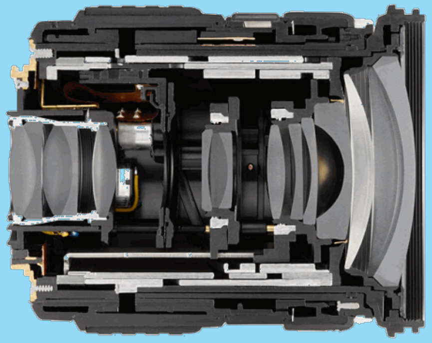
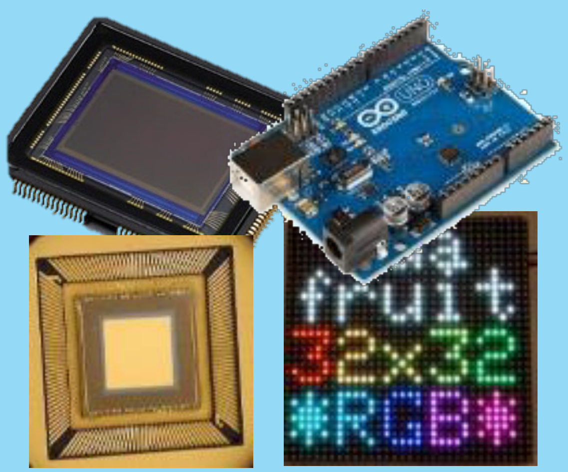


Optics

+



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COMPUTATIONAL METHODS FOR IMAGING (AND VISION)

LECTURE 17: INTRO TO INVERSE PROBLEMS (IN IMAGING)

PROF. JOHN MURRAY-BRUCE

WHERE ARE WE

WE ARE HERE!

10	15-Mar-21	Forward Models and Inverse Problems	Linear Inversion - Inverse problems - Deconvolution and Denoising	IIP 4, Appendix E	**HW 4**	
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	5-May-21	**Final Exam: 12:30 PM - 2:30 PM**				

OUTLINE

- Introduction to inverse problems
- Ill-posed and ill-conditioned inverse problems
- Non-regularized solution to deconvolution problem

LEARNING GOALS

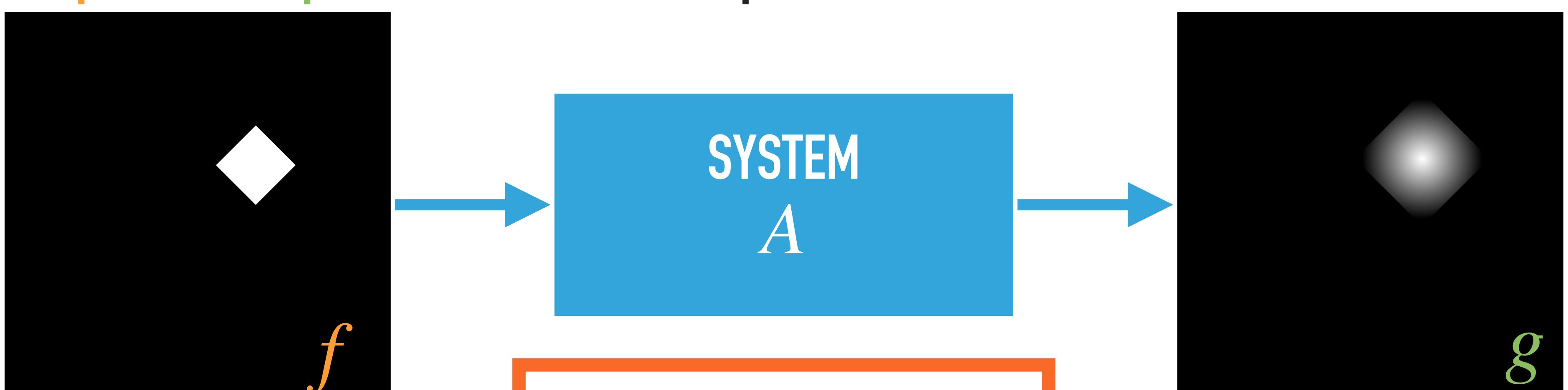
- Correctly define forward and inverse problems
- Determine the conditioning of an inverse problem
- Understand non-regularized solutions to inverse problems and limitations

READING

- IIP Chapter 4

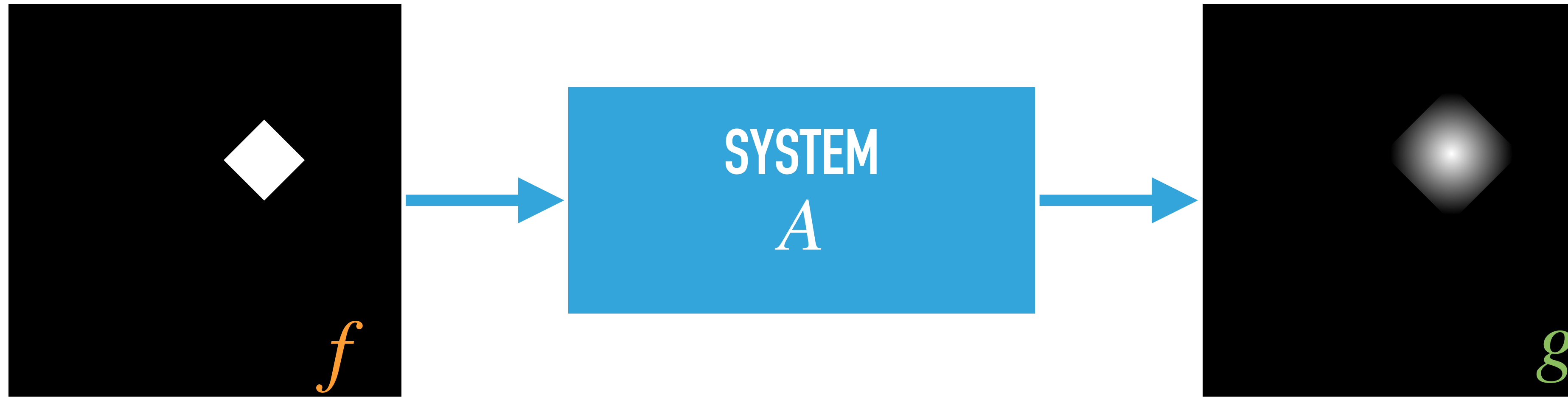
FORWARD MODELS

- **Forward models** provide a way to simulate the output of an imaging system in response to a known input
- Simply: input-output relationship



$$\boxed{g = A(f)}$$

FORWARD MODELS
TWO MAIN CLASSES



$$g = A(f)$$

- **Continuous forward models:**
  - Linear shift invariant systems, $g(x, y) = A(f) = h(x, y) \star f(x, y)$
  - Linear shift-varying models

$$g(x, y) = A(f) = \int h(x, x') f(x') \, \mathrm{d}x'$$

  - Particular form generally depends on the system, derived from physics
- **Discrete forward models:**
  - Matrix-vector expression $\mathbf{g} = \mathbf{A}\mathbf{f}$
  - Discrete LSI, $\mathbf{g} = \mathbf{a} \star \mathbf{f}$

FORWARD MODELS

FROM CONTINUOUS TO DISCRETE MODELS

Matrix-Vector relationship: $\mathbf{g} = \mathbf{A}\mathbf{f}$

To obtain a **fully discrete forward model** from continuous one

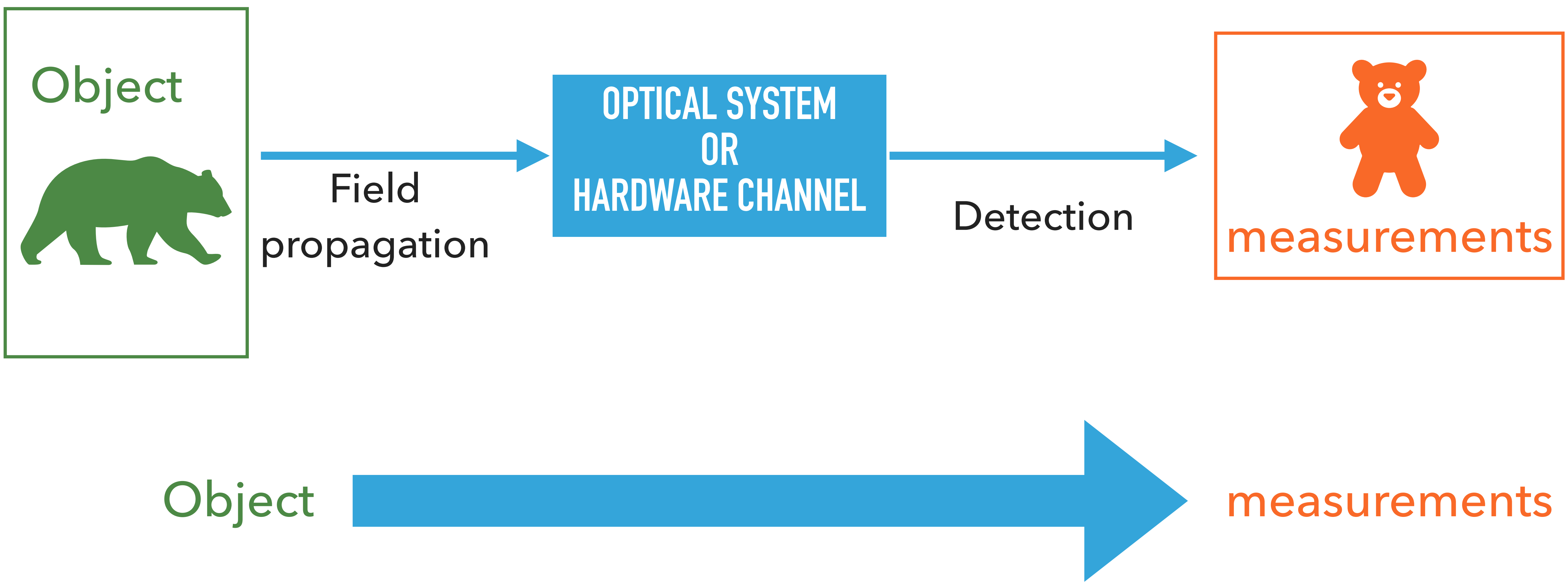
1. **Discretize object** space, continuous-domain object $f(x, y)$ is now represented by N-vector $\mathbf{f} = \begin{bmatrix} f_0 \\ \vdots \\ f_{N-1} \end{bmatrix}$

2. **Sample output** of (optical) system to obtain M-vector $\mathbf{g} = \begin{bmatrix} g_0 \\ \vdots \\ g_{M-1} \end{bmatrix}$

   (m, n) entry of matrix $\mathbf{A}$ is given by $\boxed{\mathbf{A}_{mn} = \langle A\psi_n, p_m \rangle}$

