Multivariate polynomials modulo prime powers: their roots, zeta-function and applications

M.Tech Thesis Defense

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Supervisor: Prof. Nitin Saxena

List of Papers

[CDS22] Factoring modular polynomials via Hilbert's Nullstellensatz,
Sayak Chakrabarti, Ashish Dwivedi and Nitin Saxena,
Manuscript, 2022.

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[CDS22] Factoring modular polynomials via Hilbert's Nullstellensatz,
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[CS22] Describing the roots of multivariates mod p^k and efficient computation of Igusa's local zeta function,
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  - *not* integral domain!

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- Example: $x^2 \bmod p^{2n}$– root given by $0 + p^n*$

Part I

Roots of $f(x_1, x_2)$ mod p^k in deterministic poly$((d + p + k^) d)$ time.

Describing the roots of multivariates mod p^k and efficient computation of Igusa's local zeta function

Importance of the problem

- Data-structure to give all the roots:
  - Root finding of curves: elliptic curves, Diophantine equations.
  - Root counting: cryptography, #P-complete.
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Previous work

- Restricted to univariates:
  - Lifting of roots: [BLQ13, NRS17, Pan95, DMS21].
  - Counting of roots: [DMS19, CGRW19, KRRZ20, RRZ21].
  - Igusa's LZF: [DS20, ZG03].
- $\mathbb{Z}_p$: [Chi21, DS20].
- Igusa's LZF: [Den84]

Roots of bivariates

Main ideas

- **Lifting step:** $\tilde{f}(x_1, x_2) := p^{-v} f(a_1 + px_1, a_2 + px_2)$ $(\deg(f) = d, \deg(f \bmod p) = d_1)$.

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  - Bad case: reduce to univariate root finding.

Roots of bivariates: behavior of effective degree

Theorem

Given $f(x_1, x_2)$ *of effective degree* d_1 *which has a root* $(a_1, a_2) \bmod p$ *of val-multiplitcity* v*, define* $\tilde{f}(x_1, x_2) := p^{-v} f(a_1 + px_1, a_2 + px_2)$*. Let* $d_2 := \deg(\tilde{f}(x_1, x_2) \bmod p)$*. We have the following:*

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Proof idea:

- Taylor's expansion terms,

$$f(a_1 + px_1, a_2 + px_2) = \sum_{\ell=0}^{d} \left(\sum_{|\mathbf{i}|=\ell} \frac{\partial_{\mathbf{x}^{\mathbf{i}}} f(\mathbf{a})}{\mathbf{i}!} \cdot (px_1)^{i_1} (px_2)^{i_2} \right).$$

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- When $d_1 = 1$, $\ell x_1 + m x_2 + n + p.g(x_1, x_2) \mapsto p^{-1}(\ell(a_1 + px_1) + m(a_2 + px_2) + n + p.g(a_1 + px_1, a_2 + px_2)$.
  - Fix x_1 to any value and find x_2 at every step.

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Effective degree reduction– upto linear form

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- Single val-mult. $= d_1$ root exists $\implies$ $f(x_1, x_2)$ mod p is a d_1-form: **[d_1-nonpower form]**

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- d_1-nonpower form $\not\longrightarrow$ d_1-power
- d_1-nonpower form: contiguous chain clubbed to same level; $O(k)$-many degree reducing cases.
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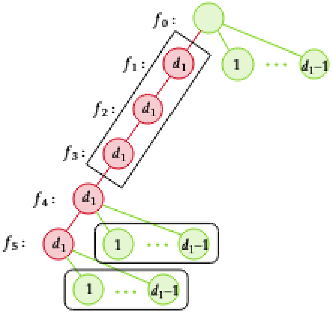

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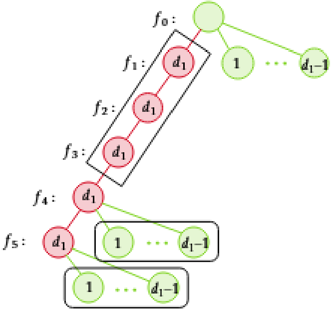

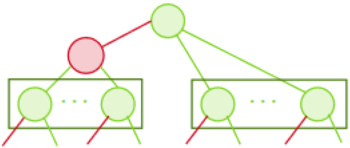

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Time Complexity: poly$((k + p + d)^d)$.

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- Linear representatives mod p^{k_0} as $\mathbf{a} \longrightarrow$ linear representatives $p^v\mathbf{a}$.

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- Find x_2 [BLQ13], find L.

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Time complexity- poly$((m + d + p)^{(2d(n-1))^{n-1}})$.

Part II

Root of $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})$ in randomized $\mathrm{poly}(m, d^{c_{nk}}, \log p)$, where $c_{nk} \leq (nk)^{O((nk)^2)}$.

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Previous work

- Roots of polynomial system over different fields: [HW99, BKW19, LPT$^+$17, Kay05, CS22].
- Factoring over fields:
  - Finite fields: [Ber67, CZ81, Kal92, KU11, vzGP01].
  - p-adics: [CG00, Chi87, Chi94, GNP12].
- Famous Hensel's lifting: [Hen18].
- Factoring achieved only for small k's: [DMS21, Sir17, vzGH96, vzGH98].

