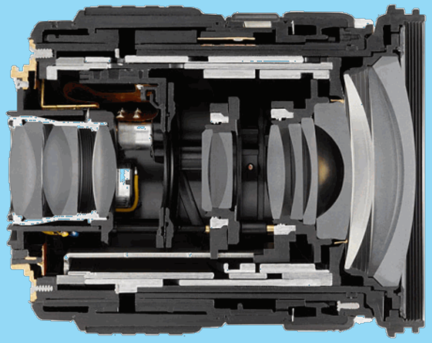
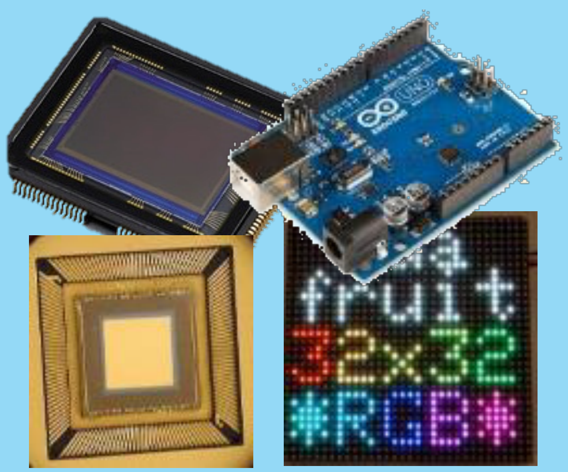


Optics

+



Sensors
&
devices

+

Signal
processing
&
algorithms

COMPUTATIONAL METHODS FOR IMAGING (AND VISION)

LECTURE 3: VECTOR SPACES (HILBERT SPACE)

PROF. JOHN MURRAY-BRUCE

(SUPER FAST) REVIEW

VECTORS

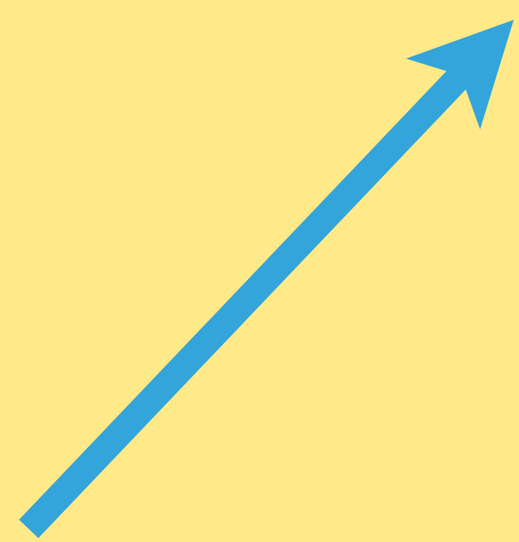
VECTORS AND VECTOR SPACES

- **Vector space** – is a set of objects (vectors, together with addition and scalar multiplication) that satisfy the following properties:
  - For all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and scalars a, b
    - $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
    - $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
    - There exists a *zero vector* $\mathbf{0}$, such that $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
    - For every $\mathbf{u}$ there exits a vector $-\mathbf{u}$, such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
    - $1\mathbf{v} = \mathbf{v}$
    - $a(b\mathbf{u}) = (ab)\mathbf{u}$
    - $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
    - $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

