



TOMOGRAPHY

Tomo - "to slice"

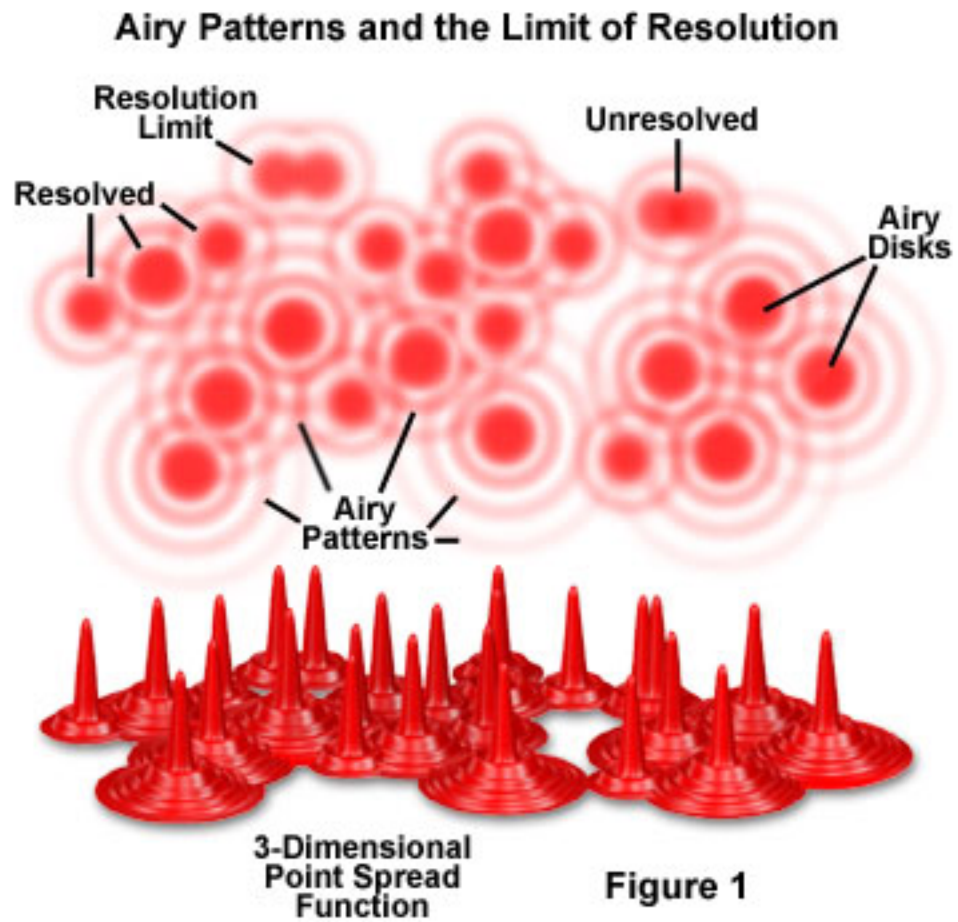
Graphy - "to write"



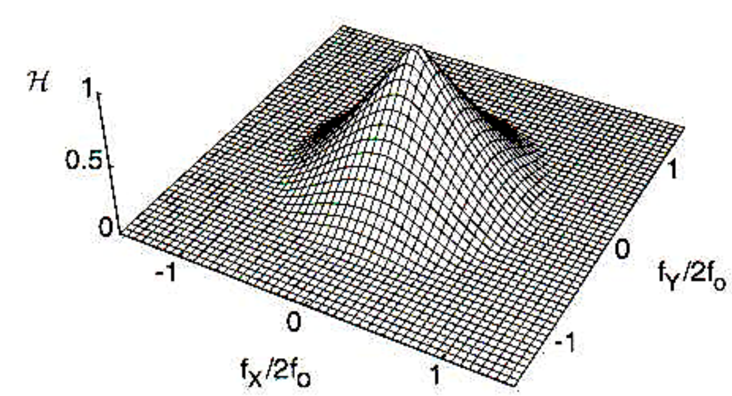
- Take many 1D projections of a 2D object
- Reconstruct original 2D object from measurement of those measured 1D projections

- The operator A maps from object space $\mathscr{X}$ to image space $\mathscr{Y}$
  - What is the object space $\mathscr{X}$ and image space $\mathscr{Y}$?
  - Is the operator A linear?

