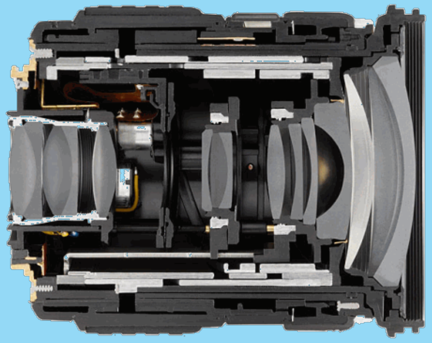
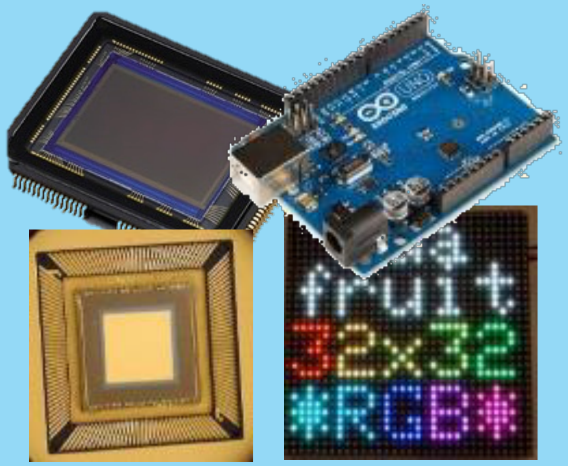


Optics

+



Sensors
&
devices

+

Signal
processing
&
algorithms

COMPUTATIONAL METHODS FOR IMAGING (AND VISION)

LECTURE 11: LSI IMAGING SYSTEMS

PROF. JOHN MURRAY-BRUCE

LINEAR SPACE INVARIANT SYSTEMS & CONVOLUTION

- Take FT, manipulate frequency spectrum of signal, and then take the inverse FT
  - **Linearity:** $\mathscr{F}\{f(x) + g(x)\} = \mathscr{F}\{f(x)\} + \mathscr{F}\{g(x)\} = F(\omega) + G(\omega)$
  - **Shift:** $\mathscr{F}\{f(x - x_0)\} = e^{-\mathrm{j}\omega x_0}\mathscr{F}\{f(x)\} = e^{-\mathrm{j}\omega x_0}F(\omega)$

- **Natural next question is, what about multiplication?**
  - $\mathscr{F}\{f(x)\}\,\mathscr{F}\{g(x)\} = F(\omega)\,G(\omega)$ **= ?**

Suppose the problem is solved and see what has to have happened!

A STAR IS BORN

CONVOLUTION

CONVOLUTION

- **Convolution:** measures overlap of $g(x)$ with another function $h(x)$ as it is (reversed and) shifted over $g(x)$

$$g(x) \star h(x) = \int_{-\infty}^{\infty} g(x')h(x - x')dx'$$

- For linear shift-invariant systems, the convolution integral relates the output g_{out} of the system to any input g_{in} to that system
- **Note:** the convolution of two functions is itself a "new" function
- The FT of the convolution of two functions in space is the product of their respective FTs, i.e. $g(x) \star h(x) \longleftrightarrow G(\omega)H(\omega)$

LINEAR SPACE INVARIANT SYSTEMS & CONVOLUTION

CONVOLUTION INTEGRAL

- **What about multiplication of spectra?**
  - $\mathscr{F}\{f(x)\}\,\mathscr{F}\{g(x)\} = F(\omega)\,G(\omega) = \mathscr{F}\{f(x) \star g(x)\}$

$$f(x) \star g(x) = \int_{-\infty}^{+\infty} f(x')g(x-x')\,\mathrm{d}x'$$

  - *(And a "star" is born!)*
- **Convolution in the space/time domain is multiplication in frequency!**
  - A very important FT property: $f(x) \star g(x) \longleftrightarrow F(\omega)\,G(\omega)$
  - Computing product in frequency domain is much faster than computing the convolution integral in the time/spatial domain

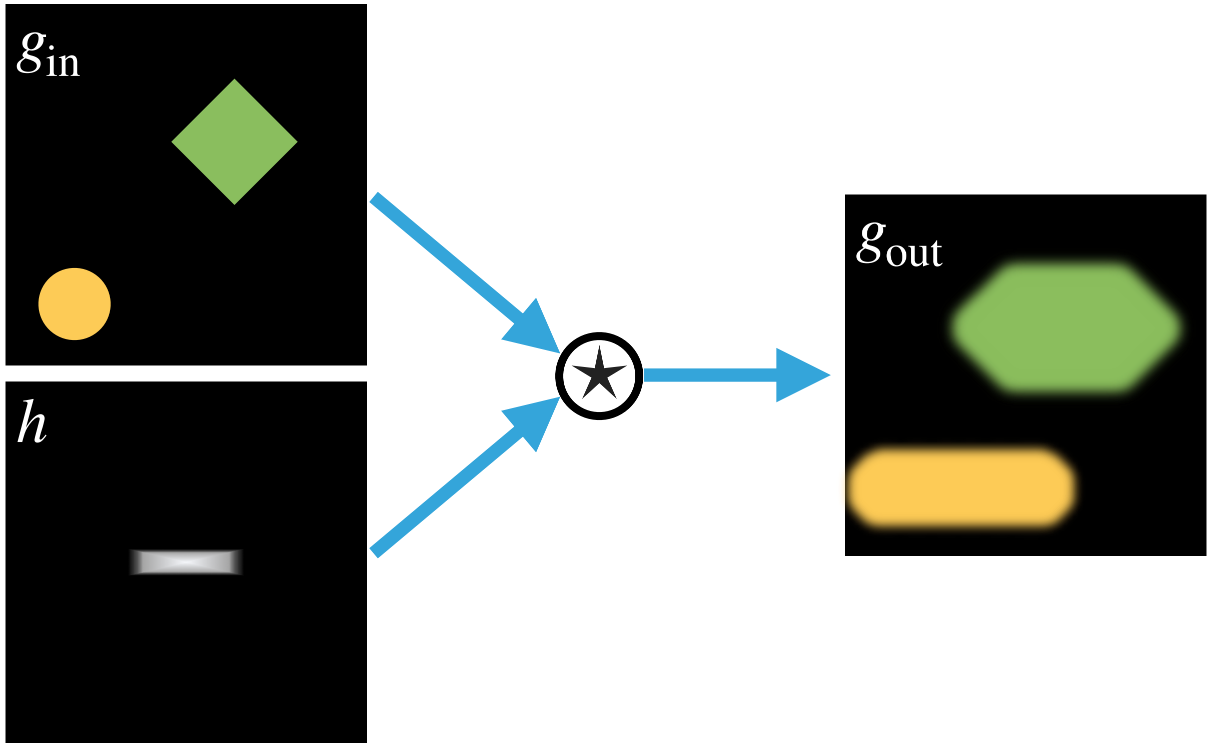
LINEAR SPACE INVARIANT IMAGING SYSTEMS & CONVOLUTION

- **LSI system:** Linear in the input and also shift invariant
  - Input described by function $g_{\text{in}}(x)$ and system by $h(x)$
- **Convolution integral (1D):**

$$g_{\text{out}}(x) \star h(x) = \int_{-\infty}^{+\infty} g_{\text{in}}(x')h(x - x')\,\mathrm{d}x'$$

  - Relates input and output of LSI imaging system
  - Convolution integral is linear in input and shift-invariant

For an imaging system, h is called the **Point Spread Function (PSF)**



LINEAR SPACE INVARIANT SYSTEMS & CONVOLUTION

2D CONVOLUTION INTEGRAL

- **2D Convolution:**

$$f(x,y) \star g(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x',y')g(x-x',y-y')\,\mathrm{d}x'\mathrm{d}y'$$

- **FT Property:** Convolution of two functions in space is equivalent to multiplication of their frequency spectra

$$f(x,y) \star g(x,y) \longleftrightarrow F(\omega_x,\omega_y)\,G(\omega_x,\omega_y)$$

WHERE ARE WE

WE ARE HERE!

