

Fuzzy Relations, Rules and Inferences

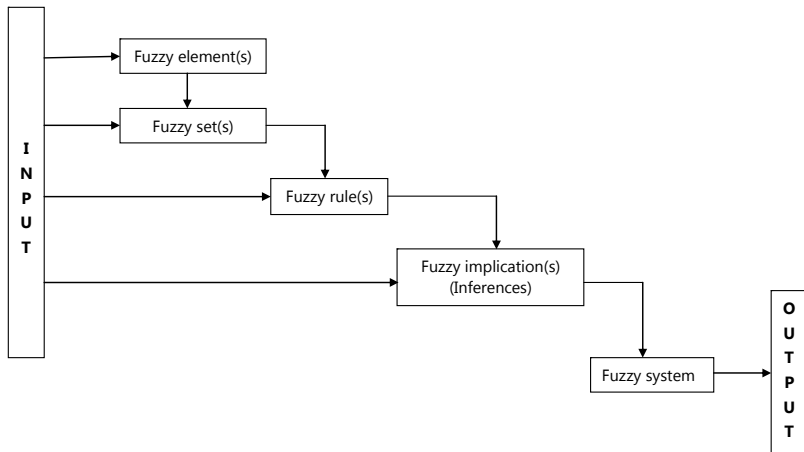
Debasis Samanta

IIT Kharagpur

dsamanta@iitkgp.ac.in

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Concept of fuzzy system



Fuzzy Relations

Crisp relations

To understand the fuzzy relations, it is better to discuss first **crisp relation**.

Suppose, A and B are two (crisp) sets. Then Cartesian product denoted as $A \times B$ is a collection of order pairs, such that

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$$

Note :

$$(1) A \times B \neq B \times A$$

$$(2) |A \times B| = |A| \times |B|$$

(3) $A \times B$ provides a mapping from $a \in A$ to $b \in B$.

The mapping so mentioned is called a **relation**.

Crisp relations

Example 1:

Consider the two crisp sets A and B as given below. $A = \{ 1, 2, 3, 4 \}$
 $B = \{ 3, 5, 7 \}$.

Then, $A \times B = \{(1, 3), (1, 5), (1, 7), (2, 3), (2, 5), (2, 7), (3, 3), (3, 5), (3, 7), (4, 3), (4, 5), (4, 7)\}$

Let us define a relation R as $R = \{(a, b) | b = a + 1, (a, b) \in A \times B\}$

Then, $R = \{(2, 3), (4, 5)\}$ in this case.

We can represent the relation R in a matrix form as follows.

$$R = \begin{matrix} & \begin{matrix} 3 & 5 & 7 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{matrix}$$

Operations on crisp relations

Suppose, $R(x, y)$ and $S(x, y)$ are the two relations define over two crisp sets $x \in A$ and $y \in B$

Union:

$$R(x, y) \cup S(x, y) = \max(R(x, y), S(x, y));$$

Intersection:

$$R(x, y) \cap S(x, y) = \min(R(x, y), S(x, y));$$

Complement:

$$\overline{R(x, y)} = 1 - R(x, y)$$

Example: Operations on crisp relations

Example:

Suppose, $R(x, y)$ and $S(x, y)$ are the two relations define over two crisp sets $x \in A$ and $y \in B$

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

Find the following:

- 1 $R \cup S$
- 2 $R \cap S$
- 3 \overline{R}

Composition of two crisp relations

Given R is a relation on X, Y and S is another relation on Y, Z . Then $R \circ S$ is called a composition of relation on X and Z which is defined as follows.

$$R \circ S = \{(x, z) | (x, y) \in R \text{ and } (y, z) \in S \text{ and } \forall y \in Y\}$$

Max-Min Composition

Given the two relation matrices R and S , the **max-min composition** is defined as $T = R \circ S$;

$$T(x, z) = \max\{\min\{R(x, y), S(y, z) \text{ and } \forall y \in Y\}\}$$

Composition: Composition

Example:

Given

$$X = \{1, 3, 5\}; Y = \{1, 3, 5\}; R = \{(x, y) | y = x + 2\}; S = \{(x, y) | x < y\}$$

Here, R and S is on $X \times Y$.

