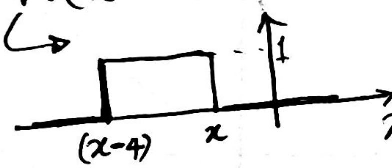
HOMEWORK #3 SOLUTIONS

1ai) $g(x) = \begin{cases} 1, & 0 \le x \le 4 \\ 0, & \text{otherwise} \end{cases}$ and $h(x) = g(x)$

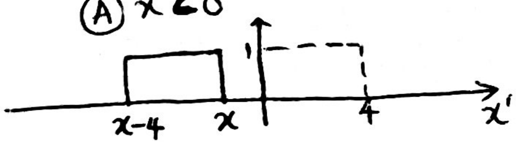
Convolution formula: $g(x) * h(x) = \int_{-\infty}^{\infty} g(x')h(x-x')\,dx'$

- ~~Pl~~ Plot $g(x')$ and $h(x-x')$



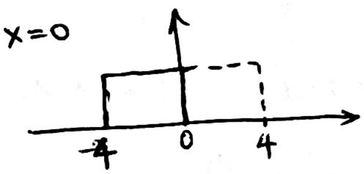
- As we slide $h(x-x')$ from left to right (i.e. increasing values of x) then have the following situations

Ⓐ $x < 0$



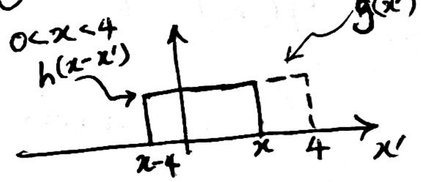
$x < 0$

Ⓑ $x = 0$

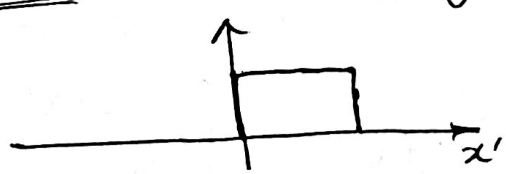


$x = 0$

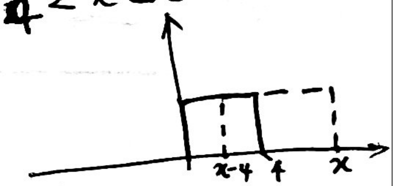
Ⓒ $0 < x < 4$



Ⓓ $x = 4$: $h(4 - x')$ and $g(x)$ fully overlap

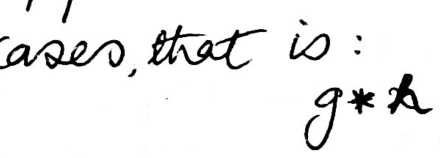


Ⓔ $4 < x < 8$

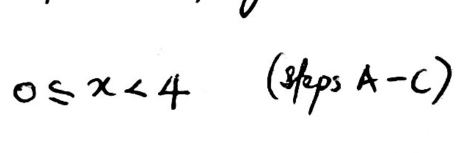


Partial overlap.

Ⓕ $x = 8$: h and g touching



Ⓖ $x > 8$ No overlap.

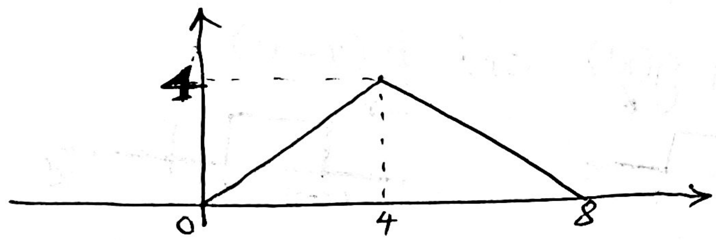


- The steps A – G means the convolution integral has two cases, that is:

$$g * h = \begin{cases} \int_0^x 1\,dx', & 0 \le x < 4 \quad \text{(steps A–C)} \\ \int_{x-4}^{4} 1\,dx', & 4 \le x < 8 \quad \text{(steps D–F)} \end{cases}$$

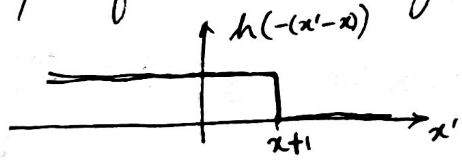
$$= \begin{cases} x, & 0 \le x < 4 \\ 8 - x, & 4 \le x < 8 \end{cases}$$

• Alternatively, we can evaluate the convolution graphically as in-class example.
Just compute the areas in the steps A – G and plot to get:



(ii) $g(x) = e^{-(x-1)} u(x-1)$ and $h(x) = u(x+1)$

- Convolution integral: $g(x) * h(x) = \int_{-\infty}^{\infty} g(x') h(x-x')\, dx'$
- Graphs of $h(x-x')$ and $g(x')$:

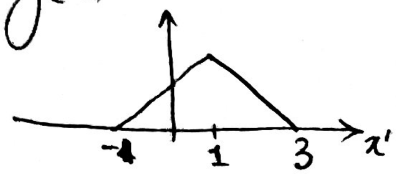


$$g(x) * h(x) = \int_1^{x+1} e^{-(x'-1)}\, dx' = \left[-e^{-(x'-1)}\right]_1^{x+1}$$

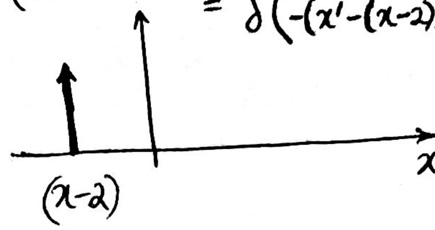
$$= -e^{-(x+1-1)} + e^{-(1-1)}$$

$$= 1 - e^{-x}$$

(iii) $g(x')$



$h(x-x') = \delta(x-x'-2) = \delta(-(x'-x+2))$
$= \delta(-(x'-(x-2)))$



- Convolution integral:

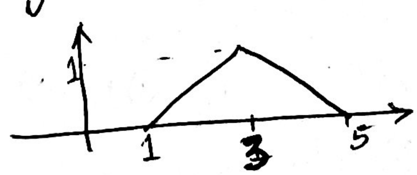
$$g*h = \int_{-\infty}^{\infty} g(x')\, h(x-x')\, dx'$$

$$= \int_{-\infty}^{\infty} g(x')\, \delta(x-x'-2)\, dx'$$

$$= \int_{-\infty}^{\infty} g(x')\, \delta(-(x'-(x-2)))\, dx'$$

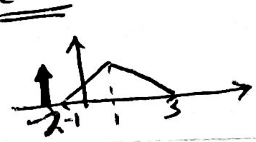
$= g(x-2)$, this step follows from the property of the Dirac delta function (see lecture 8 & 9 of the course).

Thus, $g*h = g(x-2)$, which is just $g(x)$ shifted to the right by 2 units, i.e.

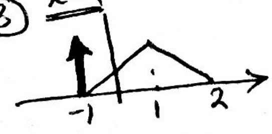


- Graphical approach; we can see what happens when we slide the delta function along (i.e. increasing values of x)

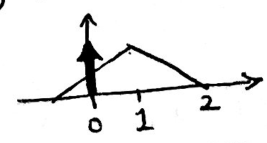
Ⓐ $x=0$



Ⓑ $x=1$



Ⓒ $x=2$



and so on

which gives the same plot $g(x-2)$

(b) $g(x) = \text{sinc}(ax)$, $h(x) = \text{sinc}(bx)$

(i) Convolution property of FT : Convolution in time/space is equivalent to multiplication in frequency

$$g(x) * h(x) \longleftrightarrow G(\omega) H(\omega)$$

for $g(x) \longleftrightarrow G(\omega)$ and $h(x) \longleftrightarrow H(\omega)$.

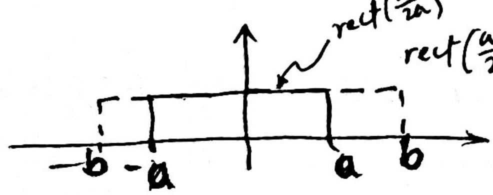
(ii) $\mathcal{F}\{\text{sinc}(x)\} = \pi \,\text{rect}(\omega/2)$.

$$g(ax) \longleftrightarrow \frac{1}{a} G(\omega/a)$$

$$\Rightarrow \text{sinc}(ax) \longleftrightarrow \frac{\pi}{a} \text{rect}\left(\frac{\omega}{2a}\right)$$

~~and~~ $\text{sinc}(bx) \longleftrightarrow \frac{\pi}{b} \text{rect}\left(\frac{\omega}{2b}\right)$

(iii) $\mathcal{F}\{g(x) * h(x)\} = G(\omega)H(\omega) = \frac{\pi^2}{ab} \text{rect}\left(\frac{\omega}{2a}\right) \text{rect}\left(\frac{\omega}{2b}\right)$



Graphically, we see that because $a \leq b$, the only thing that matters is $\text{rect}\left(\frac{\omega}{2a}\right)$

$$\Rightarrow \mathcal{F}\{g(x) * h(x)\} = \frac{\pi^2}{ab} \text{rect}(\omega/2a).$$

