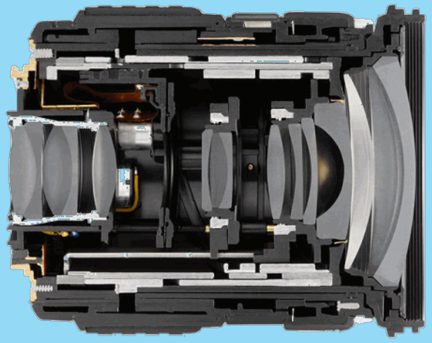
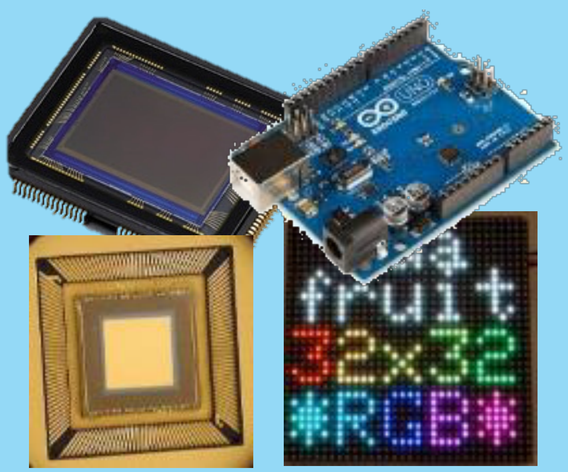


Optics

+



Sensors
&
devices

+

Signal
processing
&
algorithms

COMPUTATIONAL METHODS FOR IMAGING (AND VISION)

LECTURE 6: BASES

PROF. JOHN MURRAY-BRUCE

3 Multiplication by an orthogonal matrix

Consider the vector space $\mathbb{R}^n$ with standard norm and standard inner product. Prove that

(a) multiplication by an orthogonal matrix $\mathbf{U}$ preserves lengths, that is,

$$\|\mathbf{U}\mathbf{x}\| = \|\mathbf{x}\|,$$

for any $\mathbf{x}$.

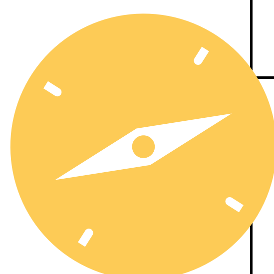
(b) multiplication by an orthogonal matrix $\mathbf{U}$ preserves angles, that is,

$$\langle \mathbf{U}\mathbf{x}, \mathbf{U}\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle,$$

for any $\mathbf{x}$ and $\mathbf{y}$.

WHERE ARE WE

WE ARE HERE!



Week	Date	Main Topic	Lecture	Readings	Homework Out	Homework Due
1	11-Jan-21		Introduction to computational imaging - Forward and Inverse problems - Common computational imaging problems			
	13-Jan-21		Vectors - Preliminaries			
2	18-Jan-21		**Dr. Martin Luther King, Jr. Holiday (no class)**			
	20-Jan-21	Mathematical preliminaries	Vectors and Vector Spaces - Subspaces, Finite dimensional spaces	IIP Appendix A; FSP 2.1 - 2.2		
3	25-Jan-21		Vector Spaces - Hilbert spaces	IIP Appendix B; FSP 2.3		
	27-Jan-21		Bases and Frames I - Orthonormal and Reisz Bases	IIP Appendix C; FSP 2.4 and 2.B	**HW 1**	
4	1-Feb-21		Bases and Frames II - Orthogonal Bases - Linear operators	IIP Appendix C; FSP 2.5 and 2.B		
	3-Feb-21		Fourier Analysis I - FT (1D and 2D) - FT properties	IIP 2.1, Appendix D; FSP 4.4		
5	8-Feb-21		Sampling and Interpolation - BL functions - Sampling	IIP 2.2, 2.3; FSP 5.4, 5.5		**HW 1**
	10-Feb-21		Fourier Analysis II (DFT)	IIP 2.4; FSP 3.6	**HW 2**	
6	15-Feb-21	Forward Modeling	LSI imaging: Forward problem I - Convolution	IIP 2.5 - 2.6, 3		
	17-Feb-21		LSI imaging: Forward problem I - Transfer functions	IIP 2.6		
7	22-Feb-21		LSI imaging: Forward problem I - Linear operators	IIP 3		
	24-Feb-21		LSI imaging: Forward problem I - Linear operators, Adjoints, and Inverses		**HW 3**	**HW 2**
8	1-Mar-21		**Mid-term Exams**			
	3-Mar-21		LSI imaging: Forward problem II - Sampling and Discretization: Matrix-vector form	IIP 2.7, 4		
9	8-Mar-21		LSI imaging: Forward problem II - Convolution matrix			
	10-Mar-21		LSI imaging: Forward problem II - Sampling and Discretization: Matrix-vector form - PSF, and Transfer functions			**HW3**

OUTLINE

- Bases (continued)
  - Riesz bases
  - Orthonormal bases
  - Biorthogonal bases
  - Matrix representation of linear operators

LEARNING GOALS

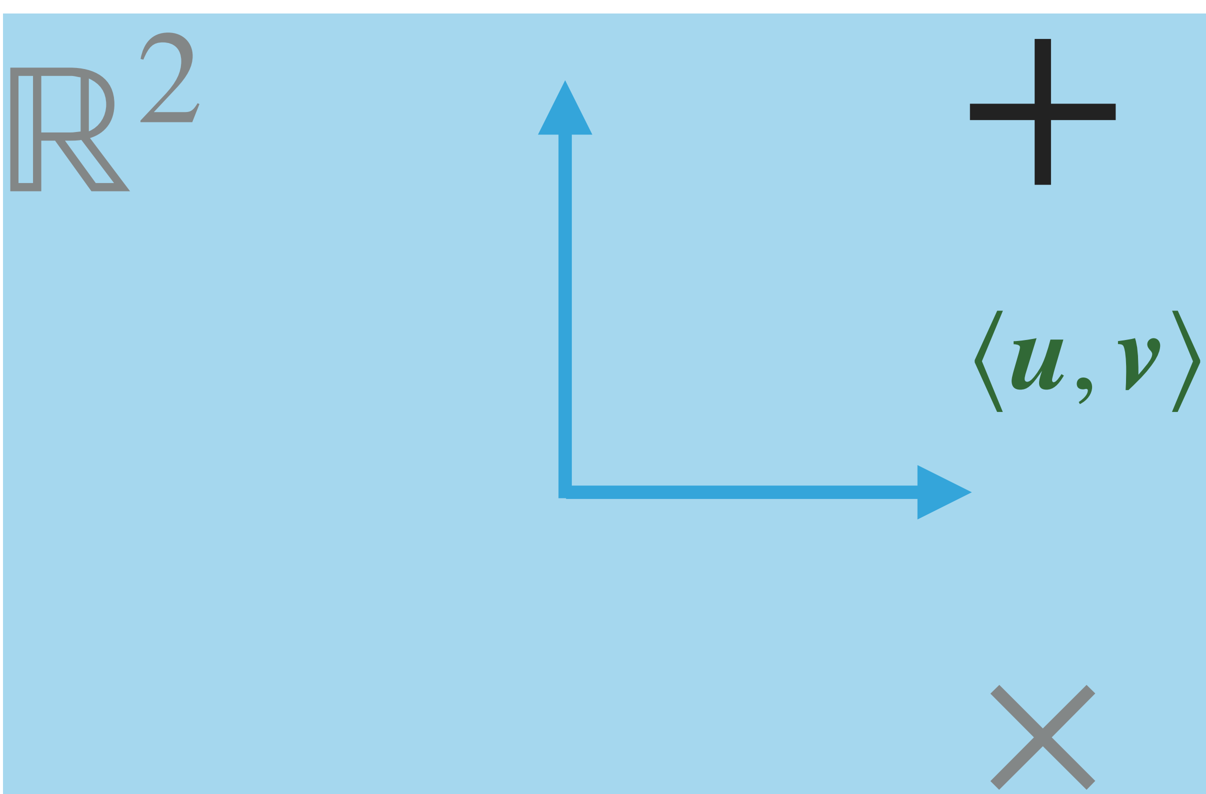
- Understand bases and related terminologies
- Linear operators and their matrix representations

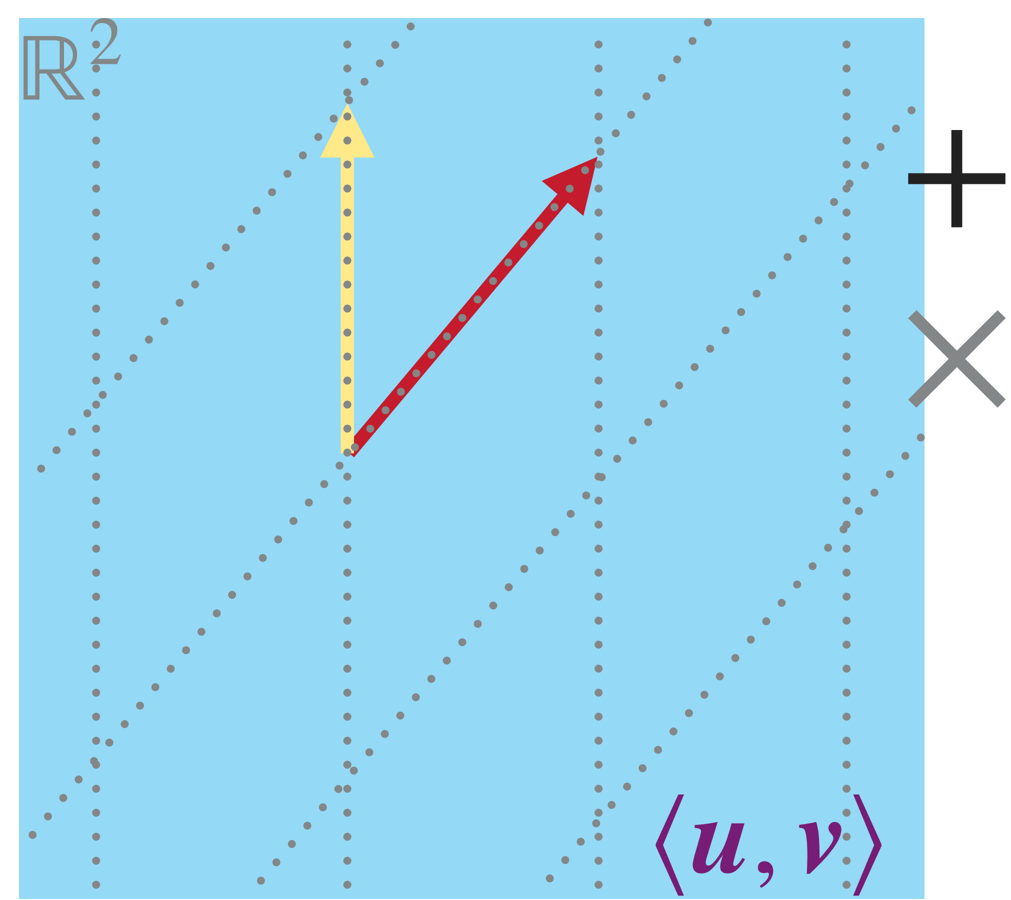
READING

- IIP Appendix C
- FSP 2.5

THOSE PESKY ARROWS (ARE BUILDING BLOCKS)

- The arrows are actually special – they form a **basis** for $\mathbb{R}^2$
  - There are called **Basis vectors**
- More generally they form a **basis** for a **Hilbert space.**





BASES

- **Problems:**
  1. What choices of arrows/vectors make a **“good” basis**?
     - ✓ **“Good”** here means **stable**!
  2. Given a **basis** how do we get the **expansion coefficients** of a vector? – **Analysis**
  3. Given expansion coefficients, how do we get back (synthesize) the original vector? – **Synthesis**