INTRO TO INVERSE PROBLEMS

FORWARD PROBLEM

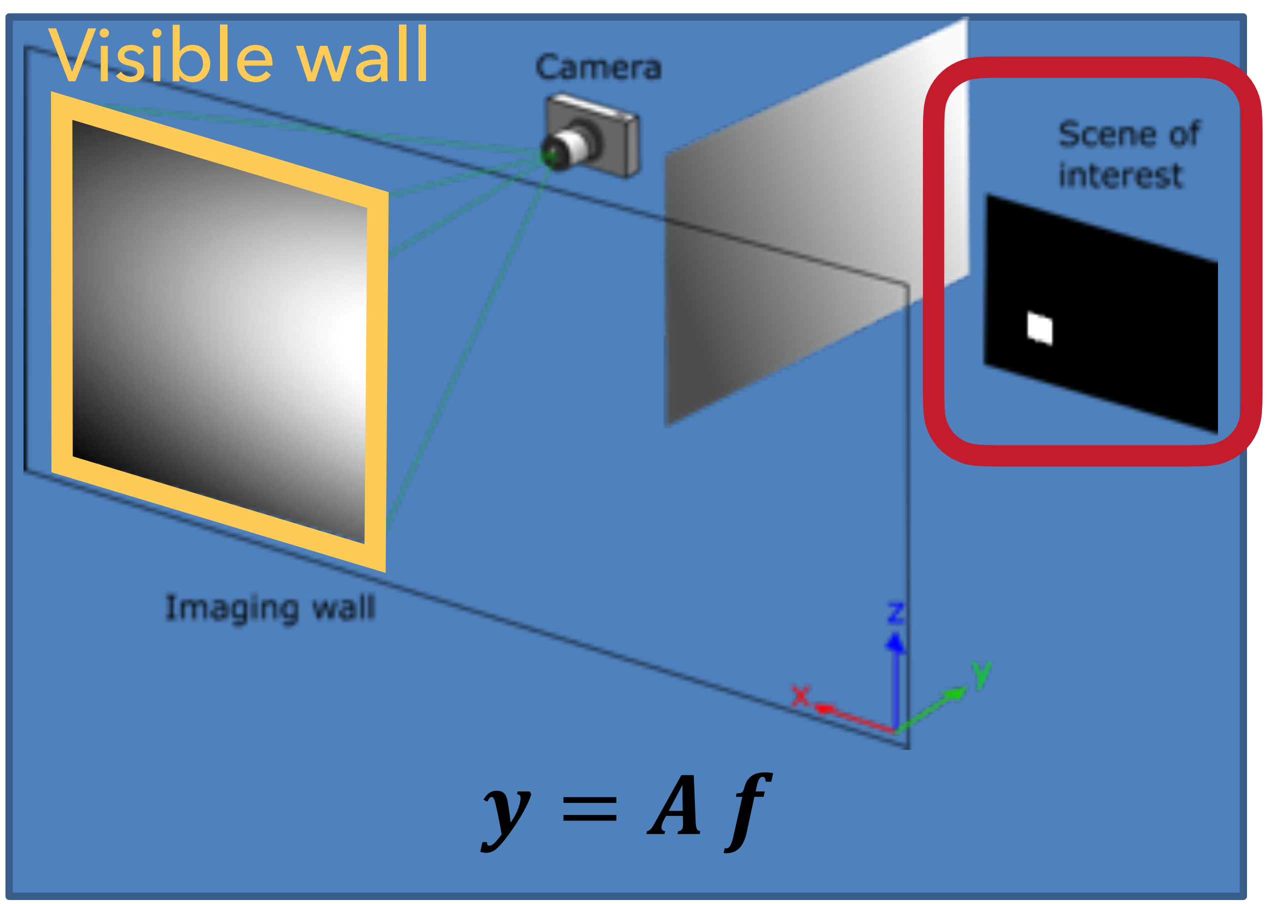


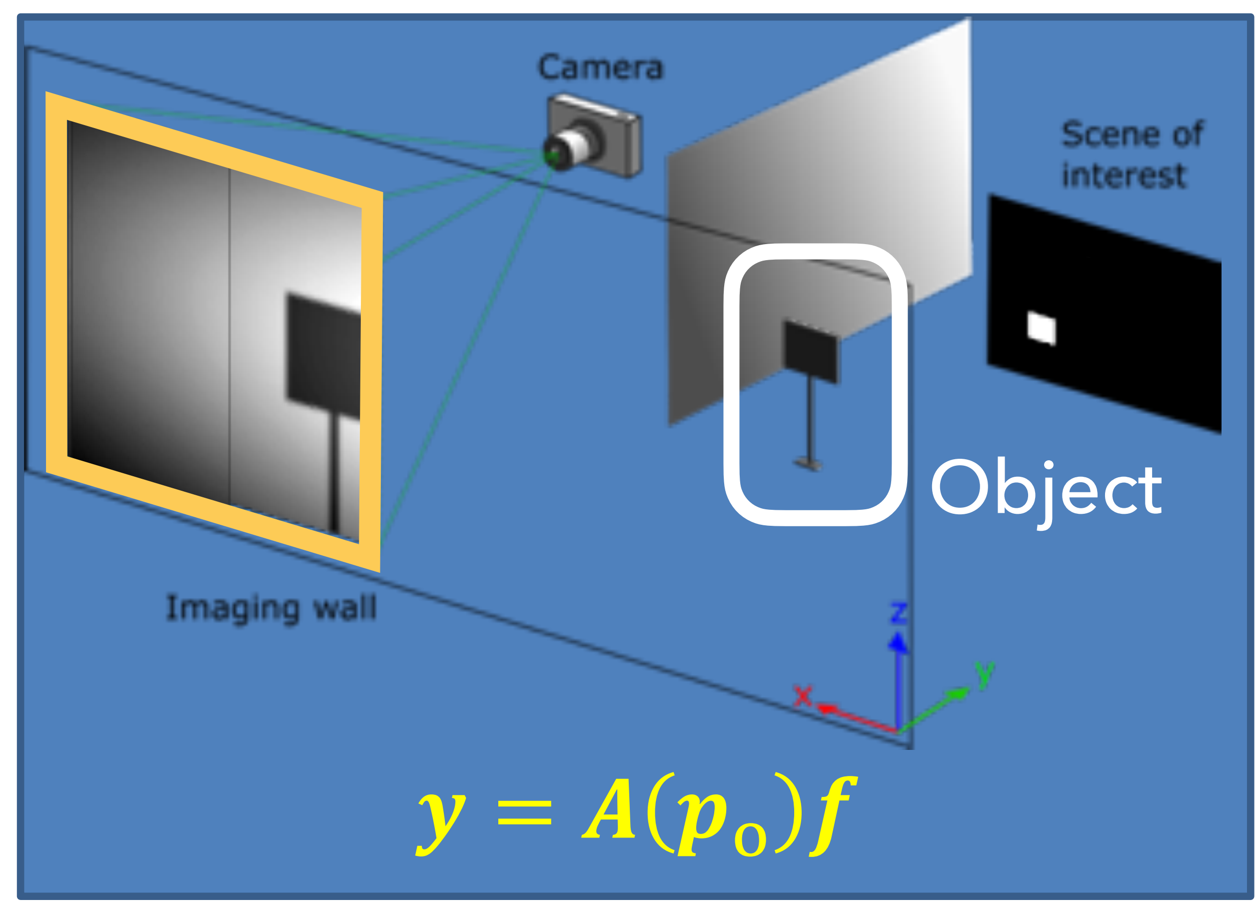
- **Forward problem** answers the question
  - Predict measurements given the object "attributes"

FORWARD PROBLEM

EXAMPLE: PROPAGATION OF LIGHT FROM A HIDDEN SPACE

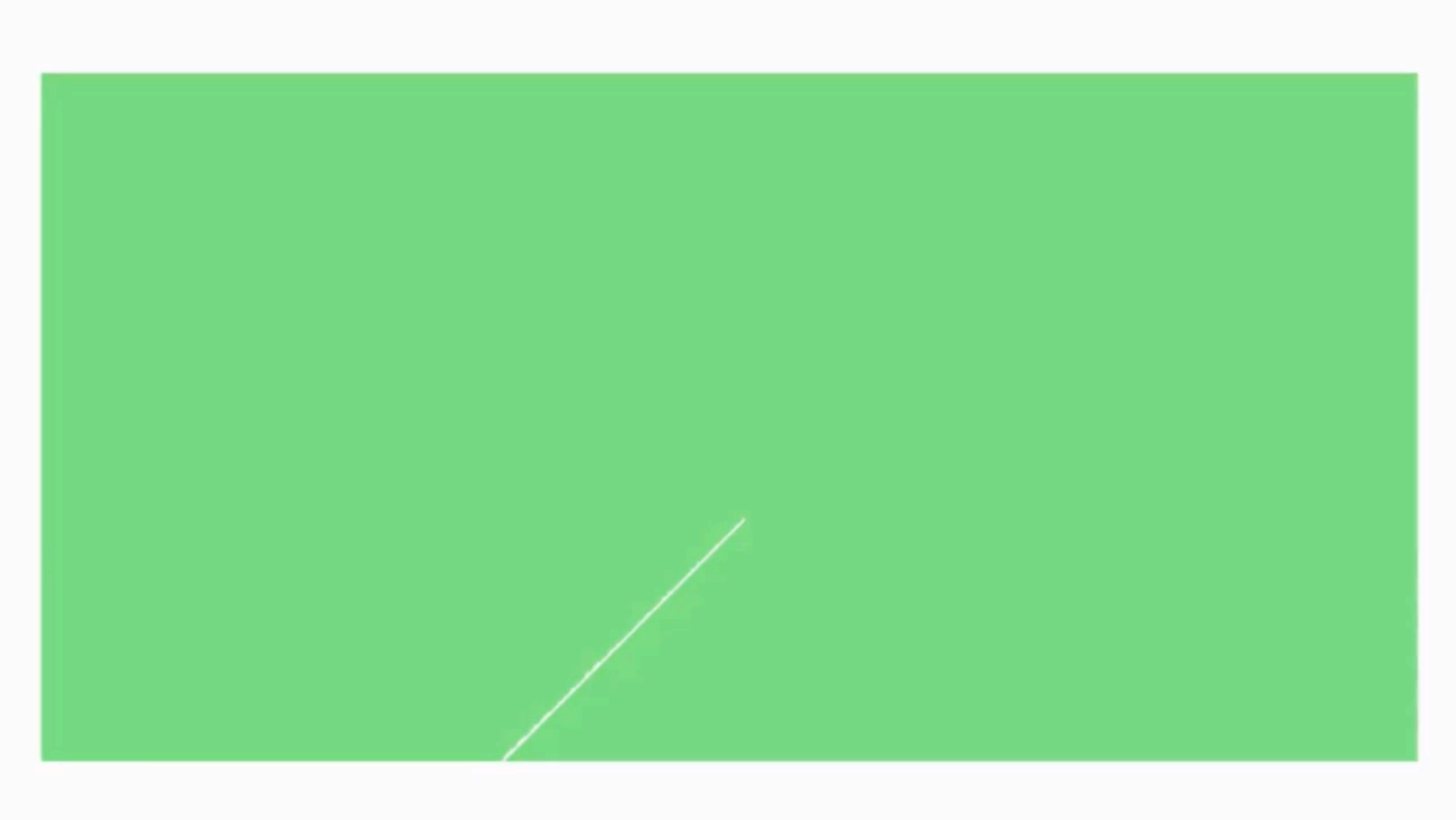
Hidden light source $\mathbf{f} = [0\ 0\ 1\ 0\ \cdots\ 0\ 1\ 0\ 0]^T$





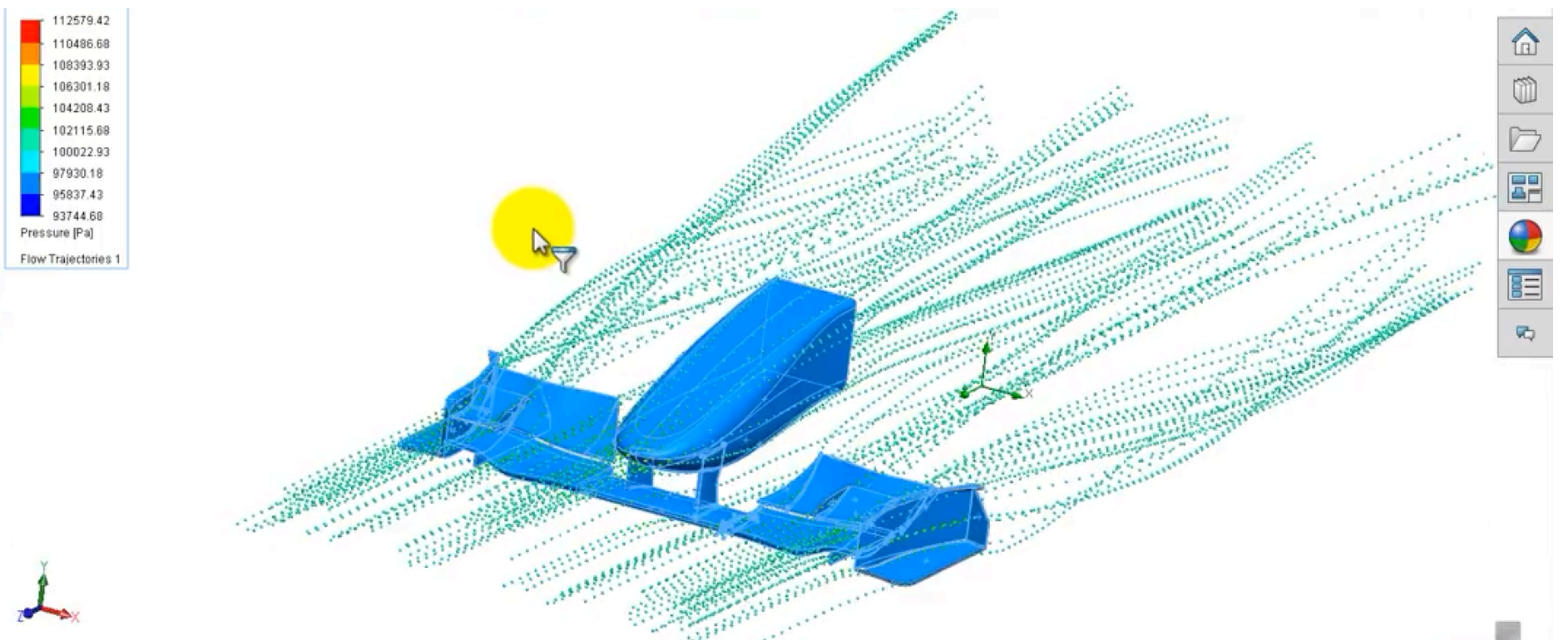
FORWARD PROBLEM

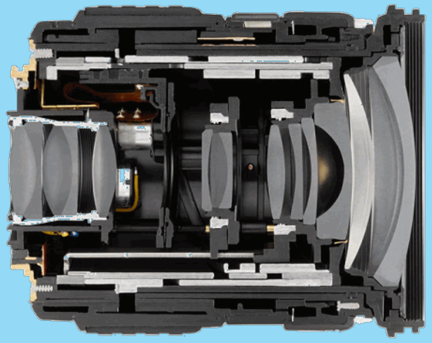
EXAMPLE: PLANE WAVE TRAVELING THROUGH CRACKED BLOCK



FORWARD PROBLEM
EXAMPLE: AIRFLOW

Airflow for the front wing of Formula 1 race car





Optics

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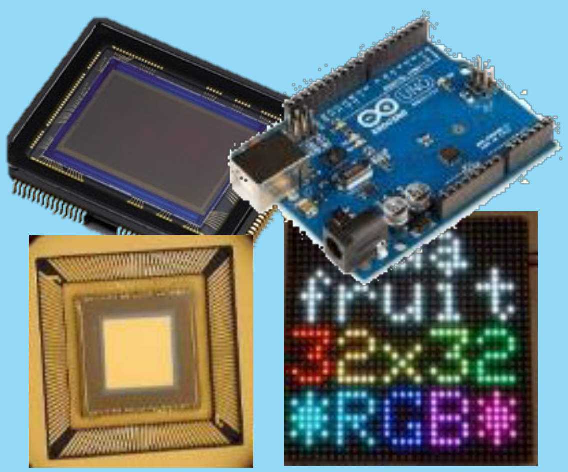

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COMPUTATIONAL METHODS FOR IMAGING (AND VISION)