Week	Date	Main Topic	Lecture	Readings	Homework Out	Homework Due
1	11-Jan-21		Introduction to computational imaging - Forward and Inverse problems - Common computational imaging problems			
	13-Jan-21		Vectors - Preliminaries			
2	18-Jan-21		**Dr. Martin Luther King, Jr. Holiday (no class)**			
	20-Jan-21	Mathematical preliminaries	Vectors and Vector Spaces - Subspaces, Finite dimensional spaces	IIP Appendix A; FSP 2.1 - 2.2		
3	25-Jan-21		Vector Spaces - Hilbert spaces	IIP Appendix B; FSP 2.3		
	27-Jan-21		Bases and Frames I - Orthonormal and Reisz Bases	IIP Appendix C; FSP 2.4 and 2.B	**HW 1**	
4	1-Feb-21		Bases and Frames II - Orthogonal Bases - Linear operators	IIP Appendix C; FSP 2.5 and 2.B		
	3-Feb-21		Fourier Analysis I - FT (1D and 2D) - FT properties	IIP 2.1, Appendix D; FSP 4.4		
5	8-Feb-21		Sampling and Interpolation - BL functions - Sampling	IIP 2.2, 2.3; FSP 5.4, 5.5		**HW 1**
	10-Feb-21		Fourier Analysis II (DFT)	IIP 2.4; FSP 3.6	**HW 2**	
6	15-Feb-21	Forward Modeling	LSI imaging: Forward problem I - Convolution	IIP 2.5 - 2.6, 3		
	17-Feb-21		LSI imaging: Forward problem I - Transfer functions	IIP 2.6		
7	22-Feb-21		LSI imaging: Forward problem I - Linear operators	IIP 3		
	24-Feb-21		LSI imaging: Forward problem I - Linear operators, Adjoints, and Inverses		**HW 3**	**HW 2**
8	1-Mar-21		**Mid-term Exams**			
	3-Mar-21		LSI imaging: Forward problem II - Sampling and Discretization: Matrix-vector form	IIP 2.7, 4		
9	8-Mar-21		LSI imaging: Forward problem II - Convolution matrix			
	10-Mar-21		LSI imaging: Forward problem II - Sampling and Discretization: Matrix-vector form - PSF, and Transfer functions			**HW3**

OUTLINE

- Computing continuous convolution
  - Examples

LEARNING GOALS

- To be able to identify LSI systems
- Identify the convolution integral
- Compute 1D and 2D convolutions

READING

- IIP 2.5 - 2.6
- IIP 3.1 - 3.3

QUICK SUMMARY

LINEAR SPACE INVARIANT SYSTEMS
&
CONVOLUTION

LINEAR SPACE INVARIANT SYSTEM

Linear: so the distributive property, $H(f+g) = H(f) + H(g)$, holds.

Space invariant: operator is the same at any point in space. H does not depend on (x, y).

Convolution of system response with input, gives the system output.

CONVOLUTION

Convolution:

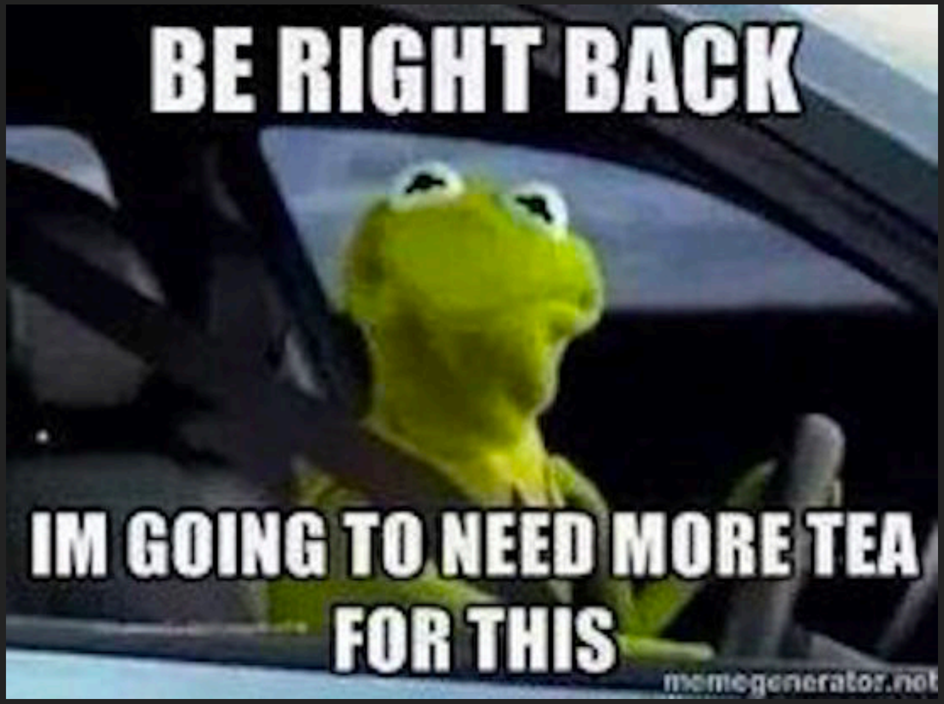
$$g(x,y) \star h(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x',y')h(x-x',y-y')\,\mathrm{d}x'\mathrm{d}y'$$

Convolution integral relates the output g_{out} of the system h to any input g_{in} to that system.

The **convolution** of two functions is itself a **"new" function**.

