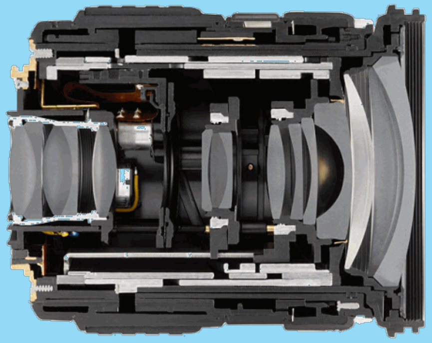
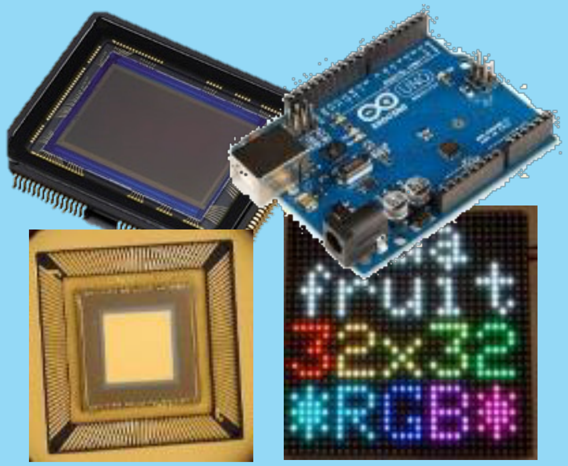


Optics

+



Sensors
&
devices

+

Signal
processing
&
algorithms

COMPUTATIONAL METHODS FOR IMAGING (AND VISION)

LECTURE 19: INVERSE PROBLEMS (REGULARIZATION)

PROF. JOHN MURRAY-BRUCE

CONTINUOUS INVERSE PROBLEM

ILL-POSED VS WELL-POSED PROBLEMS

(Definition: Well-posedness) Consider the operator $A : \mathscr{X} \longrightarrow \mathscr{Y}$. The inverse problem of solving

$$g = A(f)$$

to obtain f given measurements g is **well-posed** in the Hadamard sense, if:

1. **Existence:** a solution exists for any g
2. **Uniqueness:** a unique solution exists for any g
3. **Continuity:** f depends continuously on the measurements g

The problem is **ill-posed** if any of these above conditions are violated

DECONVOLUTION – A HOPELESS CASE?

SOLVING AN INVERSE PROBLEM
SUMMARY

- **Partially discrete forward models:** recover a continuous (i.e. infinite dimensional) object from finite samples/measurements
  - Overwhelmingly more unknowns than there are measurements **(ill-posedness)**
    - Need continuous measurements! Impossible with digital sensors, or need many, many sensors
    - Takes us back to continuous models and those issues
  - Or can discretize the object space also → **FULLY DISCRETE MODELS**

BUT DOES THAT EVEN HELP?

FULLY-DISCRETE INVERSE PROBLEMS

GIVEN MEASUREMENTS, RECOVER "OBJECT"

FULLY-DISCRETIZED FORWARD MODEL

FROM ILL-POSEDNESS TO ILL-CONDITIONING

- Remember we will have a discrete LSI system, which is a Toeplitz matrix $\mathbf{A}$, so that measurement $\mathbf{g} = \mathbf{A}\,\mathbf{f}$

- Most systems have noise (or some form of small perturbations):
  - How does small variation in measurement (call it $\delta\mathbf{g}$) affect the solution:
  - Forward model is $\delta\mathbf{g} = \mathbf{A}\,\delta\mathbf{f}$, which means $\delta\mathbf{f}_{\text{est}} = \mathbf{A}^{-1}\,\delta\mathbf{g}$
  - Looking at the size **(norm)** of this small perturbation vector $$||\delta\mathbf{f}_{\text{est}}|| \leq \frac{1}{\lambda_{\min}(\mathbf{A})}\,||\delta\mathbf{g}||$$
  - $\lambda_{\min}(\mathbf{A})$ is the smallest eigenvalue of $\mathbf{A}$

FULLY-DISCRETIZED FORWARD MODEL

FROM ILL-POSEDNESS TO ILL-CONDITIONING

- How does small measurement variation (call it $\delta\mathbf{g}$) affect the solution
  - Forward model is $\delta\mathbf{g} = \mathbf{A}\,\delta\mathbf{f}$, which means $\delta\mathbf{f}_{\text{est}} = \mathbf{A}^{-1}\,\delta\mathbf{g}$
  - Looking at the size **(norm)** of this small perturbation vector

$$\|\delta\mathbf{f}_{\text{est}}\| \leq \frac{1}{\lambda_{\min}(\mathbf{A})}\,\|\delta\mathbf{g}\| \qquad (1)$$

  - Where $\lambda_{\min}(\mathbf{A})$ is the smallest eigenvalue of $\mathbf{A}$
- Forward model is $\mathbf{g} = \mathbf{A}\,\mathbf{f}$. And the norm of the vector $\mathbf{g}$

$$\|\mathbf{g}\| \leq \lambda_{\max}(\mathbf{A})\,\|\mathbf{f}_{\text{est}}\| \qquad (2)$$

- Dividing (1)/(2) and rearranging:

$$\frac{\|\delta\mathbf{f}_{\text{est}}\|}{\|\mathbf{f}_{\text{est}}\|} \leq \frac{\lambda_{\max}(\mathbf{A})}{\lambda_{\min}(\mathbf{A})}\,\frac{\|\delta\mathbf{g}\|}{\|\mathbf{g}\|} \qquad (3)$$

FULLY-DISCRETIZED FORWARD MODEL

FROM ILL-POSEDNESS TO ILL-CONDITIONING

This describes the size of the error propagated into the recovered object

$$\frac{||\delta \mathbf{f}_{\text{est}}||}{||\mathbf{f}_{\text{est}}||} \leq \boxed{\frac{\lambda_{\max}(\mathbf{A})}{\lambda_{\min}(\mathbf{A})}} \frac{||\delta \mathbf{g}||}{||\mathbf{g}||} \qquad (3)$$

Condition number of $\mathbf{A}$

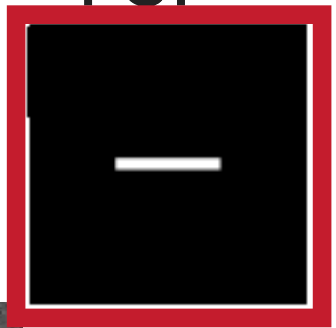
$$\text{cond}(\mathbf{A}) = \frac{\lambda_{\max}(\mathbf{A})}{\lambda_{\min}(\mathbf{A})}$$

Matlab command: 'cond'

