Homework 1

Hunter Morera

February 8, 2021

1 Gaussian Elimination

For this exercise we are asked to attempt to solve a variety of system of equations problems in the form of Ax=y.

The way to solve these types of problems is compute the following $x = A^{-1}y$ if they have a single unique solution. The catch here is that we only have a unique solution if the inverse of matrix A i.e.(A^{-1}) exists. If the inverse does not exist then there are two other outcomes, the system may have infinitely many solutions, or none at all.

I will assume that the vector x we are solving for is in the form of $x = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$.

(a)

$$\begin{bmatrix} 1 & 0 & 3 \\ 4 & 5 & 2 \\ -1 & -1 & 2 \end{bmatrix} * \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 3 \end{bmatrix}$$

To begin solving for X, Y, & Z we must check if we can invert our A matrix. A is invertible if its determinant does not equal 0.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & 5 & 2 \\ -1 & -1 & 2 \end{bmatrix}$$

Determinant of A equals 15, thus A is invertible.

$$A^{-1} = \begin{bmatrix} 0.8 & -0.2 & -1 \\ -0.67 & 0.33 & 0.67 \\ 0.67 & 0.67 & 0.33 \end{bmatrix}$$

Now we can compute values of X, Y, Z by multiplying A^{-1} * y

$$\begin{bmatrix} 0.8 & -0.2 & -1 \\ -0.67 & 0.33 & 0.67 \\ 0.67 & 0.67 & 0.33 \end{bmatrix} * \begin{bmatrix} 10 \\ 20 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Therefore X = 1, Y = 2, and Z = 3.

(b)

$$\begin{bmatrix} 1 & 0 & 2 \\ 4 & 5 & 8 \\ -1 & -1 & -2 \end{bmatrix} * \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 7 \\ 38 \\ -9 \end{bmatrix}$$

To begin solving for X, Y, & Z we must check if we can invert our A matrix. A is invertible if its determinant does not equal 0.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 4 & 5 & 8 \\ -1 & -1 & -2 \end{bmatrix}$$

Determinant of A equals 0, thus A is NOT invertible, therefore we may either have infinitely many solutions or none at all, we need to do more test to determine which it is.

The next thing to check is the rank of matrix A, we can use Matlab for this and see that the rank is 2, this is because 2 of the columns in the matrix are dependant (first and last).

Finally we need to check if the vector y belongs to the range of A. To do this I will use Matlabs row reduced echelon form function. The result is the following matrix:

$$rref(A, y) = \begin{bmatrix} 1 & 0 & 2 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$1X + 2Z = 7$
$Y = 2$

From this matrix we can see that the last number in each row can be made by a linear combination of the other 3 terms with some coefficient. This means that this system of equations has infinitely many solutions.

(c)

$$\begin{bmatrix} 1 & 0 & 2 \\ 4 & 5 & 8 \\ -1 & -1 & -2 \end{bmatrix} * \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

To begin solving for X, Y, & Z we must check if we can invert our A matrix. A is invertible if its determinant does not equal 0.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 4 & 5 & 8 \\ -1 & -1 & -2 \end{bmatrix}$$

Determinant of A equals 0, thus A is NOT invertible, therefore we may either have infinitely many solutions or none at all, we need to do more test to determine which it is.

The next thing to check is the rank of matrix A, we can use matlab for this and see that the rank is 2, this is because 2 of the columns in the matrix are dependant (first and last).

Finally we need to check if the vector y belongs to the range of A. To do this I will use Matlabs row reduced echelon form function. The result is the following matrix:

$$rref(A, y) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From this matrix we can see that the last number in the last row can NOT be made by a linear combination of the other 3 terms with some coefficient. Any coefficient multiplied by zero will be zero, and the sum of three zeros will never be one. This means that this system of equations has NO solution.

