# Fuzzy Reasoning/Inferences

# Inferring Procedure in Fuzzy

Two important inferring procedures are used in fuzzy systems :

- **Generalized Modus Ponens (GMP)**

$$\begin{array}{l} \text{If } x \text{ is } A \text{ Then } y \text{ is } B \\ x \text{ is } A' \\ \hline y \text{ is } B' \end{array}$$

- **Generalized Modus Tollens (GMT)**

$$\begin{array}{l} \text{If } x \text{ is } A \text{ Then } y \text{ is } B \\ y \text{ is } B' \\ \hline x \text{ is } A' \end{array}$$

# Fuzzy Inferring Procedure

- Here, $A, B, A^{'}$ and $B^{'}$ are fuzzy sets.
- To compute the membership function $A^{'}$ and $B^{'}$ the max-min composition of fuzzy sets $B^{'}$ and $A^{'}$ ,respectively with $R(x, y)$ (which is the known implication relation) is to be used.
- Thus,

$$B^{'} = A^{'} \circ R(x, y) \qquad \mu_B(y) = max[min(\mu_{A'}(x), \mu_R(x, y))]$$

$$A^{'} = B^{'} \circ R(x, y) \qquad \mu_A(x) = max[min(\mu_{B'}(y), \mu_R(x, y))]$$

# GMP

**Generalized Modus Ponens (GMP)**

$$P : \textbf{If } x \textbf{ is } A \textbf{ then } y \textbf{ is } B$$

Let us consider two sets of variables $x$ and $y$ be

$X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$, respectively.

Also, let us consider the following.

$A = \{(x_1, 0.5), (x_2, 1), (x_3, 0.6)\}$

$B = \{(y_1, 1), (y_2, 0.4)\}$

Then, given a fact expressed by the proposition $x$ is $A^{'}$,
where $A^{'} = \{(x_1, 0.6), (x_2, 0.9), (x_3, 0.7)\}$
derive a conclusion in the form $y$ **is** $B^{'}$ (using generalized modus ponens (GMP)).

# GMP

If $x$ is $A$ Then $y$ is $B$

$x$ is $A'$

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$y$ is $B'$

We are to find $B' = A' \circ R(x, y)$ where $R(x, y) = max\{A \times B, \overline{A} \times Y\}$

$$A \times B = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.4 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{array} \text{ and } \overline{A} \times Y = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

Note: For $A \times B$, $\mu_{A \times B}(x, y) = min(\mu_A x, \mu_B(y))$

# GMP

$$R(x,y) = (A \times B) \cup (\overline{A} \times y) = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

**Now**, $A' = \{(x_1, 0.6), (x_2, 0.9), (x_3, 0.7)\}$

Therefore, $B' = A' \circ R(x,y) =$

$$\begin{bmatrix} 0.6 & 0.9 & 0.7 \end{bmatrix} \circ \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.5 \end{bmatrix}$$

Thus we derive that $y$ is $B'$ where $B' = \{(y_1, 0.9), (y_2, 0.5)\}$

# GMT

**Generalized Modus Tollens (GMT)**

P: If $x$ is $A$ Then $y$ is $B$

Q: $y$ is $B'$

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$x$ is $A'$

# GMT Example

- Let sets of variables $x$ and $y$ be $X = \{x_1, x_2, x_3\}$ and $y = \{y_1, y_2\}$, respectively.
- Assume that a proposition **If** $x$ **is** $A$ **Then** $y$ **is** $B$ given where $A = \{(x_1, 0.5), (x_2, 1.0), (x_3, 0.6)\}$ and $B = \{(y_1, 0.6), (y_2, 0.4)\}$
- Assume now that a fact expressed by a proposition $y$ **is** $B$ is given where $B^{'} = \{(y_1, 0.9), (y_2, 0.7)\}$.
- From the above, we are to conclude that $x$ **is** $A^{'}$. That is, we are to determine $A^{'}$

# GMT Example

- We first calculate $R(x, y) = (A \times B) \cup (\overline{A} \times y)$

$$R(x, y) = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

- Next, we calculate $A' = B' \circ R(x, y)$

$$A' = \begin{bmatrix} 0.9 & 0.7 \end{bmatrix} \circ \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{array} = \begin{bmatrix} 0.5 & 0.9 & 0.6 \end{bmatrix}$$

- Hence, we calculate that $x$ is $A'$ where $A' = [(x_1, 0.5), (x_2, 0.9), (x_3, 0.6)]$