MOTION BLUR

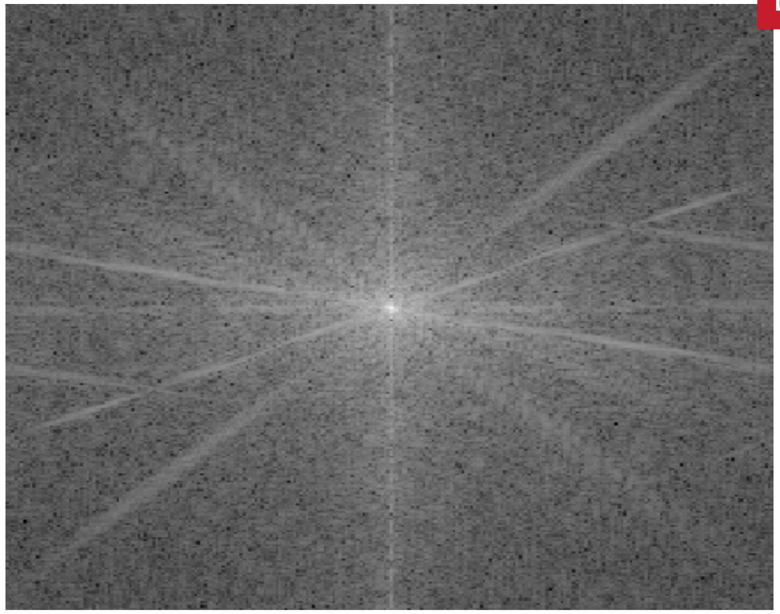
EXAMPLE

PSF



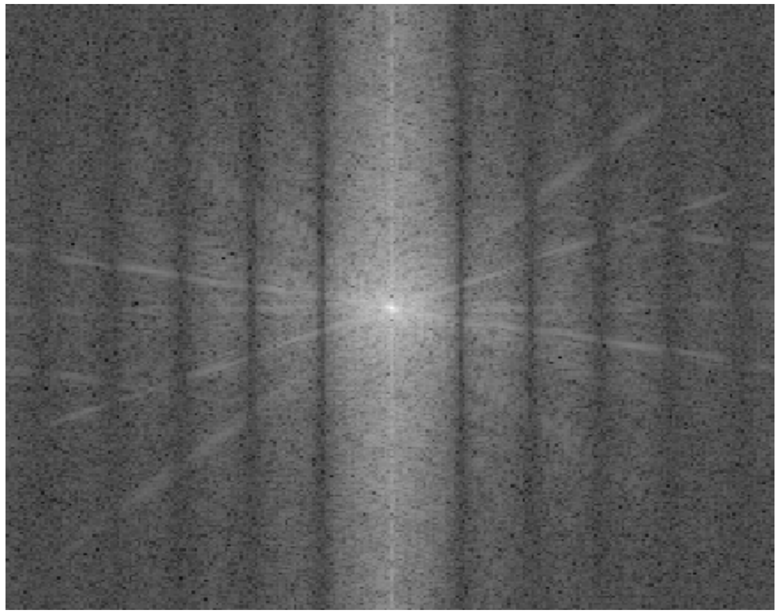
f

F



$g = A(f)$

G

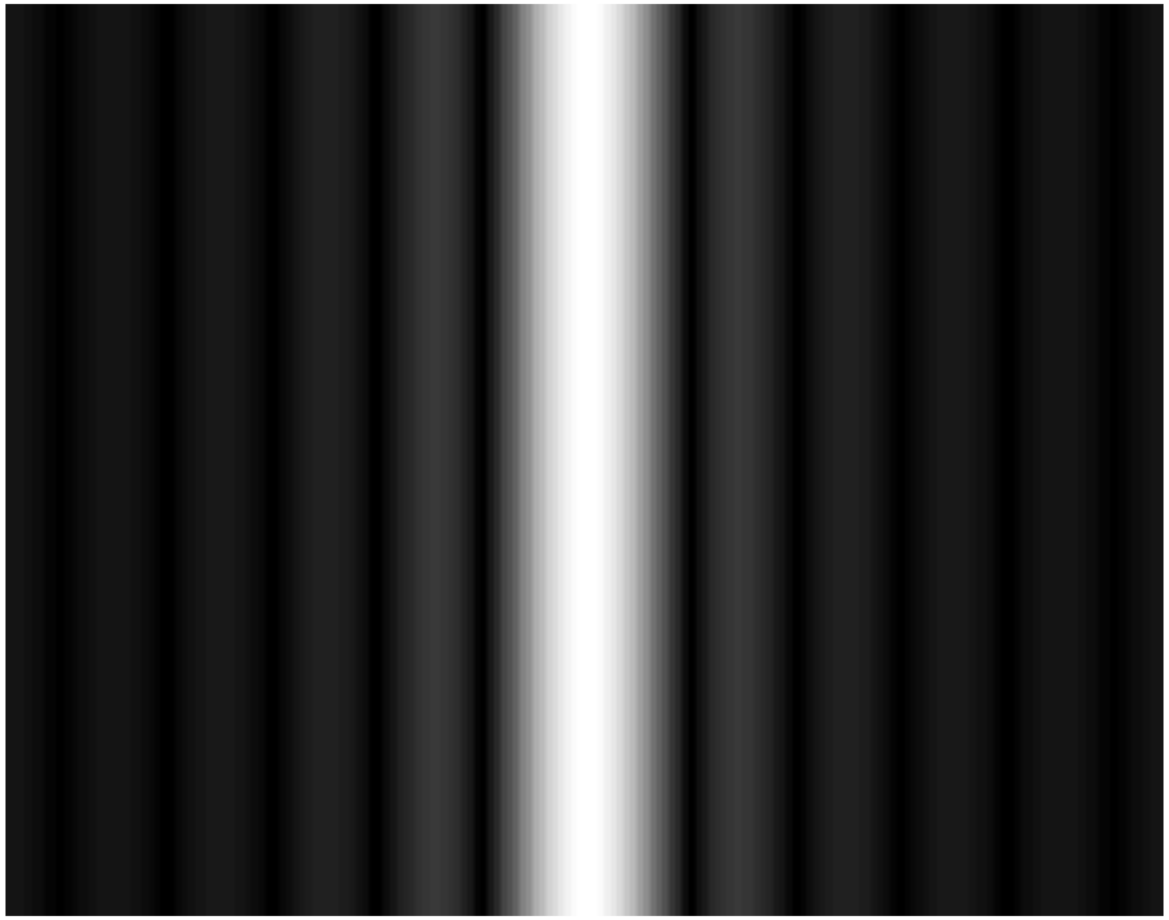


PSF

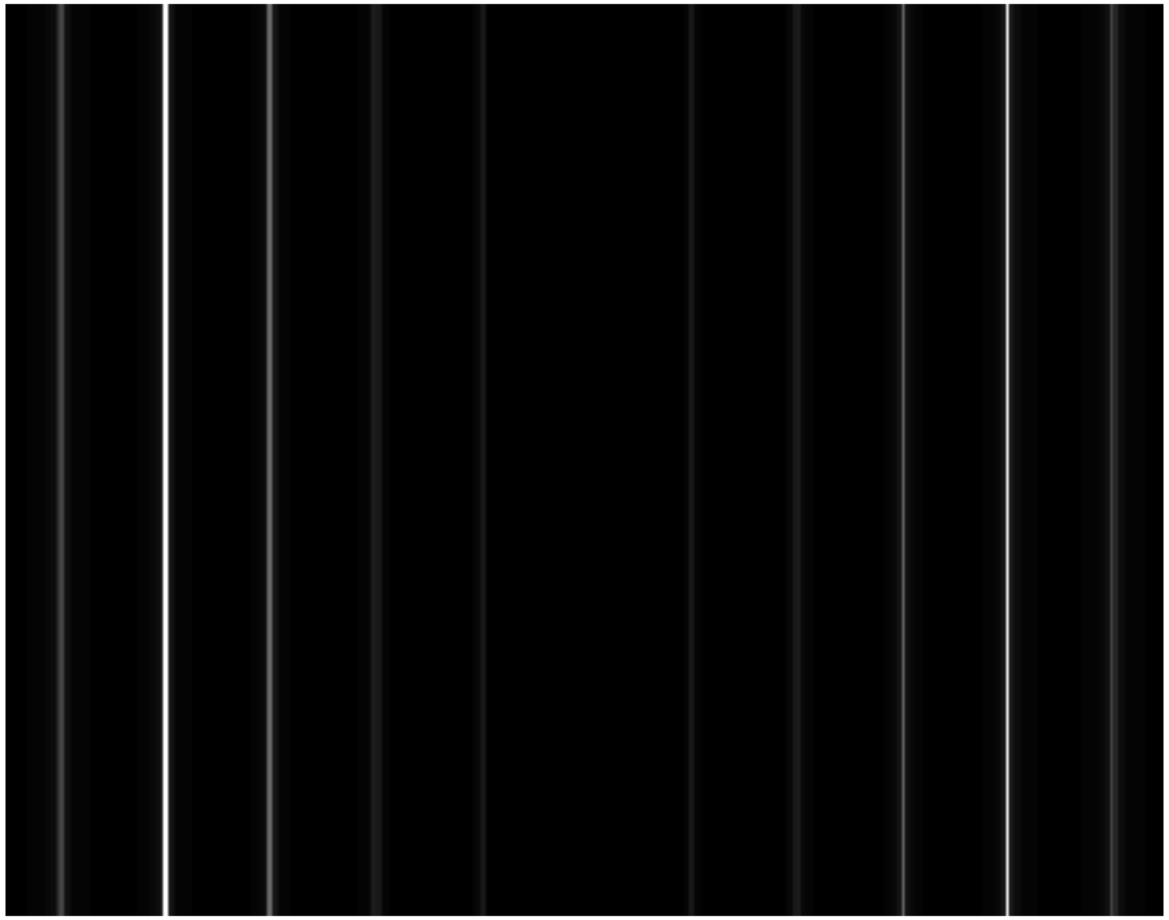
MOTION BLUR

$$\mathbf{g} = \mathbf{A}\mathbf{f} + \mathbf{n} \qquad \mathrm{cond}(\mathbf{A}) = 815$$

$|H(\omega_x, \omega_y)|$



$|H^{-1}(\omega_x, \omega_y)|$



PSF

MOTION BLUR

$$\mathbf{g} = \mathbf{A}\mathbf{f} + \mathbf{n} \qquad \mathrm{cond}(\mathbf{A}) = 815$$

f

g

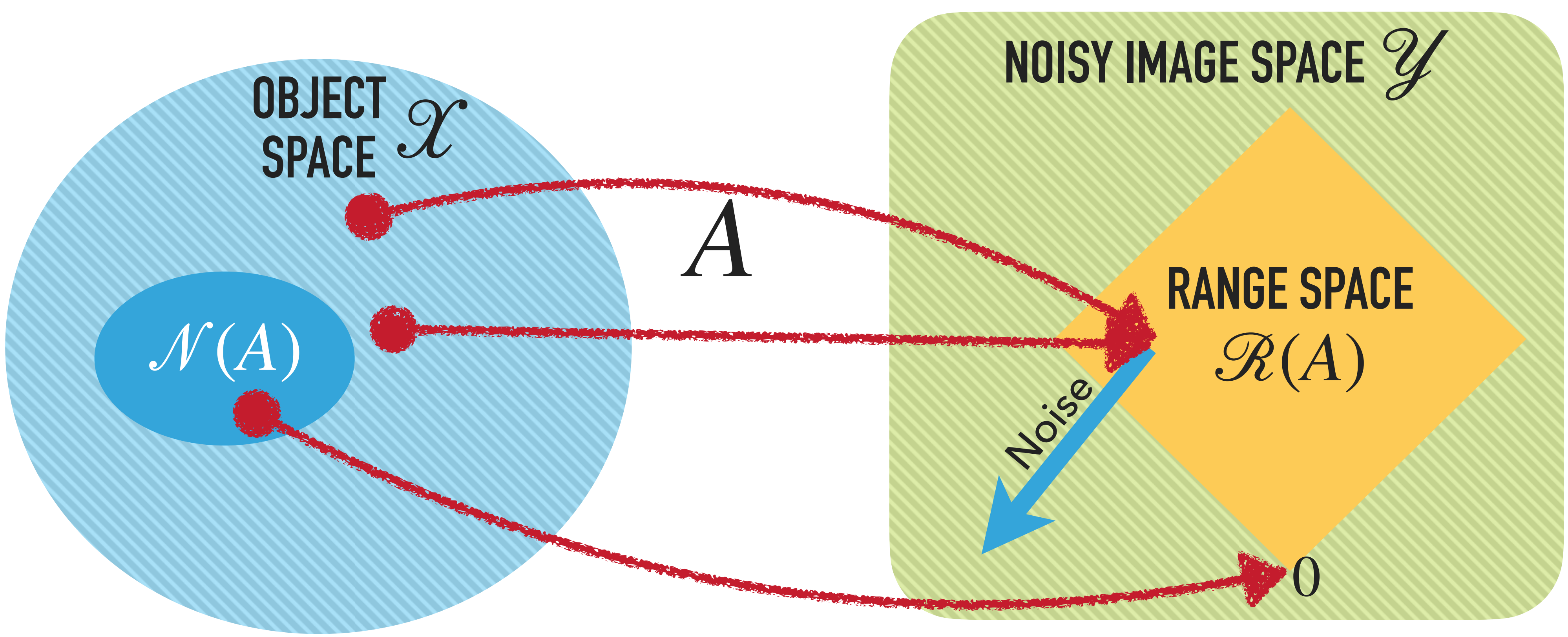
$A^{-1}g$

Noise dominates at high frequencies

SOLVING INVERSE PROBLEMS

RACE TOWARDS FINDING A CURE FOR ILL-POSEDNESS/ILL-CONDITIONING

- **No solutions:** look for **approximate solutions** instead
- **Non-uniqueness:** use **prior information**/knowledge about the object to find a good solution



MOORE-PENROSE PSEUDO-INVERSE

- Solve the minimization problem:

$$\min_{\mathbf{f}} \|\mathbf{g} - \mathbf{A}\mathbf{f}\|^2$$

- This gives the Moore-Penrose pseudo-inverse is

$$\mathbf{A}^{\dagger} = (\mathbf{A}^{H}\mathbf{A})^{-1}\mathbf{A}^{H}$$

- This is just the backslash operator '\' or '`pinv( )`' in Matlab.
- What does the optimization problem mean?
  - Give me the solution that "**best**" explains the measurement
  - Has the smallest residual error vector
  - **Caution:** no reason for this to be a (visually) good solution

MOORE-PENROSE PSEUDO-INVERSE

- Solve the minimization problem:

$$\min_{\mathbf{f}} \|\mathbf{g} - \mathbf{A}\mathbf{f}\|^2$$

- Moore-Penrose pseudo-inverse is

$$\mathbf{A}^\dagger = (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H$$

- What does the **optimization problem** mean?
  1. Pick an $\mathbf{f}_0$ and compute $\mathbf{A}\mathbf{f}_0$
  2. Examine the norm of the difference between your measurement $\mathbf{g}$ and the vector $\mathbf{A}\mathbf{f}_0$ computed in step 1. This is the error.
  3. Try all possible $\mathbf{f}_0$'s and pick the one with the smallest error vector.
  4. (How do you measure size of a vector? Use a norm, i.e. 2-norm)

