Computational Intelligence (CI)

Membership Function

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Membership Function

- A fuzzy set is completely characterized by its membership function (sometimes abbreviated as MF, and denoted as μ)

- It would be important to learn how a MF can be expressed.

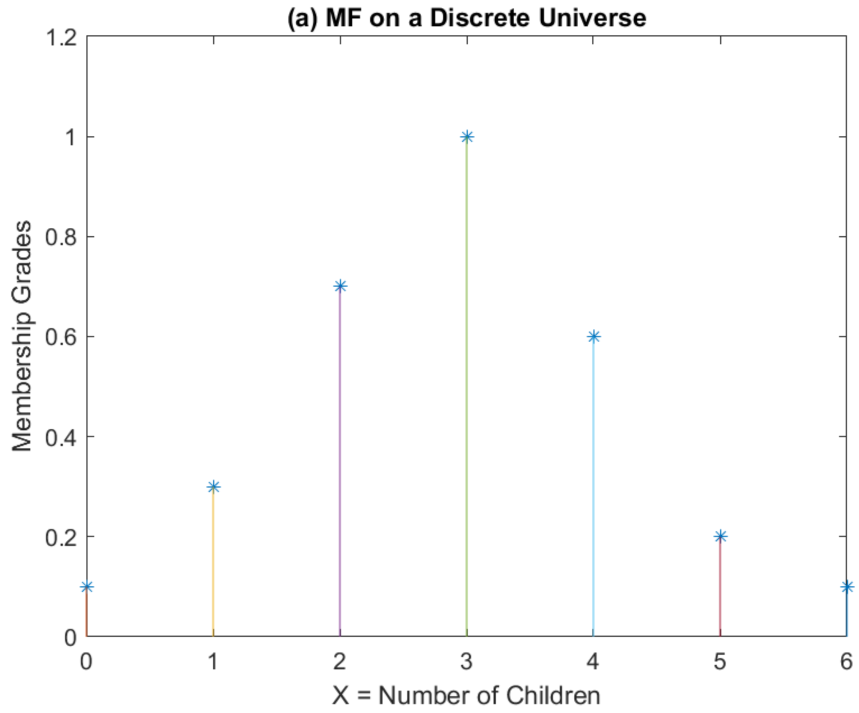
- MF can be on
  - Discrete universe of discourse
  - Continuous universe of discourse

Example (MF on a Discrete Universe)

Q1: Define a fuzzy set of “happy family” in the universe of discourse (UD) “number of children”.

X = Number of children (Universe of Discourse)

A = Fuzzy set of Happy Family



Solution (Matlab)

```
x = [0 1 2 3 4 5 6];
mf = [.1 .3 .7 1 .6 .2 .1];

plot(x, mf, '*');

% axis([-inf inf 0 1.2]);
% hold on
% for ii=1:length(x)
%   plot([x(ii) x(ii)],[0 mf(ii)], '-');
% end
% hold off

xlabel('X = Number of Children');
ylabel('Membership Grades');
title('(a) MF on a Discrete Universe');
```

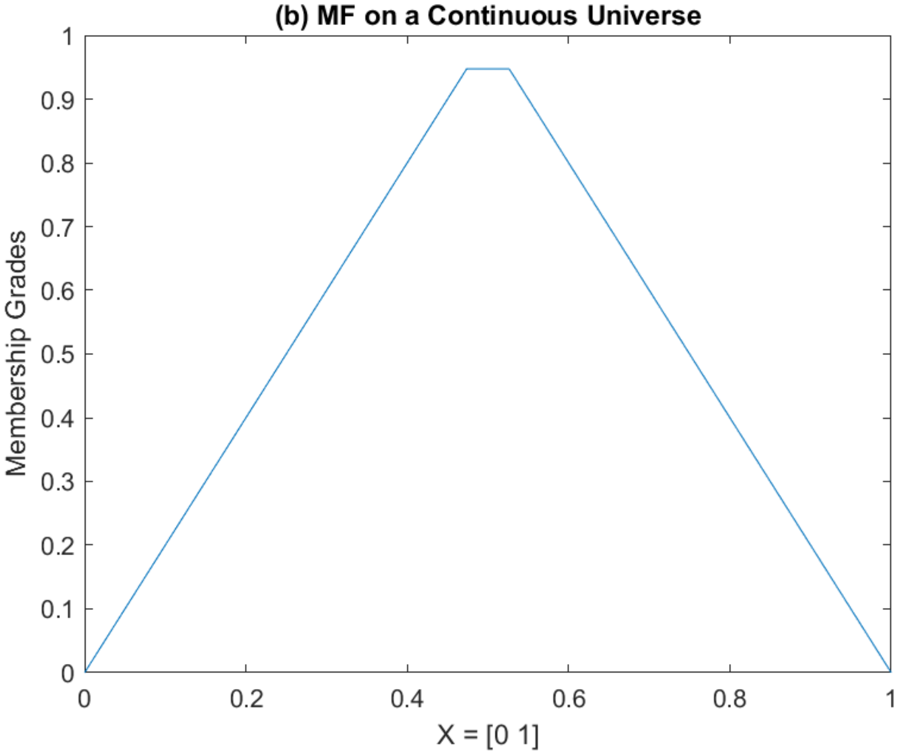
Example (MF on a Continuous Discourse)

Q2: Define a fuzzy set "close to 0.5" in the UD [0 1].

X = [0 1]