2 Eigenvalues and Eigenvectors

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$

(a)

To compute the eigenvalues of A we need to begin by solving the following equation: $0 = |A{-}\lambda I|$.

$$0 = \left|\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right|$$

$$0 = \left|\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix} - \begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}\right|$$

$$0 = \left|\begin{bmatrix}1-\lambda & 2\\2 & 1-\lambda\end{bmatrix}\right|$$

$$0 = (1-\lambda)(1-\lambda) - 2 \cdot 2$$

$$0 = \lambda^2 - 2\lambda + 1 - 4$$

$$0 = \lambda^2 - 2\lambda - 3$$

We can then solve for lambda which gives us 2 possible eigenvalues which are -1 and 3.

We then can use these eigenvalues to compute the 2 eigenvectors. We will begin with the eigenvector for the eigenvalue of 3 to show the process, then use Matlab for the -1 value.

We use the equation $Av = \lambda v$

$$\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}X\\Y\end{bmatrix} = 3\begin{bmatrix}X\\Y\end{bmatrix}$$

We turn this into a system of equations to get:

$$1X + 2Y = 3X$$
$$2X + 1Y = 3Y$$

We then move all the terms with variables to one side.

$$-2X + 2Y = 0$$
$$2X + -2Y = 0$$

Finally solve for X & Y to get our eigenvector.

The result is: $\begin{bmatrix}1\\1\end{bmatrix}$

The second eigenvector for eigenvalue of -1 would be $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now we need to find the unit norm eigenvectors by dividing each element by the norm of the vector.

The norm of these two vectors will be the same since the computation is the sum of the elements squared, then square rooting that value.

$magnitude = \sqrt{1^2 + 1^2} = 1.4142$

We then divide the elements in the two vectors by this number. The resulting unit-norm eigenvectors are:

$\begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}$ and $\begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$

Finally to determine if these two vectors are orthogonal we need to test if the dot product is 0. To do this we multiply element-wise and add.

$(0.7071 \cdot -0.7071) + (0.7071 \cdot 0.7071) = 0$

Therefore these vectors are orthogonal.

(b)

The determinant of $A = (1 \cdot 1) - (2 \cdot 2) = 1 - 2 = -1$. Since the determinant of A is -1 and thus not equal to 0, meaning we can invert A.

The inverse of A is:

$A^{-1} = \begin{bmatrix} -0.3333 & 0.6667 \\ 0.6667 & -0.3333 \end{bmatrix}$

(c)

We can compute the eigenvalues and eiganvectors for matrix B in the same way we did for matrix A. However, we will be using variables instead of finite numbers.

To compute the eigenvalues of A we need to begin by solving the following equation: $0 = |A - \lambda I|$.

$$0 = \left| \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$0 = \left| \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

$$0 = \left| \begin{bmatrix} \alpha - \lambda & \beta \\ \beta & \alpha - \lambda \end{bmatrix} \right|$$

$$0 = (\alpha - \lambda)(\alpha - \lambda) - \beta \cdot \beta$$

$$0 = (\alpha - \lambda)^2 - \beta^2$$

Finally we can put lambda in terms of a and b.

$$\lambda = \alpha - \beta$$
$$\lambda = \alpha + \beta$$

Then to compute the eigenvectors we would use the following equation for each of the two eigenvalue:

$$\begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = (\alpha - \beta) \begin{bmatrix} X \\ Y \end{bmatrix}$$

We can turn these into two separate equations that we can solve to obtain the eigenvector:

$$\alpha X + \beta Y = X(\alpha - \beta)$$
$$\beta X + \alpha Y = Y(\alpha - \beta)$$

$$\alpha X + \beta Y = X\alpha - X\beta$$
$$\beta X + \alpha Y = Y\alpha - Y\beta$$

$$\beta Y + \beta X = 0$$
$$\beta X + \beta Y = 0$$

An Eigenvector for $\lambda = (\alpha - \beta)$ would be:

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We can now compute the Eigenvector for $\lambda = (\alpha + \beta)$

$$\begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = (\alpha + \beta) \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$\alpha X + \beta Y = X(\alpha + \beta)$$
$$\beta X + \alpha Y = Y(\alpha + \beta)$$

$$\alpha X + \beta Y = X\alpha + X\beta$$
$$\beta X + \alpha Y = Y\alpha + Y\beta$$

$$\beta Y - \beta X = 0$$
$$\beta X - \beta Y = 0$$

An eigenvector for $\lambda = (\alpha - \beta)$ would be:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We can now compute the unit-norm eigenvector by dividing each element of these eigenvectors by their magnitude.