Thus, we have

$$R = \{(1, 3), (3, 5)\}$$

$$S = \{(1, 3), (1, 5), (3, 5)\}$$

$$R = \begin{matrix} & \begin{matrix} 1 & 3 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 3 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{matrix} \text{ and } S =$$

$$\begin{matrix} & \begin{matrix} 1 & 3 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 3 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

Using max-min composition $R \circ S =$

$$\begin{matrix} & \begin{matrix} 1 & 3 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 3 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

Fuzzy relations

- Fuzzy relation is a fuzzy set defined on the Cartesian product of crisp set X_1, X_2, \dots, X_n
- Here, n-tuples (x_1, x_2, \dots, x_n) may have varying degree of memberships within the relationship.
- The membership values indicate the strength of the relation between the tuples.

Example:

$X = \{ \text{typhoid, viral, cold} \}$ and $Y = \{ \text{running nose, high temp, shivering} \}$

The fuzzy relation R is defined as

	<i>runningnose</i>	<i>hightemperature</i>	<i>shivering</i>
<i>typhoid</i>	0.1	0.9	0.8
<i>viral</i>	0.2	0.9	0.7
<i>cold</i>	0.9	0.4	0.6

Fuzzy Cartesian product

Suppose

A is a fuzzy set on the universe of discourse X with $\mu_A(x) | x \in X$

B is a fuzzy set on the universe of discourse Y with $\mu_B(y) | y \in Y$

Then $R = A \times B \subset X \times Y$; where R has its membership function given by $\mu_R(x, y) = \mu_{A \times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\}$

Example :

$A = \{(a_1, 0.2), (a_2, 0.7), (a_3, 0.4)\}$ and $B = \{(b_1, 0.5), (b_2, 0.6)\}$

$$R = A \times B = \begin{array}{cc} & \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} & \left[\begin{array}{cc} 0.2 & 0.2 \\ 0.5 & 0.6 \\ 0.4 & 0.4 \end{array} \right] \end{array}$$

Operations on Fuzzy relations

Let R and S be two fuzzy relations on $A \times B$.

Union:

$$\mu_{R \cup S}(a, b) = \max\{\mu_R(a, b), \mu_S(a, b)\}$$

Intersection:

$$\mu_{R \cap S}(a, b) = \min\{\mu_R(a, b), \mu_S(a, b)\}$$

Complement:

$$\mu_{\overline{R}}(a, b) = 1 - \mu_R(a, b)$$

Composition

$$T = R \circ S$$
$$\mu_{R \circ S} = \max_{y \in Y} \{ \min(\mu_R(x, y), \mu_S(y, z)) \}$$

Operations on Fuzzy relations: Examples

Example:

$$X = (x_1, x_2, x_3); Y = (y_1, y_2); Z = (z_1, z_2, z_3);$$

$$R = \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{cc} y_1 & y_2 \\ \left[\begin{array}{cc} 0.5 & 0.1 \\ 0.2 & 0.9 \\ 0.8 & 0.6 \end{array} \right] \end{array}$$

$$S = \begin{array}{c} y_1 \\ y_2 \end{array} \begin{array}{ccc} z_1 & z_2 & z_3 \\ \left[\begin{array}{ccc} 0.6 & 0.4 & 0.7 \\ 0.5 & 0.8 & 0.9 \end{array} \right] \end{array}$$

$$R \circ S = \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{ccc} z_1 & z_2 & z_3 \\ \left[\begin{array}{ccc} 0.5 & 0.4 & 0.5 \\ 0.5 & 0.8 & 0.9 \\ 0.6 & 0.6 & 0.7 \end{array} \right] \end{array}$$

$$\begin{aligned} \mu_{R \circ S}(x_1, y_1) &= \max\{\min(x_1, y_1), \min(y_1, z_1), \min(x_1, y_2), \min(y_2, z_1)\} \\ &= \max\{\min(0.5, 0.6), \min(0.1, 0.5)\} = \max\{0.5, 0.1\} = 0.5 \text{ and so on.} \end{aligned}$$

Fuzzy relation : An example

Consider the following two sets P and D , which represent a set of paddy plants and a set of plant diseases. More precisely

$P = \{P_1, P_2, P_3, P_4\}$ a set of four varieties of paddy plants

$D = \{D_1, D_2, D_3, D_4\}$ of the four various diseases affecting the plants

In addition to these, also consider another set $S = \{S_1, S_2, S_3, S_4\}$ be the common symptoms of the diseases.