GIVE ME THE SOLUTION THAT BEST EXPLAINS THE MEASUREMENT

PSF

MOTION BLUR

MOORE-PENROSE PSEUDO-INVERSE

$$\mathbf{g} = \mathbf{A}\mathbf{f} + \mathbf{n}$$

f

g

$A^{-1}g$

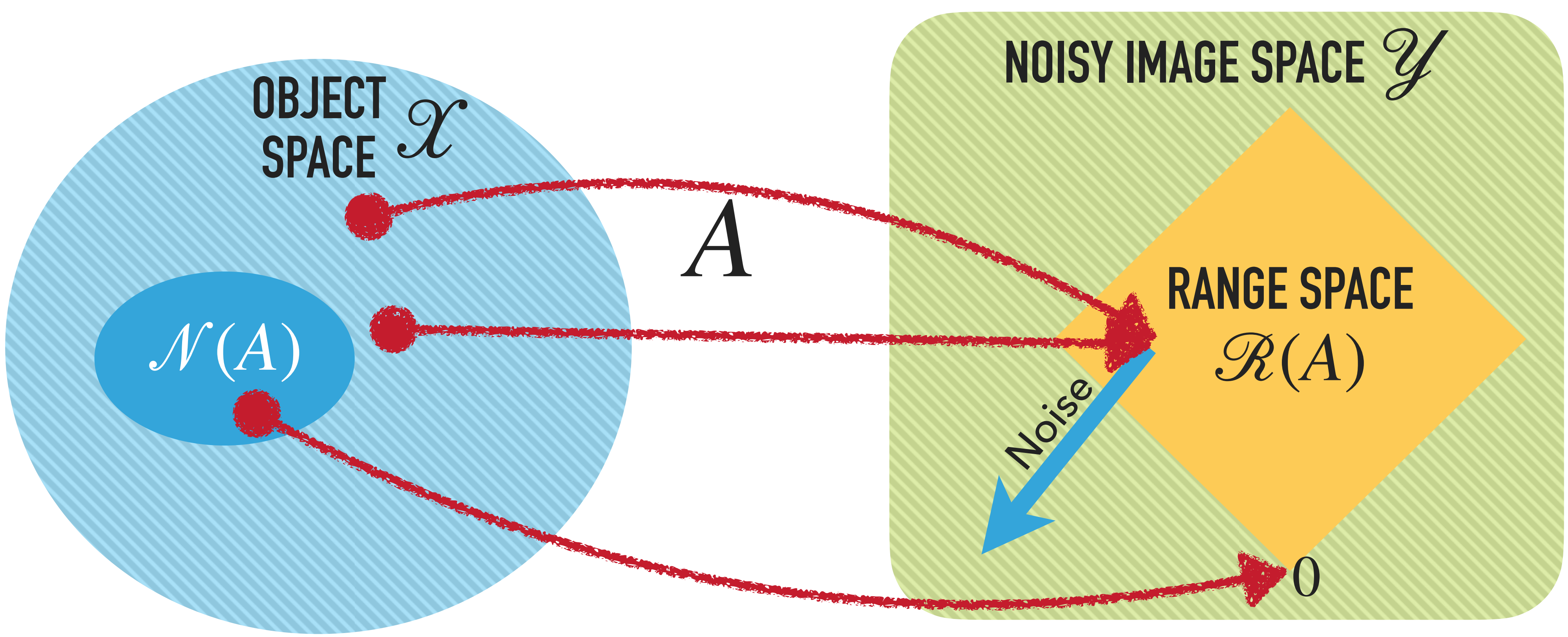
Can only recover band-limited information and not high frequency components

Cannot solve the ill-conditioned problem!

SOLVING INVERSE PROBLEMS

RACE TOWARDS FINDING A CURE FOR ILL-POSEDNESS/ILL-CONDITIONING

- **No solutions:** look for **approximate solutions** instead
- **Non-uniqueness:** use **prior information**/knowledge about the object to find a good solution



ALIASING IN TIME
VIDEO

In which direction is the train moving?

DISCRETE MODEL
THE STANDARD INVERSE

- Standard inverse (spectral form)

$$\mathbf{f}_{\text{est}} = \mathscr{F}^{-1}\left\{\frac{\mathscr{F}\{\mathbf{g}\}}{\mathscr{F}\{\mathbf{h}_{\text{sys}}\}}\right\}$$

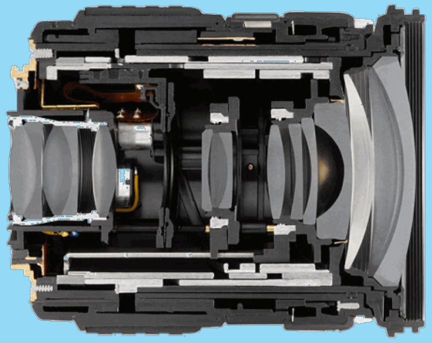
- Note that it is equivalent to: $\mathbf{f}_{\text{est}} = \mathbf{A}^{-1}\mathbf{g}$

DISCRETE MODEL
THE MOORE-PENROSE PSEUDO-INVERSE

- Moore-Penrose pseudo-inverse: solution of
$$\min_{\mathbf{f}} \|\mathbf{g} - \mathbf{A}\mathbf{f}\|^2$$
  - **Solution:**
$$\mathbf{f}_{\text{est}} = \mathbf{A}^{\dagger}\mathbf{g},$$
where $\mathbf{A}^{\dagger} = (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H$.

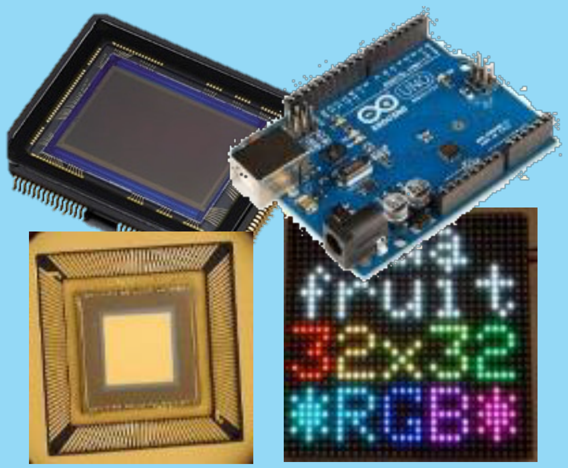
- **Spectral form:** take DFT/FFT of both sides of $\mathbf{f}_{\text{est}} = \mathbf{A}^{\dagger}\mathbf{g}$:
  - $\mathscr{F}\left\{\mathbf{f}_{\text{est}}\right\} = \mathscr{F}\left\{\mathbf{A}^{\dagger}\mathbf{g}\right\}$, because $\mathbf{A}$ is block convolution matrix:

$$\boxed{\mathscr{F}\left\{\mathbf{f}_{\text{est}}\right\} = \frac{\mathscr{F}\{\mathbf{h}_{\text{sys}}\}^H}{\mathscr{F}\{\mathbf{h}_{\text{sys}}\}^H\mathscr{F}\{\mathbf{h}_{\text{sys}}\}}\mathscr{F}\{\mathbf{g}\}}$$



Optics

+



Sensors
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devices

+

Signal
processing
&
algorithms

COMPUTATIONAL METHODS FOR IMAGING (AND VISION)

LECTURE 19: INVERSE PROBLEMS (REGULARIZATION)

PROF. JOHN MURRAY-BRUCE

WHERE ARE WE

10	15-Mar-21	Forward Models and Inverse Problems	Linear Inversion - Inverse problems - Deconvolution and Denoising	IIP 4, Appendix E	**HW 4**	
	17-Mar-21		Intro to Regularized Inversion I - Tikhonov	IIP 5, Appendix E		
11	22-Mar-21	Regularization	Intro to Regularized Inversion II - Iterative methods - Steepest descent	IIP 6		
	24-Mar-21		Statistical methods I - ML estimation - Bayesian estimation	IIP 7.1 - 7.5		
12	29-Mar-21	Forward models and Inverse Problems II	LSV imaging systems: Forward problem - SVD - Inversion	IIP 8.1, 9, 10		
	31-Mar-21	Non-linear Regularization	Beyond L_2-regularization - Sparsity (l_0- and l_1-priors) - TV prior	SMIV 1.1 - 1.5 Papers & Handout		**HW 4**
13	5-Apr-21		Algorithms overview - ISTA/FISTA - ADMM	Papers & Handout		
	7-Apr-21	Introductory Optics	Geometrical/Ray Optics - Rays & pinhole cameras - Lenless imaging and Coded apertures	IIP 8.2, 8.3, 9.5 Papers & Handout		
14	12-Apr-21 14-Apr-21	**Spring Break (no classes)**				
15	19-Apr-21	Applications of Comp. Imaging	Looking around corners (NLOS imaging)	Papers & Handout		
	21-Apr-21		Compressive Imaging and Imaging from few photons	Papers & Handout		
16	26-Apr-21 28-Apr-21	**Group Presentations (Teams)**				
17	3-May-21	***no class**				
	5-May-21	**Final Exam: 12:30 PM - 2:30 PM**				