Since the magnitude equals the sum of each term squared it will be the same for both vectors.

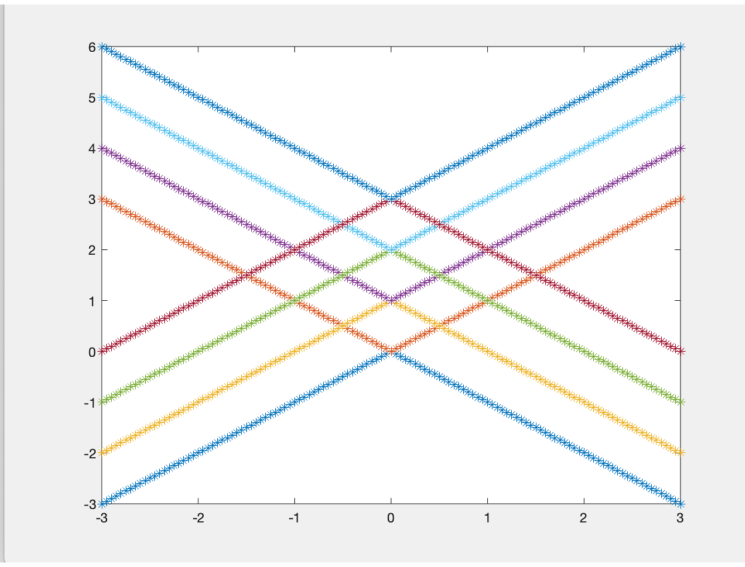
$$magnitude = \sqrt{1^2 + 1^2} = 1.4142$$

Thus we can get the two unit norm eigenvectors to be:

$$\begin{bmatrix} 0.7071 \\ -0.7071 \end{bmatrix} \text{ and } \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}$$

Using Matlab I have plotted the eigenvalues of matrix B for $\alpha \in \{0, 1, 2, 3\}$ and $\beta \in [-3, 3]$.

```
alpha = [0 1 2 3];
beta = [-3:0.05:3];


for a = alpha

    eigs = zeros([length(beta) 2]);
    i = 1;

    for b = beta

        B = [a b;
             b a];

        eigs(i,:) = eig(B);
        i = i + 1;

    end

    plot(beta,eigs, '*');
    hold on;

end
```



(d)

The determinant of matrix b can be expressed with the following equation:

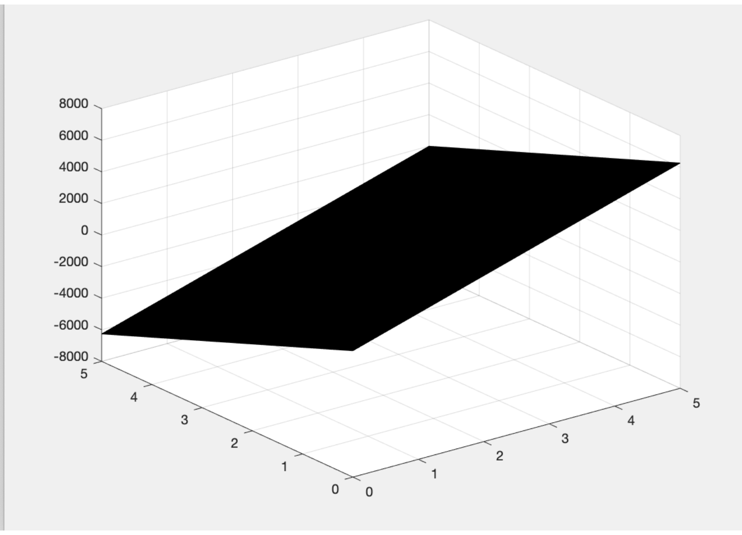
$$det(A) = (\alpha \cdot \alpha) - (\beta \cdot \beta)$$

$$det(A) = (\alpha)^2 - (\beta)^2$$

Matrix B is invertible when the determinant is NOT equal to 0, thus when $\alpha^2 \neq \beta^2$.

