Contents

Partial derivatives:

The functions which depend on more than one independent variable are called functions of several variables. Partial derivatives of a function of several variables are the derivative with respect to one of the variables when all the remaining variables are kept constant

Let $z = f(x, y)$ be a function of two variables, keeping y constant and varying x only,

the partial derivative of z with respect to x is defined as

$$\frac{\partial z}{\partial x} = \underset{\Delta x \to 0}{lt} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x},$$

which is denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or $z_x(x, y)$ or $f_x(x, y)$ or z_x or f_x

Similarly Partial Derivative of z w.r.t y can be defined as

$$\frac{\partial z}{\partial y} = \underset{\Delta y \to 0}{lt} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y},$$

which is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or $z_y(x, y)$ or $f_y(x, y)$ or z_y or f_y

Partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ or $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are known as first-order partial derivatives,

while $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ are second order partial derivatives.

Note: 1 Similarly higher order partial derivatives can be obtained for several independent variables

Note: 2 $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y\, \partial x}$

Problems

1. Find the first order Partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ for the functions

i. $u = e^x \cos y$ ii. $u = \tan^{-1}\frac{y}{x}$

Solution:

i. $u = e^x \cos y$ --- (1)

Differentiate partially with respect to x equation (1),

$$\frac{\partial u}{\partial x} = e^x \cos y$$

Differentiate partially with respect to y equation (1)

$$\frac{\partial u}{\partial y} = -e^x \sin y$$

ii. $u = \tan^{-1}\frac{y}{x}$ --- (2)

Differentiate partially with respect to x equation (2)

$$\frac{\partial u}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{-y}{x^2+y^2}$$

Differentiate partially with respect to y equation (2)

$$\frac{\partial u}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x} = \frac{x}{x^2+y^2}$$

2. **If** $u=(x-y)^4+(y-z)^4+(z-x)^4$ **, show that** $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$.

Solution:

Given $u=(x-y)^4+(y-z)^4+(z-x)^4$ **-------- (1)**

Differentiate the equation (1) partially u w.r.t., x, y and z,

$$\frac{\partial u}{\partial x}=4(x-y)^3-4(z-x)^3 \; ;$$

$$\frac{\partial u}{\partial y}=-4(x-y)^3+4(y-z)^3;$$

$$\frac{\partial u}{\partial z}=-4(y-z)^3+4(z-x)^3$$

$$LHS=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$$

$$=4(x-y)^3-4(z-x)^3-4(x-y)^3+4(y-z)^3-4(y-z)^3+4(z-x)^3$$

$$=0$$

3. **If** $u=\sin^{-1}\left(\frac{y}{x}\right)$ **then, show that** $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$.

Solution:

Given $u=\sin^{-1}\left(\frac{y}{x}\right)$ ------ **(1)**

Differentiate the equation (1) partially u w.r.t. x, and y .

$$\frac{\partial u}{\partial x}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\frac{1}{y}=\frac{y}{\sqrt{y^2-x^2}};$$

$$\frac{\partial u}{\partial y}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\frac{-x}{y^2}=\frac{-x}{\sqrt{y^2-x^2}}$$

Now consider,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{xy}{\sqrt{y^2-x^2}}-\frac{xy}{\sqrt{y^2-x^2}}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.$$

4. **If** $z=\log\sqrt{x^2+y^2}$ **then, show that** $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=1$.

Solution:

Given $z=\log\sqrt{x^2+y^2}$ -------- **(1)**

Differentiate the equation (1) partially z w.r.t. x and y .

$$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{x^2+y^2}}\frac{1}{2\sqrt{x^2+y^2}}2x;$$

$$\frac{\partial z}{\partial y}=\frac{1}{\sqrt{x^2+y^2}}\frac{1}{2\sqrt{x^2+y^2}}2y$$

Now consider,

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{x^2+y^2}{x^2+y^2}$$

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=1.$$

5. **Given that $x = r\cos\theta$ and $y = r\sin\theta$ then find r_x, r_y, θ_x and θ_y.**

$$\left(r^2 = x^2 + y^2 \;\&\; \theta = \tan^{-1}\frac{y}{x}\right)$$

Solution:

Given $x = r\cos\theta$ and $y = r\sin\theta$ can be written $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$

($x = r\cos\theta$ and $y = r\sin\theta$ are the parametric equations of $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$)

$$r = \sqrt{x^2 + y^2} \qquad \text{--- (1)}$$

$$\theta = \tan^{-1}\frac{y}{x} \qquad \text{--- (2)}$$

Differentiate (1) partially r w.r.t. x, and y .

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2 + y^2}} 2x = \frac{r\cos\theta}{r} = \cos\theta;$$

$$\frac{\partial r}{\partial y} = \frac{1}{2\sqrt{x^2 + y^2}} 2y = \frac{r\sin\theta}{r} = \sin\theta;$$

Differentiate (2) partially θ w.r.t. x, and y .

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \frac{-y}{x^2} = \frac{-y}{x^2 + y^2};$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \frac{1}{x} = \frac{x}{x^2 + y^2};$$

6. If $z = f(ax+by)$ then, show that $b\dfrac{\partial z}{\partial x} - a\dfrac{\partial z}{\partial y} = 0$.

Solution:

Given $z = f(ax+by)$ **--- (1)**

Differentiate the equation (1) partially z w.r.t. x and y.

$$\frac{\partial z}{\partial x} = f^{1}(ax+by)\,a\,;$$

$$\frac{\partial z}{\partial y} = f^{1}(ax+by)b$$

Now consider,

$$b\frac{\partial z}{\partial x} - a\frac{\partial z}{\partial y} = abf^{1}(ax+by) - baf^{1}(ax+by)$$

$$b\frac{\partial z}{\partial x} - a\frac{\partial z}{\partial y} = 0$$

7. **If** $z = e^{ax+by} f(ax-by)$, **then prove that** $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$.

Solution:

Given $z = e^{ax+by} f(ax-by)$ ----------- (*)

$$\frac{\partial z}{\partial x} = ae^{ax+by} f(ax-by) + ae^{ax+by} f^{1}(ax-by) \quad \text{-------- (1)}$$

$$\frac{\partial z}{\partial y} = be^{ax+by} f(ax-by) - be^{ax+by} f^{1}(ax-by) \quad \text{-------- (2)}$$

Now consider

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = abe^{ax+by} f(ax-by) + abe^{ax+by} f^{1}(ax-by) + abe^{ax+by} f(ax-by) - abe^{ax+by} f^{1}(ax-by)$$

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abe^{ax+by} f(ax-by)$$

Using (*) we get,

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz.$$

Homogeneous functions and Euler's theorem

A polynomial in x and y, i.e. $f(x,y)=a_0x^n+a_1x^{n-1}y+a_2x^{n-2}y^2+\dots\dots\dots\dots+a_ny^n$ is said to be homogeneous if all its terms are of same degree.

In general, A function $f(x,y)$ is said to be homogenous of degree n if it can be expressed as $x^n\varphi\left(\frac{y}{x}\right)$ or $y^n\varphi\left(\frac{x}{y}\right)$ where 'n' can be positive, negative or zero.

Euler's Theorem for homogeneous function.

Statement

If u be a homogeneous function of degree n in x and y then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$

Proof:

Since u is a homogenous function of degree n in x and y

$$u=x^n f\left(\frac{y}{x}\right) \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

Differentiate partially (1) u with respect to x

$$\frac{\partial u}{\partial x}=nx^{n-1}f\left(\frac{y}{x}\right)+x^n f^1\left(\frac{y}{x}\right)*\left(\frac{-y}{x^2}\right) \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

Differentiate partially (1) u with respect to y, we get

$$\frac{\partial u}{\partial y}=x^n f^1\left(\frac{y}{x}\right)\frac{1}{x} \qquad \ldots\ldots\ldots\ldots\ldots (3)$$

Multiply equation (2) by x and (3) by y, we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nx^n f\left(\frac{y}{x}\right)$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

Hence the Euler's theorem proved.

Problems

1.. Verify the Euler's theorem for the function $u=\dfrac{1}{\sqrt{x^2+y^2+z^2}}$.

Solution: Given $u=\dfrac{1}{\sqrt{x^2+y^2+z^2}}$ ------- (1)

$$u=\frac{1}{x\sqrt{1+\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2}}$$

$$u=x^{-1}\ f\left(\frac{y}{x},\frac{z}{x}\right)$$

u is a homogeneous function in x with degree -1.

By Euler's theorem $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=-u$. ------- (2)

Differentiate partially (1) u with respect to x, y and z, we get

$$\frac{\partial u}{\partial x}=-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\ 2x=\frac{-x}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial u}{\partial y}=-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\ 2y=\frac{-y}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial u}{\partial z}=-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\ 2z=\frac{-z}{(x^2+y^2+z^2)^{3/2}}$$

Now consider,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{-x^2}{(x^2+y^2+z^2)^{3/2}}+\frac{-y^2}{(x^2+y^2+z^2)^{3/2}}+\frac{-z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-\frac{(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-\frac{1}{\sqrt{(x^2+y^2+z^2)}}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-u \qquad \text{--- (3)}$$

From equations (2) and (3), Euler's theorem verified.

2. Verify the Euler's theorem for the function $u = ax^2 + 2hxy + b^2y^2$

Solution:

Given $u = ax^2 + 2hxy + b^2$

$$u = x^2\left(a + 2h\frac{y}{x} + b\frac{y^2}{x^2}\right) \quad \text{--- (1)}$$

$$u = x^2 \; f\left(\frac{y}{x}\right)$$

u is a homogeneous function in x with degree 2.

By Euler's theorem $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u$. --- (2)

Differentiate partially (1) u with respect to x and y , we get

$$\frac{\partial u}{\partial x} = 2ax + 2hy + 0$$

$$\frac{\partial u}{\partial y} = 0 + 2hx + 2by$$

Now consider, $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + = 2ax^2 + 2hxy + 2ay^2 + 2hxy$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + = 2(ax^2 + 2hxy + 2ay^2)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u \quad \text{--- (3)}$$

From equations (2) and (3), Euler's theorem verified.

3. Show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u\log u$ **, where** $\log u=\frac{x^3+y^3}{3x+4y}$.

Solution:

Let $z=\frac{x^3+y^3}{3x+4y}=x^2.\frac{1+\left(\frac{y}{x}\right)^3}{3+4\left(\frac{y}{x}\right)}$ where $z=\log u$ ----- (1)

z is a homogenous function of degree 2 in x and y

By Euler's theorem, we get

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z \qquad \text{------- (2)}$$

Differentiate partially (1) w.r.t. x, and y .

$$\frac{\partial z}{\partial x}=\frac{1}{u}\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y}=\frac{1}{u}\frac{\partial u}{\partial y}$$

Substituting these in equation (2), we get

$$x\frac{1}{u}\frac{\partial u}{\partial x}+y\frac{1}{u}\frac{\partial u}{\partial y}=2\ln u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u\log u\ .$$

4. **If** $u = e^{x^2y^2/(x+y)}$ **then show that** $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u\log u$.