# Different Forms of Fuzzy Rules

- Single Rule with Single Antecedent
- Single Rule with Multiple Antecedents
- Multiple Rules with Multiple Antecedents

# Single Rule with Single Antecedent

| | |
|---|---|
| premise 1 (fact): | x is $A'$, |
| premise 2 (rule): | if x is $A$ then y is $B$, |
| consequence (conclusion): | y is $B'$, |

$$B' = A' \circ R = A' \circ (A \to B)$$

Let us consider Mamdani's fuzzy implication function and classical max-min composition.

$$\begin{aligned}\mu_{B'}(y) &= \max_x \min[\mu_{A'}(x), \mu_R(x,y)] \\ &= \vee_x[\mu_{A'}(x) \wedge \mu_R(x,y)],\end{aligned}$$

# Single Rule with Single Antecedent

Let us consider Mamdani's fuzzy implication function and classical max-min composition.

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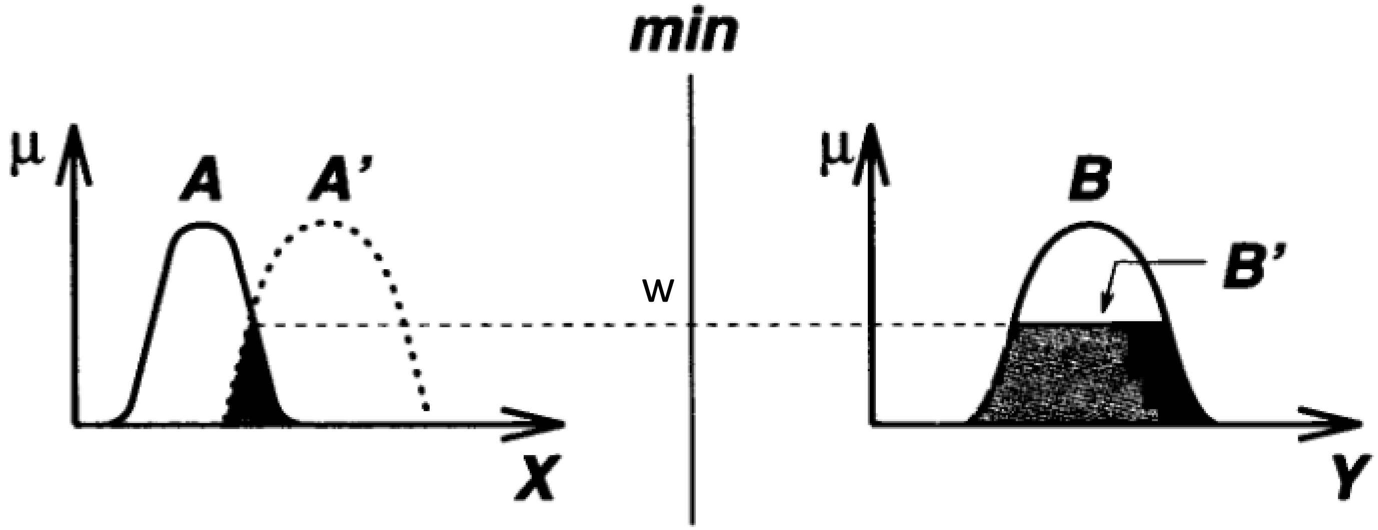
A further simplification of the equation yields,

$$\begin{aligned}\mu_{B'}(y) &= [\vee_x(\mu_{A'}(x) \wedge \mu_A(x)] \wedge \mu_B(y) \\ &= w \wedge \mu_B(y).\end{aligned}$$

# Single Rule with Single Antecedent




(Graphical interpretation of GMP using Mamdani's fuzzy implication and the max-min composition)

W indicates the degree of match (belief) that propagated by the if-then rules and results degree of belief for consequent part.