Let, R be a relation on $P \times D$, representing which plant is susceptible to which diseases, then R can be stated as

$$R = \begin{array}{c} \begin{matrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{matrix} \begin{bmatrix} \begin{matrix} D_1 & D_2 & D_3 & D_4 \end{matrix} \\ \begin{matrix} 0.6 & 0.6 & 0.9 & 0.8 \\ 0.1 & 0.2 & 0.9 & 0.8 \\ 0.9 & 0.3 & 0.4 & 0.8 \\ 0.9 & 0.8 & 0.4 & 0.2 \end{matrix} \end{bmatrix} \end{array}$$

Fuzzy relation : An example

Also, consider T be the another relation on $D \times S$, which is given by

$$S = \begin{array}{c} D_1 \\ D_2 \\ D_3 \\ D_4 \end{array} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \\ \left[\begin{array}{cccc} 0.1 & 0.2 & 0.7 & 0.9 \\ 1.0 & 1.0 & 0.4 & 0.6 \\ 0.0 & 0.0 & 0.5 & 0.9 \\ 0.9 & 1.0 & 0.8 & 0.2 \end{array} \right] \end{array}$$

Obtain the association of plants with the different symptoms of the disease using **max-min composition**.

Hint: Find $R \circ T$, and verify that

$$R \circ S = \begin{array}{c} P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \\ \left[\begin{array}{cccc} 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.7 & 0.9 \end{array} \right] \end{array}$$

Fuzzy relation : Another example

Let, $R = x$ is relevant to y

and $S = y$ is relevant to z

be two fuzzy relations defined on $X \times Y$ and $Y \times Z$, respectively, where $X = \{1, 2, 3\}$, $Y = \{\alpha, \beta, \gamma, \delta\}$ and $Z = \{a, b\}$.

Assume that R and S can be expressed with the following relation matrices :

$$R = \begin{matrix} & \alpha & \beta & \gamma & \delta \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.7 \\ 0.4 & 0.2 & 0.8 & 0.9 \\ 0.6 & 0.8 & 0.3 & 0.2 \end{bmatrix} \end{matrix} \text{ and}$$

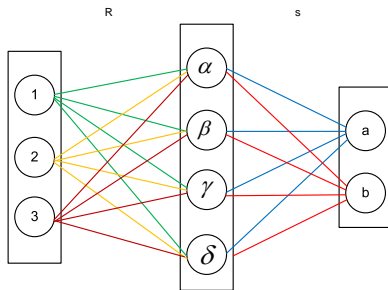
$$S = \begin{matrix} & a & b \\ \begin{matrix} \alpha \\ \beta \\ \gamma \\ \delta \end{matrix} & \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.3 \\ 0.5 & 0.6 \\ 0.7 & 0.2 \end{bmatrix} \end{matrix}$$

Fuzzy relation : Another example

Now, we want to find $R \circ S$, which can be interpreted as a derived fuzzy relation x **is relevant to** z .

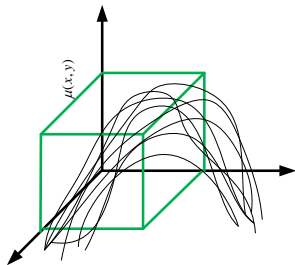
Suppose, we are only interested in the degree of relevance between $2 \in X$ and $a \in Z$. Then, using max-min composition,

$$\begin{aligned}\mu_{R \circ S}(2, a) &= \max\{(0.4 \wedge 0.9), (0.2 \wedge 0.2), (0.8 \wedge 0.5), (0.9 \wedge 0.7)\} \\ &= \max\{0.4, 0.2, 0.5, 0.7\} = 0.7\end{aligned}$$



2D Membership functions : Binary fuzzy relations

(Binary) fuzzy relations are fuzzy sets $A \times B$ which map each element in $A \times B$ to a membership grade between 0 and 1 (both inclusive). Note that a membership function of a binary fuzzy relation can be depicted with a 3D plot.



Important: Binary fuzzy relations are fuzzy sets with two dimensional MFs and so on.

2D membership function : An example

Let, $X = R^+ = y$ (the positive real line)
and $R = X \times Y = \text{"y is much greater than x"}$

The membership function of $\mu_R(x, y)$ is defined as

$$\mu_R(x, y) = \begin{cases} \frac{(y-x)}{4} & \text{if } y > x \\ 0 & \text{if } y \leq x \end{cases}$$

Suppose, $X = \{3, 4, 5\}$ and $Y = \{3, 4, 5, 6, 7\}$, then

$$R = \begin{array}{cc} & \begin{matrix} 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{matrix} 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 0.25 & 0.5 & 0.75 & 1.0 \\ 0 & 0 & 0.25 & 0.5 & 0.75 \\ 0 & 0 & 0 & 0.25 & 0.5 \end{bmatrix} \end{array}$$

Problems to ponder:

How you can derive the following?