Solution:

Given $u = e^{x^2y^2(x+y)}$

Apply ln on both sides

$$\ln u = \ln e^{x^2y^2/(x+y)}$$

$$\ln u = \frac{x^2y^2}{(x+y)}$$

Let $z = \frac{x^2y^2}{(x+y)} = \frac{x^4\left(\frac{y^2}{x^2}\right)}{x\left(1+\frac{y}{x}\right)} = x^3\frac{\left(\frac{y^2}{x^2}\right)}{\left(1+\frac{y}{x}\right)}$,

where $z = \log u$ ------- (1)

z is a homogenous function in x with degree 3.

By Euler's theorem, we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z \text{ ------- } (2)$$

Differentiate partially (1) w.r.t. x, and y .

$$\frac{\partial z}{\partial x} = \frac{1}{u}\frac{\partial u}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \frac{1}{u}\frac{\partial u}{\partial y}$$

Substituting these in equation (2), we get

$$x\frac{1}{u}\frac{\partial u}{\partial x} + y\frac{1}{u}\frac{\partial u}{\partial y} = 3\ln u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u\log u \text{ .}$$

5. **If** $u = \tan^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ **then show that** $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{4}\sin 2u$.

Solution:

Given $u = \tan^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$

$$\tan u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

Let $z = \dfrac{x+y}{\sqrt{x}+\sqrt{y}} = \dfrac{x\left(1+\dfrac{y}{x}\right)}{\sqrt{x}\left(1+\dfrac{\sqrt{y}}{\sqrt{x}}\right)} = \sqrt{x}\dfrac{\left(1+\dfrac{y}{x}\right)}{\left(1+\sqrt{\dfrac{y}{x}}\right)}$,

where $z = \tan u$ --- (1)

z is a homogenous function in x with degree 1/2.

By Euler's theorem, we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2}z \quad \text{-------- (2)}$$

Differentiate partially (1) w.r.t. x and y.

$$\frac{\partial z}{\partial x} = \sec^2 u\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \sec^2 u\frac{\partial u}{\partial y}$$

Substituting these in equation (2), we get

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\sin u}{\cos u}\frac{1}{\sec^2 u}.$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{2}{2}\sin u\cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{4}\sin 2u.$$

Extension of Euler's Theorem

Statement

If $z = f(x, y)$ is a homogeneous function of x and y of degree 'n', then

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

Proof:

By Euler's Theorem, we have

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz \qquad \text{----- (1)}$$

Differentiate equation (1) partially with respect to x

$$\frac{\partial z}{\partial x} + x \frac{\partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{\partial x \partial y} = n \frac{\partial z}{\partial x} \qquad \text{-------- (2)}$$

Differentiate equation (1) partially with respect to y

$$\frac{\partial z}{\partial y} + y \frac{\partial^2 z}{\partial y^2} + x \frac{\partial^2 z}{\partial y \partial x} = n \frac{\partial z}{\partial y} \qquad \text{------- (3)}$$

Multiply equation 2 by x and 3 by y, we get

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)z \; .$$

Problems

1. **If** $u = \tan^{-1}\sqrt{x^4 + y^4}$ **then prove that**

i. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$ **and** **ii.** $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u$

Solution:

Given $u = \tan^{-1}\sqrt{x^4 + y^4}$

$\tan u = \sqrt{x^4 + y^4}$

Let $z = \sqrt{x^4 + y^4} = \sqrt{x^4\left(1 + \left(\frac{y}{x}\right)^4\right)} = x^2\sqrt{\left(1 + \left(\frac{y}{x}\right)^4\right)}$,

where $z = \tan u$ --- (1)

z is a homogenous function in x with degree 2.

By Euler's theorem, we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \quad \text{------- (2)}$$

Differentiate partially (1) w.r.t. x, and y .

$$\frac{\partial z}{\partial x} = \sec^2 u\frac{\partial u}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \sec^2 u\frac{\partial u}{\partial y}$$

Substituting these in equation (2), we get

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{\sin u}{\cos u}\frac{1}{\sec^2 u}.$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\sin u\cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u. \quad \text{-------- (3)}$$

Differentiate partially equation (3) w.r.t x and y again, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u\frac{\partial u}{\partial x}$$

Multiply x on both sides in above equation, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u\ x\frac{\partial u}{\partial x} \qquad \text{------- (4)}$$

Similarly,

$$y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y\partial x}=2\cos 2u\frac{\partial u}{\partial y}$$

Multiply y on both sides in above equation, we get

$$y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial x}+yx\frac{\partial^2 u}{\partial y\partial x}=2\cos 2u\ y\frac{\partial u}{\partial y} \qquad \text{------- (5)}$$

Adding equations (4) and (5), we get

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 u}{\partial x\partial y}\,y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial x}+yx\frac{\partial^2 u}{\partial y\partial x}=2\cos 2u\ y\frac{\partial u}{\partial y}+2\cos 2u\ x\frac{\partial u}{\partial x}$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}\,y^2\frac{\partial^2 u}{\partial y^2}+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial x}\right)=2\cos 2u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial x}\right) \text{------- (6)}$$

Using equation (3) in (6)

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}\,y^2\frac{\partial^2 u}{\partial y^2}+\sin 2u=2\cos 2u\sin 2u$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}\,y^2\frac{\partial^2 u}{\partial y^2}=2\ \cos 2u\sin 2u-\sin 2u$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}\,y^2\frac{\partial^2 u}{\partial y^2}=\sin 4u-\sin 2u$$

2. **If** $u=\tan^{-1}\left(\frac{y}{x}\right)+y\sin^{-1}\left(\frac{x}{y}\right)$ **then prove that** $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}y^2\frac{\partial^2 u}{\partial y^2}=0$

Solution:

Given $u=\tan^{-1}\left(\frac{y}{x}\right)+y\sin^{-1}\left(\frac{x}{y}\right)$ ------ (1)

Let $u=v+w,$

where $v=\tan^{-1}\left(\frac{y}{x}\right),\ w=y\sin^{-1}\left(\frac{x}{y}\right)$ ----- (*)

Consider $v=\tan^{-1}\left(\frac{y}{x}\right)$

$$v=x^0\tan^{-1}\left(\frac{y}{x}\right)$$

Implies that v is homogeneous function in x with degree 0, then by Euler's extension theorem we can write

$$x^2\frac{\partial^2 v}{\partial x^2}+2xy\frac{\partial^2 v}{\partial x\partial y}y^2\frac{\partial^2 v}{\partial y^2}=0 \quad \text{------- (2)}$$

Again consider $w=y\sin^{-1}\left(\frac{x}{y}\right)$

$$w=x^1\ \frac{y}{x}\sin^{-1}\left(\frac{1}{\frac{y}{x}}\right)$$

Implies that w is homogeneous function in x with degree 1, then by Euler's extension theorem we can write $x^2\frac{\partial^2 w}{\partial x^2}+2xy\frac{\partial^2 w}{\partial x\partial y}y^2\frac{\partial^2 w}{\partial y^2}=1(1-1)w$

i.e. $x^2\frac{\partial^2 w}{\partial x^2}+2xy\frac{\partial^2 w}{\partial x\partial y}y^2\frac{\partial^2 w}{\partial y^2}=0$ ----- (3)

Adding (2) and (3) we get

$$x^2\frac{\partial^2 v}{\partial x^2}+2xy\frac{\partial^2 v}{\partial x\partial y}y^2\frac{\partial^2 v}{\partial y^2}+x^2\frac{\partial^2 w}{\partial x^2}+2xy\frac{\partial^2 w}{\partial x\partial y}y^2\frac{\partial^2 w}{\partial y^2}=0$$

$$x^2\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 w}{\partial x^2}\right)+2xy\left(\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial x\partial y}\right)+y^2\left(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\right)=0$$

$$x^2\frac{\partial^2}{\partial x^2}(v+w)+2xy\frac{\partial^2}{\partial x\partial y}(v+w)+y^2\frac{\partial^2}{\partial y^2}(v+w)=0$$

Using equation (*), we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}y^2\frac{\partial^2 u}{\partial y^2}=0$$

Chain Rule

Total Differentials:

Consider a function $f = f(x, y)$ of two independent variable x and y then the total differential is defined as $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ .Similarly, if a function $f = f(x, y, z)$ of three independent variable x, y and z then the total differential is defined as $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$.

Total Derivatives:

Consider a function $f = f(x(t), y(t))$ where x and y are functions of t, then total derivative of f with respect to t is defined as $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

Similarly, if a function $f = f(x(t)y(t)z(t))$ where x, y and z are functions of t then total derivative of f with respect to t is defined as $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$

Problems

1. **If** $u = x^2 + y^2 + z^2$ **and** $x = e^{2t}$, $y = e^{2t}\cos 3t$ **,** $z = e^{2t}\sin 3t$ **then, Find** $\frac{du}{dt}$ **as a total derivative and verify the result by direct substitution**

Solution:

$$\text{Given } u = x^2 + y^2 + z^2 \ , \ x = e^{2t} , y = e^{2t}\cos 3t \ \text{ and } \ z = e^{2t}\sin 3t$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= 4x\ e^{2t} + 2y(-e^{2t}3\sin 3t + 2\cos 3t\ e^{2t}) + 2z(e^{2t}\cos 3t + \sin 3t\ 2e^{2t})$$

$$= 4x\ e^{2t} - 6y\ e^{2t}\sin 3t + 4y\cos 3t\ e^{2t}) + 6z\ e^{2t}\cos 3t + 4z\ \ e^{2t}\sin 3t)$$

$$= 4xx - 6yz + 4yy + 6zy + 4zz$$

$$= 4x^2 - 6yz + 4y^2 + 6zy + 4z^2$$

$$= 4\left(x^2 + y^2 + z^2\right)$$

$$= 4u$$

$$\frac{du}{dt} = 4(e^{4t} + e^{4t}(\cos^2 t + \sin^2 t))$$

$$= 4(e^{4t} + e^{4t})$$

$$= 8e^{4t}$$

2. **If** $u=\sin^{-1}(x-y)$, $x=3t$ **and** $y=4t^3$ **then show that** $\frac{du}{dt}=3(1-t^2)^{-1/2}$.