# Single Rule with Multiple Antecedents

| | |
|---|---|
| premise 1 (fact): | $x$ is $A'$ and $y$ is $B'$, |
| premise 2 (rule): | if $x$ is $A$ and $y$ is $B$ then z is C, |
| consequence (conclusion): | z is $C'$. |

Fuzzy rule can be written in simple form    $A \times B \rightarrow C$

The resulting C' can be expressed as

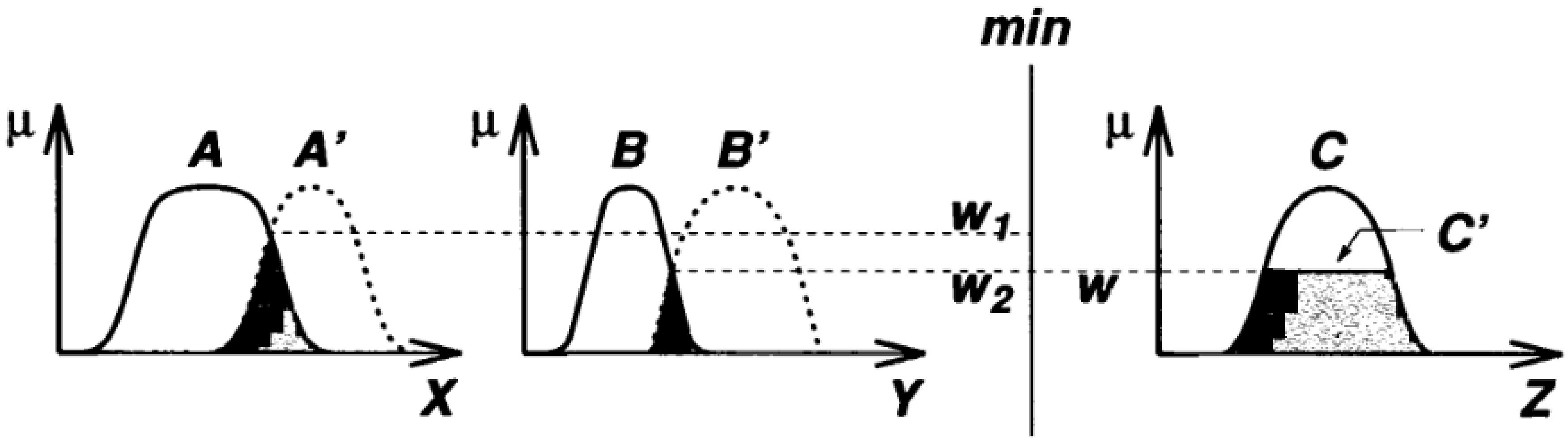
$$C' = (A' \times B') \circ (A \times B \rightarrow C)$$

# Single Rule with Multiple Antecedents

$$C' = (A' \times B') \circ (A \times B \to C).$$

$$\begin{aligned}
\mu_{C'}(z) &= \vee_{x,y}[\mu_{A'}(x) \wedge \mu_{B'}(y)] \wedge [\mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z)] \\
&= \vee_{x,y}\{[\mu_{A'}(x) \wedge \mu_{B'}(y) \wedge \mu_A(x) \wedge \mu_B(y)]\} \wedge \mu_C(z) \\
&= \underbrace{\{\vee_x[\mu_{A'}(x) \wedge \mu_A(x)]\}}_{w_1} \wedge \underbrace{\{\vee_y[\mu_{B'}(y) \wedge \mu_B(y)]\}}_{w_2} \wedge \mu_C(z) \\
&= \underbrace{(w_1 \wedge w_2)}_{\text{firing strength}} \wedge \mu_C(z),
\end{aligned}$$

# Single Rule with Multiple Antecedents




(Graphical interpretation of GMP using Mamdani's fuzzy implication and the max-min composition)

w1 and w2 represents degree of compatibility.

w= w1 ^ w2 is called firing strength or degree of fulfillment.