RIESZ BASES

ORTHOGONAL BASES

ORTHONORMAL BASES

BASIS

https://centonomy.com/you-cant-skip-the-basics/

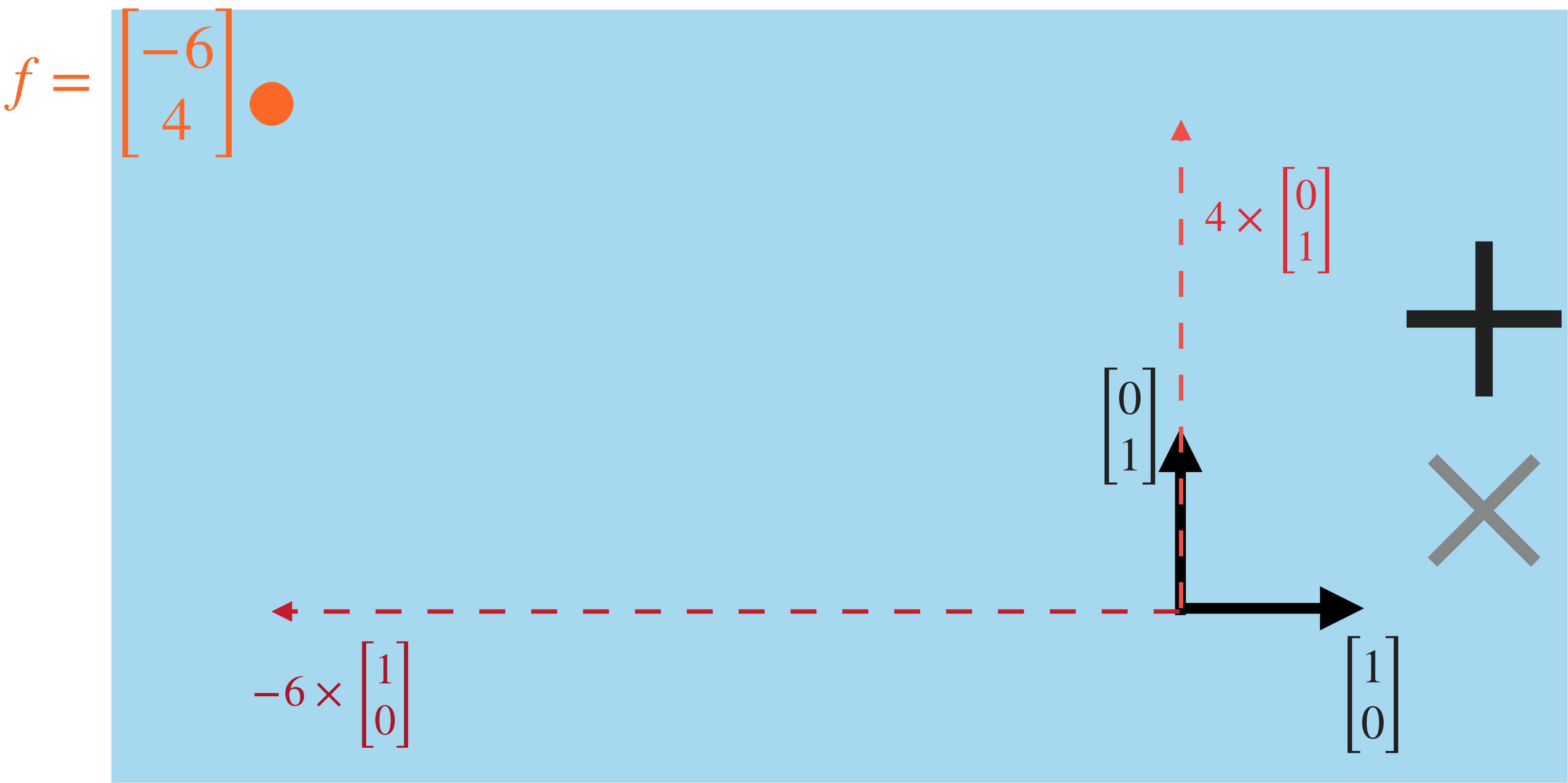
BASIS

- **Definition (Basis):** $\Phi = \{\varphi_k\}_{k\in\mathscr{K}} \subset V$ is a basis for V when
   1. Φ is **complete** in V, i.e. for any $f \in V$ there exists a sequence $\boldsymbol{\alpha} \in \mathbb{C}^{\mathscr{K}}$ such that $f = \sum_{k\in\mathscr{K}} \alpha_k \varphi_k$.
   2. For any $f \in V$, the sequence $\boldsymbol{\alpha}$ in the expansion above is **unique** with respect to $\Phi = \{\varphi_k\}_{k\in\mathscr{K}}$.

- $f = \sum_{k\in\mathscr{K}} \alpha_k \varphi_k$ is called the expansion of f in terms of $\Phi = \{\varphi_k\}_{k\in\mathscr{K}}$.
- α_k's are called the expansion coefficients

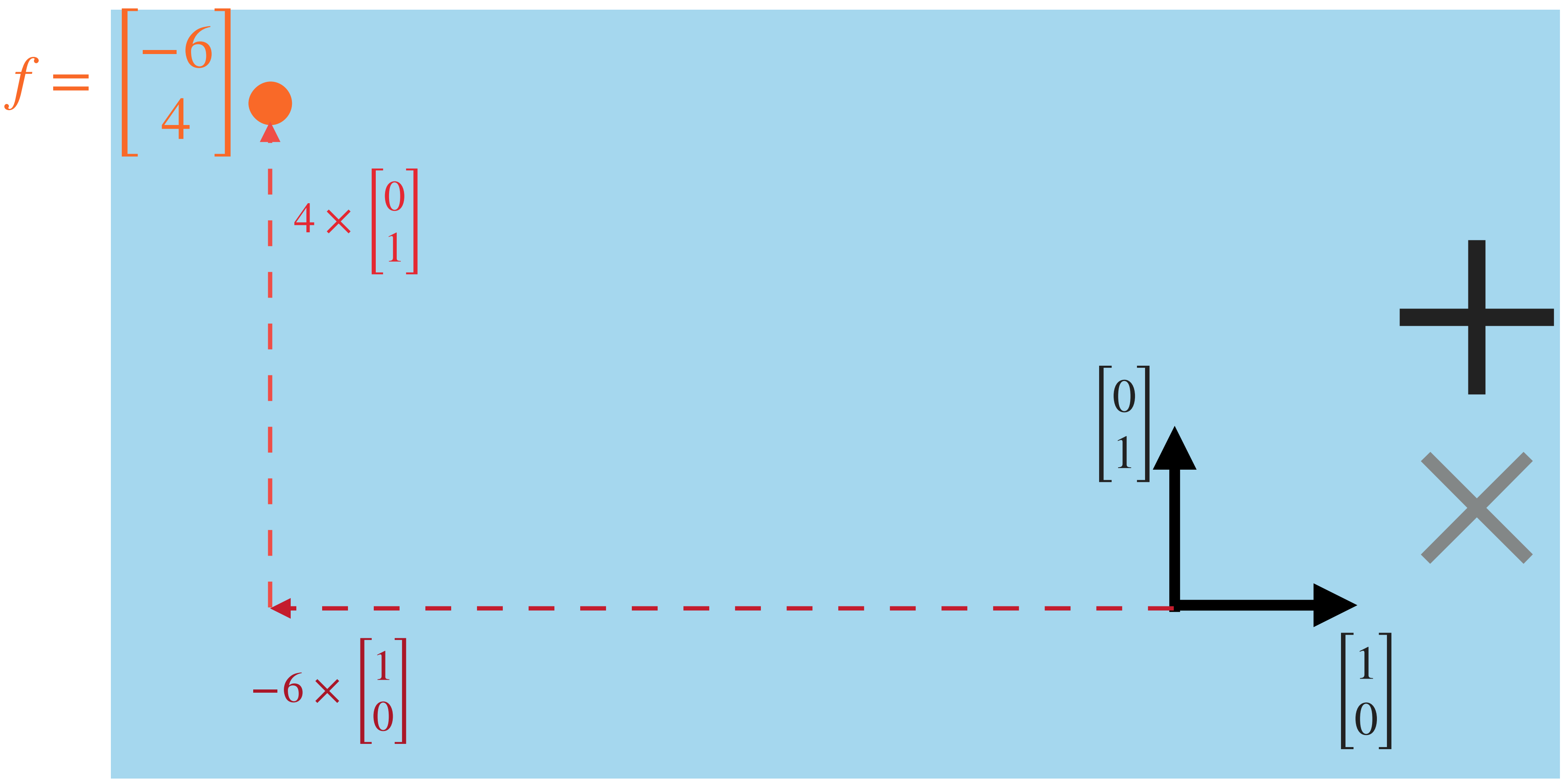
BASES
EXAMPLE

- **Canonical basis** for $\mathbb{R}^2$



BASES
EXAMPLE

- **Canonical basis** for $\mathbb{R}^2$



$$f = \sum_{k \in \mathscr{K}} \alpha_k \varphi_k = \sum_{k=1}^{2} \alpha_k \varphi_k = \alpha_1 \varphi_1 + \alpha_2 \varphi_2 = -6 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

BASES
MORE EXAMPLES

- **Standard basis for $\mathbb{R}^n$:**
  1. Basis vector is
  $$e_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad k = 1,\ldots,n$$
  2. Any vector $\boldsymbol{u} \in \mathbb{R}^n$ has the unique expansion $\boldsymbol{u} = \sum_{k=1}^{n} u_k e_k$

