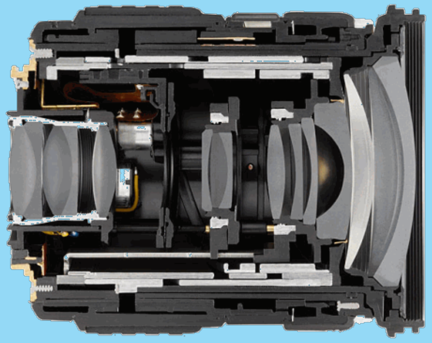

Optics

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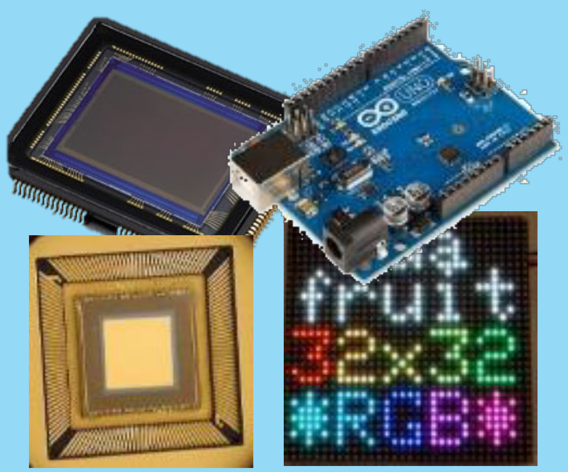

Sensors & devices

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Signal processing & algorithms

COMPUTATIONAL METHODS FOR IMAGING (AND VISION)

LECTURE 5: LINEAR OPERATORS & BASES

PROF. JOHN MURRAY-BRUCE

WHERE ARE WE

WE ARE HERE!

Week	Date	Main Topic	Lecture	Readings	Homework Out	Homework Due
1	11-Jan-21		Introduction to computational imaging - Forward and Inverse problems - Common computational imaging problems			
	13-Jan-21		Vectors - Preliminaries			
2	18-Jan-21		**Dr. Martin Luther King, Jr. Holiday (no class)**			
	20-Jan-21	Mathematical preliminaries	Vectors and Vector Spaces - Subspaces, Finite dimensional spaces	IIP Appendix A; FSP 2.1 - 2.2		
3	25-Jan-21		Vector Spaces - Hilbert spaces	IIP Appendix B; FSP 2.3		
	27-Jan-21		Bases and Frames I - Orthonormal and Reisz Bases	IIP Appendix C; FSP 2.4 and 2.B	**HW 1**	
4	1-Feb-21		Bases and Frames II - Orthogonal Bases - Linear operators	IIP Appendix C; FSP 2.5 and 2.B		
	3-Feb-21		Fourier Analysis I - FT (1D and 2D) - FT properties	IIP 2.1, Appendix D; FSP 4.4		
5	8-Feb-21		Sampling and Interpolation - BL functions - Sampling	IIP 2.2, 2.3; FSP 5.4, 5.5		**HW 1**
	10-Feb-21		Fourier Analysis II (DFT)	IIP 2.4; FSP 3.6	**HW 2**	
6	15-Feb-21	Forward Modeling	LSI imaging: Forward problem I - Convolution	IIP 2.5 - 2.6, 3		
	17-Feb-21		LSI imaging: Forward problem I - Transfer functions	IIP 2.6		
7	22-Feb-21		LSI imaging: Forward problem I - Linear operators	IIP 3		
	24-Feb-21		LSI imaging: Forward problem I - Linear operators, Adjoints, and Inverses		**HW 3**	**HW 2**
8	1-Mar-21		**Mid-term Exams**			
	3-Mar-21		LSI imaging: Forward problem II - Sampling and Discretization: Matrix-vector form	IIP 2.7, 4		
9	8-Mar-21		LSI imaging: Forward problem II - Convolution matrix			
	10-Mar-21		LSI imaging: Forward problem II - Sampling and Discretization: Matrix-vector form - PSF, and Transfer functions			**HW3**

HILBERT SPACE

- **Definition (Hilbert space):** is a complete inner product space.
  - ***Inner product space:*** a vector space, equipped with a valid inner product. [Q: What does it take for an inner product to be valid?]
  - Completeness is just a technical requirement: i.e. "any sequence of vectors whose elements become eventually arbitrarily close."
- **Examples:**
  - $\mathbb{R}^n$ equipped with the 2-norm.
  - $\ell^2(\mathbb{Z}), \mathscr{L}^2(\mathbb{R})$.

GENERALIZATION OF MATRICES

LINEAR OPERATORS

LINEAR SYSTEMS
A REVIEW OF MATRICES

- A mathematical construct used to represent various phenomena
- Yeah okay! BUT – what is it good for?
  - In linear algebra, it was used to represent a system of linear (simultaneous) equations

Solve:

$$\begin{aligned} 2x_1 - 4x_2 + x_3 &= 7 \\ -4x_1 - 1x_2 + 7x_3 &= -3 \\ 5x_1 - 13x_2 - 7x_3 &= -13 \end{aligned}$$

$$\begin{bmatrix} 2 & -4 & 1 \\ -4 & -1 & 7 \\ 5 & -13 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ -13 \end{bmatrix}$$

Can store these on a computer and apply computer a host of computer algorithms to the matrix and vectors.

LINEAR SYSTEMS
A REVIEW OF MATRICES

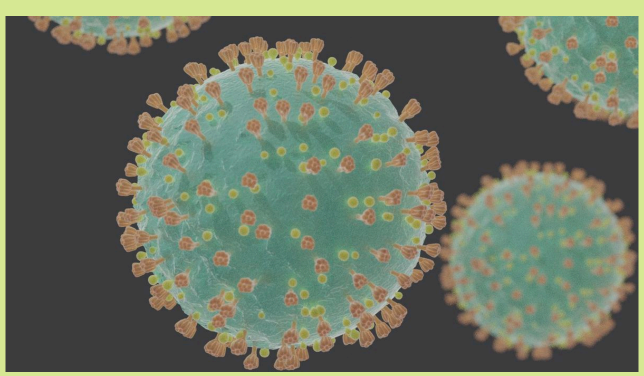
MATLAB

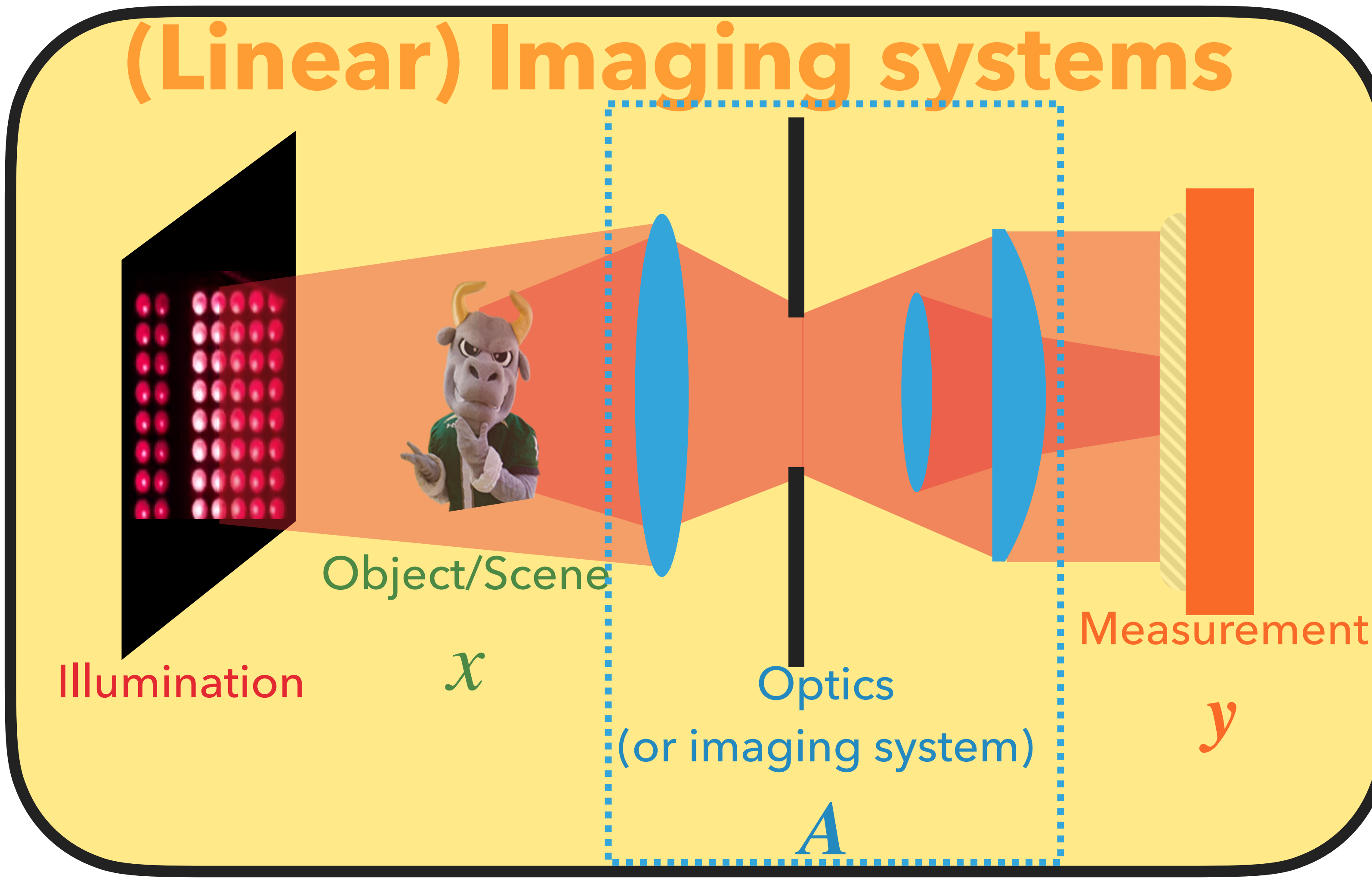
- **Operations on a matrix**
  - Addition, Scalar Multiplication, Subtraction and Matrix Multiplication
  - Transpose
- **Properties of a matrix**
  - Rank
  - Determinant
  - **Range space:** span of its column vectors
  - **Null space:** (of adjoint) is orthogonal complement of range space
  - **Norm**
  - **Condition number**
- **Eigenvalue Decomposition and Singular value decomposition (SVD)**

