Computational Intelligence (CI)

Fuzzy Set Theory

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Classical Set Theory

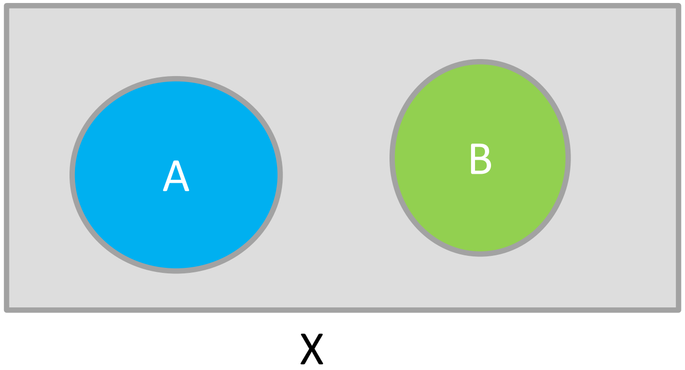
- **Set:** Well defined collection of objects. An object in a set is called an element or member of that set.

- **Universal Set/Universe of Discourse:** A set containing all possible elements.

- **Classical / Crisp Set:**
  Classical or Crisp set is a set with fixed and well-defined boundary.

- **Example:**

X: Students of KIIT
A: CSE students of KIIT
B: IT Students of KIIT



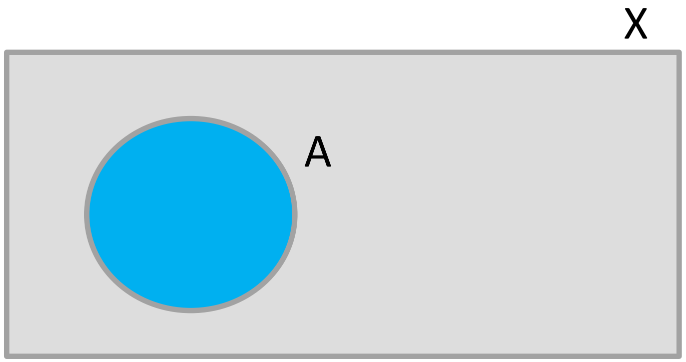
Classical Set Theory

- **Example:**

X = Students of KIIT$(x_1, x_2, x_3, \dots \dots \dots, x_n)$
A = CSE Students of KIIT

If the elements of a set **A** are subset of Universal set **X**, then the set **A** can be represented for all elements $x \in X$ by its **characteristic function**.

$$\mu_A(x) = \begin{cases} 1 & if x \in A \\ 0 & if x \notin A \end{cases}$$



Characteristic function is a function that defines an element of Universe of discourse belongs to a Crisp set or not.

Thus, $\mu_A(x)$ has any one of two values: 1 (true) or 0 (false).

Fuzzy Set Theory

- Fuzzy Set Theory is an extension of classical set theory where elements have varying degrees of membership.

- Fuzzy sets are the sets with imprecise (vague) boundaries. It allows members to have **different degree of membership (degree of truth)** by the help of membership function.

- It is to be noted that the characteristic function used in crisp set has been renamed as the membership function in fuzzy set.

- Fuzzy set theory defines Fuzzy Operators on Fuzzy Sets.

Fuzzy vs. Probability

The concept of fuzzy set was introduced by Prof. L.A. Zadeh of the University of California, USA, in 1965, although an idea related to it was visualized by Max Black, an American Philosopher, in 1937.

Prior to 1965, people used to consider **probability theory** (which works based on Aristotelian two-valued logic) as the prime agent for dealing with uncertainties.

Prof. Zadeh argued that probability theory can handle only one out of several different types of possible uncertainties. Thus, there are some uncertainties, which cannot be tackled using the probability theory.

Base Paper: L. A. Zadeh "Fuzzy Sets" *Information and Control*, 8, 338-353 (1965)

Fuzzy vs. Probability

Example: Ashish requests Priya to bring some **red apples** for him from the market.

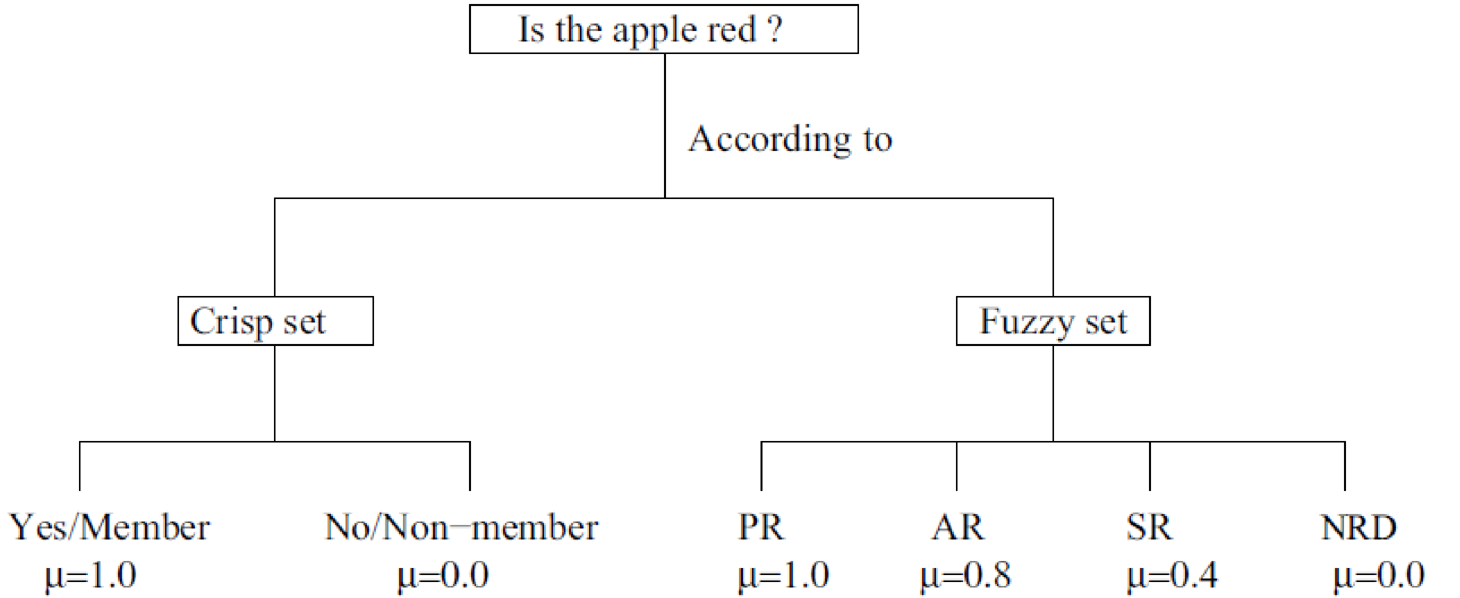
There are two uncertainties at least:
(i) availability of the apples
(ii) a guarantee that the apple is red.