RIESZ BASES

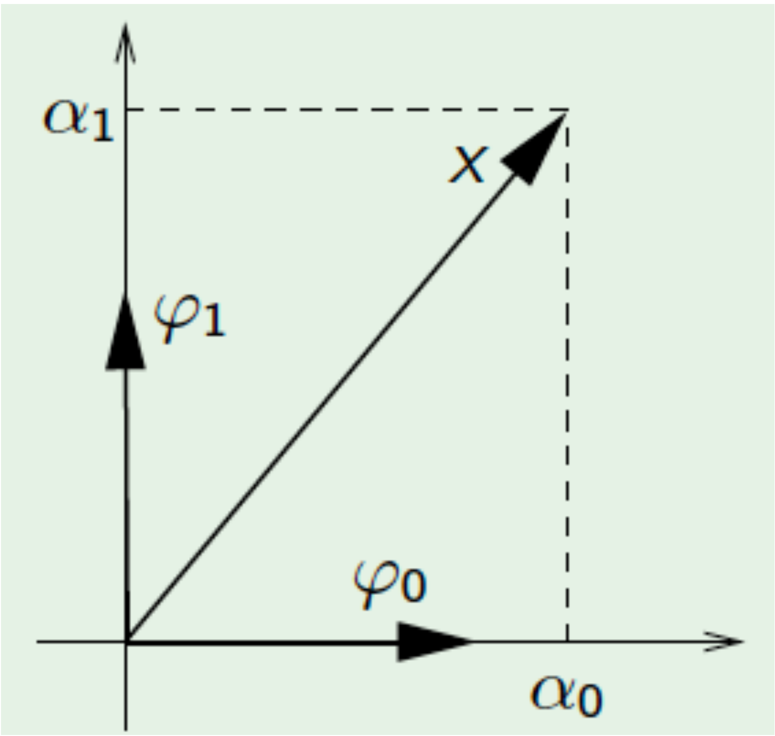
ORTHOGONAL BASES

ORTHONORMAL BASES

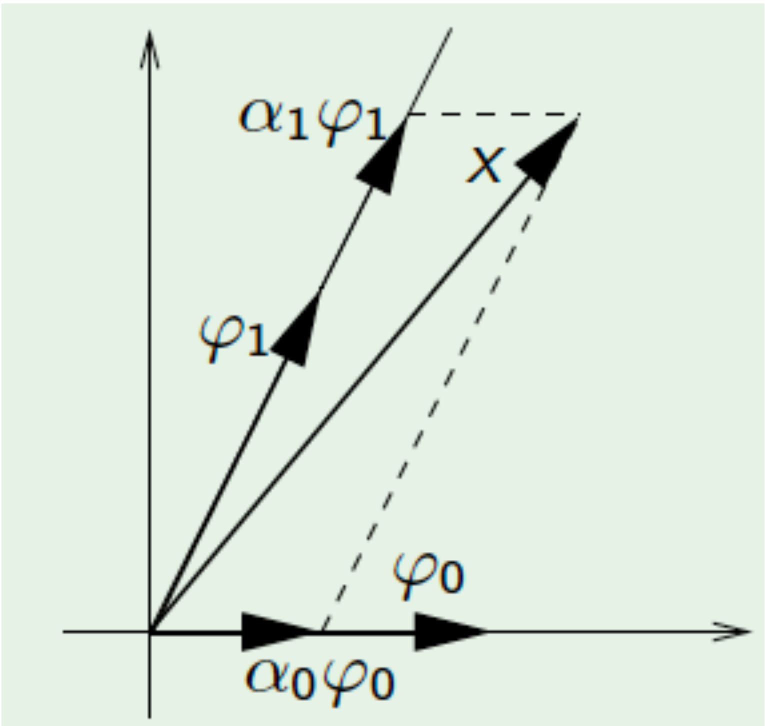
BASIS

https://centonomy.com/you-cant-skip-the-basics/

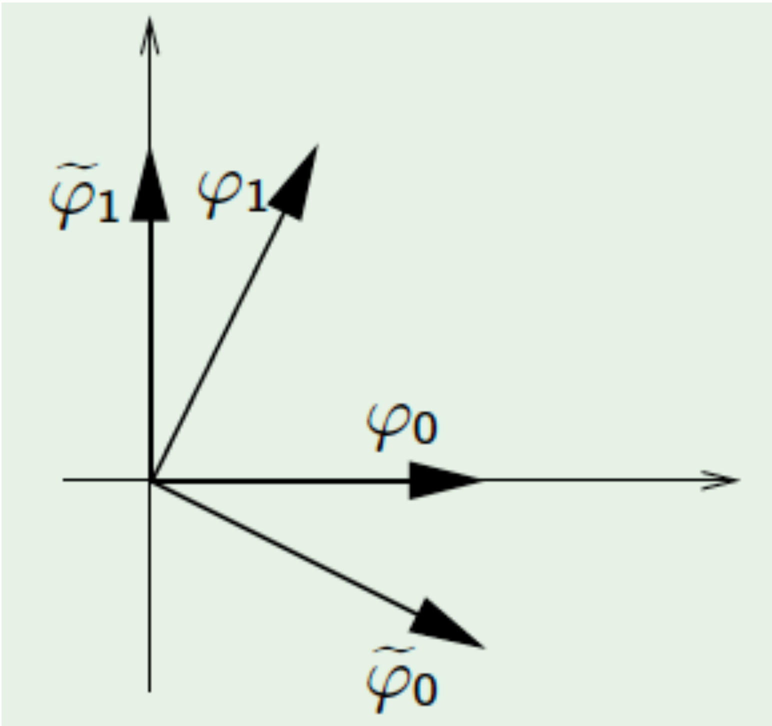
BASES
EXPANSIONS



Expansion in an orthonormal basis.



Expansion in a nonorthogonal basis.



Basis $\{\varphi_0, \varphi_1\}$ and its dual $\{\tilde{\varphi}_0, \tilde{\varphi}_1\}$.

RIESZ BASES

ORTHOGONAL BASES

ORTHONORMAL BASES

BASIS

- **Problems:**
  1. What choices of arrows/vectors make a **"good" basis**?
     - ✓ **"Good"** here means **stable**!
  2. Given a **basis** how do we get the **expansion coefficients** of a vector? – **Analysis**
  3. Given expansion coefficients, how do we get back (synthesize) the original vector? – **Synthesis**

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RIESZ BASES

- **Definition (Riesz Basis):** $\Phi = \{\varphi_k\}_{k \in \mathscr{K}}$ is a Riesz basis for the Hilbert space H when
  1. $\Phi = \{\varphi_k\}_{k \in \mathscr{K}}$ is a **Basis**
  2. There exists **stability constants** $\lambda_{\min}$ and $\lambda_{\max}$, with $0 < \lambda_{\min} \leq \lambda_{\max} < \infty$ such that for any $f \in H$ the basis expansion $f = \sum_{k \in \mathscr{K}} \alpha_k \varphi_k$ satisfies $\lambda_{\min} \|f\|^2 \leq \sum_{k \in \mathscr{K}} |\alpha_k|^2 \leq \lambda_{\max} \|f\|^2$

Numerical stability: when $\lambda_{\min} \approx \lambda_{\max}$

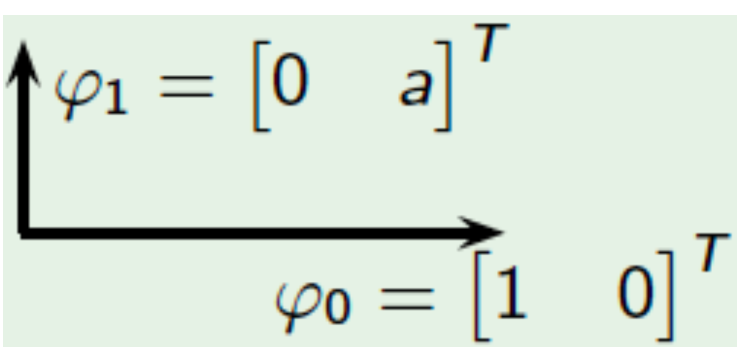
Excludes **poorly-conditioned** bases

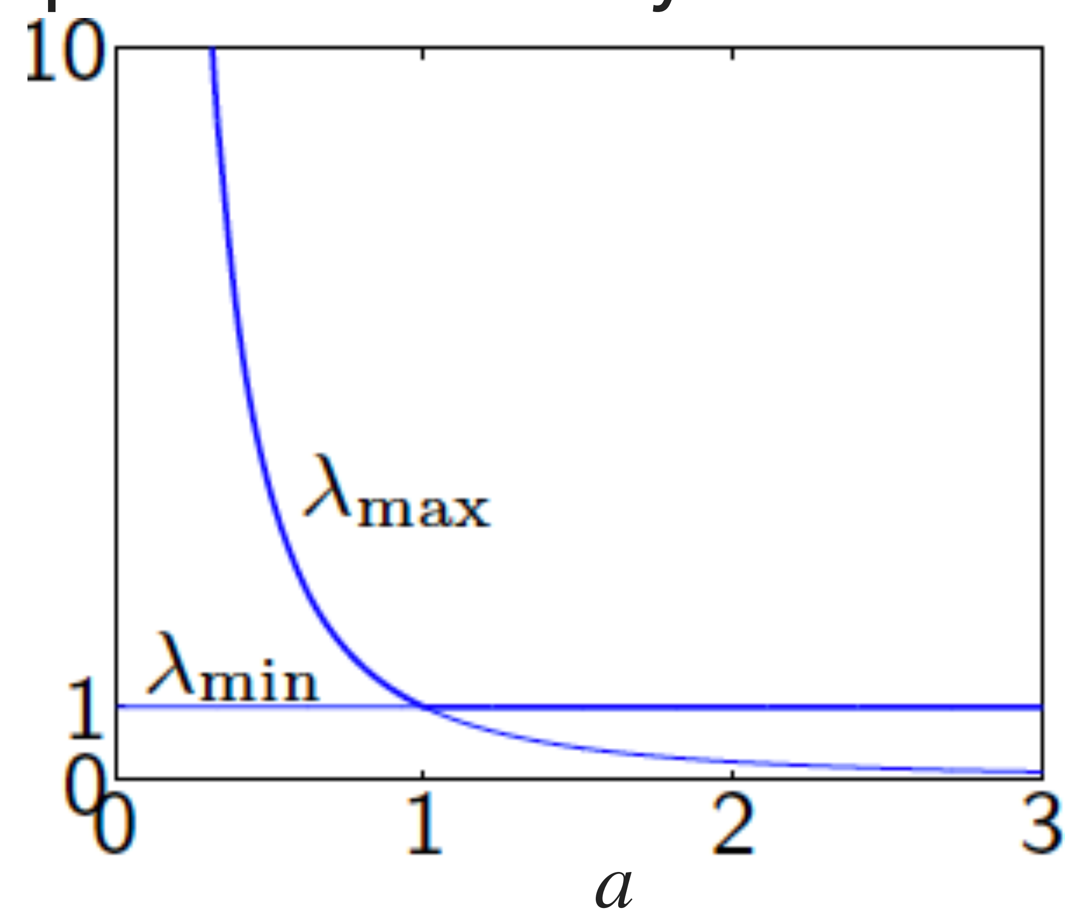
RIESZ BASES
STABILITY OF EXPANSIONS

MATLAB

Let $\Phi = \begin{bmatrix}\boldsymbol{\varphi}_1 & \boldsymbol{\varphi}_2 & \cdots & \boldsymbol{\varphi}_n\end{bmatrix}$ be a Riesz basis for $\mathbb{R}^n$ and let $G = \Phi^*\Phi$

- $1/\lambda_{\max}$ is the minimum eigenvalue of G
- $1/\lambda_{\min}$ is the maximum eigenvalue of G
- **Example:** Stability constants of bases parameterized by a





OPERATORS ASSOCIATED WITH RIESZ BASES

SYNTHESIS OPERATOR

- **Definition (Basis Synthesis Operator):** the synthesis operator of a basis for the Hilbert space

$$\Phi : \ell^2(\mathscr{K}) \to H \text{ with } \Phi\boldsymbol{\alpha} = \sum_{k \in \mathscr{K}} \alpha_k \varphi_k$$

OPERATORS ASSOCIATED WITH RIESZ BASES

ANALYSIS OPERATOR

- **Definition (Basis Analysis Operator):** For a basis $\Phi = \{\varphi_{k\in\mathscr{K}}\}$ of a Hilbert space H, the analysis operator is:

  $$\Phi^* : H \to \ell^2(\mathscr{K}) \text{ with } (\Phi^* f)_k = \langle f, \varphi_k \rangle,\ k \in \mathscr{K}$$

- **Analysis** operator is the **adjoint of the synthesis** operator. **Why?**

  Adjoint derivation: For any $\alpha \in \ell^2(\mathcal{K})$ and $y \in H$,

  $$\langle \Phi\alpha, y \rangle = \left\langle \sum_{k\in\mathcal{K}} \alpha_k \varphi_k, y \right\rangle = \sum_{k\in\mathcal{K}} \alpha_k \langle \varphi_k, y \rangle = \sum_{k\in\mathcal{K}} \alpha_k \langle y, \varphi_k \rangle^*$$

- **Recall:** adjoint is a generalization of matrix (Hermitian) transpose.