OPTICAL MICROSCOPY
EXAMPLE OF SHIFT-INVARIANT SYSTEM



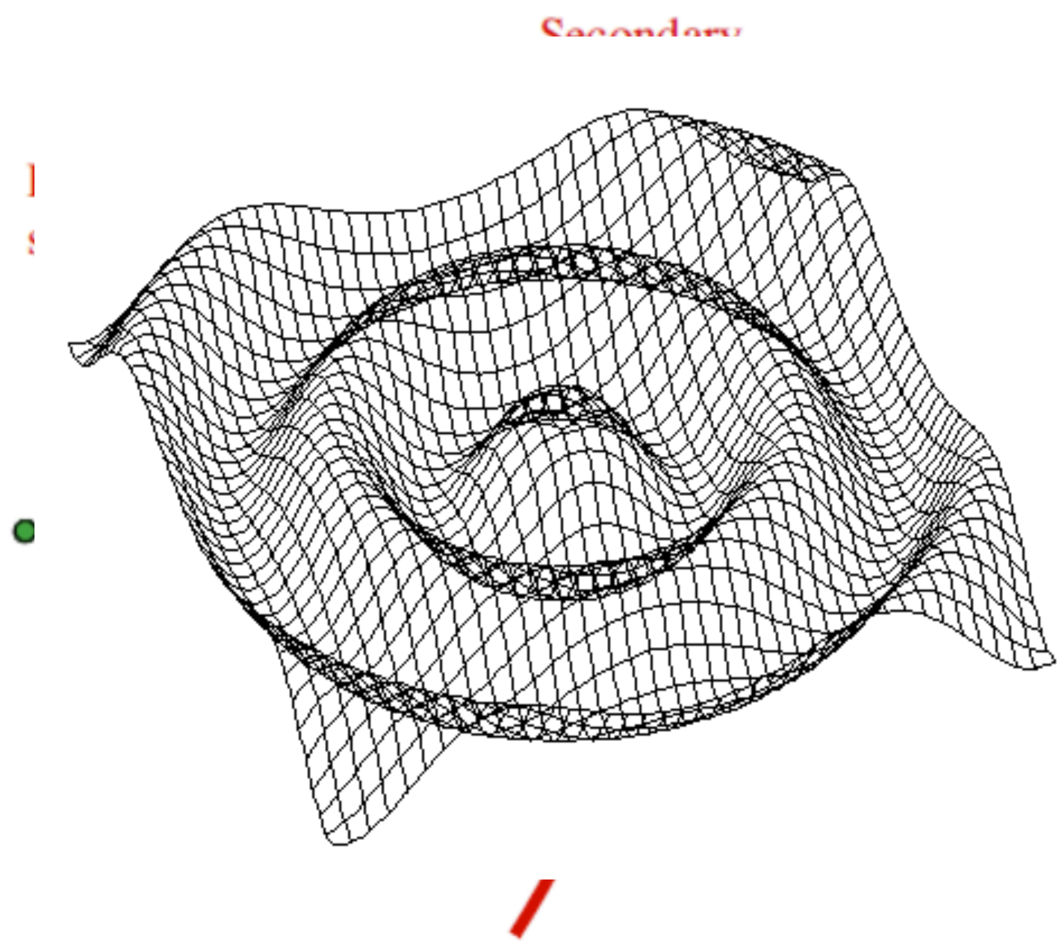
Optical Transfer function



- Operator A maps from object space $\mathscr{X}$ to image space $\mathscr{Y}$
  - What is the object space $\mathscr{X}$ and image space $\mathscr{Y}$?
  - Is the operator A linear?

