ROBUSTNESS ANALYSIS OF NETWORK

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Abstract-

The objective of this paper is to analyze the Robustness of the network. We have studied and analyzed the research papers that carries results regarding robustness of random networks. A thorough study has been carried out on robustness of Erdos-Renyi Model of networks. A variation of Duplication model is proposed by us. And then we have verified the results for our model.

Introduction

Resilience or Robustness refers to the ability of a network to avoid malfunctioning when a fraction of its constituents are damaged. Networks with a given degree distribution may be very resilient to one type of failure or attack but not to another. There may be different types of attack: Random attacks on any nodes and the targeted attacks to some special nodes in the networks.

The homogeneous structure of the random graph of Erdos-Renyi implies an invariant robustness under random node and link failures because every node forms links with others with equal probability. The heterogeneous structure of the scale-free graph of Barab´asi-Albert (preferential model), on the other hand, implies a non-trivial robustness to random node and link failures.

Parameters to judge the Robustness of the network

- **Algebraic connectivity**:

A metric, referred to as the **algebraic connectivity**, plays a special role for the robustness since it measures the extent to which it is difficult to cut the network into independent components.

Consider a graph G=(N,L) where N represents the set of nodes and L represents the set of links between them. The Laplacian matrix of G with N nodes is an N x N matrix Q = Δ - A where $\Delta = diag(d_i)$, d_i is the nodal degree of node i ϵ N and A is the adjacency matrix of G. The eigenvalues of Laplacian matrix $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_{N-1} \geq \lambda_N = 0$ are all real and non-negative. The second smallest eigenvalue of the standard Laplacian matrix, is called the Algebraic Connectivity.

- **K-connectivity:**

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For $k \geq 1$, a graph G is (node) *k*-connected if either G is a complete graph K_{k+1} or it has at least *k+2* nodes and no set of *k-1* nodes that separates it. For $k \geq 1$, a graph G is *k*-link connected if it has at least two nodes and no set of at most *k*-1 links that separates it. The maximum value of *k* for which a connected graph is *k*-connected equals the node connectivity K_N. The link connectivity K_L is defined analogously. The node (link) connectivity of an incomplete graph $G \neq K_N$ is at least as large as the algebraic connectivity $\lambda_{N-1} \leq K_N \leq K_L$. *If* $G = K_N$, *then* $\lambda_{N-1} = N > K_N = N-1$.

Analysis of Erdos-Renyi Model

Let $G_p(N)$ be a random graph. N is the number of nodes and p is the probability of having a link between any two nodes. Degree distribution is a binomial distribution, which can be replaced by a Poisson distribution, i.e.

$$Pr[D=k] = {}^{N-1}C_k p^k (1-p)^{N-1-k} \approx (m^k e^{-m})/k! \quad (1)$$

where $m = p(N-1)$ equals the mean nodal degree.

If links are added one by one to the empty random graph of N nodes in an order chosen uniformly and at random from the ${}^{N}C_2!$ possibilities, then almost surely the resulting graph becomes k-connected when it achieves a minimum degree of k. In other words, for large N,

$$Pr[G_p(N) \text{ is } k\text{-connected}] = Pr[d_{min} \geq k] \quad (2)$$

where d_{min} is the minimum nodal degree.

To understand the probability distribution of the algebraic connectivity λ_{N-1}, we will use the second theorem of connectivity to deduce the probability p of the presence of a link between any two nodes in the Erdos and Renyi random graph. From the previous equations, we have that the probability of *k*-connectivity in $G_p(N)$ equals

$$Pr[G_p(N) \text{ is } k\text{-connected}] = \left(1 - \sum_{m=0}^{k-1} \{(m)^m e^{-m}\}/m!\ \right)^N$$

Results for Erdos-Renyi model

The link probability p in $G_p(N)$ with N= 50, 100, 200 and 400, for values of the probability of being 1-connected and 10-connected of 0.5 and 0.9.

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Pr[k-connectivity]	N=50	N=100	N=200	N=400
0.5 for k=1	P= 0.0875	P=0.0503	P=0.0285	P=0.0159
0.9 for k=1	P=0.1258	P=0.0693	P=0.0379	P=0.0207
0.5 for k=10	P=0.3715	P=0.1963	P=0.1036	P=0.0546
0.9 for k=10	P=0.4378	P=0.2280	P=0.1189	P=0.0620

The Random Graph Model (A variation of Duplication Model)

First, several nodes are picked uniformly at random and then they are “duplicated” to create the new nodes that are initially connected to all the neighbors of the node of which it is a copy with probability p.

Degree Distribution of this model is given as:
$Pr[D=k]= [(N-1)p^2/N]^m+{}^{n-1}C_y p^y(1-p)^{n-1-y}$
where $y=k-[(N-1)p^2/N]^m$
where the first term is the expected number of nodes that any node will get from the new nodes and second term implies its probability of having y (de pendant on k) links from the old nodes.

$$Pr[D\geq k]= (1- \sum_{i=0}^{k-1} Pr[D=i])$$

$$Pr[\text{ degree of new nodes} \geq k]=\sum_{d=k}^{n} [e^{-(n-1)p}\{ (n-1)p\}^d/d!][\sum_{i=k}^{d} {}^dC_i p^i(1-p)^{d-i}]$$

Results

We have analyzed the trend and probability of the duplication model being 4-connected for N=20 and N=25.

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M	P=0.1	P=0.2	P=0.3
5	0.00034793	0.0337692	0.05766938
8	3.382×10^{-5}	0.007934	0.03099066
10	1.5407×10^{-7}	0.00302438	0.03099066

M	P=0.1	P=0.2	P=0.3
5	0.00012831	0.0197444	0.062554
8	6.6521×10^{-7}	0.0029808	0.0341712
10	1.99×10^{-8}	0.0008453	0.229199

Conclusion

We analyzed the trend of the graph to be k-connected in the following manner:
Along any row, if the link formation probability ‘p’ increases, the probability of the graph to remain k-connected increases, which resonates with the fact that higher the link formation probability ‘p’, higher the probability of the graph to be k-connected. Along any column, since the number of nodes that are entering increases, we will have to account for more number of links for the graph to be k-connected. Hence the lower probabilities. As the *k*-connectivity increases for any graph, the probability of failure of network due to random attacks goes on decreasing. So *k*-connectivity measures the extent of robustness of the network.

References

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