2 VARIABLES

Integral Solutions

$ax + by = C$

x	y
c/a	0

$9x + 4y = 13$

x	y
1	1

Also, $1x + 2y = 14$

	x	y		
-2	14	0	$+1$	— 2×7
	12	1		
	10	2		
	⋮	⋮		
	2	6		
	0	7		— 2×0

+ive integral solⁿ → 8 (inc. 0)

non-zero solⁿ → 6

If $ax - by = c$, ∞ +ive solⁿ (Integral)

$ax + by + cz = d$

$c > b > a$, put $z = 1, 2 \ldots$ till $d > 0$ & then solve

Quadratic Equations

$ax^2 + bx + c = 0$

$a > 0$ ∪

$a < 0$ ∩

Formula for roots → $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$D = b^2 - 4ac$

Sum of roots $= \dfrac{-b}{a}$

Product of roots $= \dfrac{c}{a}$

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Equation if a, b are roots

$$\underbrace{x^2 - (a+b)}_{\text{Sum of roots}} x + \underbrace{ab}_{\text{Product of roots}} = 0$$

$$x^2 - (\text{Sum of roots})x + \text{Product of roots} = 0$$

Trick for no. of factors

$$\frac{a}{x} + \frac{b}{y} = 1$$

Positive integral sol^n = no. of factors of ab

Eg) $\frac{4}{x} + \frac{7}{y} = 1$

$$ab = 2^2 \times 7^1$$

$$f(ab) = (2+1)(\cancel{7}+1)$$

$$= 3 \times \cancel{8}^{2} = \cancel{24}\ 6$$

Remainder Theorem

$$\frac{p(x)}{(x \pm a)} \rightarrow x \pm a = 0$$

$\therefore x = \mp a \Rightarrow$ Remainder

$\Downarrow$

Subsitute $\mp a$ in $p(x)$

Eg) $\frac{x^2 + 4x + 5}{x - 2} = \frac{p(x)}{x - a} \Rightarrow x - 2 = 0$

$x = 2$

$p(2) = 4 + 8 + 5 = 13 \rightarrow$ Remainder

Roots of Cubic equations

$$ax^3 + bx^2 + cx + d = 0$$

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

Identities

- $\rightarrow (a+b)^2 = a^2 + 2ab + b^2$
- $\rightarrow (a-b)^2 = a^2 - 2ab + b^2$
- $\rightarrow (a^2 - b^2) = (a-b)(a+b)$
- $\rightarrow (a+b)^3 = a^3 + b^3 + 3ab(a+b)$
- $\rightarrow (a-b)^3 = a^3 - b^3 - 3ab(a-b)$
- $\rightarrow a^3 + b^3 = (a+b)(a^2 + b^2 - ab)$
- $\rightarrow a^3 - b^3 = (a-b)(a^2 + b^2 + ab)$
- $\rightarrow (a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$
- $\rightarrow a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$

(I) $a^n + b^n \longrightarrow$ n is even - Not divisible by $(a+b)$ or $(a-b)$

$\downarrow$

$n \rightarrow$ Odd – Divisible by $(a+b)$ but not by $(a-b)$

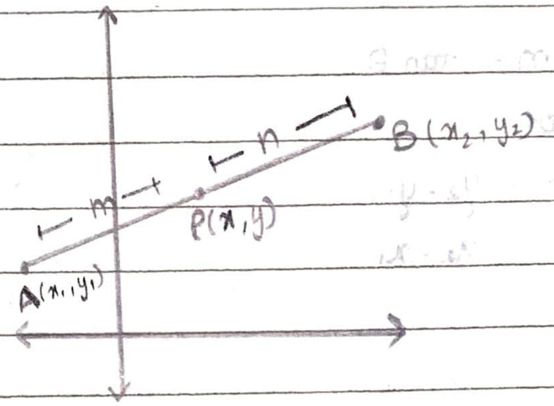
(II) $a^n - b^n \longrightarrow$ n is even - Divisible by $(a-b)$ & $(a+b)$