LINEAR SYSTEMS
MATRICES

- Here, we will use **matrices** to represent
  - Images
  - (Linear) Imaging systems

Images





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UTILITY OF MATRICES (& LINEAR OPERATORS)

- **Imaging systems** can be described by **linear operators**
  - Satisfy addition and scalar multiplication
- Properties of linear operators can be studied to reason about the imaging system
  - Range space: **span** of column vectors
  - Null space: **orthogonal complement** of range space
  - Boundedness (norm)
  - Condition number (how well-behaved is the system)

LINEAR OPERATORS
GENERALIZATION OF MATRICES

- **Definition:** $A : H_0 \to H_1$ is a linear operator when, for all $\boldsymbol{x}, \boldsymbol{y} \in H_0$ and $\alpha \in \mathbb{R}$, the following hold:
  - *Additivity*: $\boldsymbol{A}(\boldsymbol{x} + \boldsymbol{y}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{A}\boldsymbol{y}$
  - *Scalability*: $\boldsymbol{A}(\alpha\boldsymbol{x}) = \alpha(\boldsymbol{A}\boldsymbol{x})$
- **Terminology:**
  - **Null space** (a subspace of H_0): $\mathcal{N}(A) = \{\boldsymbol{x} \in H_0 \,|\, \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\}$
  - **Range space** (a subspace of H_1): $\mathcal{R}(A) = \{\boldsymbol{A}\boldsymbol{x} \in H_1 \,|\, \boldsymbol{x} = H_0\}$
  - **Boundedness**: A is bounded when $||A|| < \infty$.
    - $||A||$ is a norm of A.

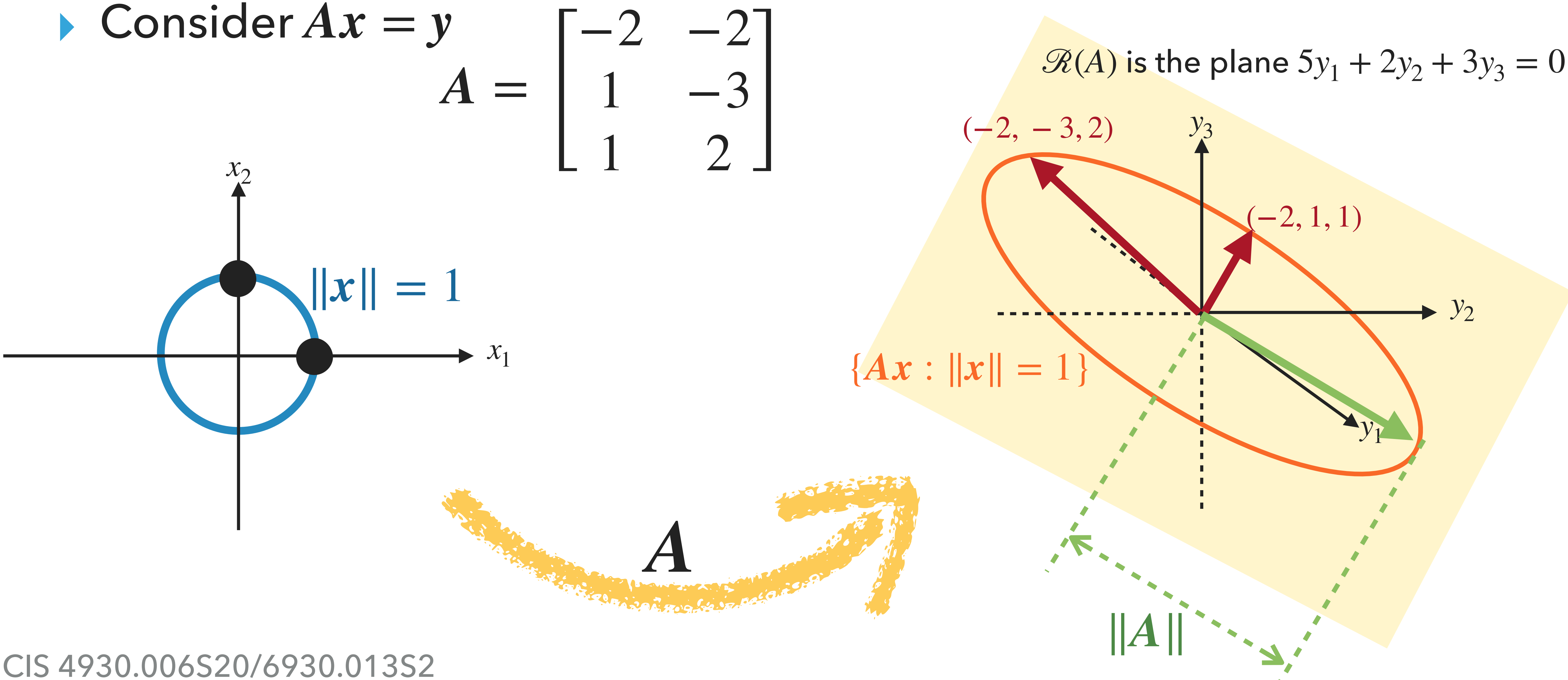
LINEAR OPERATORS
GENERALIZATION OF MATRICES

- **Operator norm:** induced by vector norms

$$\|A\| = \sup_{\|x\|=1} \|Ax\|$$

- Consider $\boldsymbol{Ax} = \boldsymbol{y}$

$$\boldsymbol{A} = \begin{bmatrix} -2 & -2 \\ 1 & -3 \\ 1 & 2 \end{bmatrix}$$

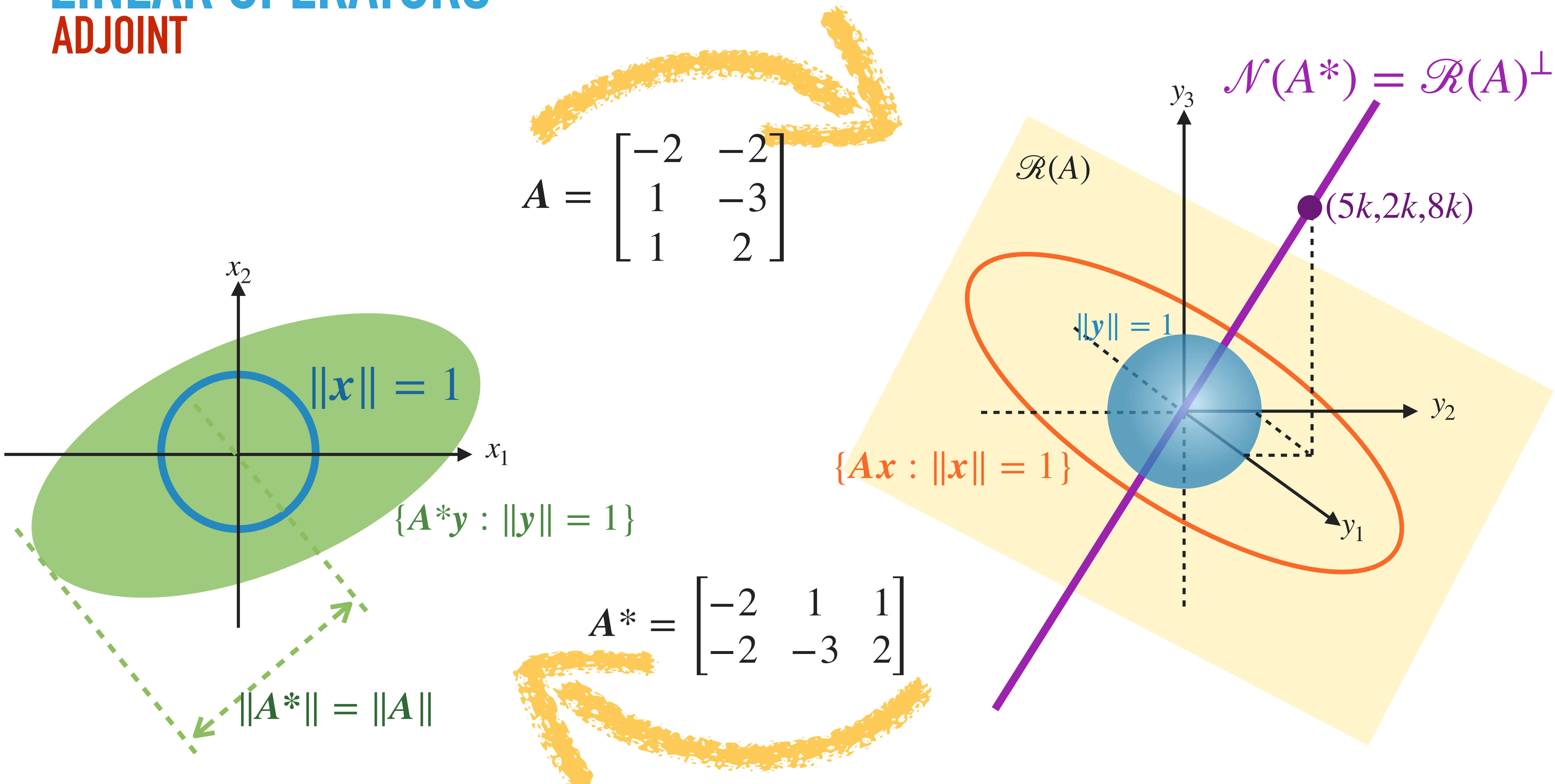


LINEAR OPERATORS
ADJOINT

- **Definition (Adjoint):** $A^* : H_1 \to H_0$ is the adjoint of linear operator $A : H_0 \to H_1$ when, $\langle \boldsymbol{Ax}, \boldsymbol{y} \rangle_{H_1} = \langle x, \boldsymbol{A}^*\boldsymbol{y} \rangle_{H_0}$ for every $\boldsymbol{x} \in H_0$ and $\boldsymbol{y} \in H_1$.
  - **Self-adjoint**: If $A = A^*$, then A is called self-adjoint (or symmetric).