Using Matlab I have plotted the determinant of matrix B for $(\alpha, \beta) \in [0, 5]^2$.

```
%{

B = [a b; b a]

determinant equation for 2x2 matrix
is multiplication of diagonals minus
each other.

So det = (a*a - b*b) for our A matrix


%}

alpha = [0:0.01:5];
beta = [0:0.01:5];

[a, b] = meshgrid(alpha,beta);

determinant = (a*a - b*b);

surf(a, b, determinant)
```



3 Multiplication by an orthogonal matrix

(a)

Prove that multiplication by an orthogonal matrix U preserves lengths, that is:

$||Ux|| = ||x||$

To prove this is true I first want to point out that one property of an orthogonal matrix is that $C^T = C^{-1}$ meaning its transpose is equal to its inverse.

To begin lets square both sides to make it easier to prove. So we are now proving that:

$||Ux||^2 = ||x||^2$

We can then expand the left hand side to get:

$||Ux||^2 = Ux \cdot Ux$

This product can then be expressed as:

$Ux \cdot Ux = (Ux)^T \cdot Ux$

Next we can apply the transpose to the U and x separately to get:

$(Ux)^T \cdot Ux = U^T \cdot x^T \cdot Ux$

Due to the property we explained earlier that $U^T = U^-1$ we can then express this as:

$U^T \cdot x^T \cdot Ux = x^T \cdot U^{-1} \cdot U \cdot x$

The inverse of U times U is equal to the identity matrix I:

$x^T \cdot U^{-1} \cdot Ux = x^T \cdot x \cdot I$

The identity matrix doesn't change the result of the multiplication thus we can get rid of it and we get:

$x^T \cdot x = x \cdot x$

Finally from the first step we know that this reduces to length of x squared.

$x \cdot x = ||x||^2$

We have now shown that:

$||Ux||^2 = ||x||^2$

We can simply take square root of both sides and we have proved that it is true that:

$||Ux|| = ||x||$

(b)

Prove that multiplication by an orthogonal matrix U preserves angles, that is:

$\langle Ux, Uy \rangle = \langle x, y \rangle$

We know that the inner product of two vectors x and y is an angle and can be expressed in the following way:

$cos(\theta) = \frac{x \cdot y}{||x|| \cdot ||y||}$

To show that the multiplication by matrix U does not change this we need to show that the inner product of Ux and Uy are equal to $cos(\theta)$. To begin we can expand the left hand side of the original equation to get:

$\langle Ux, Uy \rangle = \frac{Ux \cdot Uy}{||Ux|| \cdot ||Uy||}$

In the above question (a) we already proved that lengths are preserved so we can replace the magnitude of Ux and Uy with simply magnitude of x and y.

$\frac{Ux \cdot Uy}{||Ux|| \cdot ||Uy||} = \frac{Ux \cdot Uy}{||x|| \cdot ||y||}$

Now we can rewrite the top dot product as follows:

$\frac{Ux \cdot Uy}{||x|| \cdot ||y||} = \frac{(Ux)^T \cdot Uy}{||x|| \cdot ||y||}$

Next we can expand the top and apply the transpose to both U and x separately, we then get:

$$\frac{(Ux)^T \cdot Uy}{||x|| \cdot ||y||} = \frac{U^T \cdot U \cdot x^T \cdot y}{||x|| \cdot ||y||}$$

As we saw in the previous question that $U^T = U^-1$ Thus these two terms together give us the identity matrix I. Which again does not change the results of the multiplication so we get:

$$\frac{U^T \cdot U \cdot x^T \cdot y}{||x|| \cdot ||y||} = \frac{x^T \cdot y}{||x|| \cdot ||y||}$$

We also saw in an earlier step that the top of this fraction can be re-written as simply the product of x and y. Therefore we get:

$$\frac{x^T \cdot y}{||x|| \cdot ||y||} = \frac{x \cdot y}{||x|| \cdot ||y||}$$

And from the first step we know that this is equal to the inner product $cos(\theta)$ thus we have proved that the multiplication by an orthogonal matrix does preserve angles.