Depending on the season, there is a probability (that is, the frequency of likelihood that an element is in a class) of obtaining the apples, which varies between 0.0 and 1.0.

According to the crisp set, the apples will be either red (1) or non-red (0).

In fuzzy set, an element can be a member of the set with some membership value (that is, degree of belongingness).

Fuzzy vs. Probability



PR: Perfectly Red
AR: Almost Red
SR: Slightly Red
NRD: Not Red
μ: Membership value

Fuzzy vs. Probability

Example: Consider two bottles A and B having mixture of milk and poison. Probability of bottle A with poison is 0.7. Fuzzy membership of bottle B with poison is 0.7. Which one is safe to take?

A

B

Poison **Probability**: 0.7

Poison **plausibility/Possibility**: 0.7

Each fuzzy set is represented by some membership functions. The degree of membership or truth is not same as probability.

- Degree of membership/fuzzy truth is not likelihood of some event or condition.
- Fuzzy truth represents membership in vaguely defined sets.

Representation of a Fuzzy Set

A fuzzy set ***A***(x) **in the Universe X** is represented by a pair of two things:

- The element x
- membership value of x, $\mu_A(x)$

$$A\,(x\,) = \{\,(x, \mu_A(x) \mid x \;\in X\,\}$$

Membership Function

Definition 1: Membership function (and Fuzzy set)

If X is a universe of discourse and $x \in X$, then a fuzzy set A in X is defined as a set of ordered pairs, that is
$A = \{(x, \mu_A(x)) | x \in X\}$ where $\mu_A(x)$ is called the membership function for the fuzzy set A.

Membership function is a curve that defines how each element in the input space is mapped to a membership value/grade between 0 and 1 (both inclusive).

X may consists of discrete (ordered or unordered) items or continuous space.

Fuzzy Set with Discrete Universe

Fuzzy set may be either discrete or continuous in nature.

If a fuzzy set $A(x)$ is discrete in nature, it can be expressed using the membership values of its elements.

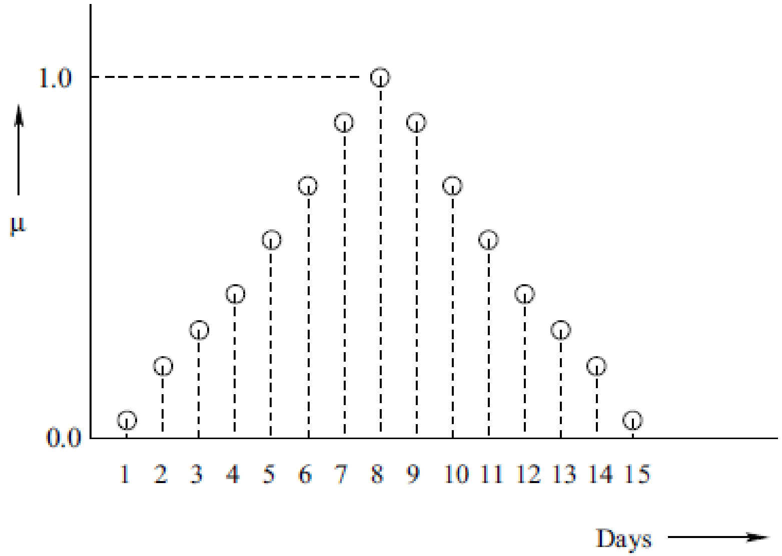
A(x) can be denoted as

$$A(x) = \sum_{i=1}^{n} \mu_A(x_i) \,/\, x_i$$

Here the + sign does not indicate any algebraic sum but collection of elements.

Fuzzy Set with Discrete Universe

Example: Atmospheric temperature during the first fifteen days of a month is found to be medium (M).



$M = 0.05/1 + 0.2/2 + 0.3/3 + 0.4/4 + 0.55/5 + 0.7/6 + 0.875/7 + 1.0/8 + 0.875/9 + 0.7/10 + 0.55/11 + 0.4/12 + 0.3/13 + 0.2/14 + 0.05/15.$

