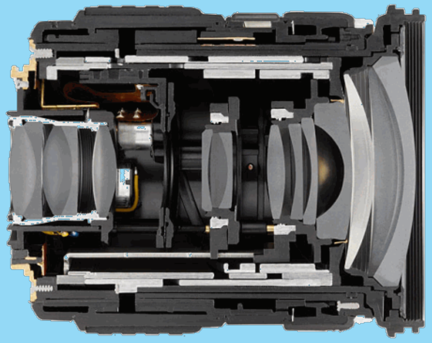
COMPUTATIONAL METHODS FOR IMAGING (AND VISION)

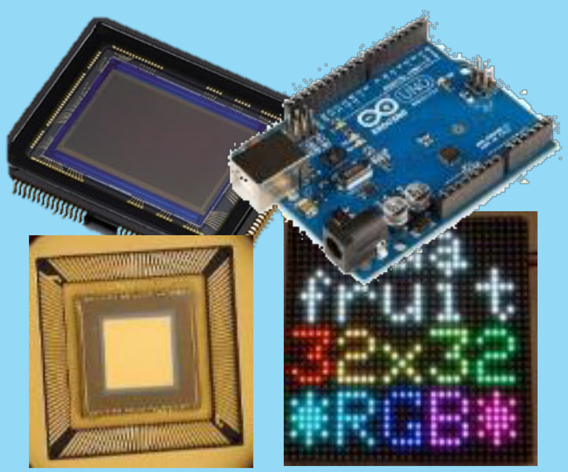
LECTURE 2: VECTORS

PROF. JOHN MURRAY-BRUCE



Optics

+

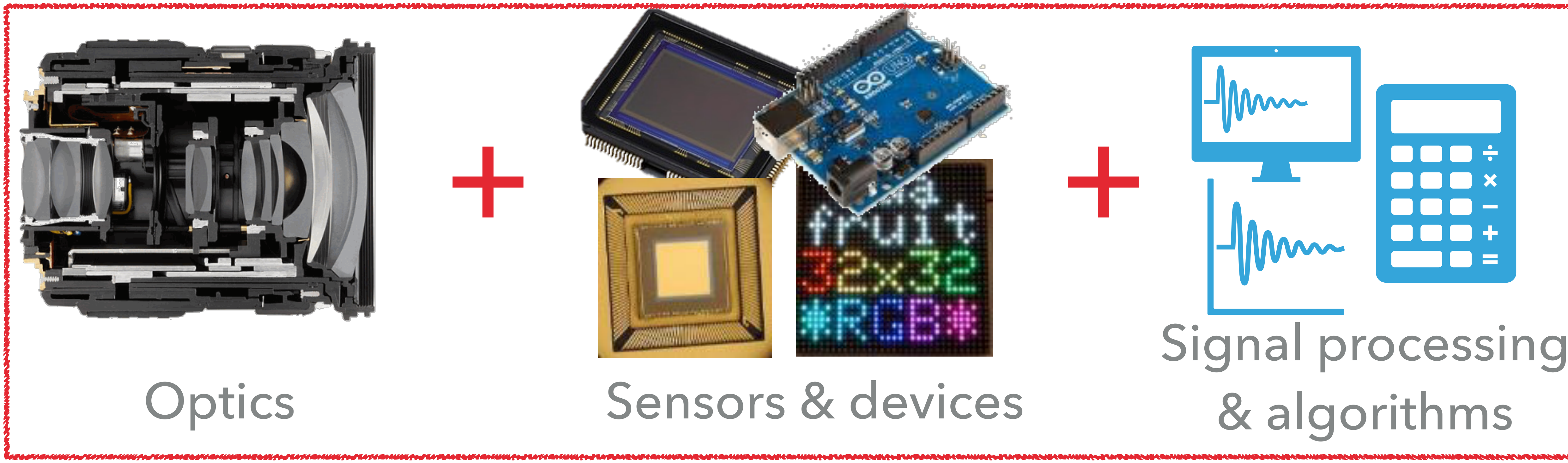


Sensors
&
devices

+

Signal
processing
&
algorithms

COMPUTATIONAL IMAGING
INTEGRATED SYSTEMS APPROACH

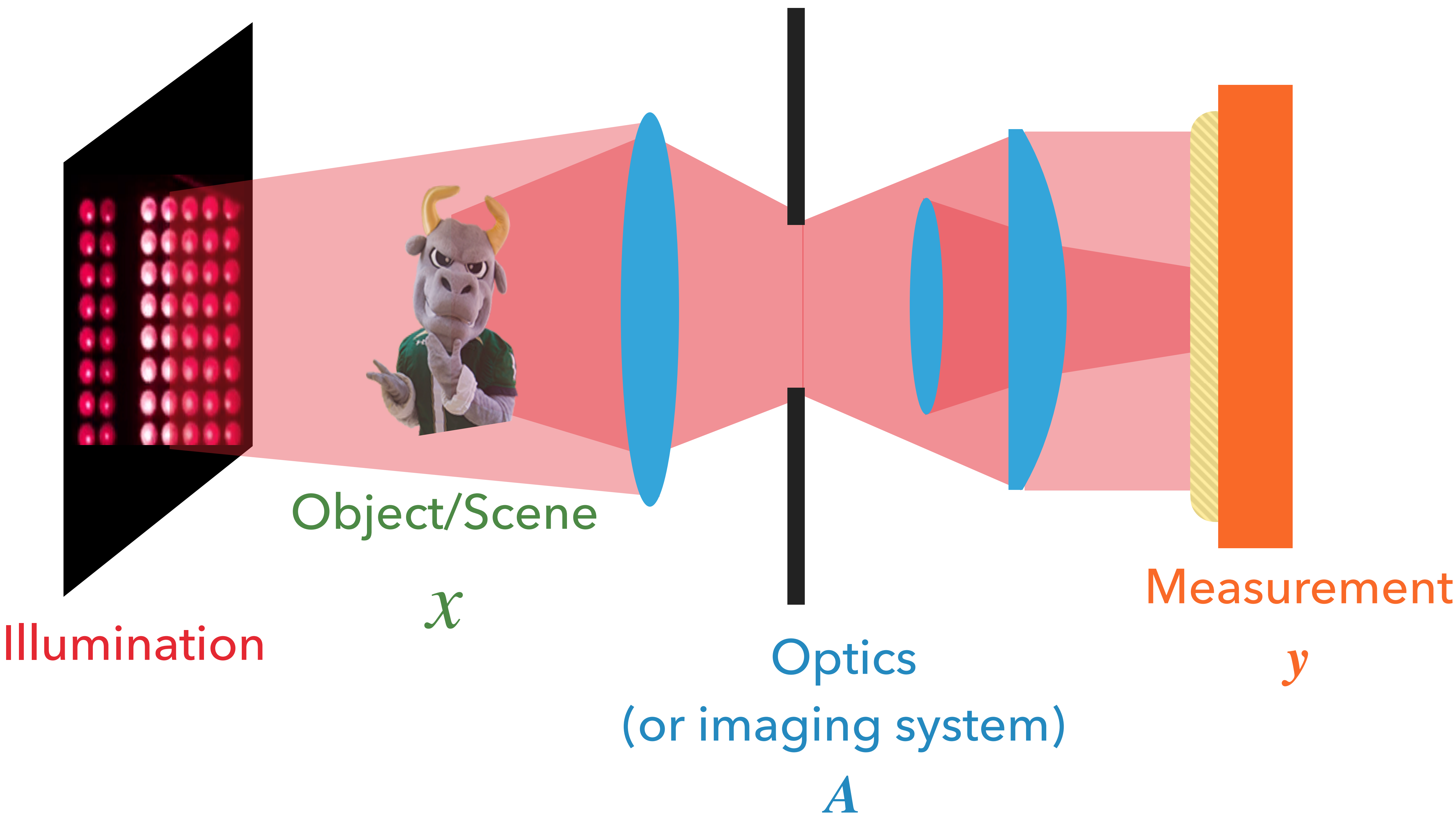


- System-level integration creates new imaging pipelines:
  - Efficiently encode information with hardware
  - Computational reconstruction (decoding)
- Design flexibility
- Enabling new capabilities, e.g. super-resolution, 3D, phase