OUTLINE

- Regularization
- Linear minimization problems
- Tikhonov regularization

LEARNING GOALS

- Identify regularization as a cure for ill-conditioning
- Formulate and identify linear minimization problems with constraints
- Identify limitations of Tikhonov regularization for imaging

READING

- IIP Chapter 5, Appendix E

A (MILDLY) QUICK REVIEW

FORWARD MODELS

$$g = A(f)$$

- **Forward models:** output as a function of input
- **Continuous models:** e.g., convolution for continuous LSI systems.
- **Discrete models:** matrix-vector relationship, $\mathbf{g} = \mathbf{A}\mathbf{f}$, in general
  - The discrete matrix representation is given by:
  $$\mathbf{A}_{mn} = \langle A\psi_n, p_m \rangle,$$
  where $\{\psi_0, \ldots, \psi_{N-1}\}$ is the object representation, while $\{p_0, \ldots, p_{M-1}\}$ is the pixel sampling
  - Discrete LSI system $\mathbf{g} = \mathbf{a} \star \mathbf{f}$ (when $\mathbf{A}$ is circulant)
  - Easier to work with discrete models for computations

SOLVING AN INVERSE PROBLEM
DECONVOLUTION CASE

- **Continuous models:** standard inverse of continuous LSI systems **(deconvolution)** is

$$(A^{-1}g)(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{G(\omega_x,\omega_y)}{H(\omega_x,\omega_y)}e^{\mathrm{j}(\omega_x x+\omega_y y)}\,\mathrm{d}\omega_x\mathrm{d}\omega_y,$$

where $g(x,y)$ is output and $h(x,y)$ is the system PSF

- **Discrete models:** the standard inverse for $\mathbf{g}=\mathbf{A}\mathbf{f}$, is

$$\mathbf{f}_{\mathrm{est}} = \mathbf{A}^{-1}\mathbf{g},$$

where $\mathbf{f}_{\mathrm{est}}$ is an estimate of $\mathbf{f}$.

  - Discrete LSI system $\mathbf{g} = \mathbf{a} \star \mathbf{f}$
  - **Discrete deconvolution:** use the convolution property of DFT
    - Then, $\mathscr{F}\{\mathbf{g}\} = \mathscr{F}\{\mathbf{h}_{\mathrm{sys}}\}\,\mathscr{F}\{\mathbf{f}\}$
    - So, $\mathbf{f}_{\mathrm{est}} = \dfrac{\mathscr{F}\{\mathbf{g}\}}{\mathscr{F}\{\mathbf{h}_{\mathrm{sys}}\}}$

THIS DIVISION HERE IS ELEMENT-WISE

DISCRETE CONVOLUTION

- **Discrete convolution** is defined for discrete sequences
- **1D discrete convolution** between two vectors:
  - $\mathbf{a} = [a_0, a_1, \cdots, a_{M-1}]^T$ and $\mathbf{f} = \left[f_0, f_1, \cdots, f_{N-1}\right]^T$

$$\mathbf{g} = \mathbf{a} \star \mathbf{f} = \sum_{k=-\infty}^{\infty} a_k f_{n-k}$$

COMPARE THIS WITH CONVOLUTION INTEGRAL

- **2D discrete convolution:** also possible for 2D sequences (e.g., images, matrices).
- Consider matrices $\mathbf{A} \in \mathbb{R}^{M_1 \times N_1}$ and $\mathbf{F} \in \mathbb{R}^{M_2 \times N_2}$

$$\mathbf{G} = \mathbf{A} \star_{2\mathrm{D}} \mathbf{F} = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} A_{i,j} F_{i-m,j-n}$$

  - Where $\star_{2\mathrm{D}}$ denotes discrete 2D convolution
  - $A_{i,j}$ and $F_{i,j}$ are the (i,j)-th elements of the matrices $\mathbf{A} \in \mathbb{R}^{M_1 \times N_1}$ and $\mathbf{F} \in \mathbb{R}^{M_2 \times N_2}$
  - The output matrix $\mathbf{G}$ has dimension $(M_1 + M_2 - 1) \times (N_1 + N_2 - 1)$

INVERTIBILITY ISSUES

- $\mathbf{A}$ may not even have an inverse!
- For example:
  - The matrix is not square
  - The matrix is rank deficient (has one or more zero eigenvalues or singular values)
  - Try computing the inverse of

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

DITCH THE URGE OF SEEKING EXACT SOLUTIONS

REGULARIZED INVERSION

LEAST SQUARES PROBLEM
MOORE-PENROSE REVISITED

May have heard about this in a machine learning course: *least squares regression*

- The least square problem:
  - Find the solution with the least/smallest square error:

$$\min_{\mathbf{f}} \|\mathbf{g} - \mathbf{A}\mathbf{f}\|_2^2$$

  - Remember that from the definition of the norm of a vector
  - $\|\mathbf{x}\|_2^2 = x_0^2 + x_1^2 + \cdots + x_{N-1}^2$
  - Let the error vector be denoted by: $\mathbf{e} = \mathbf{g} - \mathbf{A}\mathbf{f}$
  - One way to measure the side of the error is to look at its **NORM!**

LEAST SQUARES PROBLEM

MOORE-PENROSE REVISITED

- The least square problem:

$$\boxed{\min_{\mathbf{f}} \|\mathbf{g} - \mathbf{A}\mathbf{f}\|_2^2}$$

  - Error vector: $\mathbf{e} = \mathbf{g} - \mathbf{A}\mathbf{f}$
  - Size of error vector:

$$\|\mathbf{e}\|_2^2 = \|\mathbf{g} - \mathbf{A}\mathbf{f}\|_2^2$$

  - We want the solution to be the $\mathbf{f}$ that makes this error norm as small as possible!

LEAST SQUARES PROBLEM

MOORE-PENROSE REVISITED

- The least square problem:

$$\boxed{\min_{\mathbf{f}} \|\mathbf{g} - \mathbf{Af}\|_2^2}$$

  - Solution: the Moore-Penrose pseudo-inverse

$$\mathbf{A}^{\dagger} = (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H$$

- However, the Moore-Penrose pseudo-inverse still gives unsatisfactory imaging results!

