# Progress

We want to prove decidability of the Zero Problem or Infinite Zeros Problem for exponential polynomials with constant coefficients.

**Conjecture 1: [Leon Ehrenpreis]** For a given exponential polynomial of the form $f(\zeta) = \sum_{k=0}^{M} b_k e^{i\alpha_k \zeta}$, where $b_i$ are real algebraic, then we have:

$$\sum_{j=0}^{M} \left|\frac{d^{j-1} f}{dz^{j-1}}(z)\right| \geq c \frac{e^{-A|Im(z)|}}{(1+|z|)^p} \tag{1}$$

The following paragraph contains a brief explanation of this conjecture, which has been explained in greater detail in Yger's paper.
We first consider the exponential polynomial of the form

$$f(\zeta) = \sum_{k=0}^{M} b_k e^{i\alpha_k \zeta}$$

where $b_k$ are algebraic and the frequencies, $i\alpha_k$ are purely imaginary.
Consider the basis of $\{\alpha_1, \alpha_2 \dots \alpha_M\}$ over $\mathbb{Z}$. Let the basis of this be $\{\gamma_1, \gamma_2 \dots \gamma_n\}$. So for any exponential polynomial $g(e^{i\boldsymbol{\alpha} z})$ there exists another polynomial such that the same exponential polynomial can be written in the form $h(e^{i\boldsymbol{\gamma} x})$. So for every derivative of $f$ we have polynomials of the form

$$\frac{d^{j-1} f}{d\zeta^{j-1}}(z) = P_j(e^{i\boldsymbol{\gamma} z})$$

Now according to the conjecture we have the form:

$$\sum_{j=0}^{M} |P_j(e^{i\boldsymbol{\gamma} z})| = \sum_{j=0}^{M} \left|\frac{d^{j-1} f}{dz^{j-1}}(z)\right| \geq c \frac{e^{-A|Im(z)|}}{(1+|z|)^p}$$

**Conjecture 2: [We are using]** We have two polynomials $P_1(x_1, x_2, \dots x_M)$ and $P_2(x_1, x_2, \dots)$ such that $P_1(e^{i\boldsymbol{\gamma} z}) = \sum_{j=1}^{M} b_j e^{i\gamma_j z}$ and $P_2(e^{i\boldsymbol{\gamma} z}) = \sum_{j=1}^{M} c_j e^{i\gamma_j z}$, where $b_j$'s and $c_j$'s are algebraic over $\mathbb{R}$ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \dots \gamma_M)$. If their variety span a codimension of 2 then we will have a polynomial lower bound of the sum of their absolute values for all $z \in \mathbb{C}$ in the form

$$|P_1(e^{i\boldsymbol{\gamma} z})| + |P_2(e^{i\boldsymbol{\gamma} z})| \geq c \frac{e^{-A|Im(z)|}}{(1+|z|)^p} \tag{2}$$

for some constants $c, p, A > 0$ depending on $P_1, P_2$ and $\gamma_j$'s.

**Proposition 3: [Bochnak and Coste]** For two semi-algebraic functions $f : D \to R$ and $g : P \to D$ we will have $f \circ g : P \to R$ is a semi-algebraic function.

**Theorem 4:** For real algebraic $b_1, ..., b_s$ that are linearly independent over $\mathbb{Q}$ and two polynomials $P_j(x_1, ..., x_s)$ that generate a variety of dimension s-2 the $\sum_{j=,1,2} |P_j(e^{ib_1 t}, ..., e^{ib_s t})|$ is bounded below by a polynomial in $t$.
*Proof:* We are given two polynomials $P_1(x_1, x_2, \dots x_s)$ and $P_2(x_1, x_2 \dots x_s)$ such that they have a variety of $s-2$ dimensions. According to the conjecture 2, we will have

$$|P_1(e^{b_1 t}, e^{b_2 t} \dots e^{b_s t})| + |P_2(e^{b_1 t}, e^{b_2 t} \dots e^{b_s t})| \geq c \frac{e^{-A|Im(t)|}}{(1+|t|)^p}$$