4 Bases and frames of R^2

(a)

To write the matrix representations for the given vector sets we simply join all the vectors in the set to create a matrix. For $\Phi_1 \ and \ \Phi_3$ we get the following:

$$\Phi_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{\sqrt{3}}{2} & 1 \end{bmatrix}$$

$$\Phi_3 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

(b)

In order to find the dual basis we must compute the following equation for each matrix:

$$\tilde{\Phi} = \Phi \cdot (\Phi^T \cdot \Phi)^{-1}$$

To begin lets apply this to Φ_1 I will be converting to decimals in order to compute this with Matlab.

$$\Phi_1 = \begin{bmatrix} 0.5 & 0 \\ 0.866 & 1 \end{bmatrix}$$

Running this through Matlab we get the following matrix as the dual basis of Φ_1.

$$\tilde{\Phi}_1 = \begin{bmatrix} 2 & -1.732 \\ 0 & 1 \end{bmatrix}$$

Now lets apply this to Φ_3 I will be converting to decimals in order to compute this with Matlab.

$$\Phi_3 = \begin{bmatrix} 0.5 & -0.866 \\ 0.866 & 0.5 \end{bmatrix}$$

Running this through Matlab we get the following matrix as the dual basis of Φ_3.

$$\tilde{\Phi}_3 = \begin{bmatrix} 0.5 & -0.866 \\ 0.866 & 0.5 \end{bmatrix}$$

(c)

In order to check if the basis is orthonormal we must check the inner products of the two column vectors of the duals to ensure none of them equal to zero. We will begin with $\tilde{\Phi}_1$

$$\tilde{\Phi}_1 = (2 \cdot -1.732) + (0 \cdot 1) = -3.464$$

Since this values doesn't equal to 0 we can say the dual is not orthonormal.

We can repeat this for $\tilde{\Phi}_3$

$$\tilde{\Phi}_3 = (0.5 \cdot -0.866) + (0.866 \cdot 0.5) = 0$$

Since this value does equal to 0 we can say that this dual is orthonormal.

(d)

To compute the projection coefficients given that $x = \begin{bmatrix} 2, 0 \end{bmatrix}^T$ we can simply take the transpose of each $\tilde{\Phi}$ and multiply by our transposed x to get these coefficients. We will begin with $\tilde{\Phi}_1$.

$$\text{coefficients } \tilde{\Phi}_1 = \begin{bmatrix} 2 & -1.732 \\ 0 & 1 \end{bmatrix}^T \cdot \begin{bmatrix} 2, 0 \end{bmatrix}^T = \begin{bmatrix} 4.00 \\ -3.4640 \end{bmatrix}$$

Next we can apply the same steps to $\tilde{\Phi}_3$.

$$\text{coefficients } \tilde{\Phi}_3 = \begin{bmatrix} 0.5 & -0.866 \\ 0.866 & 0.5 \end{bmatrix}^T \cdot \begin{bmatrix} 2, 0 \end{bmatrix}^T = \begin{bmatrix} 1.00 \\ -1.7321 \end{bmatrix}$$

(e)

Now we need to verify the expansion formula for each. To do this we will begin with showing that $\Phi\tilde{\Phi}^T = I$, then show that $\Phi \cdot coefficients = x$.

We will begin with showing these two for Φ_1:

$\Phi_1(\tilde{\Phi}_1)^T = I$

$$\Phi_1 = \begin{bmatrix} 0.5 & 0 \\ 0.866 & 1 \end{bmatrix}$$

$$(\tilde{\Phi}_1)^T = \begin{bmatrix} 2 & 0 \\ -1.732 & 1 \end{bmatrix}$$

We then multiply together and get:

$$\Phi_1(\tilde{\Phi}_1)^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Next we check that $\Phi_1 \cdot coefficients = x$

$$\begin{bmatrix} 0.5 & 0 \\ 0.866 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -3.4640 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Therefore we verified that the expansion formula is correct for Φ_1

