CIS 4930.006S20/CIS 6930.013S20: Computational Methods for Imaging and Vision

Spring 2020
Solutions to Homework #1

The University of South Florida
Department of Computer Science and Engineering
Tampa, FL

Assigned: January 29, 2020
Due: February 10, 2020

1 Gaussian Elimination

Our aim is to solve the system of linear equations $\mathbf{A}\mathbf{x} = \mathbf{y}$. (General conditions for the existence of a solution are given in **FSP Appendix 2.B.1**.) Comment on whether a solution to each of the following systems of equations exists, and, if it does, find it.

(a)

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 \\ 4 & 5 & 2 \\ -1 & -1 & 2 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 10 \\ 20 \\ 3 \end{bmatrix}.$$

(b)

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 4 & 5 & 8 \\ -1 & -1 & -2 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 7 \\ 38 \\ -9 \end{bmatrix}.$$

(c)

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 4 & 5 & 8 \\ -1 & -1 & -2 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Solution

(a) The steps of Gaussian elimination yield

$$A' = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 5 & -10 \\ -1 & -1 & 2 \end{bmatrix}, \qquad y' = \begin{bmatrix} 10 \\ -20 \\ 3 \end{bmatrix}.$$

$$A^{(1)} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 5 & -10 \\ 0 & -1 & 5 \end{bmatrix}, \qquad y^{(1)} = \begin{bmatrix} 10 \\ -20 \\ 13 \end{bmatrix}.$$

$$A^{(2)} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 5 & -10 \\ 0 & 0 & 3 \end{bmatrix}, \qquad y^{(2)} = \begin{bmatrix} 10 \\ -20 \\ 9 \end{bmatrix}.$$

Since we have reached an upper-triangular $A^{(2)}$ with all diagonal entries nonzero, the system has a unique solution. (Any $y^{(2)}$ belongs to the range of $A^{(2)}$.) By back substitution, we solve $x_2 = 3$, $x_1 = 2$, and $x_0 = 1$.

(b) The steps of Gaussian elimination yield

$$A' = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & 0 \\ -1 & -1 & -2 \end{bmatrix}, \qquad y' = \begin{bmatrix} 7 \\ 10 \\ -9 \end{bmatrix}.$$

$$A^{(1)} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \qquad y^{(1)} = \begin{bmatrix} 7 \\ 10 \\ -2 \end{bmatrix}.$$

$$A^{(2)} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad y^{(2)} = \begin{bmatrix} 7 \\ 10 \\ 0 \end{bmatrix}.$$

Since we have reached an upper-triangular $A^{(2)}$ with a zero on the diagonal, we have either infinitely many solutions or no solutions. It is easy to see that the range of $A^{(2)}$ is the vectors in $\mathbb{R}^3$ with third component equal to zero, and the vector $y^{(2)}$ is in that range space. Those solutions are described by $x_1 = 2$ and $x_0 + 2x_2 = 7$.

(c) The steps of Gaussian elimination yield

$$A' = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & 0 \\ -1 & -1 & -2 \end{bmatrix}, \qquad y' = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}.$$

$$A^{(1)} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \qquad y^{(1)} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}.$$

$$A^{(2)} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad y^{(2)} = \begin{bmatrix} 1 \\ -2 \\ 18/5 \end{bmatrix}.$$

Like in part (b), we have either infinitely many solutions or no solutions. Again, the range of $A^{(2)}$ is the vectors in $\mathbb{R}^3$ with third component equal to zero, but this time the vector $y^{(2)}$ is not in that range space. Thus, there are no solutions.

2 Eigenvalues and Eigenvectors

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}.$$

(a) Find eigenvalues and unit-norm eigenvectors of $\mathbf{A}$. Are the eigenvectors orthogonal? Check your answer with a computer (e.g, Matlab or Python[1]).

(b) Compute the determinant of $\mathbf{A}$, i.e. $\det \mathbf{A}$. Is $\mathbf{A}$ invertible? If it is, give its inverse; if not, say why.

(c) Find eigenvalues and unit-norm eigenvectors of $\mathbf{B}$. For $\alpha \in \{0, 1, 2, 3\}$ and $\beta \in [-3, 3]$, plot the eigenvalues of $\mathbf{B}$ (with a computer). (This will be four pairs of curves that are functions of one variable.)

(d) Compute the determinant of $\mathbf{B}$. When is $\mathbf{B}$ invertible? For $(\alpha, \beta) \in [0, 5]^2$, plot $\det \mathbf{B}$ (with a computer, using Matlab or Python). (This will be a surface plot of a function of two variables.)

Solution

(a) The characteristic polynomial of A is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{bmatrix} = (\lambda - 1)^2 - 4 = (\lambda - 3)(\lambda + 1).$$

The eigenvalues of A are the roots of the characteristic polynomial: $\lambda_0 = -1$ and $\lambda_1 = 3$.