If x is A or y is B then z is C ;

Given that

- ① R_1 : If x is A then z is c [$R_1 \in A \times C$]
- ② R_2 : If y is B then z is C [$R_2 \in B \times C$]

- **Hint:**

- You have given two relations R_1 and R_2 .
- Then, the required can be derived using the union operation of R_1 and R_2

Fuzzy Propositions

Two-valued logic vs. Multi-valued logic

- The basic assumption upon which crisp logic is based - that every proposition is either TRUE or FALSE.
- The classical two-valued logic can be extended to multi-valued logic.
- As an example, three valued logic to denote true(1), false(0) and indeterminacy ($\frac{1}{2}$).

Two-valued logic vs. Multi-valued logic

Different operations with three-valued logic can be extended as shown in the following truth table:

a	b	\wedge	\vee	$\neg a$	\implies	$=$
0	0	0	0	1	1	1
0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	1	0	1	1	1	0
$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$
1	0	0	1	1	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$
1	1	1	1	1	1	1

Fuzzy connectives used in the above table are:

AND (\wedge), OR (\vee), NOT (\neg), IMPLICATION (\implies) and EQUAL ($=$).

Three-valued logic

Fuzzy connectives defined for such a three-valued logic better can be stated as follows:

Symbol	Connective	Usage	Definition
\neg	NOT	$\neg P$	$1 - T(P)$
\vee	OR	$P \vee Q$	$\max\{T(P), T(Q)\}$
\wedge	AND	$P \wedge Q$	$\min\{T(P), T(Q)\}$
\implies	IMPLICATION	$(P \implies Q)$ or $(\neg P \vee Q)$	$\max\{(1 - T(P)), T(Q)\}$
$=$	EQUALITY	$(P = Q)$ or $[(P \implies Q) \wedge (Q \implies P)]$	$1 - T(P) - T(Q) $

Fuzzy proposition

Example 1:

P : Ram is honest

- ① $T(P) = 0.0$: Absolutely false
- ② $T(P) = 0.2$: Partially false
- ③ $T(P) = 0.4$: May be false or not false
- ④ $T(P) = 0.6$: May be true or not true
- ⑤ $T(P) = 0.8$: Partially true
- ⑥ $T(P) = 1.0$: Absolutely true.

Example 2 :Fuzzy proposition

P : Mary is efficient ; $T(P) = 0.8$;

Q : Ram is efficient ; $T(Q) = 0.6$

❶ **Mary is not efficient.**

$$T(\neg P) = 1 - T(P) = 0.2$$

❷ **Mary is efficient and so is Ram.**

$$T(P \wedge Q) = \min\{T(P), T(Q)\} = 0.6$$

❸ **Either Mary or Ram is efficient**

$$T(P \vee Q) = \max\{T(P), T(Q)\} = 0.8$$

❹ **If Mary is efficient then so is Ram**

$$T(P \implies Q) = \max\{1 - T(P), T(Q)\} = 0.6$$

Fuzzy proposition vs. Crisp proposition

- The fundamental difference between crisp (classical) proposition and fuzzy propositions is in the range of their truth values.
- While each classical proposition is required to be either true or false, the truth or falsity of fuzzy proposition is a matter of degree.
- The degree of truth of each fuzzy proposition is expressed by a value in the interval $[0,1]$ both inclusive.

Canonical representation of Fuzzy proposition

- Suppose, X is a universe of discourse of five persons. Intelligent of $x \in X$ is a fuzzy set as defined below.

Intelligent: $\{(x_1, 0.3), (x_2, 0.4), (x_3, 0.1), (x_4, 0.6), (x_5, 0.9)\}$

- We define a fuzzy proposition as follows:

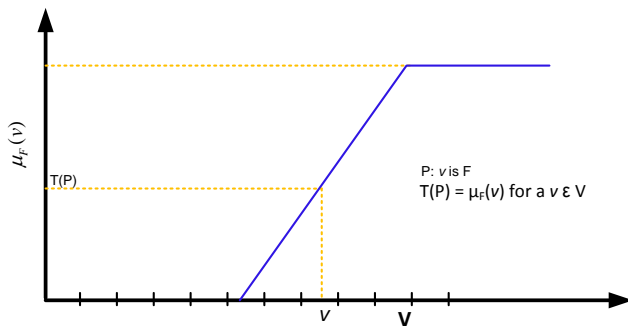
$P : x$ is intelligent

- The canonical form of fuzzy proposition of this type, P is expressed by the sentence $P : v$ is F .
- Predicate in terms of fuzzy set.