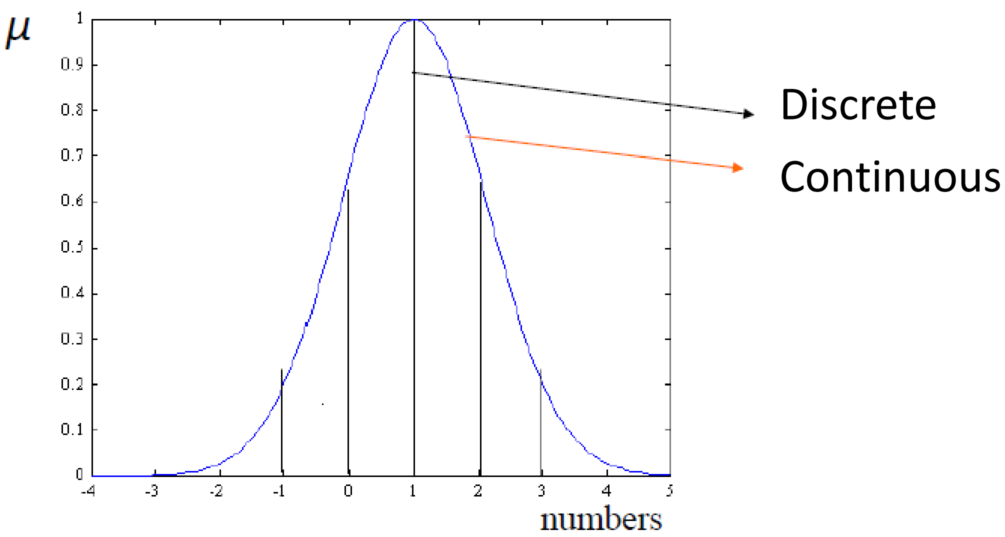
$M = \{(1, 0.05), (2, 0.2), (3, 0.3), (4, 0.4), (5, 0.55), (6, 0.7), (7, 0.875), (8, 1.0), (9, 0.875), (10, 0.7), (11, 0.55), (12, 0.4), (13, 0.3), (14, 0.2), (15, 0.05)\}.$

Fuzzy Set with Continuous Universe

A continuous fuzzy set *A*(*x*) is expressed mathematically like the following:

$$A(x) = \int_X \mu_A(x)/x.$$

It is important to mention that an integral sign is utilized here to indicate X is a continuous space.



Example

Example 2.8 *(Fuzzy Membership)*

Let a, b, c, d, and e be five students who scored 55, 35, 60, 85 and 75 out of 100 respectively in Mathematics. The students constitute the universe of discourse $U = \{a, b, c, d, e\}$ and a fuzzy set M of the students who are *good in Mathematics* is defined on U with the help of the following membership function.

$$\mu_M(x) = \begin{cases} 0, & \text{if} \quad x < 40 \\ \dfrac{x-40}{40}, & \text{if} \quad 40 \le x < 80 \\ 1, & \text{if} \quad x \ge 80 \end{cases} \tag{2.4}$$

The membership function is graphically shown in Fig. 2.8. Computing the membership value of each student with the help of the Formula 2.4 we get $M = \{(a, 0.375), (c, 0.5), (d, 1.0), (e, 0.875)\}$, or equivalently

$$M = \frac{0.375}{a} + \frac{0.5}{c} + \frac{1.0}{d} + \frac{0.875}{e}$$

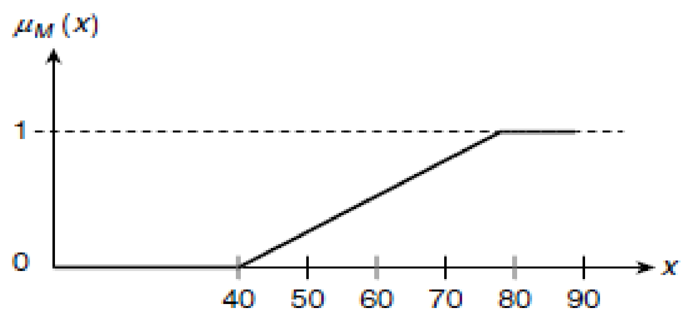


Fig. 2.8. *Membership function for students good in Mathematics.*

Example

Example 2.9 *(Fuzzy Membership)*

Let us consider the infinite set of all real numbers between 0 and 1, both inclusive, to be the universe of discourse, or, the reference set, $U = [\,0, 1\,]$. We define a fuzzy set $C0.5$ as the set of all real numbers in U that are *close* to 0.5 in the following way

$$C0.5 = \{x \in [\,0, 1\,] \mid x \text{ is } \textit{close} \text{ to } 0.5\}$$

The highest membership value would be attained by the point $x = 0.5$ because that is the number closest to 0.5, and the membership is understandably 1. On the other hand since both 0 and 1 are furthest points from 0.5 (within the interval $[\,0, 1\,]$) they should have zero membership to $C0.5$. Membership values should increase progressively as we approach 0.5 from both ends of the interval $[\,0, 1\,]$. The membership function for $C0.5$ may be defined in the following manner.

$$\mu_{C0.5}(x) = 1 - |\,2x - 1\,|, \qquad \forall x \in [\,0, 1\,] \tag{2.5}$$

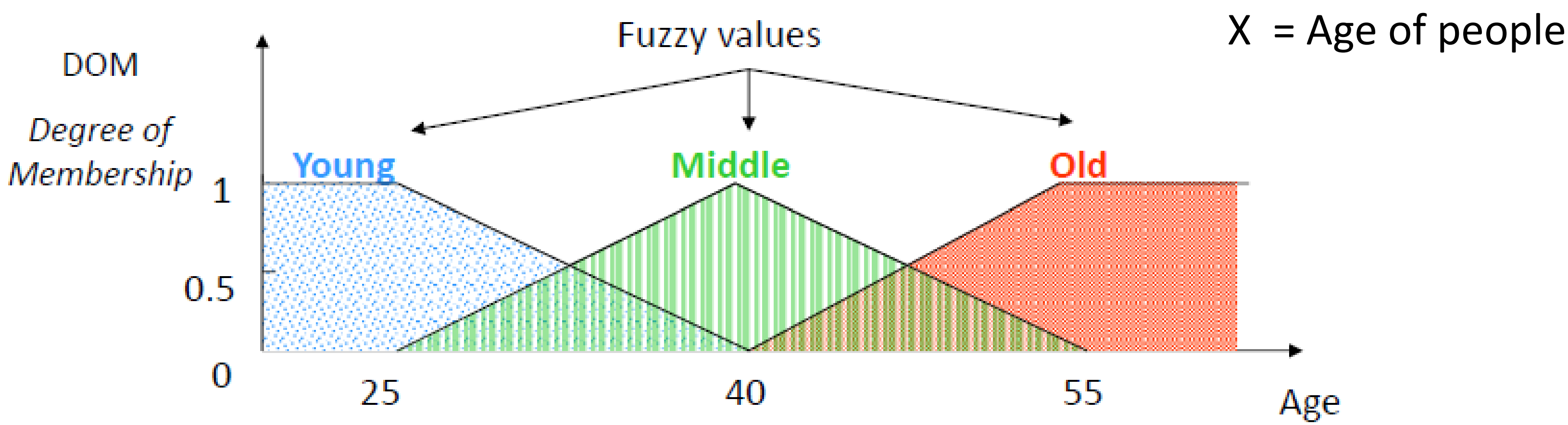
Since $C0.5$ is a fuzzy set in a continuous domain it can be expressed with the help of the notation

$$C0.5 = \int_{x \in [0,1]} \frac{\mu_{c0.5}(x)}{x} = \int_{x \in [0,1]} \frac{1 - |2x - 1|}{x} \tag{2.6}$$