For $\lambda_0 = -1$, we solve $Ax = -x$,

$$\begin{aligned} x_0 + 2x_1 &= -x_0, \\ 2x_0 + x_1 &= -x_1, \end{aligned}$$

yielding $x_1 = -x_0$. We choose $x_0 = 1$, $x_1 = -1$ and normalize. The eigenvector associated with λ_0 is thus $v_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \end{bmatrix}^\mathsf{T}$.

Similarly, for $\lambda_1 = 3$, we solve $Ax = 3x$,

$$\begin{aligned} x_0 + 2x_1 &= 3x_0, \\ 2x_0 + x_1 &= 3x_1, \end{aligned}$$

yielding $x_0 = x_1$. We choose $x_0 = x_1 = 1$ and normalize. The eigenvector associated with λ_1 is thus $v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix}^\mathsf{T}$.

The eigenvectors are orthogonal.

The following Matlab code verifies these answers:

```
A = [1, 2; 2, 1];

% Calculate eigenvalues (D) and eigenvectors (V):
[V,D] = eig(A);

% Columns of V are the eigenvectors:
v0 = V(:,1)
v1 = V(:,2)

% Inner product of eigenvectors:
inner_product = v0' * v1
```

[1] For Python, you will need `numpy`.

(b) $\det A = A_{1,1}A_{2,2} - A_{1,2}A_{2,1} = -3$, which is equivalent to the product of the eigenvalues. As the determinant is nonzero, A is an invertible matrix, and its inverse is:

$$A^{-1} = -\tfrac{1}{3}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}.$$

(c) The characteristic polynomial of B is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - \alpha & -\beta \\ -\beta & \lambda - \alpha \end{bmatrix} = (\lambda - \alpha)^2 - (-\beta^2) = (\lambda - \alpha - \beta)(\lambda - \alpha + \beta).$$

The eigenvalues of A are $\lambda_0 = \alpha - \beta$ and $\lambda_1 = \alpha + \beta$.
For $\lambda_0 = \alpha - \beta$, we solve $Ax = (\alpha - \beta)x$,

$$\begin{aligned} \alpha x_0 + \beta x_1 &= (\alpha - \beta)x_0, \\ \beta x_0 + \alpha x_1 &= (\alpha - \beta)x_1, \end{aligned}$$

yielding $x_1 = -x_0$. We choose $x_0 = 1$, $x_1 = -1$ and normalize. The eigenvector associated with λ_0 is thus $v_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \end{bmatrix}^\mathsf{T}$.
Similarly, for $\lambda_1 = \alpha + \beta$, we solve $Ax = (\alpha + \beta)x$,

$$\begin{aligned} \alpha x_0 + \beta x_1 &= (\alpha + \beta)x_0, \\ \beta x_0 + \alpha x_1 &= (\alpha + \beta)x_1, \end{aligned}$$

yielding $x_0 = x_1$. We choose $x_0 = x_1 = 1$ and normalize. The eigenvector associated with λ_1 is thus $v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix}^\mathsf{T}$.
Using these results we can plot the eigenvalues of B, as shown in Figure 1. When $\beta = 0$, the matrix B is diagonal and so the eigenvalues simply take on the value of α, due to the eigenvectors having unit norm. The following Matlab code generates Figure 1:

```
alpha = [0, 1, 2, 3];
beta = -3:3;

for a = alpha, % for each value alpha
    for b = beta, % for each value beta
        % Calculate eigenvalues (using analytical solution we found)
        eigval_1(b+4,a+1) = a-b;
        eigval_2(b+4,a+1) = a+b;
    end
end

figure;
plot( beta, eigval_1(:,1), 'r' )
hold on
plot( beta, eigval_2(:,1), 'r-.')
plot( beta, eigval_1(:,2), 'b' )
plot( beta, eigval_2(:,2), 'b-.' )
plot( beta, eigval_1(:,3), 'g' )
plot( beta, eigval_2(:,3), 'g-.' )
plot( beta, eigval_1(:,4), 'y' )
plot( beta, eigval_2(:,4), 'y-.' )
title( 'Eigenvalues of B = [\alpha, \beta; \beta, \alpha] for \alpha = {0,1,2,3}' )
xlabel( '\beta' )
ylabel( 'Eigvenvalue' )
set( gca, 'FontSize', 14 )
```

