CIS 4930.006S21/CIS 6930.013S21: Computational Methods for Imaging and Vision

Spring 2021
Homework #2

The University of South Florida
Department of Computer Science and Engineering
Tampa, FL

Assigned: February 10, 2021
Due: February 24, 2021 (at 6:00 PM)

1 Complex Conjugate Property of the Fourier Transform

Let $g(x)$ be a complex-valued function, that is $g(x)$ maps the reals x (i.e. $x \in \mathbb{R}$) to complex values; mathematically, we write:

$$g : \mathbb{R} \longmapsto \mathbb{C}.$$

If $g(x) \longleftrightarrow G(\omega)$, i.e. the Fourier transform (FT) of $g(x)$ is $G(\omega)$, then from the definition of the FT, demonstrate that,

$$g^*(x) \longleftrightarrow G^*(-\omega),$$

where $g^*(x)$ is the complex conjugate of $g(x)$. (Remember that for a complex number $x = x_\mathrm{R} + \mathrm{j}x_\mathrm{I}$, whose real part is x_R and imaginary part is x_I, the complex conjugate of x denoted by $x^* = x_\mathrm{R} - \mathrm{j}x_\mathrm{I}$. The imaginary unit $\mathrm{j} = \sqrt{-1}$.)

2 Fourier Transform of Real Functions

Let

$$u(x) = \begin{cases} 1, \text{for } x \geq 0; \\ 0, \text{otherwise}, \end{cases}$$

represent the unit step function. Using the definition of the Fourier transform, compute the FT of the following functions:

(a) $f(x) = e^{-ax}u(x)$

(b) $f(x) = \mathrm{sinc}(x) = \frac{\sin(x)}{x}$

(c) $f(x) = \mathrm{rect}(x) = \begin{cases} 1, \text{ for } -1/2 \leq x \leq 1/2; \\ 0, \text{ otherwise.} \end{cases}$

(d) $f(x) = \Lambda(x) = \begin{cases} 1 + x, \text{ for } -1 \leq x < 0; \\ 1 - x, \text{ for } 0 \leq x \leq 1; \\ 0, \text{ otherwise.} \end{cases}$

The function $\Lambda(x)$ is also known as the triangle (or 'tent') function, sketch it to see why.

3 Inverse Fourier Transforms

Using the definition of the inverse Fourier transform (IFT), compute the IFT $f(x)$ for the following functions:

(a) $F(\omega) = \mathrm{rect}(\omega) = \begin{cases} 1, \text{ for } -1/2 \leq \omega \leq 1/2; \\ 0, \text{ otherwise.} \end{cases}$

(b) $F(\omega) = e^{-a|\omega|}$, where is $a > 0$. Recall that $|\omega|$ denotes the absolute value of ω, i.e.

$$|\omega| = \begin{cases} \omega, \text{ when } \omega \geq 0; \\ -\omega, \text{ when } \omega < 0. \end{cases}$$

4 2D Fourier Transforms

In this question, we will compute the 2D FT for some important 2D functions $f(x, y)$. Two dimensional functions accept two arguments and return a scalar, mathematically $f : \mathbb{R}^2 \longmapsto \mathbb{R}$.
From the definition of the 2D FT, compute the FT of the following 2D functions:
(a) $f(x, y) = \mathrm{rect}(ax)\mathrm{rect}(by)$, where the function

$$\mathrm{rect}(x) = \begin{cases} 1, \text{ for } -1/2 \leq x \leq 1/2; \\ 0, \text{ otherwise.} \end{cases}$$

(b) $f(x, y) = \Lambda(x)\Lambda(y)$.