PSF

MOTION BLUR

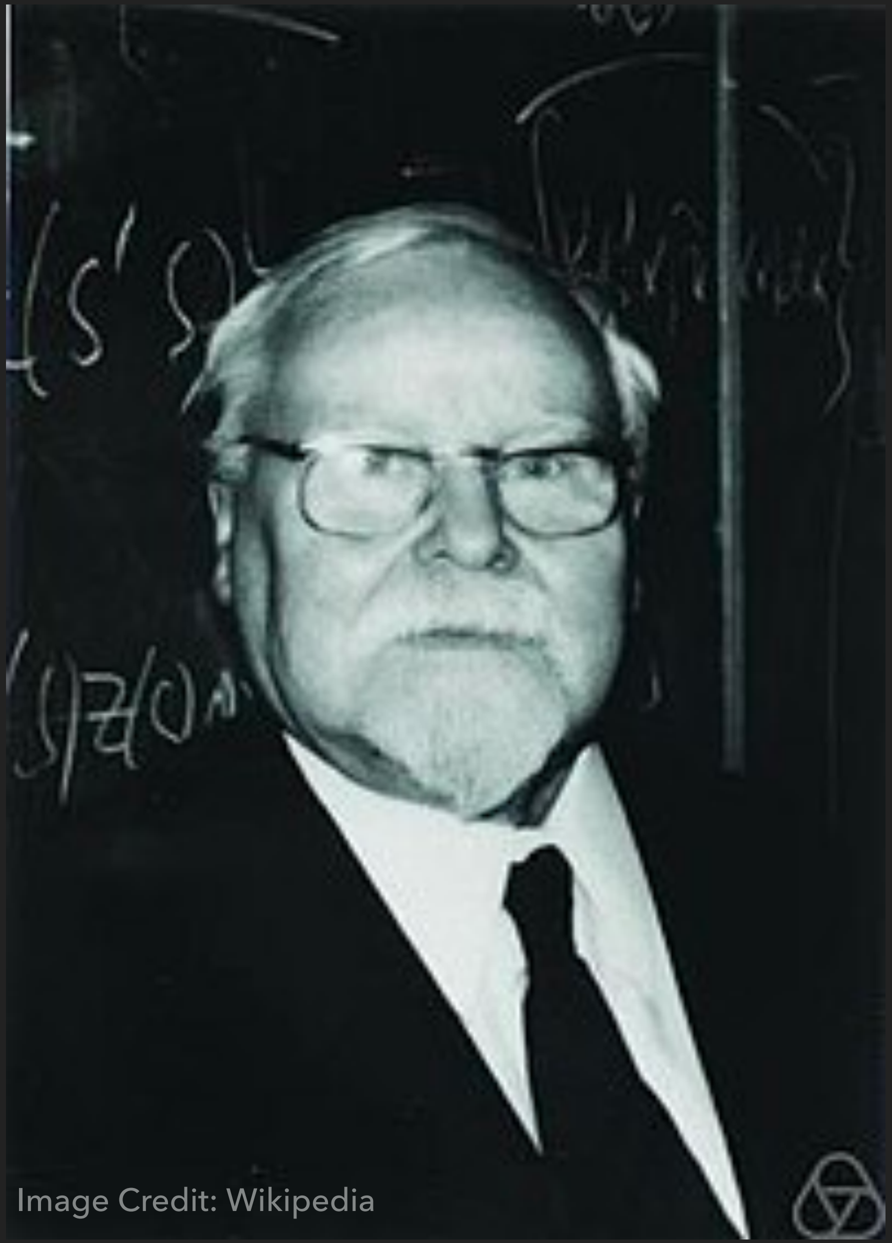
$$\mathbf{g} = \mathbf{A}\mathbf{f} + \mathbf{n} \qquad \mathrm{cond}(\mathbf{A}) = 815$$

f

g

$A^{-1}g$

Ill-conditioned: thus, noise dominates at high frequencies



TIKHONOV REGULARIZATION

INVERSE PROBLEMS
THE GOLDEN RULE

Discard the concept of exact solutions and seek approximate solutions instead

Use any **prior knowledge** we have about the object to **constrain** the set of possible **solutions**.

CONSTRAINTS

- Priors can help confine the possible candidate solutions
  - Of all possible solutions, we want a solution with the **2-norm** smaller than some threshold (E)
  - **Constrained Least Squares problem:**

$$\min_{\mathbf{f}} \|\mathbf{g} - \mathbf{A}\mathbf{f}\|_2^2 \,, \quad \text{subject to } \|\mathbf{f}\|_2^2 \leq E^2$$

  - Lagrangian form:

$$\min_{\mathbf{f}} \|\mathbf{g} - \mathbf{A}\mathbf{f}\|_2^2 + \mu\,(\|\mathbf{f}\|_2^2 - E^2)$$

  - μ is called the **regularization parameter**
  - It can be shown that the solution to:

$$\min_{\mathbf{f}} \|\mathbf{g} - \mathbf{A}\mathbf{f}\|_2^2 + \mu\,\|\mathbf{f}\|_2^2$$

    is identical to the above.

TIKHONOV REGULARIZATION

Andrey Nikolayevich Tikhonov

- **Tikhonov regularized inversion:** solves the minimization problem

$$\min_{\mathbf{f}} \overbrace{\boxed{\|\mathbf{g}-\mathbf{A}\mathbf{f}\|_2^2}}^{\text{DATA FIDELITY TERM}} + \overbrace{\boxed{\underbrace{\mu}_{\text{REGULARIZATION PARAMETER}} \|\mathbf{f}\|_2^2}}^{\text{REGULARIZATION TERM}}$$

- **Intuition:** the constraint helps to minimize large variations in the estimates of $\mathbf{f}$ caused by noise (and ill-conditioning)

TIKHONOV REGULARIZATION

- **Tikhonov regularized inversion:** solves the minimization problem

$$\min_{\mathbf{f}} \overbrace{\|\mathbf{g}-\mathbf{A}\mathbf{f}\|_2^2}^{\text{DATA FIDELITY TERM}} + \overbrace{\underbrace{\mu}_{\text{REGULARIZATION PARAMETER}} \|\mathbf{f}\|_2^2}^{\text{REGULARIZATION TERM}}$$

- **Solution:** $\mathbf{f}_\mu = (\mathbf{A}^H\mathbf{A} + \mu\mathbf{I})^{-1}\mathbf{A}^H\mathbf{g}$

- Notice that this solution is a function of μ:
  - μ controls the balance between the data fidelity and the regularization terms
  - Large μ gives highly regularized solution compared to small μ
  - When $\mu = 0$ get the usual pseudo-inverse

DISCRETE MODEL
TIKHONOV REGULARIZED INVERSION

- Moore-Penrose pseudo-inverse: solution of

$$\min_{\mathbf{f}} \|\mathbf{g} - \mathbf{A}\mathbf{f}\|^2 + \lambda \|\mathbf{f}\|_2^2$$

  - **Solution:**

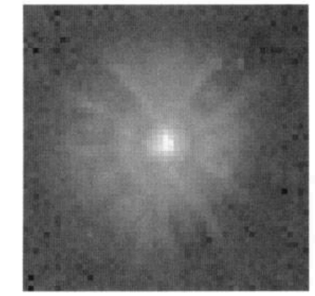
$$\mathbf{f}_{\text{est}} = \mathbf{A}^{\dagger}_{\text{Tik}}\, \mathbf{g},$$

where $\mathbf{A}^{\dagger}_{\text{Tik}} = (\mathbf{A}^H\mathbf{A} + \lambda\mathbf{I})^{-1}\mathbf{A}^H.$

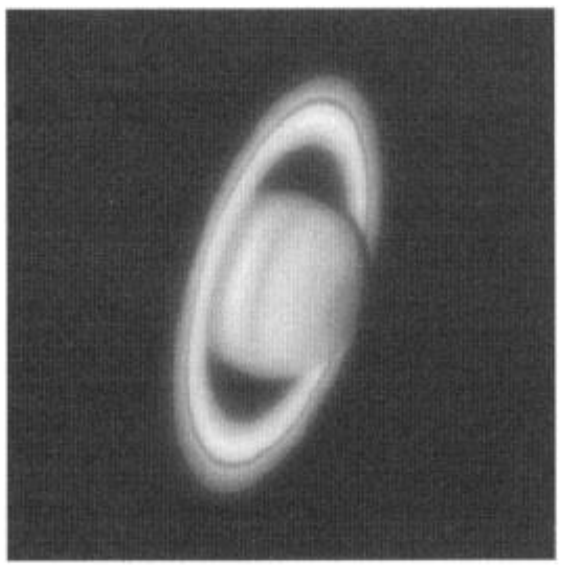
- **Spectral form:** take DFT/FFT of both sides of $\mathbf{f}_{\text{est}} = \mathbf{A}^{\dagger}\, \mathbf{g}$:
  - $\mathscr{F}\left\{\mathbf{f}_{\text{est}}\right\} = \mathscr{F}\left\{\mathbf{A}^{\dagger}\mathbf{g}\right\}$, because $\mathbf{A}$ is block convolution matrix:

$$\boxed{\mathscr{F}\left\{\mathbf{f}_{\text{est}}\right\} = \frac{\mathscr{F}\{\mathbf{h}_{\text{sys}}\}^H}{\mathscr{F}\{\mathbf{h}_{\text{sys}}\}^H \mathscr{F}\{\mathbf{h}_{\text{sys}}\} + \lambda\mathbf{I}}\, \mathscr{F}\{\mathbf{g}\}}$$