Now we repeat the same steps for Φ_3

$\Phi_3(\tilde{\Phi}_3)^T = I$

$$\Phi_3 = \begin{bmatrix} 0.5 & -0.866 \\ 0.866 & 0.5 \end{bmatrix}$$

$$(\tilde{\Phi}_3)^T = \begin{bmatrix} 0.5 & 0.866 \\ -0.866 & 0.5 \end{bmatrix}$$

We then multiply together and get:

$$\Phi_3(\tilde{\Phi}_3)^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Next we check that $\Phi_3 \cdot coefficients = x$

$$\begin{bmatrix} 0.5 & -0.866 \\ 0.866 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1.7321 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Therefore we verified that the expansion formula is correct for Φ_3

(f)

We want to know if the coefficients preserve the norm. That is that $||x|| = \sum_k |a_i, k|^2$. Essentially we want to know if the norm of x and the norm of the coefficients are equal for both Φ_1 and Φ_3.

First lets compute the magnitude of x.

$||x|| = \sqrt{2^2 + 0^2} = 2$

Now we can compute magnitude of the coefficients for Φ_1.

Φ_1 coefficient norm $= \sqrt{4^2 + -3.4640^2} = 5.2914$

Therefore for Φ_1 the norm is NOT preserved because its norm doesn't equal the norm of x, which makes since because $\tilde{\Phi}_1$ is not an orthonormal basis.

Now lets compute the magnitude of the coefficients for Φ_3.

Φ_3 coefficient norm $= \sqrt{1^2 + -1.7321^2} = 2.0$

Since the norm of Φ_3 coefficients is equal to the norm of x we can say the norm is preserved. Again this makes since because $\tilde{\Phi}_3$ is an orthonormal basis.

(g)

To write the matrix representations for the given vector sets we simply join all the vectors in the set to create a matrix. For Φ_2 *and* Φ_4 we get the following:

$$\Phi_2 = \begin{bmatrix} \frac{1}{2\sqrt{2}} & -\frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\Phi_4 = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 1 \end{bmatrix}$$

5 Inner product

It is TRUE to say that two vectors are orthogonal if their inner product is 0.

Given f(t) and g(t) we can prove they are orthogonal by taking the inner product of the two functions. This is done by taking the integral of the multiplication of the two functions. The standard test for orthogonality is the following:

$$\langle f(t), g(t)\rangle = \int_{-\infty}^{\infty} f(t) \cdot g(t)\, dt \;\; = 0$$

In this problem we are asked to prove that the inner product is orthogonal in the Hilbert space of $L^2 = [+1, -1]$. This along with the actual functions can be subbed in the above integral to get the following:

$$\langle f(t), g(t)\rangle = \int_{-1}^{1} sin(\pi nt) \cdot sin(\pi mt)\, dt$$

We must show that the above integral evaluates to 0 given any integers n and m where $n \neq m$ to show orthogonality between f(t) and g(t). We can also use a property of multiplication of sin functions to help evaluate this integral. The property is as follows:

$$sin(a) \cdot sin(b) = \tfrac{1}{2}[cos(a - b) - cos(a + b)]$$

This means we can re-write the above integral using this property to simplify the computation. When done the integral is as follows:

$$\langle f(t), g(t)\rangle = \int_{-1}^{1} \tfrac{1}{2}[cos(\pi nt - \pi mt) - cos(\pi nt + \pi mt)]\, dt$$

This integration is long and complicated. Since we were able to use Matlab for the other questions I took the liberty of using an online integral calculator for this problem (https://www.integral-calculator.com). Computing the definite integral between -1 and 1 you get the following.

$$\int_{-1}^{1} \tfrac{1}{2}[cos(\pi nt - \pi mt) - cos(\pi nt + \pi mt)]\, dt \;\; = \frac{sin(\pi(n-m))}{\pi(n-m)} - \frac{sin(\pi(n+m))}{\pi(n+m)}$$

We can see by this resulting equation that any numbers used for n & m as long as $n \neq m$ will result in a cancellation between the left and right hand side which will result in an answer of 0 for the equation. As stated above two functions who's inner product equals 0 are orthogonal, thus f(t) and g(t) must be orthogonal based on our above computation of their inner product.