VECTORS AND VECTOR SPACES

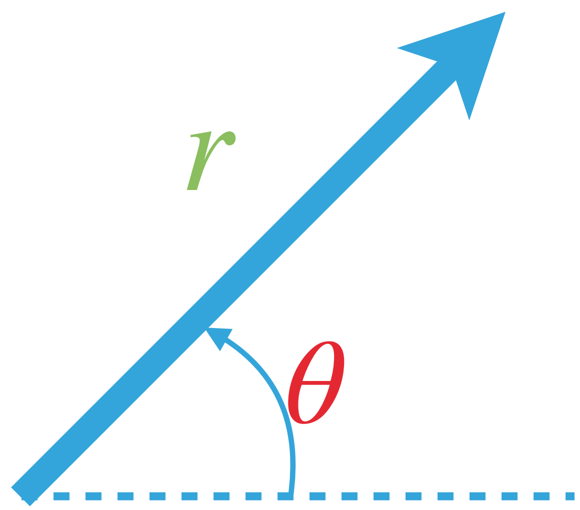
- **Vector space** – is a set of objects (vectors, together with addition and scalar multiplication) that satisfy the following properties:
  - For all vectors
    - $\mathbf{u} + \mathbf{v} = \mathbf{v} +$
    - $\mathbf{u} + (\mathbf{v} + \mathbf{w})$
    - There exists … $\mathbf{u} = \mathbf{u}$
    - For every $\mathbf{u}$ … $(-\mathbf{u}) = \mathbf{0}$
    - $1\mathbf{v} = \mathbf{v}$
    - $a(b\mathbf{u}) = (ab$
    - $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
    - $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

Intuition: think of a vector as this little arrow.



VECTORS
WHAT CAN WE MEASURE/ENCODE

- Fundamentally, vectors encode **magnitude** and **direction**



- Example: a 2D vector can be encoded as length, and an angle relative to some fixed direction.

- **Norm:** measures the “length” of a vector.

- **Inner product:** measures norm along with relative orientation. The “length” of one vector along the direction of another.

NORM
FORMAL DEFINITION

- A norm is any ***function*** that maps each vector (of a vector space) to a scalar value, and satisfies the properties:
  - $\|\mathbf{u}\| \geq 0$
  - $\|\mathbf{u}\| = 0 \iff \mathbf{u} = \mathbf{0}$
  - $\|a\mathbf{u}\| = |a|\,\|\mathbf{u}\|$
  - $\|\mathbf{u}\| + \|\mathbf{v}\| \geq \|\mathbf{u} + \mathbf{v}\|$ **(triangle inequality!)**

For a vector $\mathbf{u} \in \mathbb{R}^n$,

$$\|\mathbf{u}\| = \sqrt{\sum_{i=1}^{n} u_i^2}$$

Matlab practice: Check that each of these identities are true for some choice of vectors and scalars

INNER PRODUCT
FORMAL DEFINITION

- **Inner product:** is any function that maps any two vectors to a scalar number $\langle \mathbf{u}, \mathbf{v} \rangle$, such that the following properties hold:
  - $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
  - $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$
  - $\langle \mathbf{u}, \mathbf{u} \rangle = 0 \iff \mathbf{u} = 0$
  - $\langle a\mathbf{u}, \mathbf{v} \rangle = a\langle \mathbf{u}, \mathbf{v} \rangle$
  - $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

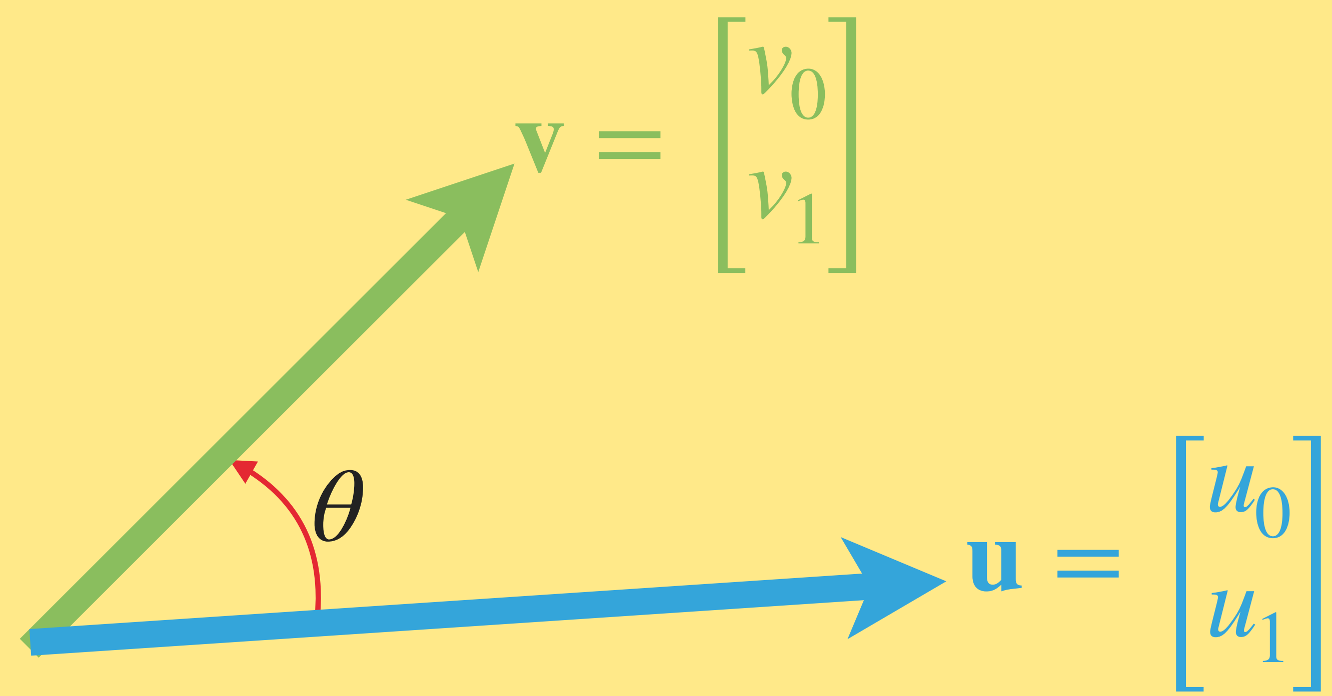
For two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, the standard inner product is