DESIRED PROPERTIES OF IMAGING SYSTEMS

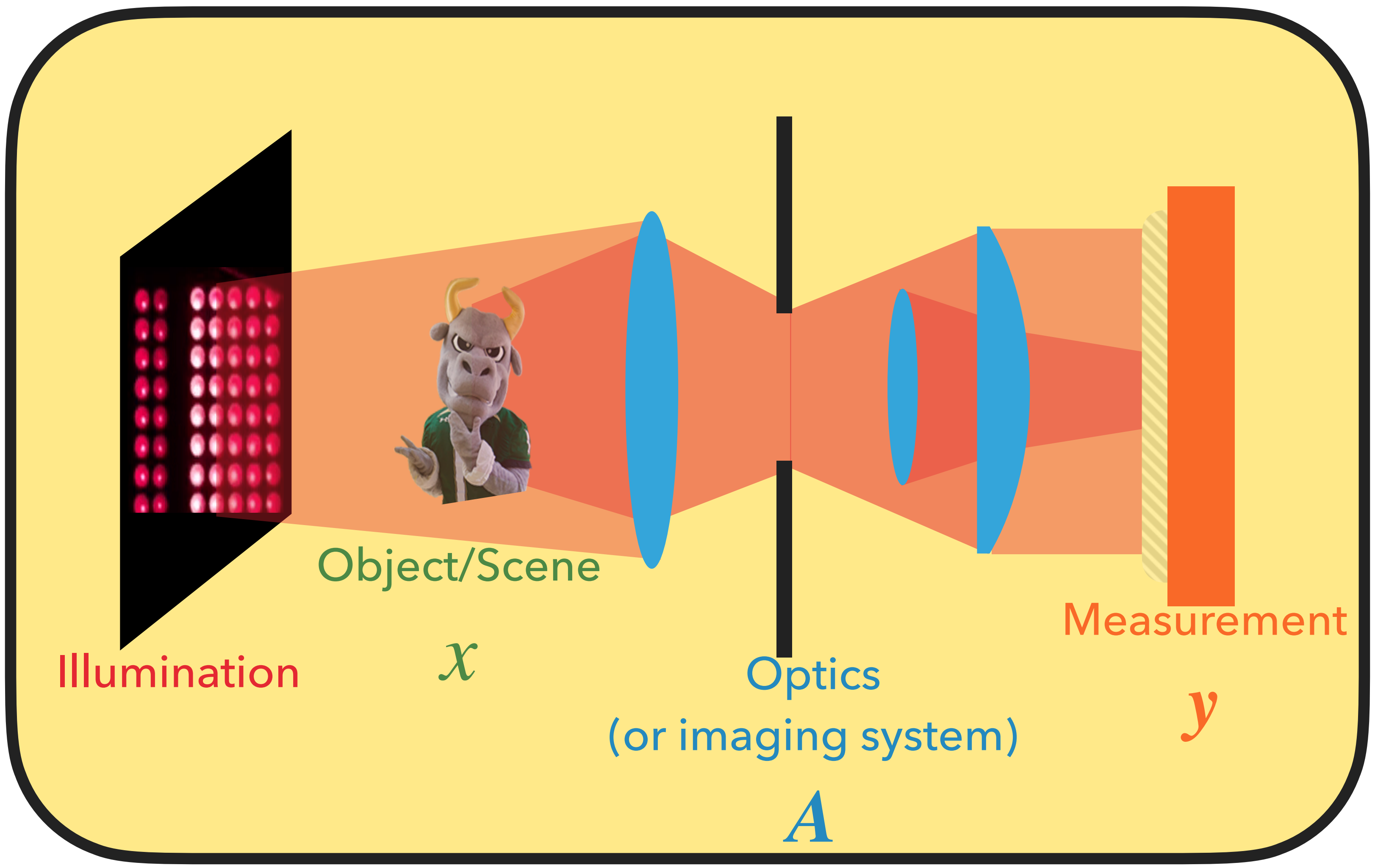
- Form images
  - With High Signal-to-Noise Ratio
  - High Contrast
  - Good resolution
- Cheap, Simple, and Robust
  - Microscopes (HIM ~$1.5m, but may be easily damaged without proper care)
  - Military applications

ABSTRACTION OF COMPUTATIONAL IMAGING SYSTEMS



ABSTRACTION OF COMPUTATIONAL IMAGING SYSTEMS

Hardware

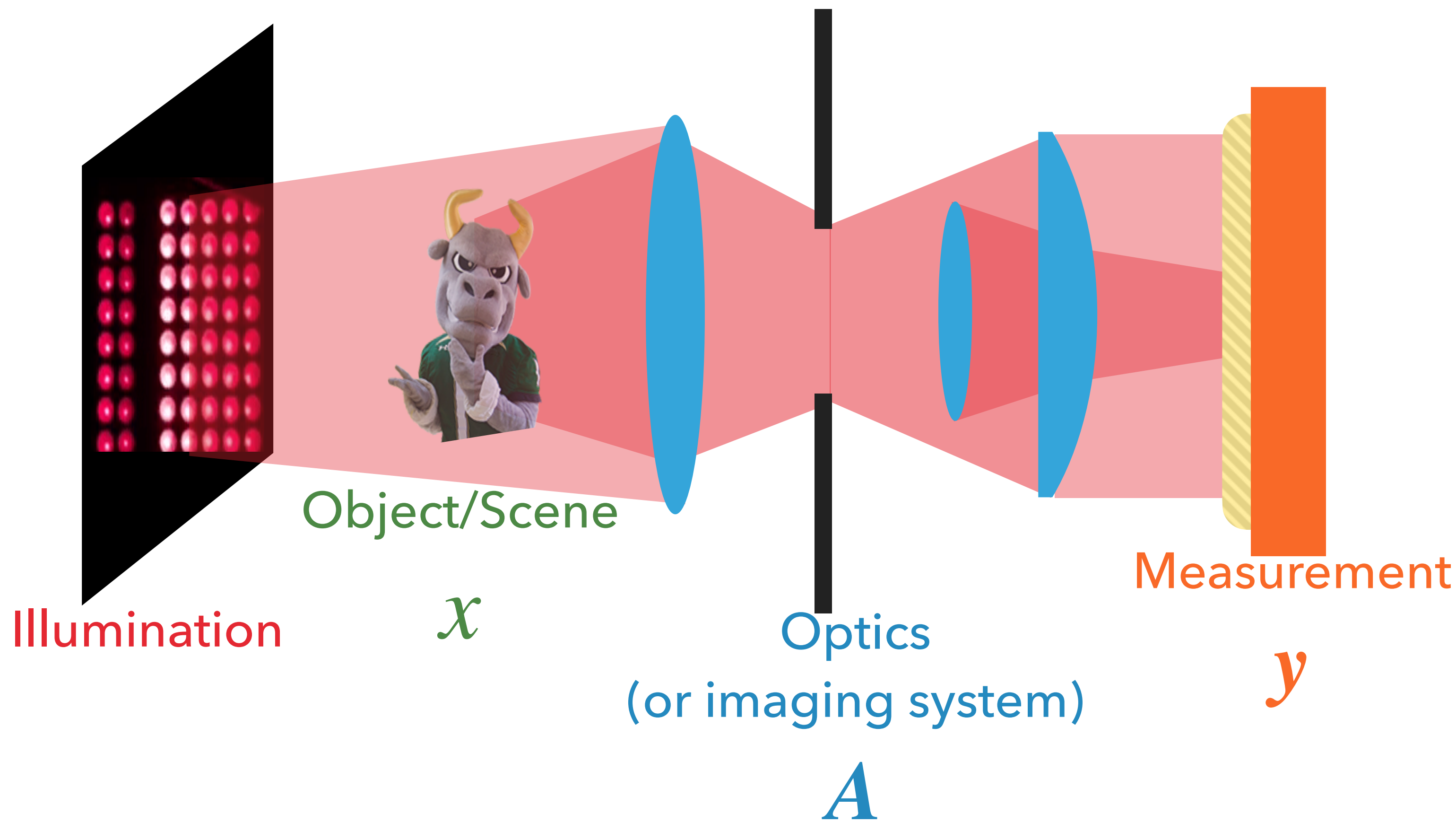


Software

Find x

Such that $Ax = y$

DESIGNING THE IMAGING SYSTEM



- **Question:** Where to introduce coding?
  - Illumination-side and detection-side coding
- **Coding domains?**
  - Spatial, temporal coding, angular, spectral, polarization and so on.