RIESZ BASES

ORTHOGONAL BASES

ORTHONORMAL BASES

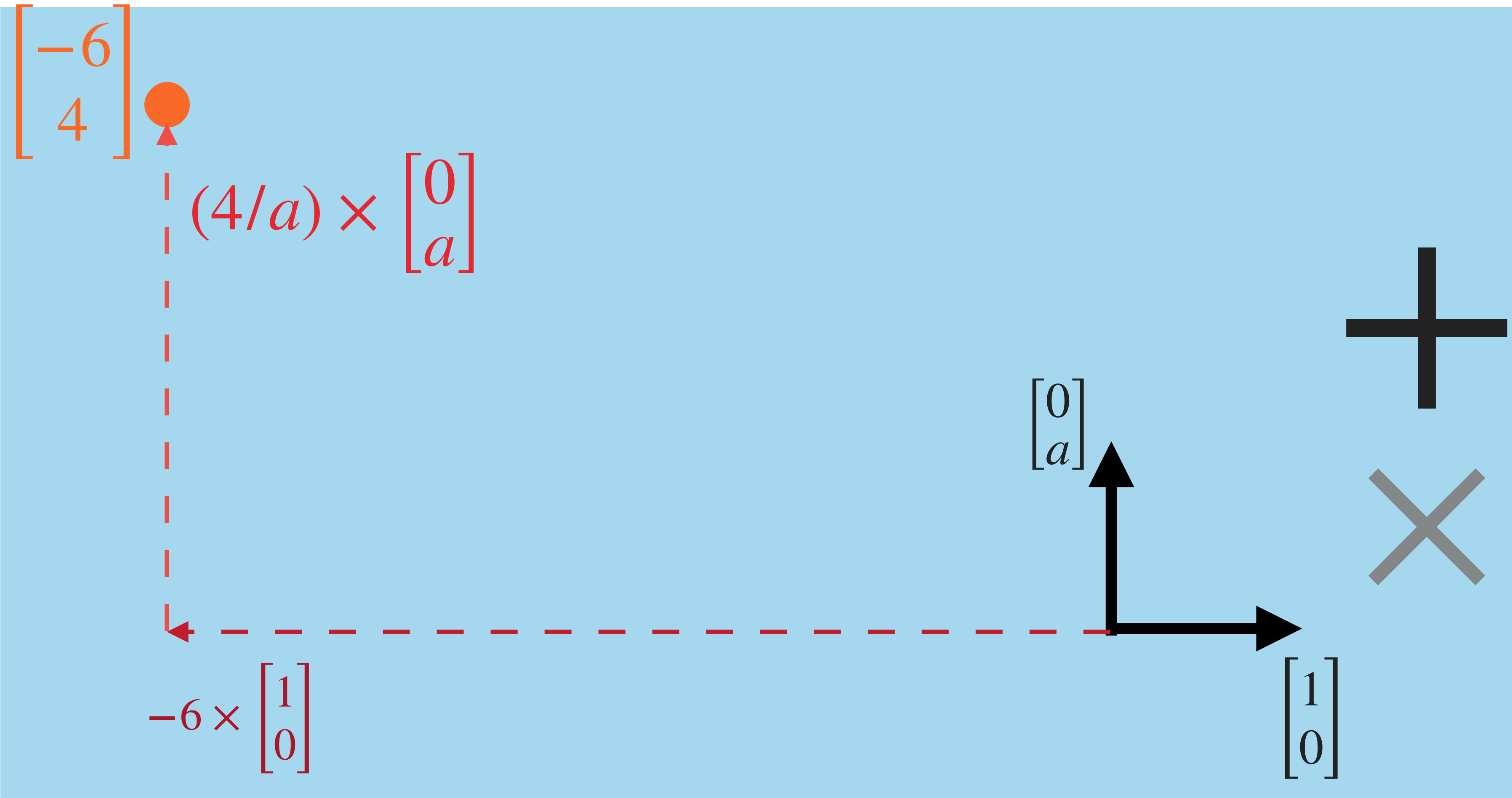
BASIS

https://centonomy.com/you-cant-skip-the-basics/

ORTHOGONAL BASIS

- **Definition (Orthogonal Basis):** $\Phi = \{\varphi_k\}_{k \in \mathcal{K}} \subset H$ is an orthogonal basis for H when
  1. Φ is a **basis** for and
  2. Φ is an **orthogonal set**, i.e. $\langle \varphi_k, \varphi_k \rangle > 0$, for all $k \in \mathcal{K}$ and $\langle \varphi_i, \varphi_k \rangle = 0$, for $i \neq k$ with $i, k \in \mathcal{K}$.
- **Key facts:**
  - If Φ is an orthogonal set, then it is linearly independent

EXAMPLE
BASIS FOR $\mathbb{R}^2$



$$\begin{bmatrix}-6\\4\end{bmatrix} = (-6)\begin{bmatrix}1\\0\end{bmatrix} + (4/a)\begin{bmatrix}0\\a\end{bmatrix}$$

Questions:

1. What are the basis vectors?
2. Which is the expansion?
3. What are the expansion coefficients?

RIESZ BASES

ORTHOGONAL BASES

ORTHONORMAL BASES

BASIS

https://centonomy.com/you-cant-skip-the-basics/

ORTHONORMAL BASIS

- **Definition (Orthonormal Basis):** $\Phi = \{\varphi_k\}_{k \in \mathscr{K}} \subset H$ is an orthonormal basis for H when
  1. Φ is a **basis** for and
  2. Φ is an **orthonormal set**, i.e. $\langle \varphi_i, \varphi_k \rangle = \delta_{i-k}$ for all $i, k \in \mathscr{K}$
- **Key facts:**
  - If Φ is an orthogonal set, then it is linearly independent
  - If $\overline{\mathrm{span}}(\Phi) = H$ and Φ is an orthogonal set, then Φ is an orthogonal basis for H
    - We have an orthonormal basis if, in addition, $\|\varphi_k\| = 1$

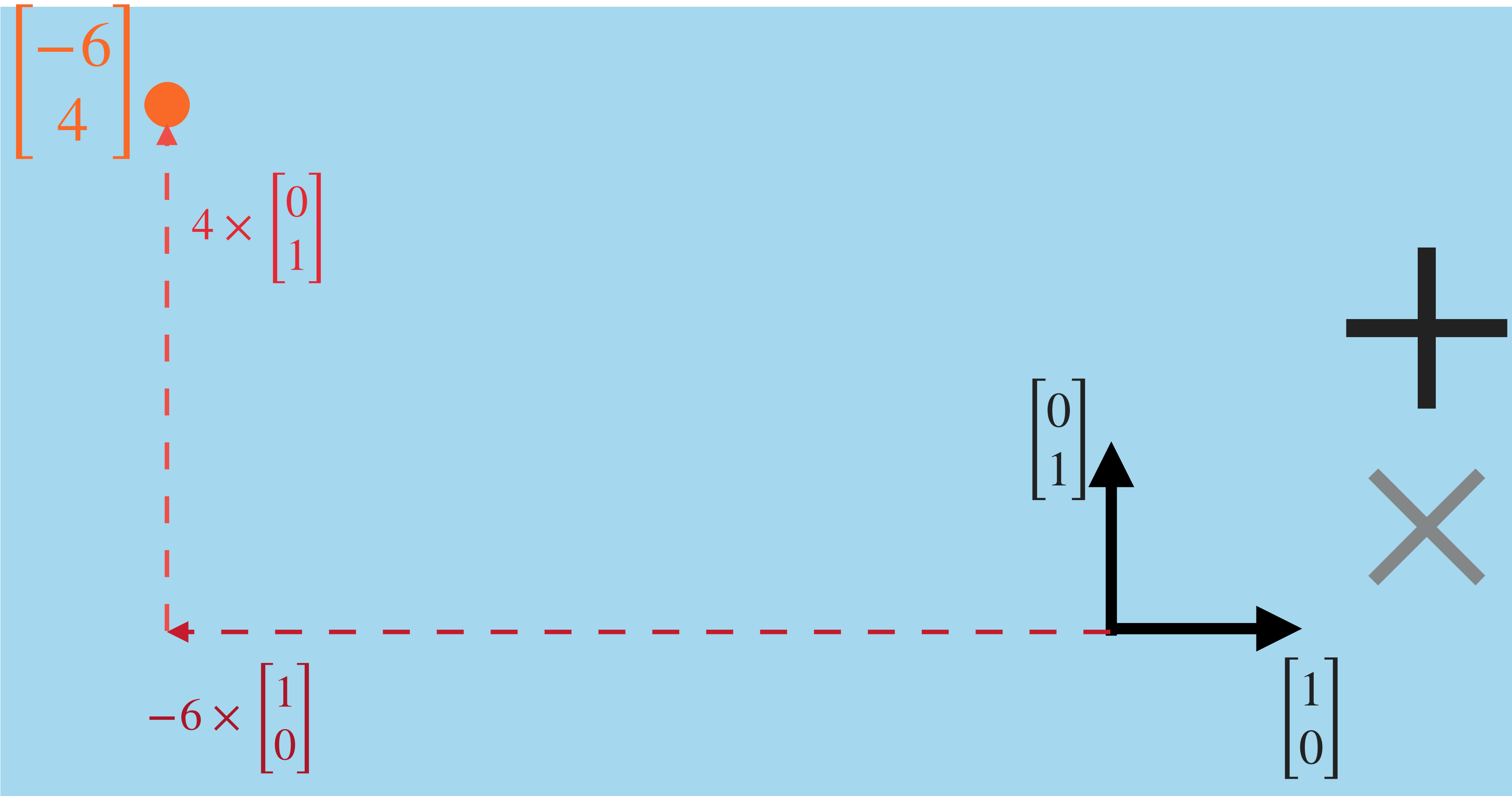
THE (KRONECKER) DELTA FUNCTION

$$\langle \varphi_i, \varphi_k \rangle = \delta_{i-k} = \begin{cases} 1, & \text{if } \ i = k; \\ 0, & \text{if } i \neq k \,. \end{cases}$$

- δ_{i-k} means that:
  - $\delta_{i-k} = 1$, when $i = k$
  - $\delta_{i-k} = 0$, otherwise.

EXAMPLE

BASIS FOR $\mathbb{R}^2$



$$\begin{bmatrix} -6 \\ 4 \end{bmatrix} = -6 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Questions:

1. What are the basis vectors?
2. Which is the expansion?
3. What are the expansion coefficients?

EXAMPLE
ANALYSIS/EXPANSION COEFFICIENTS

- How did we get those coefficients?

$$\begin{bmatrix} -6 \\ 4 \end{bmatrix} = -6 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- **Inner product:** $\alpha_k = (\Phi^* f)_k = \langle f, \varphi_k \rangle,\ k \in \mathscr{K}$

$$\langle f, \varphi_1 \rangle = \left\langle \begin{bmatrix} -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = -6$$

$$\langle f, \varphi_2 \rangle = \left\langle \begin{bmatrix} -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle = 4$$

EXAMPLE
ANALYSIS

- How did we get those coefficients?

$$\begin{bmatrix} -6 \\ 4 \end{bmatrix} = -6 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

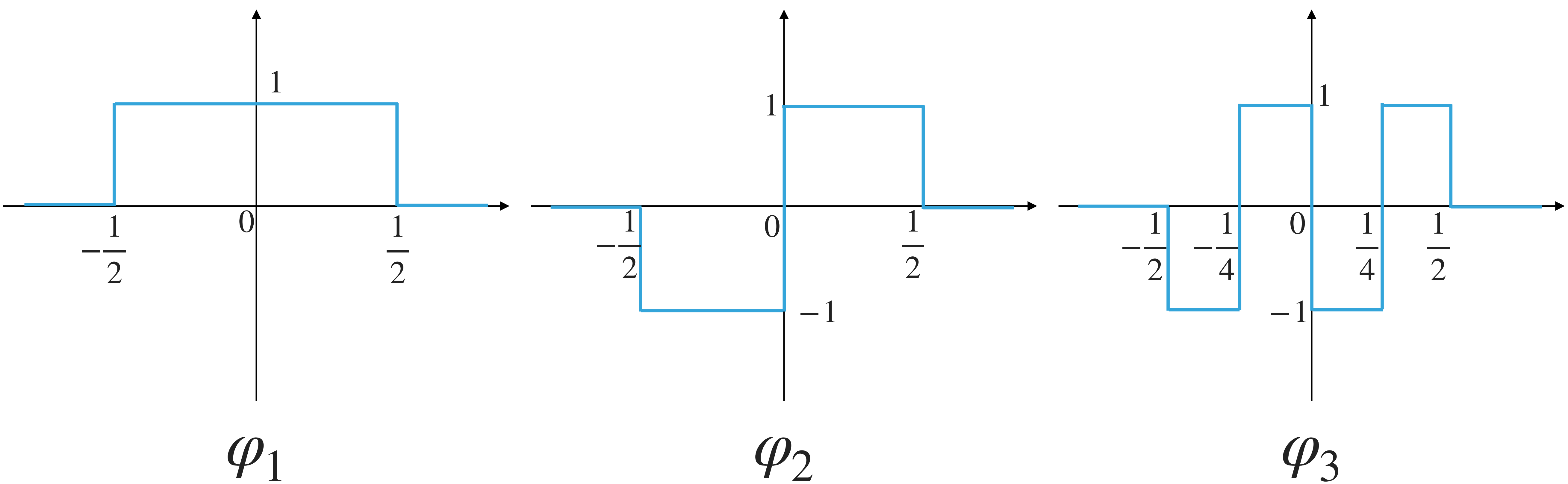
- **Operator form:** $\alpha_k = (\Phi^* f)_k,\ k \in \mathscr{K}$

$$\Phi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ then } \Phi^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\Phi^* f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -6 \\ 4 \end{bmatrix}$$