Solution:

Given $u=\sin^{-1}(x-y)$, $x=3t$ and $y=4t^3$

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$x-y=3t-4t^3$$

$$x-y=t(3-4t^2)$$

$$(x-y)^2=\left(t(3-4t^2)\right)^2$$

$$(x-y)^2=t^2(9+16t^4-24t^2)$$

$$\frac{du}{dt}=\frac{1}{\sqrt{1-(x-y)^2}}3+\frac{-1}{\sqrt{1-(x-y)^2}}12t^2$$

$$\frac{du}{dt}=\frac{3-12t^2}{\sqrt{1-(x-y)^2}}$$

$$\frac{du}{dt}=\frac{3(1-4t^2)}{\sqrt{1-9t^2-16t^6+24t^4}}$$

$$\frac{du}{dt}=\frac{3(1-4t^2)}{\sqrt{1-8t^2-t^2-16t^6+16t^4+8t^4}}$$

$$\frac{du}{dt}=\frac{3(1-4t^2)}{\sqrt{1-8t^2+16t^4-t^2-16t^6+8t^4}}$$

$$\frac{du}{dt}=\frac{3(1-4t^2)}{\sqrt{(1-8t^2+16t^4)-t^2(1-8t^2+16t^4)}}$$

$$\frac{du}{dt}=\frac{3(1-4t^2)}{\sqrt{(1-8t^2+16t^4)(1-t^2)}}$$

$$\frac{du}{dt}=\frac{3(1-4t^2)}{\sqrt{(1-4t^2)^2(1-t^2)}}$$

$$\frac{du}{dt}=\frac{3}{\sqrt{(1-t^2)}}$$

$$\frac{du}{dt}=3(1-t^2)^{-1/2}$$

Alternatively

Given $u = \sin^{-1}(x-y)$, $x = 3t$ and $y = 4t^3$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$\frac{du}{dt} = \frac{1}{\sqrt{1-(x-y)^2}}3 + \frac{-1}{\sqrt{1-(x-y)^2}}12t^2$$

$$\frac{du}{dt} = \frac{3-12t^2}{\sqrt{1-(x-y)^2}}$$

$$\frac{du}{dt} = \frac{3}{\sqrt{\frac{1-9t^2-16t^6+24t^4}{(1-4t^2)^2}}}$$

$$\frac{du}{dt} = \frac{3}{\sqrt{\frac{1-9t^2-16t^6+24t^4}{1+16t^4-8t^2}}}$$

$$\frac{du}{dt} = \frac{3}{\sqrt{(1-t^2)}}$$

$$\frac{du}{dt} = 3(1-t^2)^{-1/2}$$

$$x-y = 3t-4t^3$$

$$x-y = t(3-4t^2)$$

$$(x-y)^2 = \left(t(3-4t^2)\right)^2$$

$$(x-y)^2 = t^2(9+16t^4-24t^2)$$

$$1+16t^4-8t^2)-16t^6+24t^4-9t^2+1(-t^2+1$$

$$-16t^6+8t^4-t^2$$

$$+ \quad - \quad +$$

$$Sub \qquad 16t^4-8t^2+1$$

$$16t^4-8t^2+1$$

$$- \quad + \quad -$$

$$Sub \qquad 0$$

$$\frac{-16t^6+24t^4-9t^2+1}{1+16t^4-8t^2} = -t^2+1$$

3. **If $u=\tan^{-1}(y/x)$ where $x=e^t-e^{-t}$ and $y=e^t+e^{-t}$ then find $\dfrac{du}{dt}$**

Solution:

Given $u=\tan^{-1}(y/x)$ where $x=e^t-e^{-t}$ and $y=e^t+e^{-t}$

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$\frac{du}{dt}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{-y}{x^2}(e^t+e^{-t})+\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x}(e^t-e^{-t})$$

$$\frac{du}{dt}=\frac{-y}{x^2+y^2}y+\frac{x}{x^2+y^2}x$$

$$\frac{du}{dt}=\frac{-y^2}{x^2+y^2}+\frac{x^2}{x^2+y^2}$$

$$\frac{du}{dt}=\frac{x^2-y^2}{x^2+y^2}$$

$$x^2=e^{2t}+e^{-2t}-2$$

$$y^2=e^{2t}+e^{-2t}+2$$

$$x^2+y^2=2(e^{2t}+e^{-2t})$$

$$x^2-y^2=-4$$

$$\frac{du}{dt}=\frac{-4}{2(e^{2t}+e^{-2t})}$$

$$\frac{du}{dt}=\frac{-2}{(e^{2t}+e^{-2t})}$$

Differentiation of implicit functions

If $f(x, y) = c$ be an implicit function of x & y

then derivative of the function f w r t x is given by $\dfrac{df}{dx} = 0$ ... (1)

From the definition of total derivatives we have $\dfrac{df}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$ ... (2)

From (1) and (2) we get $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y}$.

Problems

1. **If** $u = x\log xy$ **where** $x^3 + y^3 + 3xy = 1$ **then, find** $\dfrac{du}{dx}$.

Solution:

Let $f(x, y) = x^3 + y^3 + 3xy - 1$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{(3x^2 + 3y)}{(3y^2 + 3x)} = -\frac{x^2 + y}{y^2 + x}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$\frac{du}{dx} = 1 + \log xy + \frac{\partial u}{\partial y}\left(-\frac{x^2 + y}{y^2 + x}\right)$$

$$\frac{du}{dx} = 1 + \log xy - \frac{x}{y}\left(\frac{x^2 + y}{y^2 + x}\right)$$

2. **If** $u = e^{x^2+y^2}$ **where** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **then, find** $\frac{du}{dx}$.

Solution:

Let $f(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$

$$\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y} = -\frac{2x/a^2}{2y/b^2} = -\frac{b^2}{a^2}\frac{x}{y}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$\frac{du}{dx} = 2x\, e^{x^2+y^2} + 2y\, e^{x^2+y^2}\left(-\frac{b^2}{a^2}\frac{x}{y}\right)$$

$$\frac{du}{dx} = \frac{2x}{a^2}(a^2 - b^2)e^{x^2+y^2}.$$

3. **If** $u = \cos(x^2 - y^2)$ **where** $a^2x^2 + b^2y^2 = c^2$ **then, find** $\frac{du}{dx}$.

Solution:

Let $f(x,y) = a^2x^2 + b^2y^2 - c^2$

$$\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y} = -\frac{2xa^2}{2yb^2} = -\frac{a^2}{b^2}\frac{x}{y}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$\frac{du}{dx} = -2x\sin(x^2 - y^2) - 2y\,\sin(x^2 - y^2)\left(-\frac{a^2}{b^2}\frac{x}{y}\right)$$

$$\frac{du}{dx} = -2x\left(1 + \frac{a^2}{b^2}\right)\sin(x^2 - y^2).$$

4. **If** $x^m y^n = (x+y)^{m+n}$ **prove that** $\dfrac{dy}{dx} = \dfrac{y}{x}$.

Solution:

Let $f(x,y) = x^m y^n - (x+y)^{m+n}$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{m\,x^{m-1}y^n - (m+n)(x+y)^{m+n-1}}{n\,x^m y^{n-1} - (m+n)(x+y)^{m+n-1}}$$

$$\frac{dy}{dx} = -\frac{m\,x^{m-1}y^n - \dfrac{(m+n)(x+y)^{m+n}}{(x+y)}}{n\,x^m y^{n-1} - \dfrac{(m+n)(x+y)^{m+n}}{(x+y)}}$$

$$\frac{dy}{dx} = -\frac{\dfrac{m\,x^m y^n + mx^{m-1}y^{n+1} - (m+n)(x+y)^{m+n}}{(x+y)}}{\dfrac{n\,x^{m+1}y^{n-1} + nx^m y^n - (m+n)(x+y)^{m+n}}{(x+y)}}$$

$$\frac{dy}{dx} = -\frac{m\,x^m y^n + mx^{m-1}y^{n+1} - (m+n)x^m y^n}{n\,x^{m+1}y^{n-1} + nx^m y^n - (m+n)x^m y^n}$$

$$\frac{dy}{dx} = -\frac{x^m y^n}{x^m y^n}\,\frac{m + m\dfrac{y}{x} - m - n}{n\dfrac{x}{y} + n - m - n}$$

$$\frac{dy}{dx} = -\frac{m\dfrac{y}{x} - n}{n\dfrac{x}{y} - m}$$

$$\frac{dy}{dx} = -\frac{\dfrac{my - nx}{x}}{\dfrac{-(my - nx)}{y}}$$

$$\frac{dy}{dx} = \frac{y}{x}.$$

Differentiation of Composite functions

Consider a function $f = f\big(x(u,v), y(u,v)\big)$ where f is a function of x and y further x and y is a function u and v then the differentiation of composite function f is defined as

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \text{ or } f_u = f_x x_u + f_y y_u$$

and

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \text{ or } f_v = f_x x_v + f_y y_v$$

Similarly, Consider a function $f = f\big(u(x,y), v(x,y)\big)$ where f is a function of u and v further u and v is a function of x and y then the differentiation of composite function f is defined as

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \text{ or } f_x = f_u u_x + f_v v_x \text{ and } \frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} \text{ or } f_y = f_u u_y + f_v v_y.$$

In the same manner composite function can be defined to the desired functions, here some following examples can go through

- **If** $f = f\big(x(u,v), y(u,v), z(u,v)\big)$ **then** $f_u = f_x x_u + f_y y_u + f_z z_u$; $f_v = f_x x_v + f_y y_v + f_z z_v$.
- **If** $f = f\big(x(u,v,w), y(u,v,w), z(u,v,w)\big)$

  then $f_u = f_x x_u + f_y y_u + f_z z_u$; $f_v = f_x x_v + f_y y_v + f_z z_v$; $f_w = f_x x_w + f_y y_w + f_z z_w$.
- **If** $f = f\big(x(u,v,w), y(u,v,w)\big)$ **then** $f_u = f_x x_u + f_y y_u$; $f_v = f_x x_v + f_y y_v$; $f_w = f_x x_w + f_y y_w$.