The FT of the convolution of two functions in space is the product of their respective FTs, i.e.:

$$g(x) \star h(x) \longleftrightarrow G(\omega)H(\omega)$$



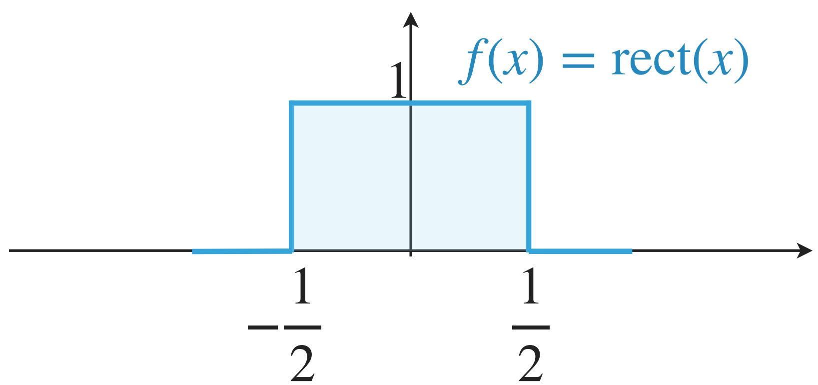
EXAMPLES

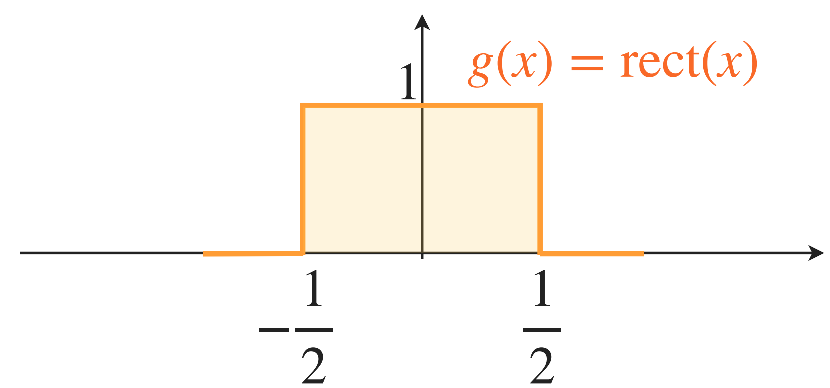
CONVOLUTION

CONVOLUTION

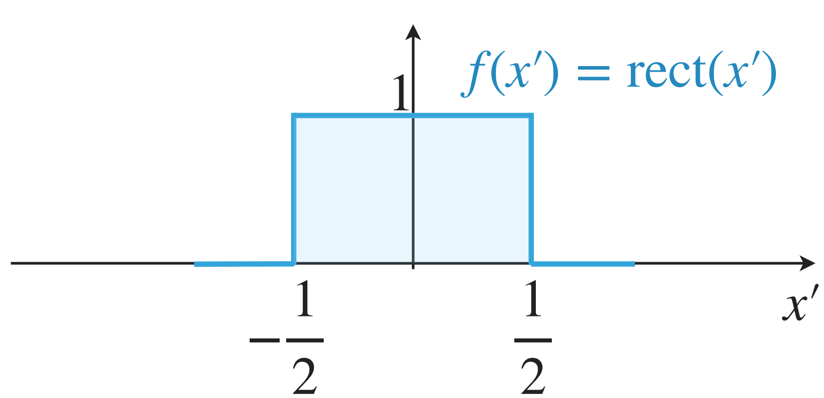
EXAMPLE 1: CONVOLUTION OF BOX-CAR FUNCTION WITH ITSELF

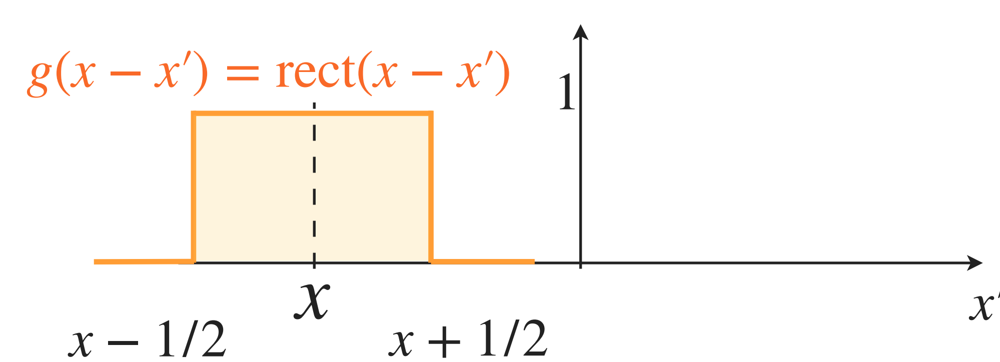
- Compute the convolution between $f(x) = \mathrm{rect}(x)$ and $g(x) = \mathrm{rect}(x)$





1. Convolution formula: $f(x) \star g(x) = \int_{-\infty}^{\infty} f(x')\, g(x - x')\, \mathrm{d}x'$
2. Plot $f(x') = \mathrm{rect}(x')$ and $g(x - x') = \mathrm{rect}(x - x') = \mathrm{rect}(-(x' - x))$





CONVOLUTION

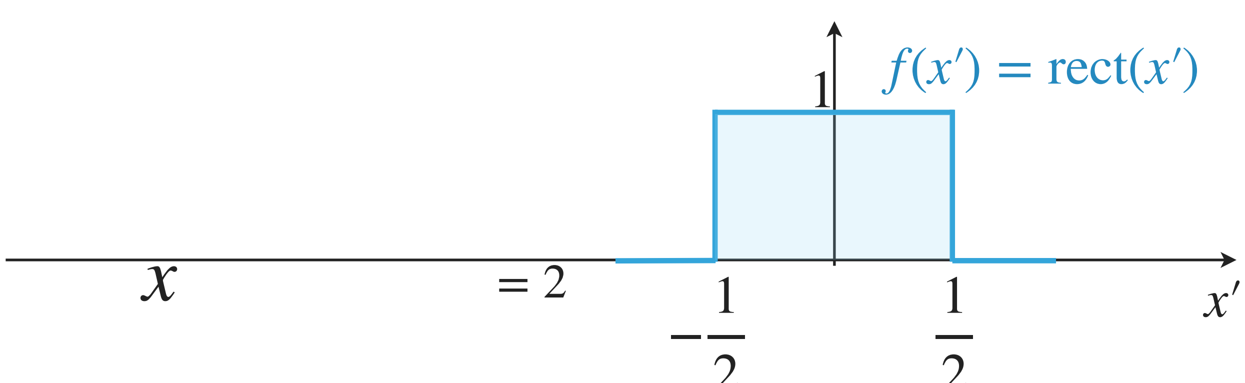
EXAMPLE 1: CONVOLUTION OF BOX-CAR FUNCTION WITH ITSELF

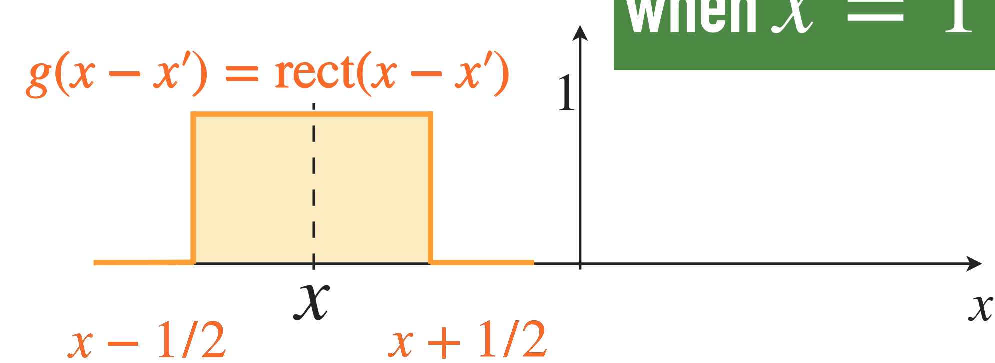
1. **Convolution formula:** $f(x) \star g(x) = \int_{-\infty}^{\infty} f(x')\, g(x - x')\, \mathrm{d}x'$
2. **Plot:** $f(x') = \mathrm{rect}(x')$ and $g(x - x') = \mathrm{rect}(x - x') = \mathrm{rect}(-(x' - x))$
3. **Limits of integral:** $f(x) \star g(x) = \int_{-\infty}^{\infty} \mathrm{rect}(x')\, \mathrm{rect}(x - x')\, \mathrm{d}x'$

$f(x) \star g(x) = \int_{-1/2}^{x+(1/2)} 1\, \mathrm{d}x'$, when $-1 \leq x \leq 0$.