$\downarrow$

n is odd – Divisible by $(a-b)$ but not $(a+b)$

Co-ordinate

Sectional formula

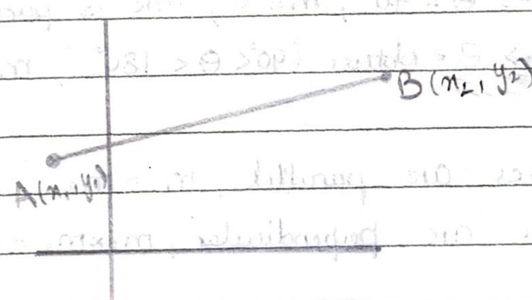


If 'P(x, y)' divides AB in the ratio m : n

$$x = \frac{mx_2 \pm nx_1}{m \pm n} \qquad y = \frac{my_2 \pm ny_1}{m \pm n}$$

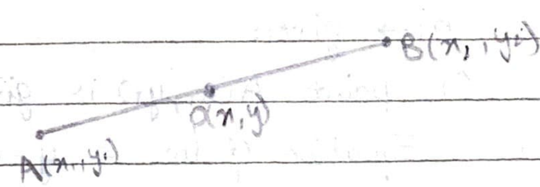
(*) -ive sign for external division, +ive sign for internal division

4) Distance formulae



$$d(AB) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Mid Point Theorem



If 'P' is the mid pt. of AB

$$x = \frac{x_1 + x_2}{2} \qquad y = \frac{y_1 + y_2}{2}$$

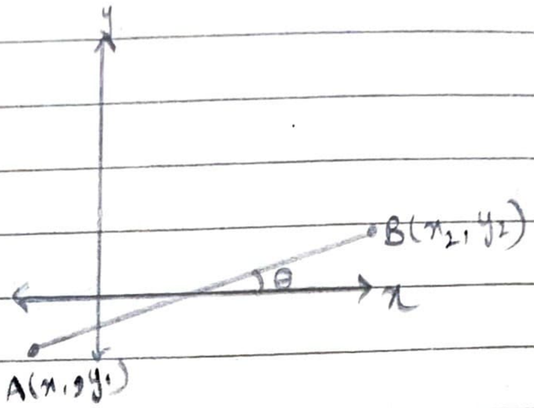
Slope of a line

$m \rightarrow$ Slope

$m = \tan\theta$

or

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$



$\tan 0° = 0$

$\tan 30° = \dfrac{1}{\sqrt{3}}$

$\tan 60° = \sqrt{3}$

$\tan 90° =$ n.d. or ∞

$\tan 45° = 1$

- $\rightarrow \theta =$ Acute $(0° < \theta < 90°)$, m is +ive
- $\rightarrow \theta = 0°$, $m = 0$, line is parallel to x-axis
- $\rightarrow \theta = 90°$, $m = \infty$, line is parallel to y-axis
- $\rightarrow \theta =$ obtuse $(90° < \theta < 180°)$, m is -ive

- 2 lines are parallel, $m_1 = m_2$
- 2 lines are perpendicular, $m_1 \times m_2 = -1$

Slope - Point form

$m \rightarrow$ given

One point $A(x_1, y_1)$ is given

Equation of line $\Rightarrow y - y_1 = m(x - x_1)$

Two - Point form

2 points of a line $A(x_1, y_1)$ & $B(x_2, y_2)$ are given

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$\therefore y - y_1 = m(x - x_1)$

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Slope - Intercept form

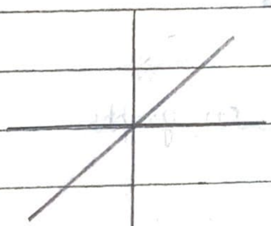
C → y-intercept is given i.e. the point (0, y)
m → Given

