Subspace Designs and Error-Correcting Codes

Sayak Chakrabarti

Subspace Designs

Definition (Subspace Designs [GK16, GX13])

Subspace designs are defined as collections of subspaces $\{H_1, H_2, \ldots, H_M\}$, where $H_i \subseteq \mathbb{F}_q^m \; \forall i \in [M]$, with the property that any "low dimensional" subspace W will have less number of intersecting points with H_i's

Error-Correcting Codes

Definition (Error-Correcting Codes)

An error-correcting code, for a distance $\delta \in [0, 1]$, is a function $E : \{0, 1\}^n \mapsto \{0, 1\}^m$ such that $\forall x \neq y$, $x, y \in \{0, 1\}^n$, $\Delta(E(x), E(y)) \geq \delta$.

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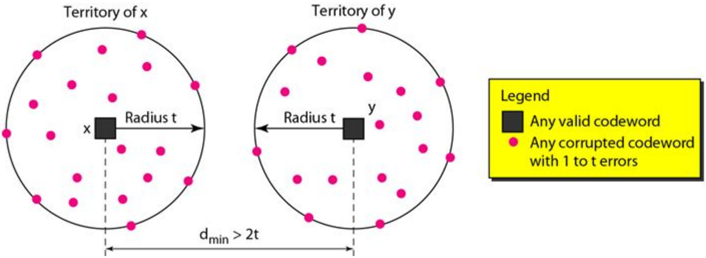

Relation to Computational Complexity Theory

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- Hardness of Approximations: Pseudorandomness [NW94]

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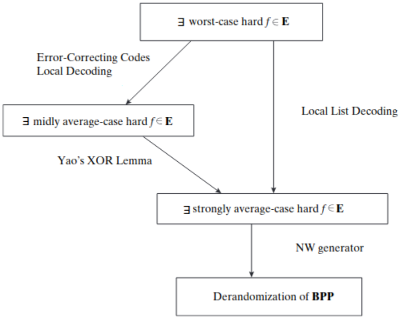

Figure 1: Relation between Hardness and Derandomization using ECC [AB09]

Error-Correcting Codes: Reed-Solomon Codes

Definition (Reed-Solomon Codes [WB99])

For a finite field $\mathbb{F}$, and integers $k \leq n \leq |\mathbb{F}|$, Reed-Solomon code is defined as a function $RS : \mathbb{F}^k \to \mathbb{F}^n$, such that on input $\overline{a} = (a_0, a_1, \ldots, a_{k-1}) \in \mathbb{F}^k$, it outputs $RS(\overline{a}) = (z_0, z_1, \ldots, z_{n-1})$, where $z_j = \sum_{i=0}^{k-1} a_i f_j^i$, for distinct $f_j \in \mathbb{F}$.

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- Error: $< 50\%$

Error-Correcting Codes: List Decoding

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- Bivariate error locator polynomial $Q(x, y)$ such that $Q(e_j, c'_j) = 0$ $\forall 0 \leq j \leq n - 1$.
- Linear equation solving such that $R(x) = Q(x, P(x)) = 0$ [Sud96].

Definition (AG Codes [Gop82, Chu04])

Given a non-singular projective curve $\mathbf{X}$ over $\mathbb{F}_q^m$, let $\mathcal{P} = \{P_1, P_2, \dots, P_n\} \subset \mathbf{X}(\mathbb{F}_q)$ be a collection of points. Let a divisor D be such that $\mathcal{P} \cap \mathrm{supp}(D) = \phi$.

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Let a divisor D be such that $\mathcal{P} \cap \mathrm{supp}(D) = \phi$. Riemann-Roch Theorem gives a unique vector space $L(D)$.

$$C(X, \mathcal{P}, D) = \{(f(P_1), f(P_2), \ldots, f(P_n)) | f \in L(D)\} \subset \mathbb{F}_q^n.$$

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- (s, A)-subspace: Every s-dimensional subspace $W \subset \mathbb{F}_q^m$ intersects with at most A-many H_i's non-trivially.
- "Well Spread-out property" [GK16]: random collection of good subspaces $\rightarrow$ good subspaces.

Subspace Designs: Motivation

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    - $f_i \in W + A_i(f_0, \ldots, f_{i-1})$, $\forall i \in 0, 1, \ldots, k-1$.

