CIS 4930.006S21/CIS 6930.013S21: Computational Methods for Imaging and Vision

Spring 2021
Homework #3

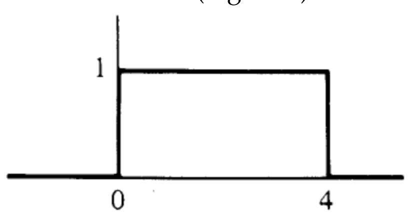
The University of South Florida
Department of Computer Science and Engineering
Tampa, FL

Assigned: February 24, 2021
Due: March 10, 2021 (6:00 PM)

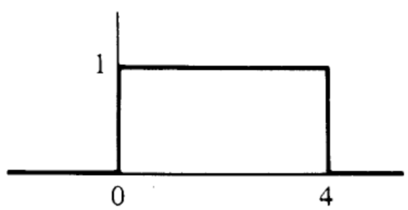
1 Convolutions

(a) Compute the (continuous) convolution of $g(x)$ and $h(x)$ for the following three cases:

(i) The functions $g(x)$ and $h(x)$ are shown below (Figure 1)



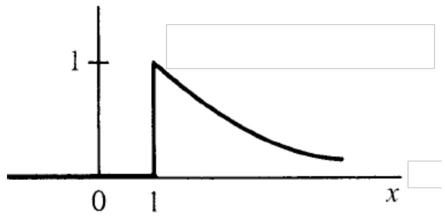
(a): $g(x)$



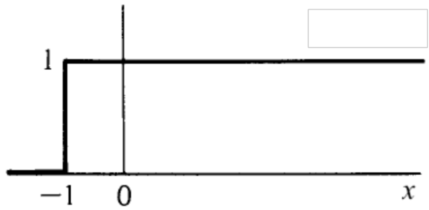
(b): $h(x)$

Figure 1: Graphs of $g(x)$ and $h(x)$.

(ii) The functions $g(x) = e^{-(x-1)}u(x-1)$ and $h(x) = u(x+1)$ are shown below (Figure 2)



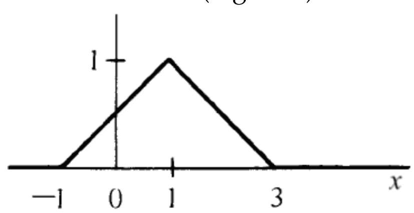
(a): $g(x) = e^{-(x-1)}u(x-1)$



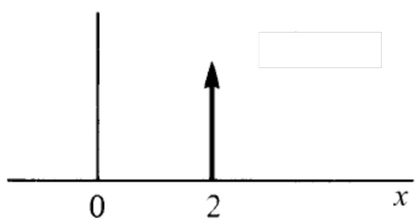
(b): $h(x) = u(x+1)$

Figure 2: Graphs of $g(x)$ and $h(x)$.

(iii) The functions $g(x)$ and $h(x)$ are shown below (Figure 3)



(a): $g(x)$



(b): $h(x)$

Figure 3: Graphs of $g(x)$ and $h(x)$.

(b) Let $g(x) = \mathrm{sinc}(ax)$ and $h(x) = \mathrm{sinc}(bx)$, where a, b are real-valued scalar quantities, with $a \leq b$. Using the **convolution property** of the Fourier transform, compute the convolution between the functions $g(x)$ and $h(x)$. To do this:

(i) State the convolution property of the Fourier transform.

(ii) Using the result $\mathcal{F}\{\mathrm{sinc}(x)\} = \pi\mathrm{rect}(\omega/2)$ (see problem 2(b) of Homework #2) and the scaling property of the Fourier transform, find the Fourier transforms for $g(x)$ and $h(x)$.

(iii) Using your answers to part (ii) above, and the convolution property of the FT, compute $\mathcal{F}(g(x) \star h(x))$. (Remember that $a \leq b$.)