ORTHONORMAL BASIS
EXAMPLE

A basis for the Hilbert space $H = \left\{ f \in \mathscr{L}^2([-1/2,\ 1/2]) \,|\, f(-t) = f(t) \right\}$

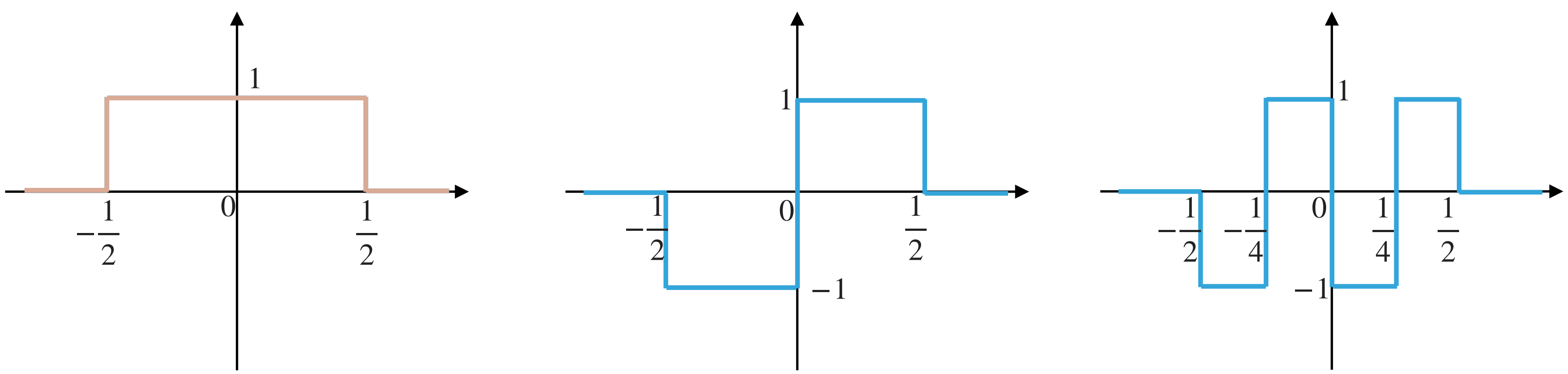


Question: How do we show that the set $\{\varphi_1,\ \varphi_2,\ \varphi_3\}$ is orthonormal?

Orthogonal **Normalized**

ORTHONORMAL BASIS
EXAMPLE

A basis for the Hilbert space $H = \left\{ f \in \mathscr{L}^2([-1/2,\ 1/2]) \,|\, f(-t) = f(t) \right\}$



φ_1 φ_2 φ_3

Orthogonal set: all pairs of vectors must be orthogonal (i.e., have zero **inner product**)

- **We ought to check that:** $\langle \varphi_1, \varphi_2 \rangle = 0$ and $\langle \varphi_2, \varphi_3 \rangle = 0$ and $\langle \varphi_1, \varphi_3 \rangle = 0$

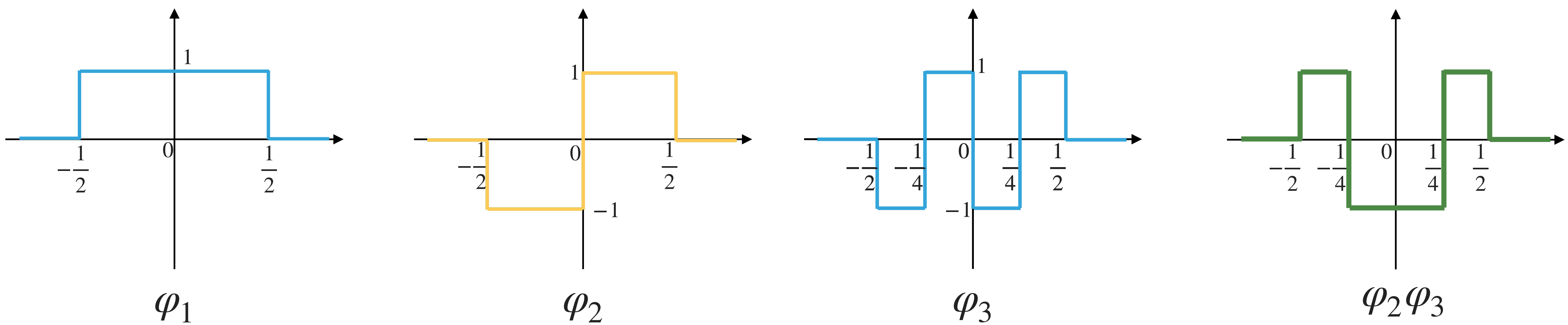
$$\langle \varphi_1, \varphi_2 \rangle = \int_{-1/2}^{1/2} \varphi_1 \varphi_2 \ \mathrm{d}t$$

$$\langle \varphi_1, \varphi_2 \rangle = \int_{-1/2}^{1/2} \varphi_2 \ \mathrm{d}t$$

$$\langle \varphi_1, \varphi_2 \rangle = \int_{-1/2}^{0} -1 \ \mathrm{d}t + \int_{0}^{1/2} 1 \ \mathrm{d}t = -1/2 + 1/2 = 0$$

ORTHONORMAL BASIS
EXAMPLE

A basis for the Hilbert space $H = \left\{ f \in \mathscr{L}^2([-1/2,\ 1/2]) \,|\, f(-t) = f(t) \right\}$



Orthogonal set: all pairs of vectors must be orthogonal (i.e., have zero **inner product**)

- **We ought to check that:** $\langle \varphi_1, \varphi_2 \rangle = 0$ and $\langle \varphi_2, \varphi_3 \rangle = 0$ and $\langle \varphi_1, \varphi_3 \rangle = 0$

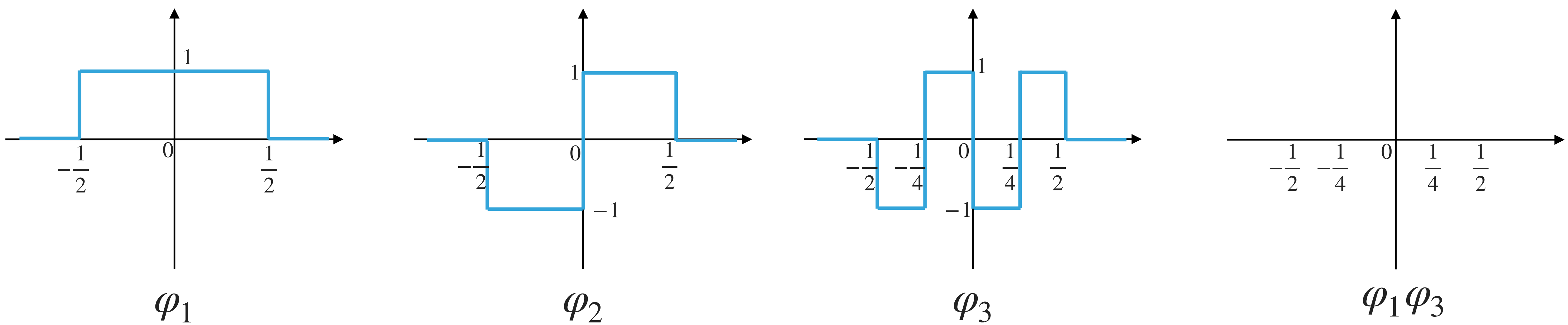
$$\langle \varphi_2, \varphi_3 \rangle = \int_{-1/2}^{1/2} \varphi_2 \varphi_3 \ \mathrm{d}t$$

$$\langle \varphi_2, \varphi_3 \rangle = \int_{-1/2}^{-1/4} 1 \ \mathrm{d}t + \int_{-1/4}^{+1/4} -1 \ \mathrm{d}t + \int_{1/4}^{1/2} 1 \ \mathrm{d}t$$

$$\langle \varphi_2, \varphi_3 \rangle = \frac{1}{4} - \frac{1}{2} + \frac{1}{4} = 0$$

ORTHONORMAL BASIS
EXAMPLE

A basis for the Hilbert space $H = \left\{ f \in \mathscr{L}^2([-1/2,\ 1/2]) \,|\, f(-t) = f(t) \right\}$



Orthogonal set: all pairs of vectors must be orthogonal (i.e., have zero **inner product**)

- **We ought to check that:** $\langle \varphi_1, \varphi_2 \rangle = 0$ and $\langle \varphi_2, \varphi_3 \rangle = 0$ and $\langle \varphi_1, \varphi_3 \rangle = 0$

$$\langle \varphi_1, \varphi_3 \rangle = \int_{-1/2}^{1/2} \varphi_1 \varphi_3 \, \mathrm{d}t$$

$$\langle \varphi_1, \varphi_3 \rangle =$$

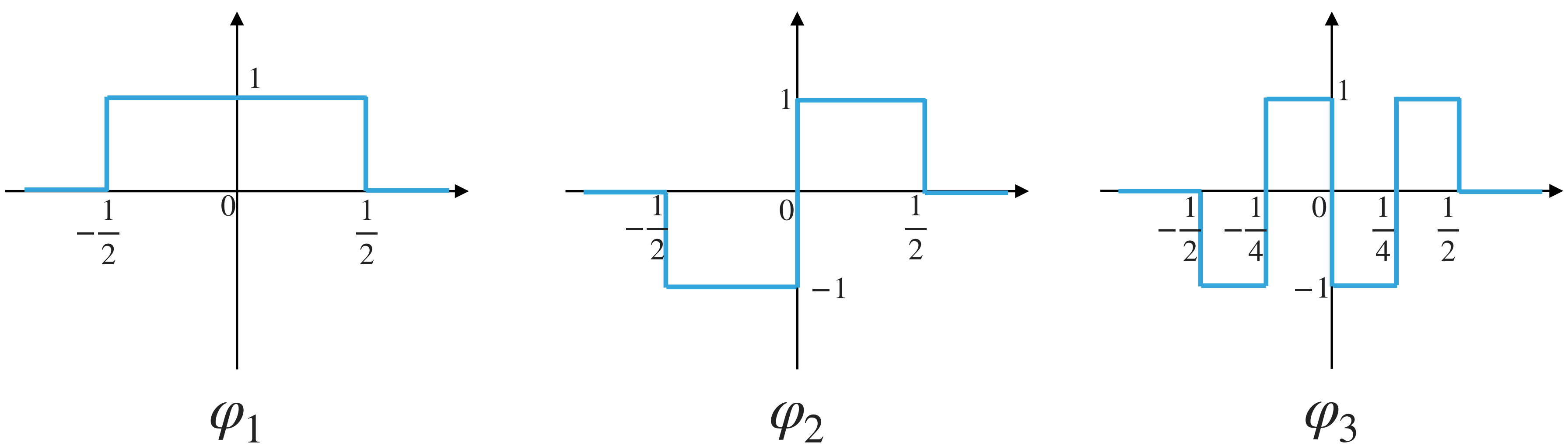
$$\langle \varphi_1, \varphi_3 \rangle = \qquad\qquad = 0$$

PLEASE DO TRY IT FOR YOURSELF !!!