EXAMPLE

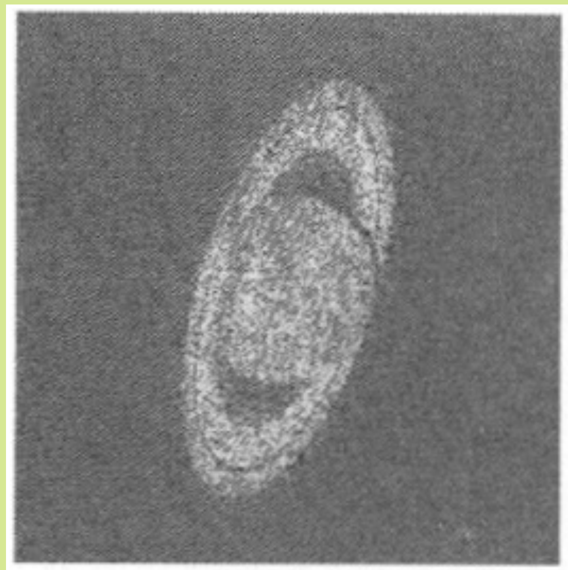



PSF

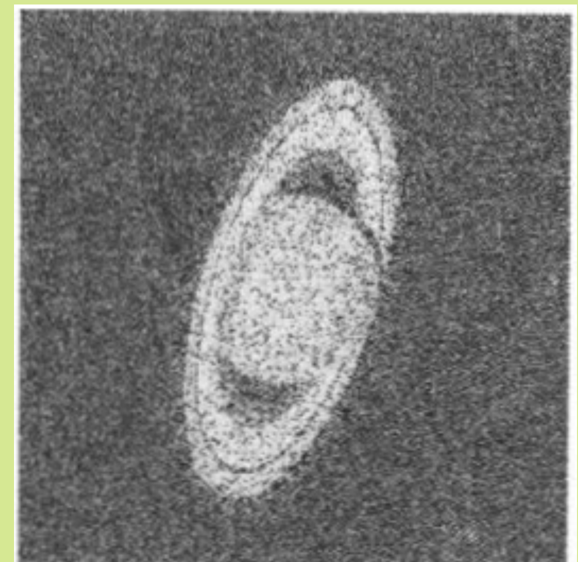


Ground truth

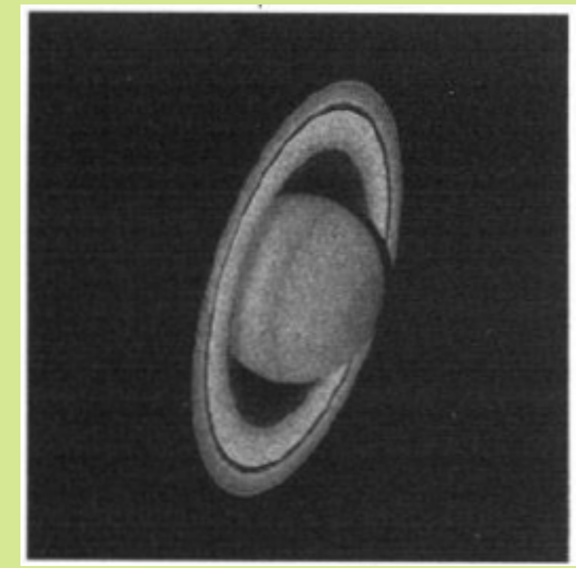
Tikhonov regularized inverses



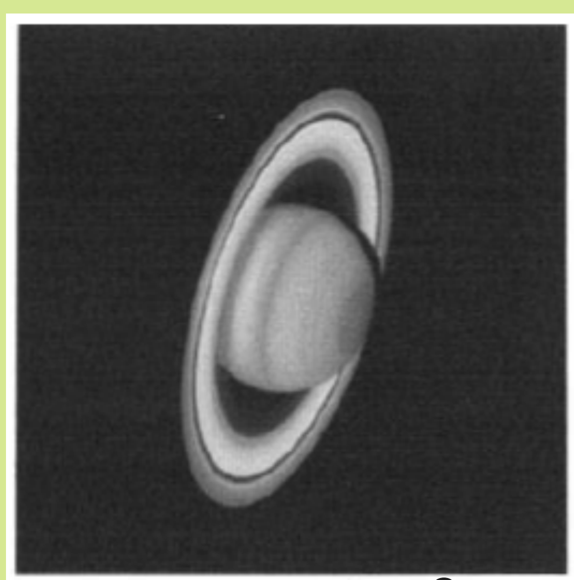
$\mu = 0$



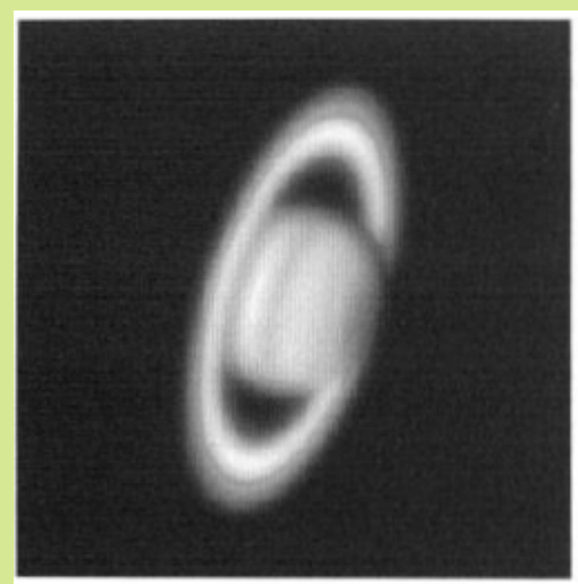
$\mu = 10^{-5}$



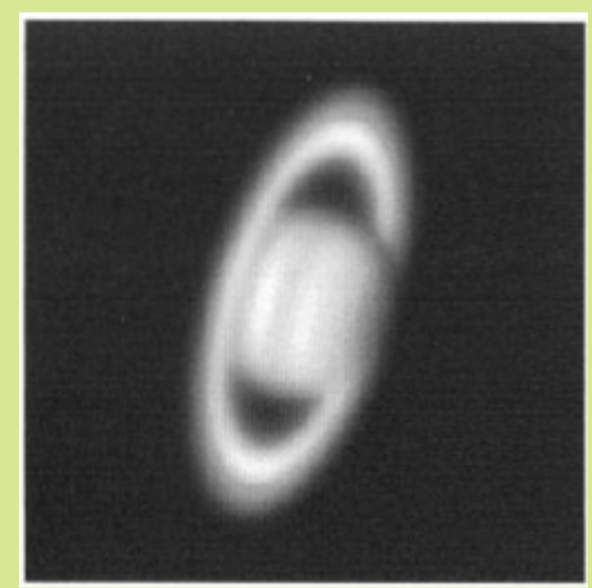
$\mu = 10^{-3}$



$\mu = 10^{-2}$



$\mu = 1$



$\mu = 1000$

LINEAR INVERSION

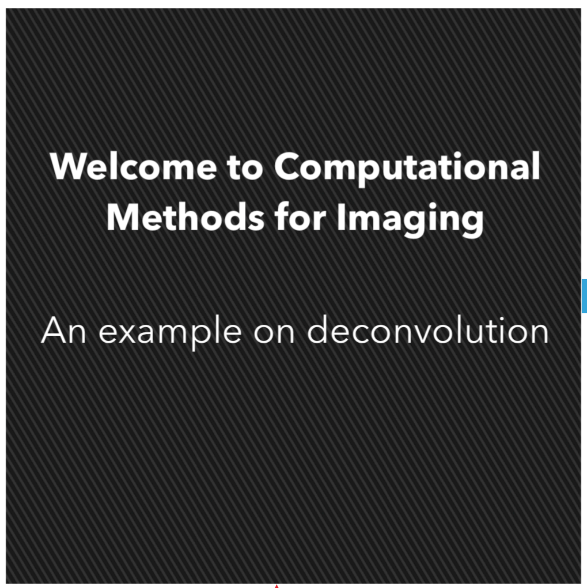
MATLAB PRACTICE 9

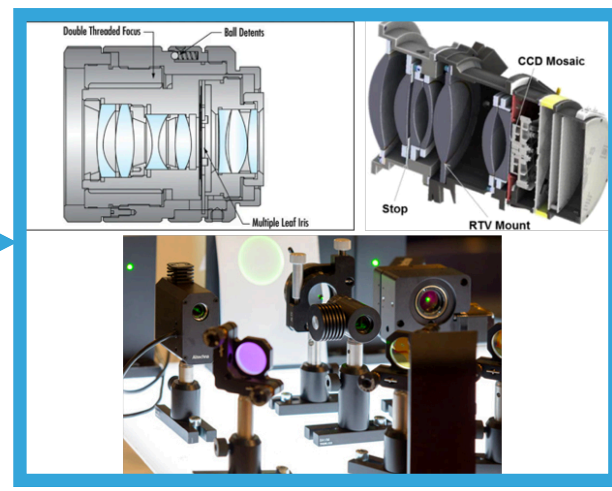
MATLAB PRACTICE

- Forward modeling: Continuous to Discrete model
- Linear Inversion
  - Standard inverse
  - Tikhonov regularized inverse

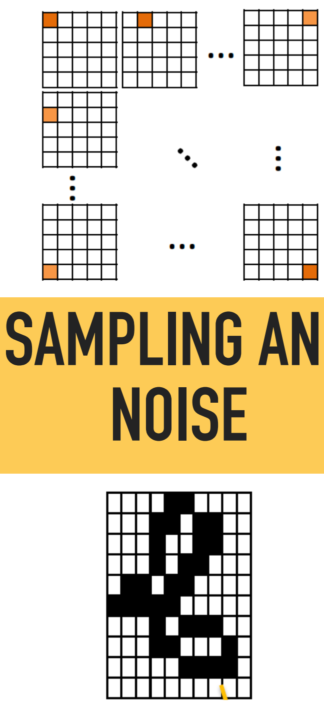
EXAMPLE: BLURRING

FORWARD MODELING AND INVERSE PROBLEM

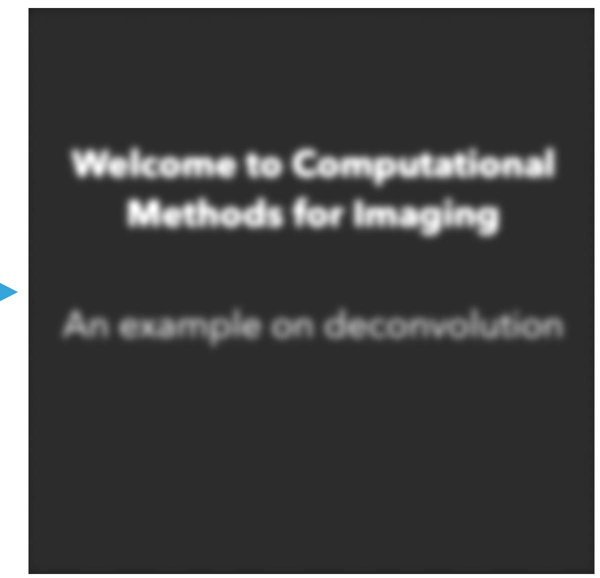




$h(x, y)$



SAMPLING AND NOISE



$$h(x, y) = e^{-\frac{x^2 + y^2}{2\sigma^2}}$$

THE INVERSE PROBLEM

PSF AND OTF

- **Operator form** of the imaging system (continuous)
- **Matrix form** for the imaging system (fully discrete form)
- Lets handle each one in turn

PSF **OTF**

$$h(x,y) = e^{-\frac{x^2+y^2}{2\sigma^2}} \iff H(\omega_x,\omega_y) = 2\pi\sigma^2 e^{-\frac{\sigma^2}{2}(\omega_x^2+\omega_y^2)}$$

HOW TO COMPUTE AND STORE THIS?

EFFICIENT APPROACH IS TO USE DFT

- We want to do computations (specifically linear inversion)
- To be able to do this, we need to program the matrix $\mathbf{A}$

$$\mathbf{A}_{m,n} = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_{0,0}(\omega_x, \omega_y) H(\omega_x, \omega_y)\, P_{0,0}(\omega_x, \omega_y)\, e^{\mathrm{j}(\omega_x(m_1\Delta - n_1\delta) + \omega_y(m_2\Delta - n_2\delta))}\, \mathrm{d}\omega_x \mathrm{d}\omega_y$$

COMES FROM TAKING THE DFT OF THE MATRIX $\mathbf{A}$

Because it is block-circulant, can represent it as one column vector ($\Delta = \delta$ and $M = N$), has only 10^6 elements.