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PROF. JOHN MURRAY-BRUCE

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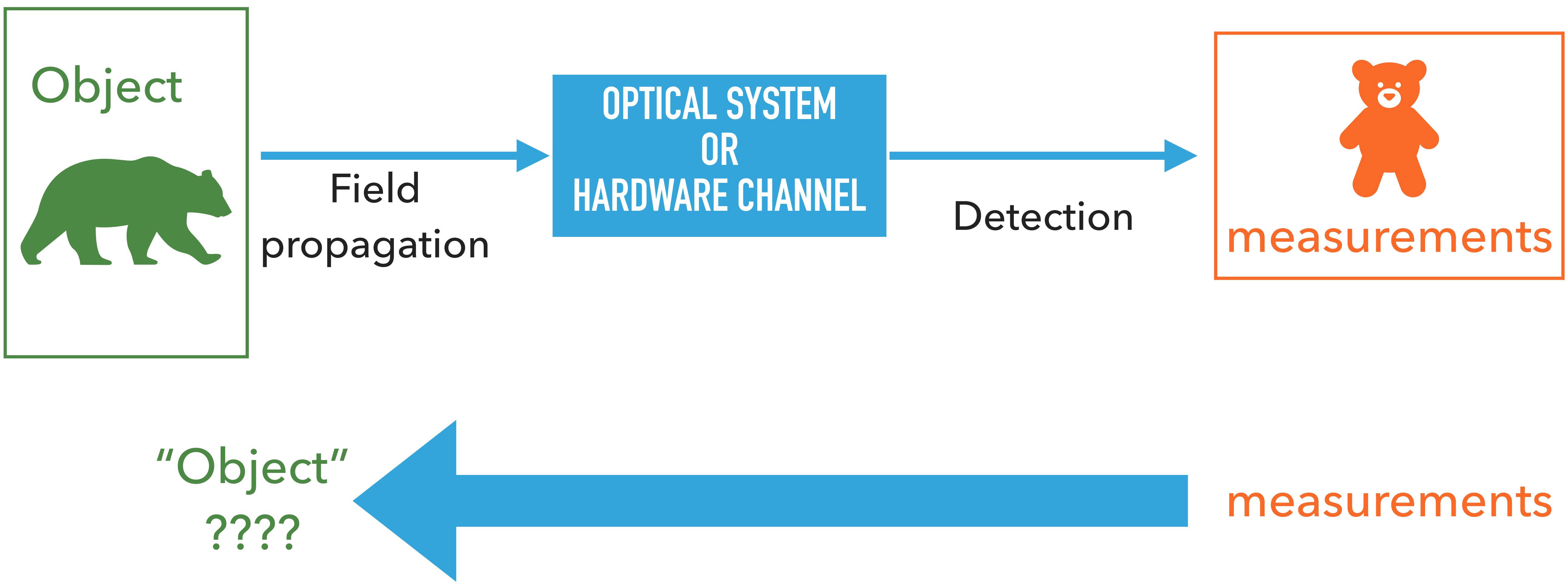
INVERSE PROBLEMS

GIVEN MEASUREMENTS, RECOVER "OBJECT"

INVERSE PROBLEM

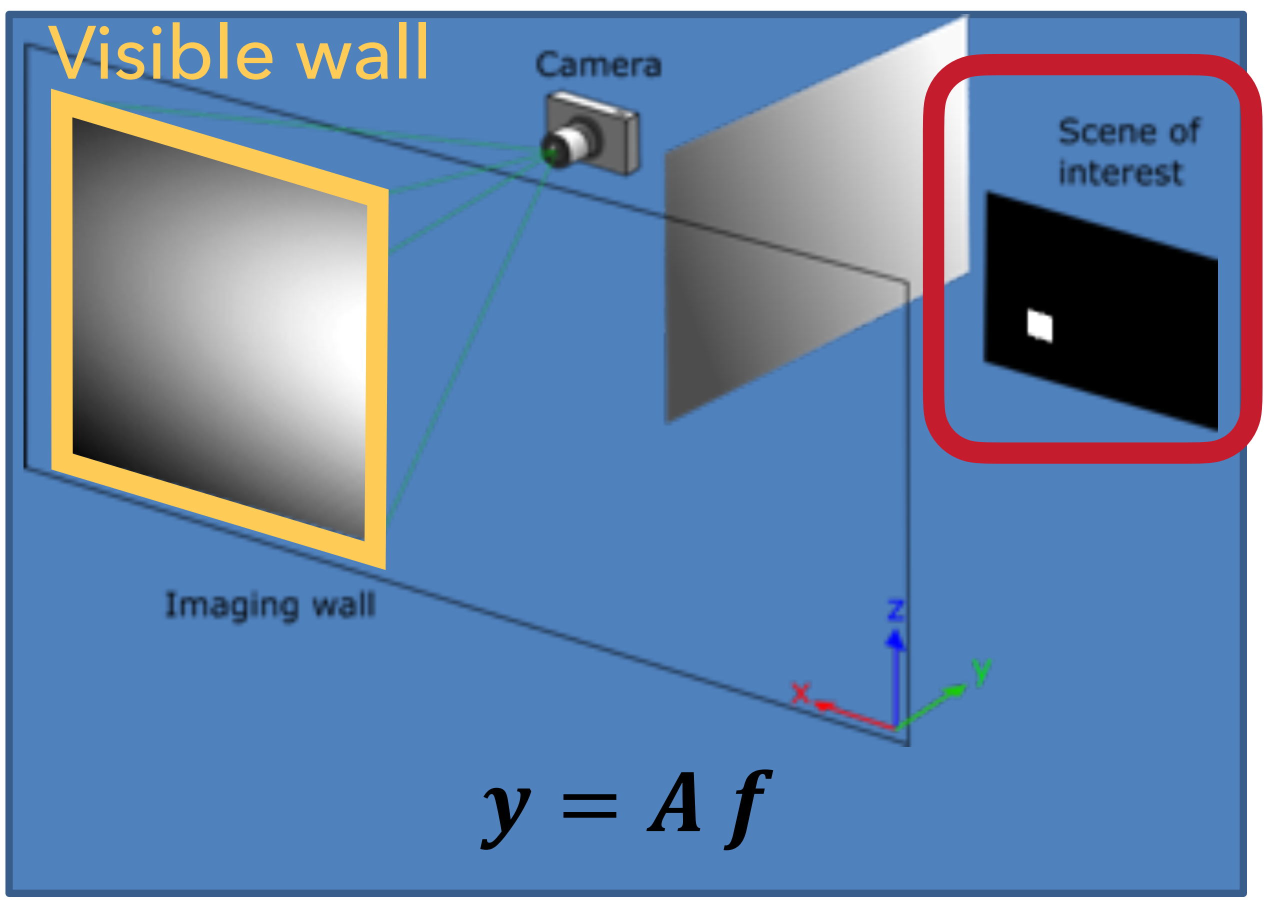


- **Inverse problem** *aims to* answer the question
    - Form an object representation given the measurements