Linguistic Variables

- Linguistic Variable/form is a variable with subjective knowledge, usually impossible to quantify (with no specific value).

Example: Describing people as "Young", "Middle-aged" and "Old"



Fuzzy Logic allows modelling of linguistic terms using linguistic variables or linguistic values.

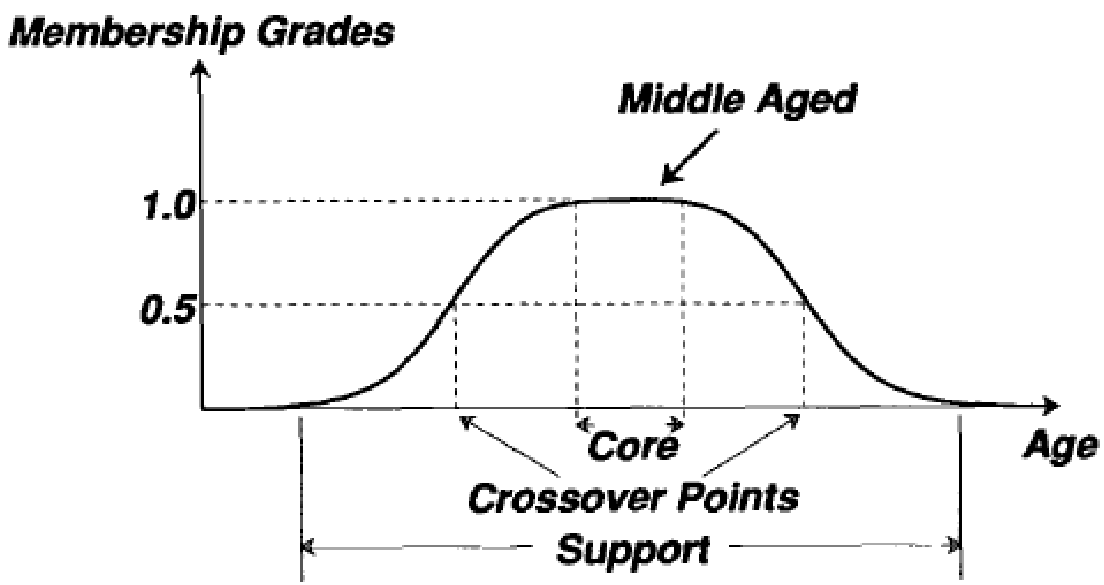
The fuzzy sets "young", "middle-aged", and "old" are fully defined by their membership functions.

Age is a linguistic variable if its values are linguistic values rather than numerical.

Fuzzy Set Terminology

Support: The support of a fuzzy set A is the set of all points x in X such that $\mu_A(x) > 0$

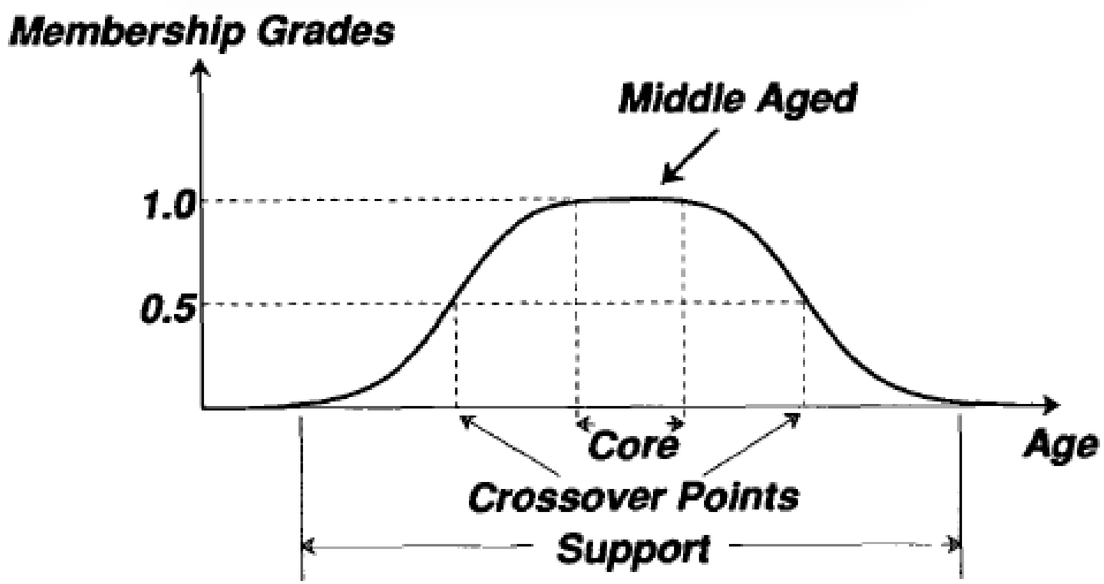
$$\text{support}(A) = \{x | \mu_A(x) > 0\}$$