Similarly, $f(x) \star g(x) = \int_{x-1/2}^{(1/2)} 1\, \mathrm{d}x'$, when $0 < x \leq 1$.

They first overlap when $x = 1$





CONVOLUTION

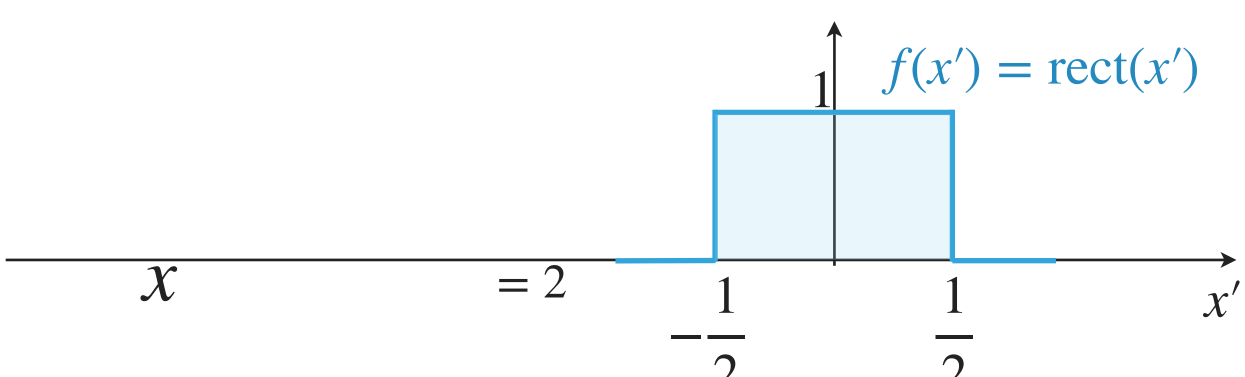
EXAMPLE 1: CONVOLUTION OF BOX-CAR FUNCTION WITH ITSELF

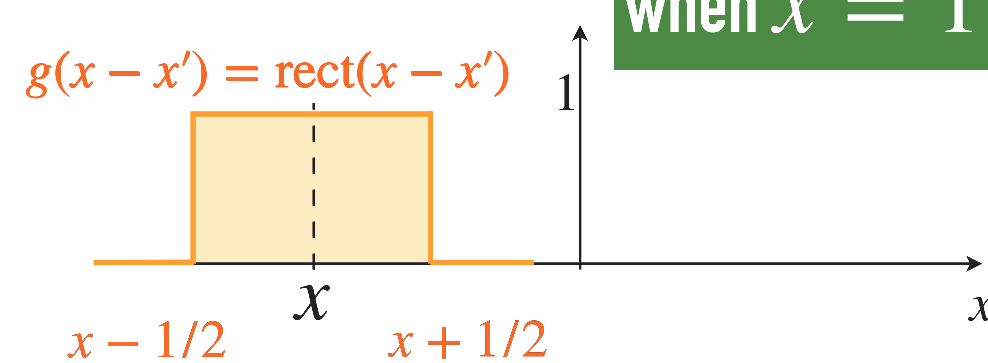
5. We thus have that:

$$f(x) \star g(x) = \begin{cases} \int_{-1/2}^{x+(1/2)} 1 \, \mathrm{d}x', & \text{when } -1 \le x \le 0; \\ \int_{x-(1/2)}^{1/2} 1 \, \mathrm{d}x', & \text{when } -0 < x \le 1; \\ 0 & \text{otherwise.} \end{cases}$$

$$f(x) \star g(x) = \begin{cases} x + 1, & \text{when } -1 \le x \le 0; \\ 1 - x, & \text{when } -0 < x \le 1; \\ 0 & \text{otherwise.} \end{cases}$$

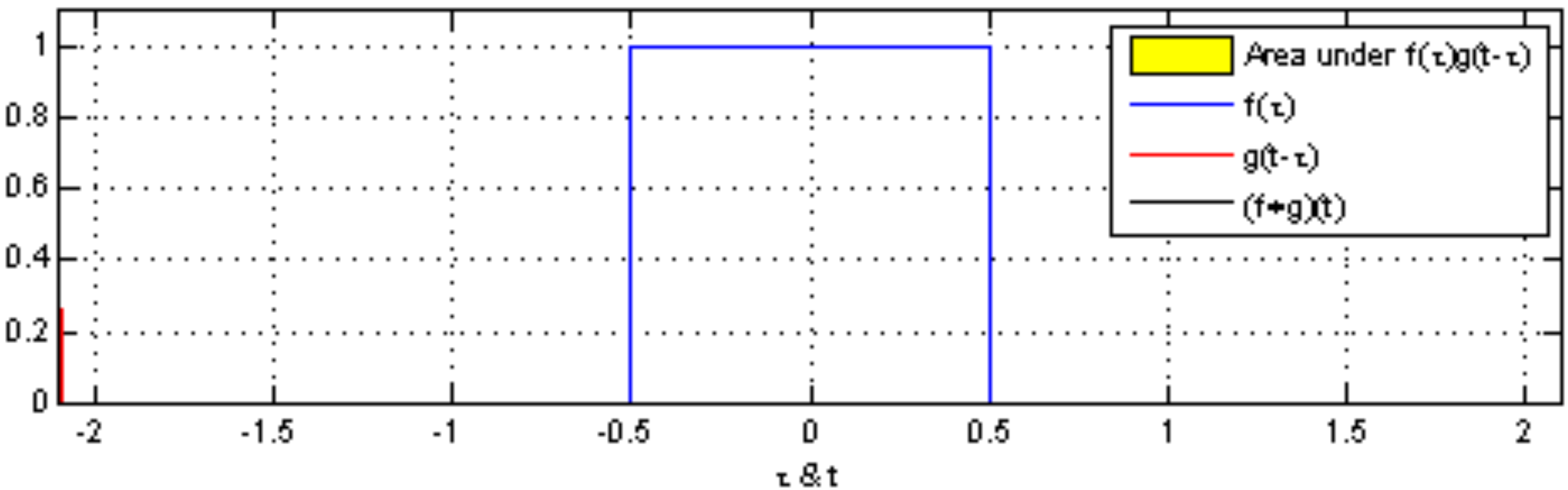
They first overlap when $x = 1$





CONVOLUTION

EXAMPLE 1: CONVOLUTION OF BOX-CAR FUNCTION WITH ITSELF (GRAPHICAL)