ORTHONORMAL BASIS

EXAMPLE

A basis for the Hilbert space $H = \{f \in \mathscr{L}^2([-1/2,\ 1/2])$, where $f(t)$ is piecewise constant with breakpoints at $-1/2,\ -1/4, 0, 1/4, 1/2\}$



Normalized: all vectors in the set must be normalized (i.e., have **norm** of **one**)

- **We ought to check that:** $\|\varphi_1\| = 1$ and $\|\varphi_2\| = 1$ and $\|\varphi_3\| = 1$

$$\|\varphi_1\| = \sqrt{\int_{-1/2}^{1/2} \varphi_1^2 \, dt} = \sqrt{\int_{-1/2}^{1/2} 1 \, dt} = \sqrt{1} = 1$$

$\|\varphi_2\| =$

$\|\varphi_3\| =$

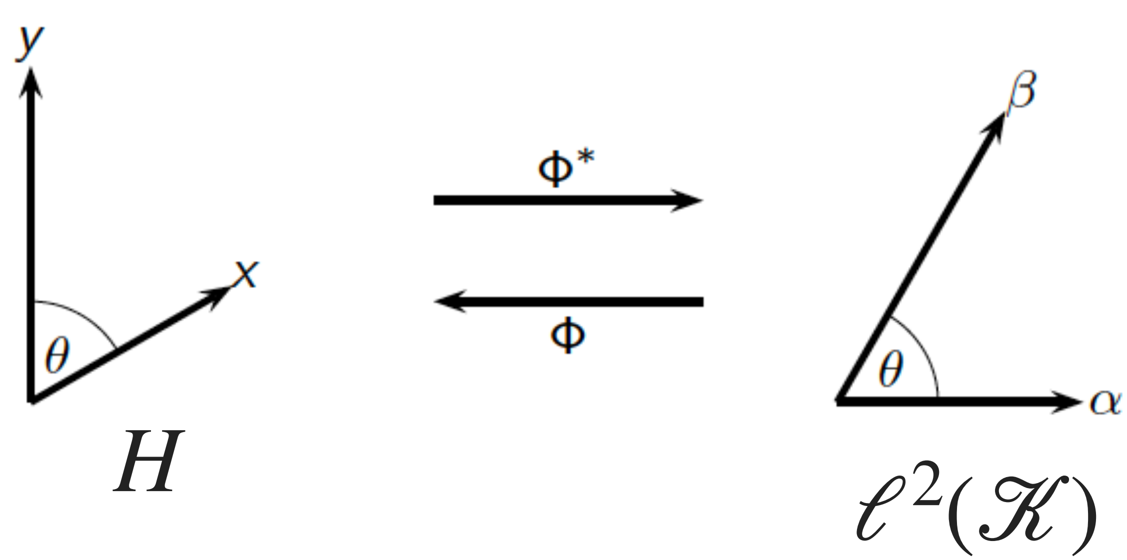
PLEASE DO TRY IT FOR YOURSELF !!!

ORTHONORMAL BASES

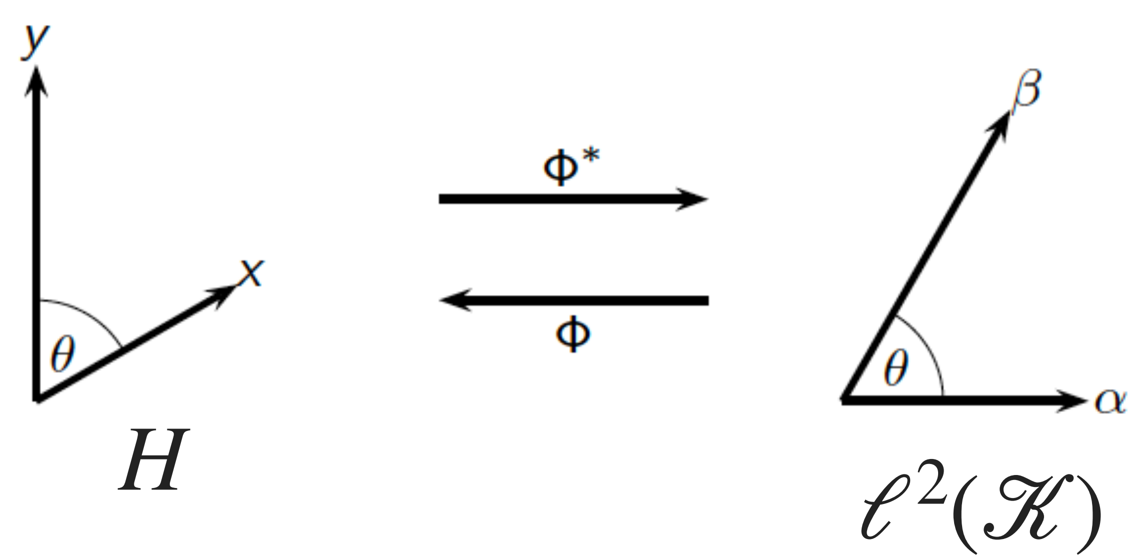
PARSEVAL EQUALITY

- The set $\Phi = \{\varphi_k\}_{k\in\mathcal{K}}$ is an orthonormal basis for H, then
$$\|f\|^2 = \sum_{k\in\mathcal{K}} \left|\langle f, \varphi_k\rangle\right|^2 = \|\Phi^* f\|^2 = \|\alpha\|^2$$

- In general:
$$\langle f, g\rangle = \langle \Phi^* f, \Phi^* g\rangle = \langle \boldsymbol{\alpha}, \boldsymbol{\beta}\rangle,$$
with $\alpha_k = \langle f, \varphi_k\rangle$ and $\beta_k = \langle g, \varphi_k\rangle$.



ORTHONORMAL BASES
PARSEVAL EQUALITY



$$\ell^2(\mathscr{K}) : \Phi^*\Phi = I$$

$$H : \Phi\Phi^* = I$$

- Perform storage and computations on the expansion coefficients
- Isometry between any separable Hilbert space H and $\ell^2(\mathbb{Z})$.
    - Angles and lengths are preserved.

WHAT WE COVERED TODAY

- **Bases** (definition)
  - Reisz basis (stability)
  - Orthogonal & Orthonormal bases
- **Analysis** and **Synthesis** with bases
  - Computing basis expansion/analysis coefficients of a vector
  - Synthesizing (making) the vector given basis and expansion/analysis coefficients

SEE YOU NEXT TIME!

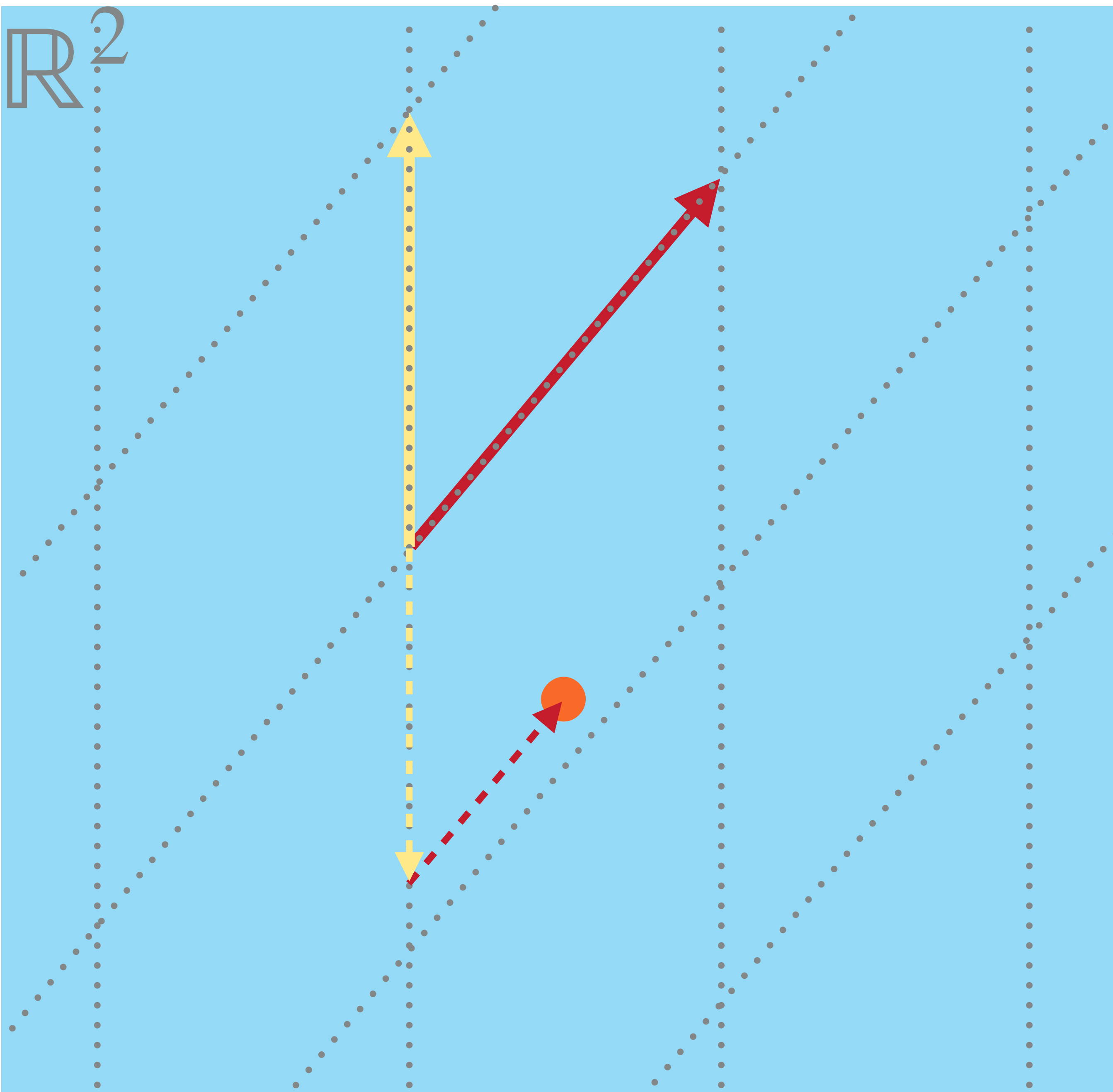
CONCLUDE BASES

BIORTHOGONAL BASES

BASIS

https://centonomy.com/you-cant-skip-the-basics/

BIORTHOGONAL BASES



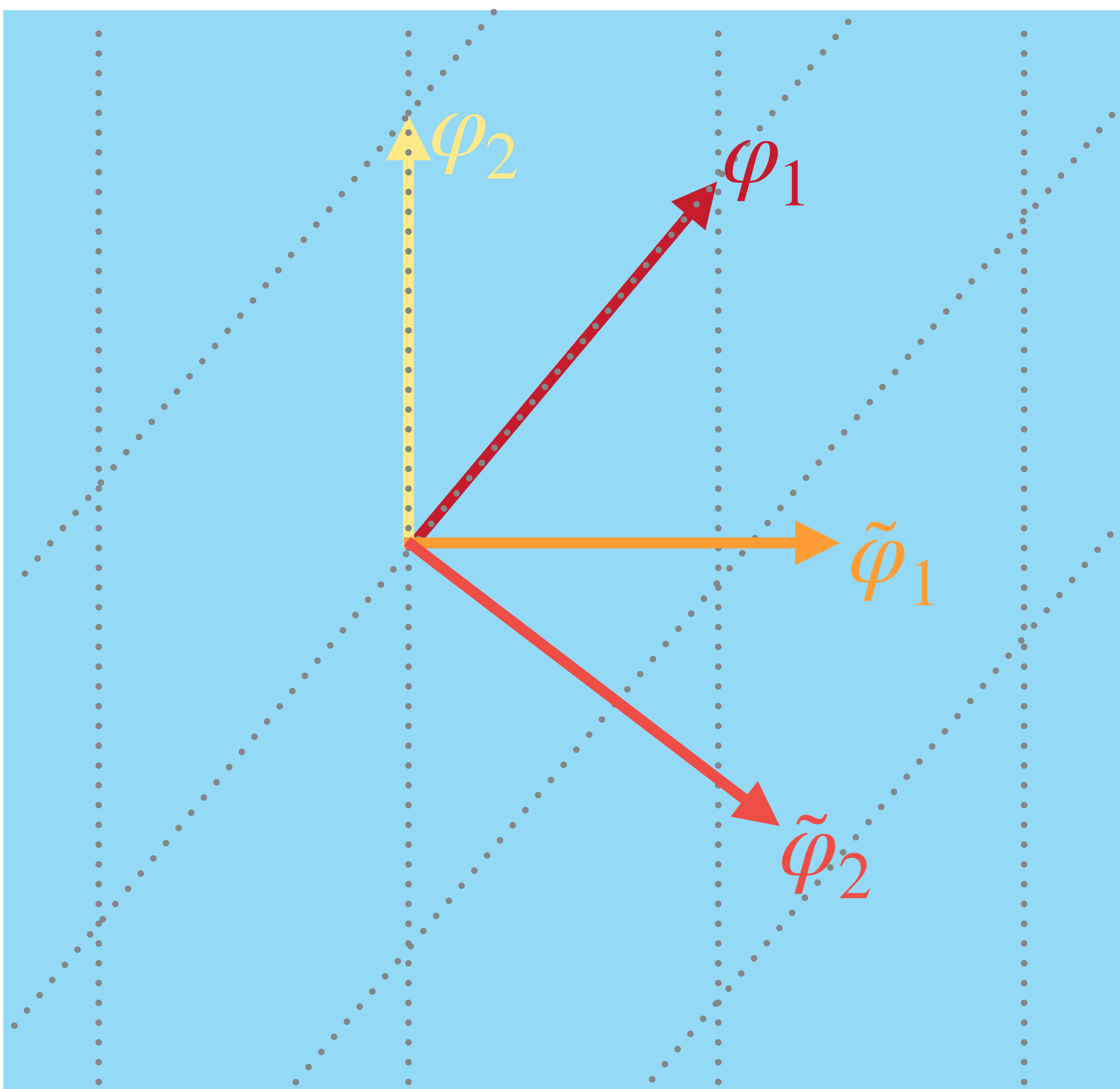
$+$

$\times$

$\langle u, v \rangle$

BIORTHOGONAL BASES

Basis $\{\varphi_0, \varphi_1\}$ and its **dual** $\{\tilde{\varphi}_0, \tilde{\varphi}_1\}$.