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^{n} u_i v_i$$

Matlab practice: Verify each of these identities by using Matlab for some choice of vectors and scalars.

INNER PRODUCT
EXAMPLE: INNER PRODUCT IN $\mathbb{R}^2$

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_0 v_0 + u_1 v_1$$



$$\langle \mathbf{u}, \mathbf{v} \rangle = \sqrt{(u_0^2 + u_1^2)(v_0^2 + v_1^2)} \cos(\theta)$$

Matlab practice: Verify the given equation, i.e. that: $\langle \mathbf{u}, \mathbf{v} \rangle = \sqrt{(u_0^2 + u_1^2)(v_0^2 + v_1^2)} \cos(\theta)$, for a few 2D vectors

OUTLINE

- **Vector spaces**
  - Norm, Inner product
- **Hilbert spaces**

LEARNING GOALS

- Understand broader meaning of vectors
- Understand basic Hilbert spaces terminology
- Understand basic manipulation of vectors using Matlab

READING

- IIP Appendix A
- FSP 2.1 - 2.4

IMPORTANT

CAUCHY-SCHWARZ INEQUALITY

CAUCHY-SCHWARZ INEQUALITY

MATLAB

- For any vectors $\mathbf{u}$ and $\mathbf{v}$, belonging to some vector space then:

$$\langle \mathbf{u}, \mathbf{v} \rangle \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$$

- **Proof (intuition):**

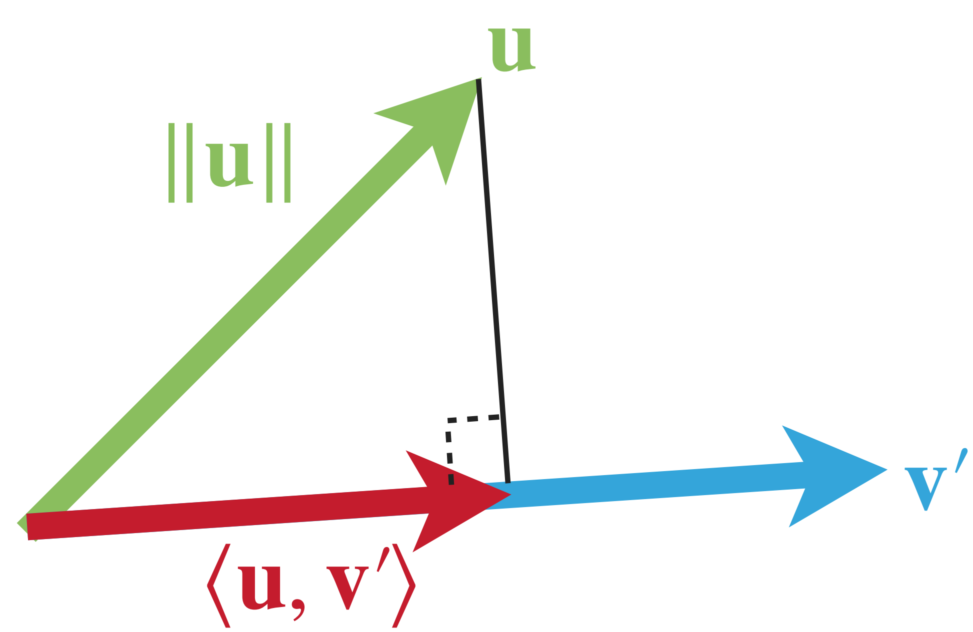
1. Let $\mathbf{v}'$ be a unit vector, i.e. $\mathbf{v}' = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$, and consider $\langle \mathbf{u}, \mathbf{v}' \rangle$.

