Programming Project: Final Phase

Multi-Variable Constraint Optimization Using Penalty Function Method

1. Introduction:

1.1. Constraint Optimization:

Constrained objective function can be defined as the variables of objective function are dependent on other equation/s, which govern/s the value of variables. Those governing equations can be of equality or inequality types. Here, general constrained objective function is given:

Minimize (x)
Subjected to, $(x) \geq 0$, $j=1,2,\ldots,J$
$h(x)=0$, $k=1,2,\ldots,K$
$(L) \leq x_i \leq x(U)$, $i=1,2,\ldots,N$

1.2. Unconstraint Optimization:

In unconstrained objective function, there is only objective function, of single or multi-variables, presents. There will be no any other equations which governs value of variables.

Minimize $f(x)$

2. Optimization Techniques used in this project:

2.1. Penalty Function Method:

Penalty function methods work in a series of sequence, each time modifying a set of penalty parameters and starting a sequence with the point obtained in the previous sequence.

$$(x,R) = (x) + \Omega(R,g(x),h(x))$$

where, 'R' is the set of penalty parameters, 'Ω' is the penalty term chosen to favor the selection of feasible point over infeasible point.

There are two types of penalty function method, one is interior penalty method and other one is exterior penalty method.
In this project, we are using Bracket operator penalty which is exterior penalty method.

2.1.1. Bracket Operator Penalty Term:

$\Omega = R * \langle g(x) \rangle^2$

Bracket operator works when $g(x)<0$, i.e., when constraints are violated.
It assigns positive value to infeasible point, that's why it is an exterior penalty term.
It starts with small value of 'R' which increases gradually.

In successive sequences of penalty function method, R value changes which depends on interior and exterior penalty terms.

Algorithm

1) Choose two termination parameters ϵ_1, ϵ_2; an initial solution ${}^{(0)}$; a penalty term Ω; and an initial penalty parameter $R^{(0)}$. Choose a parameter c to update R such that $0 < c < 1$ is used for interior penalty terms and $c \geq 1$ is used for exterior penalty terms. Set $t = 0$.

2) Form $P(x^{(t)},R^{(t)}) = f(x^{(t)}) + \Omega(R^{(t)},g(x^{(t)}),h(x^{(t)}))$.

3) Starting with ${}^{(t)}$, find ${}^{(t+1)}$ such that $(x^{(t)},R^{(t)})$ is minimum for a fixed value of $R^{(t)}$. Use ϵ_1 to terminate the unconstrained search. (Here, we use unidirectional search to find next point.)

4) Is $|(x^{(t+1)},R^{(t)}) - P(x^{(t)},R^{(t-1)})| \leq \epsilon_2$? If yes, set ${}^{(T)} = {}^{(t+1)}$ and **terminate**; else go to Step 5.

5) Choose ${}^{(t+1)} = c * {}^{(t)}$. Set $t = t + 1$ and go to Step 2.

This is the algorithm for Penalty Function method for optimization of constrained objective function. Now, we will see Powell's conjugate direction method for unconstrained multi-variable problem and in that unidirectional search as a combination of Bounding-Phase method and Golden section search method.

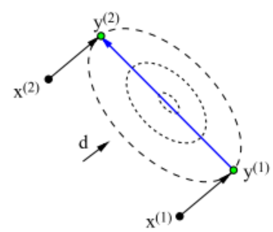
2.2. Powell's Conjugate Direction Method:

It is one of the most popular Direct search method which uses the history of previous points to create the new search directions.

It uses parallel subspace property and extended parallel subspace property to create a set of conjugate directions which we are discuss below:

Parallel Subspace Property:
Given a quadratic function q(x) = A + $B^T x$ + 0.5*x^T*C*x, of two variables (where A is scalar quantity, B is a vector & C is a 2 x 2 matrix), two arbitrary but distinct points $x^{(1)}$ and $x^{(2)}$, and a direction 'd'.

a) If $y^{(1)}$ is the solution to the problem min $q(x^1 + \lambda d)$, and $y^{(2)}$ is the solution to the problem min $q(x^2 + \lambda d)$.



b) Then, the direction $y^{(2)} - y^{(1)}$ is conjugate to 'd'.
c) For quadratic functions, the minimum of the function lies on the line joining the points $y^{(1)}$ and $y^{(2)}$.

