Exercise 1. Homogeneous coordinates.
a) The equation of a line in the plane is

$$ax + by + c = 0.$$

Show that by using homogeneous coordinates this can be written as

$$\mathbf{x}^\top \mathbf{l} = 0$$

where $\mathbf{l} = \begin{pmatrix} a & b & c \end{pmatrix}^\top$.

Let $ax+by+c = z^T l = 0$ where $z = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \wedge l = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ vector

homogenous coord

$$z l = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = ax+by+c \ (=0)$$

b) Show that the intersection of two lines $\mathbf{l}$ and $\mathbf{l}'$ is the point $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$.

Given that $l = a_1x_1 + b_1y_1 + c_1$ and $l' = a_2x_2 + b_2y_2 + c_2$
we can get a system of equations by rearrangement

$$a_1x + b_1y = -c_1$$
$$a_2x + b_2y = -c_2$$

or

$$x l = 0 \qquad x' l' = 0$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \cdot \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$

we need to solve for x, y, which satisfies both equations.

Let $x = l \times l' = \begin{bmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$

$$l \times l' = i \cdot (b_1c_2 - b_2c_1) + j \cdot (a_1c_2 - c_1a_2) + k \cdot (a_1 \cdot b_2 - b_1a_2)$$

(determinant)

triple scalar product: $(a \times b)^T c = [a, b, c]^T$

$x = l \times l'$ where x is a point (intersection point)
and $\times$ is cross product symbol

with l, l', and x we can form: triple scalar product

because cross producting 2 of ...

and $\times$ is cross product symbol

with l, l', and x we can form: triple scalar product

because cross producting 2 of the same vectors is 0

$l \times l')(l \times l') = 0$

$\Leftrightarrow l^T x = l'^T x = 0$

$\Leftrightarrow$ intersection at $x = l \times l'$

Intersection of lines. Given two lines $\mathbf{l} = (a, b, c)^\mathsf{T}$ and $\mathbf{l}' = (a', b', c')^\mathsf{T}$, we wish to find their intersection. Define the vector $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$, where $\times$ represents the vector or cross product. From the triple scalar product identity $\mathbf{l}.(\mathbf{l} \times \mathbf{l}') = \mathbf{l}'.(\mathbf{l} \times \mathbf{l}') = 0$, we see that $\mathbf{l}^\mathsf{T}\mathbf{x} = \mathbf{l}'^\mathsf{T}\mathbf{x} = 0$. Thus, if $\mathbf{x}$ is thought of as representing a point, then $\mathbf{x}$ lies on both lines $\mathbf{l}$ and $\mathbf{l}'$, and hence is the intersection of the two lines. This shows:

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Result 2.2. *The intersection of two lines* $\mathbf{l}$ *and* $\mathbf{l}'$ *is the point* $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$.

c) Show that the line through two points $\mathbf{x}$ and $\mathbf{x}'$ is $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$.

Line joining points. An expression for the line passing through two points $\mathbf{x}$ and $\mathbf{x}'$ may be derived by an entirely analogous argument. Defining a line $\mathbf{l}$ by $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$, it may be verified that both points $\mathbf{x}$ and $\mathbf{x}'$ lie on $\mathbf{l}$. Thus

Result 2.4. *The line through two points* $\mathbf{x}$ *and* $\mathbf{x}'$ *is* $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$.

d) Show that for all $\alpha \in \mathbb{R}$ the point $\mathbf{y} = \alpha\mathbf{x} + (1 - \alpha)\mathbf{x}'$ lies on the line through points $\mathbf{x}$ and $\mathbf{x}'$.

$l = x \times x'$

Due to y expressing a ratio between x and x' (like with barycentric coordinates) y is a combination of x and x', therefore y lies on the line those 2 coordinates form

Exercise 2

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Exercise 2. Transformations in 2D.
a) Use homogeneous coordinates and give the matrix representations of the following transformation groups: translation, Euclidean transformation (rotation+translation), similarity transformation (scaling+rotation+translation), affine transformation, projective transformation.

b) What is the number of degrees of freedom in these transformations?

translation:

Let $t=(t_x, t_y)$ be the vector which we want to translate with

$$X' = \begin{pmatrix} x+t_x \\ y+t_y \\ 1 \end{pmatrix} \qquad X'_{3\times1} = X_{3\times1} + t_{2\times1}$$

E transformation:

$$X'_{3\times1} = R_{3\times3} * X_{3\times1} + t_{2\times1} \qquad R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

rotation matrix (e.g.)