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  - Radius $\frac{s}{s+1}(n-k)$, $\dim(W) = s - 1$.
- Also gave a family of AG codes whose list decoded solutions are pinned down to a linear subspace.

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Theorem ([GK16])

For every $R \in (0,1)$, we can construct a family of ECCs of rate R on an alphabet set of size $(1/\epsilon)^{\mathcal{O}(1/\epsilon^2)}$. This can be list decoded in $n^{\mathcal{O}(1)}$ time with $(1 - R - \epsilon)$ errors, which outputs a list of size at most $exp_{1/\epsilon}(exp_{1/\epsilon}(exp(\mathcal{O}(\log^* n))))$.

Subspace Designs

Definition (Weak Subspace Designs [GK16])

A collection of subspaces $\mathcal{H} \subset \mathbb{F}_q^m$ is called an (s, A)-weak subspace design if, for every linear subspace $W \subset \mathbb{F}_q^m$ of dimension s, we have

$$|\{i \in [M] | \dim_{\mathbb{F}_q}(W \cap H_i) > 0\}| \leq A \tag{1}$$

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Similarly, a collection of subspaces $\mathcal{H}$ is called an (s, A)-weak subspace design if, for every linear subspace $W \subseteq \mathbb{F}_q^m$ of dimension s, we have

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Subspace Designs

Example

- $\alpha_1, \alpha_2, \ldots, \alpha_M \in \mathbb{F}_q$
- $v_{\alpha_i} = (1, \alpha_i, \alpha_i^2, \ldots, \alpha_i^{m-1}) \in \mathbb{F}_q^m$
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- (s, m)-strong subspace.

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Definition (Folded Reed-Solomon Codes)

Folded Reed-Solomon codes are a variant of Reed-Solomon Codes. The polynomial f is formed as before, and the ECC outputs

$$f(x) \mapsto (f(1), f(\gamma), f(\gamma^2), \ldots, f(\gamma^{n-1})),$$

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Definition (Multiplicity Codes [KSY14])

It is an ECC similar to Reed-Muller codes where the output is $(f(\bar{a}), \frac{\partial f(\bar{a})}{\partial x}, \frac{\partial f(\bar{a})}{\partial y})$, for $\bar{a} \in \mathbb{F}_q^2$.

Wronskian: A nice algebraic tool

Definition (Classical Wronskian)

Given polynomials $f_1(x), f_2(x), \ldots, f_s(x) \in \mathbb{F}[x]$, the Wronskian $W(f_1, f_2, \ldots, f_s)$ is defined as:

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$f_1, \dots f_s$ are linearly independent over $\mathbb{F} \iff \det(W(f_1, \dots, f_s)) \neq 0$.

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Explicit Subspace Designs: Folded Reed-Solomon Codes

Weak Subspace Construction [GK16]

For a generator γ of $\mathbb{F}_q^*$ and some t such that $s \leq t \leq m < q$, define the set $\mathcal{F} = \{\gamma^{jt} | j \in \{0, 1, \ldots, q/t\}\}$. Now, $\forall \alpha \in \mathcal{F}$, define the subspaces

$$\mathcal{H}_\alpha = \{P(x) \in \mathbb{F}_q[x]_{<m} | P(\alpha\gamma^i) = 0, \forall i \in \{0, 1, \ldots, t-1\}\}$$

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Codimension of $\mathcal{H}_\alpha$ is t.

Theorem ([GK16])

The collection of subspaces given by $\{\mathcal{H}_\alpha | \alpha \in \mathcal{F}\}$ is an $\left(s, \frac{(m-1)s}{t-s+1}\right)$-weak subspace design.

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Proof Idea:

- Consider folded Wronskian of basis of W as a polynomial $L(x)$.

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- $\deg(L) = (m-1)s$, $(t - s + 1)$-many roots.

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Improving the Construction

s, t, r, q, m are parameters such that $s \leq t \leq m < q$. Given a generator γ of $\mathbb{F}_q^*$ and an $\alpha \in \mathbb{F}_{q^r}$, define

$$S_\alpha = \{\alpha^{q^j}\gamma^i | 0 \leq j < r, 0 \leq i < t\} \subseteq \mathbb{F}_{q^r},$$

and

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- $|S_\alpha| = rt$.