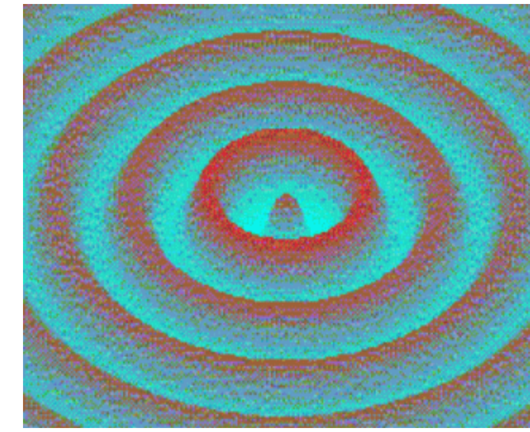
WAVE PROPAGATION

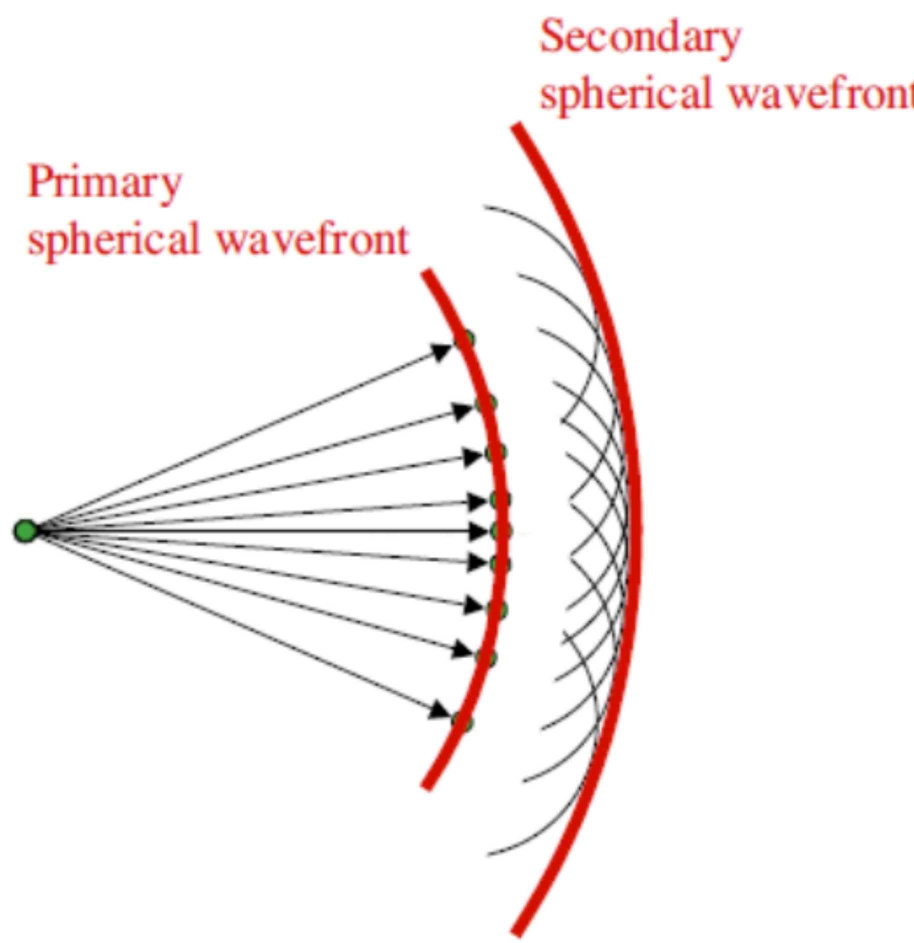
EXAMPLE OF SHIFT INVARIANT SYSTEM

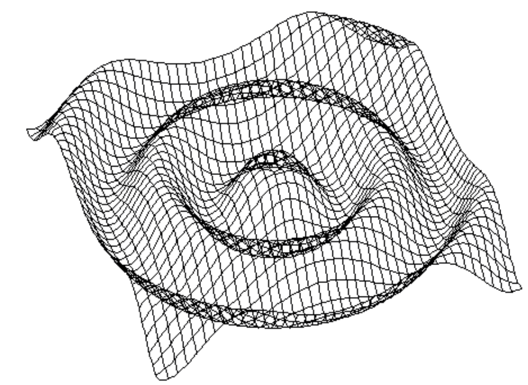


$$g_{\text{out}}(x, y; z) = \frac{e^{\text{j}kz}}{\text{j}\lambda z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_{\text{in}}(x', y') e^{\text{j}k\frac{(x-x')^2 + (y-y')^2}{2z}} \, \text{d}x' \text{d}y'$$