$$\mu_A(x) = 1 - |\, 2x - 1 \,|$$

A = Close to 0.5

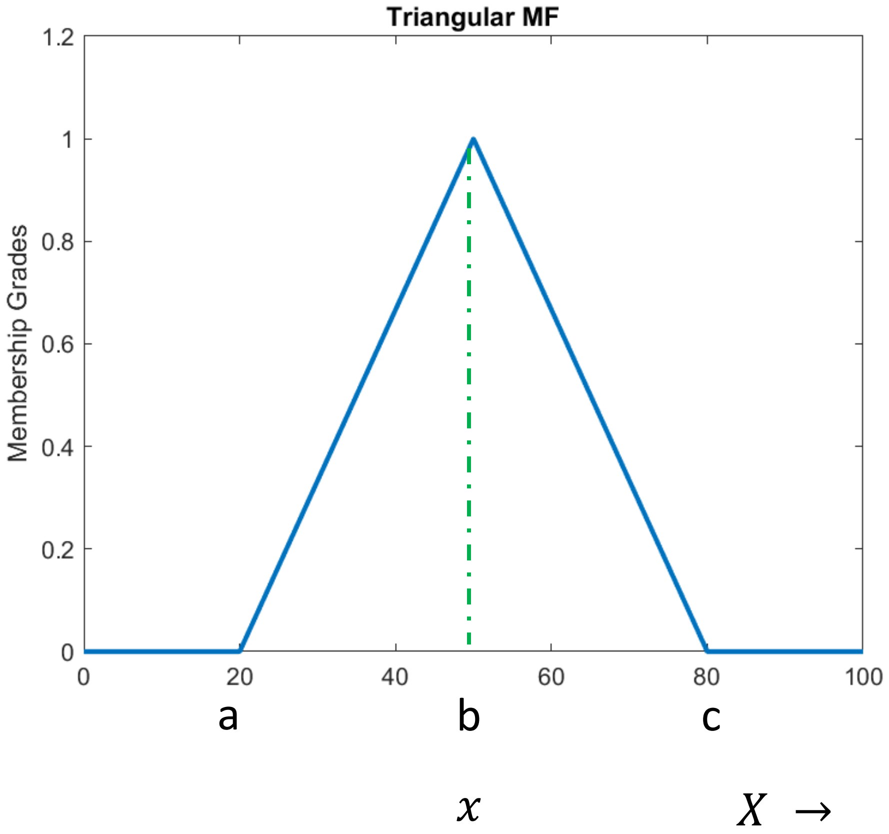


MF and Parameterization

- **One Dimensional MF:** It's a MF with single input.
- Generally, MF is expressed by parameterized function.
- Commonly used parameterized one-dimensional MF are
    - Triangular MF
    - Trapezoidal MF
    - Gaussian MF
    - Generalized Bell MF
    - Sigmoidal MF

Triangular MF

Triangular MF is defined by 3 parameters **{a, b, c}, where a<b<c**. These parameters determine the x coordinates of the three corner of the underlying MF.

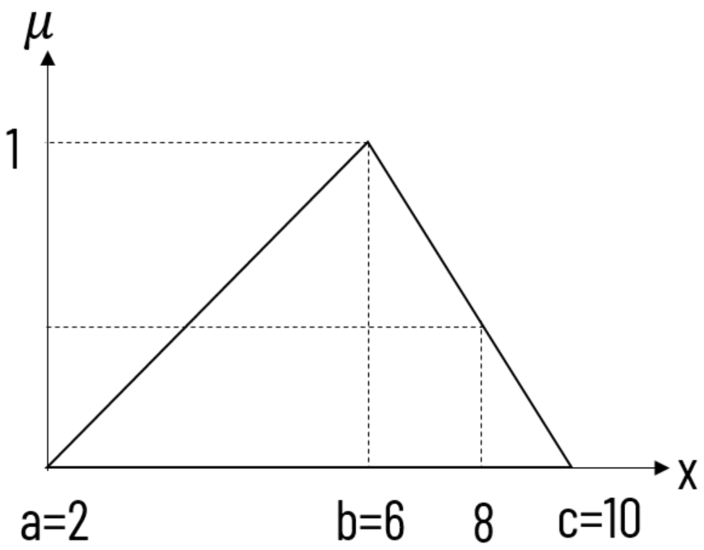


$$\text{triangle}(x;a,b,c)=\begin{cases}0, & x\le a.\\ \dfrac{x-a}{b-a}, & a\le x\le b.\\ \dfrac{c-x}{c-b}, & b\le x\le c.\\ 0, & c\le x.\end{cases}$$

$$\text{triangle}(x;a,b,c)=\max\left(\min\left(\frac{x-a}{b-a},\frac{c-x}{c-b}\right),0\right)$$

Triangular MF

Determine μ, corresponding to x = 8.0



$$\mu_{triangle}(x; a, b, c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$

$$= \max\left(\min\left(\frac{x-2}{6-2}, \frac{10-x}{10-6}\right), 0\right)$$

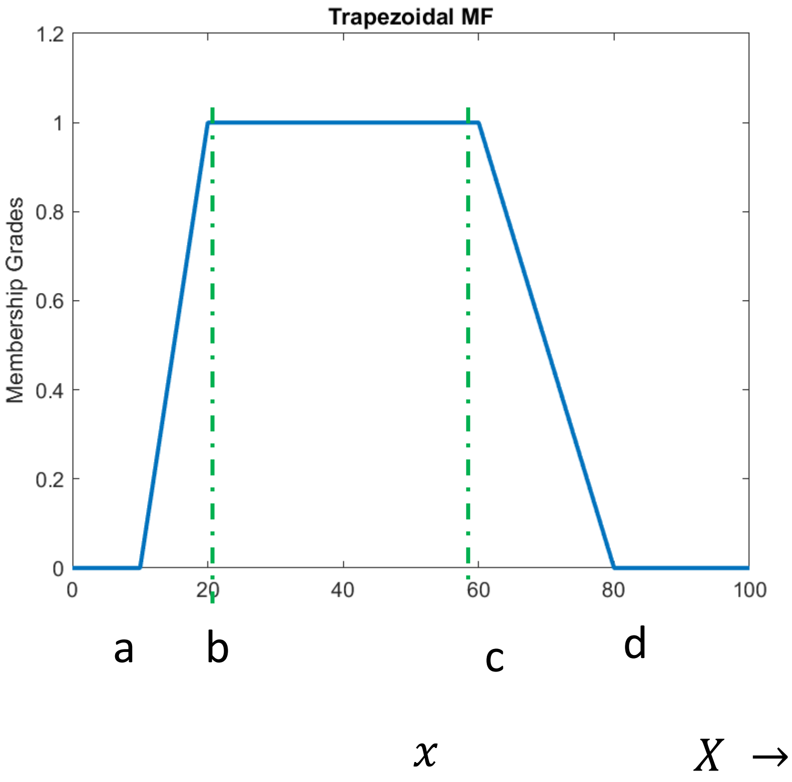
$$= \max\left(\min\left(\frac{x-2}{4}, \frac{10-x}{4}\right), 0\right)$$

$We\ put\ x = 8.0$

$$= \max\left(\min\left(\frac{3}{2}, \frac{1}{2}\right), 0\right) = \frac{1}{2} = 0.5$$

Trapezoidal MF

Trapezoidal MF is defined by four parameters **{a, b, c, d} where $\boldsymbol{a < b \leq c < d}$**. These parameters determine the x coordinates of the four corners of the underlying MF.