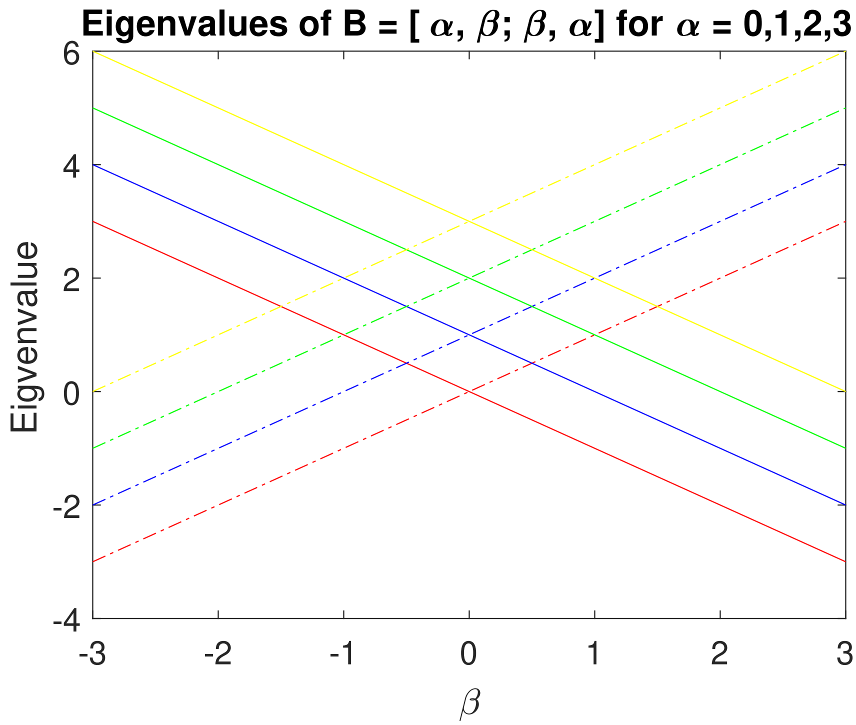

Figure 1: The eigenvalues of B for $\alpha \in \{0, 1, 2, 3\}$.

(d) We again compute the determinant as the product of the eigenvalues:

$$\det B \;=\; (\alpha - \beta)(\alpha + \beta).$$

B is not invertible if and only if $\alpha = \beta$ or $\alpha = -\beta$. The inverse is

$$B^{-1} \;=\; \frac{1}{(\alpha - \beta)(\alpha + \beta)} \begin{bmatrix} \alpha & -\beta \\ -\beta & \alpha \end{bmatrix}, \qquad \alpha \neq \pm\beta.$$

As a sanity check, when $\alpha = 1$, $\beta = 2$, then $B = A$ and $B^{-1} = A^{-1}$.

Figure 2 shows a surface plot of the determinant of B for $(\alpha, \beta) \in [0, 5]^2$. The following Matlab code generates Figure 2:

```
alpha = linspace( 0, 5, 20 );
beta  = linspace( 0, 5, 20 );

for i = 1:length(alpha),
    for j = 1:length(beta),
        determinant(i,j) = (alpha(i) - beta(j))*(alpha(i) + beta(j));
    end
end

figure;
surf( beta, alpha, determinant );
hold on;
surf( [0, 5], [0, 5], zeros(2,2) );
```

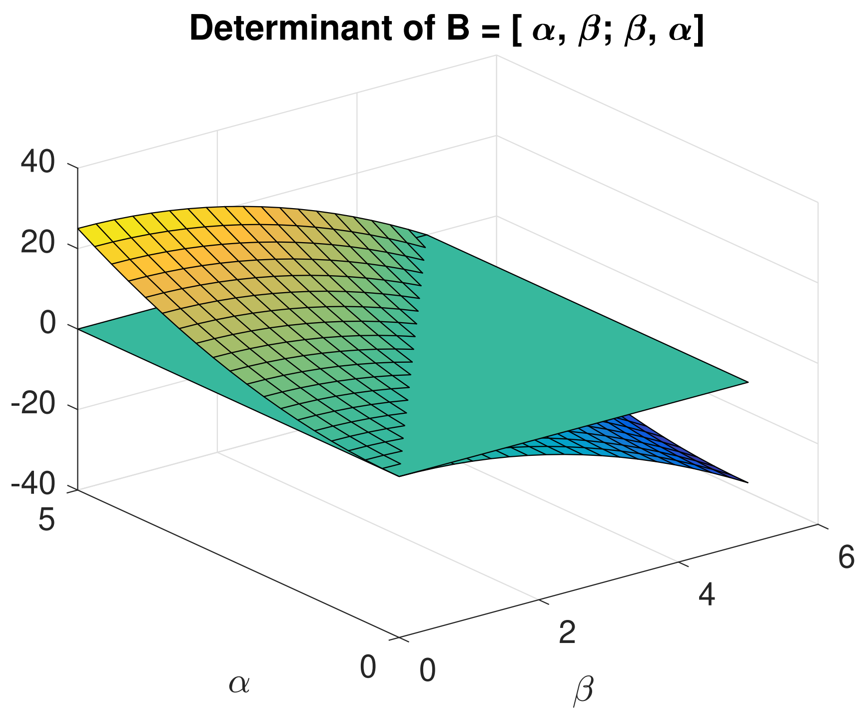

Figure 2: A surface plot of the determinant of B for $(\alpha, \beta) \in [0, 5]^2$. A plane at $z = 0$ is included in the surface plot to highlight the zero crossing.

```
xlabel( '\beta' );
ylabel( '\alpha' );
set( gca, 'FontSize', 14 );
title( 'Determinant of B = [\alpha, \beta; \beta, \alpha]' );
```