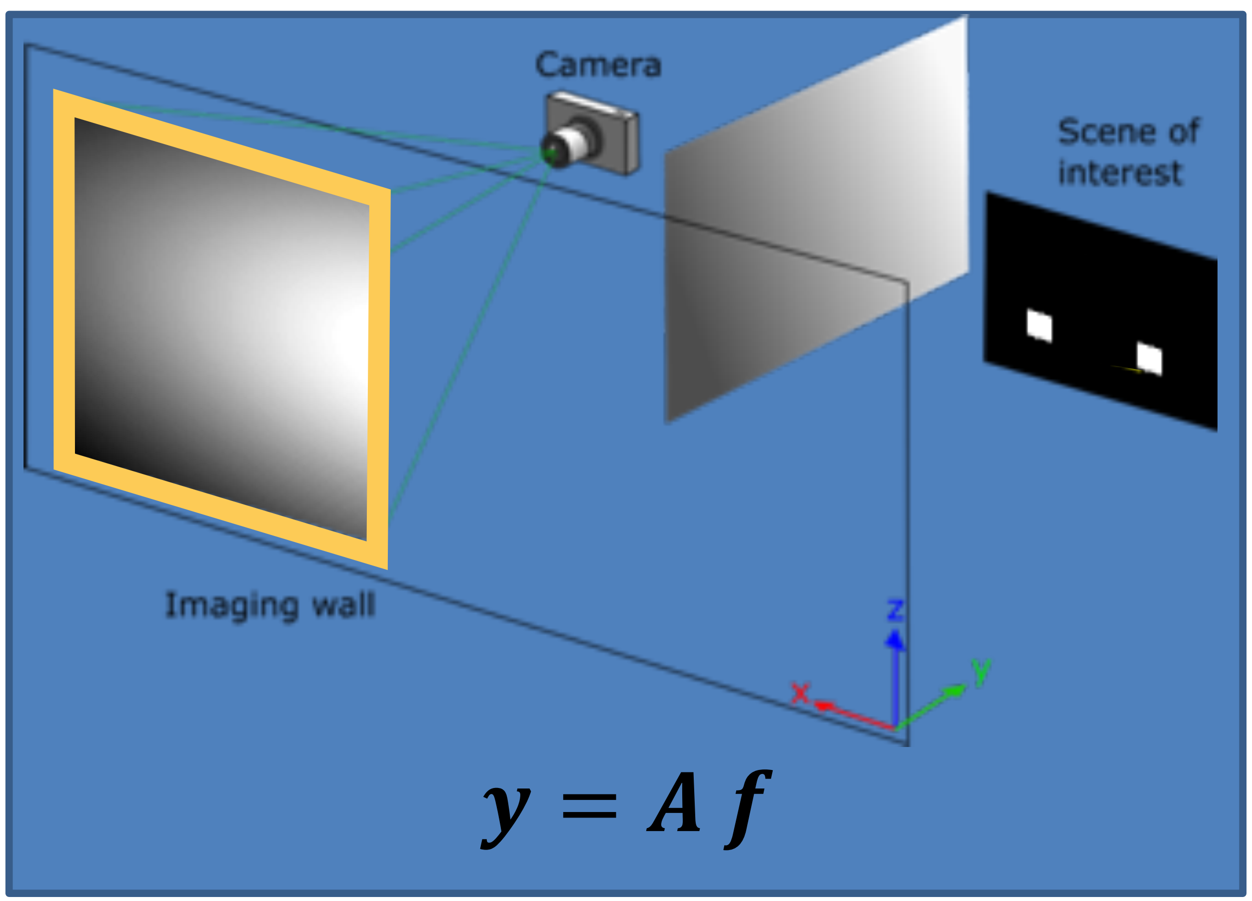
FORWARD PROBLEM

EXAMPLE: PROPAGATION OF LIGHT FROM A HIDDEN SPACE

Hidden light source



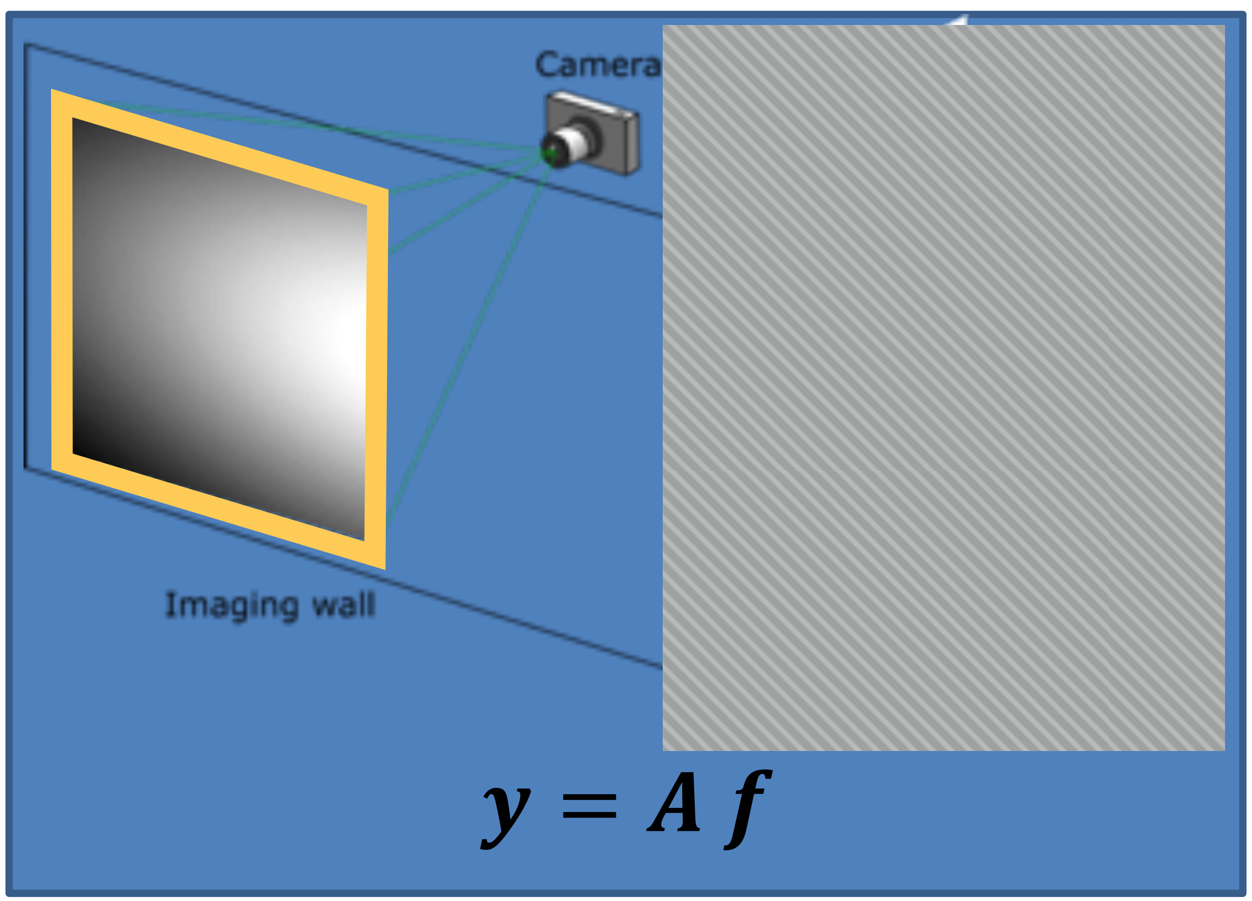
$$y = A\,f$$

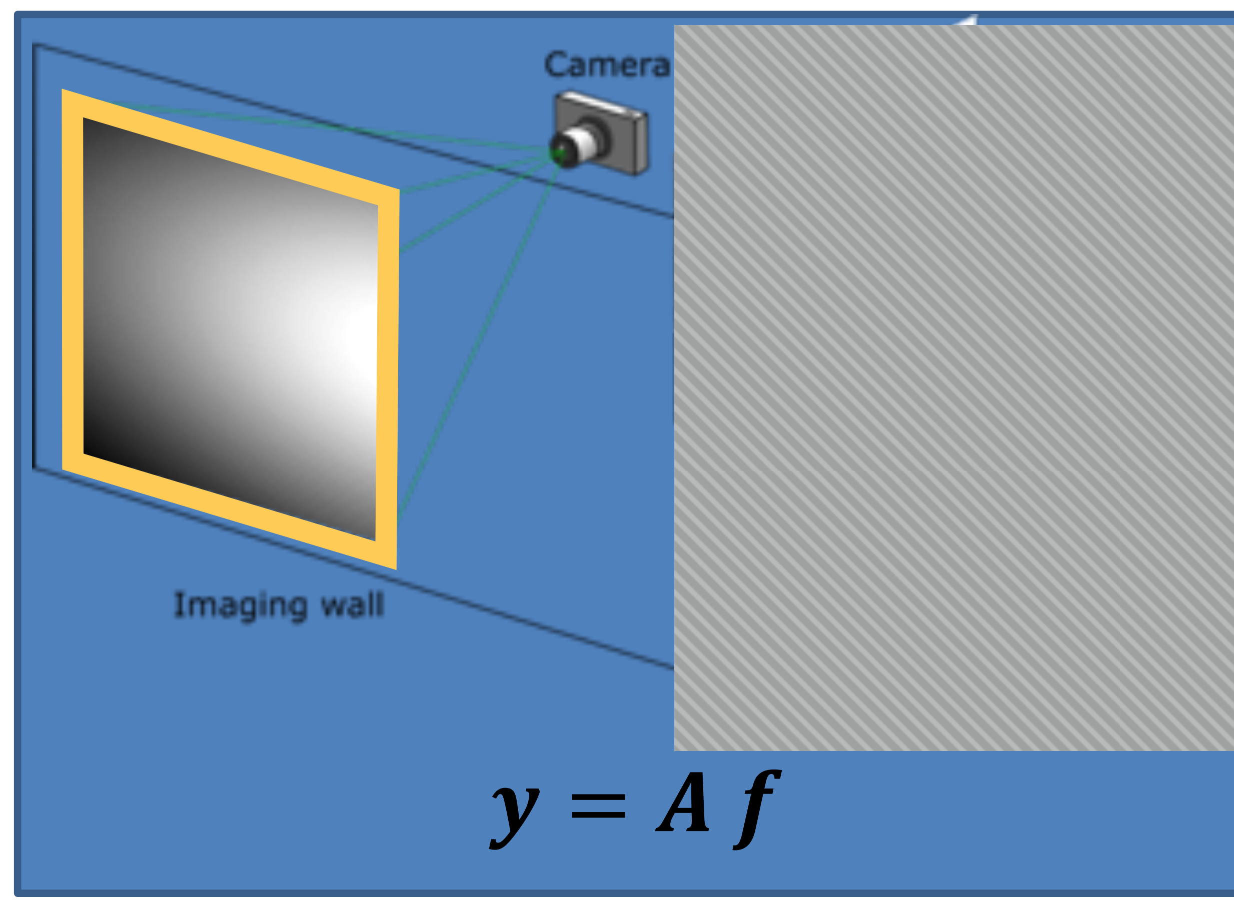


$$y = A\,f$$

FORWARD PROBLEM

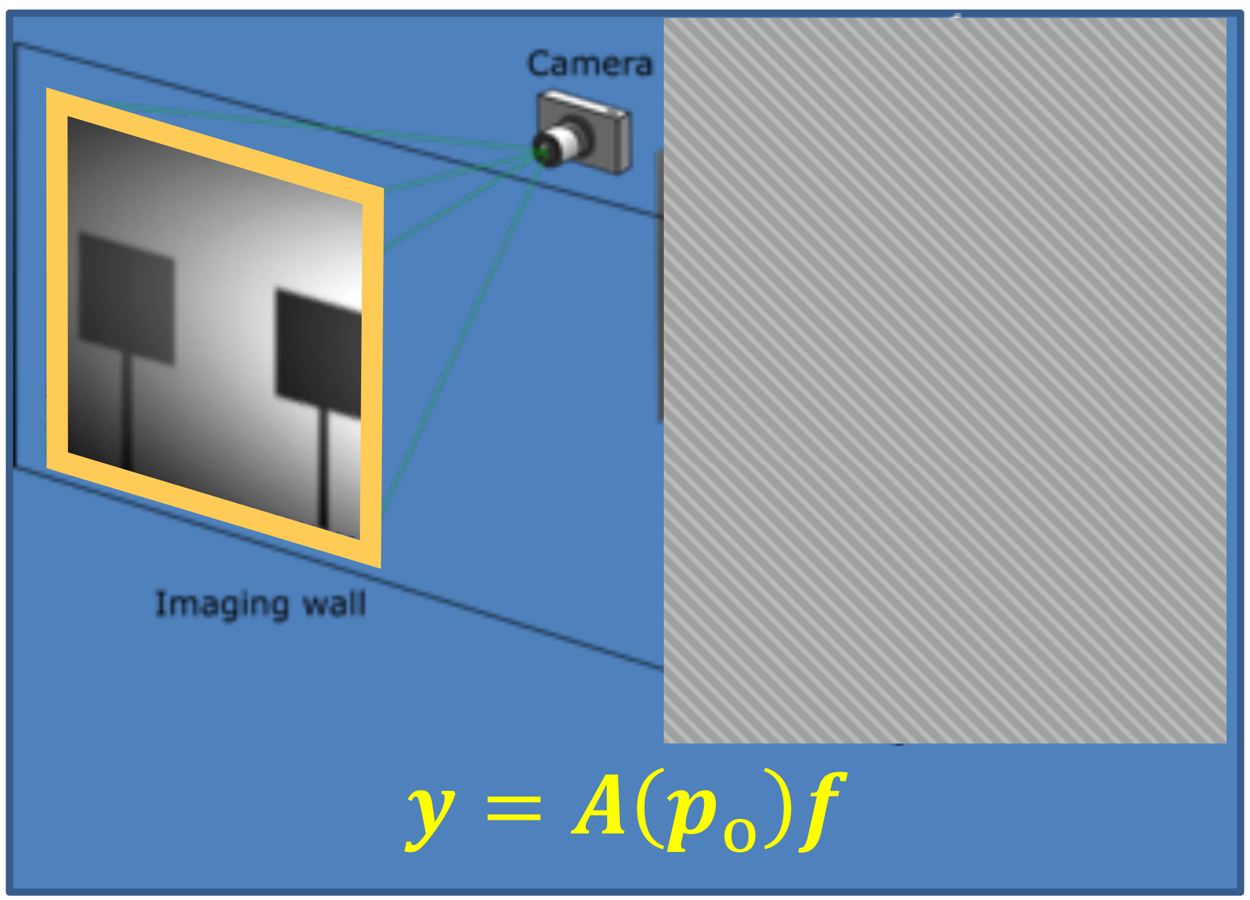
EXAMPLE: PROPAGATION OF LIGHT FROM A HIDDEN SPACE

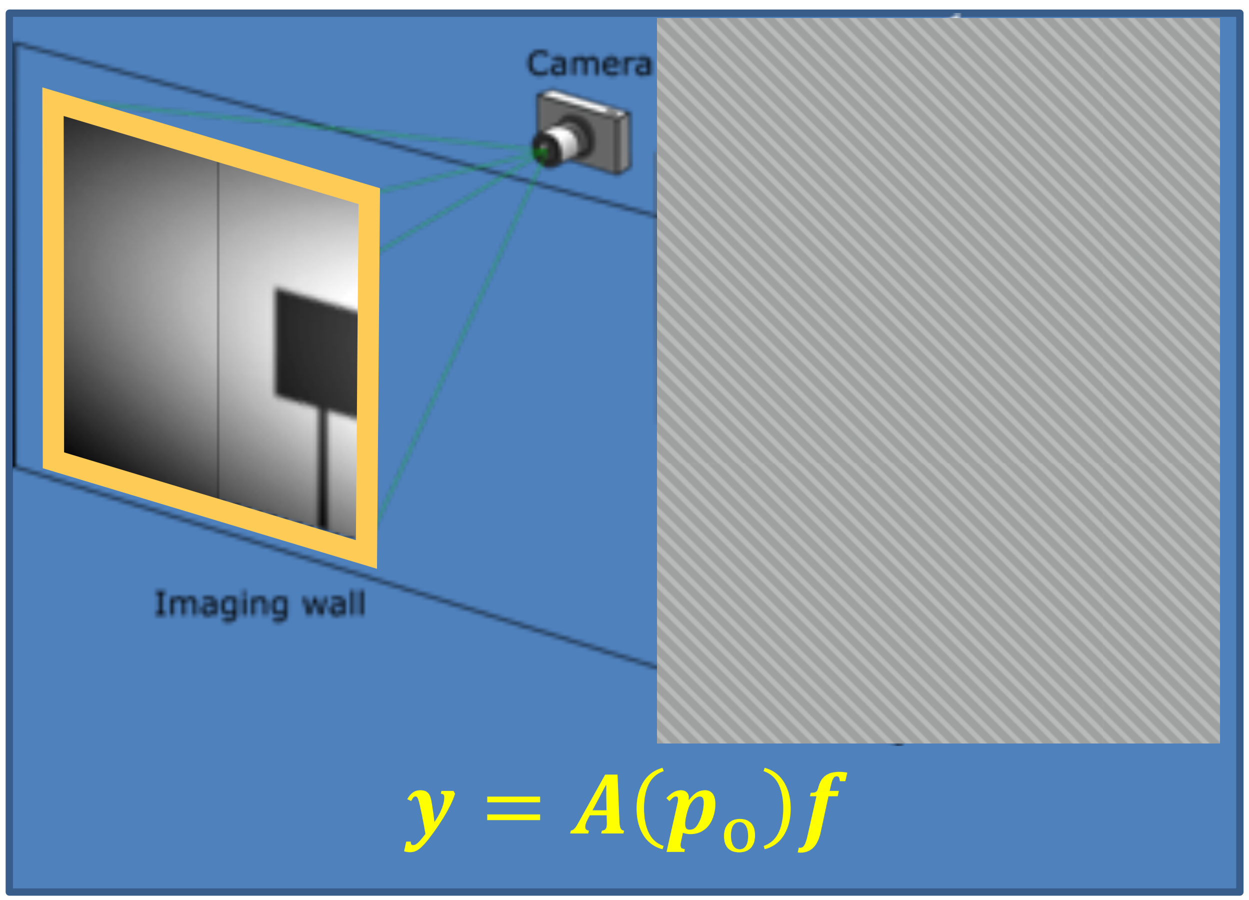




FORWARD PROBLEM

EXAMPLE: PROPAGATION OF LIGHT FROM A HIDDEN SPACE

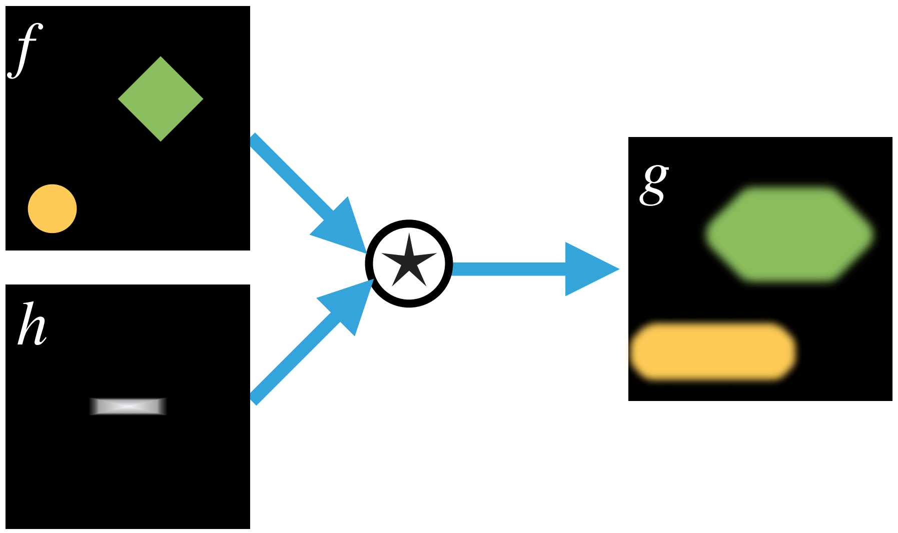




SOLVING AN INVERSE PROBLEM

- Generally achieved by inverting the forward model/operator
  - Let A denote the forward operator
  - We obtain the inverse operator A^{-1}
  - A^{-1} is then applied to the measured data
- Take the following toy example: forward model is $g = 2f + 3$
  - Forward model operator form: $g = A(f) = 3f + 2$
  - Inverse operator: $A^{-1}(g) = 0.5g - 1.5$

SOLVING AN INVERSE PROBLEM
LINEAR SHIFT INVARIANT IMAGING SYSTEMS



- LSI system (Lecture 11-13):

$$g(x,y) = h(x,y) \star f(x,y)$$
$$\Rightarrow G(\omega_x,\omega_y) = H(\omega_x,\omega_y)F(\omega_x,\omega_y)$$

- **Deconvolution is an inverse problem:**
  - Finding $f(x,y)$, given $g(x,y)$ and $h(x,y)$
  - It is **ill-conditioned**

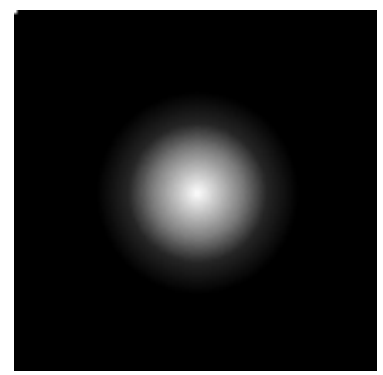
- **Inverse operator:**

$$(A^{-1}g)(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{G(\omega_x,\omega_y)}{H(\omega_x,\omega_y)}e^{\mathrm{j}(\omega_x x+\omega_y y)}\,\mathrm{d}\omega_x\mathrm{d}\omega_y$$