6 Inner product computation by expansion sequences

$f(t) = \alpha_0 + \sum_{k=1}^{\infty} \alpha_k \sqrt{2} cos(2\pi kt)$
$g(t) = \beta_0 + \sum_{k=1}^{\infty} \beta_k \sqrt{2} cos(2\pi kt)$

In order to show that $\langle f(t), g(t) \rangle = \langle \alpha, \beta \rangle$ we must first define what α and β equal and what their inner product would look like.

$\langle \alpha, \beta \rangle = \alpha_0\beta_0 + \sum_{k=1}^{\infty} \alpha_k\beta_k$

Now we need to show that the inner product between $f(t)$ and $g(t)$ is equal to the above when they are in $L^2([-\frac{1}{2}, \frac{1}{2}])$. The inner product between $f(t)$ and $g(t)$ can be represented by the following integration.

$\langle f(t), g(t) \rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} (\alpha_0 + \sum_{k=1}^{\infty} \alpha_k \sqrt{2} cos(2\pi kt)) \cdot (\beta_0 + \sum_{k=1}^{\infty} \beta_k \sqrt{2} cos(2\pi kt)) \; dt$

We can foil the above in order to simplify the integration.

$\langle f(t), g(t) \rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} (\alpha_0\beta_0) + (\alpha_0 \sum_{k=1}^{\infty} \beta_k \sqrt{2} cos(2\pi kt))) + (\beta_0 \sum_{k=1}^{\infty} \alpha_k \sqrt{2} cos(2\pi kt))) +$
$(\sum_{k=1}^{\infty} \alpha_k \sqrt{2} cos(2\pi kt)) \sum_{k=1}^{\infty} \beta_k \sqrt{2} cos(2\pi kt))) \; dt$

Now we can break up the integral into the following and then add the results together:

$\int_{-\frac{1}{2}}^{\frac{1}{2}} (\alpha_0\beta_0) \; dt = (\alpha_0\beta_0)$

$\int_{-\frac{1}{2}}^{\frac{1}{2}} \alpha_0 \sum_{k=1}^{\infty} \beta_k \sqrt{2} cos(2\pi kt) \; dt = \sqrt{2}\alpha_0 \sum_{k=1}^{\infty} \beta_k \int_{-\frac{1}{2}}^{\frac{1}{2}} cos(2\pi kt) \; dt = 0$ for all k > 0

$\int_{-\frac{1}{2}}^{\frac{1}{2}} \beta_0 \sum_{k=1}^{\infty} \alpha_k \sqrt{2} cos(2\pi kt) \; dt = \sqrt{2}\beta_0 \sum_{k=1}^{\infty} \alpha_k \int_{-\frac{1}{2}}^{\frac{1}{2}} cos(2\pi kt) \; dt = 0$ for all k > 0

$\int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{k=1}^{\infty} \alpha_k \sqrt{2} cos(2\pi kt) \cdot \sum_{k=1}^{\infty} \beta_k \sqrt{2} cos(2\pi kt) \; dt = \sqrt{2}\sqrt{2} \sum_{k=1}^{\infty} \alpha_k\beta_k \int_{-\frac{1}{2}}^{\frac{1}{2}} (cos(2\pi kt))^2$
$= 2 \cdot \sum_{k=1}^{\infty} \alpha_k\beta_k \cdot \frac{1}{2} = \sum_{k=1}^{\infty} \alpha_k\beta_k$

Now we can collect them back together by adding and get the following equation:

$\alpha_0\beta_0 + \sum_{k=1}^{\infty} \alpha_k\beta_k$

We can see that $\langle f(t), g(t) \rangle = \alpha_0\beta_0 + \sum_{k=1}^{\infty} \alpha_k\beta_k$

Therefore we have shown that $\langle \alpha, \beta \rangle = \langle f(t), g(t) \rangle$