For some constants $c, A > 0$ and a constant $p$ depending on $P_1, P_2$ and $b_1, b_2 \dots b_s$.
Now since $t$ is purely real we will have $Im(t) = 0$, which implies

$$|P_1(e^{ib_1t}, e^{ib_2t} \dots e^{ib_st})| + |P_2(e^{ib_1t}, e^{ib_2t} \dots e^{ib_st})| \geq \frac{c}{(1+t)^p} \tag{3}$$

for all $t \geq 0$.
This gives us a polynomial lower bound of the sum of $P_1(e^{ib_1t}, e^{ib_2t} \dots e^{ib_st})$ and $P_2(e^{ib_1t}, e^{ib_2t} \dots e^{ib_st})$.

**Theorem 5:** The set $\Gamma_t = \{(x, y, z) | (e^{\boldsymbol{a}t}, x, y, z) \in C_j\}$ is semi-algebraic for a fixed value of $t$.

**Theorem 6:** Given $s$ independent frequencies, there exists a parametrization of $\Gamma_t = \{(x_1, x_2 \dots x_s) | (e^{\boldsymbol{a}t}, x_1, x_2, \dots x_s) \in C_j\}$, as a continuous semi-algebraic function $h : (0,1)^n \times [0, \infty)^k \to [-1, 1]^s$ such that $h(\boldsymbol{p}, e^{a_1t}, e^{a_2t}, \dots e^{a_kt})$ gives us the set $\Gamma_t$ for all values of $\boldsymbol{p} \in (0,1)^n$.

*Proof:* Suppose we have the cell decomposition of $C_j$ as a semi-algebraic set

$$\{(\boldsymbol{u}, x_1, x_2, \dots x_s) | \cdots\} \subset \mathbb{R}^{k+s} \tag{4}$$

which as a $(i_1, i_2, \dots i_{k+s})$-cell. We inductively construct a parametrization from this given cell structure. Throughout this proof we will denote $\boldsymbol{p}_j$ as a parameter which is a tuple of $c_j$ elements each having values between $(0, 1)$.
Consider the last $s$ coordinates of the $(i_1, i_2, \dots i_{k+s})$. We assume that we have constructed a parametrization upto $j-1$ coordinates of these $s$ coordinates, and want to construct for the $j^{th}$ coordinate. By induction hypothesis we are assuming we already have constructed a continuous semi-algebraic function for parametrization $h_{j-1} : (0,1)^{c_{j-1}} \times [0, \infty)^k \to [-1, 1]^s$, where $c_{j-1} = i_{k+1} + i_{k+2} + \dots i_{k+j-1}$, such that

$$h_{j-1}(\boldsymbol{p}_{j-1}, \boldsymbol{u}) = (h_{j-1,1}(\boldsymbol{p}_1, \boldsymbol{u}), h_{j-1,2}(\boldsymbol{p}_2, \boldsymbol{u}) \dots h_{j-1,j-1}(\boldsymbol{p}_{j-1}, \boldsymbol{u})) \tag{5}$$

where each of $h_{j-1,m}$ gives us the coordinate $x_m$ from the form as in equation 4 for every $m = 1, 2, \dots (j-1)$. Each $h_{j-1,m}$ has parameter variables $\boldsymbol{p}_m$ which is a vector consisting of the first $c_m$ coordinates of $\boldsymbol{p}_{j-1}$, for a constant $c_m \leq c_{j-1}$ depending on the number of 1's in the given cell structure between coordinates $k+1$ and $k+m$ (we have $c_m = i_{k+1} + i_{k+2} + \dots i_{k+m}$ where the cell structure is $(i_1, i_2, \dots i_{k+s})$), as not every coordinate of the parameter $\boldsymbol{p}$ is required for obtaining the value of $x_i$ for some $i$.
Now we move on to finding a parametric representation of the coordinate $x_j$ as well, using the previous parametrization and the cell structure of $C_j$.