- Note that: $\mathcal{N}(\boldsymbol{A}^*) = \mathcal{R}(\boldsymbol{A})^\perp$.
- The **adjoint** is an extension of the **matrix transpose** to operators.

LINEAR OPERATORS
ADJOINT



$$\boldsymbol{A} = \begin{bmatrix} -2 & -2 \\ 1 & -3 \\ 1 & 2 \end{bmatrix}$$

$$\boldsymbol{A}^* = \begin{bmatrix} -2 & 1 & 1 \\ -2 & -3 & 2 \end{bmatrix}$$

$\mathscr{N}(A^*)$ is the line $\dfrac{1}{5}y_1 = \dfrac{1}{2}y_2 = \dfrac{1}{8}y_3$

HANDLING MATRICES

MATLAB PRACTICE 3

MATRICES IN MATLAB

- Defining a matrix
- Adding/Subtracting matrices
- Multiplying matrices by
  - Scalars
  - Vectors
  - Matrices
- Rank and determinants
- Matrix Transpose

BASIS

OUTLINE

- Bases
  - Orthogonal, Orthonormal & Reisz Bases
  - Biorthogonal bases
  - Analysis and Synthesis operators

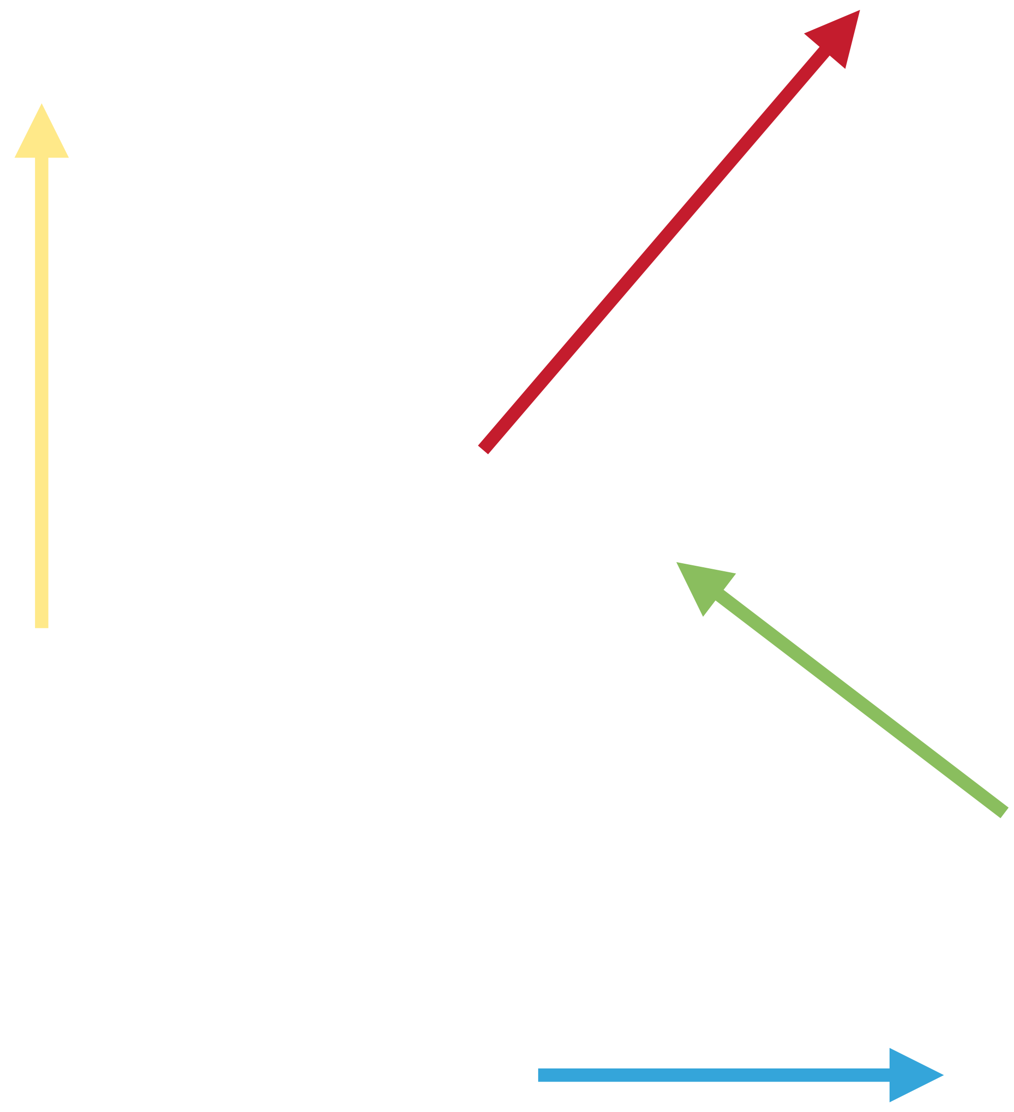
LEARNING GOALS

- Understand use and importance of bases
- Understand how to decompose vectors as sum of basis vectors
- Understand how to synthesize a vector from its coefficients