$+$

$\times$

$\langle \boldsymbol{u}, \boldsymbol{v} \rangle$

BIORTHOGONAL BASES

- **Definition (Biorthogonal Bases):** $\Phi = \{\varphi_k\}_{k\in\mathscr{K}} \subset H$ and $\widetilde{\Phi} = \{\tilde{\varphi}_k\}_{k\in\mathscr{K}} \subset H$ is a biorthogonal pair of bases when
  - Φ and $\widetilde{\Phi}$ are both **bases for** H
  - Φ and $\widetilde{\Phi}$ are **biorthogonal**, i.e.:

$$\langle \varphi_j, \tilde{\varphi}_k \rangle = \delta_{j-k} \text{ for all } j, k \in \mathscr{K}$$

- Notice that the roles of Φ and $\tilde{\Phi}$ are interchangeable

BIORTHOGONAL BASES

EXAMPLE

$$\varphi_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \varphi_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \varphi_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \Rightarrow \Phi = [\varphi_1 \quad \varphi_2 \quad \varphi_3] = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\tilde{\varphi}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad \tilde{\varphi}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \tilde{\varphi}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \Rightarrow \widetilde{\Phi} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

- Verify that
  1. Φ and $\widetilde{\Phi}$ are both bases for H (what is H?)
  2. Φ and $\widetilde{\Phi}$ are **biorthogonal**, i.e. $\langle \varphi_j, \tilde{\varphi}_k \rangle = \delta_{j-k}$ for all $j, k \in \mathscr{K}$

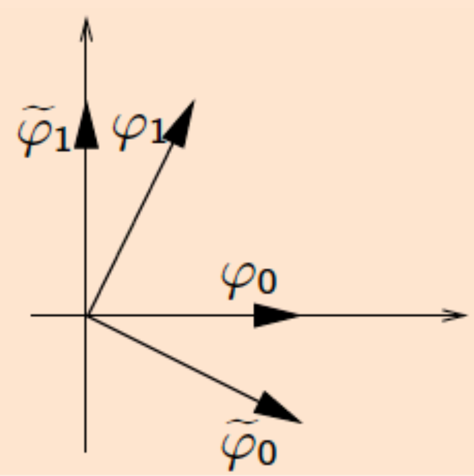
PLEASE DO TRY IT!!!

BIORTHOGONAL BASES

EXPANSION

Theorem

- $\Phi = \{\varphi_k\}_{k\in\mathcal{K}}, \widetilde{\Phi} = \{\widetilde{\varphi}_k\}_{k\in\mathcal{K}}$ *biorthogonal pair of bases for* H



BIORTHOGONAL BASES
EXPANSION

Theorem

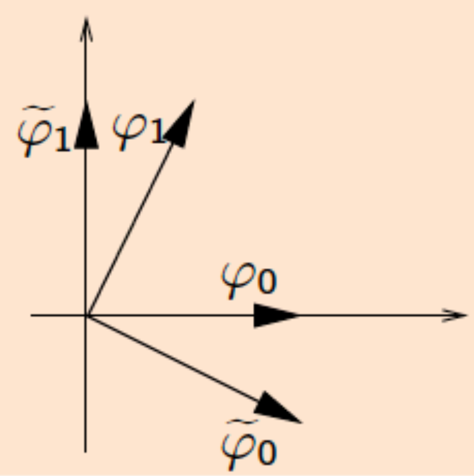
- $\Phi = \{\varphi_k\}_{k\in\mathcal{K}}, \widetilde{\Phi} = \{\widetilde{\varphi}_k\}_{k\in\mathcal{K}}$ *biorthogonal pair of bases for* H
- *Any* $x \in H$ *has expansion coefficients*

$$\alpha_k = \langle x, \widetilde{\varphi}_k\rangle,\ \ k \in \mathcal{K},\ \ \text{or } \alpha = \widetilde{\Phi}^* x$$

- *Synthesis:*

$$x = \sum_{k\in\mathcal{K}} \langle x, \widetilde{\varphi}_k\rangle \varphi_k = \Phi\alpha = \Phi\widetilde{\Phi}^* x$$

- *Also* $x = \sum_{k\in\mathcal{K}} \langle x, \varphi_k\rangle \widetilde{\varphi}_k$



DUAL BASIS

HOW DO WE COMPUTE THE DUAL Φ^*

Theorem (Dual basis)

DUAL BASIS

HOW DO WE COMPUTE THE DUAL $\widetilde{\Phi}$

Theorem (Dual basis)

- *Given a Riesz basis* $\Phi = \{\varphi_k\}_{k\in\mathcal{K}}$ *for Hilbert space* H*, the set* $\widetilde{\Phi} = \{\widetilde{\varphi}_k\}_{k\in\mathcal{K}}$ *defined via*

$$\widetilde{\varphi}_k = \sum_{\ell\in\mathcal{K}} a_{\ell,k}\varphi_\ell, \qquad \text{for each } k \in \mathcal{K},$$

$$\boxed{\widetilde{\Phi} = \Phi A = \Phi(\Phi^*\Phi)^{-1},}$$

is a basis for H*, called the dual basis, and the sets* Φ *and* $\widetilde{\Phi}$ *are a biorthogonal pair of bases*

GRAM MATRIX

- $G = \Phi^*\Phi$ is the Gram matrix

  $G_{ik} = \langle \varphi_k, \varphi_i \rangle \qquad$ for every $i, k \in \mathcal{K}$,

$$G = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \langle \varphi_{-1}, \varphi_{-1} \rangle & \langle \varphi_0, \varphi_{-1} \rangle & \langle \varphi_1, \varphi_{-1} \rangle & \cdots \\ \cdots & \langle \varphi_{-1}, \varphi_0 \rangle & \boxed{\langle \varphi_0, \varphi_0 \rangle} & \langle \varphi_1, \varphi_0 \rangle & \cdots \\ \cdots & \langle \varphi_{-1}, \varphi_1 \rangle & \langle \varphi_0, \varphi_1 \rangle & \langle \varphi_1, \varphi_1 \rangle & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}$$

- Assume $x = \Phi\alpha$, $y = \Phi\beta$ then
  $\langle x, y \rangle = \langle \Phi\alpha, \Phi\beta \rangle = \langle \Phi^*\Phi\alpha, \beta \rangle = \langle G\alpha, \beta \rangle = \beta^* G \alpha$

  The inner product in H becomes an inner product in $\ell^2(\mathcal{K})$!

  (It is a *nonstandard* inner product unless $G = I$, which happens if and only if the basis is orthonormal.)

BIORTHOGONAL BASES
EXAMPLE: COMPUTING THE DUAL

MATLAB

SOLUTIONS

1. Write the **matrix representation** (i.e., the synthesis operator) associated with the set:

$$\varphi_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \varphi_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad \varphi_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

2. Is the set an **orthonormal basis**?

3. Find the **dual** $\widetilde{\Phi}$

$$\Phi = [\varphi_1 \quad \varphi_2 \quad \varphi_3] = \begin{bmatrix}1 & 0 & 1\\1 & 1 & 1\\0 & 1 & 1\end{bmatrix}$$

Check all pairwise inner products

$$\langle \varphi_1, \varphi_2 \rangle = (1 \times 0) + (1 \times 1) + (0 \times 1) = 1$$
$$\langle \varphi_1, \varphi_3 \rangle = (1 \times 1) + (1 \times 1) + (0 \times 1) = 2$$
$$\langle \varphi_2, \varphi_3 \rangle = (0 \times 1) + (1 \times 1) + (1 \times 1) = 2$$

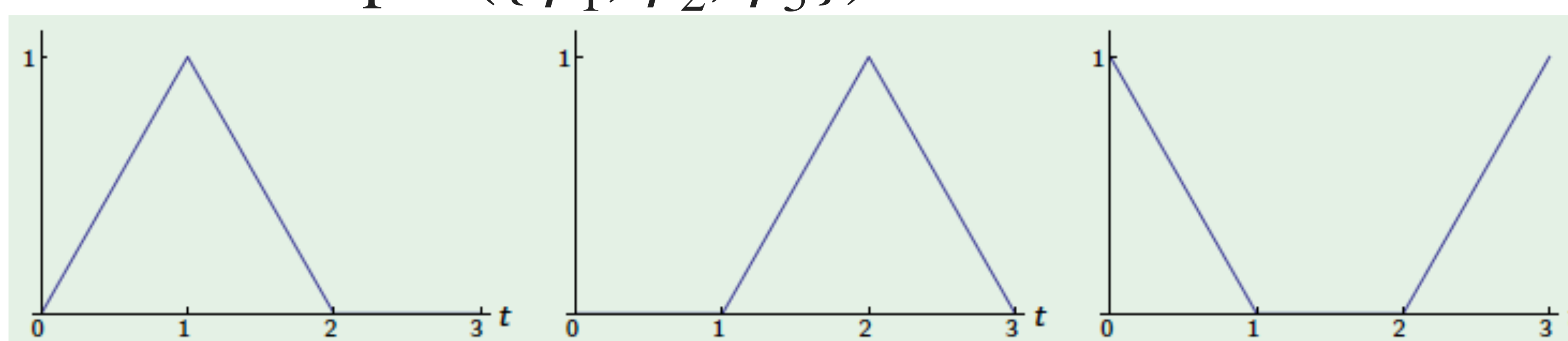
$$\widetilde{\Phi} = \Phi(\Phi^*\Phi)^{-1} = \begin{bmatrix}0 & 1 & -1\\-1 & 1 & 0\\1 & -1 & 1\end{bmatrix}$$

Also, note that: $\Phi\widetilde{\Phi}^* = \mathbf{I} \quad \Rightarrow \widetilde{\Phi}^* = \Phi^{-1}$

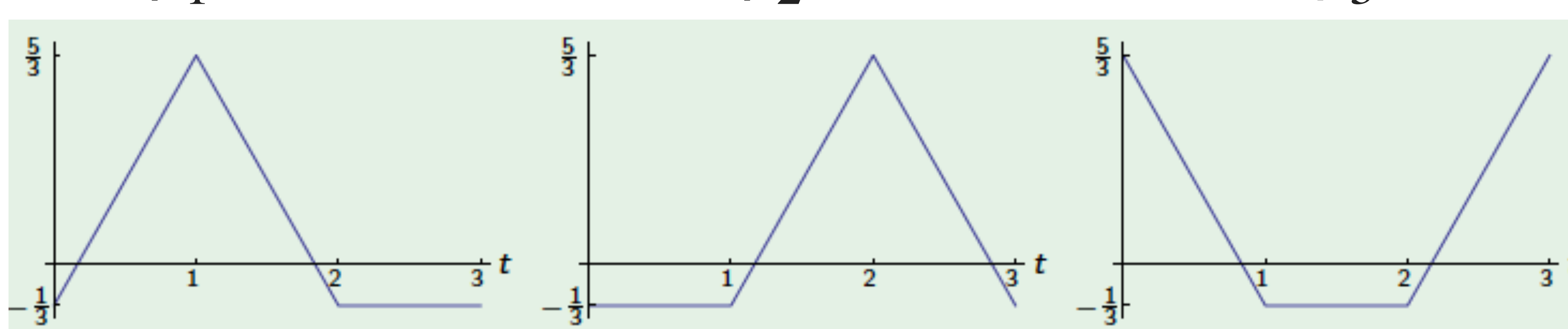
DUAL BASIS

EXAMPLE: HOW DO WE COMPUTE THE DUAL $\widetilde{\Phi}$

- Consider the function $\varphi_1 = \begin{cases} t & t \in [0,1); \\ 2-t, & t \in (1,2]; \\ 0, & t \in (2,3] \end{cases}$ in $\mathscr{L}^2([0,3])$ and its circular shifts by 1 and 2.
- $\Phi = \{\varphi_1, \varphi_2, \varphi_3\}$ is basis for $\mathrm{span}(\{\varphi_1, \varphi_2, \varphi_3\})$