2. The inner product $\langle \mathbf{u}, \mathbf{v}' \rangle$ is the "length" of $\mathbf{u}$ along $\mathbf{v}'$.

3. Geometrical reasoning (see picture), implies that the length of $\mathbf{u}$ along $\mathbf{v}'$ must be smaller than or equal to $\|\mathbf{u}\|$ (i.e. the length of $\|\mathbf{u}\|$).

4. Thus $\langle \mathbf{u}, \mathbf{v}' \rangle \leq \|\mathbf{u}\|$.

5. Complete the proof, by putting $\mathbf{v}' = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ into $\langle \mathbf{u}, \mathbf{v}' \rangle \leq \|\mathbf{u}\|$ and rearranging.



SUBSPACE

SPAN

LINEAR INDEPENDENCE

ORTHOGONALITY

USEFUL DEFINITIONS

SUBSPACE

SPAN

LINEAR INDEPENDENCE

ORTHOGONALITY

USEFUL DEFINITIONS

USEFUL DEFINITIONS

- **Definition (Subspace):** A nonempty subset $\mathscr{S}$ of a vector space $\mathscr{V}$ is a subspace when it is **closed** under the operations of **vector addition** and **scalar multiplication**.

Example: Let us consider a 3D vector space: $\mathscr{V} = \mathbb{R}^3$.

$\mathscr{S}$ is a subspace of $\mathscr{V}$.