WAVE PROPAGATION
EXAMPLE OF SHIFT INVARIANT SYSTEM

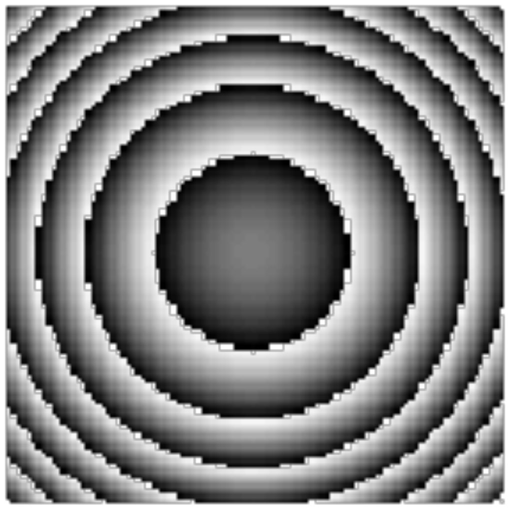






$$g_{\text{out}}(x, y; z) = \frac{e^{jkz}}{j\lambda z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_{\text{in}}(x', y') e^{jk\frac{(x-x')^2 + (y-y')^2}{2z}} \mathrm{d}x' \mathrm{d}y'$$

HOLOGRAPHY



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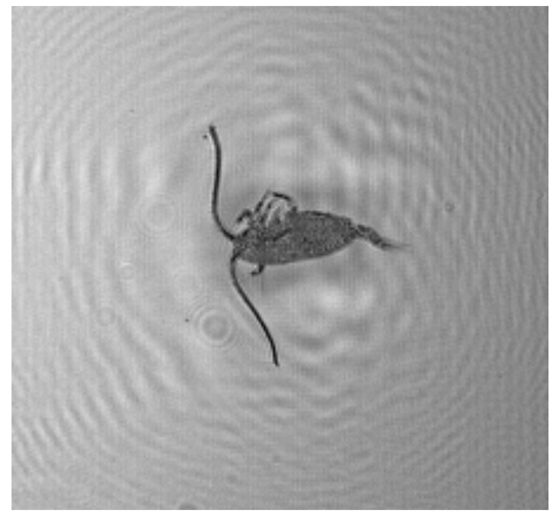

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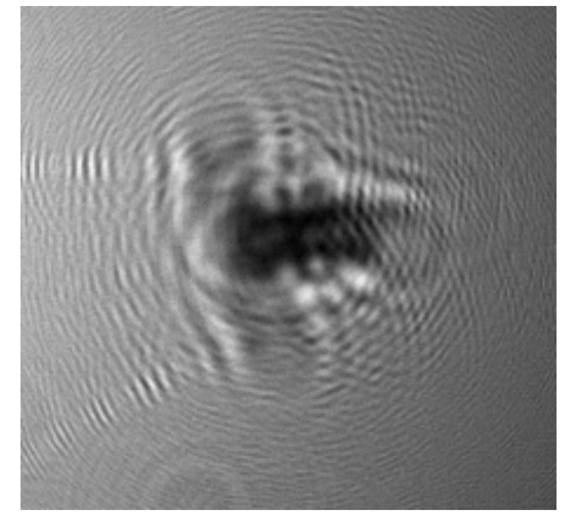


$$h_z(x, y) = \frac{e^{jkz}}{j\lambda z} e^{jk\frac{(x^2+y^2)}{2z}}$$

$g_{\text{in}}(x, y)$

$g_{\text{out}}(x, y)$

$$g_{\text{out}}(x, y; z) = \frac{e^{jkz}}{j\lambda z} \int\int g_{\text{in}}(x', y') e^{jk\frac{(x-x')^2+(y-y')^2}{2z}} \mathrm{d}x'\mathrm{d}y'$$

