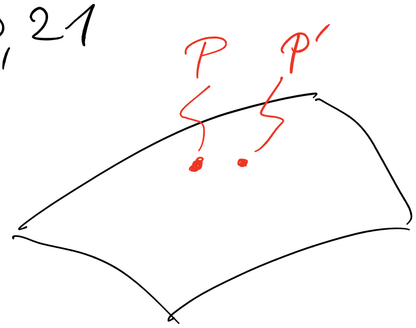
RECAP: DIFFERENTIAL GEOMETRY

Hartle, § 7, 20, 21



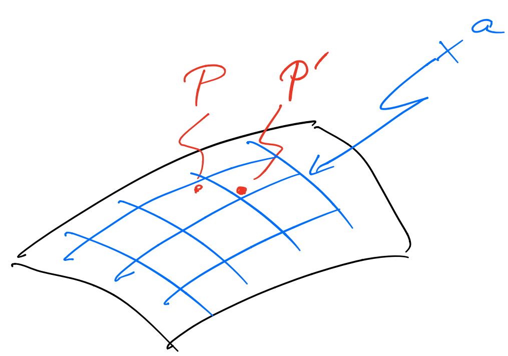
Manifold $\mathcal{M}$

- Diff Geo deals with properties of $\mathcal{M}$ which are <u>independent of coordinates</u>: "geometric" objects/properties. e.g. points, distances, gradients, Laplacian

- Often need coordinates to represent objects.

$$x^a, \quad a = 0, 1, \dots d-1 \qquad d = \text{dimension of } \mathcal{M}$$

- Point P represented by its coords x^a nearby P' by $x^a + dx^a$



- <u>Distance</u> (squared) between P and P':

$$\boxed{ds^2 = g_{ab}\, dx^a dx^b} \quad (1) \qquad \text{Line-element}$$

sum convention → $\sum_{a,b=0}^{d-1} g_{ab}\, dx^a dx^b$

- (1) generalizes Pythagoras
- Matrix g_{ab} can change from point to point g_{ab} = "Metric tensor"
- P, P' and <u>distance ds</u> <u>indep't</u> of coordinates ("invariant")

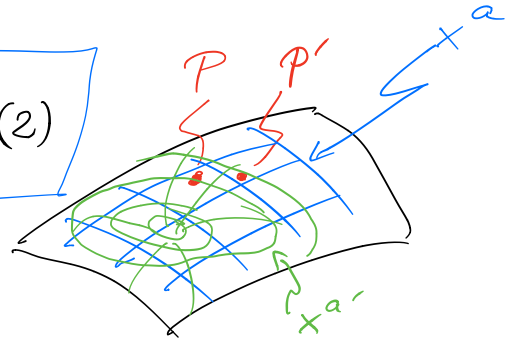
• Coordinate trafo $x^a \to x^{a'}$ to some other coords $x^{a'}$

$f(x^{a'})$

$$df = \sum_{a'=0}^{d-1} \frac{\partial f}{\partial x^{a'}} dx^{a'}$$

$x^a(x^{a'})$

$$\boxed{dx^a = \frac{\partial x^a}{\partial x^{a'}} dx^{a'} \qquad (2)}$$



ds^2 unchanged:

$$ds^2 = g_{ab}\, dx^a dx^b = g_{ab} \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} dx^{a'} dx^{b'}$$

$$\overset{!}{=} g_{a'b'}\, dx^{a'} dx^{b'}$$

works iff $\boxed{g_{a'b'} = g_{ab} \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}}}$ (3) Transformation law of metric <u>tensor</u> g_{ab}.

Tensors   Hartle §20

- geometric objects on $\mathcal{M}$
- pragmatically, defined by trafo properties like (3).
- formally, defined as certain linear operators.

e.g. vector field $\vec{v}$ on $\mathcal{M}$ is a directional derivative:

$$\vec{v} = v^a \frac{\partial}{\partial x^a}$$

· This induces a linear map

$$(f : \mathcal{M} \to \mathbb{R}) \longrightarrow (g : \mathcal{M} \to R) : \quad g = v^a \frac{\partial f}{\partial x^a}.$$

· Under coord trafo

$$\vec{v} = v^a \frac{\partial}{\partial x^a} = v^a \frac{\partial x^{a'}}{\partial x^a} \frac{\partial}{\partial x^{a'}} = v^{a'} \frac{\partial}{\partial x^{a'}}$$

$$\text{iff } v^{a'} = v^a \frac{\partial x^{a'}}{\partial x^a}$$

Trafo law of vector

basis

$$\vec{V}(x^a) = v^a(x^a) \frac{\partial}{\partial x^a}$$

geometric object

components

$$\underline{g}(x^a) = g_{ab}(x^a) \underbrace{dx^a dx^b}$$

basis $\{ dx^a dx^b \,/\, a = 0, .., d-1, \ b = 0, .. a \}$

in General Relativity

$d = 4$ space-time

g_{ab} has signature $- + + +$ everywhere

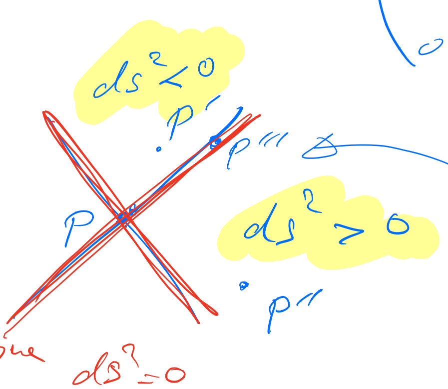
pseudo-Riemannian manifold

Minkowski-metric in special relativity

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

special case $g_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}$

Light cones



on light-cone $ds^2 = 0$

Light cone $ds^2 = 0$

line element ds^2 invariant $\Longrightarrow$ light cone structure invariant

- Upper / lower indices

V^a $\qquad g_{ab}$

- a.k.a. contravariant / covariant

- Transform differently under coord. trafo

$$V^a \to V^{a'} = \frac{\partial x^{a'}}{\partial x^a} V^a$$

$$W_b \to W_{b'} = \frac{\partial x^b}{\partial x^{b'}} W_b$$

one Jacobian per index

$$g_{a'b'} = \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} g_{ab}$$

- Contraction between one upper + one lower index invariant:

$$V^a W_a \to V^{a'} W_{a'} = V^a \underbrace{\frac{\partial x^{a'}}{\partial x^a} \frac{\partial x^b}{\partial x^{a'}}}_{= \underline{\underline{1}} = \delta_a^b \text{ dxd identity matrix}} W_b = V^a W_a$$

$\vec{V} \cdot \vec{W}$

Jacobian of inverse coord. trafo = inverse of Jacobian

$$\underbrace{\frac{\partial x^{a'}}{\partial x^a}}_{\text{dxd matrix}} = \left(\underbrace{\frac{\partial x^a}{\partial x^{a'}}}_{\text{dxd matrix}} \right)^{-1}$$

matrix inverse

- Therefore Einstein sum convention only between up + lo index

- lower an index w/ metric:

$$V_b = V^a g_{ab}$$

matrix inverse

- raise index with inverse metric $g^{ab} := (g_{ab})^{-1}$

dxd matrix

$$W^b = g^{ab} W_a$$

- in contractions, g_{ab} & g^{ab} cancel each other

$$\underline{\underline{V_b W^b}} = V^a \underbrace{g_{ab}\, g^{cb}}_{= \underline{1} = \delta_a{}^c} W_c = \underline{\underline{V^a W_a}}$$

identidy matrix

$$V^a \underbrace{\delta_a{}^c \, W_c}_{= W_a} = V^a W_a$$