COMP. METHODS FOR IMAGING

- **Focus:** the tools and techniques at the intersection between physics and digital imaging system design
  - Both equally important
- **We will cover:**
  - Mathematical tools for forward and inverse problems
  - Inversion and reconstruction algorithms
  - Introductory optics
  - Applications

KEY QUESTIONS
TO BE ANSWERED IN THIS COURSE

- **What is computational imaging (our abstraction)**
  - Forward modeling: Linear systems abstraction, i.e. $y = Ax$
  - Inverse problems: Find x given y (and A)
- **Analog signals to digital representations**
  - Vector spaces, sampling and linear operators
- **How to build A (forward problem)**
  - Study properties of A
  - Is it “well-conditioned” for inversion?
- How to exploit *prior* information about x

OUTLINE

- Linear algebra
  - Vector spaces
- Vectors
  - Norms
  - Inner products

LINEAR ALGEBRA (REVIEW)

- Why is linear algebra important for imaging?
  - Bridge between geometry, physics, optics, etc… and *digital computation*
  - Many (computational) imaging problems:
    - Derive linear algebraic expression ($y = Ax$), and
    - Use computer (Matlab) to solve $y = Ax$, for the unknown x
    - Efficient, effective & robust numerical methods and optimization techniques have made computational imaging possible

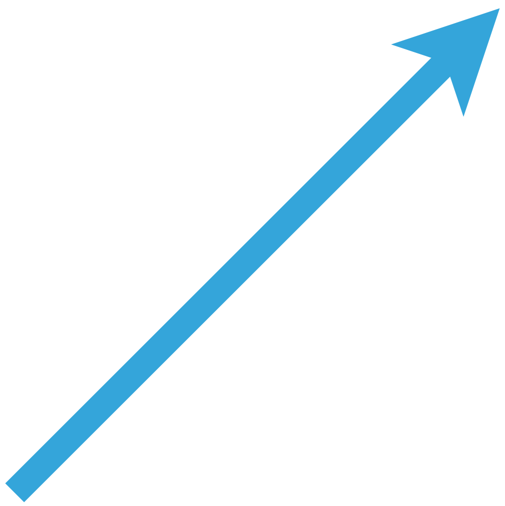
VECTORS

VECTOR SPACES

- **Definition (Linear algebra)** – the study of vector spaces and the linear maps between them.
  - **More rigorously:** For all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and scalars a, b
    - $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
    - $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
    - There exists a *zero vector* $\mathbf{0}$, such that $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
    - For every $\mathbf{u}$ there exits a vector $-\mathbf{u}$, such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
    - $1\mathbf{v} = \mathbf{v}$
    - $a(b\mathbf{u}) = (ab)\mathbf{u}$
    - $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
    - $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

VECTORS
INTUITION

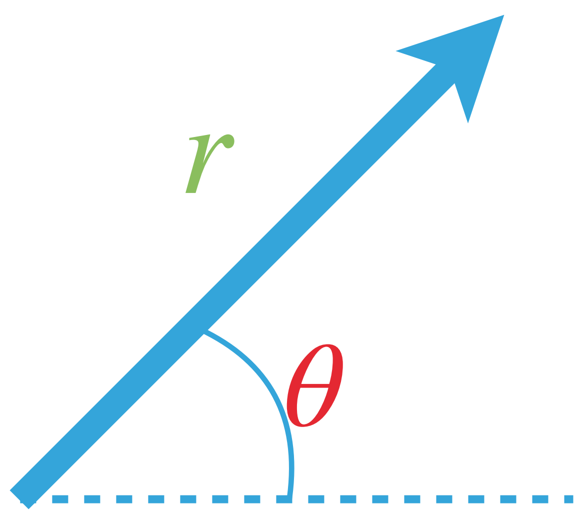
- Intuitively, think of a vector as this little arrow



- In computational imaging systems, we work with a range of entities that, may not look like arrows but, behave like vectors (or little arrows).
  - E.g.: Illumination/light sources, images, polynomials

VECTORS
WHAT CAN WE MEASURE/ENCODE

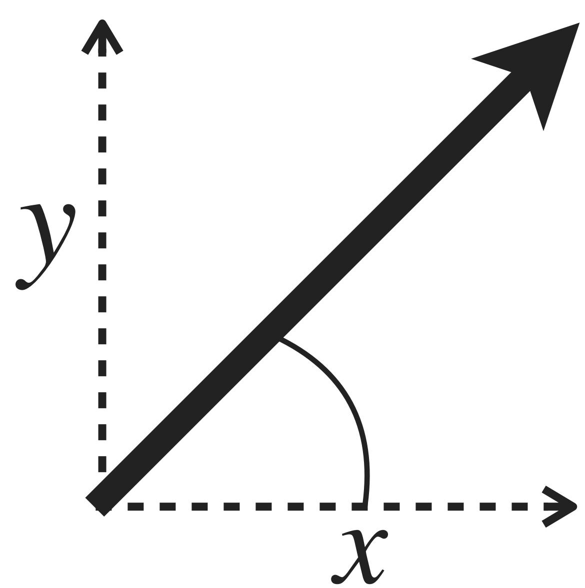
- Fundamentally, vectors have **magnitude** and **direction**



- Example: a 2D vector can be encoded as length, and an angle relative to some fixed direction
  - **Polar coordinates**
- What are other possibilities?

VECTORS
CARTESIAN COORDINATES

- Can measure components of a vector with respect to some chosen coordinate system

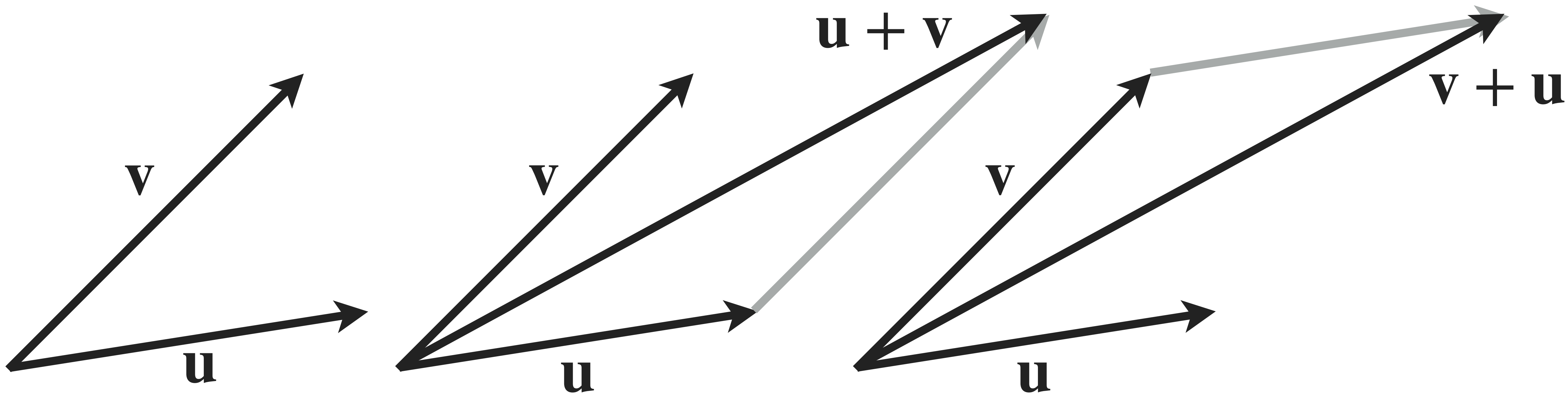


René Descartes,
Circa 1596

- Caution: Cannot compare coordinates in different systems
  - Should **not** compare (r, θ) to (x, y)

VECTORS
ADDITION

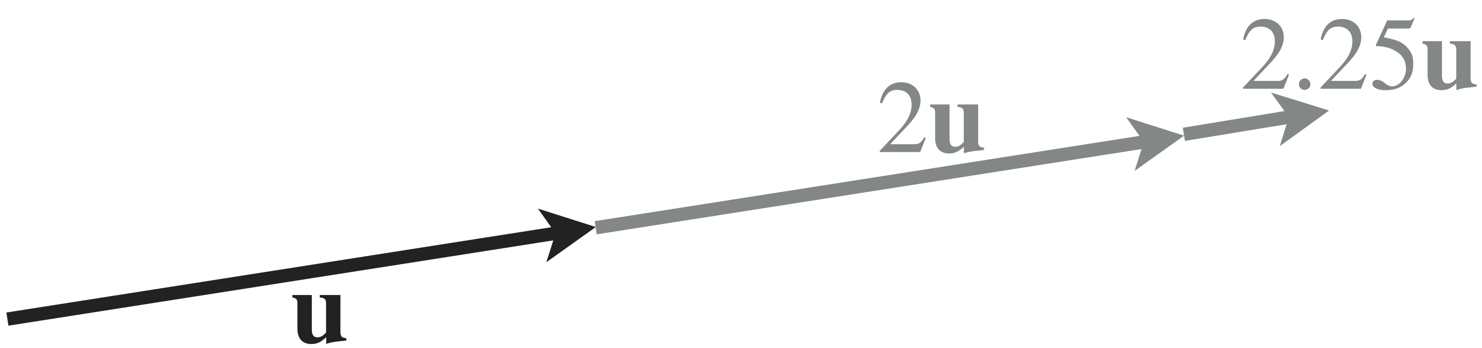
- **Addition:** connect them end to end



- Walk along $\mathbf{u}$ first and then $\mathbf{v}$
- Or, along $\mathbf{v}$ first and then $\mathbf{u}$
- The order doesn't matter: **Commutative.**