(i) $g(x) * h(x) = \mathcal{F}^{-1}\left\{\frac{\pi^2}{ab} \mathrm{rect}(\omega/2a)\right\}$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\pi^2}{ab} \mathrm{rect}\left(\frac{\omega}{2a}\right) e^{+j\omega x}\, d\omega$$

$$= \frac{\pi}{2ab}\int_{-a}^{a} e^{j\omega x}\, d\omega$$

$$= \frac{\pi}{2ab}\left[\frac{1}{jx} e^{j\omega x}\right]_{-a}^{a} = \frac{\pi}{(ab)2jx}\left(e^{jax} - e^{-jax}\right)$$

$$= \frac{a\pi}{ab} \cdot \frac{e^{jax} - e^{-jax}}{j2ax}$$

$$\left(\text{Remember that } \sin x = \frac{e^{jx} - e^{-jx}}{2j}\right)$$

$$= \frac{a\pi}{ab} \frac{\sin(ax)}{ax}$$

$$= \frac{\pi}{b} \mathrm{sinc}(ax)$$

(ii) $g(x) = \frac{1}{\sqrt{2\pi}a} e^{-\frac{x^2}{2a^2}}, \quad h(x) = \frac{1}{\sqrt{2\pi}b} e^{-\frac{x^2}{2b^2}}$

$$g(x) * h(x) = \int_{-\infty}^{\infty} g(x')\, h(x-x')\, dx' = \int_{-\infty}^{\infty} \frac{1}{(\sqrt{2\pi})^2 ab} e^{-\frac{x'^2}{2a^2}} e^{-\frac{(x-x')^2}{2b^2}}\, dx'$$

$$= \frac{1}{2\pi ab}\int_{-\infty}^{\infty} e^{-\frac{b^2x'^2 + a^2(x-x')^2}{2a^2b^2}}\, dx' = \frac{1}{2\pi ab}\int_{-\infty}^{\infty} e^{-\frac{b^2x'^2 + a^2x^2 - 2a^2xx' + a^2x'^2}{2a^2b^2}}\, dx'$$

$$= \frac{1}{2\pi ab}\int_{-\infty}^{\infty} e^{-\frac{(b^2+a^2)x'^2 - 2a^2xx' + a^2x^2}{2a^2b^2}}\, dx'$$

ab

Now, complete the square of the exponent (in terms of x')

$$g(x) * h(x) = \frac{1}{2\pi ab}\int_{-\infty}^{\infty} e^{-\frac{b^2+a^2}{2a^2b^2}\left[x'^2 - \frac{2a^2x}{a^2+b^2}x' + \frac{a^2x^2}{a^2+b^2}\right]}\, dx'$$

$$= \frac{1}{2\pi ab}\int_{-\infty}^{\infty} e^{-\frac{a^2+b^2}{2a^2b^2}\left[\left(x' - \frac{a^2x}{a^2+b^2}\right)^2 + \frac{a^2x^2}{a^2+b^2} - \left(\frac{a^2x}{a^2+b^2}\right)^2\right]}\, dx'$$

$$= \frac{1}{2\pi ab} \int_{-\infty}^{\infty} e^{-\frac{a^2+b^2}{2a^2b^2}\left(x' - \frac{a^2x}{a^2+b^2}\right)^2} \underbrace{e^{-\frac{a^2+b^2}{2a^2b^2}\left(\frac{a^2x^2}{a^2+b^2} - \left(\frac{a^2x}{a^2+b^2}\right)^2\right)}}_{\text{Does } \underline{\text{not}} \text{ depend on } x'} \, dx'$$

$$= \frac{1}{2\pi ab} e^{-\frac{a^2+b^2}{2a^2b^2}\left(\frac{a^2x^2}{a^2+b^2} - \frac{(a^2x)^2}{(a^2+b^2)^2}\right)} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{a^2+b^2}{2a^2b^2}\left(x' - \frac{a^2x}{a^2+b^2}\right)^2} dx'}_{\text{Comparing this integral with the provided hint.}}$$

$$\mu = \frac{a^2x}{a^2+b^2} \quad \text{and} \quad \cancel{2}\sigma^2 = \frac{\cancel{2}a^2b^2}{a^2+b^2}$$

$\cancel{= \frac{1}{2\pi ab} e^{-}}$ Then, $\int_{-\infty}^{\infty} e^{-\frac{(a^2+b^2)}{2a^2b^2}\left(x' - \frac{a^2x}{a^2+b^2}\right)^2} dx'$

$$= \sqrt{2\pi} \cdot \sqrt{\frac{a^2b^2}{a^2+b^2}} = \frac{ab\sqrt{2\pi}}{\sqrt{a^2+b^2}}$$