IMAGE DECONVOLUTION
INVERSE PROBLEM

$f(x, y)$

$\star$



$h(x, y)$

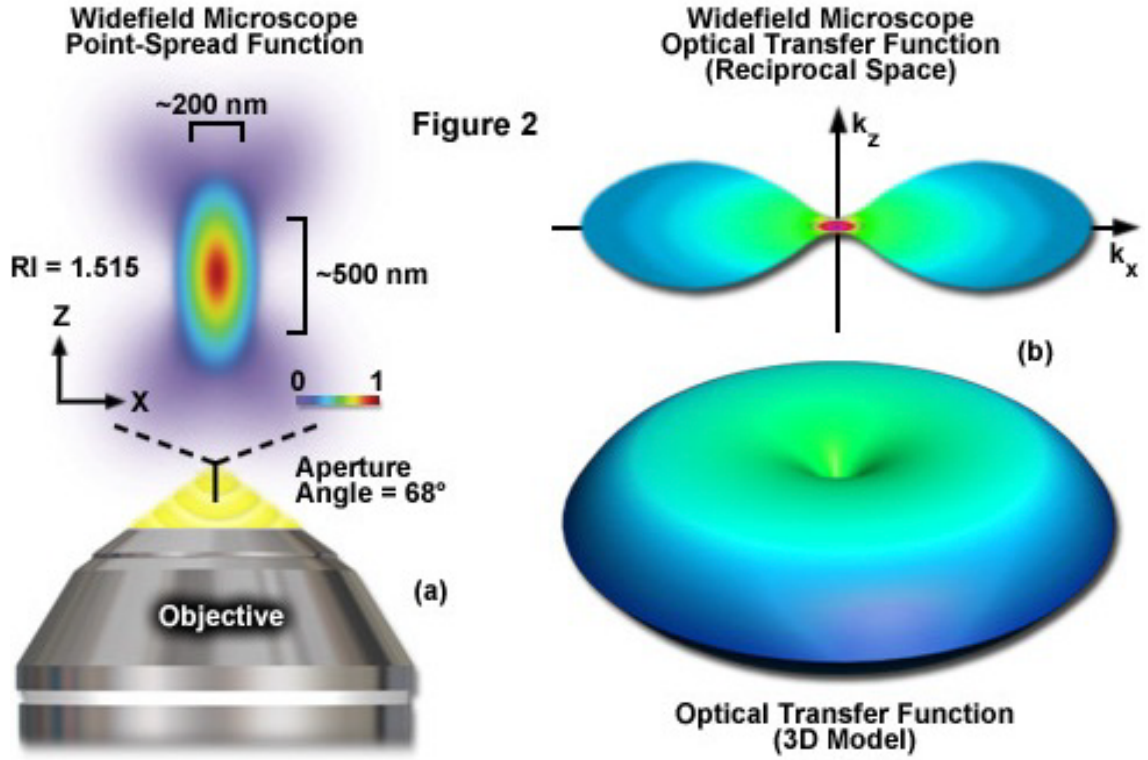
$+ \; n \; =$

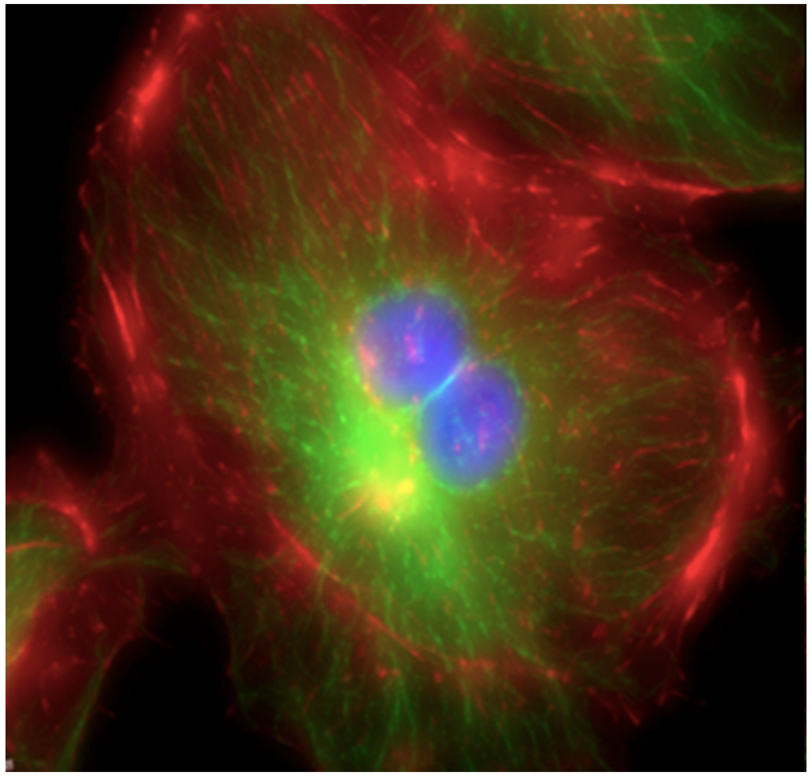
$g(x, y)$

n is unknown noise term

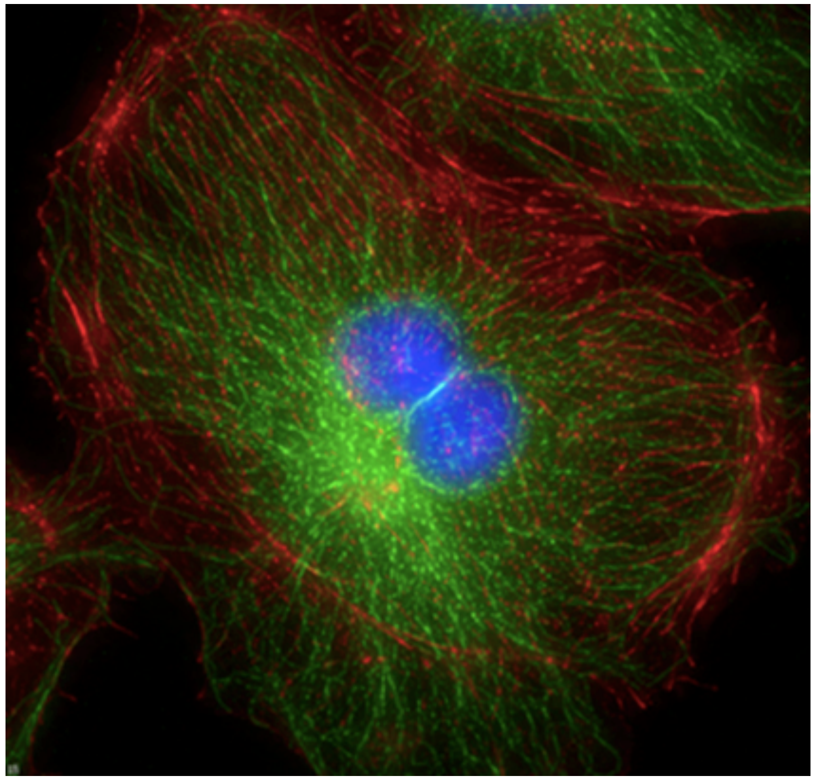
- Input goes through imaging system with PSF $h(x, y)$
- **Inverse problem:** given $g(x, y)$ and $h(x, y)$, solve for $f(x, y)$

DECONVOLUTION MICROSCOPY
EXAMPLE





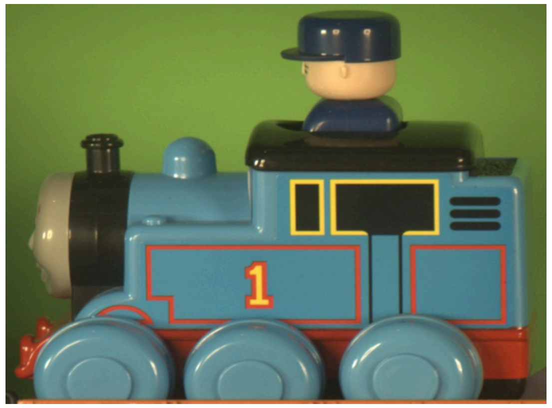
Conventional image



Deconvolved image

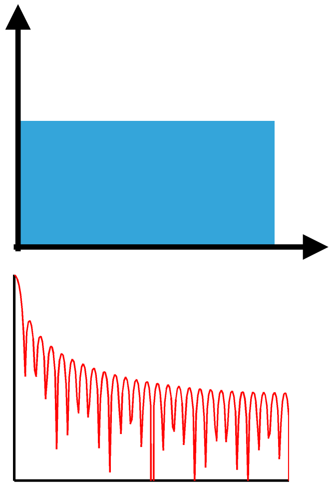
FLUTTER SHUTTER CAMERA

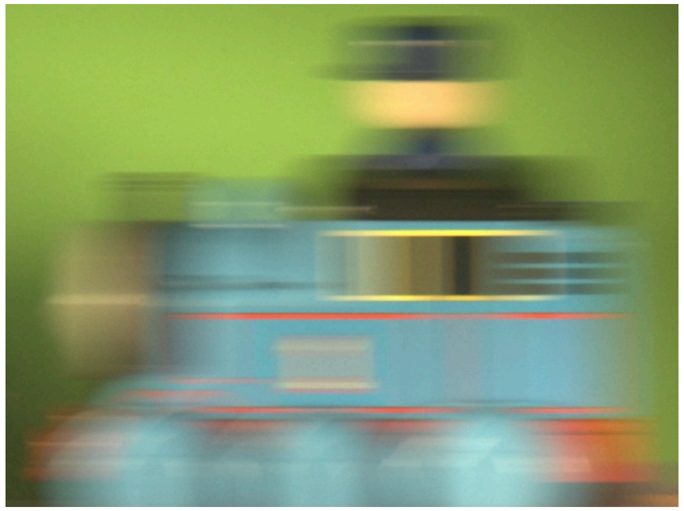
PHOTOGRAPHING A MOVING OBJECT

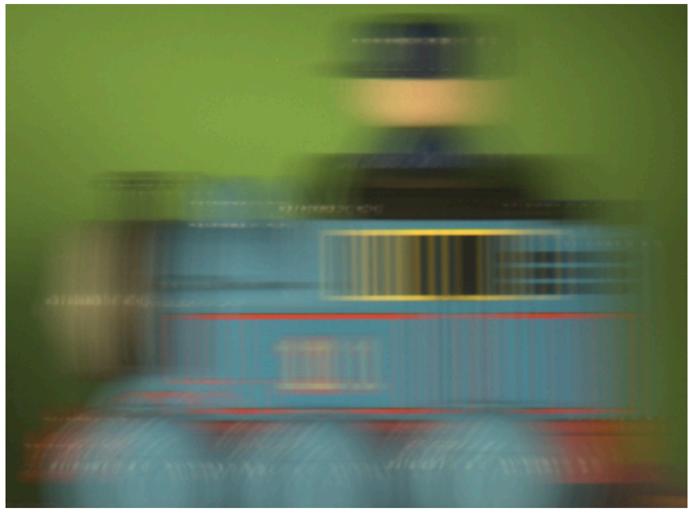


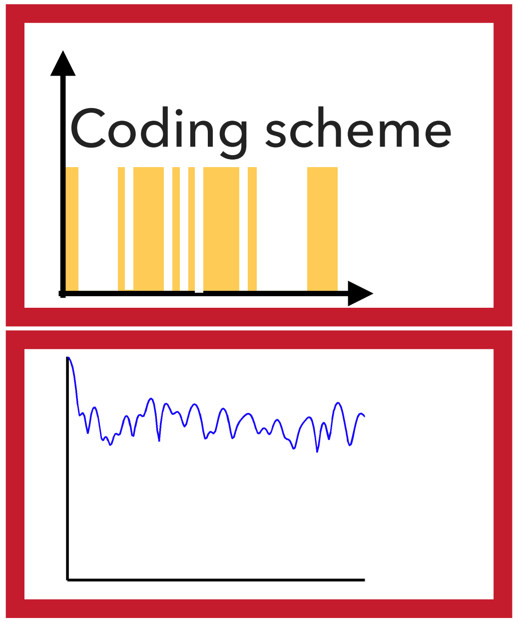
Traditional

Coded-exposure

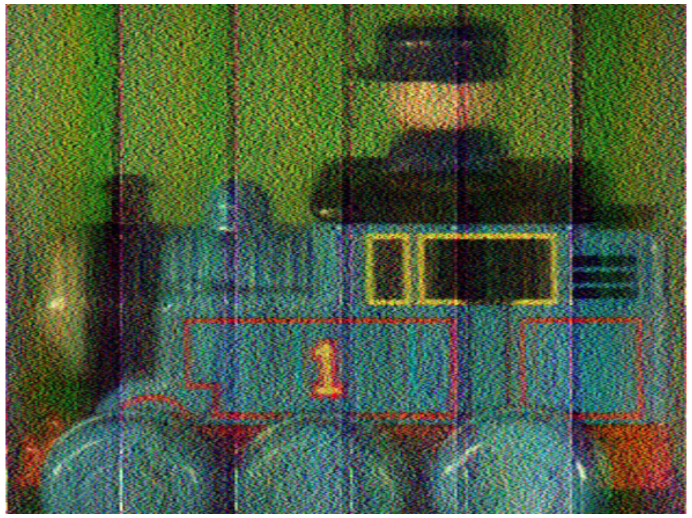


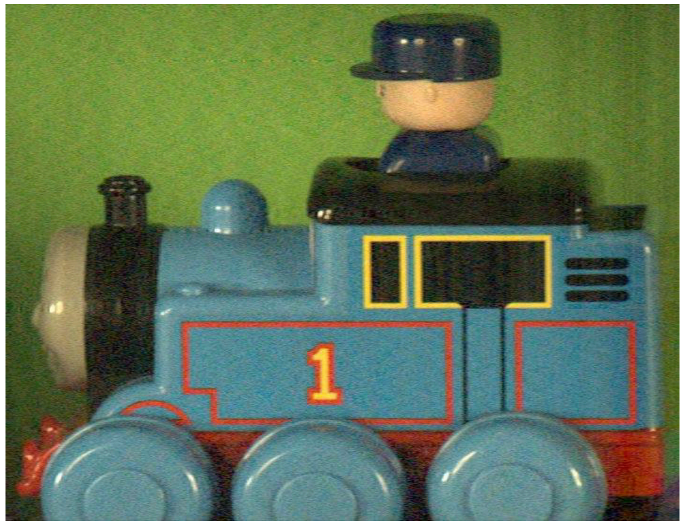






Deconvolved image





Deconvolved image

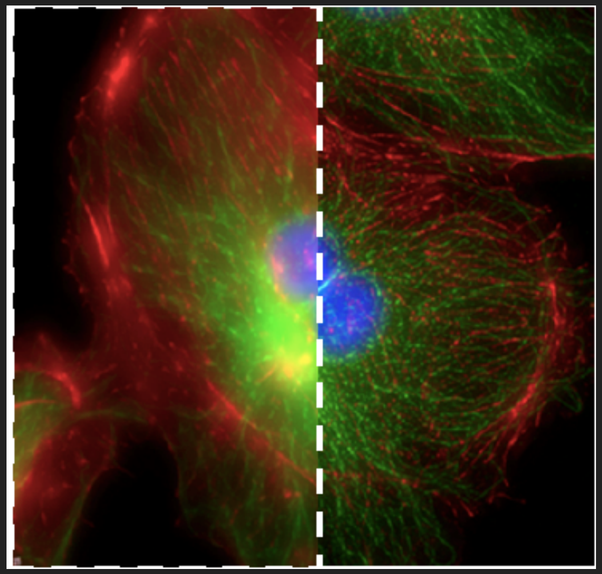
http://web.media.mit.edu/~raskar/deblur/