(iv) Take the inverse Fourier transform of your answer to (iii) above, to obtain $g(x) \star h(x)$.

(c) Let $g(x) = \frac{1}{\sqrt{2\pi}a} e^{-\frac{x^2}{2a^2}}$ and $h(x) = \frac{1}{\sqrt{2\pi}b} e^{-\frac{x^2}{2b^2}}$.

(i) Compute the convolution between the functions $g(x)$ and $h(x)$, directly, that is by using the definition of the 1D convolution integral. [Hint: At some point, you will need to use the fact that the integral $\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \sqrt{2\pi}\sigma$, where μ and σ are scalars.]

(ii) Compute the convolution between the functions $g(x)$ and $h(x)$ by applying the convolution property of the Fourier transform. [You may need the above hint.]

2 Digital Holography

Holography is a 3D imaging technique, in the sense that it allows recreate the 3D scene (optically or digitally) from a single intensity measurement. In this problem, we will explore the general idea of in-line (Gabor) holography and understand the unique feature about holography from the linear system perspective.

A schematic of the in-line holography is shown in Fig. 2. To record a hologram, a coherent light source (e.g. laser) is needed to illuminate the 3D scene. The hologram (the intensity image captured by the camera) is the result from the interference between the unperturbed illumination (reference beam) and the light scattered from the object.

The formation of the hologram from a 2D object can be approximated using the following linear shift invariant (LSI) model:

$$g_{\text{out}}(x, y) = g_{\text{in}}(x, y; z) \star h(x, y; z), \tag{1}$$

where $\star$ denotes the 2D convolution (i.e. over (x, y)), $g_{\text{out}}(x, y)$ is the output term of interest contained in the hologram, $g_{\text{in}}(x, y; z)$ is the object function, and $h(x, y; z)$ is the point spread function (PSF), determined by the free-space propagation and diffraction theory, which has the following form,

$$h(x, y; z) = \frac{1}{\mathrm{j}\lambda z} e^{\mathrm{j}k\frac{x^2+y^2}{2z}}, \tag{2}$$

and the corresponding transfer function,

$$H(f_x, f_y; z) = \mathcal{F}\{h(x, y; z)\} = e^{-\mathrm{j}\pi\lambda z(f_x^2+f_y^2)}, \tag{3}$$

where $k = 2\pi/\lambda$ is the wavenumber, λ is the wavelength of illumination, (x, y) denote the lateral coordinates and z denotes the axial direction (along which the light propagates) and (f_x, f_y) denote the spatial frequency coordinates, according to the following Fourier transform relation

$$H(f_x, f_y; z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(x, y; z) e^{-\mathrm{j}2\pi(f_x x + f_y y)}\, \mathrm{d}x \mathrm{d}y, \tag{4}$$

note that we have used the substitutions (or change of variables) $\omega_x = 2\pi f_x$ and $\omega_y = 2\pi f_y$ in the usual definition of the Fourier transform we saw in class.

(a) Write down the mathematical expression for the **operator** that represents the imaging system, in both **standard form** and **spectral form**.

(b) Find the range, null space, adjoint, and inverse of the system operator.

(c) Construct the forward models in both **matrix form** of the imaging system. Assume square pixels with size Δ and number of pixels in each spatial dimension is N. Discretization of the object space is performed on the standard basis with square pixel of size δ, and number of pixels in each spatial dimension M is chosen such that $M\delta = N\Delta$.