$$\mathscr{F}\{\mathbf{A}_{m,n}\} \approx \Psi_{0,0}\left(\frac{m_1}{M\Delta}, \frac{m_2}{M\Delta}\right) H\left(\frac{m_1}{M\Delta}, \frac{m_2}{M\Delta}\right) P_{0,0}\left(\frac{m_1}{M\Delta}, \frac{m_2}{M\Delta}\right)$$

SO HOW DO WE COMPUTE THIS?

$$h(x, y) = e^{-\frac{x^2+y^2}{2\sigma^2}}$$

$$H(\omega_x, \omega_y) = 2\pi\sigma^2 e^{-\frac{\sigma^2}{2}(\omega_x^2+\omega_y^2)}$$

- We have H, stated above in green box
- While, $\Psi_{0,0} = \mathscr{F}\{\psi_{0,0}\}$
$$= \mathscr{F}\{\mathrm{rect}(x/\Delta)\ \mathrm{rect}(y/\Delta)\}$$
$$= \mathrm{sinc}(\omega_x\Delta)\ \mathrm{sinc}(\omega_y\Delta)$$
- Similarly,
$P_{0,0} = \mathscr{F}\{p_{0,0}\} = \mathscr{F}\{\mathrm{rect}(x/\Delta)\ \mathrm{rect}(y/\Delta)\} = \mathrm{sinc}(\omega_x\Delta)\ \mathrm{sinc}(\omega_y\Delta)$

$$\mathscr{F}\{\mathbf{A}_{m,n}\} \approx \Psi_{0,0}\left(\frac{m_1}{M\Delta}, \frac{m_2}{M\Delta}\right) H\left(\frac{m_1}{M\Delta}, \frac{m_2}{M\Delta}\right) P_{0,0}\left(\frac{m_1}{M\Delta}, \frac{m_2}{M\Delta}\right)$$

- For this problem we use m_1 and m_2 go from $M/2$ to $(M/2) - 1$
  - Origin is at the center pixel of an image
- We can now program this in Matlab (vector of 10^6 terms or Matrix 1000 x 1000, either is fine)

$$\mathcal{F}\{\mathbf{A}_{m,m}\} \approx \operatorname{sinc}^2\left(\frac{m_1}{M}\right) \operatorname{sinc}^2\left(\frac{m_2}{M}\right)\, e^{-\frac{2}{\sigma^2}\left(\left(\frac{m_1}{M\Delta}\right)^2 + \left(\frac{m_2}{M\Delta}\right)^2\right)}$$

WHAT DID WE COVER

- **Introduction to inverse problems**
  - Ill-posed and ill-conditioned inverse problems
  - How to assess the ill-conditioning and ill-posedness of inverse problems
- **Non-regularized solution to deconvolution problem**
- **Regularized inversion**
- **Matlab practice**
  - Forward modeling
  - Linear Inversion

TILL NEXT TIME

CURING ILL-CONDITIONING WITH REGULARIZATION (CONTINUED) - ITERATIVE METHODS