$P : v$ is F ; where v is an element that takes values v from some universal set V and F is a fuzzy set on V that represents a fuzzy predicate.

- In other words, given, a particular element v , this element belongs to F with membership grade $\mu_F(v)$.

Graphical interpretation of fuzzy proposition



- For a given value v of variable V in proposition P , $T(P)$ denotes the degree of truth of proposition P .

Fuzzy Implications

- A fuzzy implication (also known as fuzzy If-Then rule, fuzzy rule, or fuzzy conditional statement) assumes the form :

If x is A then y is B

where, A and B are two linguistic variables defined by fuzzy sets A and B on the universe of discourses X and Y , respectively.

- Often, x **is** A is called the **antecedent** or premise, while y **is** B is called the **consequence** or conclusion.

Fuzzy implication : Example 1

- If pressure is High then temperature is Low
- If mango is Yellow then mango is Sweet else mango is Sour
- If road is Good then driving is Smooth else traffic is High
- The fuzzy implication is denoted as $R : A \rightarrow B$
- In essence, it represents a binary fuzzy relation R on the (Cartesian) product of $A \times B$

Fuzzy implication : Example 2

- Suppose, P and T are two universes of discourses representing pressure and temperature, respectively as follows.
- $P = \{ 1, 2, 3, 4 \}$ and $T = \{ 10, 15, 20, 25, 30, 35, 40, 45, 50 \}$
- Let the linguistic variable **High temperature** and **Low pressure** are given as
- $T_{HIGH} = \{(20, 0.2), (25, 0.4), (30, 0.6), (35, 0.6), (40, 0.7), (45, 0.8), (50, 0.8)\}$
- $P_{LOW} = (1, 0.8), (2, 0.8), (3, 0.6), (4, 0.4)$

Fuzzy implications : Example 2

- Then the fuzzy implication **If temperature is High then pressure is Low** can be defined as

$$R : T_{HIGH} \rightarrow P_{LOW}$$

where, $R =$

	1	2	3	4
20	0.2	0.2	0.2	0.2
25	0.4	0.4	0.4	0.4
30	0.6	0.6	0.6	0.4
35	0.6	0.6	0.6	0.4
40	0.7	0.7	0.6	0.4
45	0.8	0.8	0.6	0.4
50	0.8	0.8	0.6	0.4

Note : If temperature is 40 then what about low pressure?

Interpretation of fuzzy rules

In general, there are two ways to interpret the fuzzy rule $A \rightarrow B$ as

- A coupled with B
- A entails B

Interpretation as **A coupled with B**

$R : A \rightarrow B = A \times B = \int_{X \times Y} \mu_A(x) * \mu_B(y) |_{(x,y)}$; where $*$ is called a **T-norm operator**.

T-norm operator

The most frequently used T-norm operators are:

Minimum : $T_{min}(a, b) = \min(a, b) = a \wedge b$

Algebraic product : $T_{ap}(a, b) = ab$

Bounded product : $T_{bp}(a, b) = 0 \vee (a + b - 1)$

Drastic product : $T_{dp} = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{if } a, b < 1 \end{cases}$

Here, $a = \mu_A(x)$ and $b = \mu_B(y)$. T_* is called the function of T-norm operator.

Interpretation as **A coupled with B**

Based on the T-norm operator as defined above, we can automatically define the fuzzy rule $R : A \rightarrow B$ as a fuzzy set with two-dimensional MF:

$\mu_R(x, y) = f(\mu_A(x), \mu_B(y)) = f(a, b)$ with $a = \mu_A(x)$, $b = \mu_B(y)$, and f is the fuzzy implication function.

