Quantum Information Theory and Applications to Local Decoding

Sayak Chakrabarti

Topics

- Quantum Information Theory

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- Local Decoding

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- Local Decoding
- Random Access Codes

Quantum Information Theory: Mixed States

- States given by

$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle + \cdots + \alpha_n |n\rangle$$

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- The concept of Quantum noise
- Implementing a Quantum System,

$$\text{the device outputs} \begin{cases} |\psi_1\rangle & \text{with probability } p_1 \\ |\psi_2\rangle & \text{with probability } p_2 \\ & \vdots \\ |\psi_d\rangle & \text{with probability } p_d \end{cases}$$

Mixed States

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$$Pr[\text{observe } \left|v_i\right\rangle] = \left\langle v_i\right| \left(\sum_j p_j \left|\psi_j\right\rangle \left\langle\psi_j\right|\right) \left|v_i\right\rangle = \left\langle v_i\right| \rho \left|v_i\right\rangle$$

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- Mixed state through unitary transformation U: $U\rho U^\dagger$
- Can be written as probability distribution over orthogonal pure states using SVD

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Mixed states might not always be distinguishable.

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- **More general measurements:** Matrices M_i's satisfying $\sum_i M_i^\dagger M_i = I$
- **Projective measurements:** M_i's chosen as projector matrices

Holevo's Bound

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- How many bits does a qubit represent?

Holevo's Bound: Preliminaries

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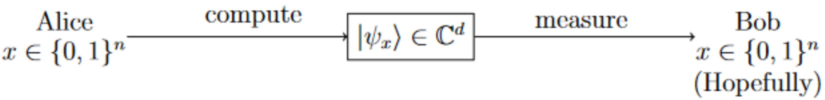

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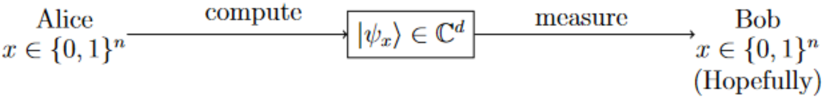

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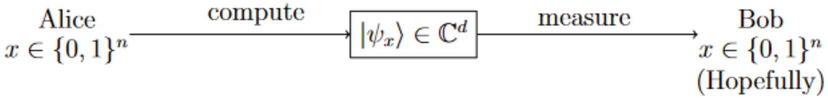

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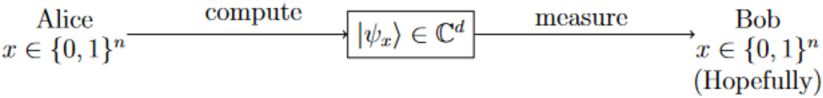

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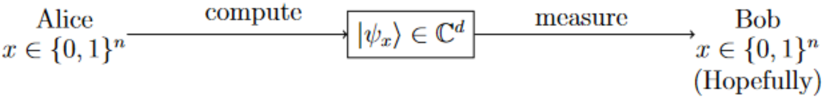

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Theorem ([Hol73])

For measurement given by POVM $E_Y = \{E_1, E_2, \dots E_n\}$ performed on ρ with measurement outcome Y, the amount of information about X possible to find from this measurement is given by

$$I_{acc}(\rho, p) \leq S(\rho) - \sum_{i=1}^{n} p_i S(\rho_i) = \chi \tag{1}$$

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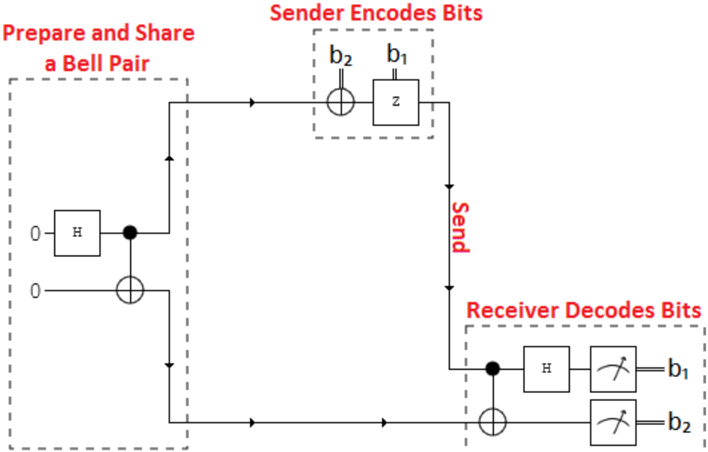
Theorem ([CvDNT98])

Suppose Alice wants to communicate a string to Bob. Then the following holds.