$\therefore \; y = mx + c$

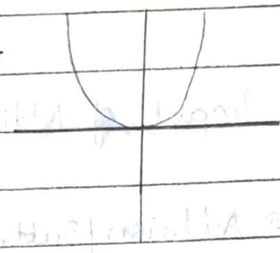
Graphs

Must Know Graphs

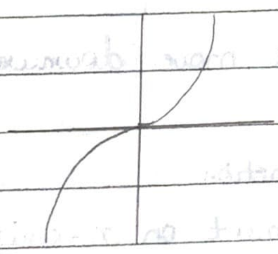
1) $y = x$



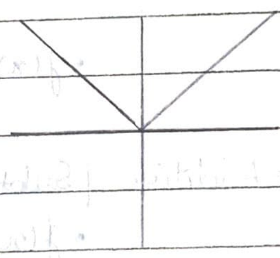
2) $y = x^2$



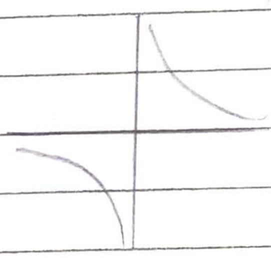
3) $y = x^3$



4) $y = |x|$



5) $y = \frac{1}{x}$



Impact of Multiplication & Division on Graphs

→ Multiplication ⇒ $y = ax, ax^2, |ax|, \frac{a}{x}, ax^3 \ldots$

The graphical representation will move towards positive y-axis

→ Division ⇒ $y = \frac{x}{a}, \frac{x^2}{a}, \left|\frac{ax}{a}\right|, \frac{1}{ax}, \frac{x^3}{a} \ldots$

The graphical representation will move towards positive x-axis

Impact of Addition & Subtraction on graphs

→ Addition | Subtraction outside the function

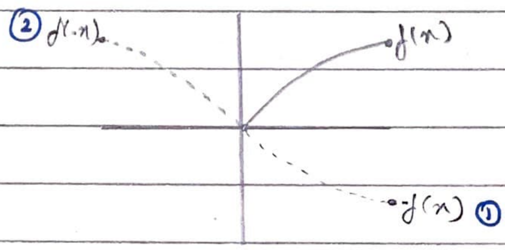
- $f(x) + K$ → y_{min} will move upwards
- $f(x) - K$ → y_{min} will move downwards

→ Addition | Subtraction in the function

- $f(x-K)$ → Right movement on x-axis
- $f(x+K)$ → Left movement on x-axis

Rotation | Mirror Image

→ $f(x)$ to $-f(x)$ — ①
→ $f(x)$ to $f(-x)$ — ②

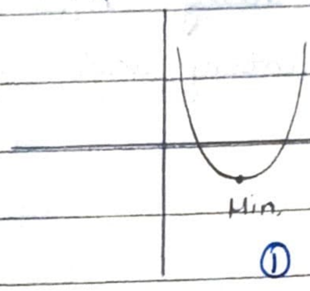


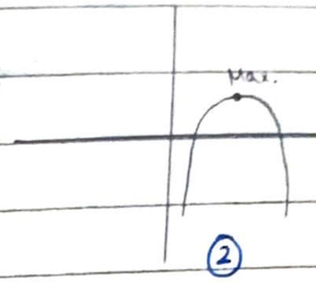
① → Mirror Image about x-axis
② → Mirror Image about y-axis