If $i_{k+j} = 0$ we will have a continuous semi-algebraic function $f_j : [0, \infty)^k \times [-1, 1]^{j-1} \to [-1, 1]$ such that $x_j = f_j(\boldsymbol{u}, x_1, x_2, \dots x_{j-1})$ (from the definition of cell decomposition for a $(\cdots, 0)$-cell). We define $h : (0,1)^{c_{j-1}} \times [0, \infty)^k \to [-1, 1]$ such that $h(\boldsymbol{p}_j, \boldsymbol{u}) = f_1(\boldsymbol{u}, h_{j-1}(p_{j-1}))$ (In this case $c_j = c_{j-1}$ as $i_{k+j} = 0$ and hence $\boldsymbol{p}_j = \boldsymbol{p}_{j-1}$).
Now we consider our parameterizing function as $h_j : (0,1)^{c_j} \times [0, \infty) \to [-1, 1]^j$ as

$$h_j(\boldsymbol{p}_j) = (h_{j,1}(\boldsymbol{p}_1, \boldsymbol{u}), h_{j,2}(\boldsymbol{p}_2, \boldsymbol{u}), \dots h_{j,j}(\boldsymbol{p}_j, \boldsymbol{u}))$$

where each of $h_{j,m} = h_{j-1,m}$ for all $m = 1, 2, \dots (j-1)$ and $h_j = h$ as defined in the previous paragraph.

However if $i_{k+j} = 1$ we will have two continuous semi-algebraic functions $f_j, g_j : [0, \infty)^k \times [-1, 1]^{j-1} \to [-1, 1]$ such that $f_j(\boldsymbol{u}, x_1, x_2, \dots x_{j-1}) < x_j < g_1(\boldsymbol{u}, x_1, x_2, \dots x_{j-1})$, where each of $x_l$ correspond to the $(l+k)^{th}$ coordinate from equation 4 (follows from the definition of $(\cdots, 1)$-cell). Now construct the continuous semi-algebraic function $h : [0, \infty)^k \times [-1, 1]^{j-1} \to [-1, 1]$ to give all the values between $f_j(\boldsymbol{u}, x_1, x_2, \dots x_{j-1})$ and $g_j(\boldsymbol{u}, x_1, x_2, \dots x_{j-1})$ in terms of parameters. We create another parameter $\lambda \in (0, 1)$ such that we get the value of $x_j$ as $\lambda f_j(\boldsymbol{u}, h_{j-1}(\boldsymbol{p}_{j-1}, \boldsymbol{u})) + (1-\lambda) g_j(\boldsymbol{u}, h_{j-1}(\boldsymbol{p}_{j-1}, \boldsymbol{u}))$, which is a convex combination to give all the values in between for values of $\lambda$. So we define $h(\boldsymbol{p}_j, \boldsymbol{u}) = \lambda f_j(\boldsymbol{u}, h_{j-1}(\boldsymbol{p}_{j-1}, \boldsymbol{u})) + (1-\lambda) g_j(\boldsymbol{u}, h_{j-1}(\boldsymbol{p}_{j-1}, \boldsymbol{u}))$ where $\boldsymbol{p}_j = (\boldsymbol{p}_{j-1}, \lambda)$. In this case another parameter $\lambda$ is added to the set of parameters $\boldsymbol{p}_{j-1}$, giving $c_j = c_{j-1} + 1$.

Now we define our parameterizing function for the $j$ coordinates by the continuous semialgebraic function $h_j : [0, \infty)^k \times [-1,1]^{c_j} \to [-1,1]^j$ as

$$h_j(\boldsymbol{p}_j, \boldsymbol{u}) = (h_{j,1}(\boldsymbol{p}_1, \boldsymbol{u}), h_{j,2}(\boldsymbol{p}_2, \boldsymbol{u}), \ldots h_{j,j}(\boldsymbol{p}_j, \boldsymbol{u}))$$

where each of $h_{j,m} = h_{j-1,m}$ for all $m = 1, 2, \ldots (j-1)$ and $h_{j,j} = h$ as defined in the previous paragraph and $c_j = c_{j-1} + 1$.

In this way we inductively construct the parameterizing continuous semi-algebraic function $h_s : [0, \infty)^k \times [-1,1]^{c_s} \to [-1,1]^s$, which gives us the parameterization of each of the coordinates $x_j$, with parameters from $(0,1)^{c_s}$. Each point in $\Gamma_t$ is given by $h_s(\boldsymbol{p}, e^{\boldsymbol{a}t})$ for a uniquely defined parameter $p \in (0,1)^{c_s}$. $\square$

Next we move on to finding exponential polynomials such that for any $(x, y, z) \in \Gamma_t$ we will have $|P_j(x, y, z)| < 2^{-A_j t}$ for some $A_j > 0$.