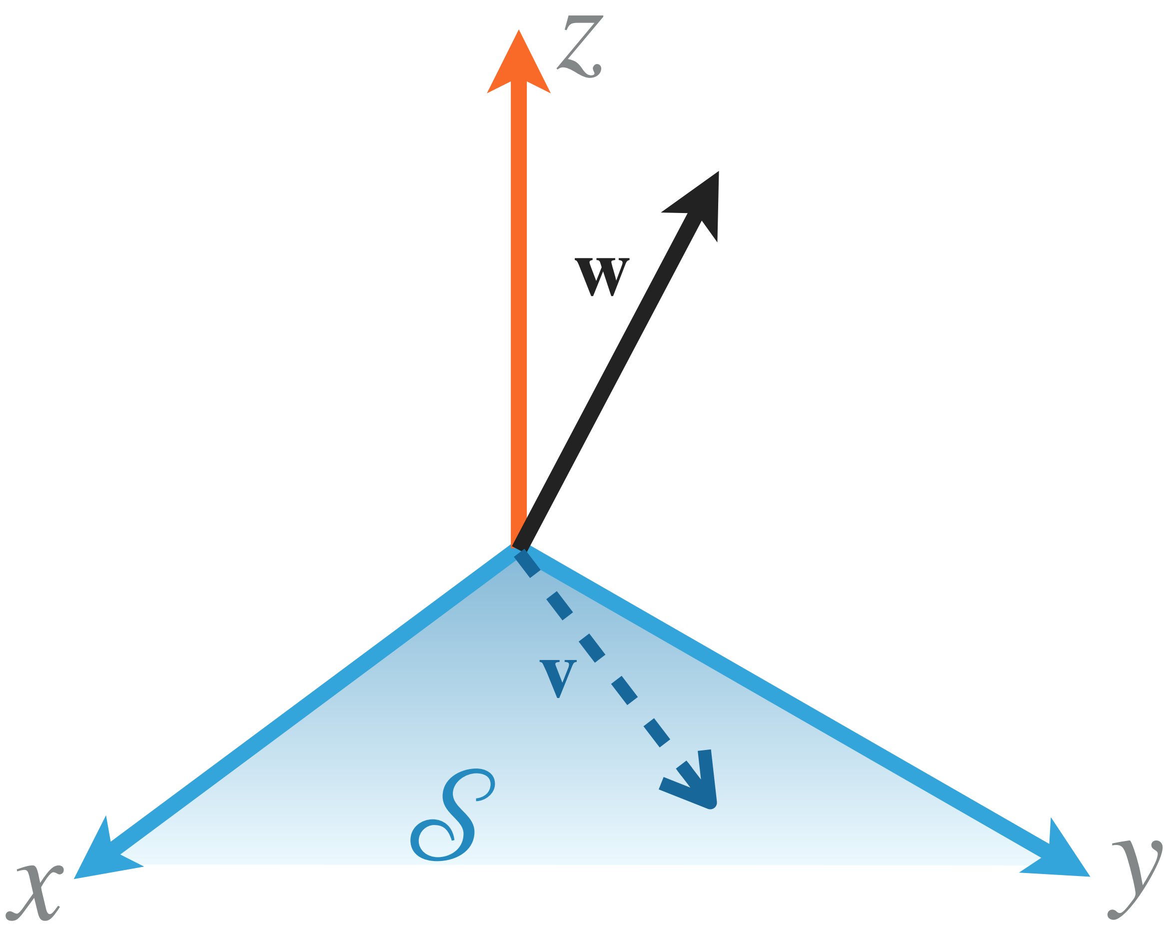
The vector $\mathbf{w} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \in \mathbb{R}^3$

The vector $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ is lives in $\mathscr{S}$

In fact, any **vector whose last entry is zero** is in $\mathscr{S}$.

Closed? - wait, but, what is that? :-/

- if we **add** any two such **vectors** that belong in $\mathscr{S}$ the result is a vector that is also in $\mathscr{S}$
- If we multiply such a vector by a scalar (say 0.5 or 1000), the result is a vector that also lives in $\mathscr{S}$



QUESTION: CAN YOU GIVE OTHER EXAMPLES OF VECTORS IN $\mathscr{S}$?

SUBSPACE

SPAN

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ORTHOGONALITY

USEFUL DEFINITIONS

USEFUL DEFINITIONS

- **Definition (Span):** The span of a set of vectors $\mathcal{S}$ is the set of all **finite linear combinations of vectors** in $\mathcal{S}$:

$$\operatorname{span}(\mathcal{S}) = \left\{ \sum_{i=1}^{n} \alpha_i \mathbf{u}_i : \alpha_i \in \mathbb{R}, \mathbf{u}_i \in \mathcal{S}, \text{ and } n \in \mathbb{N} \right\}$$

  - The $\operatorname{span}(\mathcal{S})$ is **always a subspace** (by definition).
  - Different sets can have the same span.

EXAMPLE: COMPUTING SPANS

- Write down the span of the following sets of vectors

1. $\mathcal{S}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

   ✓ **Answer:** $\text{span}(\mathcal{S}_1) = \left\{ \alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} : \alpha_1, \alpha_2 \in \mathbb{R} \right\}$

2. $\mathcal{S}_2 = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix} \right\}$

   ✓ **Answer:** $\text{span}(\mathcal{S}_2) = \left\{ \alpha_1 \begin{bmatrix} x \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ y \end{bmatrix} : \alpha_1, \alpha_2 \in \mathbb{R} \right\}$

   ✓ In fact the span of $\mathcal{S}_2$ is the same as the span of $\mathcal{S}_1$

3. $\mathcal{S}_3 = \left\{ \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} \right\}$ and additional comments?