Problems

1. If $z = \frac{\cos y}{x}$ **and** $x = u^2 - v$**,** $y = e^v$ **. Prove that** $\frac{\partial z}{\partial v} = \frac{\cos y - xy\sin y}{x^2}$ **.**

Solution:

Given $z = \frac{\cos y}{x}$ where $x = u^2 - v$, $y = e^v$

By the definition of composite function, we have

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial v} = -\frac{1}{x^2}(-\cos y) + \left(-\frac{\sin y}{x}\right)e^v$$

$$\frac{\partial z}{\partial v} = \frac{\cos y - xy\sin y}{x^2}.$$

2. If $u = x^2 - y^2$ **and** $x = 2r - 3s + 4$**,** $y = -r + 8s - 5$**. Prove that** $\frac{\partial u}{\partial r} = 4x + 2y$**.**

Solution:

Given $u = x^2 - y^2$ where $x = 2r - 3s + 4$, $y = -r + 8s - 5$

By the definition of composite function, we have

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial r} = 2x.2 + (-2y)(-1)$$

$$\frac{\partial u}{\partial r} = 4x + 2y$$

3. If $z = f(u,v)$ **where,** $u = e^x \cos y$ **and,** $v = e^x \sin y$ **, then show that** $\dfrac{\partial z}{\partial x} = u\dfrac{\partial z}{\partial u} + v\dfrac{\partial z}{\partial v}$**.**

Solution:

Given $z = f(u,v)$ where, $u = e^x \cos y$ and $v = e^x \sin y$,

By the definition of composite function, we have

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}e^x \cos y + \frac{\partial z}{\partial v}e^x \sin y$$

$$\frac{\partial z}{\partial x} = u\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}v.$$

4. If $u=\dfrac{x}{y}$, $v=\dfrac{y}{z}$, $w=z$ and $f=f(u,v,w)$ then show that $x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}+z\dfrac{\partial f}{\partial z}=w\dfrac{\partial f}{\partial w}$.

Solution:

Given $f=f(u,v,w)$ where, $u=\dfrac{x}{y}, v=\dfrac{y}{z}, w=z$,

By the definition of composite function, we have

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$$

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\left(\frac{1}{y}\right)+\frac{\partial f}{\partial v}(0)+\frac{\partial f}{\partial w}(0) \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial f}{\partial w}\frac{\partial w}{\partial y}$$

$$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\left(-\frac{x}{y^2}\right)+\frac{\partial f}{\partial v}\left(\frac{1}{z}\right)+\frac{\partial f}{\partial w}(0) \quad \text{--- (2)}$$

$$\frac{\partial f}{\partial z}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial z}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial z}+\frac{\partial f}{\partial w}\frac{\partial w}{\partial z}$$

$$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}(0)+\frac{\partial f}{\partial v}\left(-\frac{y}{z^2}\right)+\frac{\partial f}{\partial w}(1) \quad \text{--- (3)}$$

From (1), (2) and (3), we get

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=\frac{x}{y}\frac{\partial f}{\partial u}+\frac{y}{z}\frac{\partial f}{\partial v}-\frac{x}{y}\frac{\partial f}{\partial u}+z\frac{\partial f}{\partial w}-\frac{y}{z}\frac{\partial f}{\partial v}$$

$$=z\frac{\partial f}{\partial w}$$

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=w\frac{\partial f}{\partial w} \text{ (because } z=w\text{)}$$

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5. If $x = u+v+w$, $y = uv+vw+wu$, $z = uvw$ and $f = f(x,y,z)$ then show that $u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}+w\frac{\partial f}{\partial w} = x\frac{\partial f}{\partial x}+2y\frac{\partial f}{\partial y}+3z\frac{\partial f}{\partial z}$.

Solution:

Given $f = f(x,y,z)$ where, $x = u+v+w$, $y = uv+vw+wu$, $z = uvw$,

By the definition of composite function, we have

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(v+w)+\frac{\partial f}{\partial z}vw \quad \text{------ (1)}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(u+w)+\frac{\partial f}{\partial z}uw \quad \text{------ (2)}$$

$$\frac{\partial f}{\partial w} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial w}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial w}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial w}$$

$$\frac{\partial f}{\partial w} = \frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(u+v)+\frac{\partial f}{\partial z}uv \quad \text{----- (3)}$$

From equations (1), (2) and (3), we get

$$u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}+w\frac{\partial f}{\partial w} = u\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(v+w)+\frac{\partial f}{\partial z}vw\right)+v\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(u+w)+\frac{\partial f}{\partial z}uw\right)+w\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(u+v)+\frac{\partial f}{\partial z}uv\right)$$

$$u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}+w\frac{\partial f}{\partial w} = u\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(uv+uw)+\frac{\partial f}{\partial z}uvw+v\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(uv+vw)+\frac{\partial f}{\partial z}uvw+w\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(uw+vw)+\frac{\partial f}{\partial z}uvw$$

$$u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}+w\frac{\partial f}{\partial w} = (u+v+w)\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}(uv+uw+uv+vw+uw+vw)+\frac{\partial f}{\partial z}(uvw+uvw+uvw)$$

$$u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}+w\frac{\partial f}{\partial w} = (u+v+w)\frac{\partial f}{\partial x}+2(uv+vw+wu)\frac{\partial f}{\partial y}+3uvw\frac{\partial f}{\partial z}$$

$$u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}+w\frac{\partial f}{\partial w} = x\frac{\partial f}{\partial x}+2y\frac{\partial f}{\partial y}+3z\frac{\partial f}{\partial z}.$$

Maxima and Minima for functions of two variables

Let $z = f(x, y)$ be a function of two independent variables x and y .

Relative Maximum

The function $f(x, y)$ is said to have a relative maximum at a point (a,b) if $f(a,b) > f(a+h,b+k)$ for small positive or negative values of h and k .

Relative Minimum

The function $f(x, y)$ is said to have a relative minimum at a point (a,b) if $f(a,b) < f(a+h,b+k)$ for small positive or negative values of h and k .

Let $\Delta = f(a+h,b+k) - f(a,b)$

$f(a,b)$ is maximum if $\Delta < 0$ for all small values of h and k

$f(a,b)$ is minimum if $\Delta > 0$ for all small values of h and k

Extremum or Extreme Value

A maximum or minimum value of a function is called its extreme value.

Saddle Point

Saddle point is a point where function is neither maximum nor minimum. At that point f is maximum or direction while minimum in another direction.

Working rule to find the Maximum and Minimum value of $f(x,y)$

i. Find $\frac{\partial f}{\partial x}$ or f_x and $\frac{\partial f}{\partial y}$ or f_y and equate each to zero. Solve these as simultaneous equations in x and y. Let (a,b), (c,d) ,… be the pairs of values.

ii. Calculate the value of $f_{xx} = r$, $f_{xy} = s$ and $f_{yy} = t$ for each pair of values.

iii.

- **a.** If $rt - s^2 > 0$ and $r < 0$ r < 0 at (a,b) then $f(a,b)$ has maximum value
- **b.** If $rt - s^2 > 0$ and $r > 0$ at (a,b) then $f(a,b)$ has minimum value
- **c.** If $rt - s^2 < 0$ at (a,b) then $f(a,b)$ is not an extreme value, i.e., (a,b) is a saddle point.

iv. If $rt - s^2 = 0$ at (a,b), further investigation needed.

Problems

1. Show that minimum value of $u = xy + \frac{a^3}{x} + \frac{a^3}{y}$ is $3a^2$

Solution:

Let $u = xy + \frac{a^3}{x} + \frac{a^3}{y}$

$$\therefore\ p = \frac{\partial u}{\partial x} = y - \frac{a^3}{x^2};\ q = \frac{\partial u}{\partial y} = x - \frac{a^3}{y^2}$$

For maximum or minimum, we must have $\frac{\partial u}{\partial x} = 0 = \frac{\partial u}{\partial y}$

$$\frac{\partial u}{\partial x} = 0 \Rightarrow y - \frac{a^3}{y^2} = 0\ or\ x^2 y = a^3 \qquad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = 0 \Rightarrow x - \frac{a^3}{y^2} = 0\ or\ xy^2 = a^3 \qquad \text{--- (2)}$$

Solving (1) and (2), we get

$xy(x-y) = 0$ or $x = 0, y = 0$ and $x = y$

From (1) and (2) $\Rightarrow x = 0$ and $y = 0$ do not hold.

$\therefore\ x = y$ from (1) we get $x = a$

$\therefore\ x = y = a$

Now,

$$r = \frac{\partial^2 u}{\partial x^2} = \frac{2a^3}{x^3};\ s = \frac{\partial^2 u}{\partial x\,\partial y} = 1;\ t = \frac{\partial^2 u}{\partial y^2} = \frac{2a^3}{y^3}$$

at $x = y = a$, we get $r = \frac{2a^3}{a^3} = 2,\ s = 1,\ t = 2$,

$rt - s^2 = 2(2) - 1^2 = 3 > 0$

Also $r = 2 > 0$

Hence there is a minima at $x = y = a$

Hence minimum value of $u = a + a\frac{a^3}{a} + \frac{a^3}{a} = 3a^2$

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2. Show that the function $f(x,y)=x^3+y^3-3xy+1$ **is minimum at** $(1,1)$ **.**

Solution:

Let $f(x,y)=x^3+y^3-3xy+1$

$$\frac{\partial f}{\partial x}=3x^2-3y\ ;\ p=\left(\frac{\partial f}{\partial x}\right)_{(1,1)}=0 \qquad \frac{\partial f}{\partial y}=3y^2-3x\ ;\ q=\left(\frac{\partial f}{\partial y}\right)_{(1,1)}=0$$

$$\frac{\partial^2 f}{\partial x^2}=6x \quad ; \quad r=\left(\frac{\partial^2 f}{\partial x^2}\right)_{(1,1)}=6$$

$$\frac{\partial^2 f}{\partial x\partial y}=-3 \quad ; s=\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)_{(1,1)}=-3$$

$$\frac{\partial^2 f}{\partial y^2}=6y \qquad t=\left(\frac{\partial^2 f}{\partial y^2}\right)_{(1,1)}=6$$

We have, $r=6>0$ and $rt-s^2=36-9=27>0$

$\therefore\ f(x,y)$ is minimum at $(1,1)$.

3. Find the extreme values of the function $f(x,y) = x^2 + y^2 + 6x - 12$.

Solution:

Given $f(x,y) = x^2 + y^2 + 6x - 12$

$$\frac{\partial f}{\partial x} = 2x + 6, \quad \frac{\partial f}{\partial y} = 2y$$

Solving $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$

We get $x = -3$ *and* $y = 0$

Critical point is $(-3, 0)$

Now, $r = \frac{\partial^2 f}{\partial x^2} = 2$, $s = \frac{\partial^2 f}{\partial x \partial y} = 0$, $t = \frac{\partial^2 f}{\partial y^2} = 2$

We have $r = 2 > 0$ and $rt - s^2 = 2(2) - 0 = 4 > 0$

$\therefore$ $f(x,y)$ is minimum at $(-3, 0)$

$$Min\ f(x,y) = (-3)^2 + 6(-3) - 12 = -21$$

4. Show that the function $f(x, y) = xy(a - x - y),\ a > 0$ **is maximum at the point** $\left(\frac{a}{3}, \frac{a}{3}\right)$.

Solution:

Let $f(x, y) = axy - x^2 y - xy^2$

$P = \frac{\partial f}{\partial x} = ay - 2yx - y^2$, $\quad q = \frac{\partial f}{\partial y} = ax - x^2 - 2xy$,

$p = 0$ and $q = 0$ at $\left(\frac{a}{3}, \frac{a}{3}\right)$.