Then, $\cancel{f}\, g(x) * h(x) = \frac{1}{2\pi \cancel{ab}} \cdot \frac{\sqrt{2\pi}\,\cancel{ab}}{\sqrt{a^2+b^2}} \cdot e^{-\frac{\cancel{a^2}x^2}{2\cancel{a^2}b^2} + \frac{\cancel{a^4}x^2}{2\cancel{a^2}b^2(a^2+b^2)}}$

$$= \frac{1}{\sqrt{2\pi}\sqrt{a^2+b^2}} e^{-\frac{x^2}{2b^2}\left(1 - \frac{a^2}{a^2+b^2}\right)}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{a^2+b^2}} e^{-\frac{x^2}{2b^2}\left(\frac{\cancel{a^2}+b^2-\cancel{a^2}}{a^2+b^2}\right)}$$

$$= \underline{\frac{1}{\sqrt{2\pi}\sqrt{a^2+b^2}} e^{-\frac{x^2}{2(a^2+b^2)}}}$$

(ii) $g(x) = \frac{1}{\sqrt{2\pi}a} e^{-\frac{x^2}{2a^2}}$

$$G(\omega) = \int_{-\infty}^{\infty} g(x) e^{-j\omega x} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}a} e^{-\frac{x^2}{2a^2}} e^{-j\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}a} \int_{-\infty}^{\infty} e^{-\frac{x^2 + (j\omega x)(2a^2)}{2a^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}a} \int_{-\infty}^{\infty} e^{-\frac{1}{2a^2}\left(x^2 + 2j\omega a^2 x\right)} dx,$$ (Again complete the square!)

$$= \frac{1}{\sqrt{2\pi}a} \int_{-\infty}^{\infty} e^{-\frac{1}{2a^2}\left((x+j\omega a^2)^2 - (j\omega a^2)^2\right)} dx$$

$$= \frac{1}{\sqrt{2\pi}a} \int_{-\infty}^{\infty} e^{-\frac{(x+j\omega a^2)^2}{2a^2}} \underbrace{e^{+\frac{(-j)^2\omega^2 a^{4}}{2a^2}}}_{} dx$$

This factor is not dependent on x. So take it outside the integral.

$$= \frac{1}{\sqrt{2\pi}a} e^{-\frac{\omega^2 a^2}{2}} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{(x+j\omega a^2)^2}{2a^2}} dx}_{}.$$

This integral reduces ~~to~~ to $\sqrt{2\pi}a$ because of the hint provided, with $\mu = j\omega a^2$ and $2\sigma^2 = 2a^2$.

$\therefore$ $\underline{G(\omega) = e^{-\frac{\omega^2 a^2}{2}}}$, similarly $\underline{H(\omega) = e^{-\frac{\omega^2 b^2}{2}}}$

$$g(x) * h(x) = \mathcal{F}^{-1}\{G(\omega)H(\omega)\} = \mathcal{F}^{-1}\left\{e^{-\frac{\omega^2 a^2}{2}} e^{-\frac{\omega^2 b^2}{2}}\right\}$$

$$= \mathcal{F}^{-1}\left\{e^{-\frac{\omega^2(a^2+b^2)}{2}}\right\},$$ if we let $c^2 = a^2 + b^2$.

Then, if $\mathcal{F}^{-1}\left\{e^{-\frac{\omega^2 c^2}{2}}\right\} = \frac{1}{\sqrt{2\pi}c} e^{-\frac{x^2}{2c^2}}$, because of the

We can use the result we derived earlier that $\mathcal{F}\left\{\frac{1}{\sqrt{2\pi}a} e^{-\frac{x^2}{2a^2}}\right\} = e^{-\frac{\omega^2 a^2}{2}}$. (Of course, it also holds if we have 'c' instead of a.)

$$\Rightarrow \mathcal{F}\left\{\frac{1}{\sqrt{2\pi}\,c} e^{-\frac{x^2}{2c^2}}\right\} = e^{-\frac{\omega^2 c^2}{2}}$$

$$\therefore \mathcal{F}^{-1}\left\{e^{-\frac{\omega^2 c^2}{2}}\right\} = \frac{1}{\sqrt{2\pi}\,c} e^{-\frac{x^2}{2c^2}}$$

replace $c^2 = a^2 + b^2$

$$\Rightarrow g(x) * h(x) = \mathcal{F}^{-1}\left\{e^{-\frac{\omega^2(a^2+b^2)}{2}}\right\}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{a^2+b^2}} e^{-\frac{x^2}{2\sqrt{a^2+b^2}}}$$