Fuzzy Set Terminology

Core: The core of a fuzzy set A is the set of all points x in X such that $\boldsymbol{\mu_A(x) = 1}$

$$\text{core}(A) = \{x | \mu_A(x) = 1\}$$



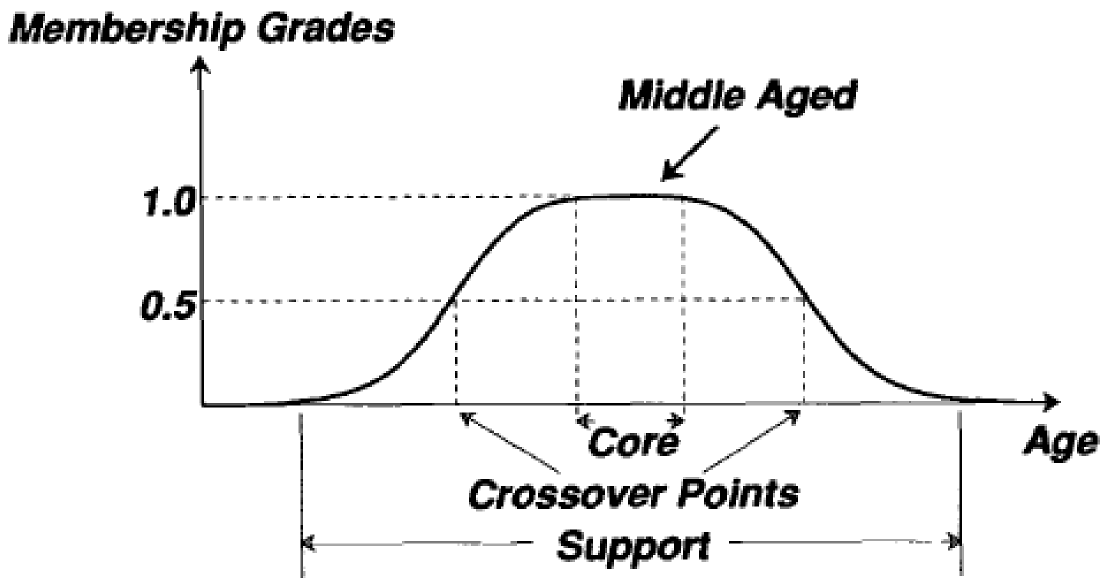
Normality: A fuzzy set A is normal if its core is non-empty.

Sub-Normality: A fuzzy set A is sub-normal if it is not normal.

Fuzzy Set Terminology

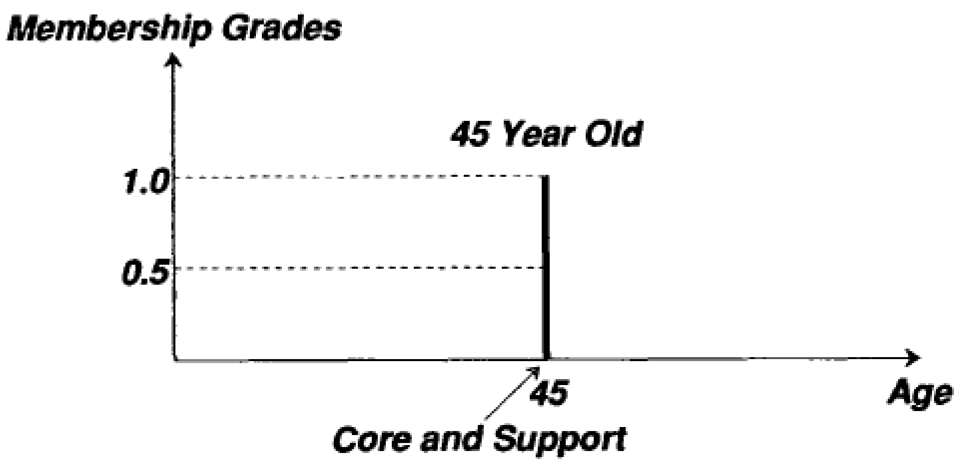
Crossover Points: A crossover point of a fuzzy set A is a point $x \in X$ at which $\boldsymbol{\mu_A(x) = 0.5}$

$$\text{crossover}(A) = \{x \mid \mu_A(x) = 0.5\}$$



Fuzzy Set Terminology

Fuzzy Singleton: A fuzzy set whose support is a **single point in X with** $\mu_A(x) = 1$