Animated

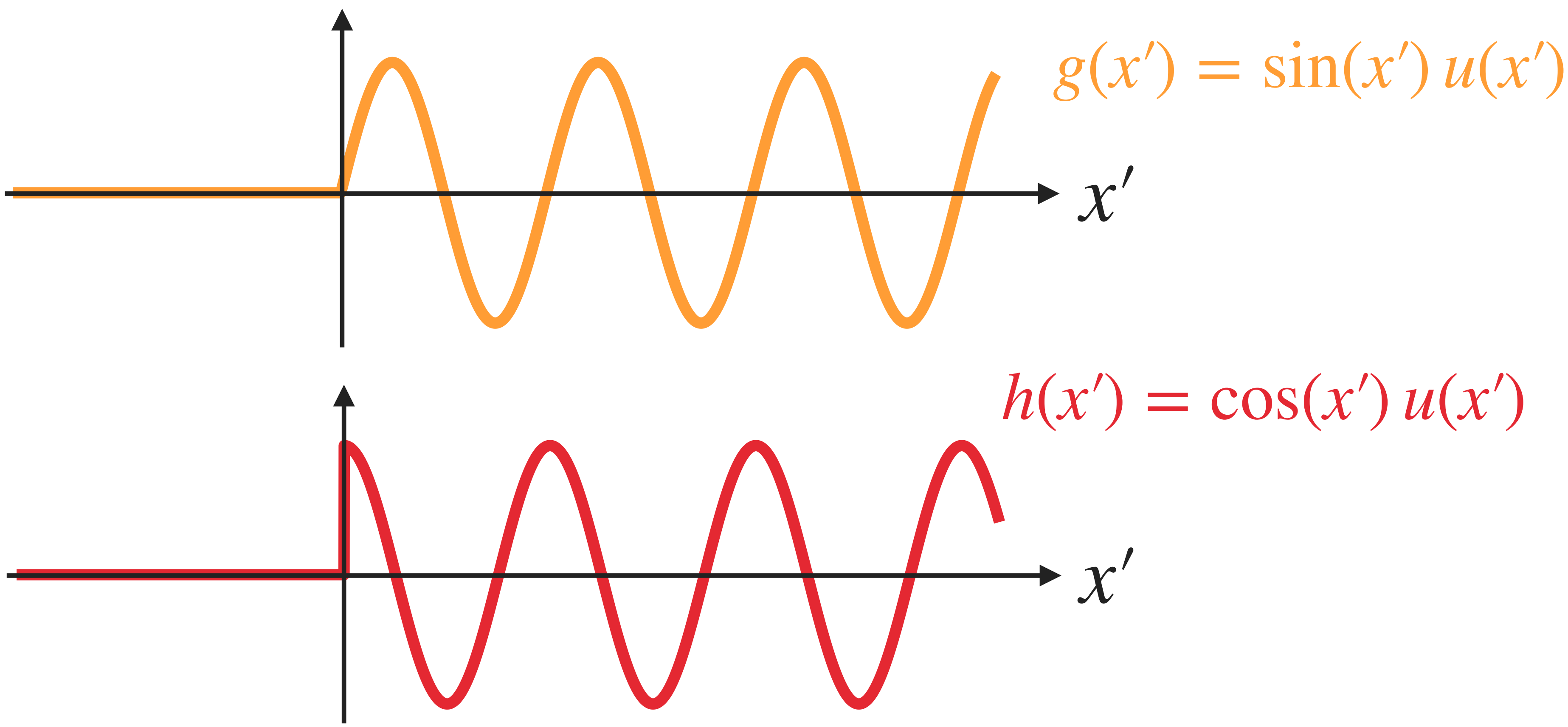
Wikipedia

CONVOLUTION

EXAMPLE 2

- Convolve $g(x) = \sin(x)\, u(x)$ and $h(x) = \cos(x)\, u(x)$
- **Solution:** Start from the definition of the convolution integral

$$\int_{-\infty}^{\infty} g(x')h(x-x')\,\mathrm{d}x' = \int_{-\infty}^{\infty} \sin(x')u(x')\ \cos(x-x')u(x-x')\,\mathrm{d}x'$$



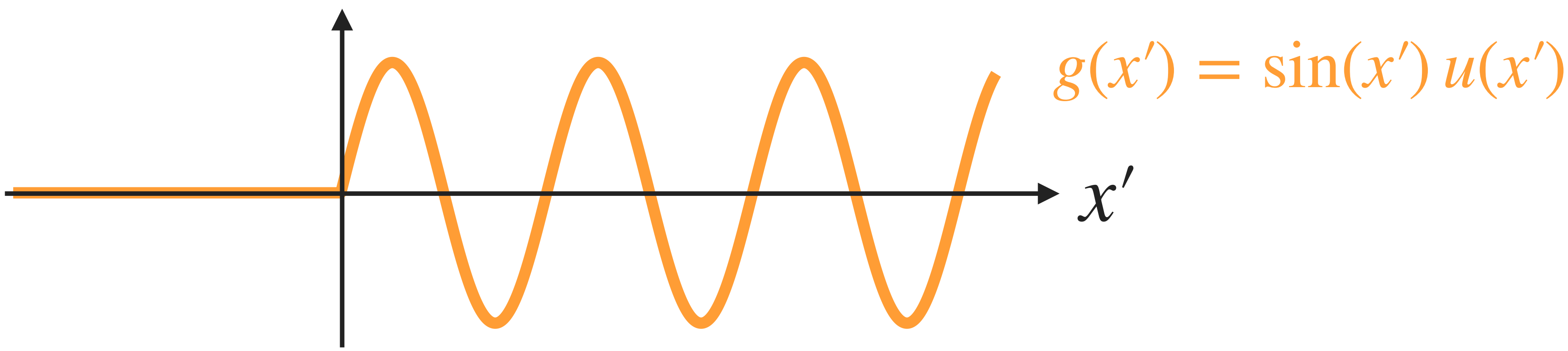
Replace x' in $h(x')$ with $-x'$

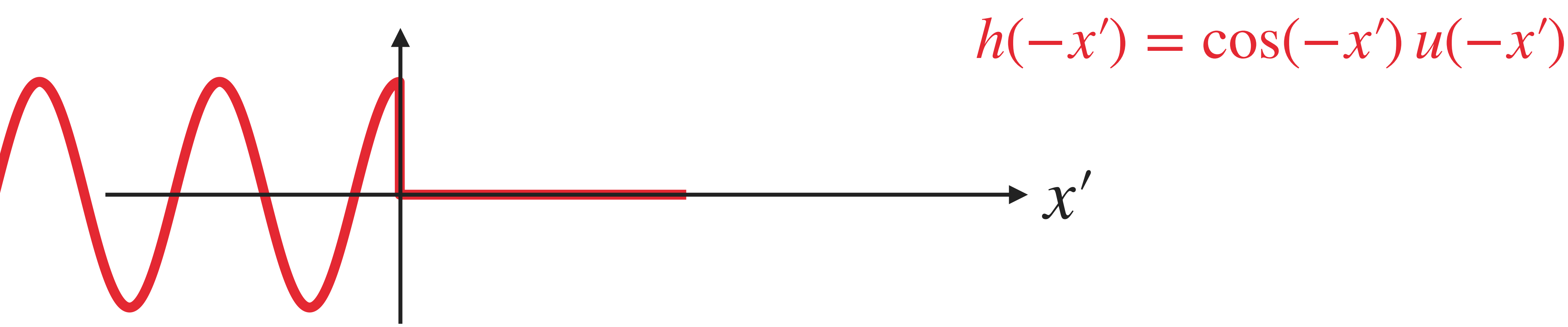
CONVOLUTION

EXAMPLE 2

- Convolve $g(x) = \sin(x)\,u(x)$ and $h(x) = \cos(x)\,u(x)$
- **Solution:** Start from the definition of the convolution integral

$$\int_{-\infty}^{\infty} g(x')h(x-x')\,\mathrm{d}x' = \int_{-\infty}^{\infty} \sin(x')u(x')\;\cos(x-x')u(x-x')\,\mathrm{d}x'$$