We have, from Theorem 6, the parametric semi-algebraic function $h(\boldsymbol{p}, \boldsymbol{u}) = (h_1(\boldsymbol{p}_1, \boldsymbol{u}), h_2(\boldsymbol{p}_2, \boldsymbol{u}), \ldots h_s(\boldsymbol{p}_s, \boldsymbol{u}))$ where $h_1, h_2, \ldots h_s$ are continuous semi-algebraic as well.
We indeed have, from Proposition 2.5.2 of Bochnak, Coste and Roy that there exists a polynomial $Q_i(\boldsymbol{x}, y)$ such that $Q_i(\boldsymbol{p}, \boldsymbol{u}, h_i(\boldsymbol{p}, \boldsymbol{u})) = 0\ \forall \boldsymbol{p}, \boldsymbol{u}$ in domains as specified in Theorem 6.
When we set $\boldsymbol{u} = (e^{a_1 t}, e^{a_2 t}, \ldots e^{a_k} t)$ we will have $Q_i(\boldsymbol{p}, e^{\boldsymbol{a}t}, h_i(\boldsymbol{p}, e^{\boldsymbol{a}t}))$ in the form:

$$Q_{i,1}(\boldsymbol{p}, h_i(\boldsymbol{p}, e^{\boldsymbol{a}t}))e^{b_1 t} + Q_{i,2}(\boldsymbol{p}, h_i(\boldsymbol{p}, e^{\boldsymbol{a}t}))e^{b_2 t} + \ldots Q_{i,m}(\boldsymbol{p}, h_i(\boldsymbol{p}, e^{\boldsymbol{a}t}))e^{b_m t} = 0$$

where $Q_{i,j}$ are polynomials with real algebraic coefficients and $b_1 > b_2 > \ldots b_m$ for some real algebraic $b_j$'s.
This can be rearranged to give:

$$|Q_{i,1}(\boldsymbol{p}, h_i(\boldsymbol{p}, e^{\boldsymbol{a}t}))| = |Q_{i,2}(\boldsymbol{p}, h_i(\boldsymbol{p}, e^{\boldsymbol{a}t}))e^{(b_2 - b_1)} + Q_{i,3}(\boldsymbol{p}, h_i(\boldsymbol{p}, e^{\boldsymbol{a}t}))e^{(b_3 - b_1)t} + \ldots Q_{i,m}(\boldsymbol{p}, h_i(\boldsymbol{p}, e^{\boldsymbol{a}t}))e^{(b_m - b_1)t}|$$

$$\implies |Q_{i,1}(\boldsymbol{p}, h_i(\boldsymbol{p}, e^{\boldsymbol{a}t}))| \leq A e^{-\epsilon t}$$

for some constants $A, \epsilon > 0$ not depending on $\boldsymbol{p}$ and $t$.
We indeed have $s$ such polynomials $Q_{i,1}$ for each $i = 1, 2, \ldots s$. However the same argument might not proceed as in Proposition 2.10 of the journal paper as in that case it was a univariate polynomial but we have several multivariate ones. However one idea is definitely to proceed by fixing some coordinates and treat this as a univariate.

Next we want to prove that $\lim_{t \to \infty} \Gamma_t$ exists and is equal to a semialgebraic set $\Gamma_*$. One idea is to show that the semi-algebraic parameterizing function can be "extended to infinity" quite like a semi-algebraic function can be extended to 0 is it is defined in an interval $(0, r]$ for some $r > 0$.

Our claim is that if $\Gamma_*$ is of codimension $\leq 1$ then we will have the fact that $(\cos b_1 t, \cos b_2 t, \ldots \cos b_s t)$ hitting $\Gamma_t$ infinitely often and hence the zero set as unbounded. Otherwise if $\Gamma_*$ has codimension $\geq 2$ we intend to prove that the zero set is indeed bounded.