Interpretation as **A coupled with B**

In the following, few implications of $R : A \rightarrow B$

Min operator:

$$R_m = A \times B = \int_{X \times Y} \mu_A(x) \wedge \mu_B(y) |_{(x,y)} \text{ or } f_{min}(a, b) = a \wedge b$$

[Mamdani rule]

Algebraic product operator

$$R_{ap} = A \times B = \int_{X \times Y} \mu_A(x) \cdot \mu_B(y) |_{(x,y)} \text{ or } f_{ap}(a, b) = ab$$

[Larsen rule]

Bounded product operator

$$R_{bp} = A \times B = \int_{X \times Y} \mu_A(x) \odot \mu_B(y) |_{(x,y)} = \int_{X \times Y} 0 \vee (\mu_A(x) + \mu_B(y) - 1) |_{(x,y)}$$

$$\text{or } f_{bp} = 0 \vee (a + b - 1)$$

Drastic product operator

$$R_{dp} = A \times B = \int_{X \times Y} \mu_A(x) \hat{\odot} \mu_B(y) |_{(x,y)}$$

$$\text{or } f_{dp}(a, b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{if otherwise} \end{cases}$$

Interpretation of **A entails B**

There are three main ways to interpret such implication:

Material implication :

$$R : A \rightarrow B = \bar{A} \cup B$$

Propositional calculus :

$$R : A \rightarrow B = \bar{A} \cup (A \cap B)$$

Extended propositional calculus :

$$R : A \rightarrow B = (\bar{A} \cap \bar{B}) \cup B$$

Interpretation of **A entails B**

With the above mentioned implications, there are a number of fuzzy implication functions that are popularly followed in fuzzy rule-based system.

Zadeh's arithmetic rule :

$$R_{za} = \bar{A} \cup B = \int_{X \times Y} 1 \wedge (1 - \mu_A(x) + \mu_B(y))|_{(x,y)}$$

or

$$f_{za}(a, b) = 1 \wedge (1 - a + b)$$

Zadeh's max-min rule :

$$R_{mm} = \bar{A} \cup (A \cap B) = \int_{X \times Y} (1 - \mu_A(x)) \vee (\mu_A(x) \wedge \mu_B(y))|_{(x,y)}$$

or

$$f_{mm}(a, b) = (1 - a) \vee (a \wedge b)$$

Interpretation of **A entails B**

Boolean fuzzy rule

$$R_{bf} = \bar{A} \cup B = \int_{X \times Y} (1 - \mu_A(x)) \vee \mu_B(x) |_{(x,y)}$$

or

$$f_{bf}(a, b) = (1 - a) \vee b;$$

Goguen's fuzzy rule:

$$R_{gf} = \int_{X \times Y} \mu_A(x) * \mu_B(y) |_{(x,y)} \text{ where } a * b = \begin{cases} 1 & \text{if } a \leq b \\ \frac{b}{a} & \text{if } a > b \end{cases}$$

Example 3: Zadeh's Max-Min rule

If x is A then y is B with the implication of Zadeh's max-min rule can be written equivalently as :

$$R_{mm} = (A \times B) \cup (\bar{A} \times Y)$$

Here, Y is the universe of discourse with membership values for all $y \in Y$ is 1, that is , $\mu_Y(y) = 1 \forall y \in Y$.

Suppose $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3, 4\}$

and $A = \{(a, 0.0), (b, 0.8), (c, 0.6), (d, 1.0)\}$

$B = \{(1, 0.2), (2, 1.0), (3, 0.8), (4, 0.0)\}$ are two fuzzy sets.

We are to determine $R_{mm} = (A \times B) \cup (\bar{A} \times Y)$

Example 3: Zadeh's min-max rule:

The computation of $R_{mm} = (A \times B) \cup (\bar{A} \times Y)$ is as follows:

$$A \times B = \begin{array}{c} \begin{matrix} & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1.0 & 0.8 & 0 \end{array} \right] \end{array} \text{ and}$$

$$\bar{A} \times Y = \begin{array}{c} \begin{matrix} & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$

Example 3: Zadeh's min-max rule:

Therefore,

$$R_{mm} = (A \times B) \cup (\bar{A} \times Y) =$$

	1	2	3	4
a	1	1	1	1
b	0.2	0.8	0.8	0.2
c	0.4	0.6	0.6	0.4
d	0.2	1.0	0.8	0

Example 3 :

$$X = \{a, b, c, d\}$$

$$Y = \{1, 2, 3, 4\}$$

$$\text{Let, } A = \{(a, 0.0), (b, 0.8), (c, 0.6), (d, 1.0)\}$$

$$B = \{(1, 0.2), (2, 1.0), (3, 0.8), (4, 0.0)\}$$

Determine the implication relation :

If x is A then y is B

Here, $A \times B =$

$$\begin{array}{c} a \\ b \\ c \\ d \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix}$$

Example 3 :

$$\text{and } \bar{A} \times Y = \begin{array}{c} a \\ b \\ c \\ d \end{array} \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \\ \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$

$$R_{mm} = (A \times B) \cup (\bar{A} \times Y) = \begin{array}{c} a \\ b \\ c \\ d \end{array} \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \\ \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.4 & 0.6 & 0.6 & 0.4 \\ 0.2 & 1.0 & 0.8 & 0 \end{array} \right] \end{array}$$

This R represents **If x is A then y is B**

Example 3 :

IF x is A THEN y is B ELSE y is C .