$\varphi_1(t)$ $\varphi_2(t)$ $\varphi_3(t)$

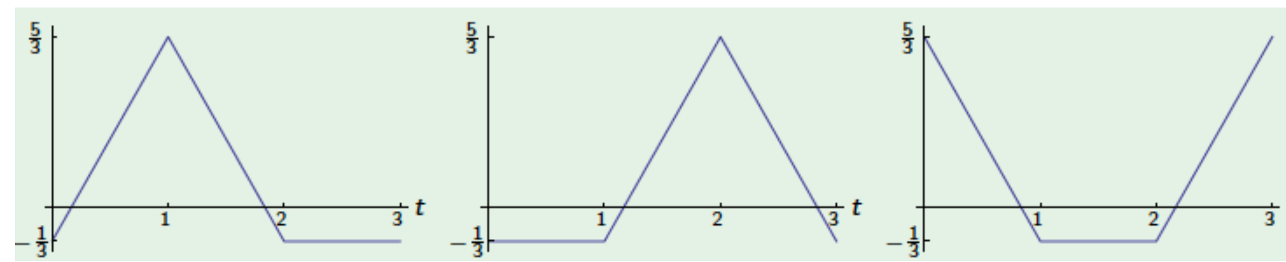


$\tilde{\varphi}_1(t)$ $\tilde{\varphi}_2(t)$ $\tilde{\varphi}_3(t)$

DUAL BASIS

EXAMPLE: HOW DO WE COMPUTE THE DUAL $\widetilde{\Phi}$

$\varphi_1(t)$ $\varphi_2(t)$ $\varphi_3(t)$



$\tilde{\varphi}_1(t)$ $\tilde{\varphi}_2(t)$ $\tilde{\varphi}_3(t)$

- $\mathrm{span}(\{\varphi_1, \varphi_2, \varphi_3\})$ is the subspace of functions $f(t)$ that are piecewise linear on $[0, 3]$ with breakpoints at 1 and 2, and satisfy $f(0) = f(3)$.

$$G = \begin{bmatrix} 2/3 & 1/6 & 1/6 \\ 1/6 & 2/3 & 1/6 \\ 1/6 & 1/6 & 2/3 \end{bmatrix}$$

- We find the dual $\widetilde{\Phi}$ by using $\widetilde{\Phi} = \Phi G^{-1}$, where

$$G^{-1} = \begin{bmatrix} 5/3 & -1/3 & -1/3 \\ -1/3 & 5/3 & -1/3 \\ -1/3 & -1/3 & 5/3 \end{bmatrix}$$

HANDLING MATRICES

MATLAB PRACTICE 4

MATRICES IN MATLAB

- Matrices
  - Matrix transpose (adjoint)
  - Matrix inverse
    - Left inverses
    - Right inverses
- Eigenvalue decomposition
  - Eigenvalues & Eigenvectors
- *Apply these concepts to obtain plot in Slide 15*
  - Homework question 2

WHAT WE COVERED TODAY

- **Bases** (definition)
  - Reisz basis (stability)
  - Orthogonal & Orthonormal bases
  - Biorthogonal bases (Basis & Dual pair)
- **Analysis** and **Synthesis** with bases
  - Computing basis expansion/analysis coefficients of a vector
  - Synthesizing (making) the vector given basis and expansion/analysis coefficients
- The **Gram matrix**

SEE YOU NEXT TIME!

INTRODUCTION TO FOURIER ANALYSIS

TECHNICAL ASIDE
CLOSED SUBSPACE

Definition (Closed subspace)

A subspace S of a normed vector space V is called *closed* when it contains all limits of sequences of vectors in S

Problem and resolution:

- Subspaces can fail to be closed ("weird" because subspaces of finite-dimensional normed vector spaces are always closed)
- Subspaces often arise from span of a set of vectors
- Span defined with *finite* linear combinations
  Span not necessarily closed
- Very often work with closure of span (which is a closed subspace):

$$\overline{\text{span}}(\{\varphi_k\}_{k\in\mathcal{K}}) = \left\{ \sum_{k\in\mathcal{K}} \alpha_k \varphi_k \mid \alpha_k \in \mathbb{C} \text{ and the sum converges} \right\}$$

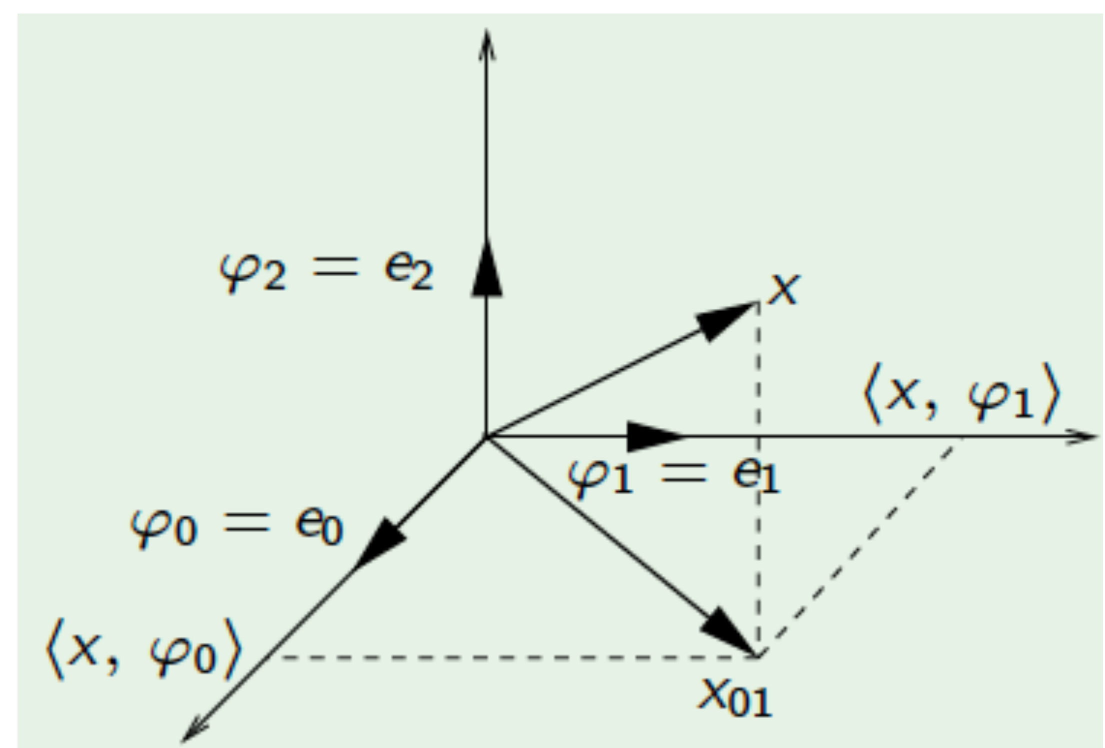
ORTHONORMAL BASIS
SYNTHESIS

- Let $\Phi = \{\varphi_k\}_{k \in \mathscr{K}}$ be an orthonormal basis for H, then for any $f \in H$ such that $\alpha_k = \langle f, \varphi_k \rangle$ for $k \in \mathscr{K}$ (or equivalently, $\boldsymbol{\alpha} = \Phi^* f$), the sequence $\boldsymbol{\alpha}$ is unique.
- **Synthesis**: how can we can get back the original f?

$$f = \sum_k \langle f, \varphi_k \rangle \varphi_k$$

$$\Phi \boldsymbol{\alpha} = \Phi \Phi^* f$$

Notice that $\Phi\Phi^*$ is identity matrix (operator).



BEST APPROXIMATION AND NORMAL EQUATIONS

Theorem (Normal equations)

- $x \in H$ and $\{\phi_k\}_{k \in \mathcal{I}}$ *a Riesz basis for a closed subspace* S
- *The closest vector to* x *in* S *is*

$$\widehat{x} = \sum_{k \in \mathcal{I}} \beta_k \phi_k = \Phi\beta$$

where β *is the unique solution to*

$$\Phi^*\Phi\beta = \Phi^* x \qquad \textit{or}$$

$$\sum_{k \in \mathcal{I}} \beta_k \langle \phi_k, \phi_i \rangle = \langle x, \phi_i \rangle \text{ for all } i \in \mathcal{I}$$

Normal equations

- $\widehat{x} = \Phi(\Phi^*\Phi)^{-1}\Phi^* x = Px$

  P *is an orthogonal projection*

- We can verify $P^2 = \Phi \underbrace{(\Phi^*\Phi)^{-1}\Phi^*\Phi}_{I}(\Phi^*\Phi)^{-1}\Phi^* = P$ and $P^* = P$

A REMINDER MATRIX-VECTOR MULTIPLICATION

$$\begin{bmatrix} -2 & -2 \\ 1 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = ?$$

$$\begin{bmatrix} -2 & -2 \\ 1 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = ?$$

APPROXIMATIONS

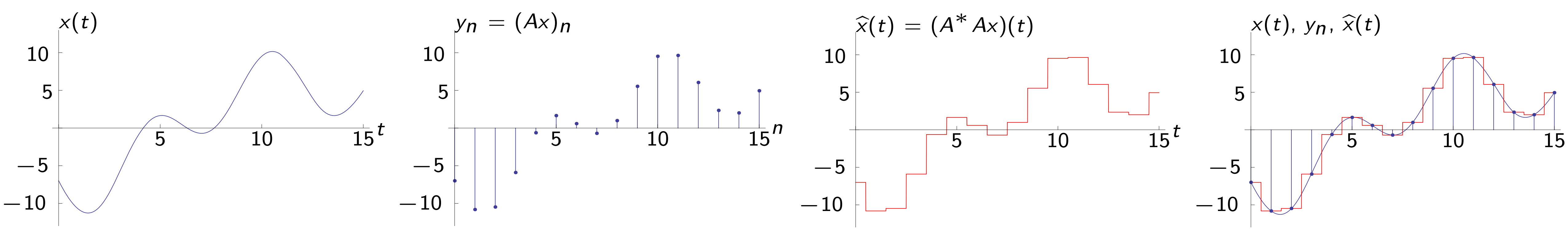
BASIS

https://centonomy.com/you-cant-skip-the-basics/

APPROXIMATION

ORTHOGONAL PROJECTION ON $\mathscr{L}^2(\mathbb{R})$

- Local averaging on $\mathscr{L}^2(\mathbb{R})$ $\quad A : \mathcal{L}^2(\mathbb{R}) \to \ell^2(\mathbb{Z}) \qquad (Ax)_n = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} x(t)\, dt$



adjoint $A^* : \ell^2(\mathbb{Z}) \to \mathcal{L}^2(\mathbb{R})$ that produces staircase function

- The local averaging operator A, and its adjoint A^* are such that $AA^* = I$, so that $P = A^*A$ is an orthogonal projection

APPROXIMATION

ORTHOGONAL PROJECTION ON $\mathscr{L}^2(\mathbb{R})$

- Orthogonal projection gives "best" approximation
  - Best here means the norm of the error is minimized!
  - Minimum mean squared error