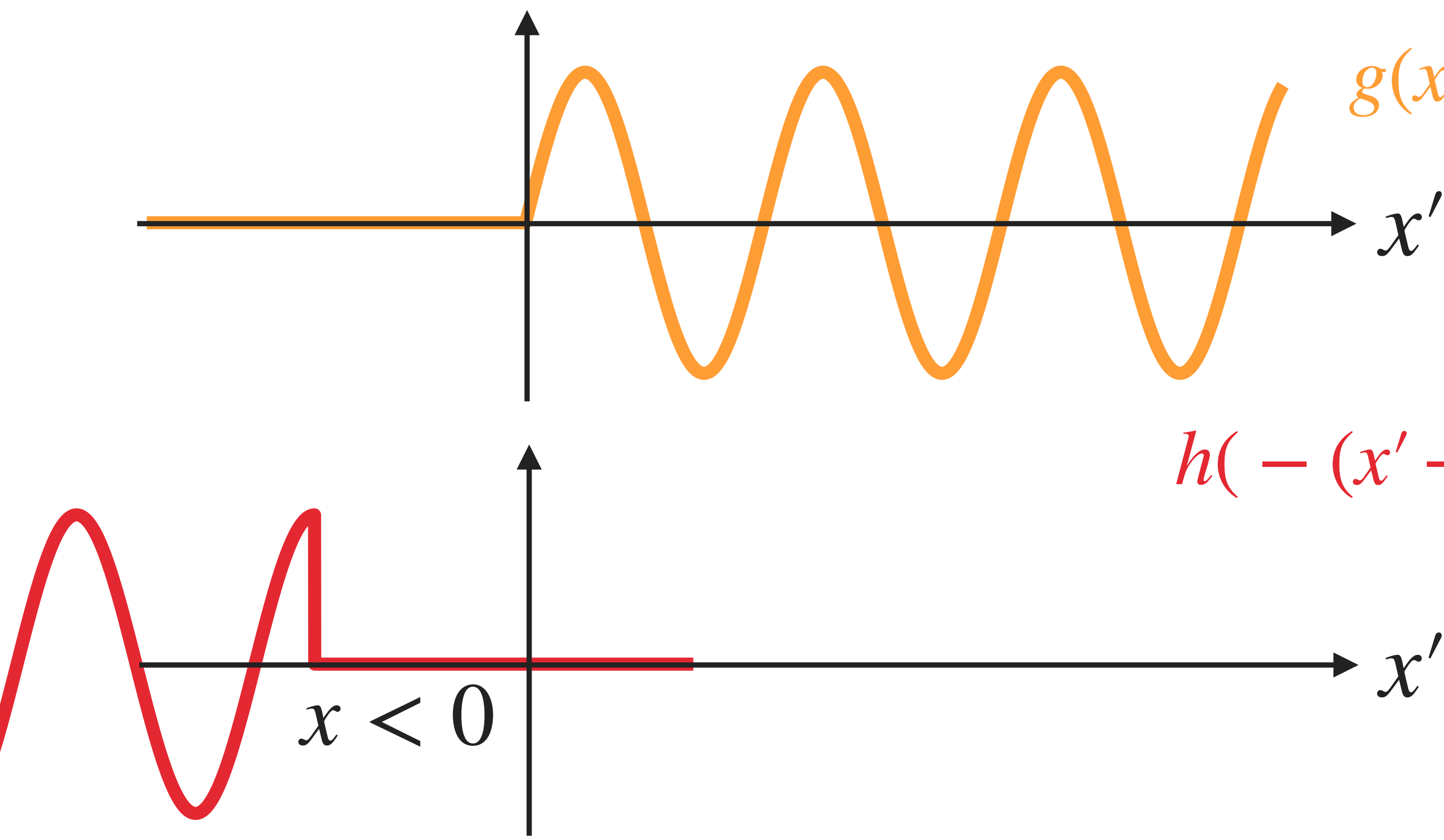
Replace x' in $h(-x')$ with $x' - x$

CONVOLUTION

EXAMPLE 2

- Convolve $g(x) = \sin(x)\, u(x)$ and $h(x) = \cos(x)\, u(x)$
- **Solution:** Start from the definition of the convolution integral

$$\int_{-\infty}^{\infty} g(x')h(x-x')\,\mathrm{d}x' = \int_{-\infty}^{\infty} \sin(x')u(x')\;\cos(x-x')u(x-x')\,\mathrm{d}x'$$



$$g(x') = \sin(x')\, u(x')$$

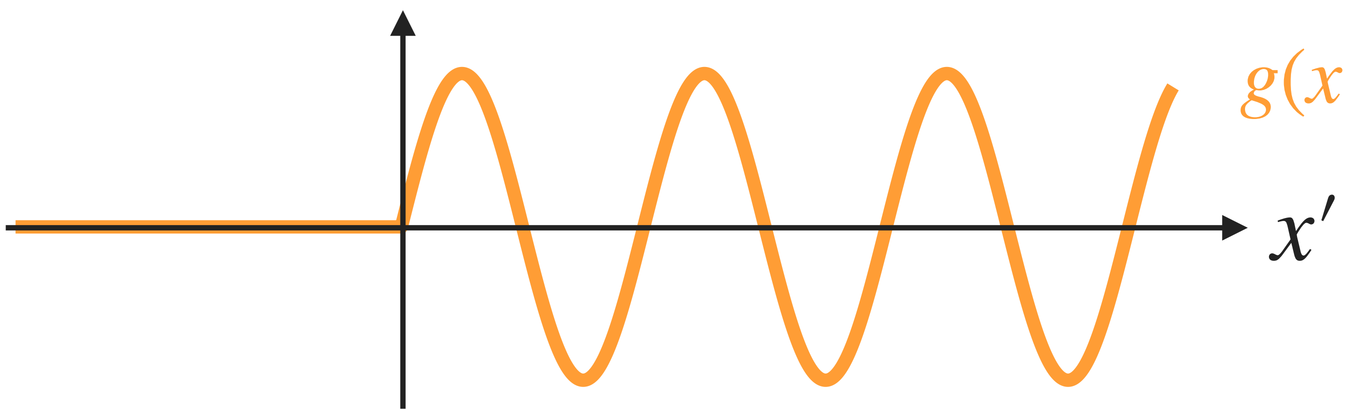
$$\begin{aligned} h(-(x'-x)) &= h(x-x') \\ &= \cos(-(x'-x))\, u(-(x'-x)) \\ &= \cos(x-x')\, u(x-x') \end{aligned}$$