READING

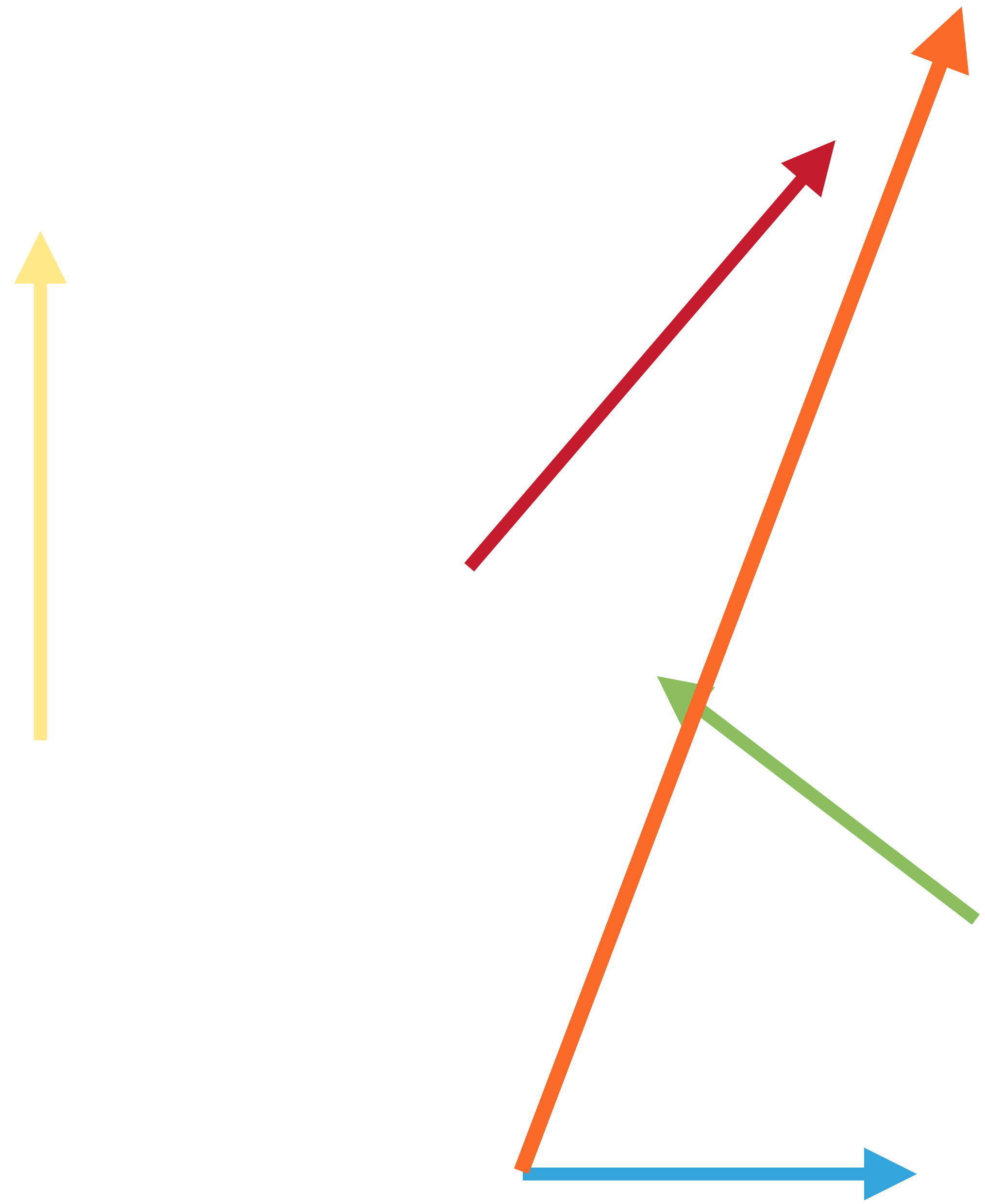
- IIP Appendix C
- FSP 2.5

VECTORS, VECTOR SPACES, SUBSPACES …



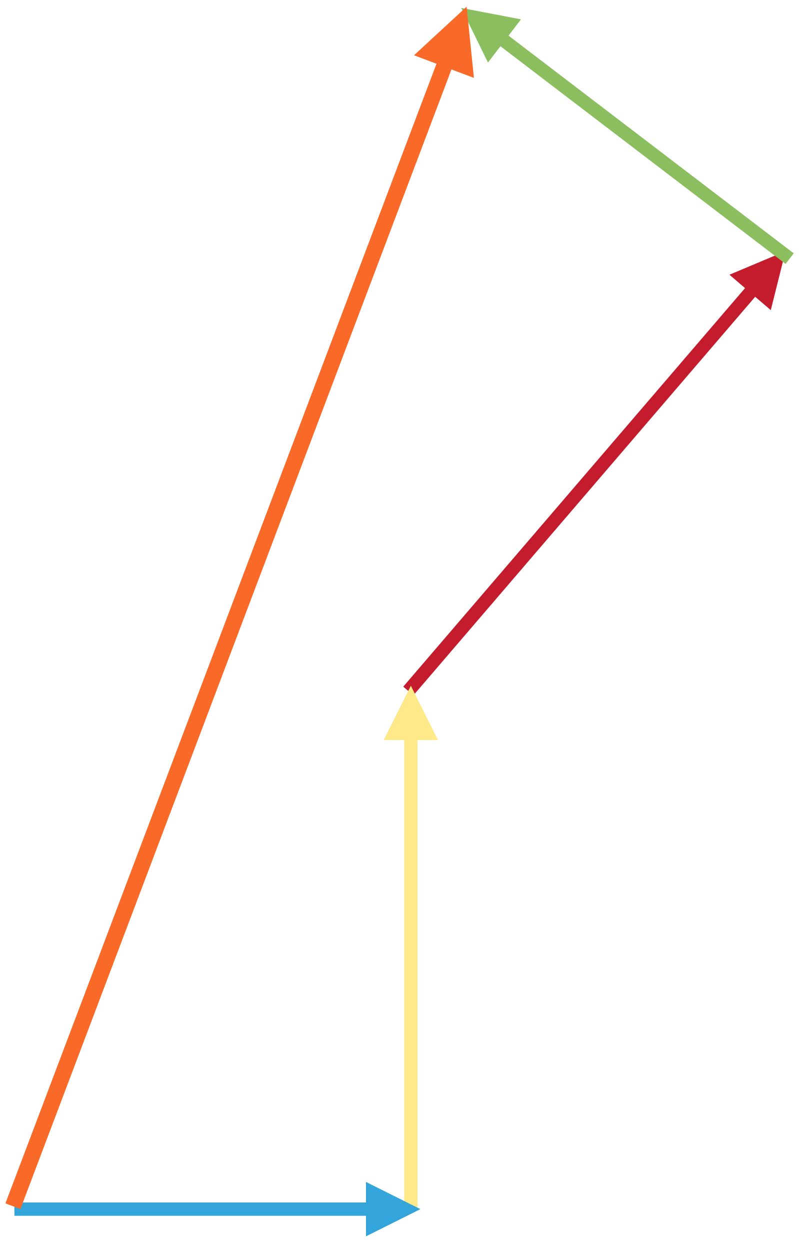
VECTORS, VECTOR SPACES, SUBSPACES …

- Vector addition



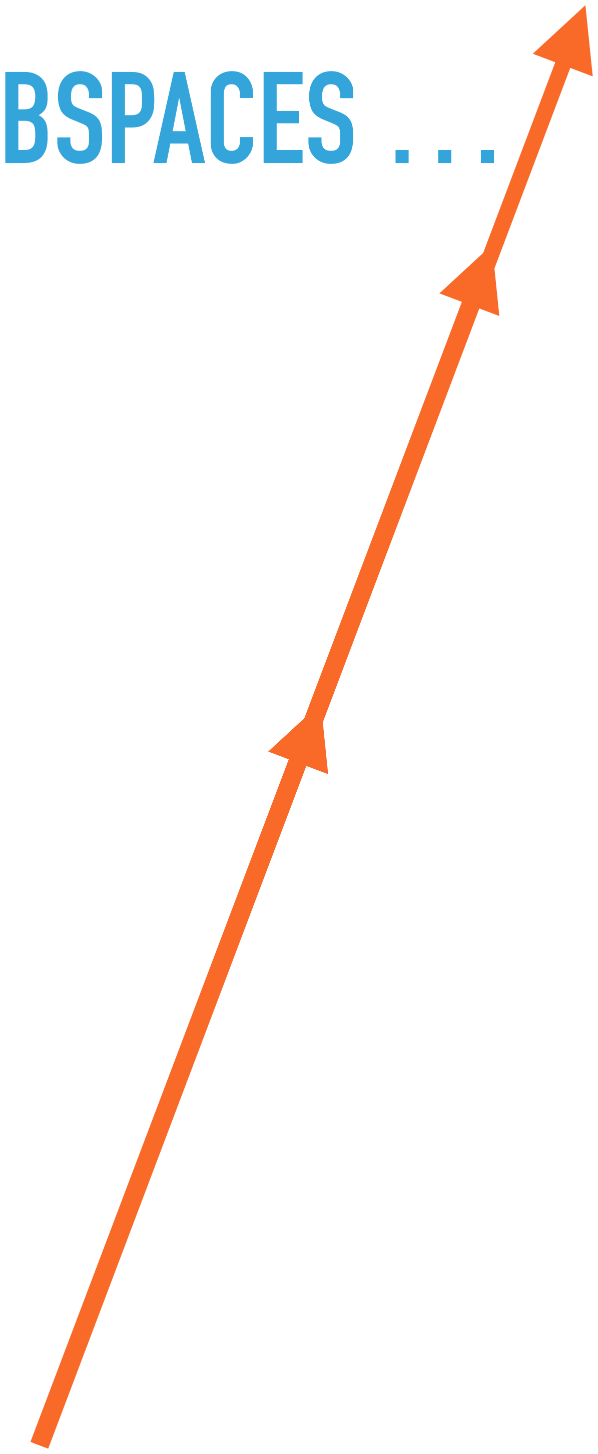
VECTORS, VECTOR SPACES, SUBSPACES …

- Vector addition

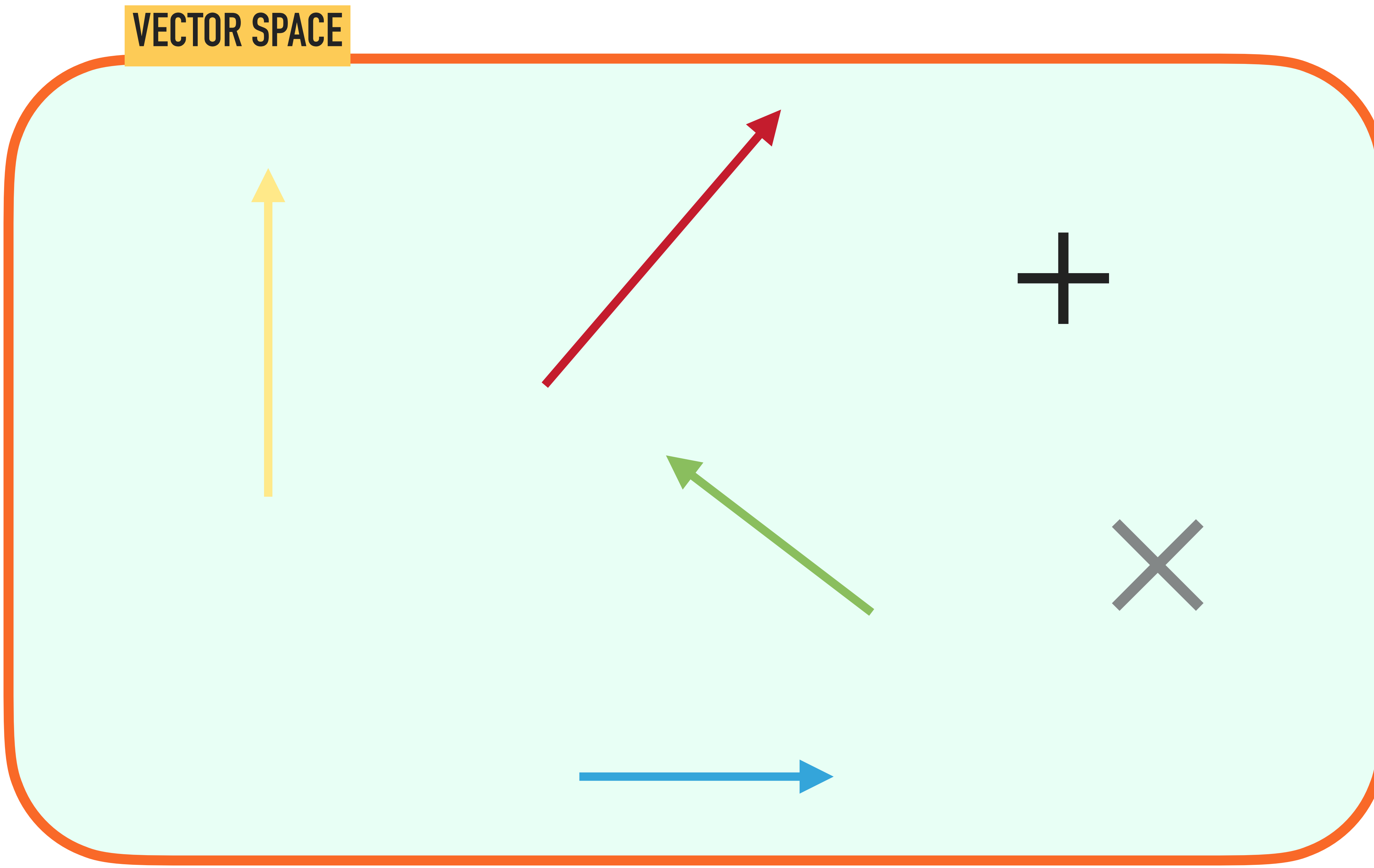


VECTORS, VECTOR SPACES, SUBSPACES …

- Vector addition