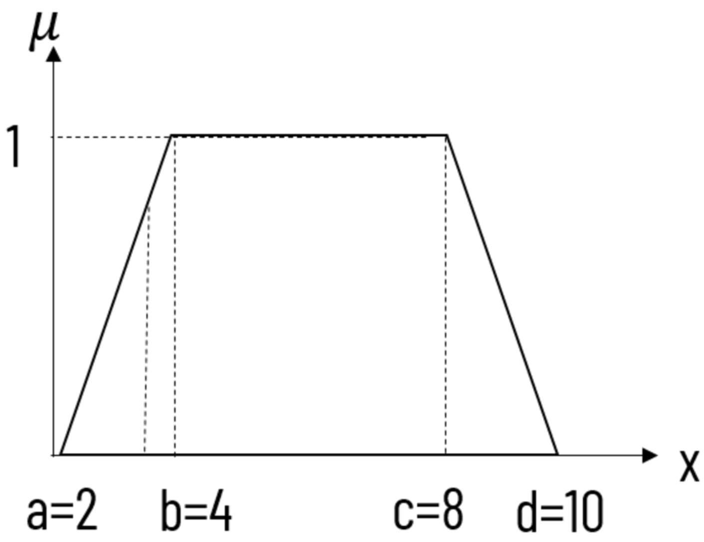


$$\text{trapezoid}(x; a, b, c, d) = \begin{cases} 0, & x \leq a. \\ \frac{x-a}{b-a}, & a \leq x \leq b. \\ 1, & b \leq x \leq c. \\ \frac{d-x}{d-c}, & c \leq x \leq d. \\ 0, & d \leq x. \end{cases}$$

$$\text{trapezoid}(x; a, b, c, d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$

Trapezoidal MF

Determine μ, corresponding to x = 3.5



$$\mu_{trapezoidal}(x;a,b,c,d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$

$$= \max\left(\min\left(\frac{x-2}{4-2}, 1, \frac{10-x}{10-8}\right), 0\right)$$

$$= \max\left(\min\left(\frac{x-2}{2}, 1, \frac{10-x}{2}\right), 0\right)$$

$We\ put\ x = 3.5$

$$= \max\left(\min\left(\frac{1.5}{2}, 1, \frac{6.3}{2}\right), 0\right)$$

$$= \max(0.75, 0) = 0.75$$

Trapezoidal MF

Fuzzy sets A is defined by trapezoid(x;12, 20, 28, 37), find its bandwidth.

Ans: 32.5-16=16.5

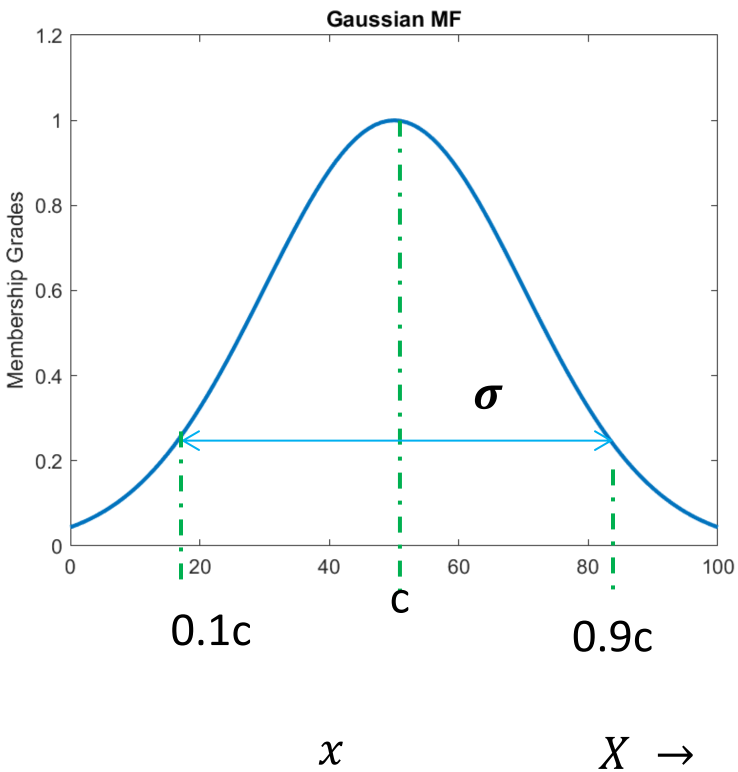
Trapezoidal MF

Create a fuzzy set "Good Students performance" in UD Age using a trapezoidal MF with parameters {10, 30, 50,60}. Find the core.

Gaussian MF

Gaussian MF is defined by two parameters **{c, σ}**
where

c = center
σ = width/ std. dev.



$$\text{gaussian}(x; c, \sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$

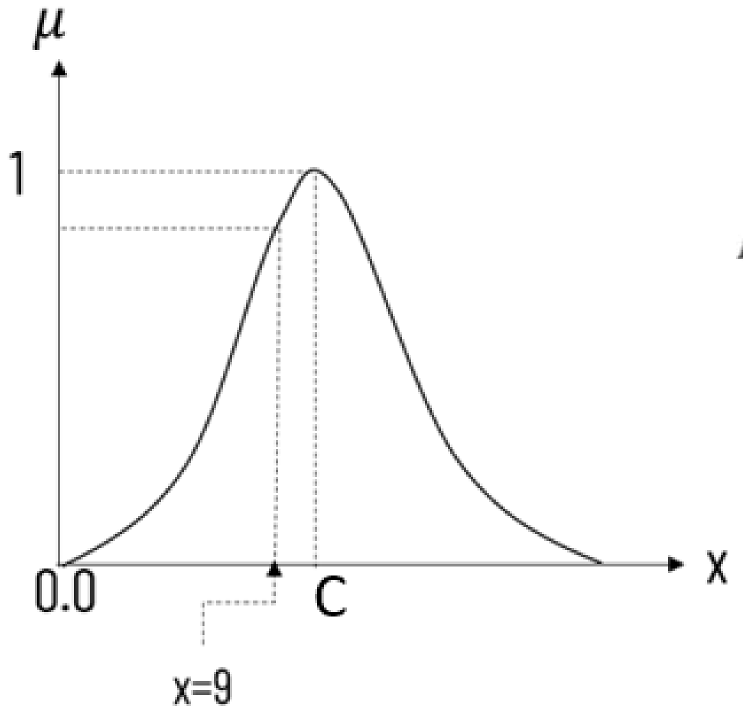
This is a more natural way of representing the data distribution, but due to mathematical complexity, it is not popular for fuzzification.

Gaussian MF

Create a fuzzy set "Average Student" based on Mark using a gaussian MF.