Algorithm

1) Choose starting point $x^{(0)}$ and a set of N linearly independent directions; possibly $s^{(i)} = e^{(i)}$ for i = 1,2,3,…,N.
2) Minimize along N unidirectional search directions using the previous minimum point to begin the next search.
3) Form a new conjugate direction 'd' using the extended parallel subspace property.
4) If $\|d\|$ is small or search directions are linearly dependents, **Terminate;** else replace $s^{(j)} = s^{(j-1)}$ for all j = N,N-1,…,2.
   Set s(1) = d / $\|d\|$ and got to step 2.

2.3. Bounding-Phase Method

Unidirectional search is performed using Bounding-Phase method and Golden Section Search method.

Algorithm

1) Take an initial guess $^{(0)}$ from the user and also increment value Δ. Set k = 0.
2) Find function value at $^{(0)}$ and at two points at Δ distance on either side of $x^{(0)}$.
   If, $f(x^{(0)} - \Delta) \geq f(x^{(0)}) \geq f(x^{(0)} + \Delta)$, then take the value of Δ as positive value,

else if $f(x^{(0)} - \Delta) \le f(x^{(0)}) \le f(x^{(0)} + \Delta)$, then take the value of Δ as negative value, else ask for the another initial guess.

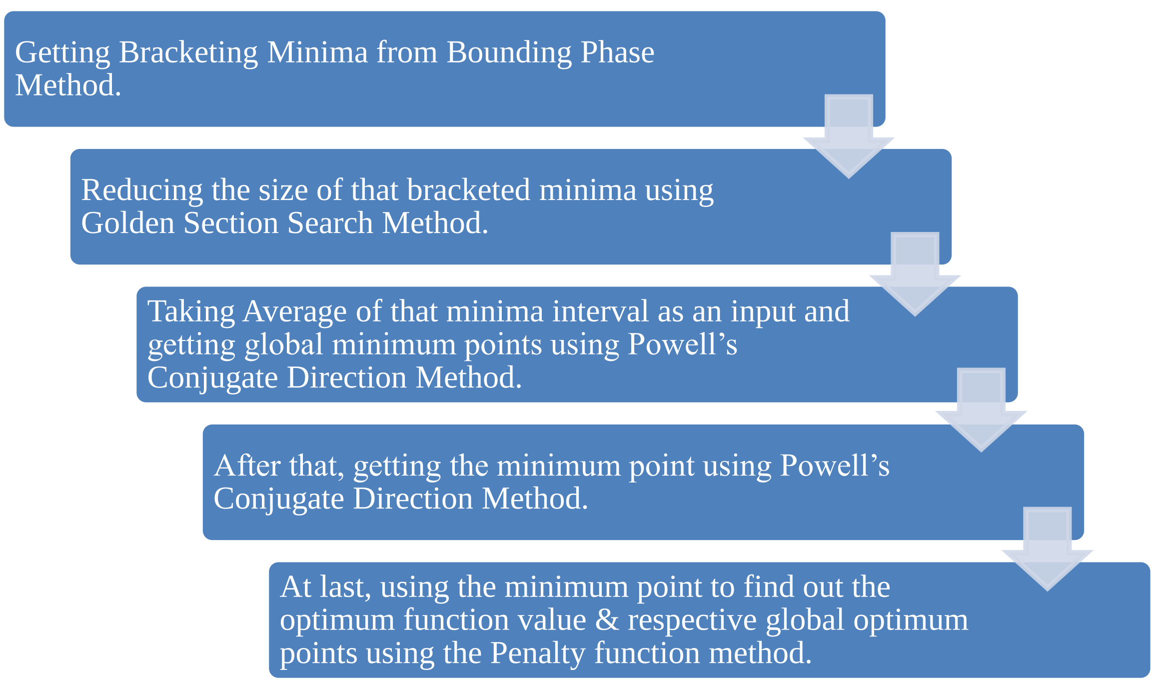
3) Calculate $^{(k+1)} = x^{(k)} + 2k * \Delta$.
4) If $f\ (^{(k+1)}) < f\ (^{(k)})$, then increase the k by 1 and go to step 3, else terminate the algorithm and our extreme point will lie in the interval $(x^{(k-1)}, x^{(k+1)})$.