$\frac{\partial^2 f}{\partial x^2} = -2y \qquad \frac{\partial^2 f}{\partial x\, \partial y} = a - 2x - 2y \qquad \frac{\partial^2 f}{\partial y^2} = -2x$

$r = \frac{-2a}{3}, \qquad s = \frac{-a}{3} \qquad t = \frac{-2a}{3}$

Now

$r = \frac{-2a}{3} < 0 \because a > 0$ and $rt - s^2 = \left(\frac{-2a}{3} . \frac{-2a}{3}\right) - \frac{a^2}{9} = \frac{a^2}{3} > 0$

$\therefore f(x, y)$ is maximum at $\left(\frac{a}{3}, \frac{a}{3}\right)$

5. **Examine the function $f(x,y) = 1 + \sin(x^2 + y^2)$ for extreme values.**

Solution:

$$f(x,y) = 1 + \sin(x^2 + y^2)$$

$$\frac{\partial f}{\partial x} = \cos(x^2 + y^2)\,(2x)\,, \quad \frac{\partial f}{\partial y} = \cos(x^2 + y^2)\,(2y)$$

Solve $\frac{\partial f}{\partial x} = 0$ *and* $\frac{\partial f}{\partial y} = 0$

$$\Rightarrow \cos(x^2 + y^2) 2x = 0$$

$\therefore\ x = 0$ or $\cos(x^2 + y^2) = 0$ and $2y\cos(x^2 + y^2) = 0$

$\therefore\ x = 0$ and $y = 0$ or $\cos(x^2 + y^2) = 0$

$\therefore (0,0)$ and (a,b) are critical points such that $a^2 + b^2 = \frac{\pi}{2}$

$$\frac{\partial^2 f}{\partial x^2} = 2\cos(x^2 + y^2) - \sin(x^2 + y^2)\,.\,4x^2$$

$$\frac{\partial^2 f}{\partial x \partial y} = -2x\sin(x^2 + y^2)\,(2y)$$

$$\frac{\partial^2 f}{\partial y^2} = 2\cos(x^2 + y^2) - \sin(x^2 + y^2)\,4y^2$$

Case (i) at $(0,0)$

$p = 0\,, q = 0\,, r = 2\,, s = 0\,,\ t = 2$

Since $r > 0$ and $rt - s^2 = 4 - 0 = 4 > 0$

$\therefore\ f(x,y)$ is minimum at $(0,0)$

Case (ii) at (a,b)

$p = \cos(a^2 + b^2)\ 2a = \cos\frac{\pi}{2}\,.\,2a\ \ p = 0$

$q = \cos(a^2 + b^2)\ 2b = 0\,,\ r = 2\cos(a^2 + b^2) - \sin(a^2 + b^2) 4a^2 = -4a^2$

$s = -4ab$ and $t = -4b^2$

Since $rt - s^2 = 0$, no conclusion can be made.

Further investigation is required.

But $f(a,b) = 1 + \sin(a^2 + b^2) = 2$

$\because\ a^2 + b^2 = \frac{\pi}{2}$

$\therefore$ Maximum value of $f(x,y)$ is 2.

6. Find the extreme value of the function $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4.$

Solution:

Let $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4.$

We get $f_x = 3x^2 + 3y^2 - 6x$ $\quad f_y = 6xy - 6y$

Solving $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

i.e., $3x^2 + 3y^2 - 6x = 0$ ---(1)

and $6xy - 6y = 0$ ---(2)

From (2) we get $\Rightarrow$ $x = 1$ or $y = 0$

Put $y = 0$ in (1) $\Rightarrow 3x^2 - 6x = 0$

$\Rightarrow 3x(x-2) = 0$

$\Rightarrow x = 0$ and $x = 2$

The critical points: $(0,0)$ and $(2,0)$

Put $x = 1$ in (1) $\Rightarrow 3y^2 - 3 = 0$

$\Rightarrow y = \pm 1$

The other critical points are $(1,1)$, $(1,-1)$

$f_{xx} = 6x - 6$, $\quad f_{xy} = 6y$, $\quad f_{yy} = 6x - 6$

To examine the nature of the critical points observe the table.

Critical point	r	$rt - s^2$	**Nature of the critical point**
$(0,0)$	$-6 < 0$	$36 > 0$	Maxima
$(2,0)$	$6 > 0$	$36 > 0$	Minima
$(1,1)$	0	-36	Saddle point
$(1,-1)$	0	-36	Saddle point

$\therefore$ Max $f(x,y) = f(0,0) = 4$ and Min $f(x,y) = f(2,0) = 0$

7. **Examine the function $f(x,y)=x^4+y^4-2(x-y)^2$ for extreme values.**

Solution:

Given $f(x,y)=x^4+y^4-2(x-y)^2$

We have, $p=f_x=4x^3-4(x-y)$, $q=f_y=4y^3-4(x-y)$,

Solving $p=0$ and $q=0$

$4x^3-4(x-y)=0$ --- (1)

$4y^3+4(x-y)=0$ --- (2)

From (1) and (2), we get

$x^3+y^3=0$

i.e. $(x+y)(x^2-xy+y^2)=0$

$\therefore\ y=-x$ or $x^2-xy+y^2=0$

Put $y=-x$ in (1) $\Rightarrow x^3=x-y$

$\Rightarrow x^3=2x$

$\Rightarrow x(x^2-2)=0$

$\Rightarrow x=0$ or $x=\pm\sqrt{2}$

When $x=0$, $y=0$

$x=\sqrt{2}$, $y=-\sqrt{2}$ and $x=-\sqrt{2}$, $y=\sqrt{2}$

$r=f_{xx}=12x^2-4$, $s=f_{xy}=4$, $t=f_{yy}=12y^2+4$

To examine the nature of the critical points the following table is considered.

Critical point	r	s	t
$(0,0)$	-4	4	-4
$(\sqrt{2},-\sqrt{2})$	20	4	20
$(-\sqrt{2},\sqrt{2})$	20	4	20

$\therefore$ Min $f(x,y)=f(-\sqrt{2},\sqrt{2})=4+4-2(8)=-8$.

8. **Examine the function $f(x,y)=x^2y^2-5x^2-8xy-5y^2$ for extreme values.**

Solution:

Let $f(x,y)=x^2y^2-5x^2-8xy-5y^2$

Solving $p=0$ and $q=0$

$2y^2x-10x-8y=0 \quad \ldots (1)$

$2x^2y-8x-10y=0 \quad \ldots (2)$

$p=f_x=2y^2x-10x-8y$, $q=f_y=2x^2y-8x-10y$

From (1) and (2)

$x^2=y^2 \Rightarrow x=\pm y$

Case 1:

Put $x=y$ in --- (1)

$\Rightarrow 2y^3-18y=0$

$\Rightarrow y^3-9y=0$

$\Rightarrow y(y^2-9)=9$

$\Rightarrow y=0,\ \pm 3$

Case 2:

Put $x=-y$ in --- (1)

$\Rightarrow -2y^3+10y-8y=0$

$\Rightarrow -2y^3+2y=0$

$\Rightarrow y^3-y=0$

$\Rightarrow y=0,\ \pm 1$

$\therefore$ The Critical points are $(0,0)$, $(3,3)$, $(-3,-3)$, $(1,-1)$, $(-1,1)$.

Consider the table for examining the nature of the critical points.

Critical point	r	s	t	$rt-s^2$
$(0,0)$	-10 < 0	-8	-10	36 > 0
$(3,3)$	8 > 0	28	8	-720 < 0
$(-3,-3)$	8 > 0	28	8	-720 < 0
$(1,-1)$	-8 < 0	-12	-8	-80 < 0

$\therefore$Max $f(x,y)=f(0,0)=0$.

Vectors

Vector is a quantity which has both magnitude and direction. Vector quantities like force, velocity, acceleration etc. have lot of significance in physical and engineering fields.

Scalars

A quantity which has only magnitude but no direction is called a scalar e.g. Length, Volume etc.

Magnitude of the vector

If $\vec{a} = \vec{AB}$ is a vector, then the magnitude of the vector is $\left|\vec{a}\right| = AB$.

The magnitude of the vector is also known as length or module of the vector.

Unit Vector

If the magnitude of the vector is one unit, then it is called a unit vector. A unit vector in the direction of the vector $\vec{a}$ is usually denoted by $\hat{a}$ and is given by $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$.

Position Vector

The position vector of a point A with respect to origin O is the vector $\vec{OA}$ which is used to specify the position of A w.r.t. O.

To find $\overrightarrow{AB}$ if the position vectors of the point A and point B are given.

If the position vectors of A and B are $\vec{a}$ and $\vec{b}$. Let the origin be O.

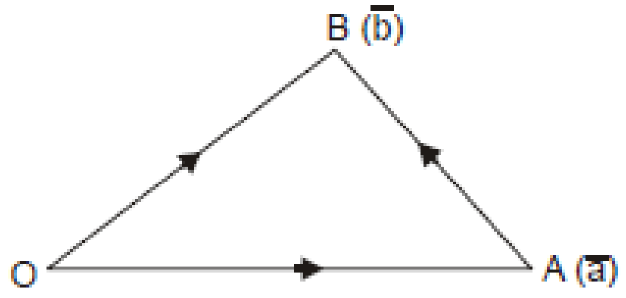
Then $\quad \vec{OA} = \vec{a}, \quad \vec{OB} = \vec{b}$

$$\vec{OA} + \vec{AB} = \vec{OB}$$

$$\vec{AB} = \vec{OB} - \vec{OA}$$

$$\Rightarrow \quad \vec{AB} = \vec{b} - \vec{a}$$

$\vec{AB}$ = **Position vector of B – Position vector of A**



Note

The position vector of a point P w.r.t. the fixed point O is the vector $\vec{r} = \overrightarrow{OP}$. Position vector of a point is also written as $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along x, y and z axes respectively.

Product of Vectors

The product of two vectors results in two different ways, the one is a number and the other is vector. So, there are two types of product of two vectors, namely scalar product and vector product. They are written as $\vec{a} \cdot \vec{b}$ and $\vec{a} \times \vec{b}$.

Scalar or Dot Product

The scalar, or dot product of two vectors $\vec{a}$ and $\vec{b}$ is defined to be $|\vec{a}||\vec{b}| \cos\theta$ *i.e.*, scalar where θ is the angle between $\vec{a}$ and $\vec{b}$.

Symbolically, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

Due to a dot between $\vec{a}$ and $\vec{b}$ this product is also called dot product.