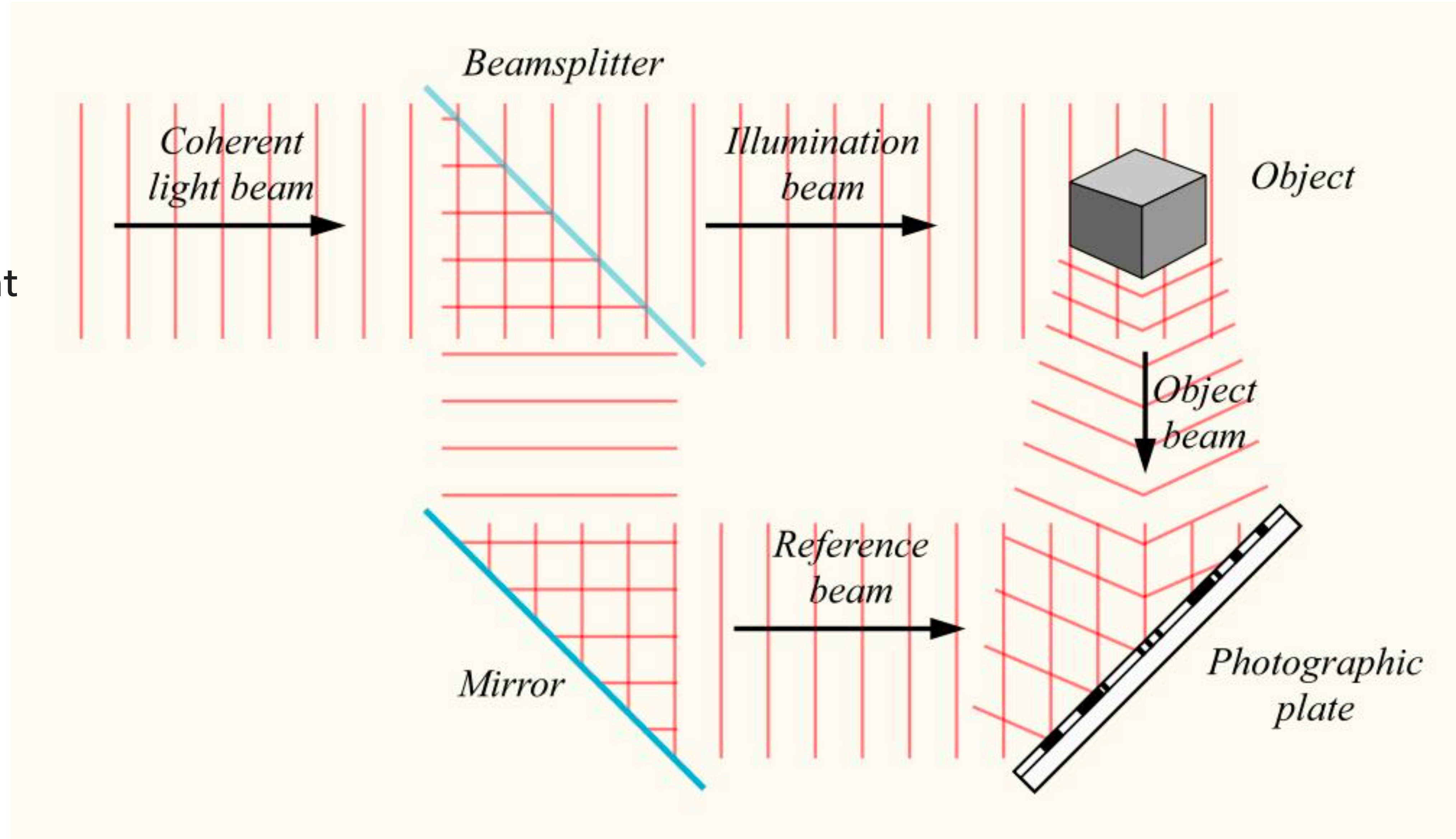
- **Optical Transfer Function:** the FT of $h_z(x, y)$

$$H_z(\omega_x, \omega_y) = e^{j2\pi z/\lambda} e^{-j\lambda z(\omega_x^2+\omega_y^2)}$$

- Range and null spaces?
- Adjoint and inverse operators?