VECTORS
SCALAR MULTIPLICATION

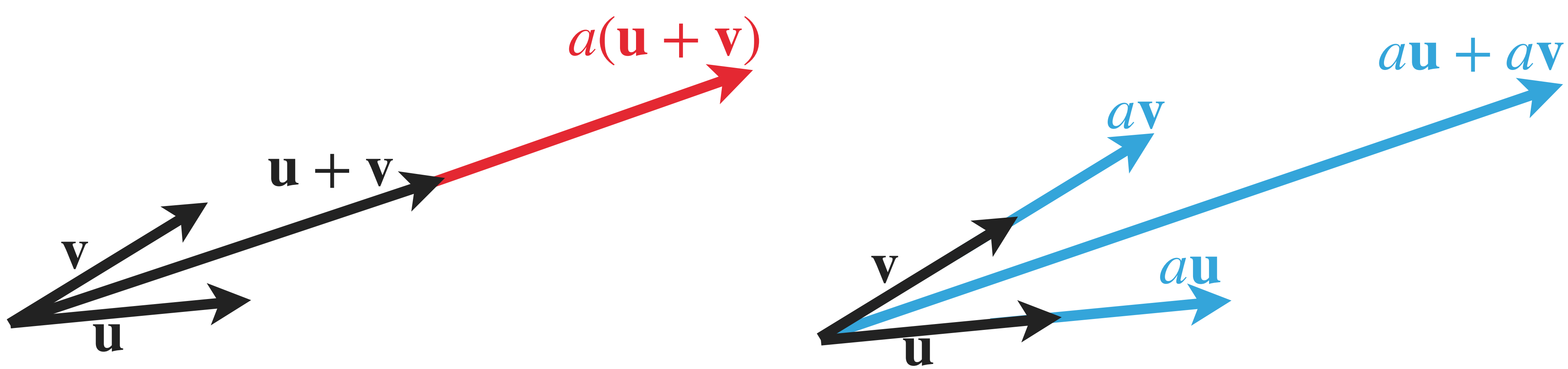
- Scaling a vector



- Multiplying a vector $\mathbf{u}$ by a scalar a gives a new vector $a\mathbf{u}$
- Similar geometric scaling of the "arrows", give: $a(b\mathbf{u}) = (ab)\mathbf{u}$

VECTORS
ADDITION AND SCALING

- Scaling the sum of two vectors



- Or addition of two scaled vectors
- Same result either way: **Distributive!**

VECTOR SPACE

- For all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and scalars a, b:
  - $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
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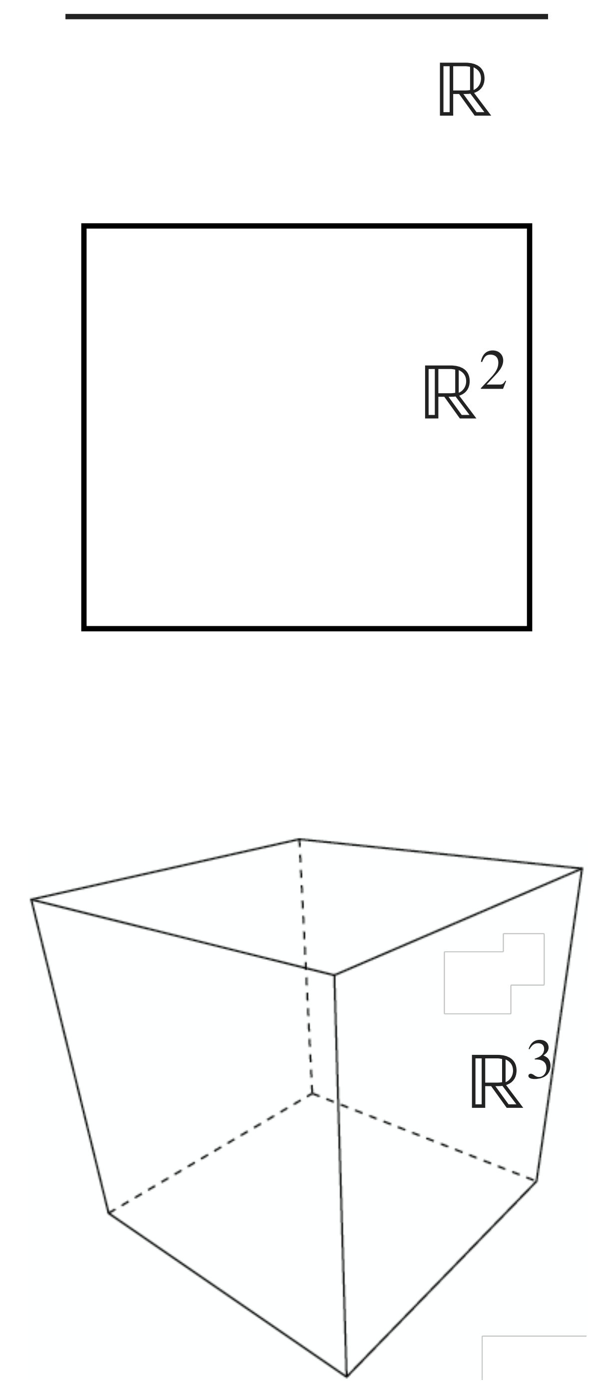
  - $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
  - $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$
- **Vector space:** collection of objects that satisfy these properties.

EUCLIDEAN VECTOR SPACE

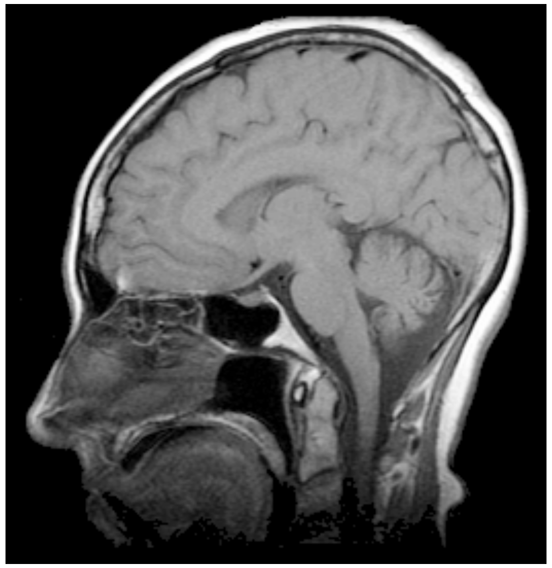
- Most common example of a vector space.
  - n-dimensional Euclidean space (denoted: $\mathbb{R}^n$)
  - E.g. $(2, -\sqrt{3}, \pi/7)$ is a point in $\mathbb{R}^3$
  - **QUESTION:** What about $(2,\ 1,\ 0,\ \pi/7,\ 0)$?
- **Can be easily encoded on a computer**
  - "Just" a list of floating-point numbers



OTHER IMPORTANT VECTORS

- Functions are a very important example of vector spaces
  - We can certainly add them
  - Or scale them
  - The other properties (of a vector space) all hold