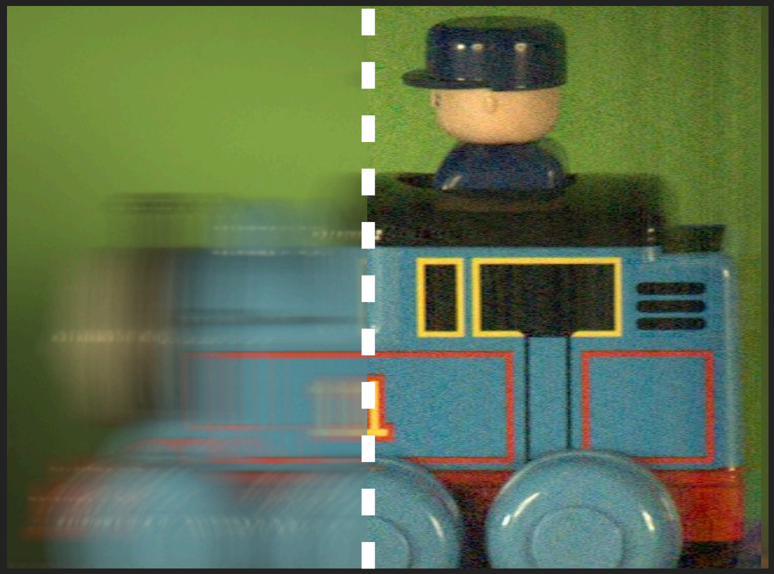
QUICK SUMMARY

- **Forward problem:** is to predict output measurements given some input object
  - Possible via forward models (derived from physics of the system)
  - Continuous forward models, $g = A(f)$, e.g. continuous LSI system
  - Discrete forward models (matrix-vector): $\mathbf{g} = \mathbf{A}\,\mathbf{f}$
- **Inverse problem:** given measurements, recover input object "representation"
  - Think of it as reversing of the forward problem: $f_{\text{est}} = A^{-1}(g)$
  - Direct inverse (spectral form) for continuous LSI system **(deconvolution)**
  - Direct matrix inverse for fully discrete system: $\mathbf{f}_{\text{est}} = \mathbf{A}^{-1}\,\mathbf{g}$

QUESTIONS

1. Is direct operator inverse or matrix inverse sufficient/effective/a good idea always?
2. What can go wrong if not, and how?
3. How can we characterize invertibility?
4. How can we cure these non-invertibility problems?





Restore High Frequency Components

Difficult/Easy Problems?

How?

ILL-POSED/CONDITIONED INVERSE PROBLEM

GENERALIZED SOLUTION (MATRIX PSEUDO-INVERSE)

DECONVOLUTION (UNREGULARIZED)

CONTINUOUS INVERSE PROBLEM
ILL-POSED VS WELL-POSED PROBLEMS

(Definition: Well-posedness) Consider the operator $A : \mathscr{X} \longrightarrow \mathscr{Y}$. The inverse problem of solving

$$g = A(f)$$

to obtain f given measurements g, is **well-posed** in the Hadamard sense, if:

1. **Existence:** a solution exists for any g
2. **Uniqueness:** a unique solution exists for any g
3. **Continuity:** f depends continuously on the measurements g

- The problem is **ill-posed** if any of these above conditions are violated

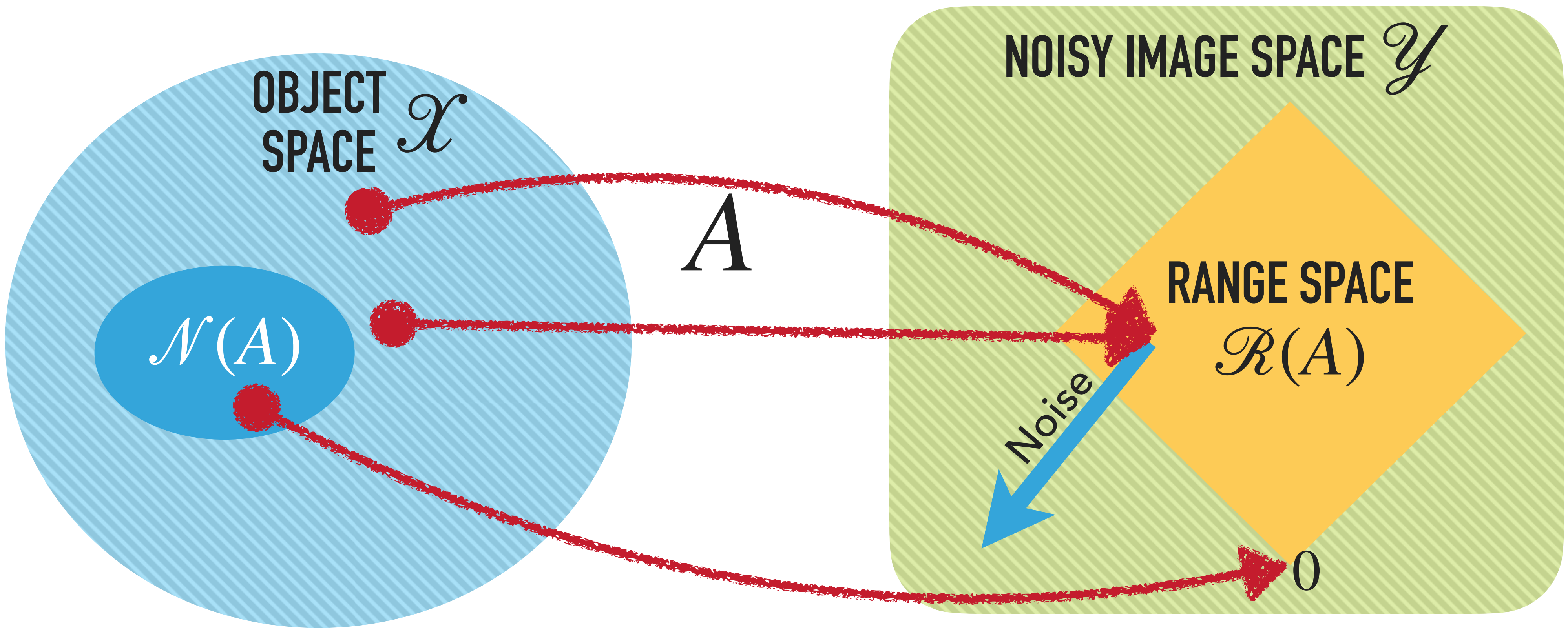
SOLVING AN INVERSE PROBLEM
DECONVOLUTION CASE

- **Continuous models:** often difficult to find a formulaic approach, have to consider each case individually
  - Inverses may not even have analytical expressions
  - Only for certain systems is this even possible:
    - We found the inverse for continuous LSI systems **(deconvolution)**:

$$(A^{-1}g)(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{G(\omega_x,\omega_y)}{H(\omega_x,\omega_y)}e^{\mathrm{j}(\omega_x x+\omega_y y)}\,\mathrm{d}\omega_x\mathrm{d}\omega_y$$

    - But how good is that solution?

CONTINUOUS INVERSE PROBLEM: WELL/ILL-POSEDNESS



- $\mathscr{X}$ is the **object space**, $\mathscr{Y}$ forms the **noisy image space**
- Operator A maps an element of a subset of $\mathscr{X}$, to element in $\mathscr{R}(A)$
  - **Noise distorts output to give noisy image**
- A linear operator is **well-posed**, if and only if,

$$\mathscr{N}(A) = \{0\}, \text{ and } \mathscr{R}(A) = \mathscr{Y}$$

CONTINUOUS INVERSE PROBLEM

ILL-POSED VS WELL-POSED PROBLEMS

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CONTINUOUS INVERSE PROBLEM: WELL/ILL-POSEDNESS

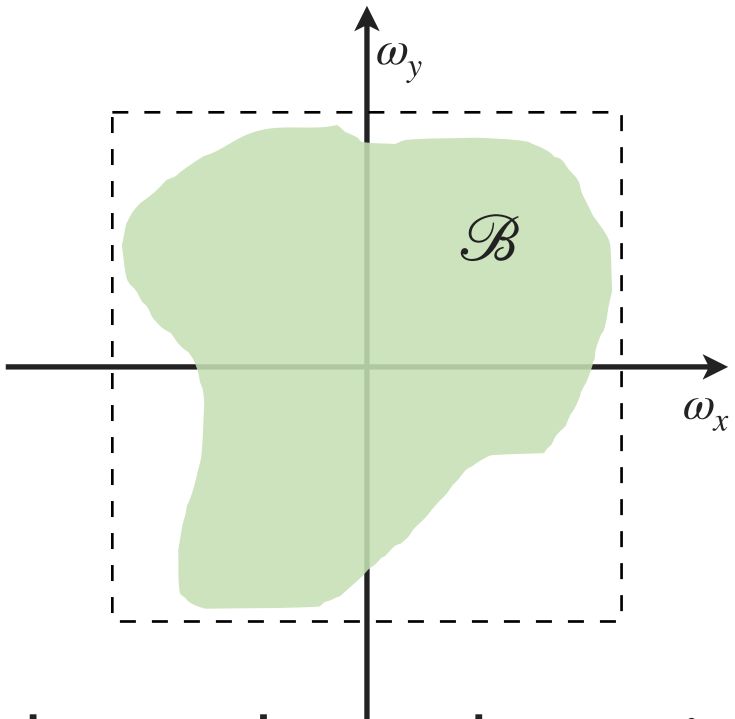
- **Linear operator is well-posed**: if and only if,

$$\mathscr{N}(A) = \{0\}, \text{ and } \mathscr{R}(A) = \mathscr{Y}$$

- This means **deconvolution/LSI inverse problem is well-posed**, if
  - The transfer function does not contain any zeros in the whole frequency space

Bandlimited LSI inverse problem is therefore always ill-posed:

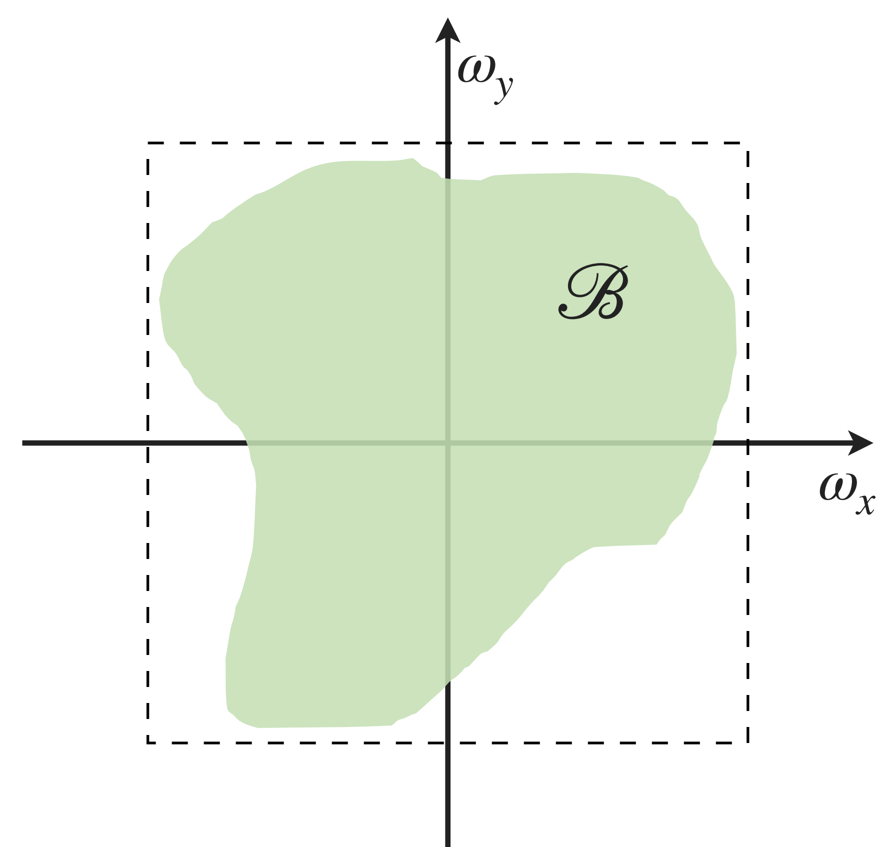
- Because transfer function is zero outside the bandwidth



The shaded area shows the region $\mathscr{B}$, i.e. the region where $H(\omega_x, \omega_y) \neq 0$.