CONVOLUTION

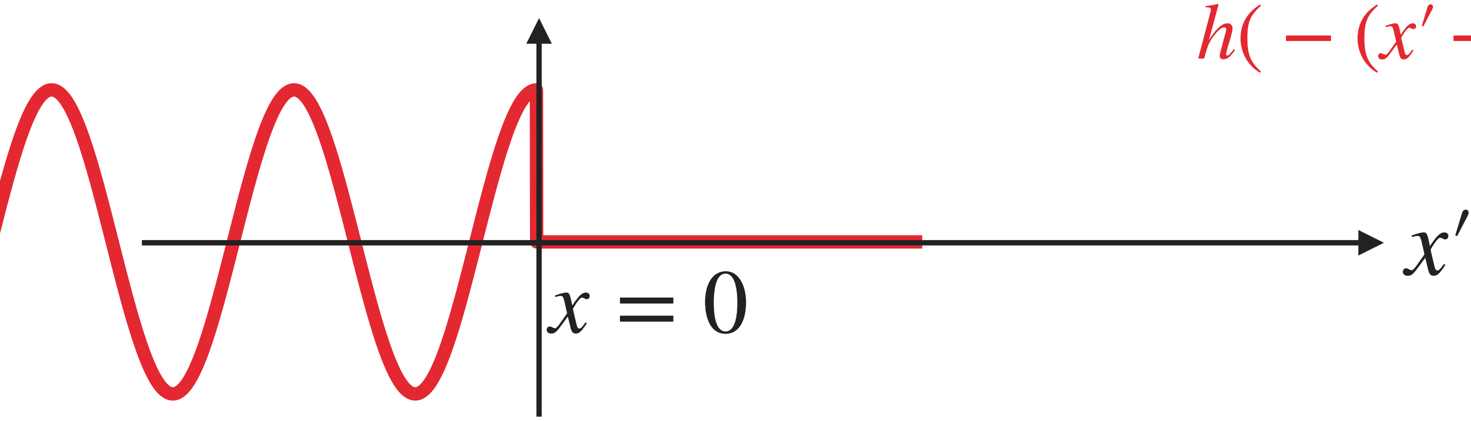
EXAMPLE 2

- Convolve $g(x) = \sin(x)\, u(x)$ and $h(x) = \cos(x)\, u(x)$
- **Solution:** Start from the definition of the convolution integral

$$\int_{-\infty}^{\infty} g(x')h(x-x')\,\mathrm{d}x' = \int_{-\infty}^{\infty} \sin(x')u(x')\ \cos(x-x')u(x-x')\,\mathrm{d}x'$$



$g(x') = \sin(x')\, u(x')$



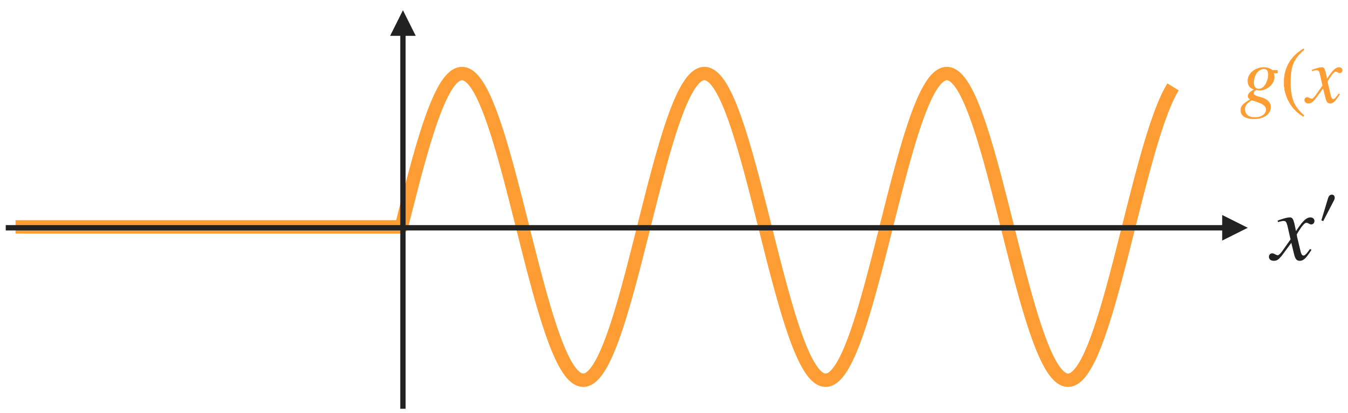
$$\begin{aligned} h(-(x'-x)) &= h(x-x') \\ &= \cos(-(x'-x))\, u(-(x'-x)) \\ &= \cos(x-x')\, u(x-x') \end{aligned}$$

CONVOLUTION

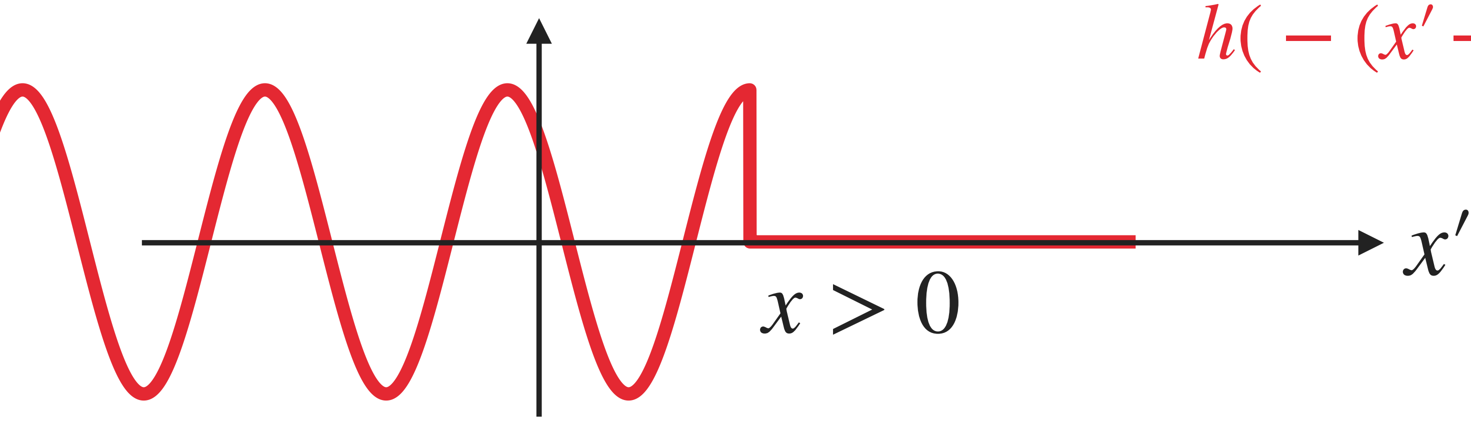
EXAMPLE 2

- Convolve $g(x) = \sin(x)\,u(x)$ and $h(x) = \cos(x)\,u(x)$
- **Solution:** Start from the definition of the convolution integral

$$\int_{-\infty}^{\infty} g(x')h(x-x')\,\mathrm{d}x' = \int_{-\infty}^{\infty} \sin(x')u(x')\ \cos(x-x')u(x-x')\,\mathrm{d}x'$$



$$g(x') = \sin(x')\,u(x')$$



$$\begin{aligned} h(-(x'-x)) &= h(x-x') \\ &= \cos(-(x'-x))\,u(-(x'-x)) \\ &= \cos(x-x')\,u(x-x') \end{aligned}$$