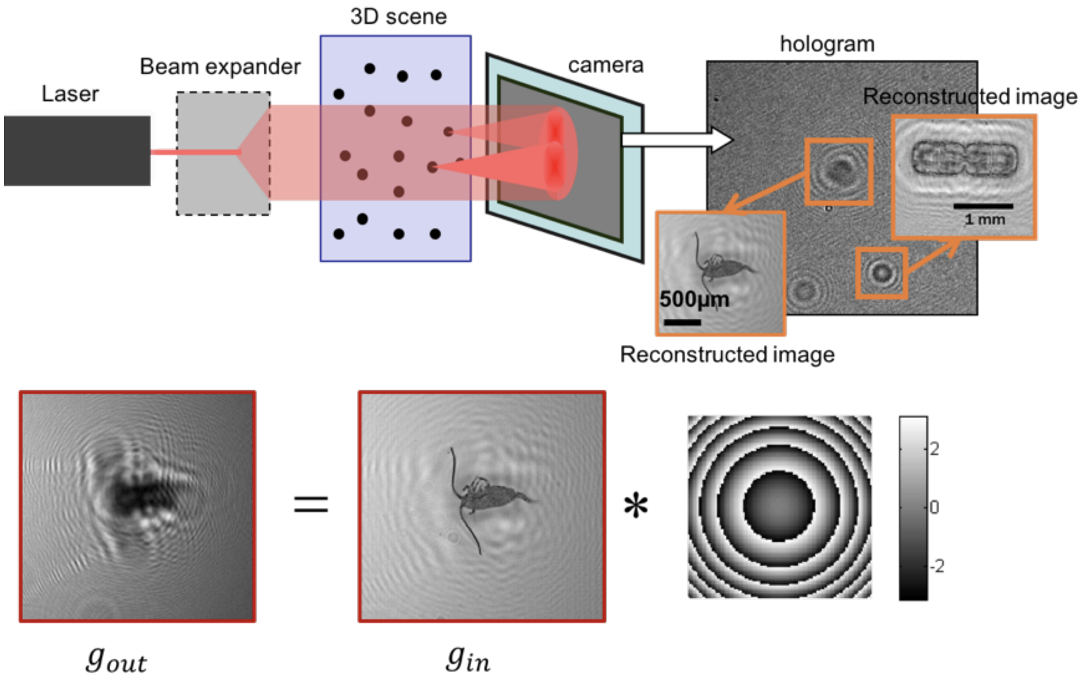


Figure 4: Caption

(d) Find the range, null space, adjoint, and inverse of the discretized system.

(e) Write a Matlab/Python script for the forward model in (c).

(f) Plot the system transfer functions with the following parameters: $\lambda = 0.5\ \mu\text{m}$, $N = 1000$.

  (i) Keep $z = 50$ mm fixed, $\delta = \Delta = 2\ \mu\text{m}, 5\ \mu\text{m}, 10\ \mu\text{m}, 20\ \mu\text{m}$. How does sampling affect the performance of digital holography?

  (ii) Keep $\delta = 5\ \mu\text{m}$ fixed, while z ranges from 30 mm to 70 mm (use a small step-size). How does object distance affect the performance of digital holography?

  (iii) Keep $\Delta = 5\ \mu\text{m}$ and $z = 50$ mm fixed, while δ ranges from $1\ \mu\text{m}$ to $10\ \mu\text{m}$ in $2\ \mu\text{m}$ step size. How does discretization of the object affect the performance of digital holography?

(Optional for Extra credit: 3D) Consider the 3D holographic imaging problem. The formation of the hologram from an extended scene can be approximated using the following linear model

$$g_{\text{out}}(x, y) = \int_{z_l} g_{\text{in}}(x, y; z_l) \star h(x, y; z_l)\,\mathrm{d}z_l \tag{5}$$

where $\star$ denotes the 2D convolution in (x, y), $g_{\text{in}}(x, y; z_l)$ is the object function at the distance z_l.

(g) Construct the forward models in both **operator and matrix forms**. Assume that the discretization of the object space is performed on the standard basis with lateral spacing $\delta_x = \delta_y = \delta$, and axial spacing δ_z.

(h) Is the system shift invariant? If so, plot the system transfer function. If not, plot the singular values of the system.

(i) Find the image space, object space, range, null space, and adjoint of the system. Does the inverse exist?

(j) Write a Matlab/Python script for the forward model in part (g).