Gaussian MF

Determine μ corresponding to x = 9, c = 10 and σ = 3.0



$$\mu_{gaussian} = e^{-\frac{1}{2}\left(\frac{x-10}{3}\right)^2}$$

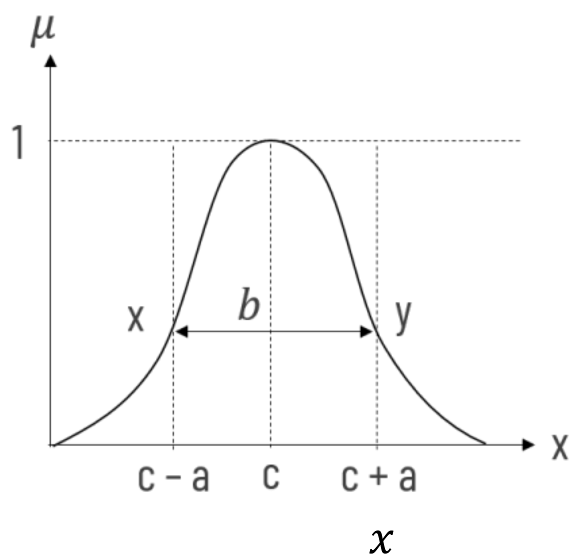
$$Put\ x = 9$$

$$\mu_{gaussian} = e^{-\frac{1}{2}\left(\frac{9-10}{3}\right)^2} = 0.9459$$

Generalized Bell MF

GBell MF is defined by three parameters **{a, b, c}**
where

a: controls the width,
b: controls the slope at crossover point,
c = center.



Slope at x $= \frac{b}{2a}$

Slope at y $= -\frac{b}{2a}$

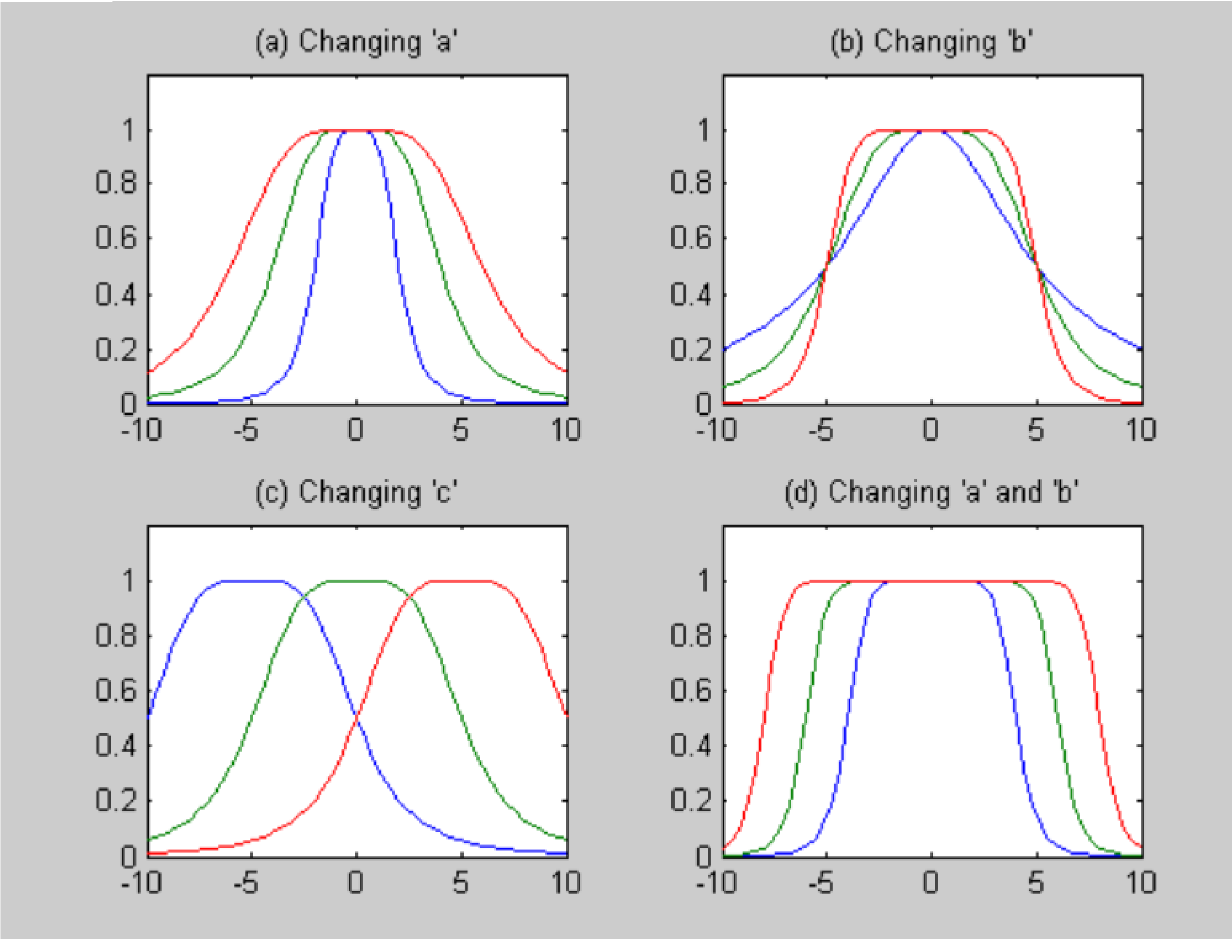
$X \rightarrow$

$$\text{bell}(x; a, b, c) = \frac{1}{1 + \left|\frac{x - c}{a}\right|^{2b}}$$

It is called generalized MF, because by changing the parameters a, b and c, we can produce a family of different membership functions.