4. $\mathcal{S}_4 = \left\{ \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right\}$

SUBSPACE

SPAN

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USEFUL DEFINITIONS

USEFUL DEFINITIONS

- **Definition (Linear independence):** A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is **linearly independent** when $\sum_{i=1}^{n} \alpha_i \mathbf{u}_i = \mathbf{0}$ if and only if $\alpha_i = 0$ for all i.
  - Otherwise the set is **linearly dependent**.

Question: Is the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ linearly independent?

- **Infinite sets** (infinite collection of vectors):
  - an **infinite set** is linearly independent, if every finite subset is linearly independent.

EXAMPLES

MATLAB

- Classify each of these sets as linearly independent or linearly dependent

A. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} \right\}$

SUBSPACE

SPAN

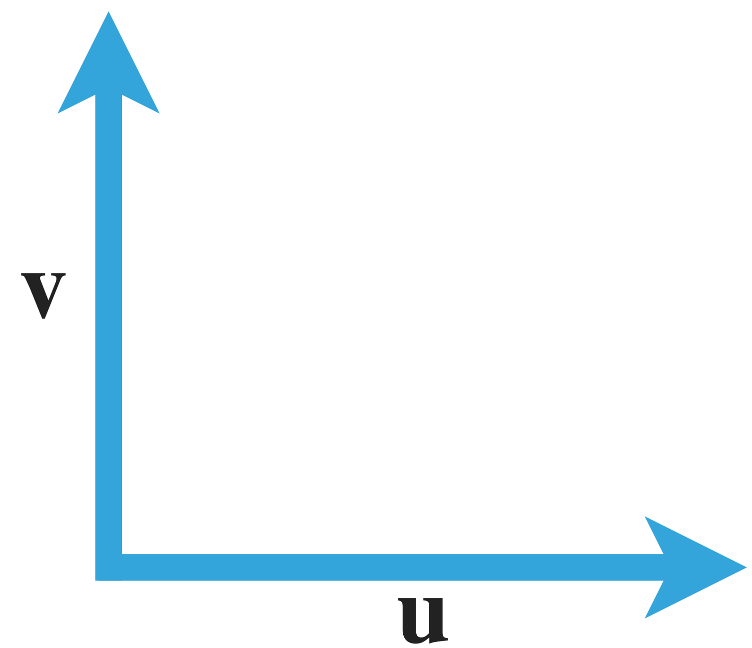
LINEAR INDEPENDENCE

ORTHOGONALITY

USEFUL DEFINITIONS

USEFUL DEFINITIONS
ORTHOGONALITY

- Vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** when $\langle \mathbf{u}, \mathbf{v} \rangle = 0$
  - Again, think of the little arrows



  - We will write this as $\mathbf{u} \perp \mathbf{v}$
- **Orthogonality** can be extended to subspaces (collection of vectors) as well