Fuzzy Set Terminology

α-cut or α-level set: It is a crisp set defined by

$$A_\alpha = \{x | \mu_A(x) \geq \alpha\}$$

Strong α-cut or Strong α-level set: It is a crisp set defined by

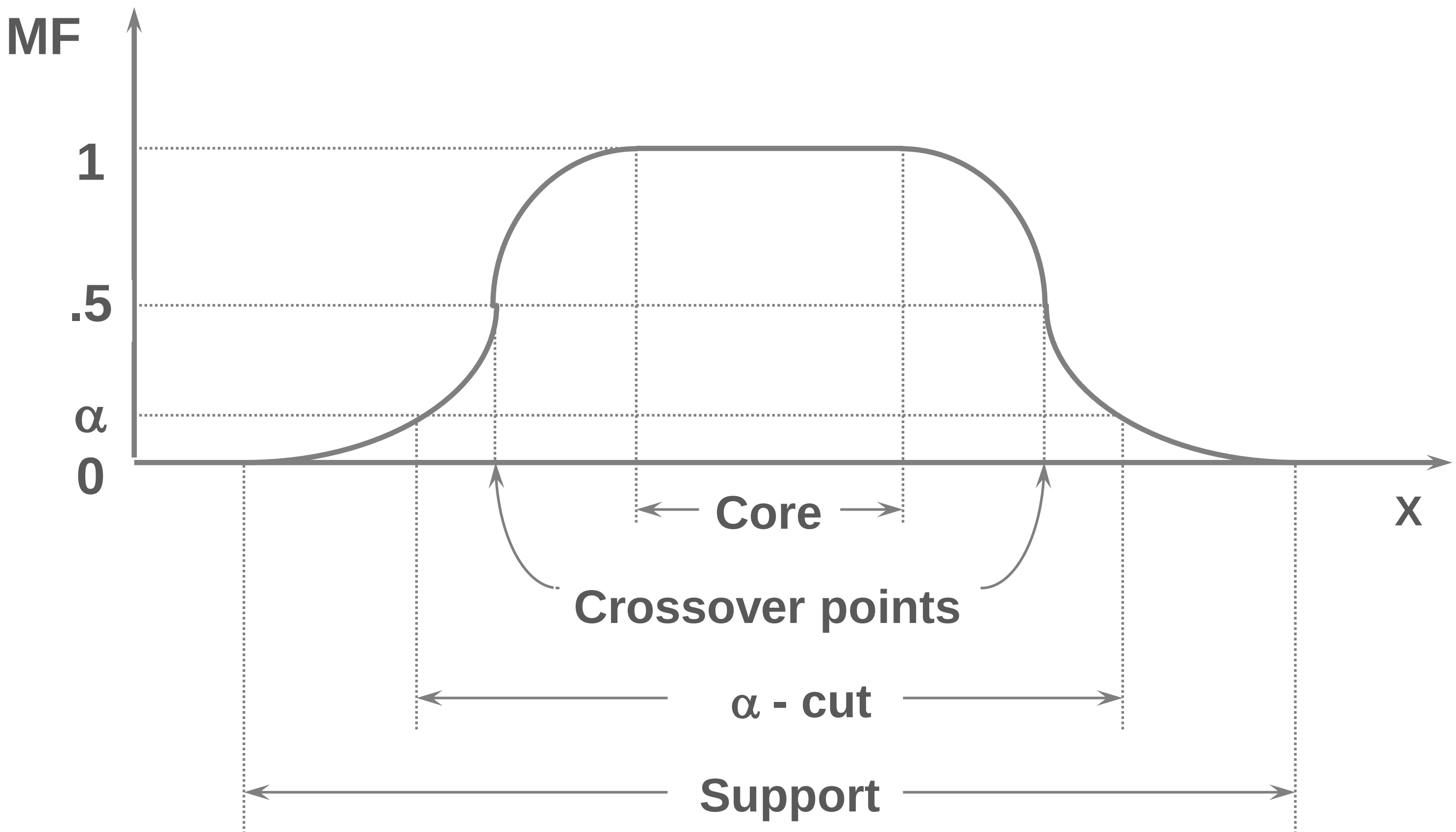
$$A'_\alpha = \{x | \mu_A(x) > \alpha\}$$

Using level set, support and core of a fuzzy set A can be defined as

$$\text{support}(A) = A'_0$$

$$\text{core}(A) = A_1$$

Overview

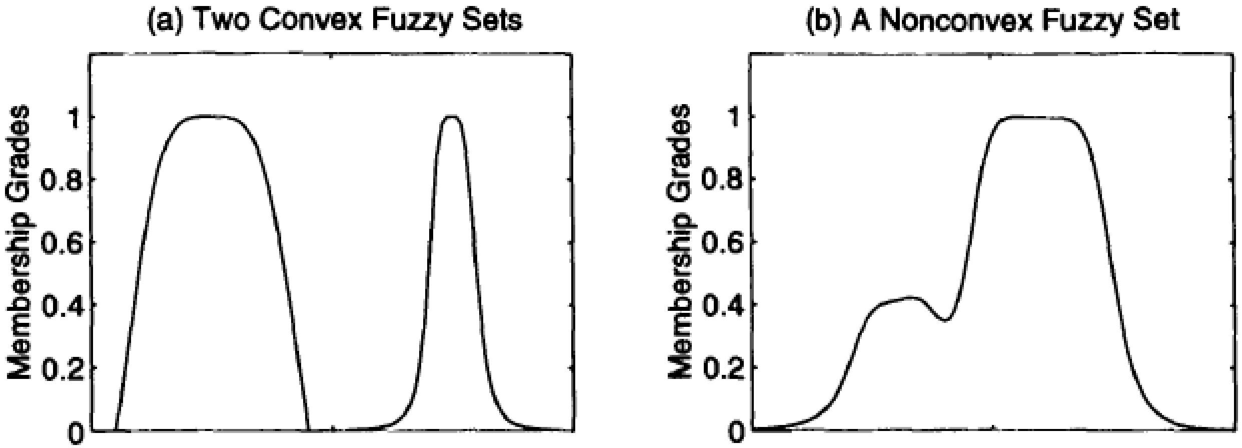


Fuzzy Set Terminology

Convexity: Fuzzy set A is convex if and only if for any $x_1, x_2 \in X$ and $\lambda \in [0, 1]$,

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}$$

Note# A is convex if all its α-level sets are convex.



Fuzzy Set Terminology

Bandwidth/Width: It is the distance between two unique crossover points.

$$\text{width}(A) = |x_2 - x_1|, \quad \text{where } \mu_A(x_1) = \mu_A(x_2) = 0.5.$$

Symmetry: Fuzzy set A is symmetric if its MF is symmetric around a certain point $x = c, where$