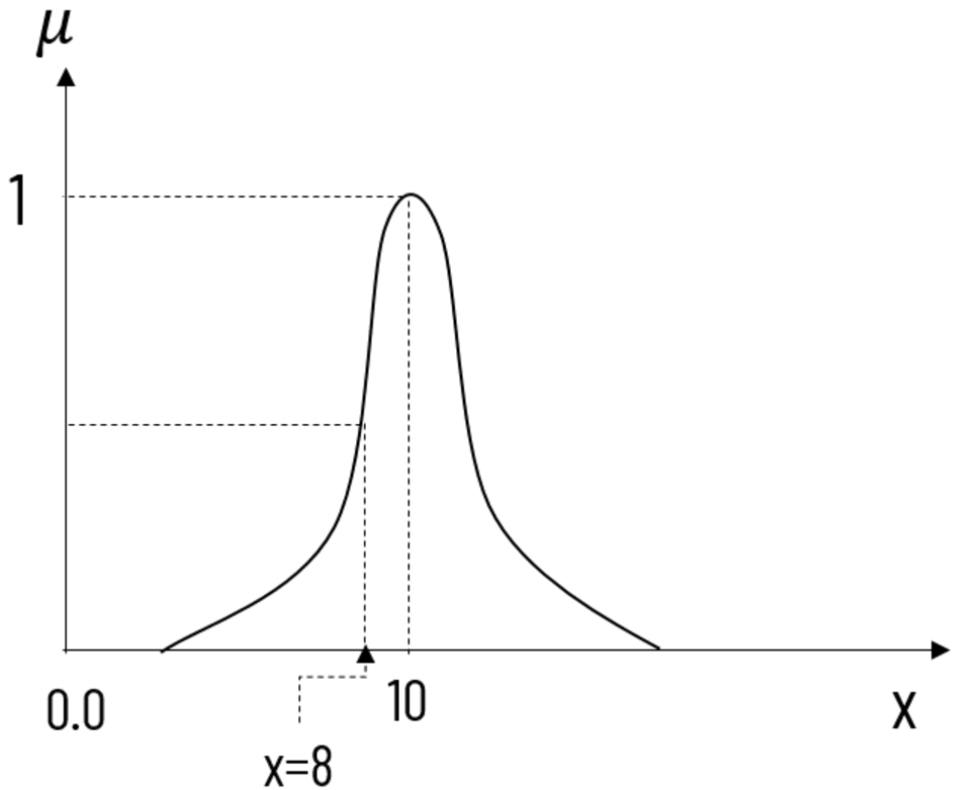
Generalized Bell MF

$$\text{bell}(x; a, b, c) = \frac{1}{1 + \left|\frac{x-c}{a}\right|^{2b}}$$



Generalized Bell MF

Determine μ corresponding to x = 8, where a=2, b=3 and c = 10, in bell MF.



$$\mu_{bell}(x; a, b, c) = \frac{1}{1+\left|\frac{x-c}{a}\right|^{2b}}$$

$$Take\ c\ = 10, a = 2, b = 3$$

$$\mu_{bell} = \frac{1}{1+\left|\frac{x-10}{2}\right|^{6}}$$

$$Put\ x = 8$$

$$\mu_{bell} = \frac{1}{1+\left|\frac{8-10}{2}\right|^{6}} = 0.5$$

Generalized Bell MF

Fuzzy sets A is defined by bell(x;10,20,30), find its **core**.

a=10   b=20   c=30 ➔ **core=30**

Generalized Bell MF

Fuzzy sets A is defined by bell(x;16,20, 50), find its bandwidth.

At crossover points,

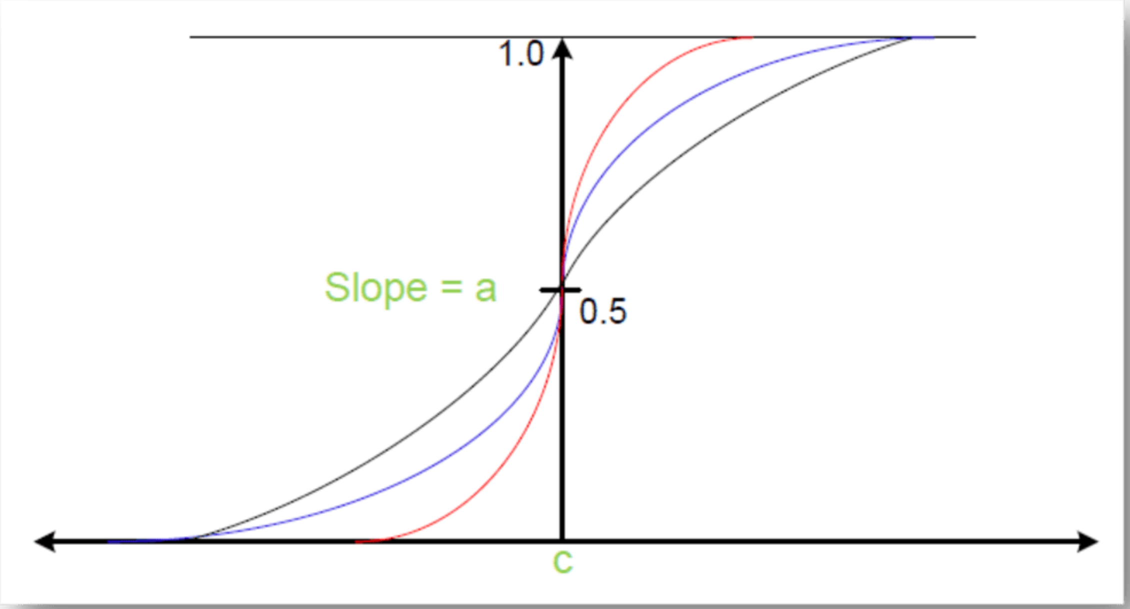
$$\mu_A(x) = \frac{1}{1+\left|\frac{x-50}{16}\right|^{2\times 20}} = 1/2 \Rightarrow 2 = 1+\left|\frac{x-50}{16}\right|^{2\times 20} \Rightarrow \frac{x-50}{16} = \pm 1 \Rightarrow x = 66 \text{ or } 34$$

Bandwidth = 66 -34 = 32.

Sigmoidal MF

Sigmoidal MF is defined by two parameters **{a, c} where c = crossover point** and **a controls the slope at crossover point 'c'.**

$$\mu_{sigmoid}(x;a,c) = \frac{1}{1+e^{-a(x-c)}}$$



Sigmoidal MF (Matlab)

Write a function to define a Sigmoidal MF (SIG_MF.m). Create a fuzzy set “heavy smoker” based on #of cigarettes using SIG_MF.

```
x = (-10:0.4:10)';

mf = SIG_MF(x, [1, -5]);
% mf = SIG_MF(x, [2, 5]);
% mf = SIG_MF(x, [-2, 5]);
```

```
function y = SIG_MF(x, parameter)

a = parameter(1); c = parameter(2);
y = 1./(1 + exp(-a*(x-c)));

end
```

Two-Dimensional MF

2D MF: MFs with **two inputs**, each in a different universe of discourse.