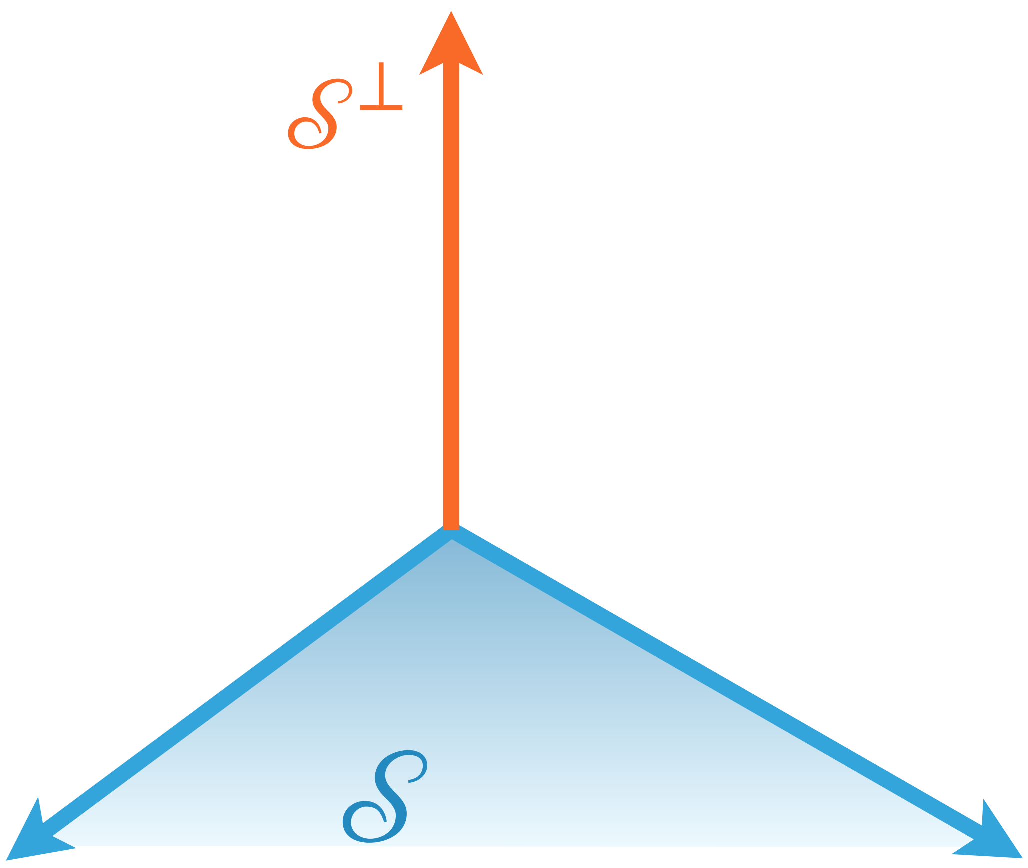
USEFUL DEFINITIONS
ORTHOGONALITY

- The **set of vectors** $\mathscr{S}$ are orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ for every $\mathbf{u}, \mathbf{v} \in \mathscr{S}$.
- **Orthonormal set** $\mathscr{S}$: is orthogonal, and $\langle \mathbf{u}, \mathbf{u} \rangle = 1$ for every $\mathbf{u} \in \mathscr{S}$.
- A <u>vector</u> $\mathbf{u}$ is orthogonal to the <u>set of vectors</u> $\mathscr{S}$, if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ for every $\mathbf{v} \in \mathscr{S}$. Written as $\mathbf{u} \perp \mathscr{S}$.
- Two sets of vectors $\mathscr{S}_0$ and $\mathscr{S}_1$ are orthogonal when every vector in $\mathscr{S}_0$ is orthogonal to every vector in $\mathscr{S}_1$.

USEFUL DEFINITIONS

ORTHOGONALITY

- $\mathscr{S}^{\perp}$ is the **orthogonal complement** of a subspace $\mathscr{S}$, of a vector space $\mathscr{V}$, i.e. the set $\mathscr{S}^{\perp} = \left\{ \mathbf{u} \in \mathscr{V} : \mathbf{u} \perp \mathscr{S} \right\}$.



