Quantum Galton Boards: Concepts, Mathematics, and Implementation

1 Classical vs. Quantum Galton Boards

A *Galton board* (bean machine) is a mechanical device in which balls fall through a triangular array of pegs, producing a binomial (approximately Gaussian) distribution of outcomes. For an n-level board with unbiased pegs ($p = q = 0.5$), the probability to reach the k-th output bin is

$$P(k) = \binom{n}{k} p^k q^{n-k}, \quad p = q = \tfrac{1}{2},$$

which approaches a normal distribution in the large-n limit via the De Moivre–Laplace theorem [1].

In the *quantum* Galton board (QGB), the classical ball is replaced by a quantum walker that propagates through a network of “quantum pegs” (coherent beam splitters or controlled gates). Each peg applies a quantum coin operation to create a superposition, followed by a conditional shift in position. All possible paths interfere, yielding output distributions with nonclassical features such as ballistic spreading and non-Gaussian shapes [2, 3].

2 Quantum Walk Formalism

The QGB can be formulated as a discrete-time coined quantum walk on a one-dimensional lattice. The Hilbert space is

$$\mathcal{H} = \mathcal{H}_{\text{coin}} \otimes \mathcal{H}_{\text{pos}},$$

where the coin subspace $\{|L\rangle, |R\rangle\}$ controls left/right movement and $\mathcal{H}_{\text{pos}}$ is spanned by position basis states.

A single step applies:

$$U_{\text{step}} = S\,(C \otimes I),$$

where C is the coin operator (e.g. Hadamard H or rotation $R_x(\theta)$), and S is the conditional shift:

$$S\,|x, L\rangle = |x-1, L\rangle, \quad S\,|x, R\rangle = |x+1, R\rangle.$$

For the Hadamard coin:

$$H|L\rangle = \frac{|L\rangle + |R\rangle}{\sqrt{2}}, \quad H|R\rangle = \frac{|L\rangle - |R\rangle}{\sqrt{2}}.$$

After n steps, the walker’s state is a coherent superposition over 2^n paths, with interference determined by C.

3 Circuit-Level Implementation

Carney and Varcoe [3] describe a modular quantum circuit for the QGB. Each *quantum peg* consists of:

1. Initializing all qubits in $|0\rangle$.
2. Applying a coin flip (Hadamard or $R_x(\theta)$) to a *coin* qubit.
3. Using an X gate on a “ball” qubit to mark occupancy.
4. Applying a controlled-SWAP (Fredkin) gate to route the ball to left or right *position registers*, conditioned on the coin state.
5. Resetting the coin via an inverted CNOT to prepare for the next layer.

By cascading such pegs, an n-level QGB is realized. The construction uses $O(n^2)$ gates to simulate all 2^n classical trajectories, giving an exponential computational advantage over explicit classical simulation of all paths.

4 Distribution and Control

For a balanced QGB, aggregating many shots yields a binomial/normal-shaped distribution, consistent with the classical board. However, quantum interference can produce markedly different *single-shot* patterns: e.g., two dominant peaks at the edges, characteristic of ballistic spreading in quantum walks [2].

Biasing the QGB is achieved by replacing the Hadamard with $R_x(\theta)$, tuning the relative amplitudes for left/right movement [3]. Experiments on IBM Quantum hardware show that, despite NISQ noise, the dominant output bins match the expected central peaks in biased boards, validating the principle of quantum path superposition.

5 Experimental Realizations

QGBs and related quantum walks have been realized in:

- **Integrated photonics:** On-chip beam splitter networks coupled to superconducting nanowire detectors [5].
- **Fiber optics:** Time-multiplexed photonic walks showing clear non-Gaussian output distributions [4].
- **Superconducting qubits:** Circuit-based QGBs with controlled-SWAP gates [3].

These platforms confirm that coherent interference, absent in classical Galton boards, produces qualitatively new statistical outcomes.

References

References

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