3 Multiplication by an orthogonal matrix

Consider the vector space $\mathbb{R}^n$ with standard norm and standard inner product. Prove that

(a) multiplication by an orthogonal matrix $\mathbf{U}$ preserves lengths, that is,

$$\|\mathbf{U}\mathbf{x}\| = \|\mathbf{x}\|,$$

for any $\mathbf{x}$.

(b) multiplication by an orthogonal matrix $\mathbf{U}$ preserves angles, that is,

$$\langle \mathbf{U}\mathbf{x}, \mathbf{U}\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle,$$

for any $\mathbf{x}$ and $\mathbf{y}$.

Solution

(a) To prove that multiplication by an orthogonal matrix preserves lengths, we write

$$\|\mathbf{U}\mathbf{x}\|^2 = \langle \mathbf{U}\mathbf{x}, \mathbf{U}\mathbf{x} \rangle = (\mathbf{U}\mathbf{x})^{\mathsf{T}}(\mathbf{U}\mathbf{x}) = \mathbf{x}^{\mathsf{T}}\mathbf{U}^{\mathsf{T}}\mathbf{U}\mathbf{x} = \mathbf{x}^{\mathsf{T}}\mathbf{x} = \langle \mathbf{x}, \mathbf{x} \rangle = \|\mathbf{x}\|^2,$$

where (a) follows from $\mathbf{U}^{\mathsf{T}}\mathbf{U} = \mathbf{I}$.

(b) To prove that multiplication by an orthogonal matrix preserves angles, we write

$$\langle \mathbf{U}\mathbf{x}, \mathbf{U}\mathbf{y} \rangle = (\mathbf{U}\mathbf{x})^{\mathsf{T}}(\mathbf{U}\mathbf{y}) = \mathbf{x}^{\mathsf{T}}\mathbf{U}^{\mathsf{T}}\mathbf{U}\mathbf{y} \overset{(a)}{=} \mathbf{x}^{\mathsf{T}}\mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle,$$

where again (a) follows from $\mathbf{U}^{\mathsf{T}}\mathbf{U} = \mathbf{I}$.

4 Bases and frames of $\mathbb{R}^2$

Given the following sets of vectors:

$$\Phi_1 = \{\varphi_{1,0},\, \varphi_{1,1}\} = \left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \tag{1}$$

$$\Phi_2 = \{\varphi_{2,0},\, \varphi_{2,1},\, \varphi_{2,2},\, \varphi_{2,3}\} = \left\{ \begin{bmatrix} \frac{1}{2\sqrt{2}} \\ \frac{\sqrt{3}}{2\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{3}}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\} \tag{2}$$

$$\Phi_3 = \{\varphi_{3,0}, \varphi_{3,1}\} = \left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} \right\} \tag{3}$$

$$\Phi_4 = \{\varphi_{4,0}, \varphi_{4,1}\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \tag{4}$$

For each of the sets of vectors, Φ_1 and Φ_3, do the following:

(a) Write the matrix representation for the set, that is, the synthesis operator associated with the set.

(b) Find the dual basis. Sketch the original sets and their duals.

(c) Specify whether it is an orthonormal basis.

(d) For $\mathbf{x} = [2,\, 0]^\mathsf{T}$, write down the projection coefficients, $\alpha_{i,k} = \langle x, \widetilde{\varphi}_{i,k} \rangle$.

(e) For the same $\mathbf{x}$, verify the expansion formula $\Phi\widetilde{\Phi}^\mathsf{T} = \mathbf{I}$.

(f) Specify whether the expansion preserves the norm, that is, whether it is true that $\|\mathbf{x}\| = \sum_k |\alpha_{i,k}|^2$.

For each of the sets of vectors, Φ_2 and Φ_4, write the matrix representation for the set, that is, the synthesis operator associated with the set.