USEFUL DEFINITIONS
ORTHOGONALITY

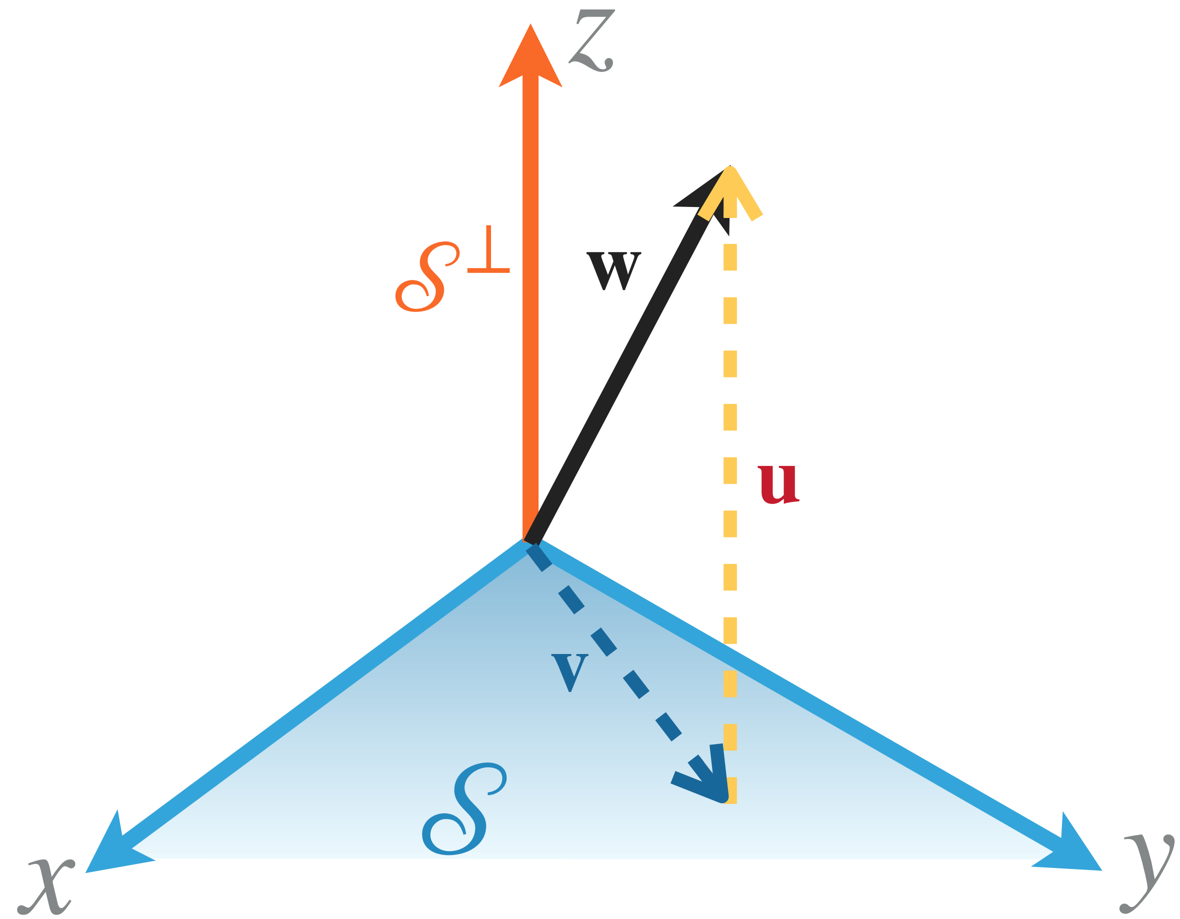
- $\mathscr{S}^\perp$ is the **orthogonal complement** of a subspace $\mathscr{S}$, of a vector space $\mathscr{V}$, i.e. the set $\mathscr{S}^\perp = \left\{\mathbf{u} \in \mathscr{V} : \mathbf{u} \perp \mathscr{S}\right\}$.

Example: Let us consider a 3D vector space: $\mathscr{V} = \mathbb{R}^3$.

And the vector $\mathbf{w} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \in \mathbb{R}^3$

The vector $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ is lives in $\mathscr{S}$,

while vector $\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}$ lives in $\mathscr{S}^\perp$, i.e. orthogonal complement of $\mathscr{S}$.



RECAP

Subspace: is a subset of a vector space that is closed under addition and scalar multiplication

Span: is the set of all finite linear combinations of vectors of a vector space.

Linear independence: if any vector in the set can be written as a linear combination of the remaining vectors the set is said to be linearly dependent.

Orthogonal: two vectors are orthogonal if their inner product is zero. That is,

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0$$

VECTORS

MATLAB PRACTICE 1

MATLAB CHECKLIST

- **Creating vectors**
- **Manipulating vectors**
  - Scalar multiplication
  - Vector addition
  - Indexing
  - Inner products and Norms
- **Checking linear independence**
  - Check **rank** or **determinant** of matrix representation of the set of vectors
  - Can also look at **reduced row echelon form** of matrix representation of the vectors

WHAT WE COVERED TODAY

- Vectors and Vector spaces
  - Norms
  - Inner products
- Hilbert space
- Matlab
  - Manipulating vectors

SEE YOU NEXT TIME!