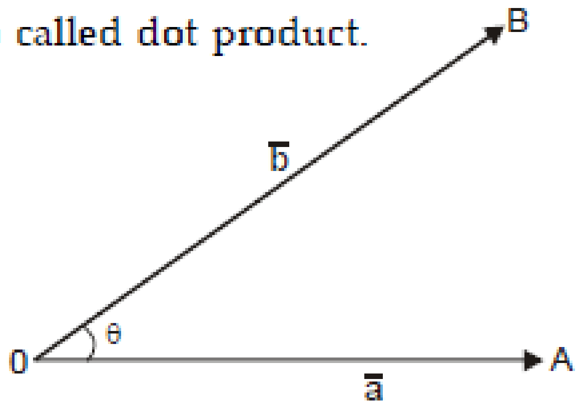
The scalar product is commutative

To Prove. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

Proof. $\vec{b} \cdot \vec{a} = |\vec{b}||\vec{a}| \cos(-\theta)$

$= |\vec{a}||\vec{b}| \cos\theta$

$= \vec{a} \cdot \vec{b}$ **Proved.**



Geometrical interpretation. The scalar product of two vectors is the product of one vector and the length of the projection of the other in the direction of the first.

Useful Results

$\hat{i} \cdot \hat{i} = (1)(1)\cos 0° = 1$ Similarly, $\hat{j} \cdot \hat{j} = 1$, $\hat{k} \cdot \hat{k} = 1$

$\hat{i} \cdot \hat{j} = (1)(1)\cos 90° = 0$ Similarly, $\hat{j} \cdot \hat{k} = 0$, $\hat{k} \cdot \hat{i} = 0$

Note. If the dot product of two vectors is zero then vectors are prependicular to each other.

Vector Product/ Cross product

The vector, or cross product of two vectors $\vec{a}$ and $\vec{b}$ is defined to be a vector

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$$

, where $\hat{n}$ is the unit vector $\perp$ to both $\vec{a}$ and $\vec{b}$

Useful Results

Since $\hat{i}, \hat{j}, \hat{k}$ are three mutually perpendicular unit vectors, then

$$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$$

$$\hat{i}\times\hat{j} = -\hat{j}\times\hat{i} = \hat{k}$$

$$\hat{j}\times\hat{k} = -\hat{k}\times\hat{j} = \hat{i}$$

$$\hat{k}\times\hat{i} = -\hat{i}\times\hat{k} = \hat{j}$$

and

$$\hat{j}\times\hat{i} = -\hat{i}\times\hat{j}$$

$$\hat{k}\times\hat{j} = -\hat{j}\times\hat{k}$$

$$\hat{i}\times\hat{k} = -\hat{k}\times\hat{i}$$

Vector Point Function

Let the position vector of a point $P(x, y, z)$ in space be $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. If x, y, z are all functions of a single parameter 't', then $\vec{r}$ is said to be a vector function of 't' (also called vector point function) denoted as $\vec{r} = \vec{r}(t)$.

Ex:

- Velocity at different points within a moving fluid
- Acceleration at different points within a moving fluid

Derivative of a Vector Function

The **derivative of the vector** $\vec{r}(t)$ is denoted and is defined as follows.

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{\delta t \to 0} \frac{\vec{r}(t+\delta t) - \vec{r}(t)}{\delta t}$$

Velocity and Acceleration

Since $\vec{r} = x\,i + y\,j + z\,k$ represents the position vector of a point moving along a curve, $x,\ y,\ z$ will be a function of the time variable t and accordingly $\vec{r}$ is a function of the time variable t.

$\therefore \qquad \vec{v} = \dfrac{d\vec{r}}{dt}$ gives the **velocity of the particle at time** t

Further $\vec{a} = \dfrac{d\vec{v}}{dt} = \dfrac{d}{dt}\left(\dfrac{d\vec{r}}{dt}\right) = \dfrac{d^2\vec{r}}{dt^2}$ represents the rate of change of velocity $\vec{v}$ and is called the **acceleration of the particle at time** t

Scalar point function:

A function that assigns a real number/ scalar to each point of some region of space. i.e. If each point *(x,y,z)* of a region *R* in space there is assigned a real number $u = \phi(x, y, z)$ then ϕ is called a scalar point function.

Ex

- Temperature distribution within some body at a particular point of time

Vector differential operator:

The vector differential operator is denoted by (∇) (read as del) and is defined as

$$\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

We must note that this operator has no meaning by itself unless it operates on some function suitably

Gradient of a scalar point function

If $\phi\ (x, y, z)$ be a scalar function then $\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$ is called the gradient of the scalar function ϕ.

And is denoted by grad ϕ.

Thus,

$$\text{grad}\ \phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\text{gard}\ \phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi(x, y, z)$$

$$\text{gard}\ \phi = \nabla\ \phi \qquad (\nabla \text{ is read del or nebla})$$

Note:

Gradient tells us the direction to move to increase the temperature, but not give us the coordinates of where to go

Properties

If f and g are two scalar functions, then

1. $gard(f \pm g) = grad\ f \pm grad\ g$

2. $grad(fg) = f(grad g) + g(grad f)$

3. If c is a constant, $grad(cf) = c(grad f)$

4. $grad\left(\dfrac{f}{g}\right) = \dfrac{g(grad\ f) - f(grad\ g)}{g^2}$, $\quad (g \neq 0)$

Normal vector

The normal vector is a vector which is perpendicular to the surface at a given point. i.e. If f is surface, then $grad\ f\ (or)\ \nabla f$ is the normal vector to the given surface.

Unit Normal Vector

The unit vector obtained by normalizing the normal vector $grad\ f\ (or)\ \nabla f$ is called unit normal vector. It can be computed by using the following formula

$$\frac{\nabla f}{|\nabla f|}$$

Problems

1. If $f(x, y, z) = 2x + yz - 3y$ then find $grad\ f$.

Solution:

Given $f(x, y, z) = 2x + yz - 3y$

Then $grad\ f = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$$= \hat{i}\frac{\partial}{\partial x}(2x + yz - 3y^2) + \hat{j}\frac{\partial}{\partial y}(2x + yz - 3y^2) + \hat{k}\frac{\partial}{\partial z}(2x + yz - 3y^2)$$

$$= 2\hat{i} + (z - 6y)\hat{i} + y\hat{k}$$

2. Find $grad\ f$ where

a). $f = x^3 + y^3 + 3xyz$ b). $f = x^2 y + y^2 x + z^2$ c). $f = x^3 - y^3 + x^2 z$

at the point $(1,1,-1)$

Solution:

a) Given $f(x,y,z) = x^3 + y^3 + 3xyz$

Then $grad\ f = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$$= \hat{i}\frac{\partial}{\partial x}(x^3 + y^3 + 3xyz) + \hat{j}\frac{\partial}{\partial y}(x^3 + y^3 + 3xyz) + \hat{k}\frac{\partial}{\partial z}(x^3 + y^3 + 3xyz)$$

$$= \hat{i}(3x^2 + 3yz) + \hat{j}(3y^2 + 3xz) + \hat{k}(3xy)$$

$$grad\ f = [\nabla f]_{((1,1,-1))} = -3\hat{i} - 3\hat{j} + 3\hat{k}$$

3. Find the unit vector normal to the surface $xy^3z^2 = 4$ at $(-1, -1, 2)$

Solution:

Let $f = xy^3z^2$ $Grad\, f$ (or ∇f) is a vector normal to the surface.

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = y^3z^2\,\hat{i} + 3xy^2z^2\,\hat{j} + 2xy^3z\,\hat{k}$$

$$\therefore (\nabla f)_{(-1,-1,2)} = -4\hat{i} - 12\hat{j} + 4\hat{k} = -4(\hat{i} + 3\hat{j} - \hat{k})$$

Hence the required unit vector normal $\hat{n} = \dfrac{\nabla f}{|\nabla f|}$ i.e., $\hat{n} = \dfrac{-4(\hat{i} + 3\hat{j} - \hat{k})}{\sqrt{4^2(1^2 + 3^2 + 1^1)}} = -\dfrac{(\hat{i} + 3\hat{j} - \hat{k})}{\sqrt{11}}$

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4. Find the unit vector normal to the surface $x^2y + 2xz = 4$ at $(2, -2, 3)$

Solution:

Let the given surface be $f = x^2y + 2xz - 4$

We know that *Grad f* (or ∇f) is a vector normal to the surface.

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = \hat{i}(2xy + 2z) + \hat{j}(x^2) + \hat{k}(2x)$$

$$\therefore (\nabla f)_{(2,-2,3)} = -2\hat{i} + 4\hat{j} + 4\hat{k}$$

Hence the required unit vector normal $\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{-2\hat{i} + 4\hat{j} + 4\hat{k}}{\sqrt{4+16+16}} = \frac{(-\hat{i} + 2\hat{j} + 2\hat{k})}{3}$

5. Find the unit normal vector to the surface $x^2+y^2+2z^2 = 26$ at $(2, 2, 3)$ (Homework)

6. Find the unit normal vector to the surface $z=x^2+y^2$ at $(-1,-2, 5)$ (Homework)

Tangent vector and Unit tangent vector

Let $\vec{r} = x\,\hat{i}+y\,\hat{j}+z\,\hat{k}$ be a vector, then the tangent vector to $\vec{r}$ is denoted by $\vec{T}$ and defined as $\vec{T} = \frac{dr}{dt}$

and **unit tangent vector** can be obtained using $\hat{T} = \frac{\vec{T}}{|\vec{T}|}$

1. If $x = t^2 + 1,\ y = 4t - 3, z = 2t^2 - 6t$ represents the parametric equation of the curve, then find the tangent vector and unit tangent vector at any point on the curve.

Solution

We know that, vector equation of the curve is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = (t^2+1)\hat{i} + (4t-3)\hat{j} + (2t^2-6t)\hat{k}$$

Then the tangent vector to $\vec{r}$ is $\vec{T} = \dfrac{dr}{dt} = (2t)\hat{i} + 4\hat{j} + (4t-6)\hat{k}$ and unit tangent vector can be

obtained using $\hat{T} = \dfrac{\vec{T}}{|\vec{T}|} = \dfrac{(2t)\hat{i} + 4\hat{j} + (4t-6)\hat{k}}{\sqrt{4t^2 + 16 + (4t-6)^2}} = \dfrac{(2t)\hat{i} + 4\hat{j} + (4t-6)\hat{k}}{\sqrt{20t^2 - 48t + 52}}$

2. Find the unit tangent vector to the curve $\vec{r} = (4\sin t)\hat{i} + (4\cos t)\hat{j} + (3t)\hat{k}$ at any point on the curve.