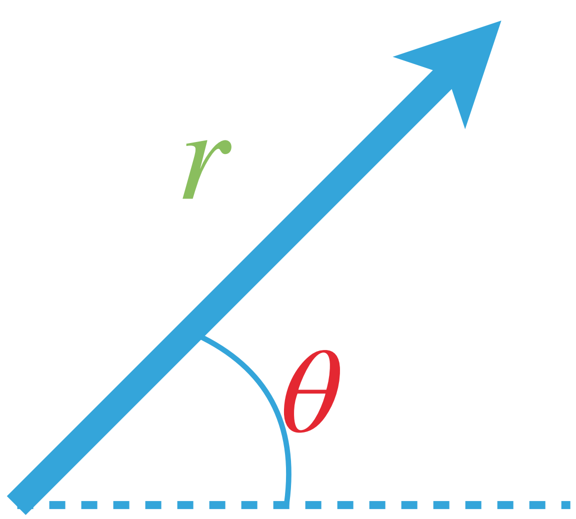
Images

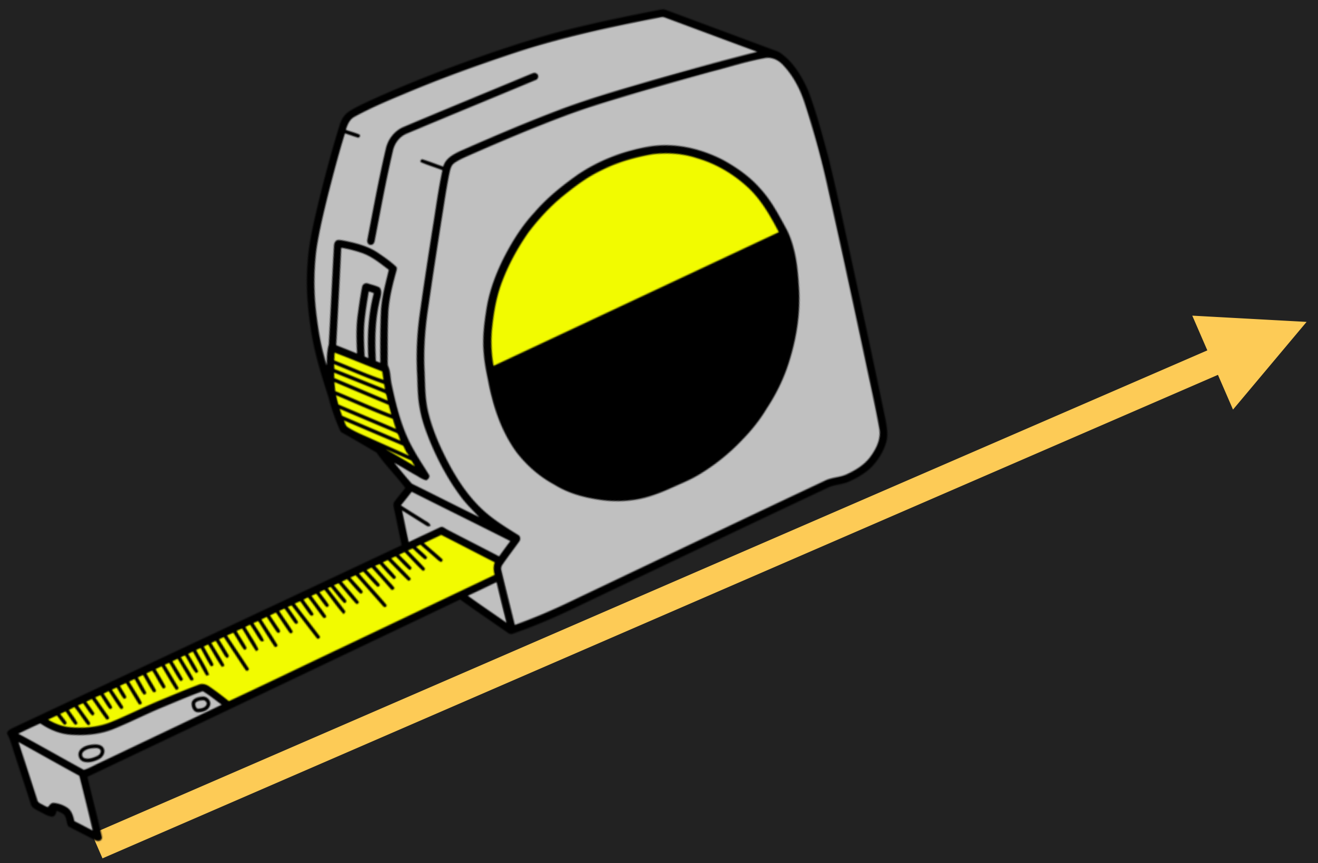


Irradiance from light sources

MEASURING VECTORS

- **Question:** What information do vectors encode?
  - Magnitude and direction
- How can we *measure* these quantities





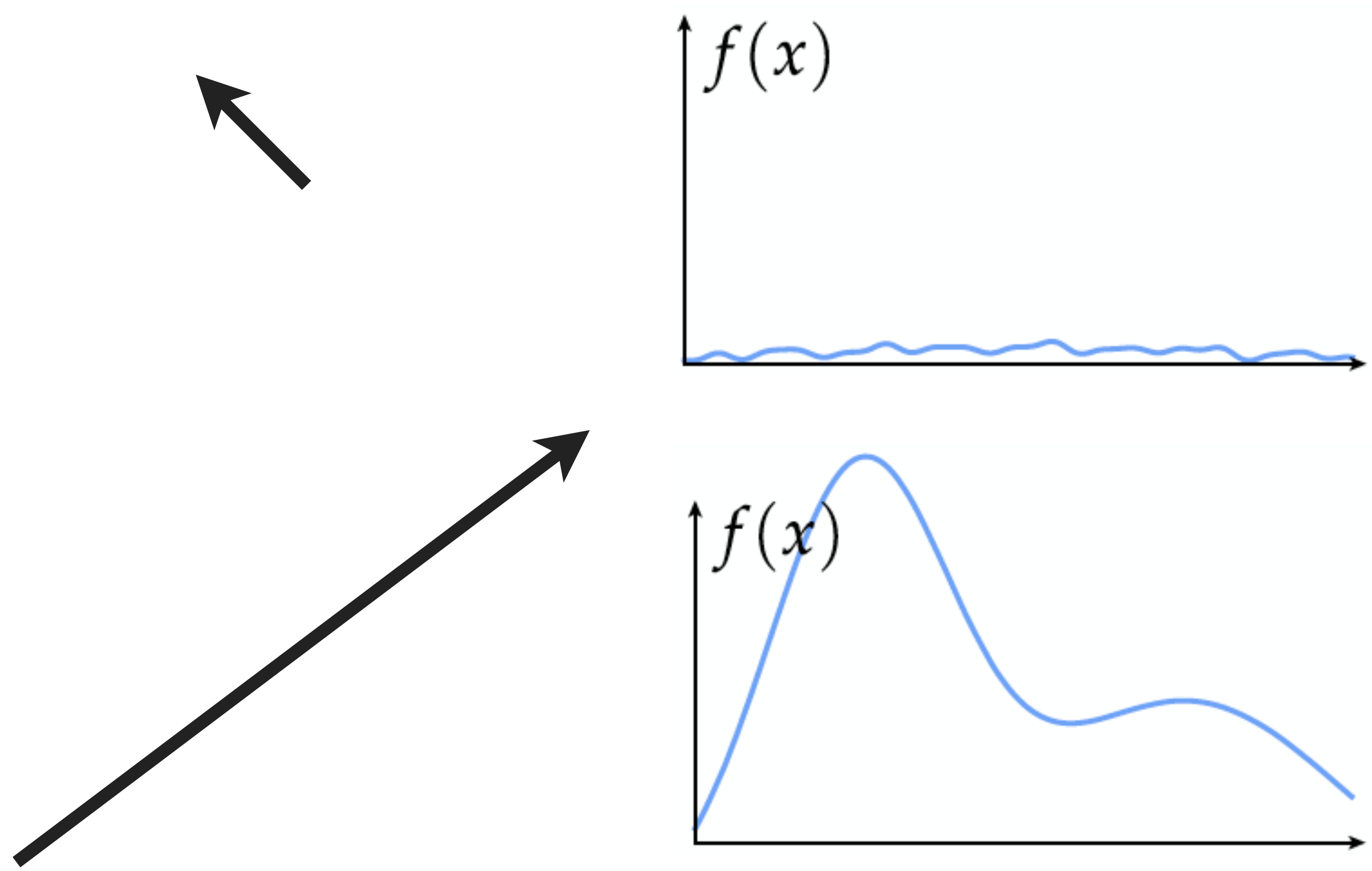
NORMS

NORM OF A VECTOR

- **Norm:** magnitude, or length or size of a vector
    - Intuitively, captures some notion of how “large” the vector is

SMALL NORM

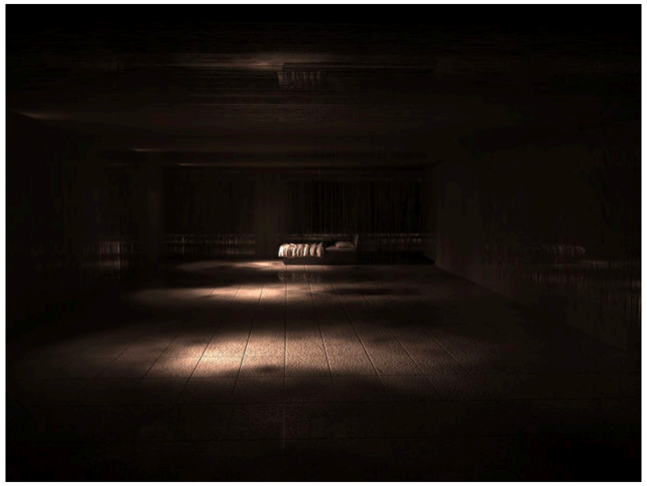
LARGE NORM



NORM OF A VECTOR

- **Norm:** magnitude, or length or size of a vector
  - Intuitively, captures some notion of how "large" the vector is

Question: Which has the larger norm?