$$\mu_A(c + x) = \mu_A(c - x) \text{ for all } x \in X$$

Fuzzy Set Terminology

Open left: Fuzzy set A is open left if,

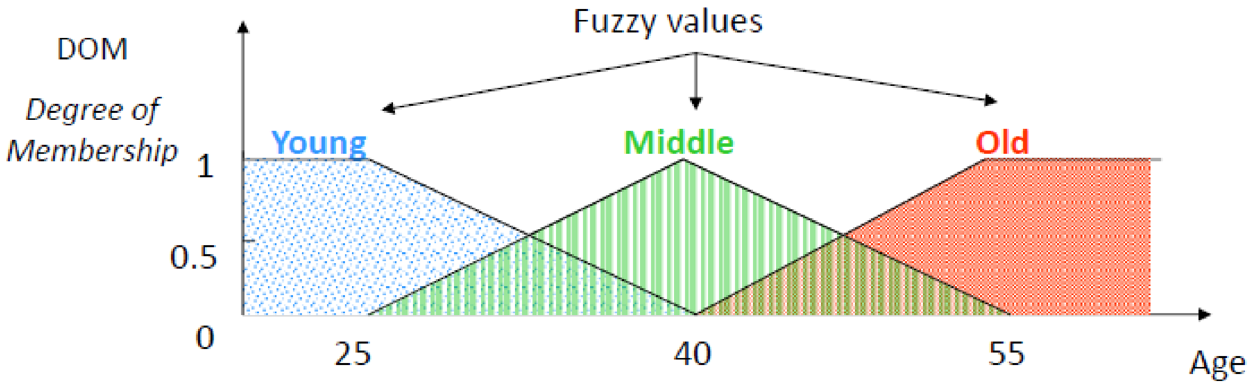
$$\lim_{x\to-\infty}\mu_A(x) = 1 \text{ and } \lim_{x\to+\infty}\mu_A(x) = 0$$

Open right: Fuzzy set A is open right if,

$$\lim_{x\to-\infty}\mu_A(x) = 0 \text{ and } \lim_{x\to+\infty}\mu_A(x) = 1$$

Closed: Fuzzy set A is closed if,

$$\lim_{x\to-\infty}\mu_A(x) = \lim_{x\to+\infty}\mu_A(x) = 0$$



Young: Open Left

Middle: Closed

Old: Open right

Fuzzy Set Terminology

Height: The height of a fuzzy set is defined as the maximal membership value attained by its elements.

$$Height\ (A) = Max_{x\,\in\,X}\,\mu_A(x)$$

Note# *A is normal fuzzy set if* ***Height (A) = 1*** *and sub-normal if* ***Height(A) < 1*** *.*

Cardinality: The sum of all membership values of the members of Fuzzy set A is said to be cardinality of A.

$$Cardinality\ (A) = \sum_{x\,\in\,X} \mu_A(x)$$

Problem

Problem 2.9 Let *F* be a fuzzy set of *matured* persons where the maturity is measured in terms of age in years. The fuzzy membership function followed is given below

$$\mu_F(x) = \begin{cases} 0, & if \quad x \le 5 \\ \left(\dfrac{x-5}{20}\right)^2, & if \quad 5 \le x \le 25 \\ 1, & if \quad x \ge 25 \end{cases}$$

The universe consists of the individuals Sunny, Moon, Pikoo, Gina, Osho, Chang, Paul, Lalu, Lila, and Poly whose ages are 15, 20, 10, 27, 32, 12, 18, 24, 3, and 8 years respectively. Find the *normalcy* of the set as well as *Height* (*F*), *Support* (*F*), *Core* (*F*), and *Cardinality* (*F*).

Hints:

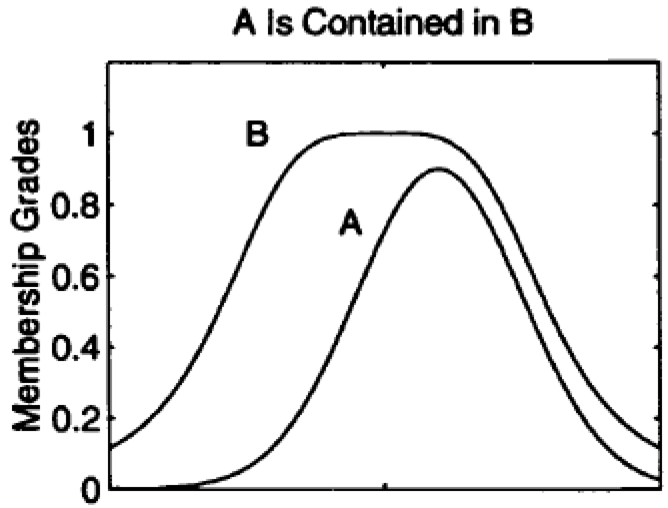
$$F = \frac{0.25}{Sunny} + \frac{0.5625}{Moon} + \frac{0.0625}{Pikoo} + \frac{1.0}{Gina} + \frac{1.0}{Osho} + \frac{0.1225}{Chang} + \frac{0.4225}{Paul} + \frac{0.9025}{Lalu} + \frac{0}{Lila} + \frac{0.0225}{Poly}$$

The set is normal because there are two members, Gina and Osho, who attain full memberships. Obviously, *Height* (*F*) = 1.0. *Support* (*F*) = {*Sunny*, *Moon*, *Pikoo*, *Gina*, *Osho*, *Chang*, *Paul*, *Lalu*, *Poly*}, *Core* (*F*) = {*Gina*, *Osho*}, and *Cardinality* (*F*) = 0.25 + 0.5625 + 0.0625 + 1.0 + 1.0 + 0.1225 + 0.4225 + 0.9025 + 0 + 0.0225 = 4.345.

Set-Theoretic Operations

Subset/Containment: Fuzzy set A is in B or A is a subset of B iff

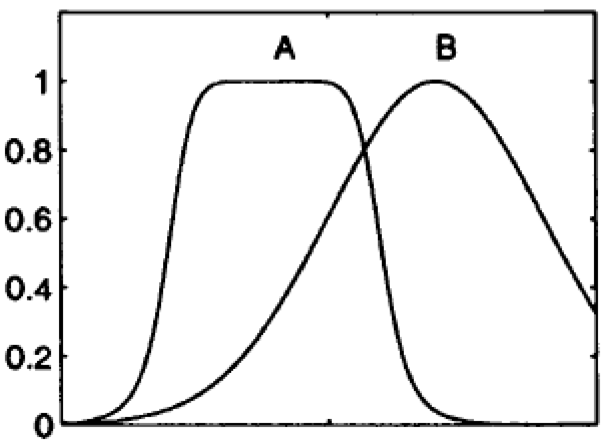
$$A \subseteq B \Longleftrightarrow \mu_A(x) \leq \mu_B(x)$$