Maxima - Minima

1) Square of a variable

$$y = ax^2 + bx + c$$





- If $a > 0$

  Graph faces upwards 1

  $y_{max} = \infty$, $y_{min} = \frac{-\Delta}{4a}$ at $x = \frac{-b}{2a}$

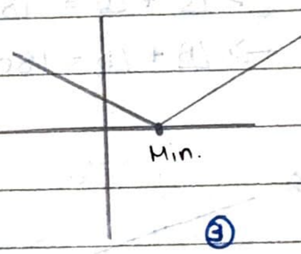
- If $a < 0$

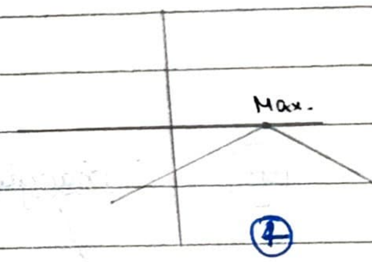
  Graph faces downwards 2

  $y_{min} = -\infty$, $y_{max} = \frac{-\Delta}{4a}$ at $x = \frac{-b}{2a}$

2) Modulus of a Variable

$$y = |ax|$$





- If $a > 0$,

  Graph faces upwards 3

  $y_{max} = \infty$

  $y_{min} \Rightarrow$ Subsititute value of x such that modulus function becomes 0

- If $a < 0$,

  Graph faces downwards 4

  $y_{min} = -\infty$

  $y_{max} \Rightarrow$ Subsitute value of x such that modulus function becomes 0

$F(x)$

Functions

$y = f(x) \rightarrow y$ is a function of x

Eg) $f(x) = \dfrac{x^2+2}{6}$

$$f(x+1) = \frac{(x+1)^2+2}{6}$$

$$f(x^3) = \frac{(x^3)^2+2}{6} = \frac{x^6+2}{6}$$

Composite functions

$$fog \rightarrow f(g(x))$$

Eg) $f(x) = 2x+1 \qquad g(x) = x^2$

$$gofog \rightarrow g(f(g(x))) = g(f(x^2))$$
$$= g(2x^2+1)$$
$$= (2x^2+1)^2$$

Eg) $f(x) = 2x^3 \qquad g(x) = \dfrac{\sqrt{x}}{2}$

$$fog = K(gof)$$

$$f(g(x)) = K[g(f(x))]$$
$$f\left(\frac{\sqrt{x}}{2}\right) = K \times g(2x^3)$$

$$2 \times \left(\frac{\sqrt{x}}{2}\right)^3 = K \times \frac{\sqrt{2x^3}}{2}$$

$$\frac{2 \times x\sqrt{x}}{\cancel{8}\ 4\ 2} = \frac{K\sqrt{2x}\,.\,x}{2}$$

$$\cancel{K = \frac{2}{\sqrt{2}}} \ \cancel{or} \ K = \frac{1}{2\sqrt{2}}$$

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Even & Odd Functions

- **Even function** → $f(-x) = f(x)$

  Eg → x^2

  $f(x) = x^2$ $f(-x) = (-x)^2 = x^2$

  → $|x|$

  $f(x) = |x| = x$ $f(-x) = |-x| = x$

- **Odd function** → $f(-x) = -f(x)$

  Eg → x^3

  $f(x) = x^3$ $f(-x) = (-x)^3 = -x^3$

  → $\frac{1}{x}$

  $f(x) = \frac{1}{x}$ $f(-x) = \frac{1}{-x} = -\frac{1}{x}$

Cos x is an even function
Sin x is an odd function

Recursive Function

Eg) $f(x+y) = f(x) \times f(y)$

$f(4) = 3$ $f(-8) = ?$

$f(-4) \Rightarrow f(4-8) = f(4) \times f(-8)$

$f(-4) = 3f(-8) \Rightarrow f(-8) = \frac{1}{3} f(-4)$

$f(-8) \Rightarrow f(-4-4) = f(-4) \times f(-4)$

$\cancel{f(-8)} = 3\cancel{f(-8)} \times 3f(-8)$

$f(-8) = \frac{1}{9}$

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Eg) $f_n(x) = \begin{cases} f_{n-1}(x) + 1 & n \to \text{odd} \\ f_{n-1}(x) + 2 & n \to \text{even} \end{cases}$