Cylindrical Extension:

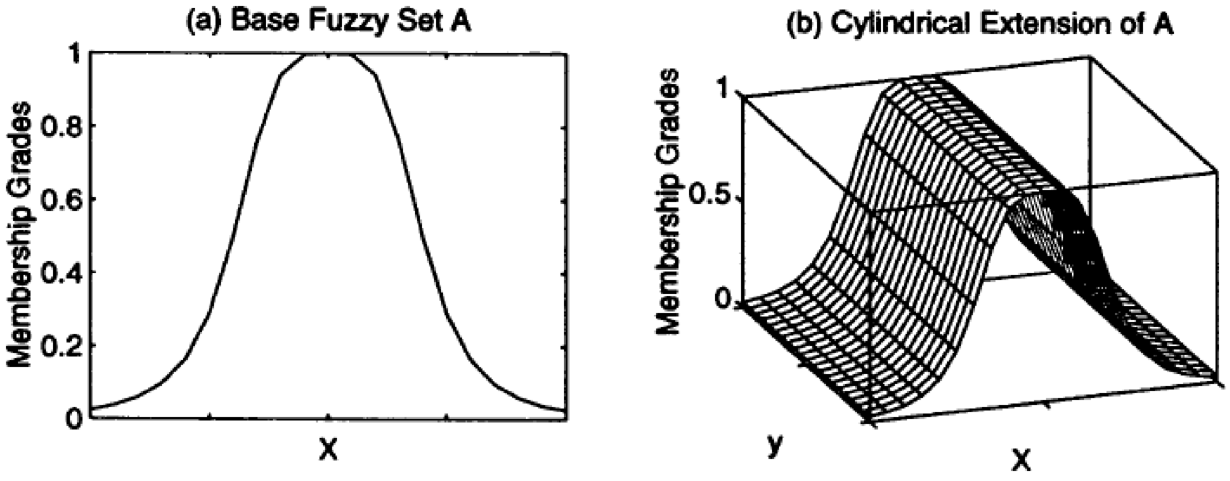
One dimensional MF can be extended to two-dimensional MF using cylindrical extension.

Cylindrical Extension

Cylindrical Extension: One-dimensional MF →Two-dimensional MF

If A is a fuzzy set in X, then its cylindrical extension in X x Y is a fuzzy set c(A) defined by

$$c(A) = \int_{X\times Y} \mu_A(x)/(x,y).$$



Cylindrical Extension

Example: A = { (x1, 1) (x2, 0.8) (x3,1) }

	y1	**y2**	**y3**
x1	1	1	1
x2	0.8	0.8	0.8
x3	1	1	1

c(A) = { (x1, y1, 1), (x1, y2, 1), ……
(x2, y1, 0.8), (x2, y2, 0.8), ……
(x3, y1, 1), (x3, y2, 1), ……

}

Projection

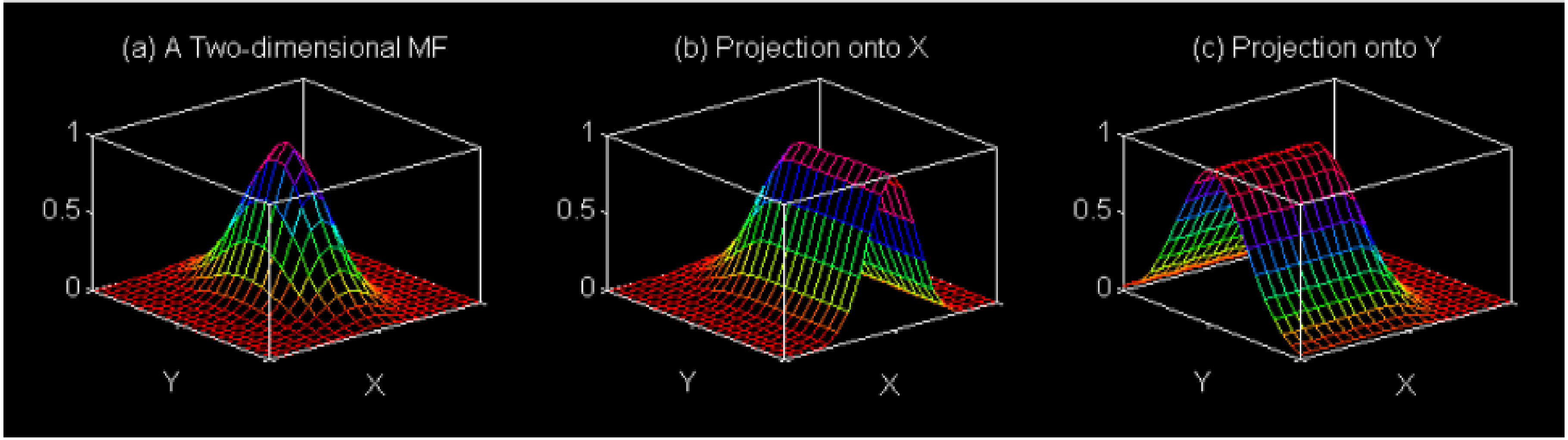
Projection brings back a multidimensional MF to 1D MF.

Let R be a two-dimensional fuzzy set on **X x Y**. Then the **projection of R onto X and Y** are defined as

$$R_X = \int_X [\max_y \mu_R(x, y)]/x$$

$$R_Y = \int_Y [\max_x \mu_R(x, y)]/y,$$

Projection


(a) A Two-dimensional MF
(b) Projection onto X
(c) Projection onto Y
1
0.5
0
Y
X

Projection: Example

Projection: 2D MF → 1D MF.

Y

X

	y1	y2	y3	y4	
x1	0.8	1	0.1	0.7	1
x2	0	0.8	0	0	0.8
x3	0.9	1	0.7	0.8	1
	0.9	1	0.7	0.8	

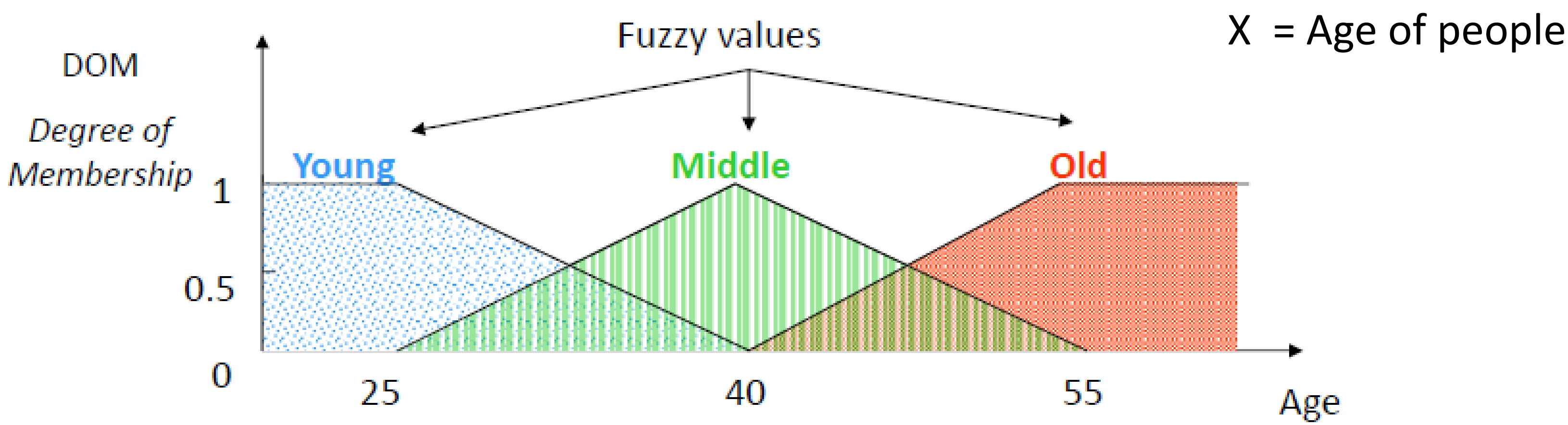
Projection on X: { (x1, 1) (x2, 0.8) (x3, 1) }