9 DOF

$$X' = \begin{pmatrix} \cos\theta \cdot x - \sin\theta \cdot y + t_x \\ \sin\theta \cdot x + \cos\theta \cdot y + t_y \\ 1 \end{pmatrix}$$

6 DOF

similarity transformation:

Let scaling vector be $s=(s_x, s_y)$

using the previous X' matrix as X

$$X_{3\times1} \cdot s_{2\times1}$$

$$X' = \begin{pmatrix} x \cdot s_x \\ y \cdot s_y \\ 1 \end{pmatrix}$$

2 DOF + 6 DOF (previously)

affine transformation

$$A = \begin{bmatrix} a_{11} & a_{12} & t_1 \\ a_{21} & a_{22} & t_2 \\ 0 & 0 & 1 \end{bmatrix}$$

6 DOF

$$X' = X \cdot A = \begin{pmatrix} a_{11}x + a_{12}y + t_1 \\ a_{21}x + a_{22}y + t_2 \\ 1 \end{pmatrix}$$

projective transformation

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

9 degrees of freedom

− 1 due to the question in c)?

$$X' = X \cdot P = (P_{11}x + P_{12}y + P_{13})$$

$$X' = X \cdot P = \begin{pmatrix} p_{11}x + p_{12}y + p_{13} \\ p_{21}x + p_{22}y + p_{23} \\ p_{31}x + p_{32}y + p_{33} \end{pmatrix}$$

Exercise 3. Planar projective transformation.
The equation of a line on a plane, $ax + by + c = 0$, can be written as $\mathbf{l}^\top \mathbf{x} = 0$, where $\mathbf{l} = [a\ b\ c]^\top$ and $\mathbf{x}$ are homogeneous coordinates for lines and points, respectively. Under a planar projective transformation, represented with an invertible 3×3 matrix $\mathbf{H}$, points transform as

$$\mathbf{x}' = \mathbf{H}\mathbf{x}.$$

a) Given the matrix $\mathbf{H}$ for transforming points, as defined above, define the line transformation (i.e. transformation that gives $\mathbf{l}'$ which is a transformed version of $\mathbf{l}$).

2.3.1 Transformations of lines and conics

Transformation of lines. It was shown in the proof of theorem 2.10 that if points $\mathbf{x}_i$ lie on a line $\mathbf{l}$, then the transformed points $\mathbf{x}'_i = \mathtt{H}\mathbf{x}_i$ under a projective transformation lie on the line $\mathbf{l}' = \mathtt{H}^{-\mathsf{T}}\mathbf{l}$. In this way, incidence of points on lines is preserved, since $\mathbf{l}'^{\mathsf{T}}\mathbf{x}'_i = \mathbf{l}^{\mathsf{T}}\mathtt{H}^{-1}\mathtt{H}\mathbf{x}_i = 0$. This gives the transformation rule for lines:
Under the point transformation $\mathbf{x}' = \mathtt{H}\mathbf{x}$, a line transforms as

$$\mathbf{l}' = \mathtt{H}^{-\mathsf{T}}\mathbf{l}. \tag{2.6}$$

One may alternatively write $\mathbf{l}'^{\mathsf{T}} = \mathbf{l}^{\mathsf{T}}\mathtt{H}^{-1}$. Note the fundamentally different way in which lines and points transform. Points transform according to $\mathtt{H}$, whereas lines (as rows) transform according to $\mathtt{H}^{-1}$. This may be explained in terms of “covariant” or “contravariant” behaviour. One says that points transform *contravariantly* and lines transform *covariantly*. This distinction will be taken up again, when we discuss tensors in chapter 15 and is fully explained in appendix 1(*p*562).

– passage from Multiple View Geometry in Computer Vision