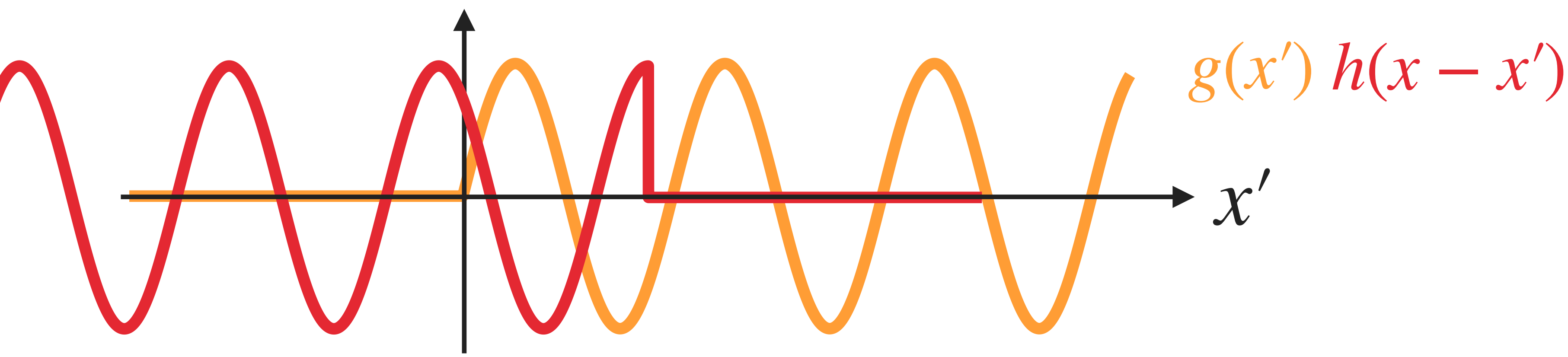
Multiply both functions

CONVOLUTION

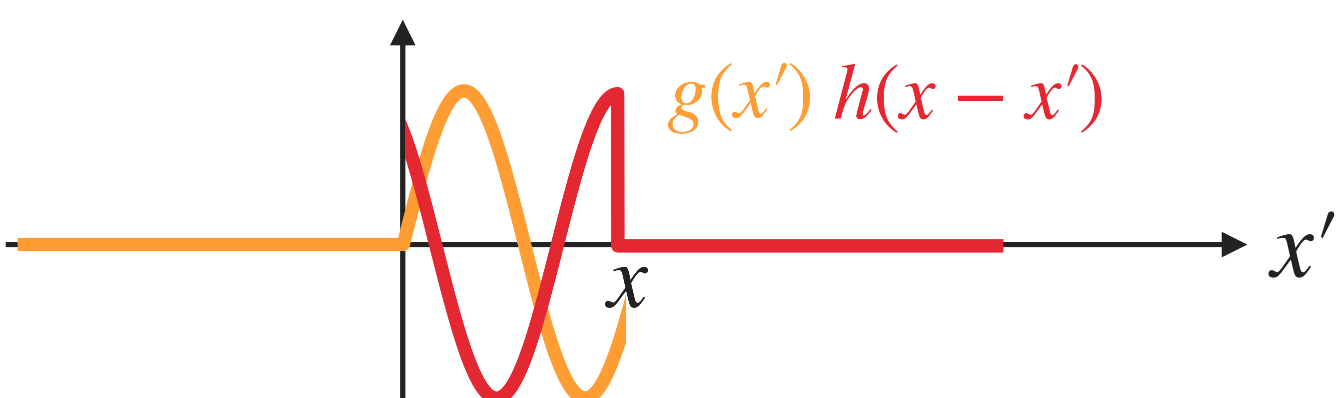
EXAMPLE 2

- Convolve $g(x) = \sin(x)\, u(x)$ and $h(x) = \cos(x)\, u(x)$
- **Solution:** Start from the definition of the convolution integral

$$\int_{-\infty}^{\infty} g(x')h(x-x')\,\mathrm{d}x' = \int_{-\infty}^{\infty} \sin(x')u(x')\;\cos(x-x')u(x-x')\,\mathrm{d}x'$$



$x > 0$



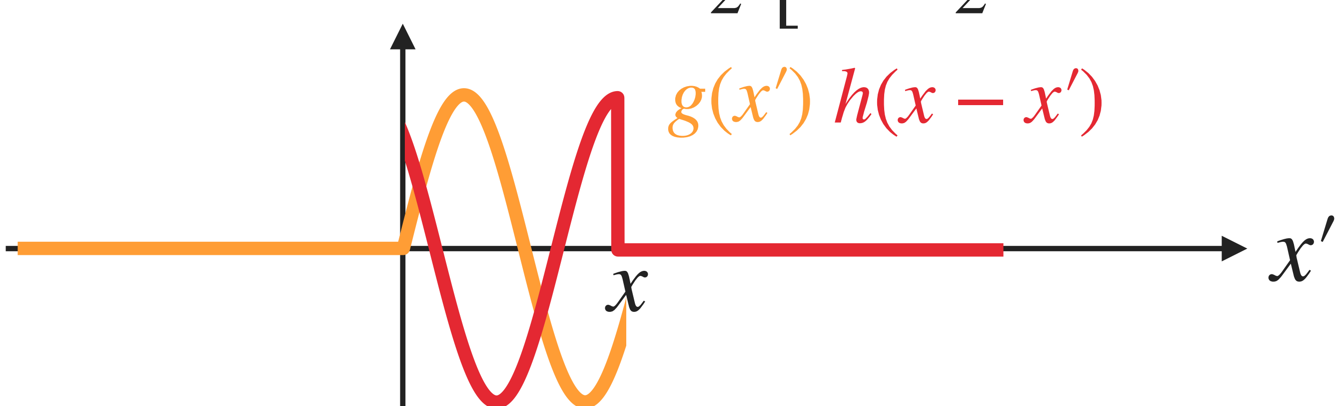
This gives us the limits for our integral

CONVOLUTION

EXAMPLE 2

- Convolve $g(x) = \sin(x)\, u(x)$ and $h(x) = \cos(x)\, u(x)$
- **Solution:** Start from the definition of the convolution integral

$$
\begin{aligned}
\int_{-\infty}^{\infty} g(x')h(x-x')\,\mathrm{d}x' &= \int_{-\infty}^{\infty} \sin(x')u(x')\ \cos(x-x')u(x-x')\,\mathrm{d}x' \\
&= \int_{0}^{x} \sin(x')u(x')\ \cos(x-x')u(x-x')\,\mathrm{d}x' \\
&= \int_{0}^{x} \sin(x')\ \cos(x-x')\,\mathrm{d}x' \\
&= \frac{1}{2}\int_{0}^{x} \sin(x-2x') + \sin(x)\,\mathrm{d}x' \\
&= \frac{1}{2}\left[\frac{\cos(x-2x')}{2} + x'\sin(x)\right]_{0}^{x} = \frac{1}{2}x\sin(x)
\end{aligned}
$$



WHAT WE COVERED TODAY

- **Convolution (definition)**
- Examples

NEXT TIME!

MORE LSI SYSTEMS