2.4. Golden Section Search Method:

Algorithm

1) Set the interval of x, that is (a,b) and a small number ε.
Normalize the variable x by using the equation $\omega = (x - a) / (b - a)$. Thus $a_{\omega} = 0$, $b_{\omega} = 1$, and $L_{\omega} = 1$. Set k = 1.
2) Set $\omega_1 = a_{\omega} + 0.618\ L_{\omega}$ and $\omega_2 = b_{\omega} - 0.618\ L_{\omega}$. Compute $f(\omega_1)$ or $f(\omega_2)$, depending on whichever of the two was not evaluated earlier.
Eliminate region. Set new a_{ω}, b_{ω} and $L_{\omega} = b_{\omega} - a_{\omega}$.
3) Is mod of $L_{\omega} \le \varepsilon$ small? If no, set k = k+1, go to step2;
Else **Terminate.**

3. Project Flow Chart:



4. Problem Statements:

Problem1:

Min $f(x) = (x_1 – 10)^3 + (x_2 – 20)^3$

Subject to $g_1(x) = (x_1 – 5)^2 + (x_2 – 5)^2 – 100 \geq 0,$
$g_2(x) = (x_1 – 6)^2 + (x_2 – 5)^2 – 82.81 \leq 0,$
$13 \leq x_1 \leq 20,\ 0 \leq x_2 \leq 4$

Problem2:

Max $f(x) = \sin^3(2\pi\, x_1) \sin(2\pi\, x_2) / x^3(x_1 + x_2)$

Subject to $g_1(x) = x_1^2 - x_2 + 1 \leq 0,$
$g_2(x) = 1 – x_1 + (x_2 – 4)^2 \leq 0,$
$0 \leq x_1 \leq 10,\ 0 \leq x_2 \leq 10$

Problem3:

Min $f(x) = x_1 + x_2 + x_3$

Subject to $g_1(x) = -1 + 0.0025(x_4 + x_6) \leq 0,$
$g_2(x) = -1 + 0.0025(-x_4 + x_5 + x_7) \leq 0,$
$g_3(x) = -1 + 0.01(-x_6 + x_8) \leq 0,$
$g_4(x) = 100\, x_1 - x_1\, x_6 + 833.33252\, x_4 – 83333.333 \leq 0,$
$g_5(x) = x_2\, x_4 – x_2\, x_7 - 1250\, x_4 + 1250\, x_5 \leq 0,$
$g_6(x) = x_3\, x_5 – x_3\, x_8 - 2500\, x_5 + 1250000 \leq 0$

$100 \leq x_1 \leq 10000,$
$1000 \leq x_i \leq 10000,\ i = 2,3$
$10 \leq x_i \leq 1000,\ i = 4,5,\ldots,8$

5. *Results & Discussion:*

Problem1:

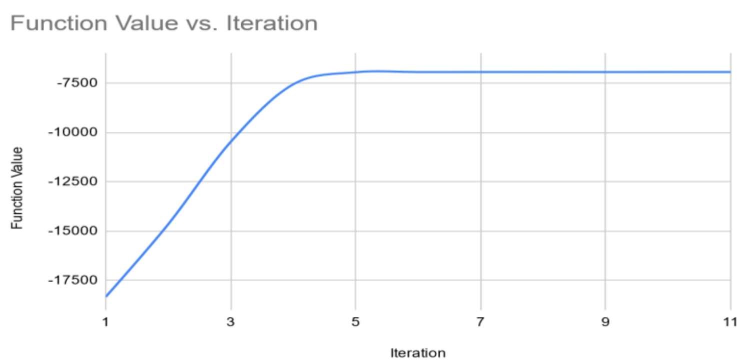
We have used exterior penalty term, value of R, starting from $^{(0)}$ =0.1, is increased by factor c = 10 after performing each sequence and with ε = 0.0001