Projection on Y: { (y1, 0.9) (y2, 1) (y3, 0.7) (y4, 0.8) }

Linguistic Variables

- Linguistic Variable/form is a variable with subjective knowledge, usually impossible to quantify (with no specific value).

Example: Describing people as "Young", "Middle-aged" and "Old"



Fuzzy Logic allows modelling of linguistic terms using linguistic variables or linguistic values.

The fuzzy sets "young", "middle-aged", and "old" are fully defined by their membership functions.

Age is a linguistic variable. Its values are linguistic values rather than numerical.

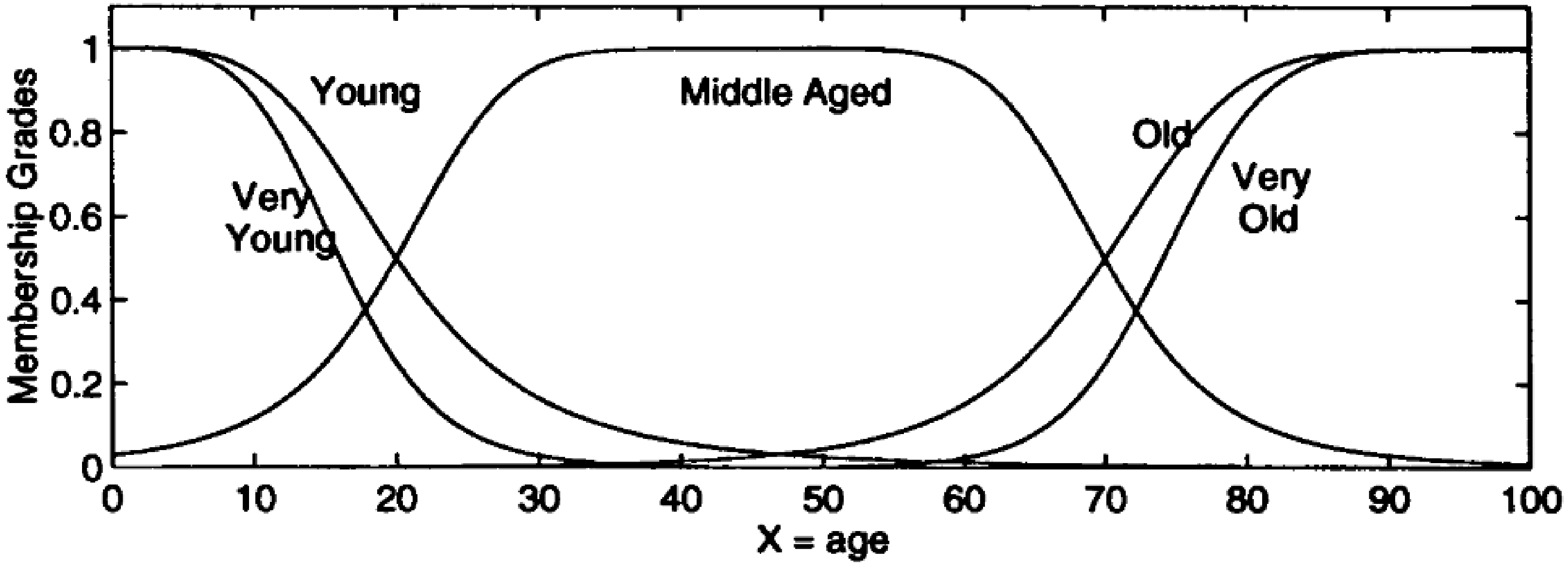
Linguistic Variables

A linguistic variable is characterized by a quintuple

$$(x, T(x), X, G, M)$$

x : name of the variable
$T(x)$: term set of x (linguistic values or terms)
X : universe of discourse
G : syntactic rule which generates the terms in $T(x)$
M : semantic rule which associates linguistic value A with its meaning M(A), M(A) is a fuzzy set in X.

Example



Membership functions of the term set T(age)

Example

If age is interpreted as a linguistic variable, then its term set T(age)

$$
\begin{aligned}
T(age) \quad = \quad \{ \quad & young,\ not\ young,\ very\ young,\ not\ very\ young,\ \ldots, \\
& middle\ aged,\ not\ middle\ aged,\ \ldots, \\
& old,\ not\ old,\ very\ old,\ more\ or\ less\ old,\ not\ very\ old,\ \ldots, \\
& not\ very\ young\ and\ not\ very\ old,\ \ldots \},
\end{aligned}
$$

Universe of discourse X: [0 100]

Syntactic rule (G) refers to the way the linguistic values in the term set T(age) are generated.

The Semantic rule(M) defines the membership function of each linguistic value of the term set.

Example

$$
\begin{aligned}
T(age) \;=\; \{ \quad & young,\ not\ young,\ very\ young,\ not\ very\ young,\ \ldots, \\
& middle\ aged,\ not\ middle\ aged,\ \ldots, \\
& old,\ not\ old,\ very\ old,\ more\ or\ less\ old,\ not\ very\ old,\ \ldots, \\
& not\ very\ young\ and\ not\ very\ old,\ \ldots \},
\end{aligned}
$$

In this example term set T(age) consists of

Primary terms : young, middle-age, old

Primary terms modified by the

negation (not)

hedges (very, more or less, quite, ….

Primary terms linked by connectives (and, or, either, neither,….)

Note# A linguistic hedge is *an operation that modifies the meaning of a fuzzy set*.

