Random Matrix and Potential Theory.

Lecture 1 A random matrix is a matrix whose entries are random.

For example,

$$X = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \end{pmatrix}_{2\times 3}$$ is a 2×3 random matrix.

Here, X_{ij}'s are random variables defined on the same probability space.

(X_{ij} maybe real valued or complex valued)

We will be concerned with square random matrices.

The main observable of interest in that case are eigenvalues (also eigenvectors)

$$\text{Matrix entries} \xleftrightarrow[\text{non-linear}]{} \text{Eigenvalues}$$

Due to this non-linear relations, interesting limit theorems emerge in this area. Note in CLT, we are interested in sum of given random variables; such a relation is linear.

Wigner matrices $\{Z_{i,j}\}_{1\leqslant i<j}$ complex-valued iid $\mathbb{E}Z_{1,2}=0$; $\mathbb{E}|Z_{1,2}|^2=1$.

$\{Y_i\}_{i\geqslant 1}$ real valued iid. $\mathbb{E}Y_1=0$; $\mathbb{E}Y_1^2<\infty$

$$X_N = \begin{pmatrix} Y_1 & Z_{12} & Z_{13} & \cdots \\ \overline{Z}_{12} & Y_2 & Z_{23} & \\ \overline{Z}_{13} & \overline{Z}_{23} & Y_3 & \\ & & & \end{pmatrix}_{N\times N}$$

→ Wigner matrix

→ random Hermitian matrix

Since X_N is Hermitian, it has real eigenvalues :→

$$\lambda_1^N \leqslant \lambda_2^N \leqslant \cdots \leqslant \lambda_N^N$$

$$\sum_{i=1}^N (\lambda_i^N)^2 = \mathrm{Tr}(X_N^2) = \sum_{i=1}^N Y_i^2 + 2\sum_{i<j} |Z_{ij}|^2 = O(N^2).$$

So, we expect $\lambda_i^N = O(N^{1/2})$.

Consider the empirical distribution of eigenvalues :→

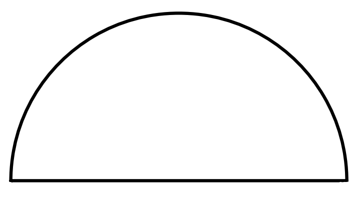
$$L_N = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i^N/\sqrt{N}} \quad \to \text{ random probability measure.}$$

$$L_N(A) = \frac{1}{N}\sum_{i=1}^N \mathbb{1}\left\{\frac{\lambda_i^N}{\sqrt{N}} \in A\right\} \qquad \text{for } A \subseteq \mathbb{R}.$$

$$\int f(x)\, dL_N(x) = \frac{1}{N}\sum_{i=1}^N f\left(\lambda_i^N/\sqrt{N}\right)$$

Note if $U_1, \ldots, U_N$ are iid from μ, $\tilde{L}_N = \frac{1}{N}\sum_{i=1}^N \delta_{U_i} \longrightarrow \mu$ weakly in probability.

<ins>Theorem</ins> (Wigner) For all bounded continuous function f



$$\int f(x)\, dL_N(x) \xrightarrow{p} \int f(x)\, \sigma(x)\, dx$$

where $\sigma(x) = \frac{1}{2\pi}\sqrt{4-x^2}\, \mathbb{1}_{|x|\leqslant 2}$ is the density of the semicircle distribution.

In other words, L_N converges weakly in probability to the semicircle distribution.

Wigner's proof is based on moment computations and involves interesting combinatorics. See [AGZ, Section 2]

I will focus on a particular Gaussian class of matrices and prove this theorem using Potential theory. The primary goal is to show you the connections to Potential theory and how it can be used to prove various aspects of L_N and beyond.

Gaussian ensembles

Gaussian orthogonal ensemble (GOE) [$\beta = 1$]
$$\begin{cases} Z_{i,j} \sim N(0,1) \\ Y_i \sim N(0,2) \end{cases}$$

Gaussian unitary ensemble (GUE) [$\beta = 2$]
$$\begin{cases} Z_{i,j} = (\xi_{i,j} + i\eta_{i,j})/\sqrt{2} \\ \xi_{i,j}, \eta_{i,j}, Y_i \sim N(0,1). \end{cases}$$

Joint density :→ (GUE)

$$C_N \exp\left(-\sum_{i<j} \frac{\xi_{ij}^2}{2} - \sum_{i<j} \frac{\eta_{ij}^2}{2} - \sum_{i=1}^{N} \frac{y_i^2}{2}\right)$$

$$= C_N \exp\left(-\frac{1}{2}\left\{\sum_{i=1}^{N} y_i^2 + 2\sum_{i<j} |z_{ij}|^2\right\}\right)$$

$$= C_N \exp\left(-\frac{1}{2}\left\{\underbrace{\operatorname{tr} X_N^2}\right\}\right)$$

↳ depends only on the eigenvalues.

For GOE :→

$$\text{Jt density} \propto \exp\left(-\frac{1}{4}\left\{\operatorname{tr} X_N^2\right\}\right).$$

Distribution of eigenvalues :→

$X_N = UDU^*$ where D is diagonal matrix → n free parameters

and U is a unitary / orthogonal matrix.

↳ $N(N-1)$ or $\frac{N(N-1)}{2}$ free parameters.

$$\mathbb{R}^{\beta\frac{N(N-1)}{2}} \times \mathbb{R}^N \xrightarrow{\hat{T}} \mathcal{H}_N^\beta \qquad (\hat{T} \text{ is "bijective"})$$

$$z \times \lambda \longrightarrow X_N$$

Then, jt density of (z, λ) :→ $\propto \exp\left(-\frac{\beta}{4}\sum_{i=1}^N \lambda_i^2\right) \text{Jacobian}(\hat{T})$

The $\hat{T}$ map :→

$$T(U) = \left(\frac{U_{1,2}}{U_{1,1}}, \dots, \frac{U_{1,N}}{U_{1,1}}, \frac{U_{2,3}}{U_{2,2}}, \dots, \frac{U_{2,N}}{U_{2,2}}, \dots, \frac{U_{N-1,N}}{U_{N-1,N-1}}\right)$$

$\downarrow$
unitary / orthogonal

T is "bijective". $\quad \hat{T}(z, \lambda) := T^{-1}(z)\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}(T^{-1}(z))^*$

$$\text{Jacobian}(\hat{T}) = g(z) \prod_{i<j} (\lambda_i - \lambda_j)^\beta$$

(can be computed or can be argued that $\lambda_i - \lambda_j$ has to be a factor).

Thus, jt density of eigenvalues :→

$$\mathbb{1}_{\lambda_1 < \lambda_2 < \dots < \lambda_N} \prod_{i<j} (\lambda_i - \lambda_j)^\beta \exp\left(-\frac{\beta}{4}\sum_{i=1}^N \lambda_i^2\right)$$

Caution :→ $T, \hat{T}$ are not bijective. But they turn out to be bijective after discarding some Lebesgue zero measure set and introducing the orders in eigenvalues. For a complete proof, see [AGZ, Section 2.5]