RECORDING A HOLOGRAM

Laser beam is coherent

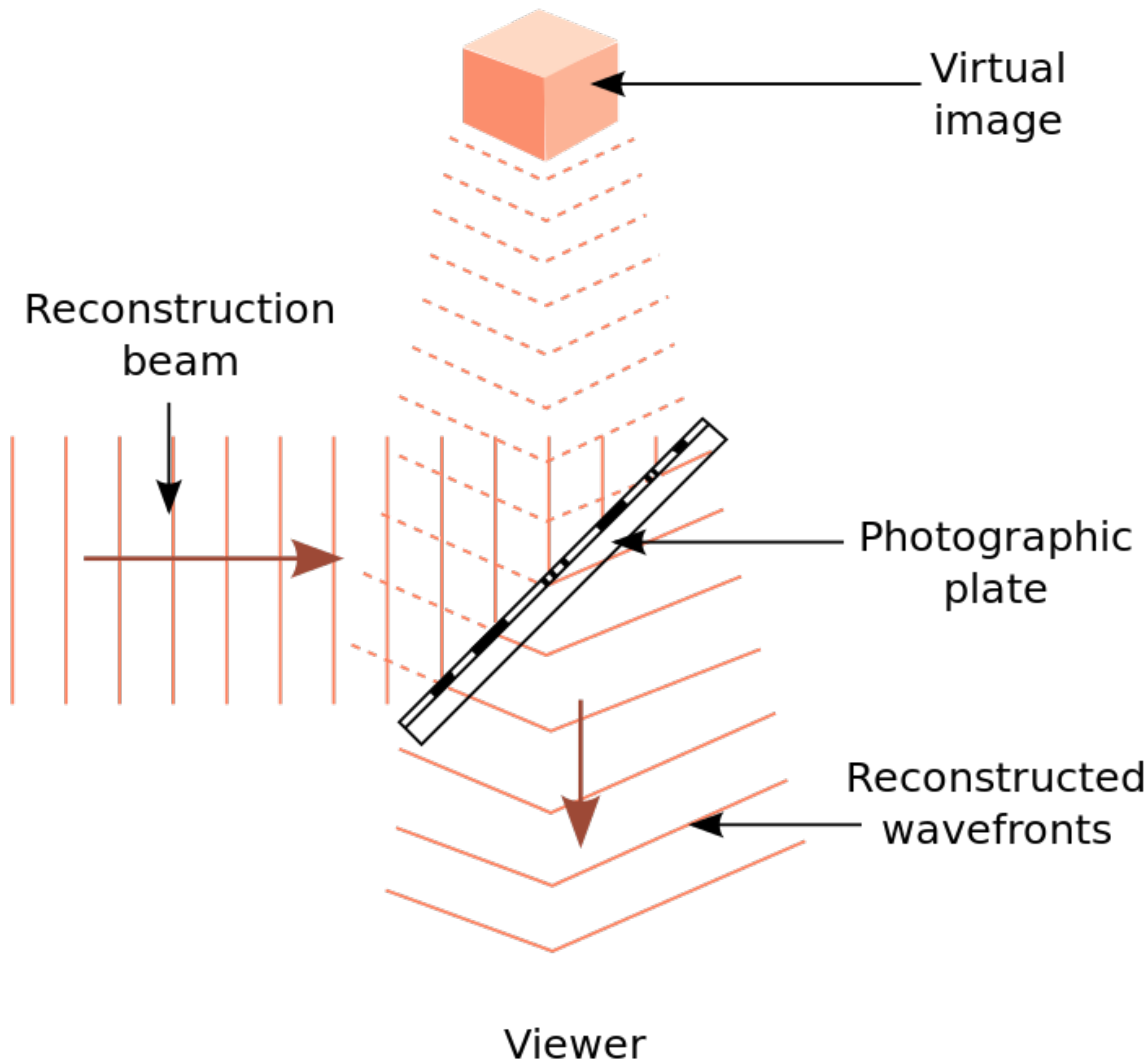


$$H_z(\omega_x, \omega_y) = e^{j2\pi z/\lambda} e^{-j\lambda z(\omega_x^2+\omega_y^2)}$$