Solution

(a) Concatenate the vector elements, in each set, to get a matrix. For example, we will use Φ_1, the approach is the same for the others. Thus, for

$$\Phi_1 = \{\varphi_{1,0},\, \varphi_{1,1}\} = \left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \tag{5}$$

the matrix representation is

$$\Phi_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{\sqrt{3}}{2} & 1 \end{bmatrix} \tag{6}$$

(b) Finding the dual basis or a dual frame is easiest using matrices. As long as each matrix above is of full rank (rank 2), we will be able to find the inverse (for bases/square matrices) or a right inverse (for frames/rectangular matrices),

$$\Phi\widetilde{\Phi}^\mathsf{T} = \mathbf{I}.$$

Thus, $\widetilde{\Phi}^\mathsf{T} = \Phi^{-1}$ (i.e. the inverse of Φ), and $\widetilde{\Phi} = (\Phi^{-1})^\mathsf{T}$

$$\widetilde{\Phi}_1 = \begin{bmatrix} 2 & -\sqrt{3} \\ 0 & 1 \end{bmatrix}. \tag{7}$$

Similarly,

$$\widetilde{\Phi}_3 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}. \tag{8}$$

(c) Φ_1 is a basis, and it is not orthonormal because it is not equal to its dual. (Alternatively, its two elements are not orthogonal.) Φ_3 is a basis, and it is orthonormal because it is equal to its dual. (Alternatively, its two elements are orthogonal and have unit norm.)

(d) These projection coefficients can be computed as $\alpha_k = \widetilde{\Phi}_k^{\mathsf{T}} \mathbf{x}$. So just (pre-)multiply the vector $\mathbf{x}$ by $\widetilde{\Phi}^{\mathsf{T}}$:

$$\boldsymbol{\alpha}_1 = \begin{bmatrix} 4 \\ -2\sqrt{3} \end{bmatrix},$$

while

$$\boldsymbol{\alpha}_3 = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}.$$

(e) Same as in (b).

(f) The norm of $\mathbf{x}$ is 2. The norms of the expansion vectors are $\|\boldsymbol{\alpha}_1\| = 2\sqrt{7}$ and $\|\boldsymbol{\alpha}_3\| = 2$. The orthonormal basis preserves the norm, as predicted by the Parseval equality, while the other does not.

5 Inner product

True or False, two vectors, say $f(t)$ and $g(t)$, are **orthogonal** if their inner product is zero. Using your response to the above prove that $f(t) = \sin(\pi nt)$ and $f(t) = \sin(\pi mt)$ are orthogonal in the Hilbert space $\mathcal{L}^2[-1, +1]$, for any integers $n \neq m$ (i.e., when $n, m \in \mathbb{Z}$ and $n \neq m$).

Solution

True! They are also linearly independent. Further if $f(t)$ and $g(t)$ have unit norm—i.e. if $\|f(t)\| = 1$ and $\|g(t)\| = 1$—then we have an orthonormal set.

$$f(t) = \sin(\pi nt), g(t) = \sin(\pi mt) \mid n, m \in \mathbb{Z}, \tag{9}$$

If orthogonal, their inner-product will equal 0.

$$\langle f(t), g(t) \rangle = \int_{-1}^{+1} \sin(\pi nt) \sin(\pi mt) \, \mathrm{d}t, \tag{10}$$

We can expand $\sin\alpha \sin\beta$, to simplify integral

$$\sin(\pi nt) \sin(\pi mt) = \frac{\cos(\pi t(n-m)) - \cos(\pi t(n+m))}{2} \tag{11}$$

and substitute (3) back into (2):

$$\langle f(t), g(t) \rangle = \frac{1}{2} \int_{-1}^{+1} \cos(\pi t(n-m)) - \cos(\pi t(n+m)) \, \mathrm{d}t, \tag{12}$$

This yields the two definite integrals:

$$\langle f(t), g(t) \rangle = \frac{1}{2} \left[\int_{-1}^{+1} \cos(\pi t(n-m)) dt - \int_{-1}^{+1} \cos(\pi t(n+m)) \, \mathrm{d}t \right], \tag{13}$$

Since the antiderivative of $\cos()$ is $\sin()$, the definite integral may be expanded to:

$$\langle f(t), g(t) \rangle = \frac{1}{2} \left[\frac{\sin(\pi(n-m))}{\pi(n-m)} - \frac{\sin(\pi(m-n))}{\pi(n-m)} - \frac{\sin(\pi(n+m))}{\pi(n+m)} + \frac{\sin(-\pi(n+m))}{\pi(n+m)} \right], \tag{14}$$

$$\langle f(t), g(t) \rangle = \frac{\sin(\pi(n-m))}{\pi(n-m)} - \frac{\sin(\pi(n+m))}{\pi(n+m)}. \tag{15}$$

And since terms of $(n-m), (n+m) \in \mathbb{Z}$ all arguments of the $\sin$ terms all operate on positive or negative integral multiples of π, which makes all terms $\sin((n-m)\pi) = \sin((n+m)\pi) = 0$. Thus,

$$\langle f(t), g(t) \rangle = 0. \quad \square \tag{16}$$

6 Inner product computation by expansion sequences

Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be sequences in $\ell^2(\mathbb{N})$. Then, the functions

$$f(t) = \alpha_0 + \sum_{k=1}^{\infty} \alpha_k \sqrt{2}\cos(2\pi k t),$$
$$g(t) = \beta_0 + \sum_{k=1}^{\infty} \beta_k \sqrt{2}\cos(2\pi k t),$$

are in $\mathcal{L}^2\left(\left[-\frac{1}{2}, +\frac{1}{2}\right]\right)$. Demonstrate that the standard inner product between the functions, $f(t)$ and $g(t)$ can be written as the standard inner product between the sequences $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. That is, show that $\langle f(t), g(t)\rangle = \langle \boldsymbol{\alpha}, \boldsymbol{\beta}\rangle$.