Concentration

Let A be a linguistic value characterized by a fuzzy set with membership function. The operation concentration is defined as

$$CON\,(A) = A^2$$

```
x = 0:100;

mfg = GAUSS_MF(x, [50, 20]);

mfc = mfg .^2;

plot(x,mfg,'g*',x,mfc,'bo')
```

Dilution

Task: Demonstrate the effect of dilution on a fuzzy membership function.

$$DIL\ (A\ ) = \ A^{0.5}$$

```
x = 0:100;

mfg = GAUSS_MF(x, [50, 20]);

mfc = mfg .^0.5;

plot(x,mfg,'g*',x,mfc,'bo')
```

Contrast Intensification

Task: Demonstrate the effect of contrast intensification on a fuzzy membership function.

$$INT\,(A\,) = \; 2A^2, \qquad \text{if } 0 \leq \mu_A(x) \leq 0.5$$

$$\neg 2(\neg A)^2 \quad \text{if } \; 0.5 \; \leq \; \mu_A(x) \leq 1$$

```
x = 0:100;

mfg = GAUSS_MF(x, [50, 20]);

mfint = INT(mfg);

plot(x,mfg,'g*',x,mfint,'b+')
```

```
function new_mf = INT(mf)

    index1 = find(mf < 0.5);
    index2 = find(mf >= 0.5);

    tmp = mf(index1);
    mf(index1) = 2*tmp.*tmp;
    tmp = 1-mf(index2);
    mf(index2) = 1-2*tmp.*tmp;
    new_mf = mf;
end
```

Contrast Intensification Example

Solution: $$INT(A) = \begin{cases} 2A^{(2)} & \text{for } 0 \le \mu_A(x) \le 0.5 \quad \forall x \in X \\ \neg 2(\neg A)^{(2)} & \text{for } 0.5 \le \mu_A(x) \le 1 \quad \forall x \in X \end{cases}$$

$$A = 0.7/1 + 0.6\,/\,2 + 0.1\,/\,3 + 0.5\,/\,4 + 0.3\,/\,5$$

For $x = 1 \Rightarrow \mu_A(1) = 0.7$ i.e. $0.5 \le \mu_A(1) \le 1$

$$\neg 2(\neg A)^{(2)} = \neg 2(\neg(0.7))^2/1 = 0.82/1$$ 1-(2*(1-.7).*(1-.7))

For $x = 2 \Rightarrow \mu_A(2) = 0.6$ i.e. $0.5 \le \mu_A(2) \le 1$

$$\neg 2(\neg A)^{(2)} = \neg 2(\neg(0.6))^2/2 = 0.68/2$$ 1-(2*(1-.6).*(1-.6))

For $x = 3 \Rightarrow \mu_A(3) = 0.1$ i.e. $0 \le \mu_A(3) \le 0.5$

$$2A^{(2)} = 2(0.1)^2/3 = 0.02/3$$ 2*(0.1).*(0.1)

For $x = 4 \Rightarrow \mu_A(4) = 0.5$ i.e. $0.5 \le \mu_A(4) \le 1$

$$\neg 2(\neg A)^{(2)} = \neg 2(\neg 0.5)^2/4 = 0.5/4$$ 1-(2*(1-.5)*(1-.5))

For $x = 5 \Rightarrow \mu_A(5) = 0.3$ i.e. $0 \le \mu_A(5) \le 0.5$

$$2A^{(2)} = 2(0.3)^2/5 = 0.18/5$$ 2*(0.3).*(0.3)

Linguistic Hedges

Hedges	Mathematical Expression
A little	$[\mu_A\,(x)]^{1.3}$
Slightly	$[\mu_A\,(x)]^{1.7}$
Very	$[\mu_A\,(x)]^{2}$ / CON (A)
Extremely	$[\mu_A\,(x)]^{8}$
Outstanding	$[\mu_A\,(x)]^{8}$
Very Very	$[\mu_A\,(x)]^{4}$ / CON(CON(A))
More or less	$[\mu_A\,(x)]^{0.5}$ / DIL (A)
Some What	$[\mu_A\,(x)]^{0.5}$ / DIL (A)
Indeed	INT (A)

Composite Linguistic Term

Let linguistic term **Young** and **Old** be defined by the following membership function.

$$\mu_{\text{young}}(x) = \text{bell}(x, 20, 2, 0) = \frac{1}{1 + (\frac{x}{20})^4},$$

$$\mu_{\text{old}}(x) = \text{bell}(x, 30, 3, 100) = \frac{1}{1 + (\frac{x-100}{30})^6},$$

Where x is the age of a person with interval [0, 100] as the universe of discourse.

$$\textit{more or less old} = \text{DIL}(\textit{old}) = \textit{old}^{0.5}$$
$$= \int_X \sqrt{\frac{1}{1 + (\frac{x-100}{30})^6}} \Big/ x.$$

Composite Linguistic Term

$$\mu_{\text{young}}(x) = \text{bell}(x, 20, 2, 0) = \frac{1}{1 + (\frac{x}{20})^4}, \qquad \mu_{\text{old}}(x) = \text{bell}(x, 30, 3, 100) = \frac{1}{1 + (\frac{x-100}{30})^6},$$

$$\begin{aligned} &\textit{not young and not old} = \neg \textit{young} \cap \neg \textit{old} \\ &= \int_X \left[1 - \frac{1}{1 + (\frac{x}{20})^4}\right] \wedge \left[1 - \frac{1}{1 + (\frac{x-100}{30})^6}\right] \Big/ x. \end{aligned}$$

$$\begin{aligned} &\textit{young but not too young} = \textit{young} \cap \neg \textit{young}^2 \\ &= \int_X \left[\frac{1}{1 + (\frac{x}{20})^4}\right] \wedge \left[1 - \left(\frac{1}{1 + (\frac{x}{20})^4}\right)^2\right] \Big/ x. \end{aligned}$$

$$\begin{aligned} &\textit{extremely old} \\ &= \text{CON}(\text{CON}(\text{CON}(\textit{old}))) = ((\textit{old}^2)^2)^2 = \int_X \left[\frac{1}{1 + (\frac{x-100}{30})^6}\right]^8 \Big/ x. \end{aligned}$$

Composite Linguistic Term

Task: Student's performance is defined by following membership function, Where x is the percentage of marks obtained in all subjects.

$$\mu_{\text{average}}(x) = \text{BELL}(x; 50, 2, 0)$$

$$\mu_{good}(x) = \text{GAUSSIAN}(x; 60, 15)$$

$$\mu_{\text{excellent}}(x) = \text{BELL}(x; 60, 3, 100)$$

Construct the MF for the following composite linguistic term.

a. More or less good
b. Not average and not good
c. Very good but not excellent
d. Average but not below average
e. Outstanding

Thank you