The Knapsack Problem

A Survey of Solution Approaches

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Abstract

This paper surveys existing literature for different approaches to solve the knapsack problem. The Knapsack Problem is a combinatorial optimization problem in which one has to maximize the profits gained by packing a set of objects in a knapsack without exceeding its capacity. The problem is $\mathcal{NP}$-hard, thus there is no known polynomial time algorithm for a large input.

Specifically, we take a look at the *0-1 Knapsack Problem* and provide a qualitative comparison between the two well-known approaches towards solving the problem: dynamic programming and backtracking algorithms.

1 Introduction

The *Knapsack Problem* is an optimization problem, which at a high level is to choose the most profitable subset from a collection of available items without overloading the knapsack. The problem is formally defined as follows:

Given a knapsack of maximum capacity C and n items each weighing w_i and with an associated profit of p_i, the *Knapsack Problem* is to choose a subset of the items such to maximize $\sum_{i=1}^{n} p_i x_i$ on the condition $\sum_{i=1}^{n} w_i x_i \leq C, i = 1, ..., n$ where x_i is the number of copies of each item.

The *0-1* knapsack problem restricts the copy count to either zero or one, meaning the object is either included in the knapsack or not, *i.e.* $x_i \in \{0, 1\}$.

2 Algorithms

A naïve brute force approach would be to consider all 2^n possible combinations of items for the knapsack and choose the one that yields te maximum profit. Such an approach would lead to exponential complexity and hence is not desirable.

2.1 Dynamic Programming

Dynamic programming solves optimization problems by breaking it into smaller subproblems and then solving those subproblems to find the overall solution. In this approach, we first define a function *knap(1, n, C)* which finds the optimal solution, $f_n(C)$ for a knapsack of capacity C using objects from *1* to *n*. We divide this into subproblems denoted by *knap(1, j, y)* which finds the optimal solution for a knapsack of capacity y using objects from *1* to *j*. Let the solution to this be defined by $f_j(y)$.

At any point in the problem state, the solution depends on making a decision on whether to use the current object or not. So, we obtain the top-down recurrence relation

$$f_j(y) = max \quad \{f_{j-1}(y), f_{j-1}(y - w_j) + p_j\}, y \geq w_j \tag{1}$$

Let us consider an example with a knapsack of capacity $C = 6$, three given objects with weights 2,3,4 and profits 1,2,5 respectively. Using the relation 1 we calculate all possible values and store them in a table as follows

	0	1	2	3	4	5	6
$f_1(y)$	0	0	1	1	1	1	1
$f_2(y)$	0	0	1	2	2	3	3
$f_3(y)$	0	0	1	2	5	5	6

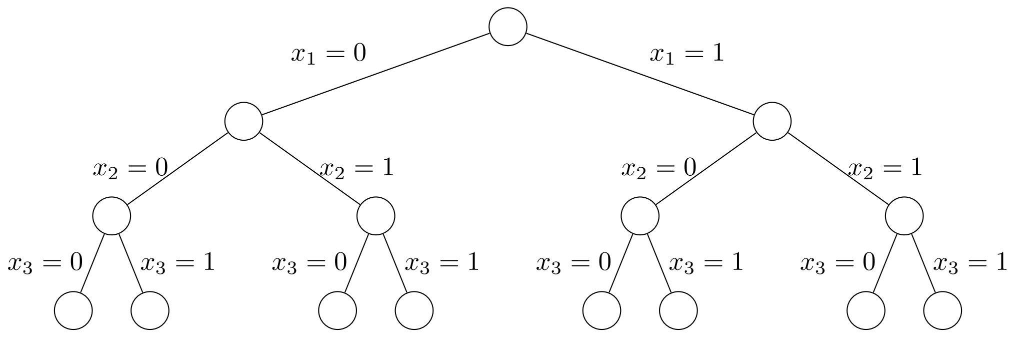
So, we have a matrix of size n x C which we fill in row-wise manner as values in a row depend on the previous row values. The running time is *O(nC)* and the space requirement is *O(C)*. Below is the pseudo-code:

Algorithm 1 DynamicKnapsack($p,\ w,\ n,\ C$)

```
for $j$ from 0 to $C$ do
    $result[0, j] \leftarrow 0$
end for
for $i$ from 1 to $n$ do
    for $j$ from 0 to $C$ do
        if $w[i-1] > j$ then
            $result[i, j] \leftarrow result[i-1, j]$
        else
            $result[i, j] \leftarrow max(result[i-1, j], result[i-1, j-w[i-1]] + p[i-1])$
        end if
    end for
end for
```

2.2 Backtracking

A backtracking algorithm builds a set of possible solution all the while discarding any partial solution candidates which it determines to be not feasible. As discussed earlier the solution space for *0-1* Knapsack problem consists of 2^n ways of assigning 0 or 1 to x_i. Below is a possible solution space when $n = 3$.



This space is searched in depth-first manner from the start node. From each E-node we check if the next node will result in a feasible solution. If yes, the next node becomes the E-node. If no, we backtrack to the previous node and kill the E-node. To do this, we use an upper bound on the best solution that can be achieved from this node. We obtain this bound by relaxing the constraint from $x_i \in \{0,\ 1\}$ to $0 \leq x_i \leq 1$ and using the greedy algorithm (of sorting the items in non-decreasing order of profit per unit weight and adding to the knapsack) for the rest of the problem. The pseudo-code for the bounding algorithm and the backtracking solution is given below.

3 Qualitative Comparison

As we saw above, the brute force approach has a complexity of *O(n2^n)*. Although it is quite simple to code, it can be used only for very small input due to its exponential complexity.

The dynamic programming approach performs much better with a running time of *O(nC)*. It also has a space complexity of *O(C)*. This stems from the fact that in a dynamic programming approach we

Algorithm 2 GreedyBound(p, w, n, C, k, p_c, w_c)

```
for $i$ from $k+1$ to $n$ do
    $w_c \leftarrow w_c + w_i$
    if $w_c < C$ then
        $p_c \leftarrow p_c + p_i$
    else
        return $p_c + (1 - (w_c - C)/w_i) * p_i$
    end if
end for
return $p_c$
```

Algorithm 3 BacktrackingKnapsack(p, w, n, C, k, p_c, w_c)

```
if $w_c + w_k \leq C$ then
    $y_k \leftarrow 1$
    if $k < n$ then
        BacktrackingKnapsack($p$, $w$, $n$, $C$, $k$+1, $p_c + p_k$, $w_c + w_k$)
    end if
    if $p_c + p_k > p_t$ and $k = n$ then
        $p_t \leftarrow p_c + p_k$
        $w_t \leftarrow w_c + w_k$
        for $j$ from 1 to $k$ do
            $x_j \leftarrow y_j$
        end for
    end if
end if
if GreedyBound($p$, $w$, $n$, $C$, $k$, $p_c$, $w_c$) $\geq p_t$ then
    $y_k \leftarrow 0$
    if $k < n$ then
        BacktrackingKnapsack($p$, $w$, $n$, $C$, $k$+1, $p_c$, $w_c$)
    end if
    if $p_c > p_t$ and $k = n$ then
        $p_t \leftarrow p_c$
        $w_t \leftarrow w_c$
        for $j$ from 1 to $k$ do
            $x_j \leftarrow y_j$
        end for
    end if
end if
```

need to store two rows of a C-column array as values in the current row are calculated using those from the previous row. From the perspective of coding complexity, it is also quite easy to implement.

The backtracking algorithm will create the complete state space tree with $2^{n-1} - 1$ nodes in the worst case. In theory it is hard to say how much of the search tree is pruned out by the bounding heuristics however in practice it does not generate all the possible nodes, so there is substantial speed increase. The implementation in code is a bit more difficult in this case compared to the dynamic programming method.

4 Conclusion

5 References

Horowitz, Ellis and Sahni, Sartaj and Rajasekaran, Sanguthevar, 2007, *Computer Algorithms*, Silicon Pr., 773p