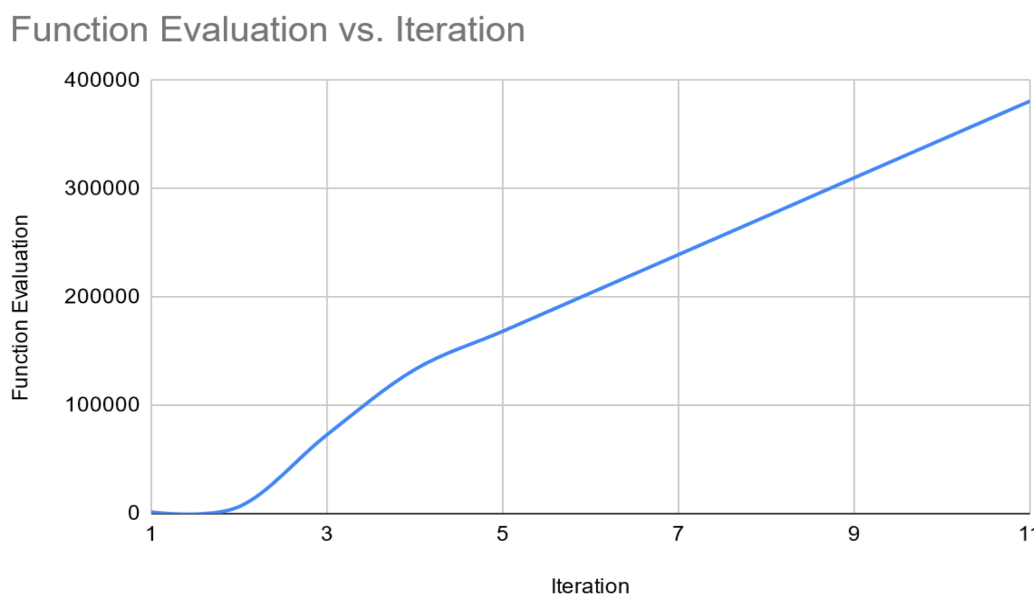
S.No.	Initial Guess 1	Initial Guess 2	X_1^T	X_2^T	No. of iterations	Function value
1.	14.00	1.00	14.105091	0.863265	10	-6938.973965
2.	14.00	2.00	14.105350	0.863213	11	-6939.018263
3.	14.00	3.00	14.104698	0.862428	9	-6939.913622
4.	14.00	3.50	14.105390	0.863843	10	-6938.324200
5.	15.00	2.00	14.105194	0.863430	12	-6938.787885
6.	15.00	3.00	14.104351	0.861774	10	-6940.649806
7.	16.00	2.00	14.105421	0.863917	10	-6938.241653
8.	16.00	3.00	14.104816	0.862702	10	-6939.607191
9.	17.00	2.00	14.104553	0.862179	10	-6940.194268
10.	18.00	3.00	14.104267	0.861641	10	-6940.800281