The relation R is equivalent to

$$R = (A \times B) \cup (\bar{A} \times C)$$

The membership function of R is given by

$$\mu_R(x, y) = \max[\min\{\mu_A(x), \mu_B(y)\}, \min\{\mu_{\bar{A}}(x), \mu_C(y)\}]$$

Example 4:

$$X = \{a, b, c, d\}$$

$$Y = \{1, 2, 3, 4\}$$

$$A = \{(a, 0.0), (b, 0.8), (c, 0.6), (d, 1.0)\}$$

$$B = \{(1, 0.2), (2, 1.0), (3, 0.8), (4, 0.0)\}$$

$$C = \{(1, 0), (2, 0.4), (3, 1.0), (4, 0.8)\}$$

Determine the implication relation :

If x is A then y is B else y is C

Here, $A \times B =$

$$\begin{array}{c} \begin{matrix} & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1.0 & 0.8 & 0 \end{array} \right] \end{array}$$

Example 4:

$$\text{and } \bar{A} \times C =$$

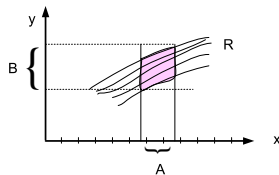
	1	2	3	4
a	0	0.4	1.0	0.8
b	0	0.2	0.2	0.2
c	0	0.4	0.4	0.4
d	0	0	0	0

$$R =$$

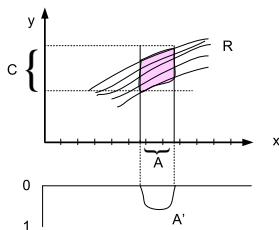
	1	2	3	4
a	0	0.4	1.0	0.8
b	0.2	0.8	0.8	0.2
c	0.2	0.6	0.6	0.4
d	0.2	1.0	0.8	0

Interpretation of fuzzy implication

If x is A then y is B



If x is A then y is B else y is C



Fuzzy Inferences

Fuzzy inferences

Let's start with propositional logic. We know the following in propositional logic.

① Modus Ponens : $P, P \implies Q, \quad \Leftrightarrow Q$

② Modus Tollens : $P \implies Q, \neg Q \quad \Leftrightarrow, \neg P$

③ Chain rule : $P \implies Q, Q \implies R \quad \Leftrightarrow, P \implies R$

An example from propositional logic

Given

$$① \quad C \vee D$$

$$② \quad \sim H \implies (A \wedge \sim B)$$

$$③ \quad C \vee D \implies \sim H$$

$$④ \quad (A \wedge \sim B) \implies (R \vee S)$$

From the above can we infer $R \vee S$?

Similar concept is also followed in fuzzy logic to infer a fuzzy rule from a set of given fuzzy rules (also called fuzzy rule base).

Inferring procedures in Fuzzy logic

Two important inferring procedures are used in fuzzy systems :

- **Generalized Modus Ponens (GMP)**

If x is A Then y is B

x is A'

y is B'

- **Generalized Modus Tollens (GMT)**

If x is A Then y is B

y is B'

x is A'

Fuzzy inferring procedures

- Here, A, B, A' and B' are fuzzy sets.
- To compute the membership function A' and B' the max-min composition of fuzzy sets B' and A' , respectively with $R(x, y)$ (which is the known implication relation) is to be used.
- Thus,

$$B' = A' \circ R(x, y) \qquad \mu_{B'}(y) = \max[\min(\mu_{A'}(x), \mu_R(x, y))]$$

$$A' = B' \circ R(x, y) \qquad \mu_{A'}(x) = \max[\min(\mu_{B'}(y), \mu_R(x, y))]$$

Generalized Modus Ponens

Generalized Modus Ponens (GMP)

P : If x is A then y is B

Let us consider two sets of variables x and y be

$X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$, respectively.

Also, let us consider the following.