Solution

Simply recall that $\langle f(t), g(t)\rangle = \displaystyle\int_{-1/2}^{+1/2} f(t)g(t)\mathrm{d}t$. Now,

$$\langle f(t), g(t)\rangle = \int_{-1/2}^{+1/2} \left(\alpha_0 + \sum_{k=1}^{\infty} \alpha_k \sqrt{2}\cos(2\pi k t)\right)\left(\beta_0 + \sum_{l=1}^{\infty} \beta_l \sqrt{2}\cos(2\pi l t)\right) \mathrm{d}t$$

where we have changed the index in the second summation to avoid confusion (note that changing the index letter from k to l, does not alter the value of the sum or the integral). Now expanding the brackets gives:

$$\langle f(t), g(t)\rangle = \int_{-1/2}^{+1/2} \alpha_0\beta_0 + \beta_0 \sum_{k=1}^{\infty} \alpha_k \sqrt{2}\cos(2\pi k t) + \alpha_0 \sum_{l=1}^{\infty} \beta_l \sqrt{2}\cos(2\pi l t)$$
$$+ \sum_{k=1}^{\infty} \alpha_k \sqrt{2}\cos(2\pi k t) \sum_{l=1}^{\infty} \beta_l \sqrt{2}\cos(2\pi l t)\, \mathrm{d}t.$$

There are four terms in the integrals:

- **(Term 1)** $\int_{-1/2}^{+1/2} \alpha_0\beta_0 \,\mathrm{d}t = \alpha_0\beta_0$.
- **(Term 2)**
$$\int_{-1/2}^{+1/2} \beta_0 \sum_{k=1}^{\infty} \alpha_k \sqrt{2}\cos(2\pi k t)\,\mathrm{d}t = \beta_0 \sum_{k=1}^{\infty} \alpha_k \sqrt{2} \int_{-1/2}^{+1/2} \cos(2\pi k t)\,\mathrm{d}t = 0.$$
This is because, after swapping the order of the summation and integral signs, the integral $\int_{-1/2}^{+1/2} \cos(2\pi k t)\,\mathrm{d}t = 0$ for integer values of $k \neq 0$.
- **(Term 3)** Similarly,
$$\int_{-1/2}^{+1/2} \alpha_0 \sum_{l=1}^{\infty} \beta_l \sqrt{2}\cos(2\pi l t)\,\mathrm{d}t = \alpha_0 \sum_{l=1}^{\infty} \beta_l \sqrt{2} \int_{-1/2}^{+1/2} \cos(2\pi l t)\,\mathrm{d}t = 0.$$
This is because, after swapping the order of the summation and integral signs, the integral $\int_{-1/2}^{+1/2} \cos(2\pi l t)\,\mathrm{d}t = 0$ for integer values of $l \neq 0$.

- **(Term 4)**

$$\begin{aligned}
\int_{-1/2}^{+1/2} \sum_{k=1}^{\infty} \alpha_k \sqrt{2}\cos(2\pi kt) \sum_{l=1}^{\infty} \beta_l \sqrt{2}\cos(2\pi lt)\,\mathrm{d}t &= \int_{-1/2}^{+1/2} \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} 2\alpha_k\beta_l\cos(2\pi kt)\cos(2\pi lt)\,\mathrm{d}t \\
&= 2\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \alpha_k\beta_l \int_{-1/2}^{+1/2} \cos(2\pi kt)\cos(2\pi lt)\,\mathrm{d}t \\
&= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \alpha_k\beta_l \int_{-1/2}^{+1/2} \cos\left(2\pi(k+l)t\right) + \cos\left(2\pi(k-l)t\right)\,\mathrm{d}t,
\end{aligned}$$

where the last line follows because $\cos(a)\cos(b) = \frac{1}{2}(\cos(a+b) - \cos(a-b))$. Now, because for $(k+l) > 0$ for all values of $k \geq 1$ and $l \geq 1$, then the integral $\int_{-1/2}^{+1/2} \cos\left(2\pi(k+l)t\right)\,\mathrm{d}t = 0$, for all integer values of $k \geq 1$ and $l \geq 1$.