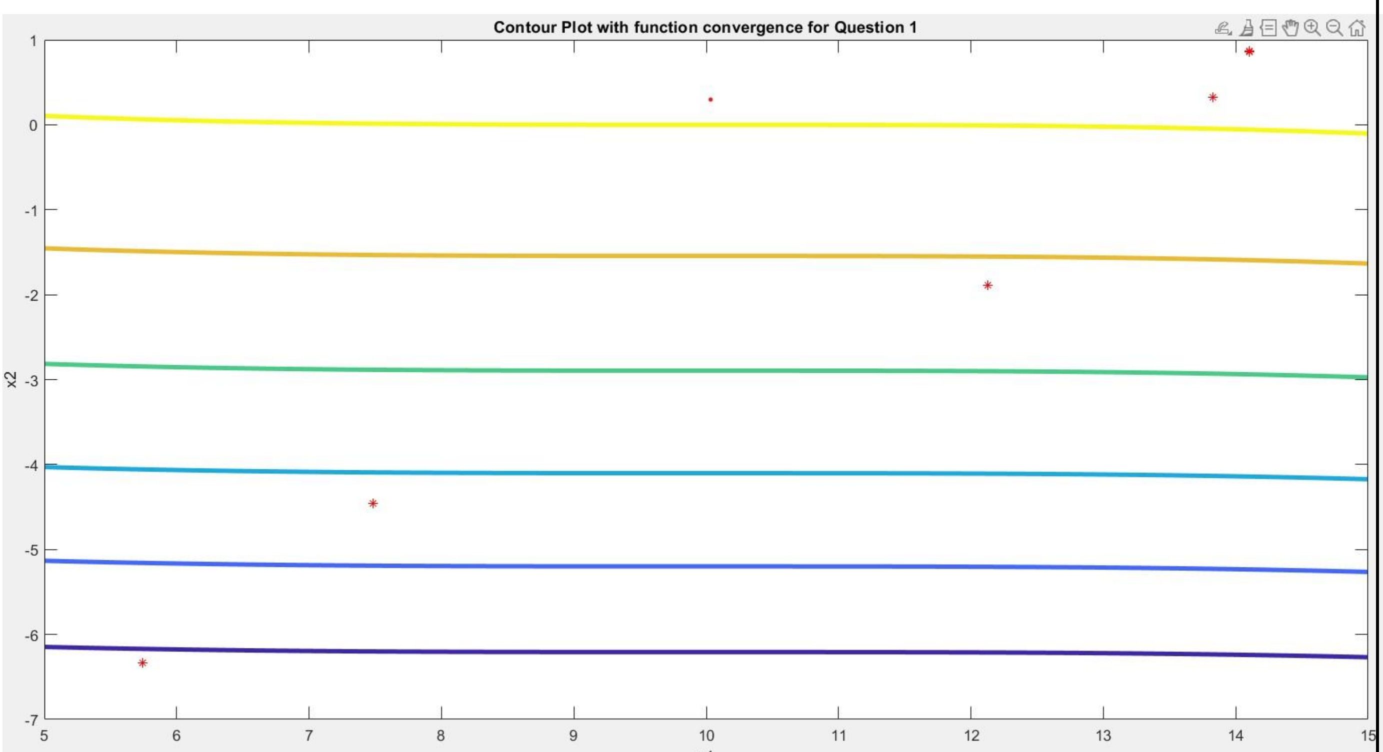
Our optimum value of objective function is -6961.8139, at $x1$=14.0950, $x2$=0.8430

Solutions	X_1^T	X_2^T	Function value
Best	14.1042670	0.8616410	-6940.800281
Worst	14.1054210	0.8639170	-6938.241653
Mean	14.1049131	0.8628392	-6939.451113
Median	14.1049535	0.8629575	-6936.312727
Standard Deviation	0.00043532	0.000818769	0.92106826

Thus, our best solution which is obtained by MATLAB code is very near to the optimum solution.




Function Evaluation vs. Iteration
400000
300000
200000
100000
0
Function Evaluation
1
3
5
7
9
11
Iteration


Contour Plot with function convergence for Question 1
1
0
-1
-2
-3
-4
-5
-6
-7
x2
5
6
7
8
9
10
11
12
13
14
15
x1

Problem2:

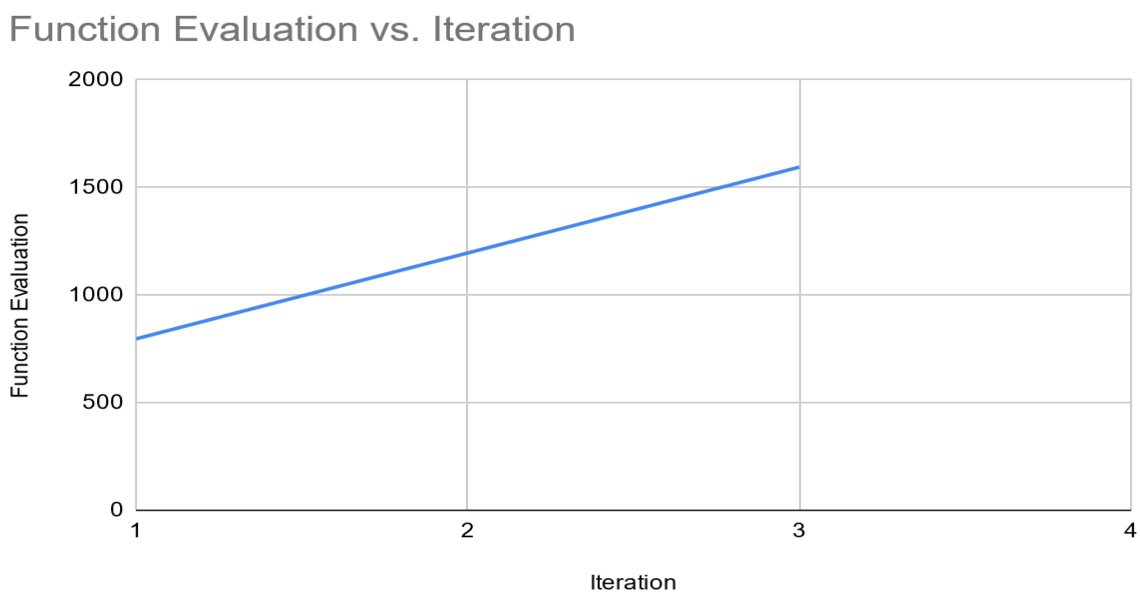
We have used exterior penalty term, value of R, starting from $^{(0)}$ =0.1, is increased by factor c = 10 after performing each sequence and with ε = 0.00001

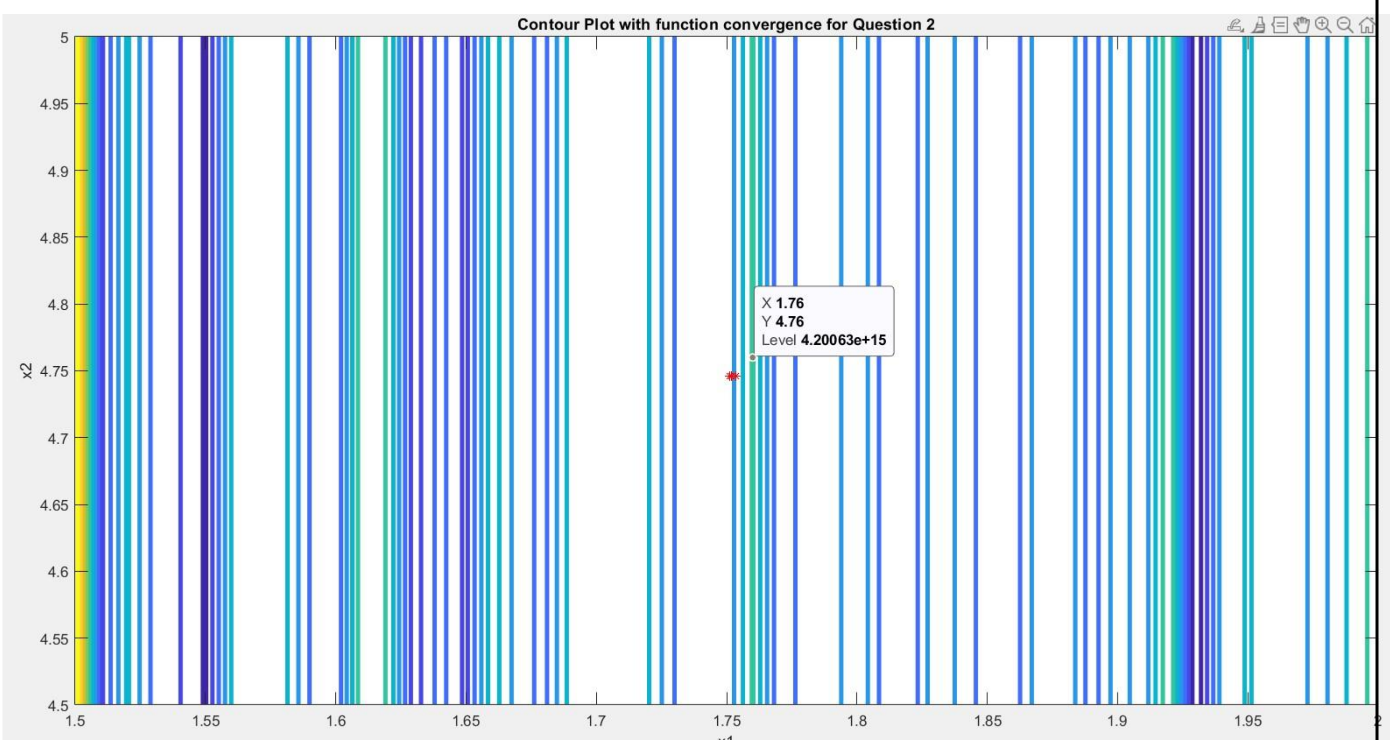
S.No.	Initial Guess 1	Initial Guess 2	X_1^T	X_2^T	No. of iterations	Function value
1.	3.00	2.00	1.751288	4.745958	3	0.028643
2.	3.00	4.00	1.733274	4.746389	1	0.029142
3.	3.00	5.00	1.741835	4.746001	2	0.029042
4.	4.00	2.00	1.751330	4.745958	3	0.028640
5.	5.00	2.00	1.751273	4.745958	3	0.028644
6.	5.00	4.00	1.733168	4.746389	1	0.029142
7.	5.00	4.50	1.234616	4.245102	1	0.095575
8.	5.50	4.50	1.234161	4.245102	1	0.095608
9.	6.00	4.50	1.234327	4.245102	1	0.095596
10.	6.50	4.50	1.234428	4.245102	1	0.095589