- Scalar multiplication

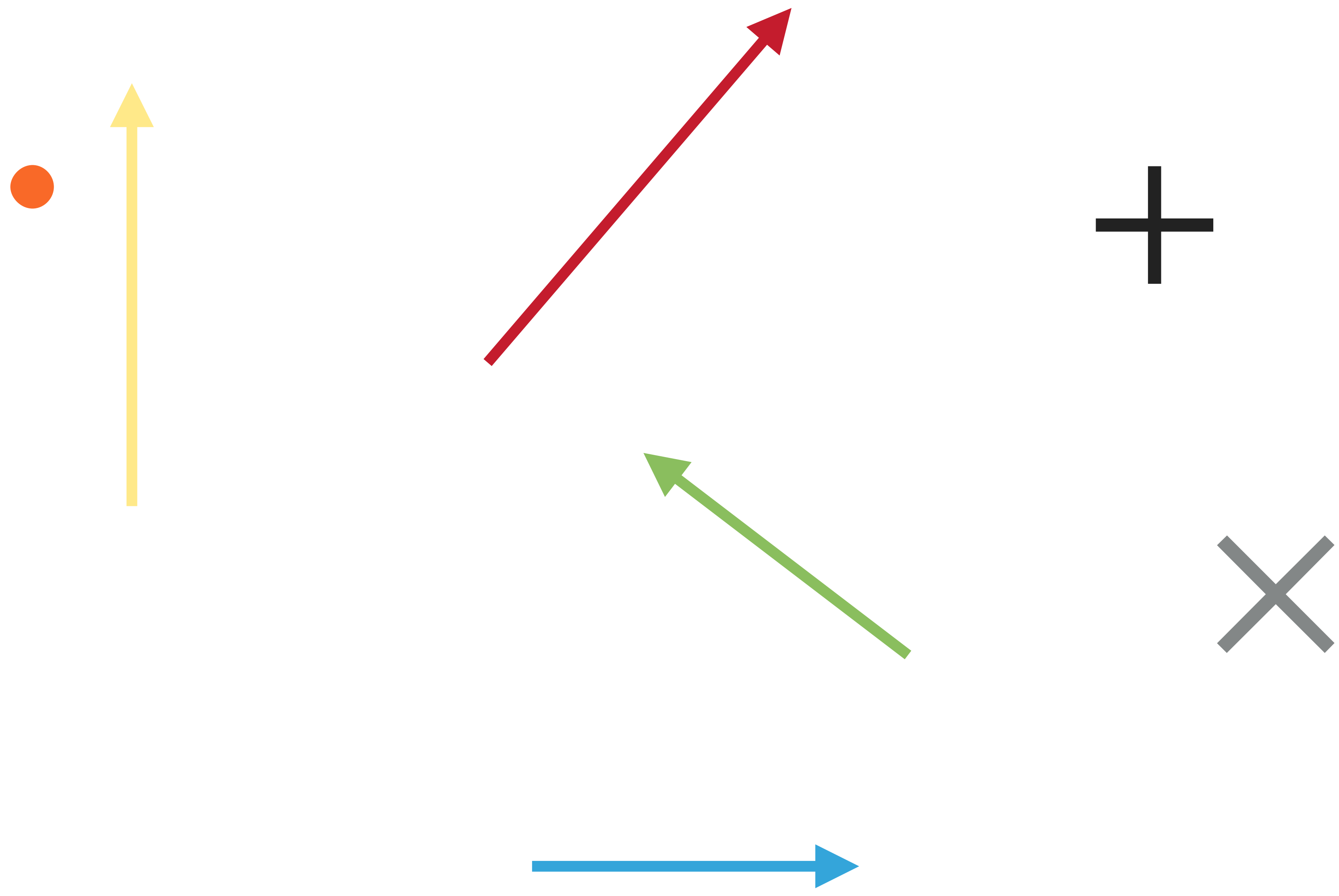


VECTORS, VECTOR SPACES, SUBSPACES …

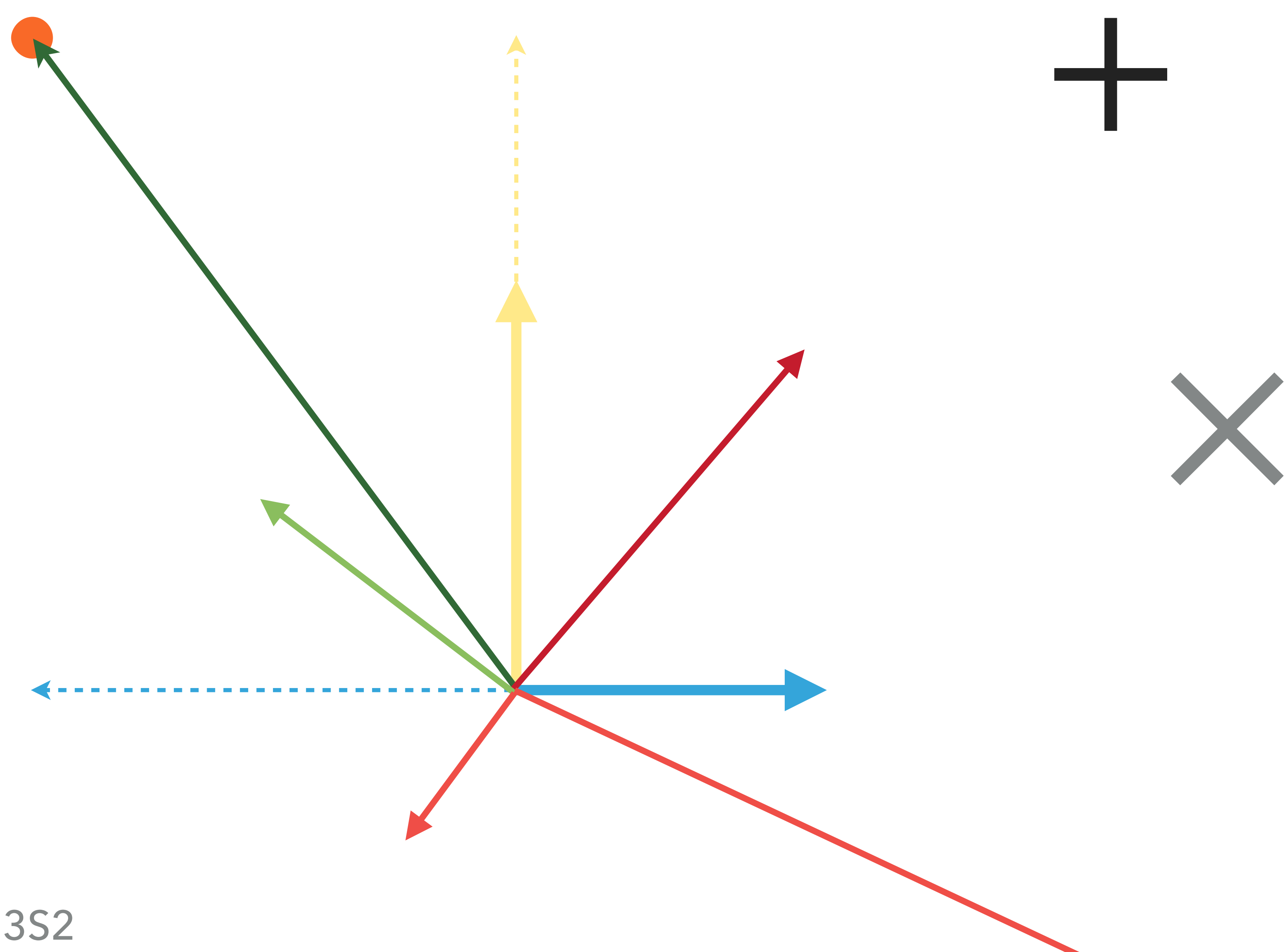


× means scalar multiplication

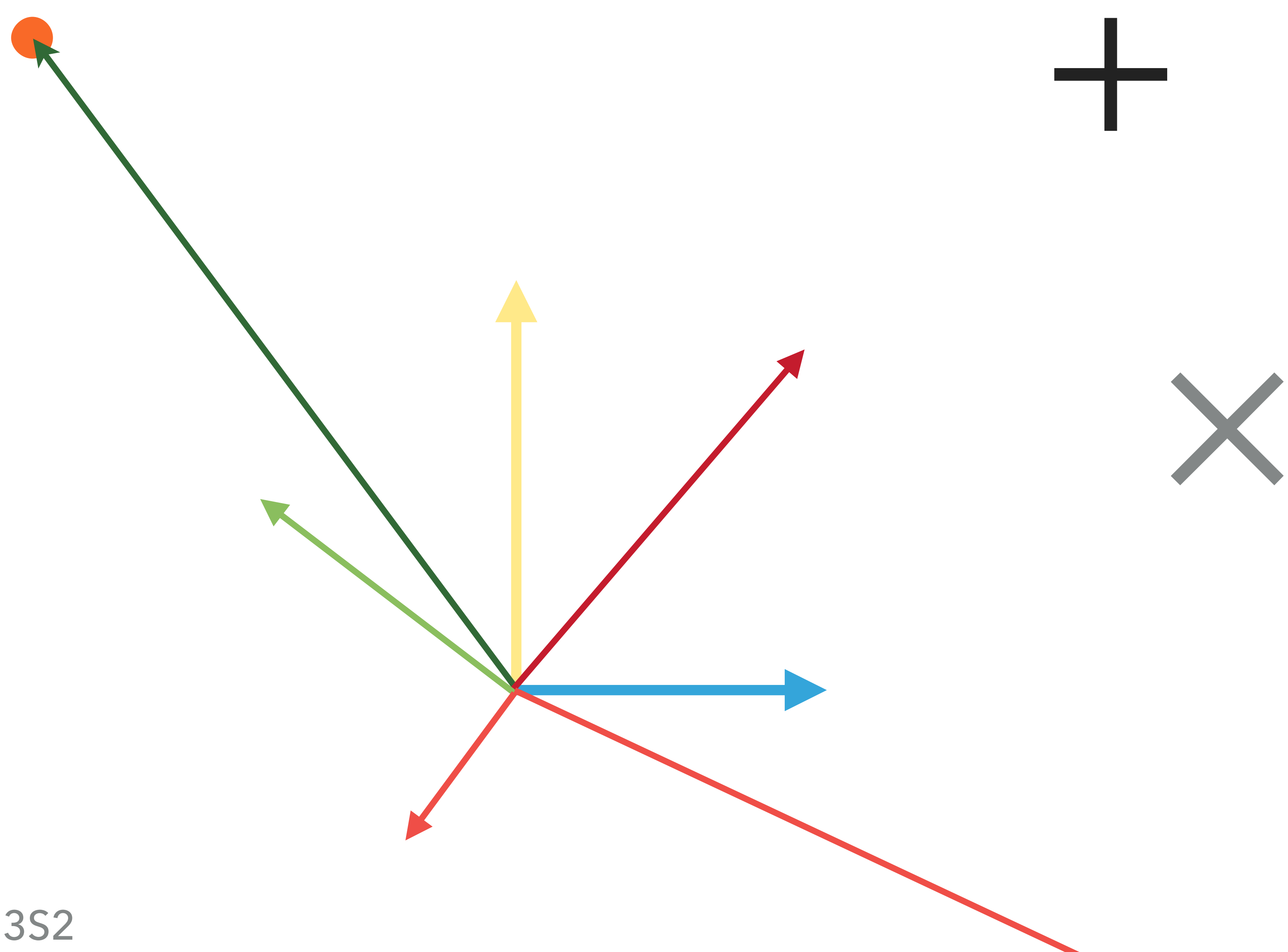
VECTORS, VECTOR SPACES, SUBSPACES …



VECTORS, VECTOR SPACES, SUBSPACES …

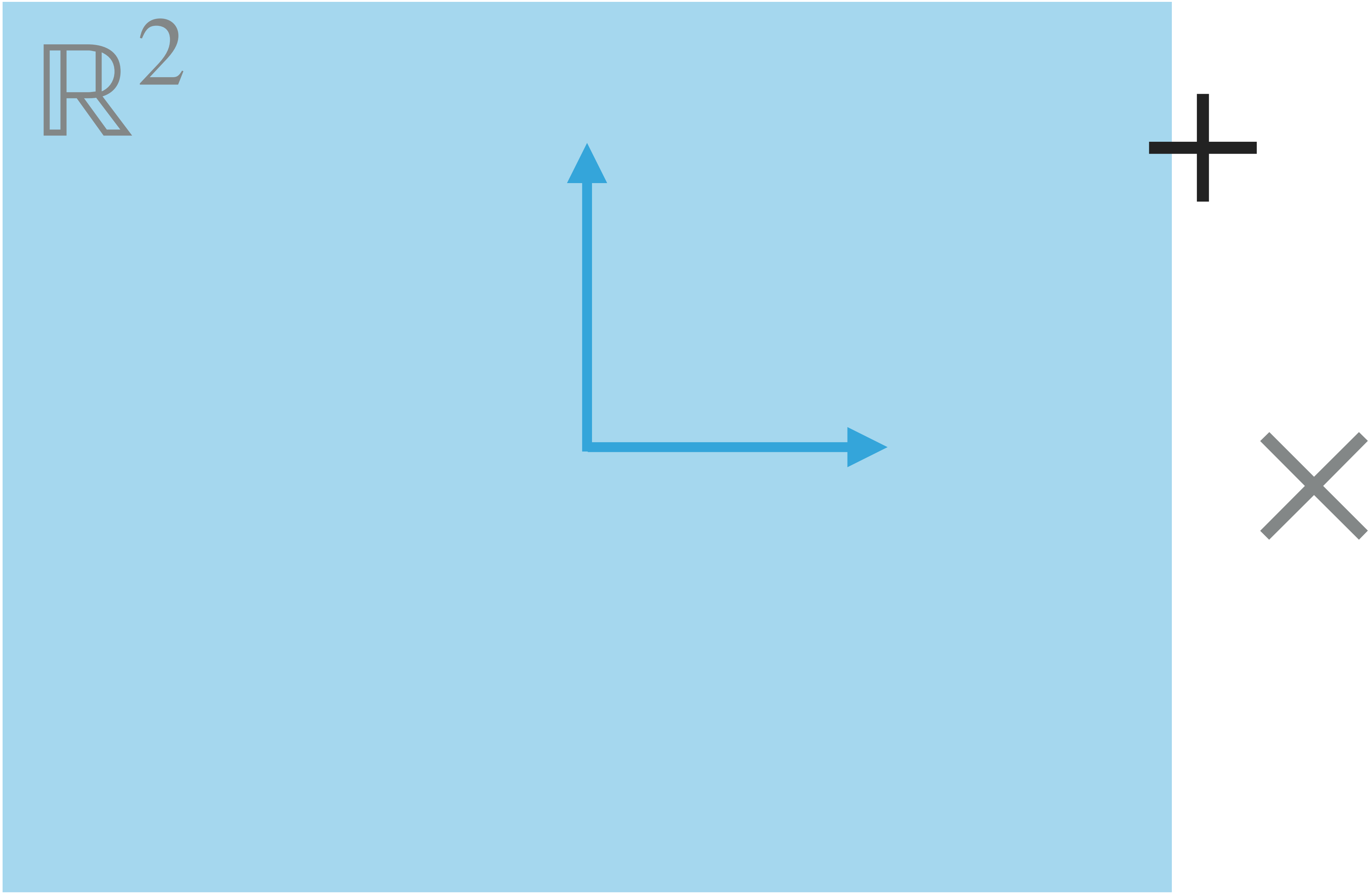


VECTORS, VECTOR SPACES, SUBSPACES …

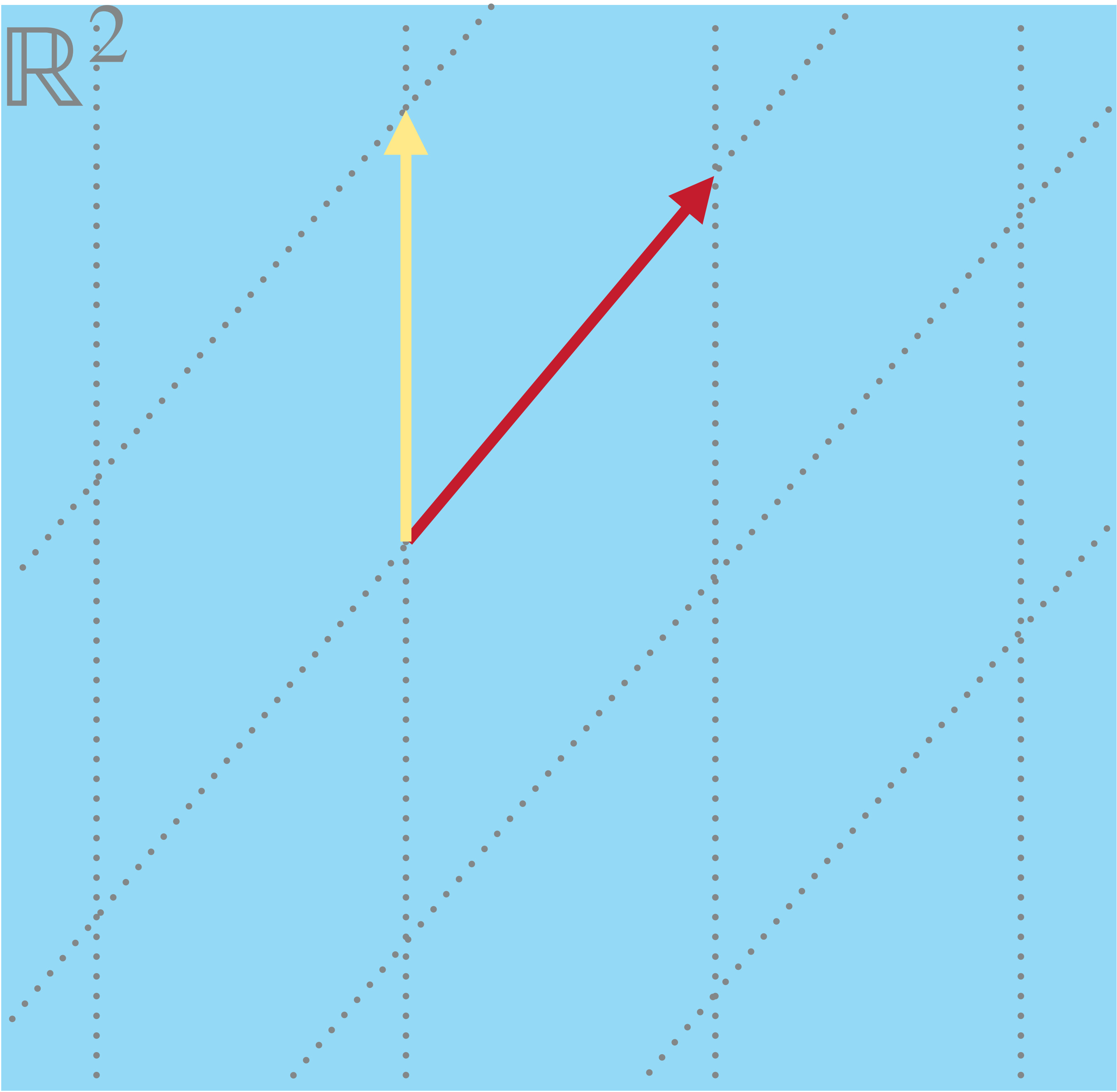