**Proposition 7:** [**Bochnak, Coste, Roy**, Proposition 2.5.3] Let $\phi : (0, r] \to R$ be a bounded continuous semi-algebraic function defined on an interval $(0, r] \subset R$. Then $\phi$ can be continuously extended to 0.

We intend to use this proposition for multivariates, namely extending a bounded semi-algebraic function $\phi : (0, r_1] \times (0, r_2] \times \ldots (0, r_n] \to \mathbb{R}$ to $(0, 0, \ldots 0)$.

**Proposition 8:** Given a bounded continuous semi-algebraic function $\phi : (0, r_1] \times (0, r_2] \times \ldots (0, r_n] \to \mathbb{R}$, with $r_i \in \mathbb{R}\ \forall\ i$, the function can be continuously extended to $(0, 0, \ldots 0)$.
*Proof:* We prove this using induction. First we consider the semi-algebraic function $\phi(X_1, X_2 \ldots X_n)$ and assume that we already have extended it to zero for the last $n - i$ variables, i.e. have have a value of $\phi(x_1, x_2, \ldots x_i, 0, 0, \ldots 0)$ for every value of $x_1 \in (0, r_1], x_2 \in (0, r_2] \ldots x_i \in (0, x_i]$.

For the base case we show that for every $x_1, x_2 \dots x_{n-1} \in (0, r_1] \times (0, r_2] \times \dots (0, r_{n-1}]$, the function $\phi(x_1, x_2, \dots x_{n-1}, X)$ as a bounded continuous semi-algebraic function in $X$ can be continuously extended to 0.
We use a proof similar to that given in Bochnak, Coste and Roy Proposition 2.5.3. Let $f \in R[X_1, X_2, \dots X_n, Y]$ be a polynomial such that $\forall x_1, x_2 \dots x_n \in (0, r_1] \times (0, r_2] \times \dots (0, r_n]$ we have $f(x_1, x_2 \dots x_n, \phi(x_1, x_2, \dots x_n)) = 0$. We use induction on the degree, say $d$, of $Y$ in $f$ to prove the base case.
If $d = 1$, we will have $\phi(X_1, X_2, \dots X_{n-1}, X) = \frac{N(X_1, X_2 \dots X_{n-1}, X)}{D(X_1, X_2, \dots X_{n-1}, X)}$ where $N$ and $D$ are relatively coprime wrt $X$ and $X$ does not divide $D(x_1, x_2, \dots x_{n-1}, X)$ since the absolute value of $\phi$ is bounded. (**Obstruction:** It might so be that for some non-zero $x_1, x_2, \dots x_{n-1}$ in the domain of $\phi$, $D(x_1, x_2, \dots x_{n-1}, 0) = 0$, can this be disproved from the fact that $\phi$ is bounded? Infact for the entire proof to go through using this method, we need the fact that $D(x_1, x_2, \dots, x_{i-1}, 0, x_{i+1}, \dots x_n) \neq 0$ for any $i$ and all other non-zero values of $x_j$'s).