LARGE NORM

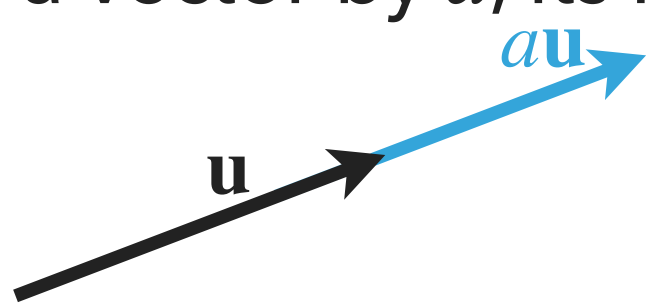
SMALL NORM

PROPERTIES OF A NORM I

- What properties should we expect the norm (length) of a vector should satisfy?
  - **Non-negativity:**
    - $\|\mathbf{u}\| \geq 0$
  - It should be zero *if and only if* the vector it itself the zero vector:
    - $\|\mathbf{u}\| = 0 \iff \mathbf{u} = \mathbf{0}$

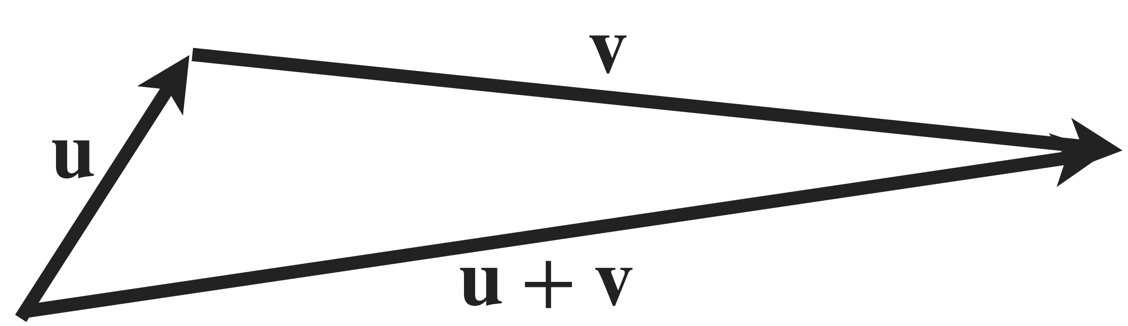
PROPERTIES OF A NORM II

- **Scaling** – if we scale a vector by a, its norm is also scaled by a



$$\|a\mathbf{u}\| = |a|\,\|\mathbf{u}\|$$

- **Triangle inequality** – shortest point between any two points is a straight line



$$\|\mathbf{u}\| + \|\mathbf{v}\| \geq \|\mathbf{u}+\mathbf{v}\|$$

NORM

FORMAL DEFINITION

- A norm is any *function* that maps each vector (of a vector space) to a scalar value, and satisfies the properties:
  - $\|\mathbf{u}\| \geq 0$
  - $\|\mathbf{u}\| = 0 \iff \mathbf{u} = \mathbf{0}$
  - $\|a\mathbf{u}\| = |a| \, \|\mathbf{u}\|$
  - $\|\mathbf{u}\| + \|\mathbf{v}\| \geq \|\mathbf{u} + \mathbf{v}\|$

EUCLIDEAN NORM IN CARTESIAN COORDINATES

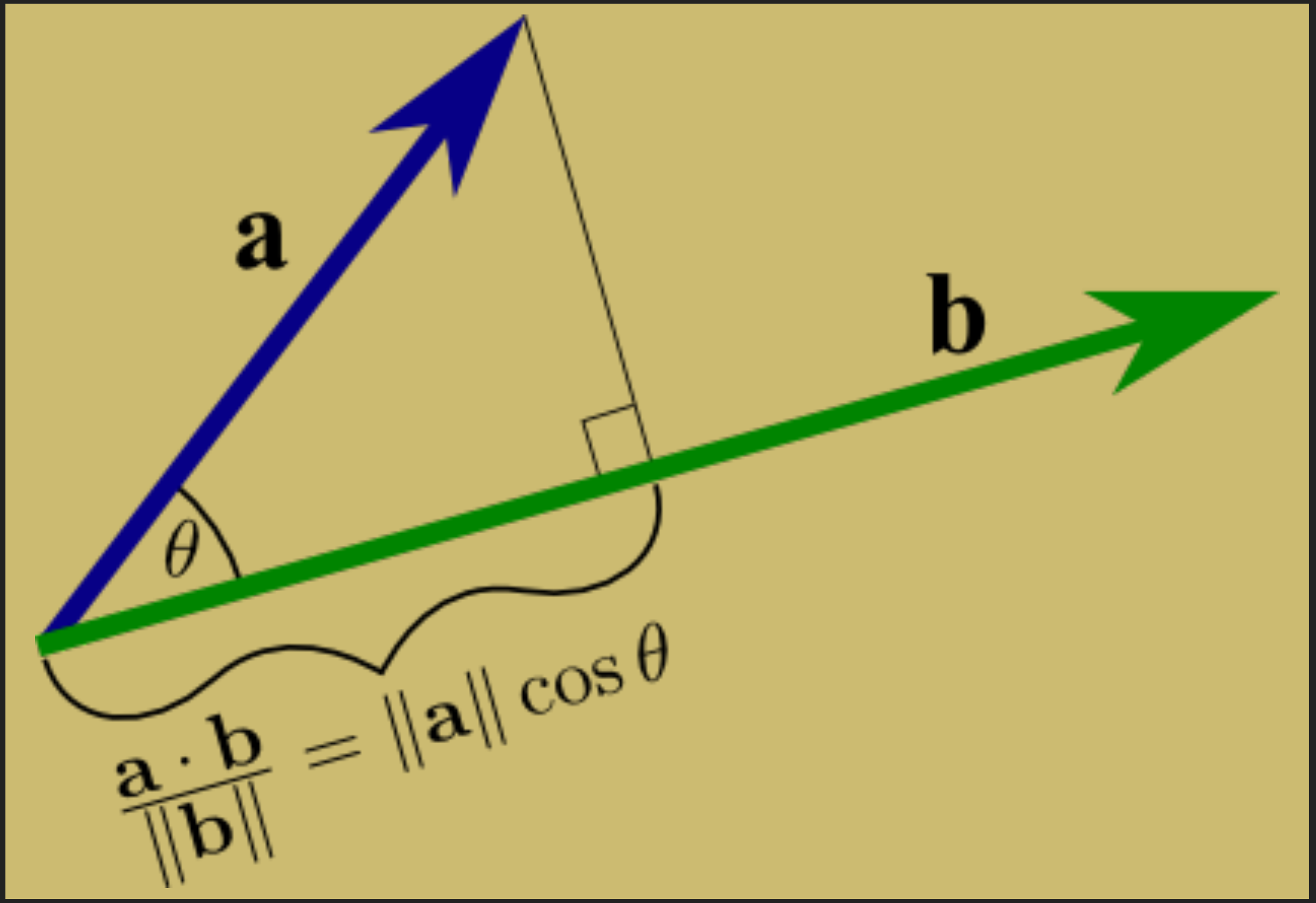
- The Euclidean norm for n-dimensional vector space
  - For $\mathbf{u} \in \mathbb{R}^n$, we use the notation: $\mathbf{u} = (u_1, u_2, \ldots, u_n)$

$$\boxed{\|\mathbf{u}\| = \|(u_1, u_2, \ldots, u_n)\| := \sqrt{\sum_{i=1}^{n} (u_i)^2}}$$

- **Example:** Compute $\|\mathbf{u}\|$ for $\mathbf{u} = (3,4)$, and $\mathbf{u} = (3,0,0,4)$
  - We will see definitions of other **norms** and **semi-norms**
  - Can be extended to **functions**

MORE EXAMPLES

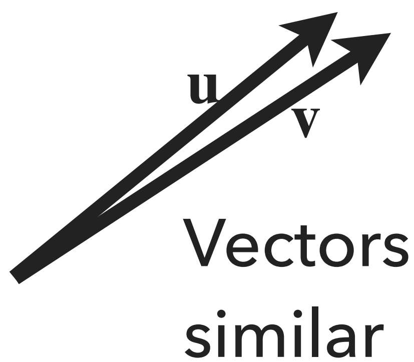
- Find $\|\boldsymbol{u}\|$ for the following
  - $\boldsymbol{u} = (1, 2, 3)$
  - $\boldsymbol{u} = (0, 0, 0)$
  - $\boldsymbol{u} = (0, 0, 1, 1, 2, 1)$