Solution

Given $\vec{r} = (4\sin t)\hat{i} + (4\cos t)\hat{j} + (3t)\hat{k}$

Then the tangent vector to $\vec{r}$ is $\vec{T} = \frac{dr}{dt} = (4\cos t)\hat{i} - 4\sin t\,\hat{j} + 3\hat{k}$ and unit tangent vector can be obtained using

$$\hat{T} = \frac{\vec{T}}{|\vec{T}|} = \frac{(4\cos t)\hat{i} - 4\sin t\,\hat{j} + 3\hat{k}}{\sqrt{16\cos^2 t + 16\sin^2 t + (3)^2}} = \frac{(4\cos t)\hat{i} - 4\sin t\,\hat{j} + 3\hat{k}}{\sqrt{16(\cos^2 t + \sin^2 t) + 9}}$$

$$= \frac{(4\cos t)\hat{i} - 4\sin t\,\hat{j} + 3\hat{k}}{5}$$

$$= \frac{1}{5}(4\cos t\,\hat{i} - 4\sin t\,\hat{j} + 3\hat{k})$$

3. Find the unit tangent vector to the curve $\vec{r} = (t^2+1)\hat{i} + (4t-3)\hat{j} + (2t^2-6t)\hat{k}$ at $t=2$ (Homework)

4. Find the unit tangent vector to the curve $\vec{r} = \left(t - \frac{t^3}{3}\right)\hat{i} + t^2\hat{j} + \left(t + \frac{t^3}{3}\right)\hat{k}$ at $t = \pm 3$ (Homework)

Directional derivative

The rate of change of a scalar point function $f(x, y, z)$ at a point $p(x, y, z)$ in the direction of the unit vector $\hat{n}$ is called the directional derivative and is defined as $\hat{n}.\nabla f$

1. Find the directional derivative of $f = x^2yz + 4xz^2$ at $(1,-2,-1)$ along $2\hat{i} - \hat{j} - 2\hat{k}$

Solution:

Let $f = x^2yz + 4xz^2$

$$\therefore \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = (2xyz + 4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y + 8xz)\hat{k}$$

$$(\nabla f)_{(1,-2,-1)} = 8\hat{i} - \hat{j} - 10\hat{k}$$

The unit vector in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$ is $\hat{n} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4+1+4}} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{3}$

$\therefore$ The required directional derivative is $\nabla f . \hat{n} = (8\hat{i} - \hat{j} - 10\hat{k}).\frac{(2\hat{i} - \hat{j} - 2\hat{k})}{3}$

$$= \frac{37}{3}$$

2. Find the directional derivative of $f = xy + yz + zx$ *at* $(1,2,0)$ in the direction of the vector $\hat{i}+2\hat{j}+2\hat{k}$

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Solution:

Let $f = xy + yz + zx$

$$\therefore \nabla f = \frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k} = (y+z)\hat{i}+(z+x)\hat{j}+(x+y)\hat{k}$$

$$(\nabla f)_{(1,2,0)} = 2\hat{i}+\hat{j}+3\hat{k}$$

The unit vector in the direction of $\hat{i}+2\hat{j}+2\hat{k}$ is

$$\hat{n}=\frac{\hat{i}+2\hat{j}+2\hat{k}}{\sqrt{1+4+4}}=\frac{\hat{i}+2\hat{j}+2\hat{k}}{3}=\frac{1}{3}(\hat{i}+2\hat{j}+2\hat{k})$$

$\therefore$The required directional derivative is $\hat{n}.\nabla f = \frac{1}{3}(\hat{i}+2\hat{j}+2\hat{k}).(2\hat{i}+\hat{j}+3\hat{k}) = \frac{10}{3}$

3. Find the directional derivative of $f = 2xy + z^2$ *at* $(1,-1,3)$ in the direction of the vector $\hat{i}+2\hat{j}+3\hat{k}$ (Homework)

Note:

The greatest value of the directional derivative of $f(x, y, z)$ at a point $p(x, y, z)$ is $|\nabla f|$ at that point $p(x, y, z)$

1. In which direction the directional derivative of x^2yz^3 is maximum at $(2,1,-1)$ and find the magnitude of this maximum.

Solution:

We know that the directional derivative is maximum along the normal vector ∇f.

Let $f = x^2yz^3$ $\therefore \nabla f = 3xyz^3\,\hat{i} + x^2z^3\,\hat{j} + 3x^2yz^2\,\hat{k}$

$[\nabla f]_{(2,1,-1)} = -4i - 4j + 12k$ which is the direction in which the directional derivative is maximum and its magnitude is $\sqrt{4^2(1+1+9)} = 4\sqrt{11}$.

2. If the temperature at any point in space is given by $t = xy + yz + zx$, find the direction in which the temperature changes most rapidly with distance from the point $(1,1,1)$ and find the maximum rate of change.

Solution:

We know that the directional derivative is maximum along the normal vector ∇f.

Let $t = xy + yz + zx$ $\therefore \nabla t = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$

$[\nabla t]_{(1,1,1)} = 2\hat{i} + 2\hat{j} + 2\hat{k}$ which is the direction in which the directional derivative is maximum and its magnitude is $\sqrt{4+4+4} = 2\sqrt{3}$.

Hence, at the point $(1,1,1)$ the temperature changes most rapidly in the direction given by the vector $2\hat{i} + 2\hat{j} + 2\hat{k}$ and maximum rate of increase is $2\sqrt{3}$

Angle between the normal to the surfaces

1. Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2, -1, 2)$.

Solution:

The angle between the surfaces is the angle between their normals.

We know that ∇f is a vector normal to the surface.

Equation of two surfaces be $f_1=x^2+y^2+z^2$ and $f_2=x^2+y^2-z$

$$\therefore \nabla f_1 = 2x\hat{i}+2y\hat{j}+2z\hat{k} \text{ and } \nabla f_2 = 2x\hat{i}+2y\hat{j}-\hat{k}$$

$$(\nabla f_1)_{(2,-1,2)} = 4\hat{i}-2\hat{j}+4\hat{k} \text{ and } (\nabla f_2)_{(2,-1,2)} = 4\hat{i}-2\hat{j}-\hat{k} = 2(2\hat{i}-\hat{j}+2\hat{k})$$

Let θ be the angle between the normals,

$$\therefore \cos\theta = \frac{\nabla f_1.\nabla f_2}{|\nabla f_1||\nabla f_2|} = \frac{2(8+2-2)}{\sqrt{2^2(4+1+4)}\sqrt{(16+4+1)}} = \frac{8}{3\sqrt{21}}$$

(or) $\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$

2. Find the angle between the spheres $x^2+y^2+z^2=29$ and $x^2+y^2+z^2+4x-6y-8z-47=0$ at the point $(4,-3,2)$.

Solution:

The angle between the surfaces is the angle between their normals.

We know that ∇f is a vector normal to the surface.

Equation of two spheres be

$f_1=x^2+y^2+z^2-29=0$ and $f_2=x^2+y^2+z^2+4x-6y-8z-47=0$

$\therefore \nabla f_1 = 2x\,\hat{i}+2y\,\hat{j}+2z\,\hat{k}$ and $\nabla f_2=(2x+4)\,\hat{i}+(2y-6)\,\hat{j}-(2z-8)\,\hat{k}$

$(\nabla f_1)_{(4,-3,2)}=8\,\hat{i}-6\,\hat{j}+4\,\hat{k}$ and $(\nabla f_2)_{(4,-3,2)}=12\,\hat{i}-12\,\hat{j}-4\,\hat{k}$

Let θ be the angle between the normals,

$$\therefore \cos\theta=\frac{\nabla f_1 . \nabla f_2}{|\nabla f_1||\nabla f_2|}=\frac{152}{\sqrt{116}\sqrt{304}}$$

(or) $\quad \theta=\cos^{-1}\left(\sqrt{\dfrac{19}{29}}\right)$

3. Find the angle between the normals to the surface $xy=z^2$ at the points $(4,1,2)$ & $(3,3,-3)$.

Divergence of a vector point function

The divergence of a vector point function $\vec{F}$ is denoted by *div* F and is defined as below.

Let $\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$

$$\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\hat{i}F_1 + \hat{j}F_2 + \hat{k}F_3) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

It is evident that div F is scalar function.

Solenoidal vector

A vector $\vec{v}$ whose divergence is zero is called a solenoidal vector i.e. $div\,\vec{v} = 0$

Curl of a vector point function

The curl of a vector point function F is defined as below

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F} \qquad (\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k})$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \times (F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) - \hat{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

Curl $\vec{F}$ is a vector quantity.

Irrotational vector

A vector $\vec{v}$ is said to be Irrotational if $curl\, \vec{v} = 0$

Ex: A hurricane, Tornado

Problems

1. If $\overrightarrow{v} = 3xz\,\hat{i} + 2xy\,\hat{j} - yz^2\,\hat{k}$ then find $div\,\overrightarrow{v}$.

Solution:

Given vector $\overrightarrow{v} = 3xz\,\hat{i} + 2xy\,\hat{j} - yz^2\,\hat{k}$

We know that

$$div\,\vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$= 3z + 2x - 2yz$$

2. If $\vec{f} = xy^2\,\hat{i} + 2x^2yz\,\hat{j} - 3yz^2\,\hat{k}$ then find $div\,\vec{v}$ at $(1,-1,1)$.

Solution:

Given vector $\vec{f} = xy^2\,\hat{i} + 2x^2yz\,\hat{j} - 3yz^2\,\hat{k}$

We know that

$$div\,\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}, \text{where } f_1 = xy^2,\ f_2 = 2x^2yz,\ f_3 = -3yz^2$$

$$div\,\vec{f} = y^2 + 2x^2z - 6yz$$

$$\left(div\,\vec{f}\right)_{(1,-1,1)} = 1 + 2 + 6 = 9$$

3. Show that $3y^4z^2\,\hat{i} + x^2z^2\,\hat{j} - 3x^2y^2\hat{k}$ is solenoidal.

Solution:

Let $\vec{f} = 3y^4z^2\,\hat{i} + x^2z^2\,\hat{j} - 3x^2y^2\hat{k}$

We know that

$$div\,\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}, \text{where } \; f_1 = 3y^2z^2,\; f_2 = x^2z^3,\quad f_3 = -3x^2y^2$$

$$div\,\vec{f} = 0$$

$\because\;\; \vec{f}$ is Solenoidal vector

4. If $\vec{f} = (x+3y)\,\hat{i} + (y-2z)\,\hat{j} + (x+pz)\hat{k}$ is solenoidal, then find p.

Solution:

Given $\vec{f} = (x+3y)\,\hat{i} + (y-2z)\,\hat{j} + (x+pz)\hat{k}$

We know that

$$div\,\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}, \text{where } \; f_1 = x+3y,\; f_2 = y-2z,\quad f_3 = x+pz$$

$$div\,\vec{f} = 2 + p$$

Since, $\vec{f} = (x+3y)\,\hat{i} + (y-2z)\,\hat{j} + (x+pz)\hat{k}$ is solenoidal,

we have $div\,\vec{f} = 0 \Rightarrow 2 + p = 0 \Rightarrow p = -2$

5. If $\vec{v} = yz\,\hat{i} + 3z\,\hat{j} + z\,\hat{k}$ find $Curl\,\vec{v}$ with respect to right-handed Cartesian coordinates.