Now let us assume that we have extended the last coordinate of $\phi$ to zero whenever degree of $f$ is less than or equal to $d - 1$, and want to prove it for degree $d$. (**Obstruction:** For the univariate case it says that we can suppose $f$ is never divisible by $X$.) We consider a slicing $(A_i, (\xi_{i,j})_{j=1,2,\dots l_i})$ of $(f(X_1, X_2, \dots X_{n-1}, X, Y), \frac{\partial f(X_1, X_2, \dots X_{n-1}, X, Y)}{\partial Y})$ with $A_1 = (0, r]$ for some small enough $r$ and $\phi = \xi_{1,j_0}$ for some $j_0$ (the fact that the interval $(0, r]$ is semi-algebraically connected can be used to see that one of $\xi_{1,j}$ coincides with $\phi$).
If $\phi(X_1, X_2, \dots X_{n-1}, X)$ is a root of $\frac{\partial f}{\partial Y}$ for every value of $X$ in $(0, r]$ and values of $x_1, x_2 \dots x_i$, then it can be used from the induction hypothesis to extend $\phi(x_1, x_2, \dots x_{n-1}, X)$ to $X = 0$. Otherwise, WLOG let us assume that $\frac{\partial f(x_1, x_2, \dots x_{n-1}, X, Y)}{\partial Y}|_{Y=\phi(x_1, x_2, \dots x_{n-1}, X)} > 0$ for all $X \in (0, r]$ and $x_i \in (0, r_i]$. Now we select two continuous semi-algebraic function $\rho$ and $\theta$ from $(0, r_1] \times (0, r_2] \times \dots (0, r_{n-1}] \times [0, r]$ to $\mathbb{R}$, such that for every $x_1, x_2, \dots x_{n-1} \in (0, r_1] \times (0, , r_2] \times \dots (0, r_{n-1}]$ and every $x$ in $(0, r]$ we will have $\rho(x_1, x_2, \dots x_{n-1}, x) < \phi(x_1, x_2, \dots x_{n-1}, x) < \theta(x_1, x_2, \dots x_{n-1}, x)$ and $\frac{\partial f(x_1, x_2, \dots x_{n-1}, X, y)}{\partial Y} > 0$ for every $y$ in $(\rho(x_1, x_2, \dots x_{n-1}, x), \theta(x_1, x_2, \dots x_{n-1}, x))$ (The existence of these two functions has been shown in Bochnak, Coste and Roy Prop. 2.5.3).
Now if for some $(x_1, x_2, \dots x_{n-1})$, $\rho(x_1, x_2, \dots x_{n-1}, 0) = \theta(x_1, x_2, \dots x_{n-1}, 0)$ then we define $\phi(x_1, x_2, \dots x_{n-1}, 0) = \rho(x_1, x_2, \dots x_{n-1}, 0)$.
However if $\rho(x_1, x_2, \dots x_{n-1}, X) < \theta(x_1, x_2, \dots x_{n-1}, X)$ and $\frac{\partial f(x_1, x_2, \dots x_{n-1}, 0, y)}{\partial y}$ is never $< 0$ on the interval $[\rho(x_1, x_2, \dots x_{n-1}, 0), \theta(x_1, x_2, \dots x_{n-1}, 0)]$. We have