BANDLIMITED CONTINUOUS LSI IMAGING SYSTEM

- $\mathcal{N}(A) \neq \{0\}$, specifically
- Specifically, $\mathcal{N}(A) \supset \left\{ f_0 \in \mathcal{L}^2(\mathbb{R}) : F_0(\omega_x, \omega_y) = 0, (\omega_x, \omega_y) \in \mathcal{B} \right\}$
- Can two different objects be mapped to the same point?
  - **YES!**



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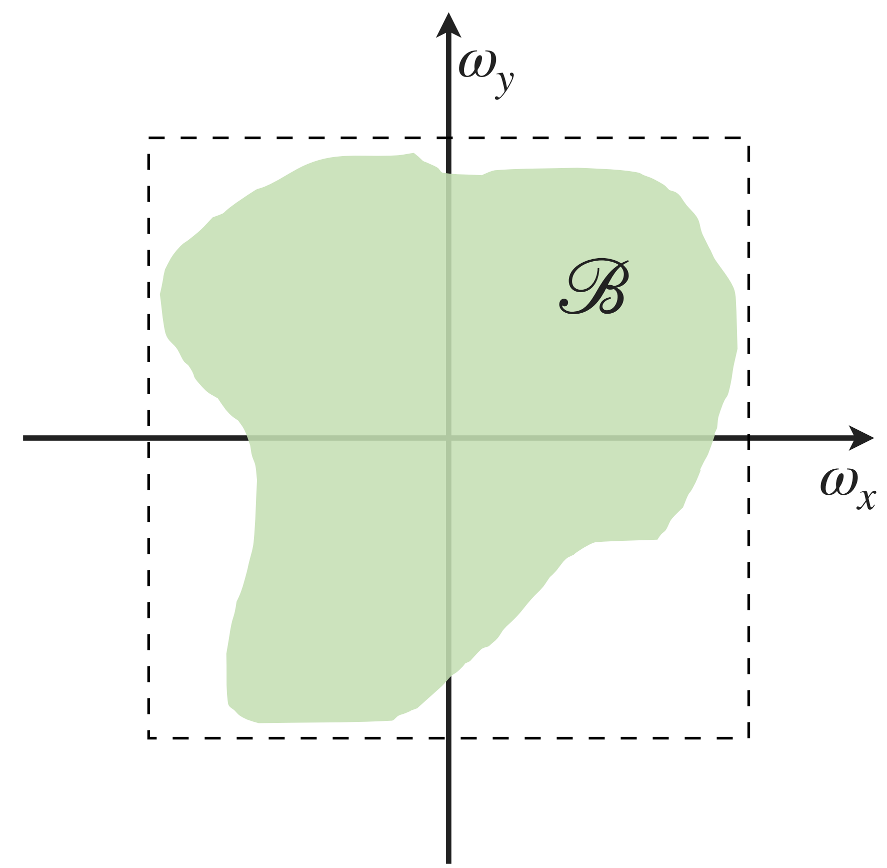
BANDLIMITED CONTINUOUS LSI IMAGING SYSTEM

ILL-POSEDNESS: DUE TO NON-UNIQUENESS

- Can two different objects generate to the same measurements?
  - **YES:** Consider two objects f_1 and f_2, where:
    - $f_2 = f_1 + f_0$, and $f_0 \in \mathcal{N}(A)$
  - Then $A(f_1) = g_1$, but

$$A(f_2) = A(f_1 + f_0)$$
$$A(f_2) = A(f_1) + A(f_0)$$
$$A(f_2) = A(f_1) + 0$$
$$A(f_2) = A(f_1) = g_1$$

  - **NON-UNIQUENESS!!!**



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BANDLIMITED CONTINUOUS LSI IMAGING SYSTEM

ILL-POSED: DUE TO DISCONTINUITY

- **For noiseless image:** $G(\omega_x, \omega_y) = F_{\text{in}}(\omega_x, \omega_y)H(\omega_x, \omega_y)$
  - Direct inverse gives estimated image of original *input* object:

$$F_{\text{est}}(\omega_x, \omega_y) = \frac{G(\omega_x, \omega_y)}{H(\omega_x, \omega_y)}$$

- **For noisy image:** $G(\omega_x, \omega_y) = F_{\text{in}}(\omega_x, \omega_y)H(\omega_x, \omega_y) + n(\omega_x, \omega_y)$
  - Direct inverse gives estimated image of original *input* object:

$$F_{\text{est}}(\omega_x, \omega_y) = F_{\text{in}}(\omega_x, \omega_y) + \frac{n(\omega_x, \omega_y)}{H(\omega_x, \omega_y)}$$

  - Highlighted term blows up for values where $H(\omega_x, \omega_y) = 0$
  - *The noise is random and not known*
- **Solution is not continuous under small perturbations from noise**

CONTINUOUS INVERSE PROBLEM
ILL-POSED VS WELL-POSED PROBLEMS

(Definition: Well-posedness) Consider the operator $A : \mathscr{X} \longrightarrow \mathscr{Y}$. The inverse problem of solving

$$g = A(f)$$

to obtain f given measurements g, is **well-posed** in the Hadamard sense, if:

1. **Existence:** a solution exists for any g
2. **Uniqueness:** a unique solution exists for any g
3. **Continuity:** f depends continuously on the measurements g

The problem is **ill-posed** if any of these above conditions are violated

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DECONVOLUTION – A HOPELESS CASE?

SOLVING AN INVERSE PROBLEM
SUMMARY

- **Partially discrete forward models:** recover a continuous (i.e. infinite dimensional) object from finite samples/measurements
  - Overwhelmingly more unknowns than there are measurements **(ill-posedness)**
    - Need continuous measurements! Impossible with digital sensors, or need many, many sensors
    - Takes us back to continuous models and those issues
  - Or can discretize the object space also → **FULLY DISCRETE MODELS**

BUT DOES THAT EVEN HELP?

FULLY-DISCRETE INVERSE PROBLEMS

GIVEN MEASUREMENTS, RECOVER "OBJECT"

FULLY-DISCRETIZED FORWARD MODEL

FROM ILL-POSEDNESS TO ILL-CONDITIONING

- Remember we will have a discrete LSI system, which is a Toeplitz matrix $\mathbf{A}$, so that measurement $\mathbf{g} = \mathbf{A}\,\mathbf{f}$
- Most systems have noise (or some form of small perturbations):
  - How does small variation in measurement (call it $\delta\mathbf{g}$) affect the solution:
  - Forward model is $\delta\mathbf{g} = \mathbf{A}\,\delta\mathbf{f}$, which means $\delta\mathbf{f}_{\mathrm{est}} = \mathbf{A}^{-1}\,\delta\mathbf{g}$
  - Looking at the size **(norm)** of this small perturbation vector $$||\delta\mathbf{f}_{\mathrm{est}}|| \leq \frac{1}{\lambda_{\min}(\mathbf{A})}\,||\delta\mathbf{g}||$$
  - $\lambda_{\min}(\mathbf{A})$ is the smallest eigenvalue of $\mathbf{A}$

FULLY-DISCRETIZED FORWARD MODEL
FROM ILL-POSEDNESS TO ILL-CONDITIONING

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  - Forward model is $\delta\mathbf{g} = \mathbf{A}\,\delta\mathbf{f}$, which means $\delta\mathbf{f}_{\text{est}} = \mathbf{A}^{-1}\,\delta\mathbf{g}$
  - Looking at the size **(norm)** of this small perturbation vector

$$||\delta\mathbf{f}_{\text{est}}|| \leq \frac{1}{\lambda_{\min}(\mathbf{A})}\,||\delta\mathbf{g}|| \qquad (1)$$

  - Where $\lambda_{\min}(\mathbf{A})$ is the smallest eigenvalue of $\mathbf{A}$
- Forward model is $\mathbf{g} = \mathbf{A}\,\mathbf{f}$. And the norm of the vector $\mathbf{g}$

$$||\mathbf{g}|| \leq \lambda_{\max}(\mathbf{A})\,||\mathbf{f}_{\text{est}}|| \qquad (2)$$

- Dividing (1)/(2) and rearranging:

$$\frac{||\delta\mathbf{f}_{\text{est}}||}{||\mathbf{f}_{\text{est}}||} \leq \frac{\lambda_{\max}(\mathbf{A})}{\lambda_{\min}(\mathbf{A})}\,\frac{||\delta\mathbf{g}||}{||\mathbf{g}||} \qquad (3)$$

FULLY-DISCRETIZED FORWARD MODEL
FROM ILL-POSEDNESS TO ILL-CONDITIONING

This describes the size of the error propagated into the recovered object

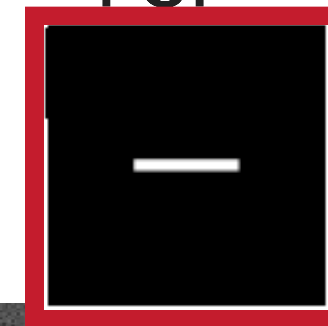
$$\frac{||\delta \mathbf{f}_{\text{est}}||}{||\mathbf{f}_{\text{est}}||} \leq \boxed{\frac{\lambda_{\max}(\mathbf{A})}{\lambda_{\min}(\mathbf{A})}} \frac{||\delta \mathbf{g}||}{||\mathbf{g}||} \qquad (3)$$

Condition number of $\mathbf{A}$

$$\text{cond}(\mathbf{A}) = \frac{\lambda_{\max}(\mathbf{A})}{\lambda_{\min}(\mathbf{A})}$$

Matlab command: 'cond'