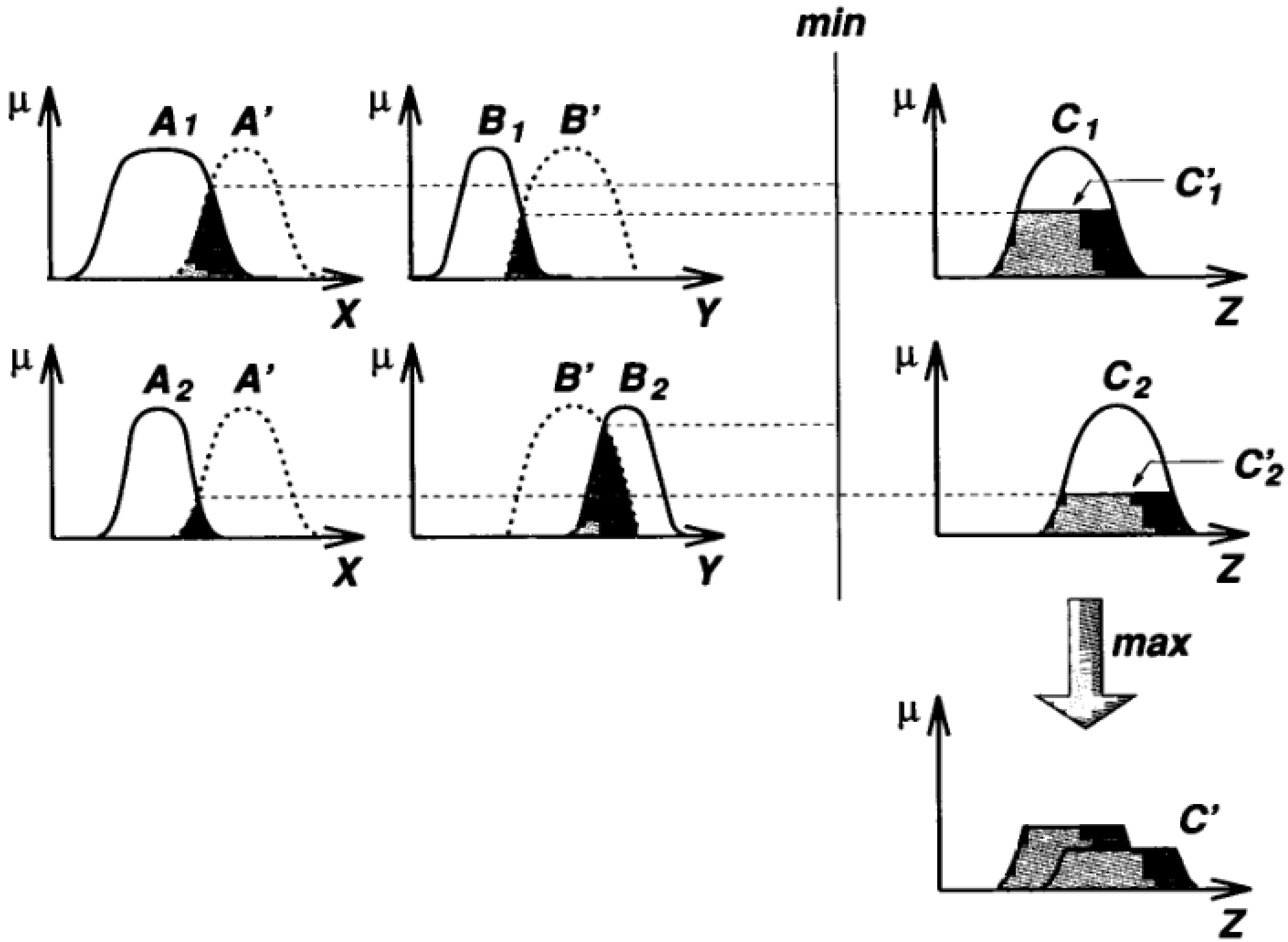
# Multiple Rules with Multiple Antecedents

| | |
|---|---|
| premise 1 (fact): | $x$ is $A'$ and $y$ is $B'$, |
| premise 2 (rule 1): | if $x$ is $A_1$ and $y$ is $B_1$ then z is $C_1$, |
| premise 3 (rule 2): | if $x$ is $A_2$ and $y$ is $B_2$ then z is $C_2$, |
| consequence (conclusion): | z is $C'$, |

The resulting C' can be expressed as

$$\begin{aligned} C' &= (A' \times B') \circ (R_1 \cup R_2) \\ &= [(A' \times B') \circ R_1] \cup [(A' \times B') \circ R_2] \\ &= C_1' \cup C_2', \end{aligned}$$

# Multiple Rules with Multiple Antecedents




(Fuzzy Reasoning for multiple rules with multiple antecedents)

# Fuzzy Reasoning Steps

**Degrees of compatibility** Compare the known facts with the antecedents of fuzzy rules to find the degrees of compatibility with respect to each antecedent MF.

**Firing strength** Combine degrees of compatibility with respect to antecedent MFs in a rule using fuzzy AND or OR operators to form a firing strength that indicates the degree to which the antecedent part of the rule is satisfied.

**Qualified (induced) consequent MFs** Apply the firing strength to the consequent MF of a rule to generate a qualified consequent MF. (The qualified consequent MFs represent how the firing strength gets propagated and used in a fuzzy implication statement.)

**Overall output MF** Aggregate all the qualified consequent MFs to obtain an overall output MF.

Thank You