[Image credit: Bob Mellish / Creative Commons]

RECORDING A HOLOGRAM

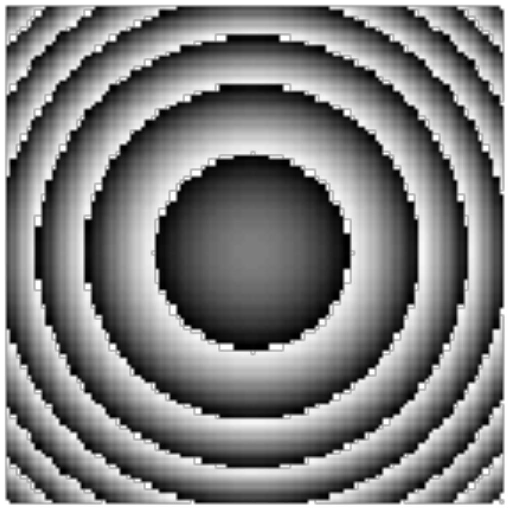
RECONSTRUCTION



[Image credit: Wikimedia]

HOLOGRAPHY: DIGITAL BACK-PROJECTION

MATLAB



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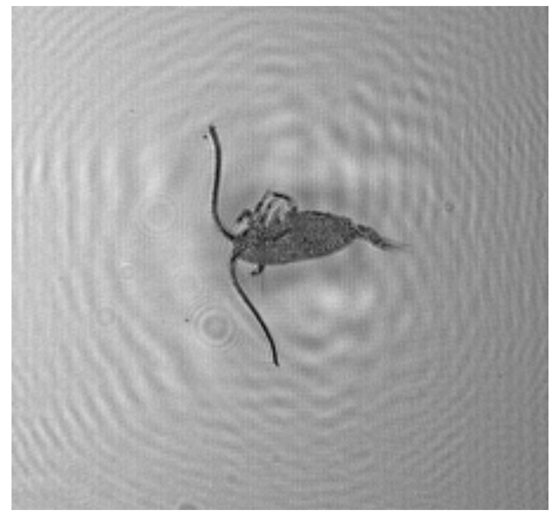

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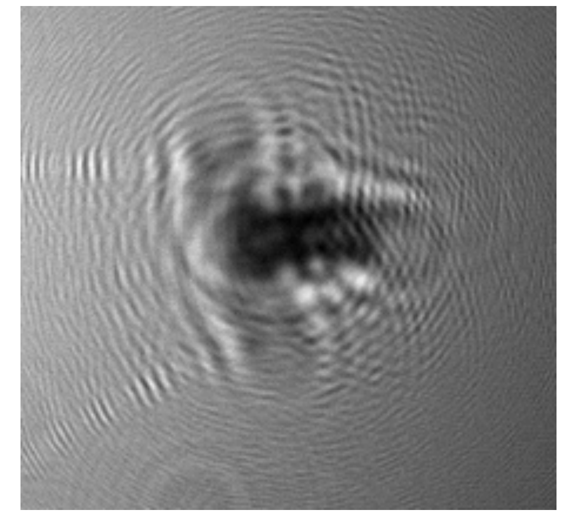


$$h_z(x, y) = \frac{e^{jkz}}{j\lambda z} e^{jk\frac{(x^2+y^2)}{2z}}$$

$g_{\text{in}}(x, y)$ $g_{\text{out}}(x, y)$

$$g_{\text{out}}(x, y; z) = \frac{e^{jkz}}{j\lambda z} \iint g_{\text{in}}(x', y') e^{jk\frac{(x-x')^2+(y-y')^2}{2z}} \, dx'dy'$$