Hilbert's Nullstellensatz $\bmod p^k$

Main ideas

- Store local roots in ideals (~~enumeration~~)
- Virtual roots at i-th step $\mathbf{y}$.
- **Lifting:** $f_j(\mathbf{x}) := p^{-v_j} f_j(\mathbf{y} + p\mathbf{x}) \ \forall j \in [m]$.
- Virtual roots such that $f_j(\mathbf{y}) \equiv 0 \bmod p$.
- $\mathtt{I} \leftarrow \mathtt{I} + \langle f_j(\mathbf{y}) \bmod p \rangle$.

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- $\mathbb{Z}/p^k\mathbb{Z} \to \mathbb{F}_p \to \mathbb{Z}_p \to \mathbb{Z}/p^k\mathbb{Z}$.

Remark

- p-adic roots not unique.
- $a + pb$, replace $a := a - pt$, $b := b + t$.

Birationally equivalent hypersurface

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- $\phi_1 : \mathtt{H} \to \mathbf{V}(\mathtt{I})$; $\phi_2 : \mathbf{V}(\mathtt{I}) \to \mathtt{H}$.
- 'Most' points of $\mathbf{V}(\mathtt{I})$ are mapped.

$\mathbb{F}_p$ coordinates to $\mathbb{Z}_p$ roots

Lifting $\mathbb{F}_p$ roots to $\mathbb{Z}_p$

- I vanishes over $\mathbb{F}_p$, need to vanish over $\mathbb{Z}_p$.

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- Lift to $\mathbb{Z}_p$.
- Need for irreducible components $\mathtt{I} \rightarrow \mathtt{H} \rightarrow \mathtt{C}$.

$\mathbb{F}_p$ coordinates to $\mathbb{Z}_p$ roots

$$
\begin{tikzcd}
\widehat{\mathbb{G}}(\ell_1,\ldots,\ell_r)[\ell_{r+1},\ldots,\ell_N]/\hat{\mathtt{C}} \arrow[d, "\bmod p"'] & & \widehat{\mathbb{G}}(\ell_1,\ldots,\ell_r)[Y]/\langle\hat{h}\rangle \arrow[ll, "\hat{\psi}_2"'] \arrow[d, "\bmod p"] \\
\mathbb{F}_q(\ell_1,\ldots,\ell_r)[\ell_{r+1},\ldots,\ell_N]/\mathtt{C} \arrow[rr, shift left, "\psi_1"] & & \mathbb{F}_q(\ell_1,\ldots,\ell_r)[Y]/\langle h\rangle \arrow[ll, shift left, "\psi_2"]
\end{tikzcd}
$$

$\mathbb{F}_p$ coordinates to $\mathbb{Z}_p$ roots

$$\boxed{\mathtt{C} \longrightarrow \hat{\mathtt{C}}}$$

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$$\boxed{\texttt{C} \longrightarrow \hat{\texttt{C}}}$$

- Compute Gröbner basis of $\texttt{C}$.
- Integral lift to $\mathbb{Z}_p$: $\hat{\texttt{C}}$.
- Roots nicely commute.

$\mathbb{F}_p$ coordinates to $\mathbb{Z}_p$ roots: Absolutely irreducible components

- Hensel's lifting of roots (non-singular roots).

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$\mathbb{F}_p$ coordinates to $\mathbb{Z}_p$ roots: Absolutely irreducible components

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- Lesser dimension absolutely irreducible components.

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  - Backtracking.

Hilbert's Nullstellensatz mod p^k: Algorithm

1. If k exhausted, return ideals.

Hilbert's Nullstellensatz $\bmod p^k$: Algorithm

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    1. $\tilde{f}_j(\mathbf{x}) := p^{-1} f_j(\mathbf{y}_\ell + p\mathbf{x}) \bmod \hat{\mathtt{C}}$ over $\mathbb{Z}_p$.
    2. Recursively return root on $\tilde{f}_j$'s.

- Ideals exactly capture *all* roots.

Hilbert's Nullstellensatz mod p^k: Algorithm

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- Find one [HW99].

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- Size of tree $d^{(nk)^{O((nk)^2)}}$
- Time complexity poly$(m, d^{c_{nk}}, \log p)$, where $c_{nk} \leq (nk)^{O((nk)^2)}$.

Application: Factoring $\bmod p^k$

- Hensel's lifting [Hen18] fails when $f(x) \equiv \varphi(x)^e \bmod p$.

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- Solved for constant a.

Questions?

Describing roots of multivariates:

- Small $\log p, d, n$.
- Iteratively finding each coordinate.
- Main ideas:
  - Degree reduction,
  - Reduction to $n - 1$ variates.
- Output:
  - Representative roots,
  - Linear-representative roots.

Hilbert's Nullstellensatz $\bmod p^k$

- Small n, k.
- Storing coordinates in ideals in $\mathbb{Z}_p$.
- Main ideas:
  - Reduction to $\mathbb{F}_p$ system solving,
  - Virtual roots in ideals.
- Output:
  - Ideals containing roots,
  - Coordinates given as $\mathbb{Z}_p$ points.

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Thank You!