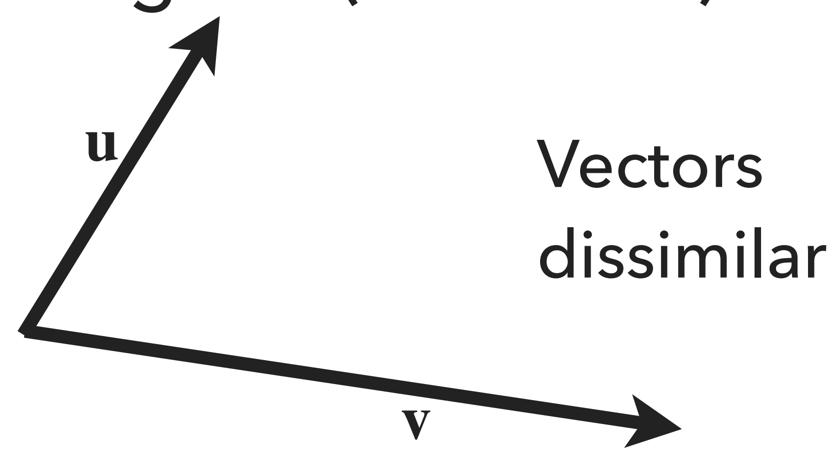


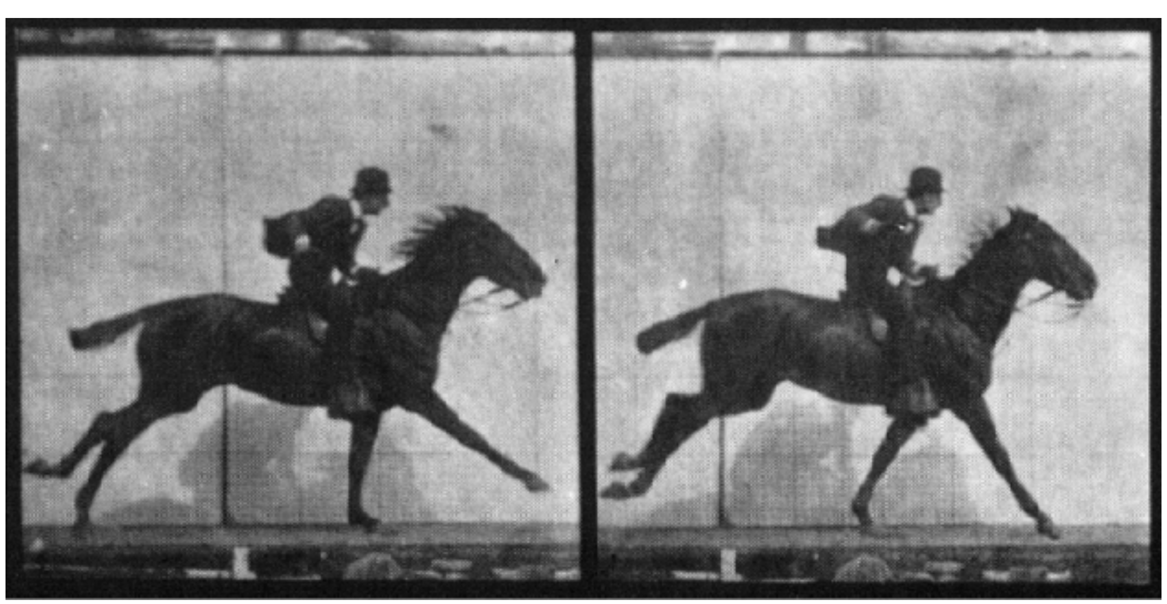
INNER PRODUCTS

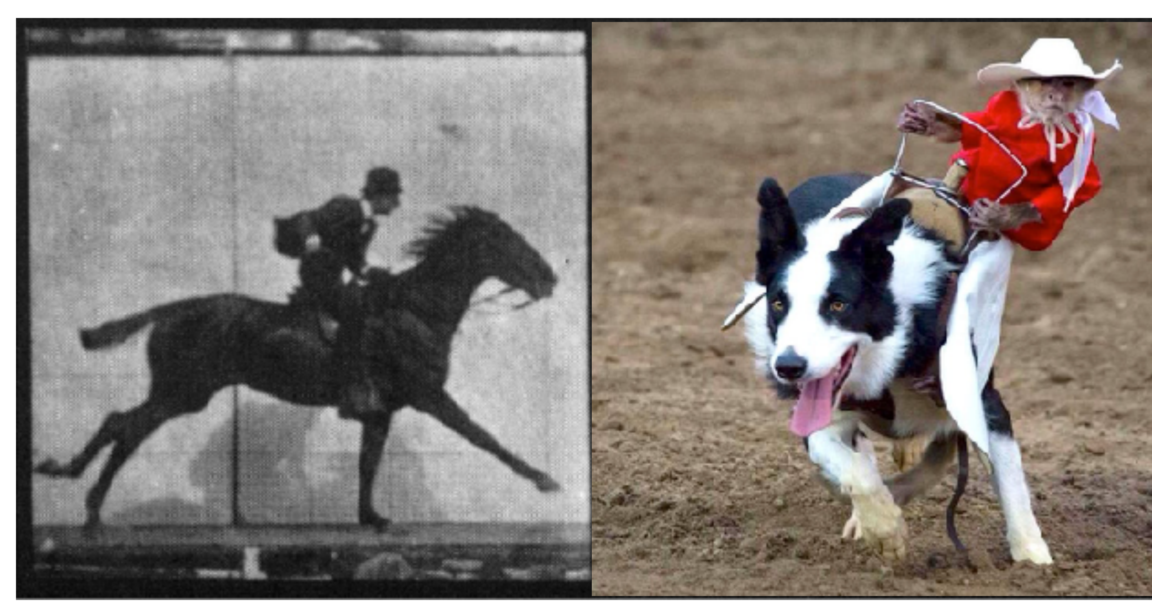
INNER PRODUCT

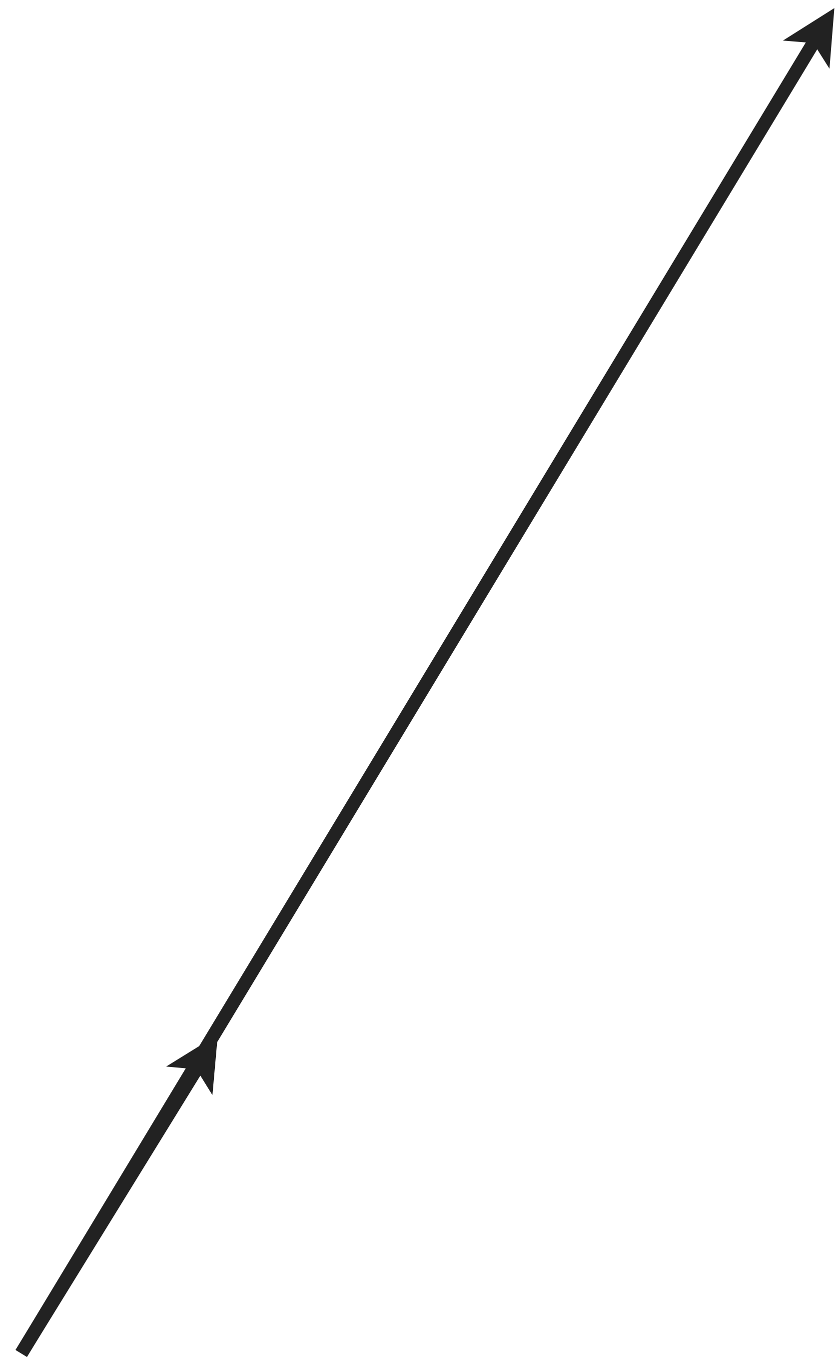
- Vectors have direction (orientation)
  - **Inner product:** measures how aligned (or similar) two vectors are