$$f(x_1, x_2, \dots x_{n-1}, 0, \rho(x_1, x_2, \dots x_{n-1}, 0)) \leq 0 \leq f(x_1, x_2, \dots x_{n-1}, 0, \theta(x_1, x_2, \dots x_{n-1}, 0))$$

and $f(x_1, x_2, \dots x_{n-1}, 0, Y)$ is increasing in the interval, implying that it has only one root $y_0 \in [\rho(x_1, x_2, \dots x_{n-1}, 0), \theta(x_1, x_2, \dots x_{n-1}, 0)]$. We define $\phi(x_1, x_2, \dots x_{n-1}, 0) = y_0$. It can be shown for a fixed $x_1, x_2, \dots x_{n-1}$, $\phi(x_1, x_2, \dots x_{n-1}, X)$ is continuous in a similar fashion as shown in Bochnak, Coste, Roy. However we are left with proving continuity for every one of the variables $X_1, X_2, \dots X_{n-1}$. Let us consider $X_i$. We will have small constants such that for every small $\epsilon > 0$, $f(x_1, \dots x_{i-1}, x_i + \epsilon, x_{i+1}, \dots x_{n-1}, 0, y_0') = 0$ where $y_0'$ is similarly obtained the described procedure by replacing $x_i$ by $x_i + \epsilon$. Now we indeed have $\rho$ and $\theta$ as continuous and $\exists \delta_1, \delta_2 > 0$ for $\epsilon$ such that $|\rho(x_1, \dots x_{i-1}, x_i + \epsilon, x_{i+1}, \dots x_{n-1}, 0) - \rho(x_1, \dots x_{i-1}, x_i, x_{i+1}, \dots x_{n-1}, 0)| < \delta_1$ and $|\theta(x_1, \dots x_{i-1}, x_i + \epsilon, x_{i+1}, \dots x_{n-1}, 0) - \theta(x_1, \dots x_{i-1}, x_i, x_{i+1}, \dots x_{n-1}, 0)| < \delta_2$. (**Note** I am being vague in the following argument as I cannot concretely understand if this is the right way, so please let me know if anything can be built upon this). So we have the intervals $[\rho(x_1, \dots x_{i-1}, x_i + \epsilon, x_{i+1}, \dots x_{n-1}, 0), \theta(x_1, \dots x_{i-1}, x_i + \epsilon, x_{i+1}, \dots x_{n-1}, 0)]$ and $[\rho(x_1, \dots x_{i-1}, x_i, x_{i+1}, \dots x_{n-1}, 0), \theta(x_1, \dots x_{i-1}, x_i, x_{i+1}, \dots x_{n-1}, 0)]$ as roughly similar and both $y_0$ and $y_0'$ belong to these. And now since $f$ is a continuous polynomial we will have, for the roots of $f(x_1, \dots x_{i-1}, x_i + \epsilon, x_{i+1}, \dots x_{n-1}, 0, y)$ and $f(x_1, \dots x_{i-1}, x_i, x_{i+1}, \dots x_{n-1}, 0, y)$ as $y_0$ and $y_0'$, there exists a constant $\delta$ depending on $\epsilon$ such that $|y_0' - y_0| < \delta$.
Another thing that can be probably used to show continuity is probably the fact that it is a slicing, and for small $\epsilon$ if $f(x_1, x_2, \dots x_{i-1}, x_i, x_{i+1}, \dots, x_{n-1}, Y)$ and $f(x_1, x_2, \dots x_{i-1}, x_i + \epsilon, x_{i+1}, \dots, x_{n-1}, Y)$ have roots which are not close then from Rolle's theorem we can find a point where the derivative changes sign, from which a contradiction can be brought as it is a slicing of $(f, \frac{\partial f}{\partial Y})$.
**Edit:** Continuity can be shown from 3.10 of Basu, Pollack, Roy if I am not wrong in my understanding of that theorem. $\square$

Now for the induction step we assume that we have continuously extended $\phi : (0, r_1] \times (0, r_2] \dots (0, r_i] \times [0, r_{i+1}] \times [0, r_{i+2}] \dots [0, r_n] \to \mathbb{R}$. To extend it to $\phi : (0, r_1] \times (0, r_2] \dots [0, r_i] \times [0, r_{i+1}] \times [0, r_{i+2}] \dots [0, r_n] \to \mathbb{R}$, we apply the same proof as of the base case and fix $x_{i+1}, x_{i+2}, \dots x_n$ to 0. In this way we can continuously extend a multivariate bounded continuous semi-algebraic function to $\mathbf{0}$.

**Theorem 9:** Given that we have a parametrization of $\Gamma_t$ as a semi-algebraic function as in Theorem 4, $lim_{t \to \infty} \Gamma_t$ exists and is semi-algebraic.
*Proof idea:* We already have the fact that a semi-algebraic function extends to zero. So if we consider the multivariate polynomials in the boolean expression corresponding to the map of the semi-algebraic function $h(\cdots)$ and consider their reverse (something like $x_1^m x_2^n f(1/x_1, 1/x_2)$ where $m$ and $n$ are the degrees of $x_1$ and $x_2$ in $f$ respectively) to show that "extending $h$ to infinity" is same as extending another semi-algebraic function to zero, which can be done, showing that the limit of $\Gamma_t$ exists. □

Next we intend to show that roots are unbounded when the codimension of $Gamma_t$ is small. Dimension of $Gamma_t$ is $d$ when $Gamma_t$ is homeomorphic to the cylinder $(0, 1)^d$. This degree can be found from the parameterization which is again obtained from the cell decomposition. When $C_j$ is a cell of the form $(i_1, i_2, \dots i_{k+s})$, the parameterizing function $h$ is homeomorphic to $c_s$, with notation same as that in proof of Theorem 6. Now for small codimension, most of the $i_j$'s in the cell decomposition will be 1.
We have these intervals for each coordinate in the output of $h$ and need to check if for unbounded infinitely many $t$ $\cos b_j t$ is included in the interval. For low codimension, these intervals would be fixed points and it might be easier to decide if these coincide with $\cos b_j t$ or not. This is an idea of proceeding with the proof.