$A = \{(x_1, 0.5), (x_2, 1), (x_3, 0.6)\}$

$B = \{(y_1, 1), (y_2, 0.4)\}$

Then, given a fact expressed by the proposition x is A' ,

where $A' = \{(x_1, 0.6), (x_2, 0.9), (x_3, 0.7)\}$

derive a conclusion in the form y is B' (using generalized modus ponens (GMP)).

Example: Generalized Modus Ponens

If x is A Then y is B

x is A'

y is B'

We are to find $B' = A' \circ R(x, y)$ where $R(x, y) = \max\{A \times B, \bar{A} \times Y\}$

$$A \times B = \begin{matrix} & y_1 & y_2 \\ x_1 & \begin{bmatrix} 0.5 & 0.4 \end{bmatrix} \\ x_2 & \begin{bmatrix} 1 & 0.4 \end{bmatrix} \\ x_3 & \begin{bmatrix} 0.6 & 0.4 \end{bmatrix} \end{matrix} \text{ and } \bar{A} \times Y = \begin{matrix} & y_1 & y_2 \\ x_1 & \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \\ x_2 & \begin{bmatrix} 0 & 0 \end{bmatrix} \\ x_3 & \begin{bmatrix} 0.4 & 0.4 \end{bmatrix} \end{matrix}$$

Note: For $A \times B$, $\mu_{A \times B}(x, y) = \min(\mu_A(x), \mu_B(y))$

Example: Generalized Modus Ponens

$$R(x, y) = (A \times B) \cup (\bar{A} \times y) = \begin{matrix} & y_1 & y_2 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} & \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{matrix}$$

Now, $A' = \{(x_1, 0.6), (x_2, 0.9), (x_3, 0.7)\}$

$$\text{Therefore, } B' = A' \circ R(x, y) = \\ [0.6 \quad 0.9 \quad 0.7] \circ \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} = [0.9 \quad 0.5]$$

Thus we derive that y is B' where $B' = \{(y_1, 0.9), (y_2, 0.5)\}$

Example: Generalized Modus Tollens

Generalized Modus Tollens (GMT)

P: If x is A Then y is B

Q: y is B'

x is A'

Example: Generalized Modus Tollens

- Let sets of variables x and y be $X = \{x_1, x_2, x_3\}$ and $y = \{y_1, y_2\}$, respectively.
- Assume that a proposition **If x is A Then y is B** given where $A = \{(x_1, 0.5), (x_2, 1.0), (x_3, 0.6)\}$ and $B = \{(y_1, 0.6), (y_2, 0.4)\}$
- Assume now that a fact expressed by a proposition y **is B** is given where $B' = \{(y_1, 0.9), (y_2, 0.7)\}$.
- From the above, we are to conclude that x **is A'** . That is, we are to determine A'

Example: Generalized Modus Tollens

- We first calculate $R(x, y) = (A \times B) \cup (\bar{A} \times y)$

$$R(x, y) = \begin{matrix} & y_1 & y_2 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} & \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{matrix}$$

- Next, we calculate $A' = B' \circ R(x, y)$

$$A' = [0.9 \quad 0.7] \circ \begin{matrix} & y_1 & y_2 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} & \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{matrix} = [0.5 \quad 0.9 \quad 0.6]$$

- Hence, we calculate that x is A' where $A' = [(x_1, 0.5), (x_2, 0.9), (x_3, 0.6)]$

Apply the fuzzy GMP rule to deduce **Rotation is quite slow**

Given that :

- If temperature is High then rotation is Slow.
- temperature is Very High

Let,

$X = \{30, 40, 50, 60, 70, 80, 90, 100\}$ be the set of temperatures.

$Y = \{10, 20, 30, 40, 50, 60\}$ be the set of rotations per minute.

Practice

The fuzzy set High(H), Very High (VH), Slow(S) and Quite Slow (QS) are given below.

$$H = \{(70, 1), (80, 1), (90, 0.3)\}$$

$$VH = \{(90, 0.9), (100, 1)\}$$

$$S = \{(30, 0.8), (40, 1.0), (50, 0.6)\}$$

$$QS = \{(10, 1), (20, 0.8)\}$$

- 1 If temperature is High then the rotation is Slow.

$$R = (H \times S) \cup (\bar{H} \times Y)$$

- 2 temperature is Very High

Thus, to deduce "rotation is Quite Slow", we make use the composition rule $QS = VH \circ R(x, y)$

Any questions??