Solution:

Let $\vec{v} = yz\,\hat{i} + 3z\,\hat{j} + z\,\hat{k}$ then $curl\,\vec{v} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & 3zx & z \end{vmatrix}$

$$= -3x\,\hat{i} + y\,\hat{j} + 2z\,\hat{k}$$

6. Find $div\,\vec{v}$ and $Curl\,\vec{v}$ where $\vec{v} = \nabla(x^3 + y^3 + z^3 - 3xyz) = grad(x^3 + y^3 + z^3 - 3xyz)$.

Solution:

Let $f = x^3 + y^3 + z^3 - 3xyz$

$$\therefore \vec{v} = \nabla f = (3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)\hat{k}$$

$$div\,\vec{v} = \nabla . \vec{v}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left\{(3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)\hat{k}\right\}$$

$$= \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy).$$

$$\therefore div\,\vec{v} = 6x + 6y + 6z = 6(x + y + z)$$

Also $curl\,\vec{v} = \nabla \times \vec{v}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x^2 - 3yz) & (3y^2 - 3xz) & (3z^2 - 3xy) \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3y^2 - 3xz)\right\} - \hat{j}\left(\frac{\partial}{\partial x}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3x^2 - 3yz)\right)$$

$$+ \hat{k}\left(\frac{\partial}{\partial x}3y^2 - 3xz) - \frac{\partial}{\partial y}(3x^2 - 3yz)\right)$$

$$= \hat{i}\{-3x - (-3x)\} - \hat{j}\{-3y - (-3y)\} + \hat{k}\{-3z - (-3z)\}$$

$$= \vec{0}$$

7. Find *divergence* and *curl* of $\vec{v} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$ *at* $(2, -1, 1)$

Solution. Here, we have

$$\vec{v} = (x\,y\,z)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$$

$$\text{Div. } \vec{v} = \nabla\phi$$

$$\text{Div } \vec{v} = \frac{\partial}{\partial x}(x\,y\,z) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$$

$$= yz + 3x^2 + 2x\,z - y^2 \qquad = -1 + 12 + 4 - 1 = 14 \text{ at } (2, -1, 1)$$

$$\text{Curl } \overline{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix} \qquad = -2yz\,\hat{i} - (z^2 - xy)\hat{j} + (6xy - xz)\hat{k}$$

$$= -2yz\,\hat{i} + (xy - z^2)\hat{j} + (6xy - xz)\hat{k}$$

Curl at (2, –1, 1)

$$= -2(-1)(1)\hat{i} + \{(2)(-1) - 1\}\hat{j} + \{6(2)(-1) - 2(1)\}\hat{k}$$

$$= 2\hat{i} - 3\hat{j} - 14\hat{k}$$

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8. Prove that $(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$ is both solenoidal and Irrotational.

Solution. Let $\vec{F} = (y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$

For solenoidal, we have to prove $\vec{\nabla}.\vec{F} = 0$.

Now, $\vec{\nabla}.\vec{F} = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\cdot\left[(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}\right]$

$$= -2 + 2x - 2x + 2 = 0$$

Thus, $\vec{F}$ is solenoidal. For irrotational, we have to prove Curl $\vec{F} = 0$.

Now, Curl $\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$

$$= (3z + 2y - 2y + 3z)\hat{i} - (-2z + 3y - 3y + 2z)\hat{j} + (3z + 2y - 2y - 3z)\hat{k}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = 0$$

Thus, $\vec{F}$ is irrotational.

Hence, $\vec{F}$ is both solenoidal and irrotational. **Proved.**

9. Prove that, for every field $\vec{V}$, $div.\,curl\,\vec{V} = 0$

Solution. Let $V = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

$$\text{div}\,(\overline{\text{curl}}\,\vec{V}) = \vec{\nabla}.(\vec{\nabla}\times\vec{V})$$

$$= \vec{\nabla}.\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left[\hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) - \hat{j}\left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$$= \frac{\partial^2 V_3}{\partial x\,\partial y} - \frac{\partial^2 V_2}{\partial x\,\partial z} - \frac{\partial^2 V_3}{\partial y\,\partial x} + \frac{\partial^2 V_1}{\partial y\,\partial z} + \frac{\partial^2 V_2}{\partial z\,\partial x} - \frac{\partial^2 V_1}{\partial z\,\partial y}$$

$$= \left(\frac{\partial^2 V_1}{\partial y\,\partial z} - \frac{\partial^2 V_1}{\partial z\,\partial y}\right) + \left(\frac{\partial^2 V_2}{\partial z\,\partial x} - \frac{\partial^2 V_2}{\partial x\,\partial z}\right) + \left(\frac{\partial^2 V_3}{\partial x\,\partial y} - \frac{\partial^2 V_3}{\partial y\,\partial x}\right)$$

$$= 0$$

10. If $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then show that

a). $grad\, r = \dfrac{\vec{r}}{r}$ b). $grad\left(\dfrac{1}{r}\right) = -\dfrac{\vec{r}}{r^3}$

Solution. (*i*) $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow r = \sqrt{x^2 + y^2 + z^2} \Rightarrow r^2 = x^2 + y^2 + z^2$

$$\therefore \quad 2r\frac{\partial r}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$\text{grad } r = \nabla r = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) r = \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}$$

$$= \hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{\overline{r}}{r}$$ **Proved.**

$$(ii)\ \text{grad}\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right) = \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= \hat{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + \hat{j}\left(\frac{-1}{r^2}\frac{\partial r}{\partial y}\right) + \hat{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= \hat{i}\left(-\frac{1}{r^2}\frac{x}{r}\right) + \hat{j}\left(-\frac{1}{r^2}\frac{y}{r}\right) + \hat{k}\left(-\frac{1}{r^2}\frac{z}{r}\right) = -\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r^3} = -\frac{\overline{r}}{r^3}$$ **Proved.**

11. If $\vec{A}$ is a constant vector, prove that div $(\vec{A}\times\vec{r}) = 0$.

12. Prove that $\mathrm{Curl}(r^n \vec{r}) = 0$. **(Homework)**

Solution:

Let $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ be a constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\therefore \vec{A}\times\vec{r} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \sum \hat{i}(a_2 z - a_3 y)$$

$$\therefore div(\vec{A}\times\vec{r}) = \nabla.(\vec{A}\times\vec{r}) = \left(\sum \frac{\partial}{\partial x} i\right).\sum i(a_2 z - a_3 y) = \sum \frac{\partial}{\partial x}(a_2 z - a_3 y) = 0$$

$$\therefore div(\vec{A}\times\vec{r}) = 0$$

13. Determine the constants a and b such that curl of a vector

$\vec{A} = (2xy + 3yz)\hat{i} + (x^2 + axz - 4z^2)\hat{j} - (3xy + byz)\hat{k}$ is zero.

Solution. $$\text{Curl } A = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times [(2xy + 3yz)\hat{i} + (x^2 + axz - 4z^2)\hat{j} - (3xy + byz)\hat{k}]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + 3yz & x^2 + axz - 4z^2 & -3xy - byz \end{vmatrix}$$

$$= [-3x - bz - ax + 8z]\hat{i} - [-3y - 3y]\hat{j} + [2x + az - 2x - 3z]\hat{k}$$

$$= [-x(3 + a) + z(8 - b)]\hat{i} + 6y\,\hat{j} + z(-3 + a)\hat{k}$$

$$= 0 \qquad \text{(given)}$$

i.e., $3 + a = 0$ and $8 - b = 0$, $\quad -3 + a = 0$

$a = -3, 3 \qquad b = 8 \qquad \Rightarrow \qquad a = 3$ **Ans.**

Note

If $\overline{f}$ is Irrotational, then there will always exists a scalar potential function $\phi(x, y, z)$ such that $\overline{f} = grad\,\phi$ (or) $\nabla\phi$

1. Show that $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ is Irrotational or a conservative force field. Find its Scalar potential.

Solution:

$$curl\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (x^2 - yz) & (y^2 - zx) & (z^2 - xy) \end{vmatrix} = 0$$

Hence $\vec{F}$ is a conservative force field or irrational. Then $\exists\ \phi$ *such that* $\overline{f} = \nabla\phi$

Now consider $\overline{f} = \nabla\phi$, where φ a scalar potential

To determine, we have

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$$

Therefore,

$$\frac{\partial \varphi}{\partial x} = x^2 - yz \Rightarrow \phi = \int (x^2 - yz)dx = \frac{x^3}{3} - xy + c_1 \qquad (1)$$

$$\frac{\partial \varphi}{\partial y} = y^2 - zx \Rightarrow \phi = \int (y^2 - zx)dy = \frac{y^3}{3} - xyz + c_2 \qquad (2)\ ,$$

$$\frac{\partial \varphi}{\partial z} = z^2 - xy \Rightarrow \phi = \int (z^2 - xy)dz = \frac{z^3}{3} - xyz + c_3 \qquad (3)$$

From (1-3), we have

$$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + c, \quad \text{where c be a constant}$$

Which is the required scalar potential

2. Show that $\vec{F} = (x^2 + xy^2)\hat{i} + (y^2 + x^2 y)\hat{j}$ is Irrotational or a conservative force field. Find its Scalar potential.

Solution:

Consider

$$curl\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 + xy^2) & (y^2 + x^2 y) & 0 \end{vmatrix} = 0$$

Hence $\vec{F}$ is a conservative force field or irrational. Then $\exists\ \phi\ \ such\ that\ \bar{f} = \nabla\phi$

Now consider $\bar{f} = \nabla\phi$, where φ a scalar potential

To determine, we have

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}\ \ \vec{F} = (x^2 + xy^2)\hat{i} + (y^2 + x^2\ y)\hat{j}$$

Therefore,

$$\frac{\partial\varphi}{\partial x} = x^2 - xy^2 \Rightarrow \phi = \int (x^2 - xy^2)dx = \frac{x^3}{3} - \frac{x^2y^2}{3} + c_1 \qquad (1)$$

$$\frac{\partial\varphi}{\partial y} = y^2 + x^2y \Rightarrow \phi = \int (y^2 + x^2y)dy = \frac{y^3}{3} + \frac{x^2y^2}{2} + c_2 \qquad (2),$$

$$\frac{\partial\varphi}{\partial z} = 0 \Rightarrow \phi = c_3 \qquad (3)$$

From (1-3), we have

$$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{x^2y^2}{2} + c, \quad \text{where c be a constant}$$

Which is the required scalar potential

Thank You