VECTORS, VECTOR SPACES, SUBSPACES …

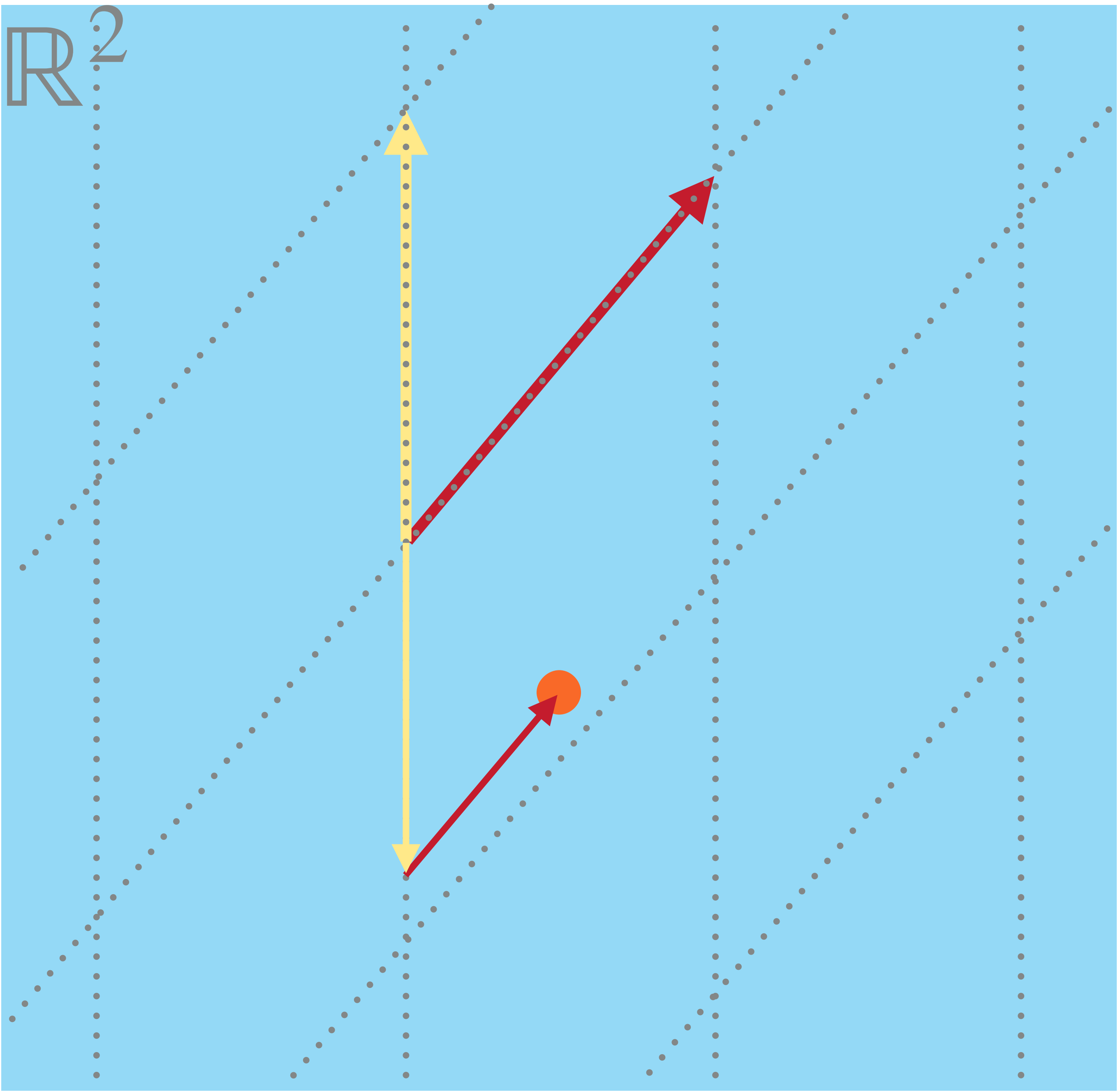
THE TWO ARROWS ARE ORTHOGONAL



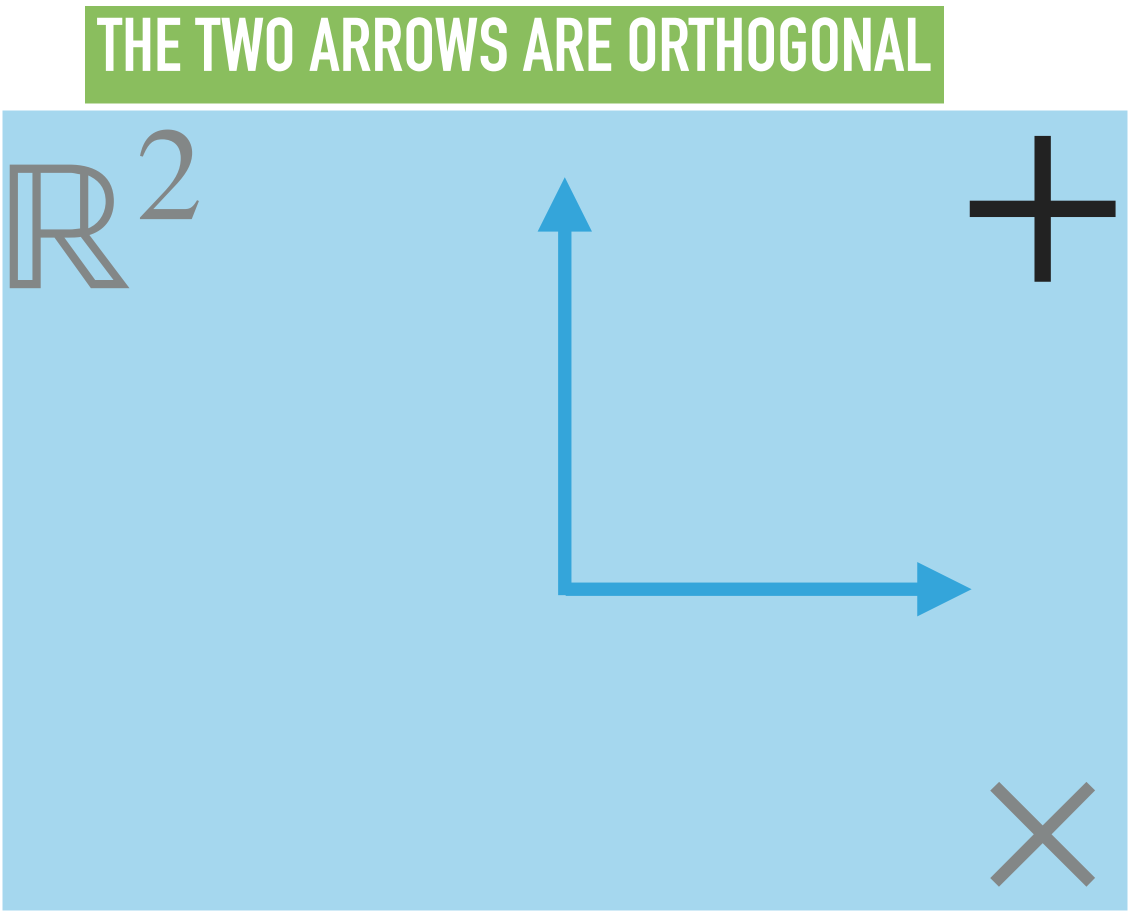
VECTORS, VECTOR SPACES, SUBSPACES …

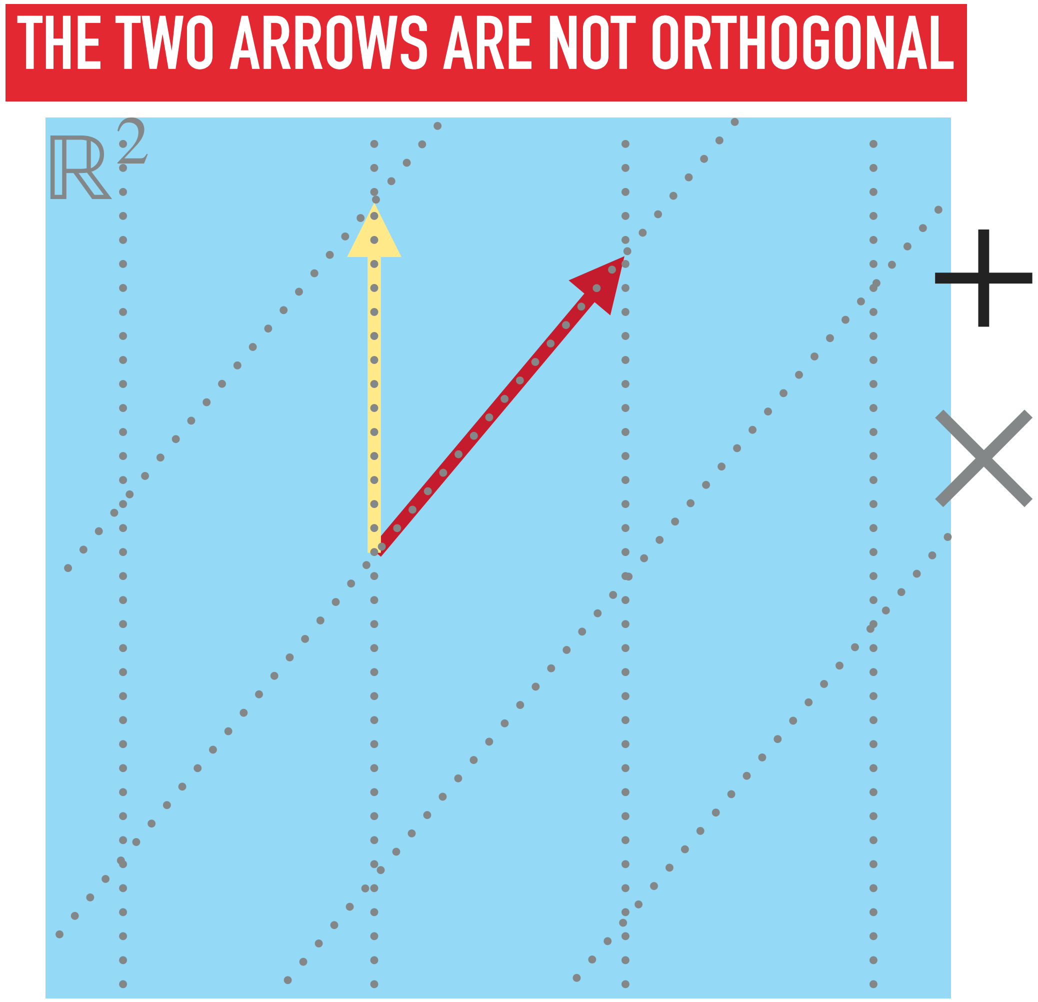


VECTORS, VECTOR SPACES, SUBSPACES …

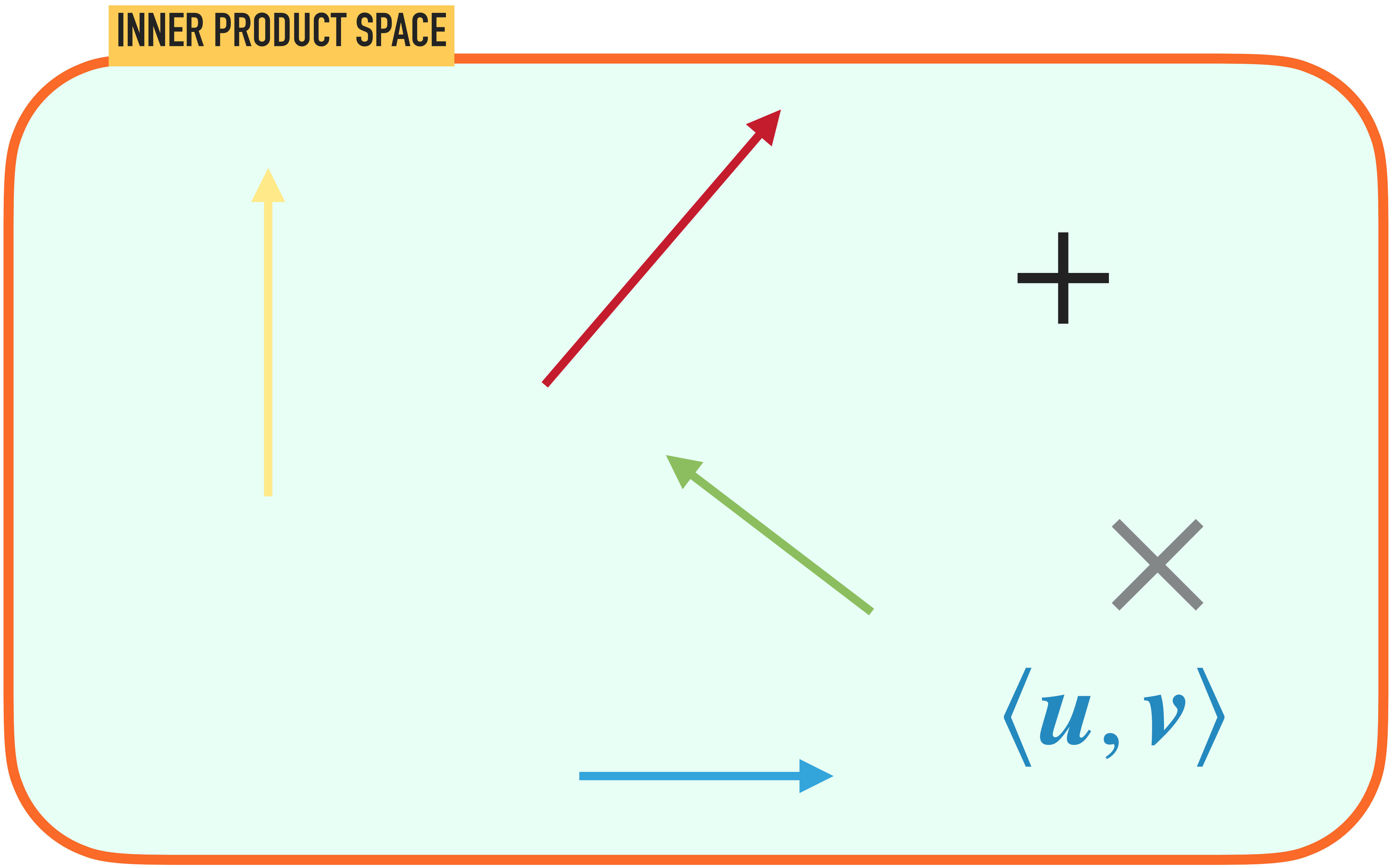


VECTORS, VECTOR SPACES, SUBSPACES, …





VECTORS, VECTOR SPACES, SUBSPACES …



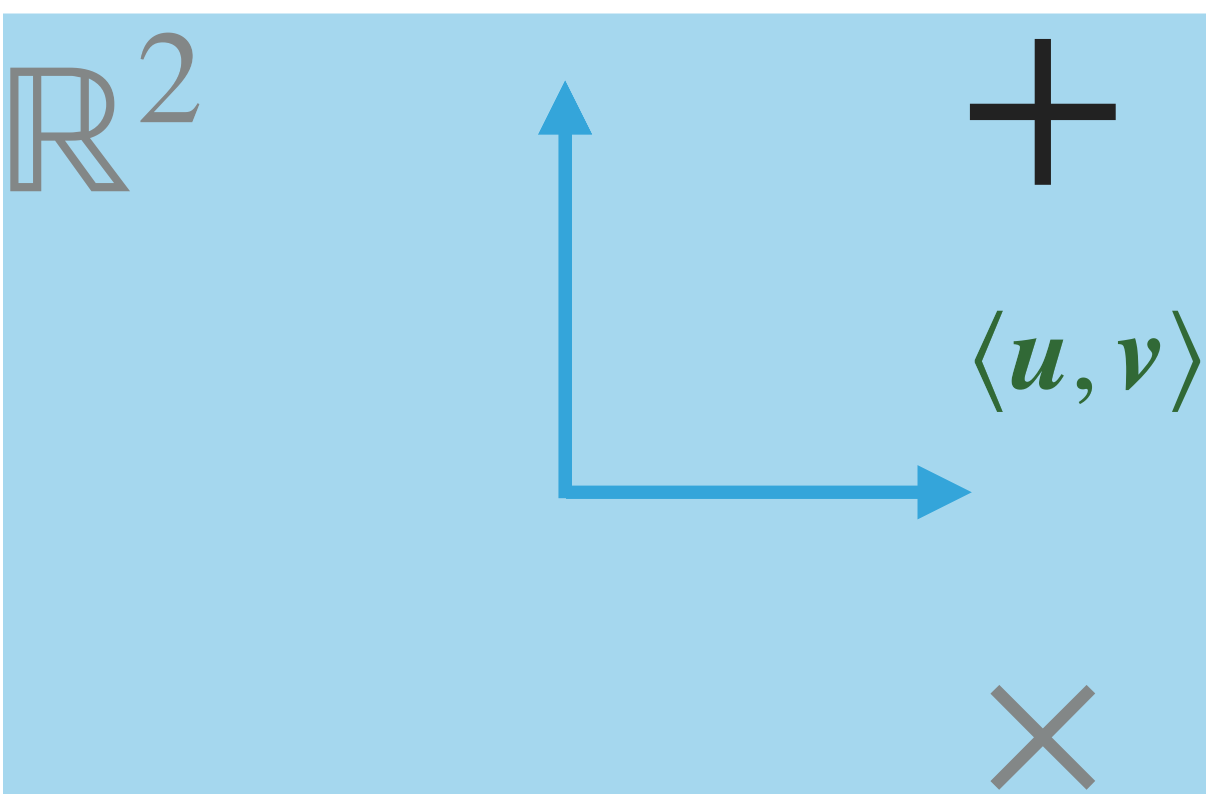