However, for $(k-l)$, there are two cases to consider: The first is when $k = l$, then $k - l = 0$ and thus $\int_{-1/2}^{+1/2} \cos\left(2\pi(k-l)t\right)\,\mathrm{d}t = \int_{-1/2}^{+1/2} 1\,\mathrm{d}t = 1$. The second case, is when $k \neq l$, this means $(k-l) \neq 0$ and so, as before $\int_{-1/2}^{+1/2} \cos\left(2\pi(k-l)t\right)\,\mathrm{d}t = 0$. Thus, it follows that,

$$\begin{aligned}
\int_{-1/2}^{+1/2} \sum_{k=1}^{\infty} \alpha_k \sqrt{2}\cos(2\pi kt) \sum_{l=1}^{\infty} \beta_l \sqrt{2}\cos(2\pi lt)\,\mathrm{d}t &= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \alpha_k\beta_l \int_{-1/2}^{+1/2} \cos\left(2\pi(k+l)t\right) + \cos\left(2\pi(k-l)t\right)\,\mathrm{d}t \\
&= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \alpha_k\beta_l, \text{ for } k = l \\
&= \sum_{k=1}^{\infty} \alpha_k\beta_k
\end{aligned}$$

Summing the results of all the integrals, terms 1 to 4, gives

$$\langle f(t), g(t)\rangle = \alpha_0\beta_0 + \sum_{k=1}^{\infty} \alpha_k\beta_k = \sum_{k=0}^{\infty} \alpha_k\beta_k = \langle \boldsymbol{\alpha}, \boldsymbol{\beta}\rangle.$$

As required!

7 Linear Independence (Optional, for extra credit.)

Find the values of the parameter $a \in \mathbb{C}$ such that the following set is linearly independent:

$$U = \left\{ \begin{bmatrix} 0 & a^2 \\ 0 & \mathrm{j} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & a-1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \mathrm{j}a & 1 \end{bmatrix} \right\}.$$

For $a = \mathrm{j}$, express the matrix

$$\begin{bmatrix} 0 & 5 \\ 2 & \mathrm{j}-2 \end{bmatrix}$$

as a linear combination of the elements of U. [*Note that* j *denotes the imaginary unit, i.e.* $\mathrm{j} = \sqrt{-1}$*, so that* $c\mathrm{j} \times d\mathrm{j} = c \times d \times \mathrm{j}^2 = c \times d \times -1 = -cd$.]

Solution

For any set to be a linearly independent set, then the linear combination of its elements is zero if and only if the weights of the linear. sum is zero. Specifically, for the set U to be linearly independent, it is necessary and sufficient that:

$$\lambda_1 \begin{bmatrix} 0 & a^2 \\ 0 & \mathrm{j} \end{bmatrix}, +\lambda_2 \begin{bmatrix} 0 & 1 \\ 1 & a-1 \end{bmatrix}, +\lambda_3 \begin{bmatrix} 0 & 0 \\ \mathrm{j}a & 1 \end{bmatrix} = \mathbf{0}$$

for $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ have the unique solution $\lambda_1 = \lambda_2 = \lambda_3 = 0$. This is equivalent to the system of equations:

$$\begin{aligned} a^2\lambda_1 + \lambda_2 &= 0, \\ \lambda_2 + \mathrm{j}a\lambda_3 &= 0, \\ \mathrm{j}\lambda_1 + (a-1)\lambda_2 + \lambda_3 &= 0. \end{aligned}$$

By combining the first two equations, it follows that $a^2\lambda_1 = -\lambda_2 = \mathrm{j}a\lambda_3$. Multiplying the last equation by a^2 and substituting for λ_1 and λ_2 using the first two equations, we get

$$\begin{aligned} \mathrm{j}a^2\lambda_1 + (a-1)a^\lambda_2 + a^2\lambda_3 &= \mathrm{j}(\mathrm{j}a\lambda_3) + (a-1)a^2(\mathrm{j}a\lambda_3) + a^2\lambda_3 \\ &= (-a - \mathrm{j}(a-1)a^3 + a^2)\lambda_3 \\ &= a(1 - \mathrm{j}a^2)(a-1)\lambda_3 \\ &= a(1-ak)(1+ak)(a-1)\lambda_3 = 0, \end{aligned}$$

where $k = \sqrt{\mathrm{j}} = (1+\mathrm{j})/\sqrt{2}$. When $a \in \{1, 1, -1/k, 1/k\}$ it must be that $\lambda_3 = 0$. In addition, we have that $a^2\lambda_1 = -\lambda_2 = \mathrm{j}a\lambda_3$, which ensures that $\lambda_1 = \lambda_2 = 0$. Hence U is an independent set if and only if the complex number a is not equal to any of the values in the set $\{0,\ 1,\ (1-\mathrm{j})/\sqrt{2},\ -(1-\mathrm{j})/\sqrt{2}\}$.