- **Optical Transfer Function:** the FT of $h_z(x, y)$

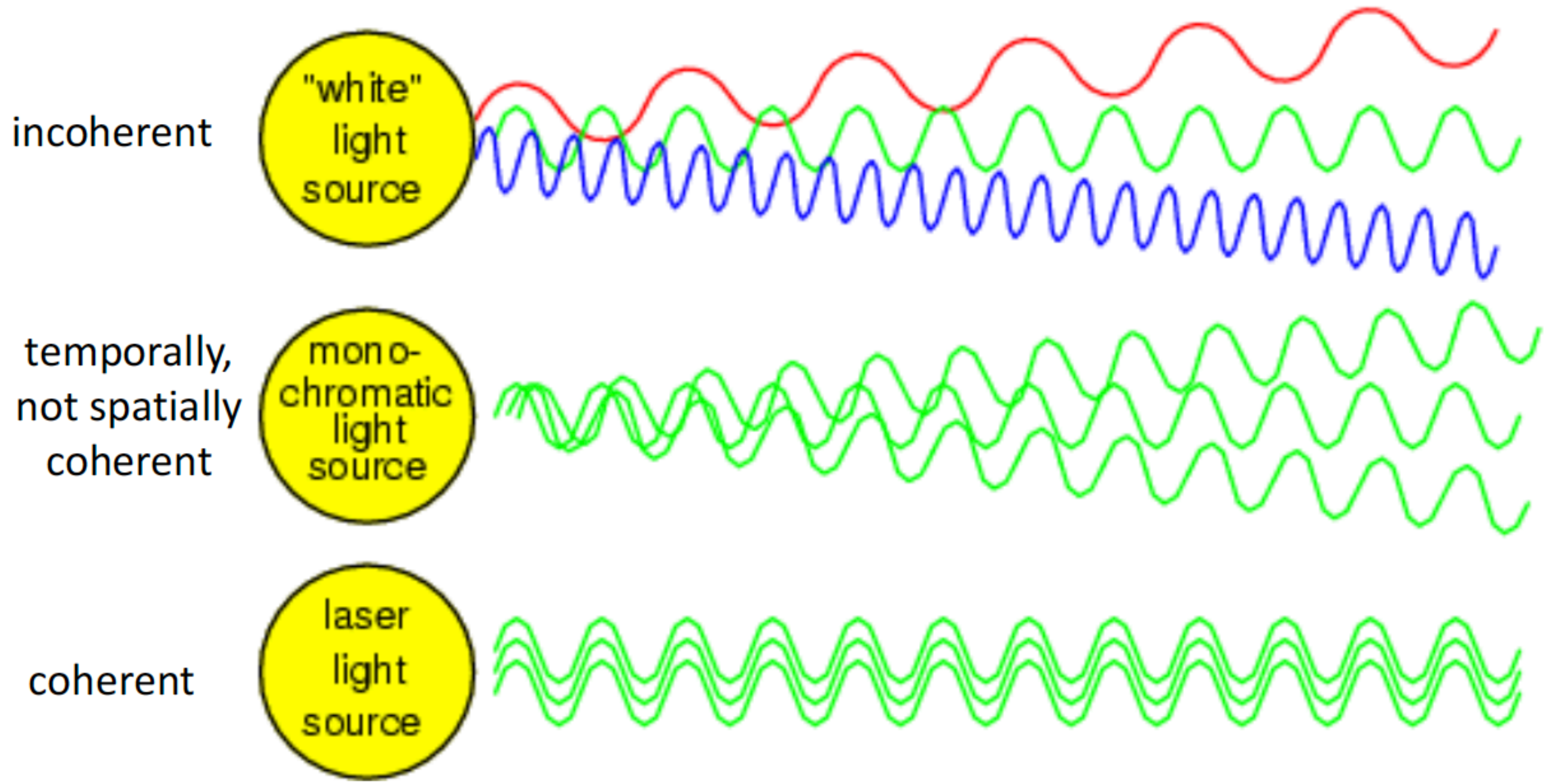
$$H_z(\omega_x, \omega_y) = e^{j2\pi z/\lambda} e^{-j\lambda z(\omega_x^2+\omega_y^2)}$$

- Range and null spaces?
- Adjoint and inverse operators?

MAKE
$z \longrightarrow (-z)$

RECONSTRUCTION

TYPES OF COHERENCE



[Image credit: Lei Tian (BU)]

WHAT WE COVERED TODAY

- **Convolution operator**
  - Properties - function spaces view
- **LSI imaging systems**
  - Examples: defocus, tomography, digital holography
- **Matlab**
  -

NEXT TIME!

FORWARD PROBLEM IN MATRIX FORMS