● means scalar multiplication

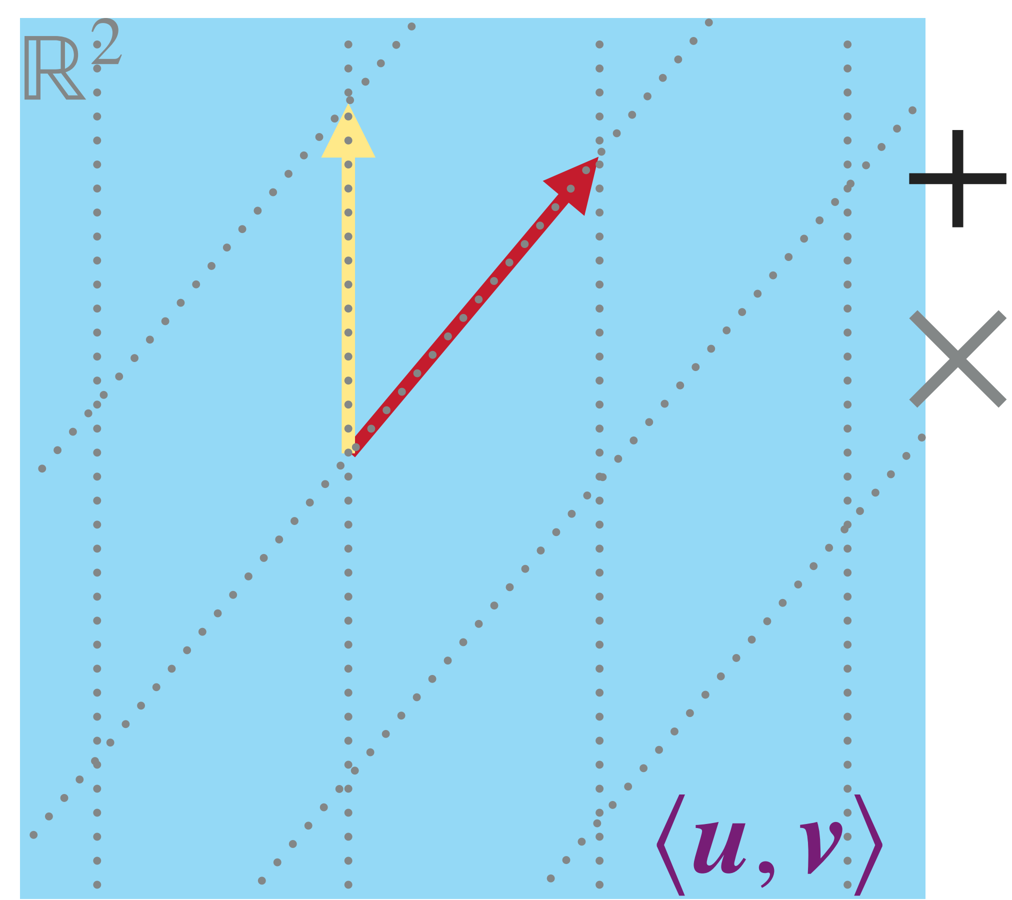
TO INFINITY

- We can do this for higher dimensions
  - Say $\mathbb{R}^3, \mathbb{R}^4, \mathbb{R}^{20}, \ldots$
- **Infinite dimensions**
  - An additional requirement is **completeness**
  - **Hilbert space:** an inner product space that is complete.

THOSE PESKY ARROWS (BUILDING BLOCKS)

- The arrows are actually special – they form a **basis** for $\mathbb{R}^2$
  - There are called **Basis vectors**
- More generally they form a **basis** for a **Hilbert space.**





SEE YOU NEXT TIME!

BASES (CONTINUED)

A REMINDER MATRIX-VECTOR MULTIPLICATION

$$\begin{bmatrix} -2 & -2 \\ 1 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = ?$$

$$\begin{bmatrix} -2 & -2 \\ 1 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = ?$$