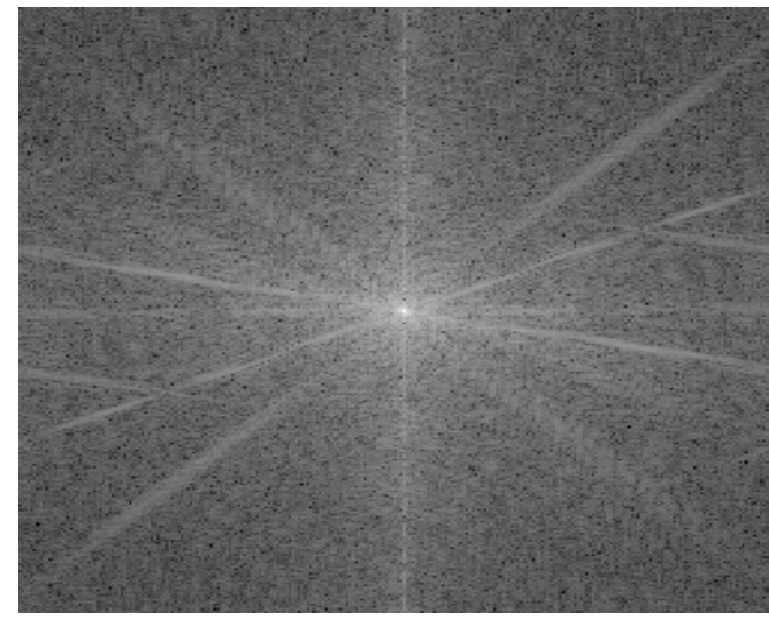
PSF



MOTION BLUR
EXAMPLE

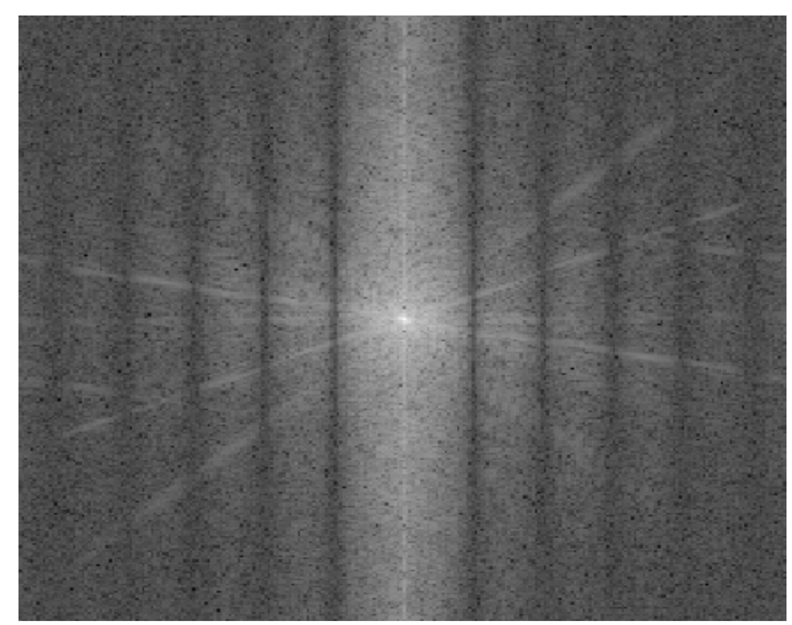
f

F



$g = A(f)$

G

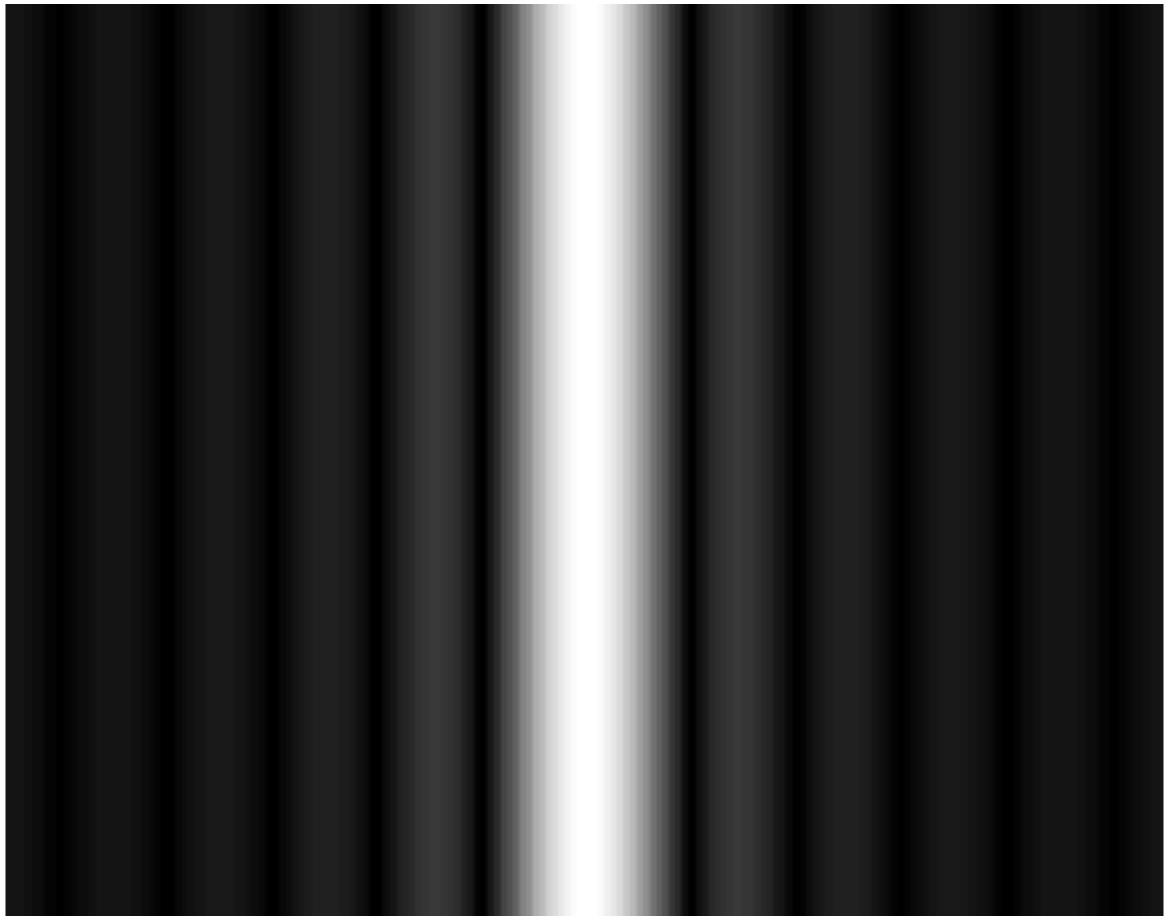


PSF

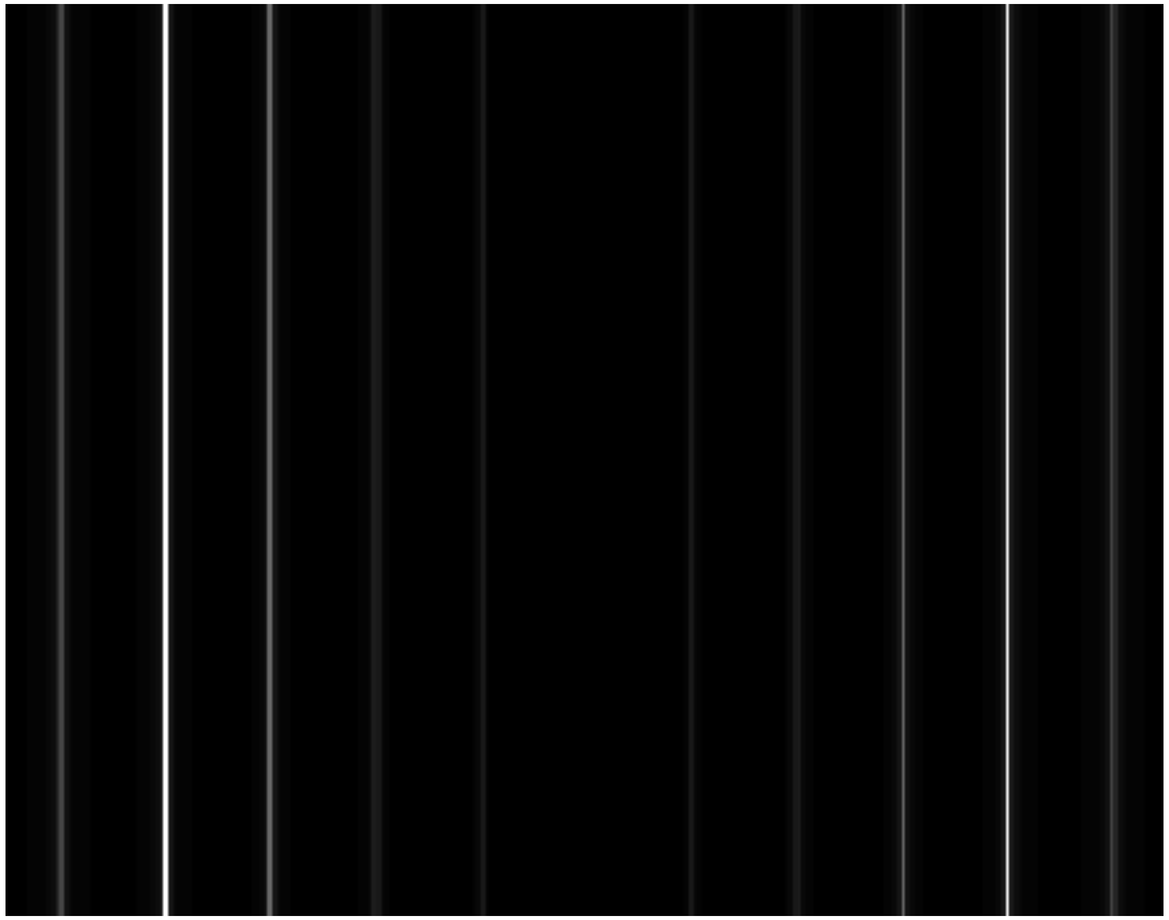
MOTION BLUR

$$\mathbf{g} = \mathbf{A}\mathbf{f} + \mathbf{n} \qquad \mathrm{cond}(\mathbf{A}) = 815$$

$|H(\omega_x, \omega_y)|$



$|H^{-1}(\omega_x, \omega_y)|$



PSF

MOTION BLUR

$$\mathbf{g} = \mathbf{A}\mathbf{f} + \mathbf{n} \qquad \text{cond}(\mathbf{A}) = 815$$

f

g

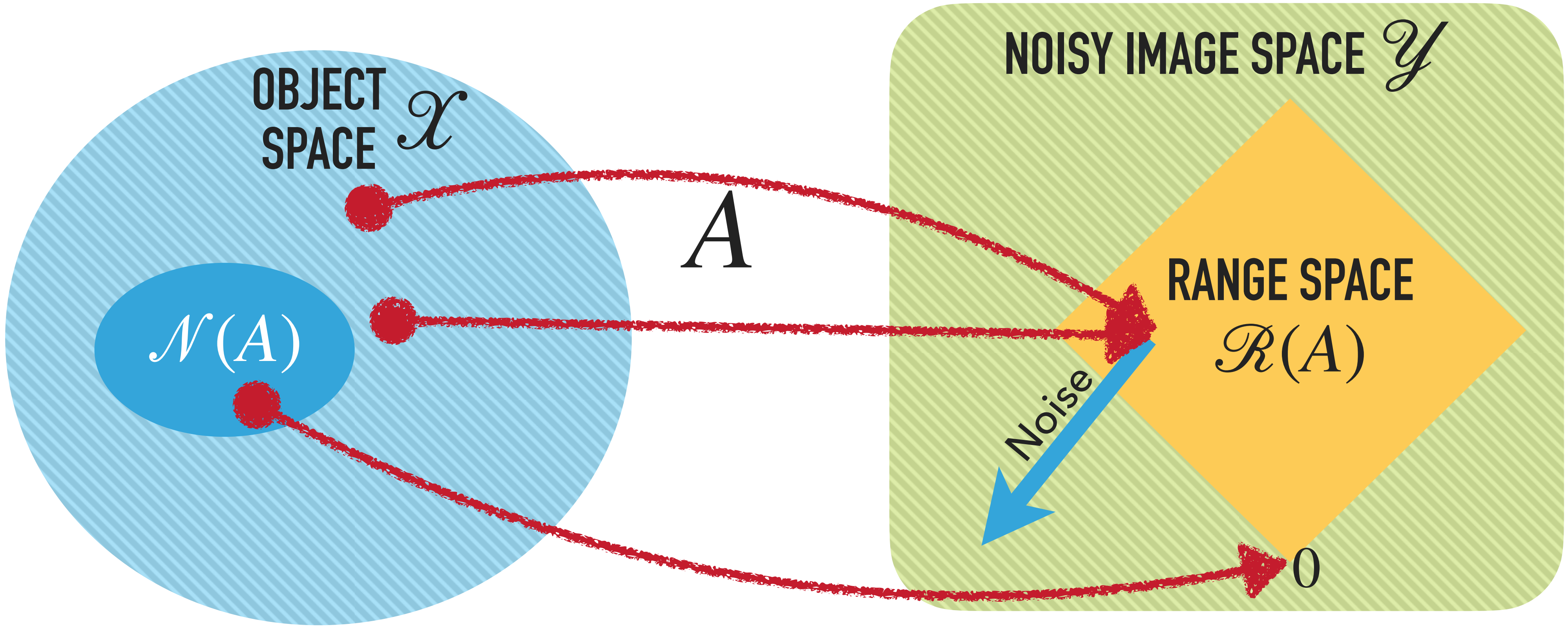
$A^{-1}g$

Noise dominates at high frequencies

SOLVING INVERSE PROBLEMS

RACE TOWARDS FINDING A CURE FOR ILL-POSEDNESS/ILL-CONDITIONING

- **No solutions:** look for **approximate solutions** instead
- **Non-uniqueness:** use **prior information**/knowledge about the object to find a good solution



MOORE-PENROSE PSEUDO-INVERSE

- Solve the minimization problem:

$$\min_{\mathbf{f}} \|\mathbf{g} - \mathbf{A}\mathbf{f}\|^2$$

- This gives the Moore-Penrose pseudo-inverse is

$$\mathbf{A}^{\dagger} = (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H$$

- This is just the backslash operator '\' or '`pinv( )`' in Matlab.
- What does the optimization problem mean?
  - Give me the solution that "**best**" explains the measurement
  - Has the smallest residual error vector
  - **Caution:** no reason for this to be a (visually) good solution

MOORE-PENROSE PSEUDO-INVERSE

- Solve the minimization problem:

$$\min_{\mathbf{f}} \|\mathbf{g} - \mathbf{A}\mathbf{f}\|^2$$

- Moore-Penrose pseudo-inverse is

$$\mathbf{A}^\dagger = (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H$$

- What does the **optimization problem** mean?
  1. Pick an $\mathbf{f}_0$ and compute $\mathbf{A}\mathbf{f}_0$
  2. Examine the norm of the difference between your measurement $\mathbf{g}$ and the vector $\mathbf{A}\mathbf{f}_0$ computed in step 1. This is the error.
  3. Try all possible $\mathbf{f}_0$'s and pick the one with the smallest error vector.
  4. (How do you measure size of a vector? Use a norm, i.e. 2-norm)

GIVE ME THE SOLUTION THAT BEST EXPLAINS THE MEASUREMENT

PSF

MOTION BLUR

MOORE-PENROSE PSEUDO-INVERSE

$$\mathbf{g} = \mathbf{A}\mathbf{f} + \mathbf{n}$$

f

g

$A^{-1}g$

Can only recover band-limited information and not high frequency components

Cannot solve the ill-conditioned problem!

SOLVING INVERSE PROBLEMS

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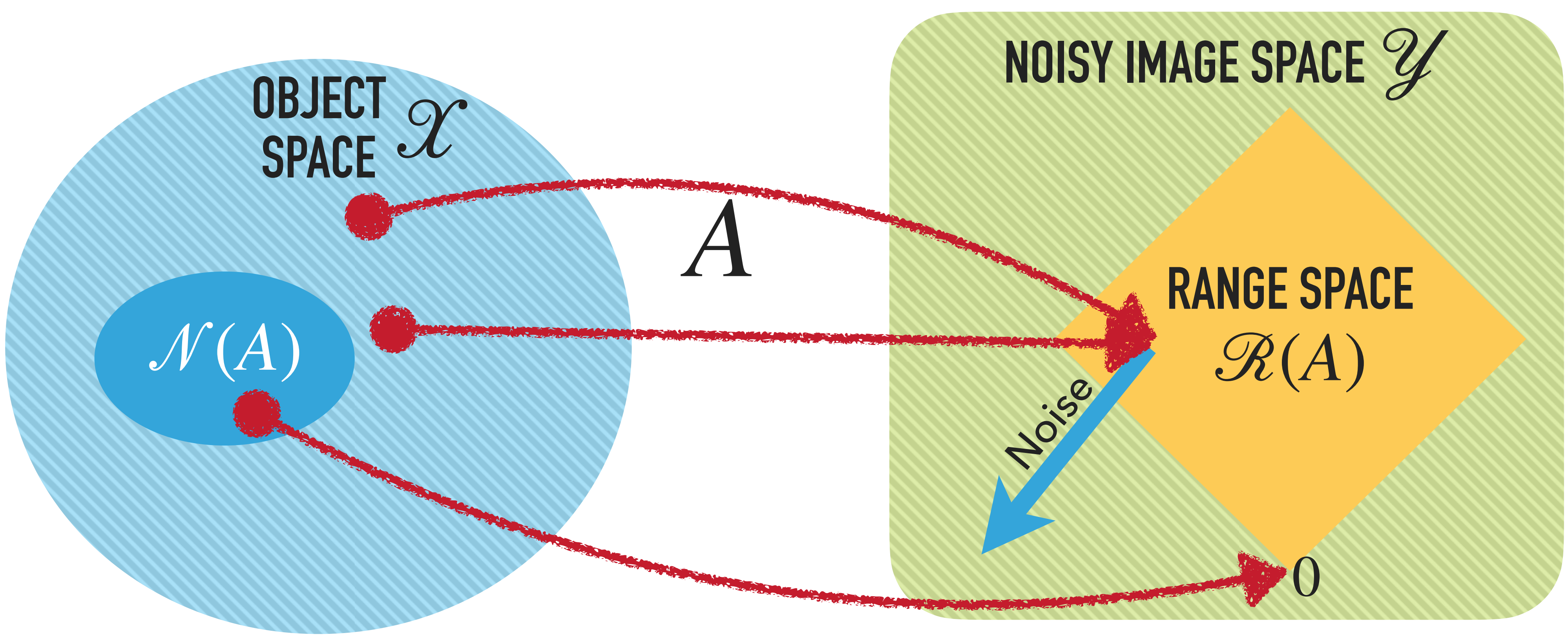

ALIASING IN TIME
VIDEO

In which direction is the train moving?

WHAT DID WE COVER

- Introduction to inverse problems
  - Ill-posed and ill-conditioned inverse problems
  - How to assess the ill-conditioning and ill-posedness of inverse problems
- Non-regularized solution to deconvolution problem

TILL NEXT TIME

CURING ILL-CONDITIONING WITH REGULARIZATION