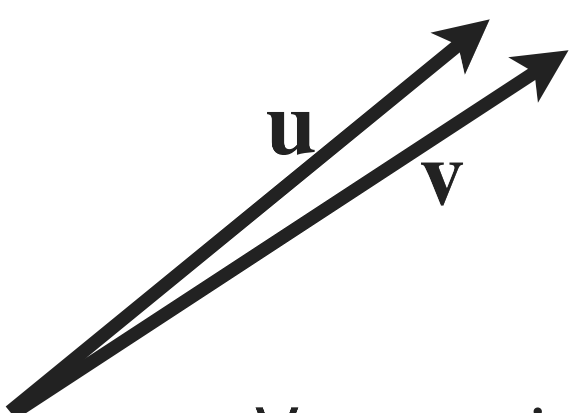




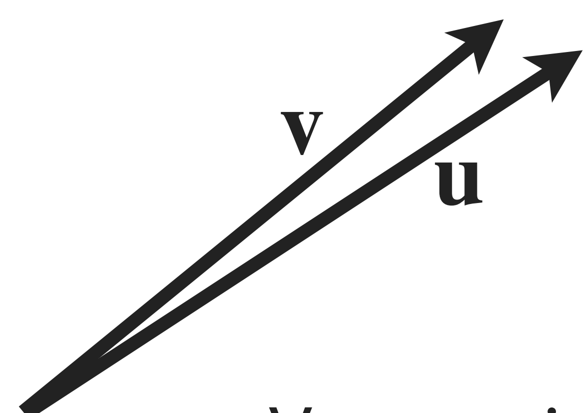


INNER PRODUCT

- We will denote the inner product between vectors $\mathbf{u}$, and $\mathbf{v}$
    - $\langle \mathbf{u}, \mathbf{v} \rangle$
- **Properties:**
    - Order should not matter: $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$



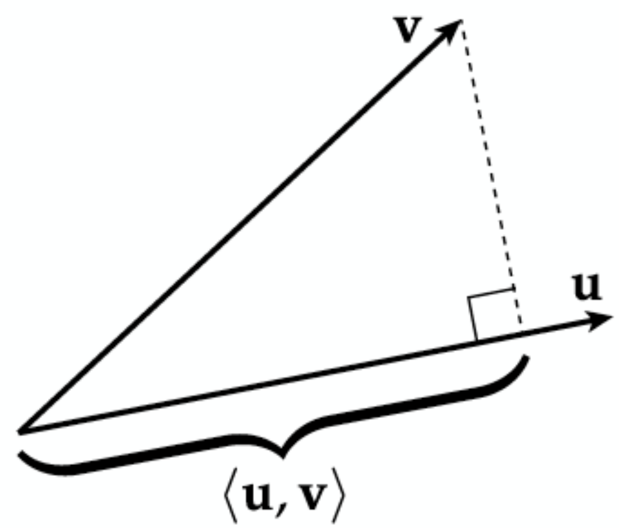
Vectors similar



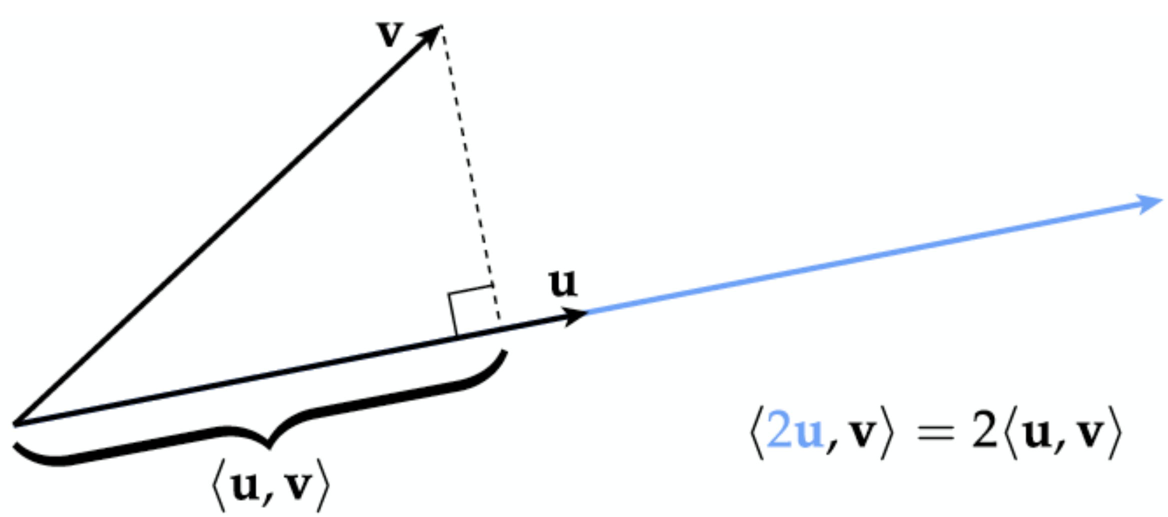
Vectors similar

INNER PRODUCT
PROJECTION AND SCALING

- For unit vectors $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$, i.e. vectors of length one, an **inner product measures the extent of one along the direction of the other**.



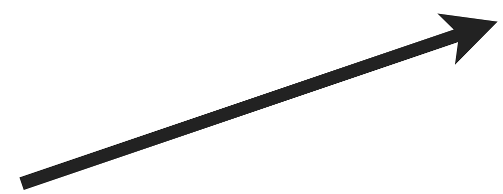
- Scale either vector and the inner product scales accordingly:



INNER PRODUCT
PROJECTION AND SCALING

- A vector should be aligned with itself
  - Inner product with itself is positive (or at least, non-negative)

$$\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$$



- For unit length vector $\mathbf{u}$, must have:

$$\langle \mathbf{u}, \mathbf{u} \rangle = 1$$

- **Question:** For general vector $\mathbf{u}$, what must $\langle \mathbf{u}, \mathbf{u} \rangle$ be?

INNER PRODUCT
FORMAL DEFINITION

- **Inner product:** is any function that maps any two vectors to a scalar number $\langle \mathbf{u}, \mathbf{v} \rangle$, such that the following properties hold:
  - $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
  - $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$
  - $\langle \mathbf{u}, \mathbf{u} \rangle = 0 \iff \mathbf{u} = 0$
  - $\langle a\mathbf{u}, \mathbf{v} \rangle = a\langle \mathbf{u}, \mathbf{v} \rangle$
  - $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

INNER PRODUCT
CARTESIAN COORDINATES

- **Inner product:** for two n-vectors

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle (u_1, \ldots, u_n), (v_1, \ldots, v_n) \rangle := \sum_{i=1}^{n} u_i v_i$$

- **Examples:** for the following pairs of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, find the inner products:
  - $\boldsymbol{u} = (1, 2, 3)$, $\boldsymbol{v} = (1, 2, 3)$
  - $\boldsymbol{u} = (1, 2, 3)$, $\boldsymbol{v} = (0, 0, 0)$

WHAT WE COVERED TODAY

- Importance of linear algebra in computational imaging
- Vectors and Vector spaces
  - Norms
  - Inner products

Please Install Matlab for next lesson

SEE YOU NEXT TIME!

HILBERT SPACE