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We take $|\mathcal{F}| = \Omega(\frac{q^r}{rt})$. Also, $|S_\alpha| = r(t - s + 1)$

Explicit Subspace Designs: Folded Reed-Solomon Codes

Strong Subspace Construction [GK16]

For every $\alpha \in \mathcal{F}$, define the subspaces

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Theorem ([GK16])

The collection $(\mathcal{H}_\alpha)_{\alpha \in \mathcal{F}}$ is an $\left(s, \frac{(m-1)s}{r(t-s+1)}\right)$-strong subspace.

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- The matrix $M(\alpha) \in \mathbb{F}_{q^r}^{t \times s}$ satisfies $\mathrm{rank}(M(\alpha)) \leq s - \dim_{\mathbb{F}_q}(W \cap \mathcal{H}_\alpha)$.

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- For each $\beta \in S'_\alpha$, we have $\dim(W \cap H_\alpha) \leq \mathrm{mult}(L, \beta) \leq s(m-1)$

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- For each $\beta \in S'_\alpha$, we have $\dim(W \cap H_\alpha) \leq \text{mult}(L, \beta) \leq s(m-1)$
- Sum over all α and β.

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Strong Subspace Construction [GK16]

For each $\alpha \in \mathbb{F}_q$, consider the subspaces

$$\mathcal{H}_\alpha = \{P(x) \in \mathbb{F}_q[x]^{<m} | \mathrm{mult}(P, \alpha) \geq t\}$$

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Theorem ([GK16])

The collection $(\mathcal{H}_\alpha)_{\alpha \in \mathcal{F}}$ is an $\left(s, \frac{(m-1)s}{(t-s+1)}\right)$-strong subspace.

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Improving the Construction

s, t, r, q, m are parameters such that $s \leq t \leq m < \mathrm{char}(\mathbb{F}_q)$.

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- Choose one element from each of these sets to form $\mathcal{F}$ with $|\mathcal{F}| \approx q^r/r$

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- $(m-1).s \geq \sum_{\alpha \in \mathcal{F}_0} \mathrm{mult}(L, \alpha) = \sum_{\alpha \in \mathcal{F}} r.\mathrm{mult}(L, \alpha) \geq r.(t - s + 1) \sum_{\alpha \in \mathcal{F}} \dim(W \cap H_\alpha)$.

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- AG codes: Do in blocks [GX13, GK16].

Sanjeev Arora and Boaz Barak.

Computational complexity: a modern approach.

Cambridge University Press, 2009.

Key One Chung.

Goppa codes, December 2004.

Department of Mathematics, Iowa State University.

Venkatesan Guruswami and Swastik Kopparty.

Explicit subspace designs.

Combinatorica, 36(2):161–185, 2016.

Valerii Denisovich Goppa.

Algebraico-geometric codes.

Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya, 46(4):762–781, 1982.

Venkatesan Guruswami and Chaoping Xing.

List decoding reed-solomon, algebraic-geometric, and gabidulin subcodes up to the singleton bound.

In *Proceedings of the forty-fifth annual ACM symposium on Theory of computing*, pages 843–852, 2013.

Swastik Kopparty, Noga Ron-Zewi, Shubhangi Saraf, and Mary Wootters.

Improved decoding of folded reed-solomon and multiplicity codes.

In *2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 212–223. IEEE, 2018.

Swastik Kopparty, Shubhangi Saraf, and Sergey Yekhanin.

High-rate codes with sublinear-time decoding.

Journal of the ACM (JACM), 61(5):1–20, 2014.

Noam Nisan and Avi Wigderson.

Hardness vs randomness.

Journal of computer and System Sciences, 49(2):149–167, 1994.

Madhu Sudan.

Maximum likelihood decoding of reed solomon codes.

In *Proceedings of 37th Conference on Foundations of Computer Science*, pages 164–172. IEEE, 1996.

Madhu Sudan.

Decoding of reed solomon codes beyond the error-correction bound.

Journal of complexity, 13(1):180–193, 1997.

Lloyd R Welch and Elwyn R Berlekamp.

Error correction for algebraic block codes, December 30 1986.

US Patent 4,633,470.

Stephen B Wicker and Vijay K Bhargava.

Reed-Solomon codes and their applications.

John Wiley & Sons, 1999.

Thank You!