- *If Alice sents m qubits to Bob, and they* do not *have any prior entangled state, then Bob receives at most m bits of information about x.*
- *If Alice sents m qubits to Bob, and they* do *have some prior entangled state, then Bob receives at most $2m$ bits of information about x.*
- *If Alice sents m classical bits to Bob, and even if they have some prior entangled state, then Bob receives at most m bits of information about x.*

Superdense coding



From Holevo's Theorem to Low-dimensional Encodings

- Let $X \equiv U_N$, having values from $[N]$
- Encoding $E : x \in X \mapsto \rho_x$, some d-dimensional density matrix.
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- Encode n uniformly random bits into m qubits, success probability after decoding is $\frac{2^m}{2^n}$

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- For each $x \in \{0,1\}^n$, we want $Tr(M_i \rho_x) \geq p$ for $x_i = 1$ and $Tr(M_i \rho_x) \leq 1 - p$ for $x_i = 0$.

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- Nayak [Nay99] tightened this by showing $m \geq (1 - H(p))n$.

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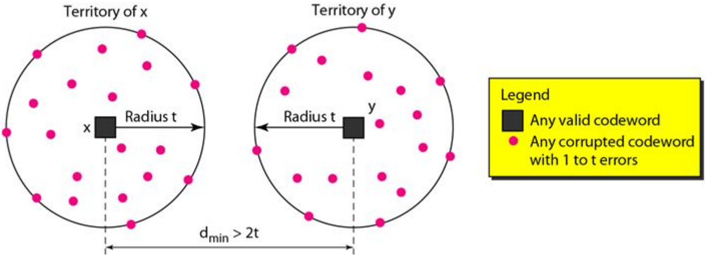

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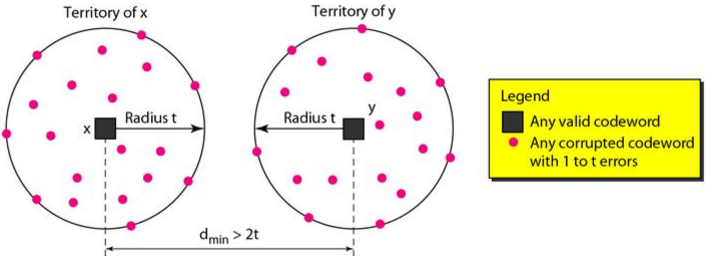

Definition (Local Decoding)

A map $E : \{0,1\}^n \to \{0,1\}^m$ is called an (q, δ, ϵ)-LDC if there exists a classical randomized decoding algorithm D satisfying the properties:

- For each $x \in \{0,1\}^n$ and $\forall y \in \{0,1\}^m$, $Ham(E(x), y) \leq \delta n$, we have $\forall i \in [n]$, $Pr[D(y,i) = x_i] \geq \frac{1}{2} + \epsilon$
- D makes at most non-adaptive q queries to y

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Example $n = 2$. For $x = 01$,
$E(x) = (x \cdot 00, x \cdot 01, x \cdot 10, x \cdot 11) = 0101.$

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- Now, we have $E(x)_z$ and $E(x)_{z \oplus e_i}$. From this,

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- WH is $(2, \delta, \frac{1}{2} - 2\delta)$-LDC.

Local Decoding: Lower Bounds

Theorem ([KT00])

For every (q, δ, ϵ)-LDC with encoding $E : \{0,1\}^n \to \{0,1\}^m$, and each $i \in [n]$, there exists a set $\mathcal{M}_i$ of $\Omega(\delta\epsilon m/q^2)$-many disjoint tuples. Each tuple $t \in \mathcal{M}_i$ consist of q indices from $[m]$, and a bit $a_{i,t}$ such that

$$Pr_{x\in\{0,1\}^n}\left[x_i = a_{i,t} \oplus \sum_{j\in t} E(x)_j\right] \geq \frac{1}{2} + \Omega(\frac{\epsilon}{2^q})$$

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Probability boosted by enumerating over t's randomly.

Local Decoding: Lower Bounds and connections to Quantum Computing

- Given $(2, \delta, \epsilon)$-LDC with encoding function $E : \{0, 1\}^n \mapsto \{0, 1\}^m$.
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- Number of qubits $= \log m$

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- Given $(2, \delta, \epsilon)$-LDC with encoding function $E : \{0,1\}^n \mapsto \{0,1\}^m$.
- **Spoiler:** Proof using QC to prove m is exponential in n!
- **Strategy:** m-dimensional quantum encoding is a QRAC for x, with success probability $p > \frac{1}{2}$

$$x \mapsto |\phi_x\rangle = \frac{1}{\sqrt{m}} \sum_{j=1}^{m} (-1)^{E(x)_j} |j\rangle$$

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- Put them together! $N \geq 2^{\Omega(\delta^2 \epsilon^4 n)}$

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- Success probability $p \geq \frac{r}{2} + \left(\frac{1}{2} + \Omega(\epsilon)\right)(1-r) = \frac{1}{2} + \Omega(\delta\epsilon^2)$

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Thank You!