$f_1(x) = 0 \qquad f_{100}(x) = ?$

$x = 2$

$f_2(x) = f_1(x) + 2$

$x = 3$

$f_3(x) = f_2(x) + 1 = f_1(x) + 2 + 1$

$x = 4$

$f_4(x) = f_3(x) + 2 = f_1(x) + 2 + 1 + 2$

Now, $x = 100$

$f_{100}(x) = f_1(x) + 2 + 1 + 2 + 1 + 2 + 1 \ldots + 2$

$f_1(x) \downarrow 0 \qquad 2 \to 50 \qquad 1 \to 49$

$\therefore f_{100}(x) = 0 + 100 + 49 = 149$

Inequalities

If $x > y$, Inequality sign doesn't change except -

→ K is a -ive number

- $Kx < Ky$
- $\frac{x}{K} < \frac{y}{K}$

→ $\frac{1}{x} < \frac{1}{y}$ (both x & y have same sign)

Egs) 1) $4 < 6$ $(\times -2)$

$-8 > -12$

2) $4 < 6$ $(\div -2)$

$\cancel{\frac{-4}{2}} \cdot \cancel{\frac{-6}{3}}$ $-2 > -3$

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3) $4 < 6$ $\qquad\qquad$ $-4 > -6$

$\frac{1}{4} > \frac{1}{6}$ $\qquad\qquad$ $\frac{-1}{4} < \frac{-1}{6}$

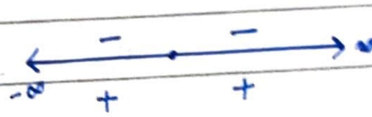
Quadratic Inequations

( ) used for > or <

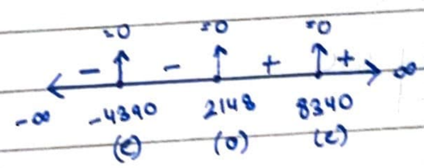
[ ] used for ≥ or ≤

$(x \pm a)^{odd}$ → Alternate Signs

$(x \pm a)^{even}$ → Same Signs



Eg - $(x-2148)^{4937}(x-8340)^{4000}(x+4390)^{100048} > 0$



Range ⇒ $x > 2148$ & $x \neq 8340$

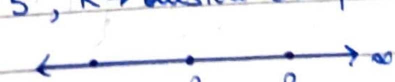
If Ques. was .... < 0 the $x < 2148$

Inequations Using Modulus

$|a| \geq 0$ ∴ If a question asks $|a| >$ -ive no., it is true for all values

→ $|x-3| > 5$

- Make modulus as 0 $(x = 3)$
- Mark 2 other pts. at a dist. of 5 from $x = 3$
- If $|a| > 5$, R → outside the pts. else if $|a| < 5$, R → b/w the points



R → $(-\infty, -2) \cup (8, \infty)$

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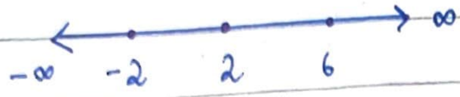
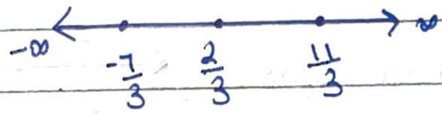
- $|x-2| \leqslant 4$



$R \rightarrow -2 \leqslant x \leqslant 6$

- $|3x-2| < 9$

$3\left|x-\frac{2}{3}\right| < 9$; $\left|x-\frac{2}{3}\right| < 3$



$R \rightarrow -\frac{7}{3} < x < \frac{11}{3}$

- $|-2x-5| < 10$

$-2\left|x-\frac{5}{2}\right| < 10$

$\left|x-\frac{5}{2}\right| > -5$ (True for all values)

- $1 < |4x-3| < 13$

$|4x-3| > 1$

$4\left|x-\frac{3}{4}\right| > 1$

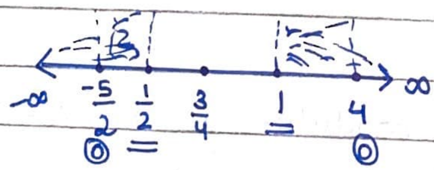
$\left|x-\frac{3}{4}\right| > \frac{1}{4}$