5 Fourier Transform of some Non-Square-Integrable Functions

In this question, we aim to compute the Fourier transform of functions $g(x)$ that are not in the space $\mathcal{L}^2(\mathbb{R})$, i.e. non-square-integrable functions. Examples include the signum[1] function, defined as

$$\mathrm{sign}(x) = \begin{cases} 1, \text{ for } x > 0; \\ 0, \text{ for } x = 0; \\ -1, \text{ for } x < 0; \end{cases} \quad ,$$

and the unit step function

$$u(x) = \begin{cases} 1, \text{for } x \geq 0; \\ 0, \text{otherwise.} \end{cases}$$

For such functions, the Fourier transform integral does not exist because it is not finite. However, we can sometimes derive, so called, *generalized Fourier transforms* $G(\omega)$, that are still very useful.

To begin, consider the function

$$f_a(x) = \begin{cases} e^{-ax}, \text{ for } x > 0; \\ 0, \text{ for } x = 0; \\ -e^{ax}, \text{ for } x < 0, \end{cases}$$

where a is strictly positive.
(a) Sketch the function $f_a(x)$, for some $a > 0$.

[1] The signum function simply returns the sign of its argument/input.

(b) Let $g(x) = \lim_{a\to 0} f_a(x)$. Sketch $g(x)$ and argue that it is equal to the signum function. That is, argue that as a goes to zero, the function $f_a(x)$ goes tends to $\mathrm{sign}(x)$.

(c) Using the result in part (b), above, prove that the Fourier transform of the signum function,

$$\mathcal{F}\{\mathrm{sign}(x)\} = \frac{2}{\mathrm{j}\omega}.$$

(d) By noting that the unit step function $u(x)$ may be written as,

$$u(x) = 1/2\,(1 + \mathrm{sign}(x)),$$

show that the Fourier transform of $u(x)$ is

$$U(\omega) = \pi\delta(\omega) + \frac{1}{\mathrm{j}\omega}.$$

[Hint: you may use the following Fourier transform pair: $1 \longleftrightarrow 2\pi\delta(\omega)$. To be explicit, this means that the FT of 1 is $2\pi\delta(\omega)$.]

6 Fast Fourier Transform (Computational Question)

(a) Consider the time-varying signal $g(t) = 3\sin(200\pi t) + \sin(40\pi t)$.

(i) Using Matlab, obtain samples of the signal $g(t)$ at a frequency $f_\mathrm{s} = 1000$ Hz, for 1 s. How many samples should you have?

(ii) Using Matlab, compute the FFT of the sampled signal, and plot the magnitude spectrum in Hz.

(iii) Comment on the position and amplitude/size of the peaks, in the plot.

7 2D Fourier Transform of the Circle Function (Optional, for extra credit)

In general, imaging systems have an aperture through which light passes in order to enter the imaging system. The circle function is a good model for such apertures and it's Fourier transform can be important when attempting to derive a mathematical model for the imaging systems. Given that the circle function can be written mathematically as

$$\mathrm{circ}\left(\sqrt{x^2+y^2}\right) = \begin{cases} 1, & \text{when } x^2+y^2 < 1; \\ 1/2, & \text{when } x^2+y^2 = 1; \\ 0, & \text{otherwise} \end{cases}$$

Show that the Fourier transform of the circle function is

$$\mathcal{F}\left\{\mathrm{circ}\left(\sqrt{x^2+y^2}\right)\right\} = \frac{2\pi}{\rho} J_1(\rho) \frac{2\pi}{\sqrt{\omega_x^2+\omega_y^2}} J_1\left(\sqrt{\omega_x^2+\omega_y^2}\right),$$

where $J_1(x)$ is the first order Bessel function of the first kind, given by

$$\int_0^x \xi J_0(\xi)\,\mathrm{d}\xi = xJ_1(x),$$

and $J_0(a) = \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{j}a\cos(\theta-\phi)}\,\mathrm{d}\theta$ is the zero order Bessel function of the first kind. [Hint: Use the substitutions $x = r\cos(\theta)$, $y = r\sin(\theta)$, $\omega_x = \rho\cos(\phi)$, and $\omega_y = \rho\sin(\phi)$.]