For $a = \mathrm{j}$, notice that:

$$(-2) \begin{bmatrix} 0 & -1 \\ 0 & \mathrm{j} \end{bmatrix}, +3 \begin{bmatrix} 0 & 1 \\ 1 & \mathrm{j}-1 \end{bmatrix}, + \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 2 & \mathrm{j}-2 \end{bmatrix},$$

as required.

8 Vector space $\mathbb{C}^n$ (Optional, for extra credit.)

Prove that $\mathbb{C}^n$ is a vector space.

Solution

To prove that $\mathbb{C}^n$ is a vector space, we need to check that the conditions stated in (Lecture 2, Definition 2.1). Specifically, we need to prove Commutativity, Associativity, Distributivity, Additive identity, Additive inverse, Multiplicative identity.

(i) **Commutativity**:

$$\begin{aligned}\mathbf{x}+\mathbf{y} &= \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} + \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} x_0+y_0 \\ x_1+y_1 \\ \vdots \\ x_{n-1}+y_{n-1} \end{bmatrix} \overset{(a)}{=} \begin{bmatrix} y_0+x_0 \\ y_1+x_1 \\ \vdots \\ y_{n-1}+x_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} + \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} = \mathbf{y}+\mathbf{x},\end{aligned}$$

where step (a) follows from the commutative property of addition on $\mathbb{C}$.

(ii) **Associativity**:

$$\begin{aligned}(\mathbf{x}+\mathbf{y})+\mathbf{z} &= \begin{bmatrix} x_0+y_0 \\ x_1+y_1 \\ \vdots \\ x_{n-1}+y_{n-1} \end{bmatrix} + \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \end{bmatrix} = \begin{bmatrix} (x_0+y_0)+z_0 \\ (x_1+y_1)+z_1 \\ \vdots \\ (x_{n-1}+y_{n-1})+z_{n-1} \end{bmatrix} \\ &\overset{(a)}{=} \begin{bmatrix} x_0+(y_0+z_0) \\ x_1+(y_1+z_1) \\ \vdots \\ x_{n-1}+(y_{n-1}+z_{n-1}) \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} + \begin{bmatrix} (y_0+z_0) \\ (y_1+z_1) \\ \vdots \\ (y_{n-1}+z_{n-1}) \end{bmatrix} \\ &= \mathbf{x}+(\mathbf{y}+\mathbf{z}),\end{aligned}$$

where step (a) follows from the associative property of addition on $\mathbb{C}$. And

$$\begin{aligned}(\alpha\beta)\mathbf{x} &= \begin{bmatrix} (\alpha\beta)x_0 \\ (\alpha\beta)x_1 \\ \vdots \\ (\alpha\beta)x_{n-1} \end{bmatrix} \\ &\overset{(a)}{=} \begin{bmatrix} \alpha(\beta x_0) \\ \alpha(\beta x_1) \\ \vdots \\ \alpha(\beta x_{n-1}) \end{bmatrix} = \alpha(\beta\mathbf{x}),\end{aligned}$$

where (a) follows from the associative property of multiplication on $\mathbb{C}$.

(iii) **Distributivity**: One can show that the distributive properties $\alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x}+\alpha\mathbf{y}$, and $(\alpha+\beta)\mathbf{x} = \alpha\mathbf{x}+\beta\mathbf{x}$ follow by using a similar approach to the two properties above.

(iv) **Additive identity**: The element $\mathbf{0} = [0\ 0\ \cdots\ 0]^\mathsf{T} \in \mathbb{C}^n$ is the additive identity, since

$$\mathbf{x} + \mathbf{0} = \begin{bmatrix} x_0 + 0 \\ x_1 + 0 \\ \vdots \\ x_{n-1} + 0 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} = \mathbf{x},$$

and $\mathbf{0} + \mathbf{x} = \mathbf{x}$ follows similarly.

(v) **Additive inverse**: For any $\mathbf{x} \in \mathbb{C}^n$, the element

$$(-\mathbf{x}) = \begin{bmatrix} -x_0 \\ -x_1 \\ \vdots \\ -x_{n-1} \end{bmatrix} \in \mathbb{C}^n,$$

is the unique additive inverse, since

$$\begin{aligned} \mathbf{x} + (-\mathbf{x}) &= \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} + \begin{bmatrix} -x_0 \\ -x_1 \\ \vdots \\ -x_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} x_0 + (-x_0) \\ x_1 + (-x_1) \\ \vdots \\ x_0 + (-x_{n-1}) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}, \end{aligned}$$

$(-\mathbf{x}) + \mathbf{x} = \mathbf{0}$ follows similarly. Uniqueness follows from the uniqueness of additive inverses in $\mathbb{C}$.

(vi) **Multiplicative identity**: This property, i.e. for $\mathbf{x} \in \mathbb{C}^n$ then $1\mathbf{x} = \mathbf{x}$, follows similarly to the additive identity proof above.