Our optimum value of objective function is 0.0958, at $x1$=1.227, $x2$=4.245

Solutions	X_1^T	X_2^T	Function value
Best	1.234161	4.245102	0.095608
Worst	1.751330	4.745958	0.028640
Mean	1.539970	4.545706	0.055562
Median	1.733221	4.745958	0.029142
Standard Deviation	0.263091	0.258718	0.034452

Thus, our best solution which is obtained by MATLAB code is very near to the optimum solution.


Function Evaluation vs. Iteration
2000
1500
1000
500
0
Function Evaluation
1
2
3
4
Iteration


Contour Plot with function convergence for Question 2
X 1.76
Y 4.76
Level 4.20063e+15
x2
x1
5
4.95
4.9
4.85
4.8
4.75
4.7
4.65
4.6
4.55
4.5
1.5
1.55
1.6
1.65
1.7
1.75
1.8
1.85
1.9
1.95
2

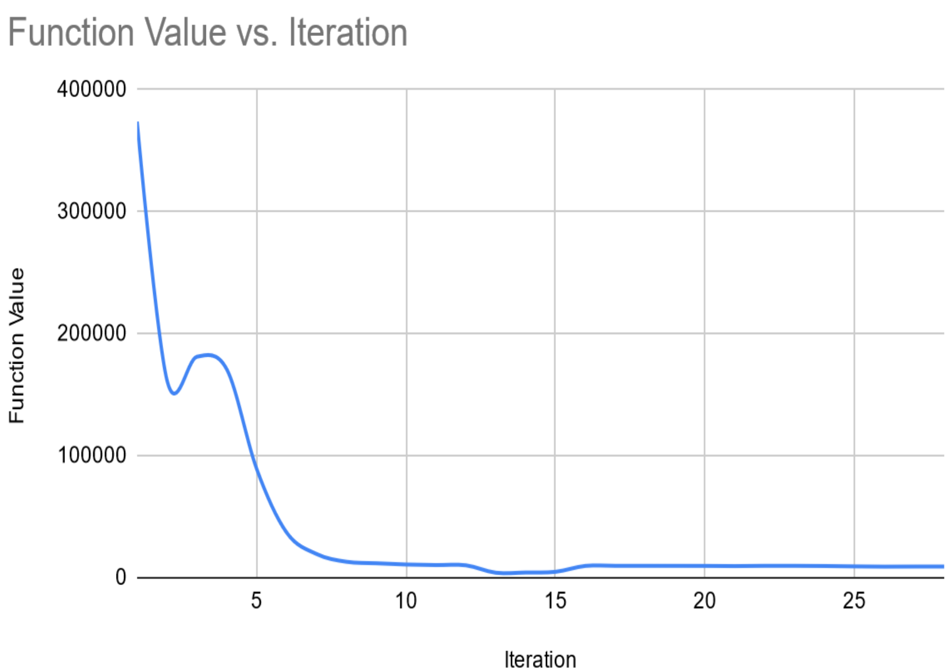
Problem3:

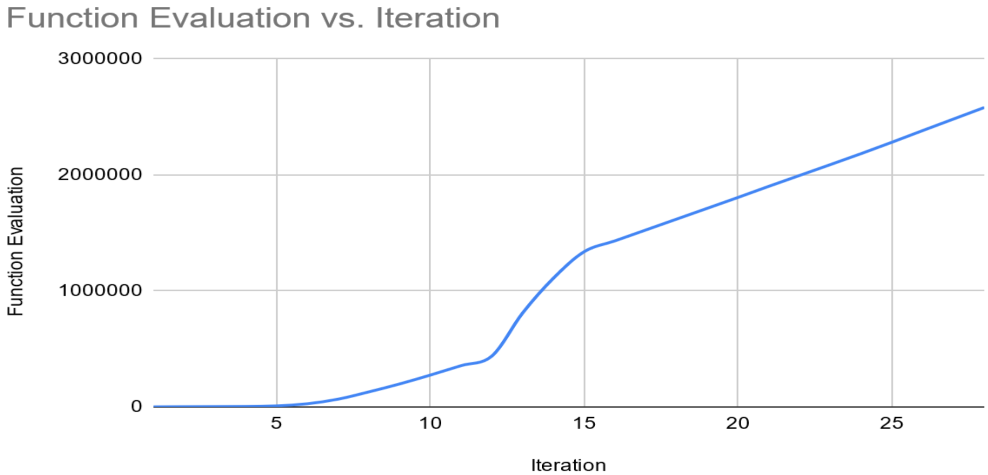
We have used exterior penalty term, value of R, starting from $^{(0)}$ =0.1, is increased by factor c = 10 after performing each sequence & with ε = 0.0001

S.No. →	1 (worst)	2 (Best)	3
X_1^T	100.051522	100.043514	100.107294
X_2^T	1000.000054	1000.000052	1000.000023
X_3^T	3324.798549	5947.767101	3507.117307
X_4^T	121.414446	106.620908	119.042563
X_5^T	10.002361	10.000293	10.000374
X_6^T	278.512659	156.250000	259.580000
X_7^T	151.230000	136.450000	104.780000
X_8^T	378.488309	216.250000	359.580000
No. of iterations	30	29	6
Function Value	4424.850124	7047.810666	4607.224624

The global minima: x∗ = (579.3167, 1359.943, 5110.071, 182.0174, 295.5985, 217.9799, 286.4162,395.5979), f (x∗) = 7049.3307

As we can see that for the second set of values we are getting the function value which is very close to the optimum function value.





6. *Conclusion:*

6.1. Penalty Function method is computationally expensive as it includes Powell's Conjugate direction search method for unconstrained multi-variable problem.

6.2. For the 1st and 2nd problem, we are getting close solution to the optimum solution, most of the times. Sometimes it is not converging to the optimum solution, one of the reason for that might be that solution is converging to the other local optimum point.

6.3. From these 3 problems, we can say that on increasing constrained equation on variables. It becomes difficult to get the optimum point.

6.4. In the 3rd problem, many times we are getting different function values which are deviating from our optimum function value and this deviation might be because in-spite of getting global optimum values, we are actually getting local optimum values which can be vary with the initial guesses.

6.5. Compared to 1st and 2nd problems, 3rd problem is taking much more time to print output results.

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