$|4x-3| < 13$

$4\left|x-\frac{3}{4}\right| < 13$

$\left|x-\frac{3}{4}\right| < \frac{13}{4}$

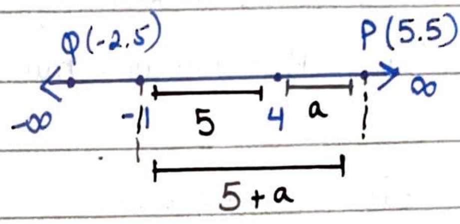
$\frac{1}{2}$ 1



$R \rightarrow \left(\frac{-5}{2}, \frac{1}{2}\right) \cup (1, 4)$

Multiple Mod Inequalities

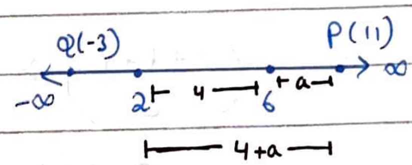
- $|x+1| + |x-4| < 8$



$$5 + a + a = 8$$
$$2a = 3$$
$$a = \frac{3}{2} \text{ or } 1.5$$

$$P = 4 + a = 5.5$$
$$Q = -1 - a = -2.5$$
Range $\rightarrow -2.5 < x < 5.5$

- $|x-2| + |x-6| > 14$



$$4 + a + a = 14$$
$$a = 5$$
$$P = 6 + 5 = 11$$
$$Q = 2 - 5 = -3$$
Range $\rightarrow (-\infty, -3) \cup (11, \infty)$

Inequalities - AM & GM

This is possible for only positive real no.s

$AM(a, b) = \frac{a+b}{2}$ $\quad$ $GM(a, b) = \sqrt{ab}$ or $(ab)^{1/2}$

$AM(a, b, c) = \frac{a+b+c}{3}$ $\quad$ $GM(a, b, c) = (ab)^{1/3}$

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AM $\geq$ GM & AM = GM when no.s are same

- When AM is known, max value of GM can be found & when GM is known, min value of AM can be found

Eg) $a+b+c = 24$ $(abc)_{max} = ?$

$$\frac{a+b+c}{3} \geq (abc)^{1/3}$$

$$8 \geq (abc)^{1/3}$$

$$512 = (abc)_{max}$$

Eg) $(abc)_{max} = 1331$

$$\left(\frac{a+b+c}{3}\right) \geq (1331)^{1/3}$$

$$\frac{a+b+c}{3} \geq [(11)^3]^{1/3}$$

$$a+b+c \geq 33$$

$$(a+b+c)_{min} = 33$$

- If 2 values are in the form of $x + \frac{1}{x}$, then the min. value is always equal to 2

$$\frac{x+\frac{1}{x}}{2} \geq \left(x \times \frac{1}{x}\right)^{1/2}$$

$$\therefore \left(x+\frac{1}{x}\right)_{min} = 2$$

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Eg) - $\left(2p+\frac{1}{2p}\right)\left(3q+\frac{3}{q}\right)$ min = ?

$$= 2\times 3\left(q+\frac{1}{q}\right)$$

$$= 2\times 3\times 2$$

$$= 12$$

Eg) $\frac{x^2-8}{\sqrt{x^2-9}} = \frac{x^2-9+1}{\sqrt{x^2-9}}$

$$= \frac{x^2-9}{\sqrt{x^2-9}} + \frac{1}{\sqrt{x^2-9}}$$

$$= \sqrt{x^2-9} + \frac{1}{\sqrt{x^2-9}} = 2$$

Eg) $2x+3y=10$ $\qquad (x^2y^3)_{max} = ?$

$$\frac{x+x+y+y+y}{5} \geq (x.x.y.y.y)^{1/5}$$

$$2 \geq (x^2y^3)^{1/5} \Rightarrow 32 \geq x^2y^3$$

$$32 = (x^2y^3)_{max}$$