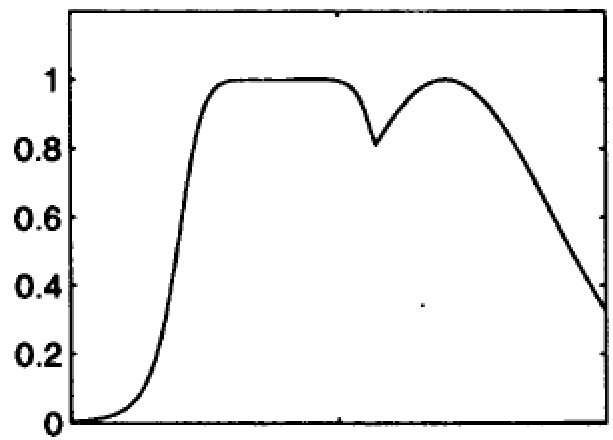
Set-Theoretic Operations

Union (disjunction): Union of two fuzzy sets A and B is a fuzzy set C iff

$$\mu_C(x) = \max(\mu_A(x), \mu_B(x)) = \mu_A(x) \vee \mu_B(x)$$



Fuzzy Sets A and B



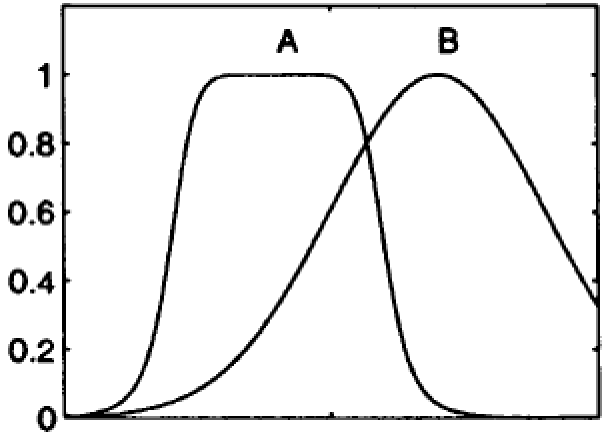
C = A U B

C = A OR B

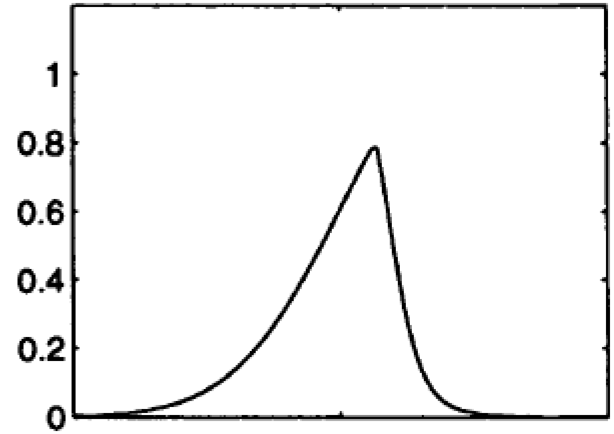
Set-Theoretic Operations

Intersection (Conjunction): Intersection of two fuzzy sets A and B is a fuzzy set C iff

$$\mu_C(x) = \min(\mu_A(x), \mu_B(x)) = \mu_A(x) \wedge \mu_B(x)$$



Fuzzy Sets A and B



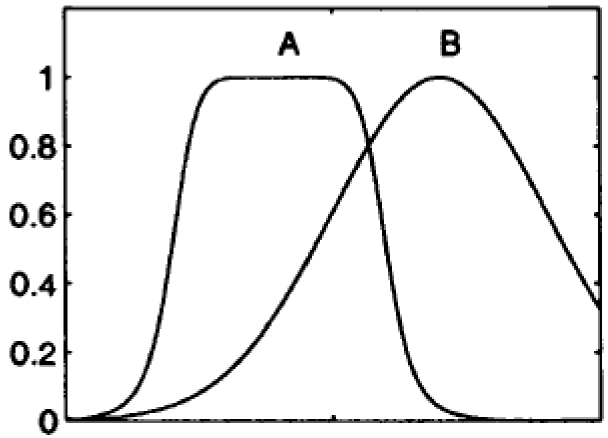
C = A ∩ B

C = A **AND** B

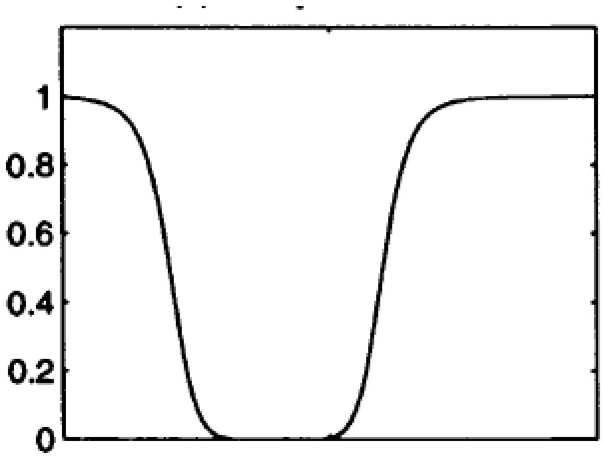
Set-Theoretic Operations

Complement (negation): The complement of fuzzy set A, denoted by $\overline{A}$ ($\neg A$, NOT A)

$$\mu_{\overline{A}}(x) = 1 - \mu_A(x)$$



Fuzzy Sets A and B



$\overline{A}$ ($\neg A$, NOT A)

Set-Theoretic Operations

Cartesian Product ($A \times B$):

let A and B be fuzzy sets in X and Y, respectively. Cartesian product of A and B, denoted by $A \times B$, is a fuzzy set in the product space $X \times Y$ with the membership function

$$\mu_{A\times B}(x,y) = \min(\mu_A(x), \mu_B(y))$$

Cartesian Co-Product ($A + B$) is a fuzzy set with membership function

$$\mu_{A+B}(x,y) = \max(\mu_A(x), \mu_B(y))$$

Note# $A \times B$ and $A + B$ are characterized by two-dimensional membership function.

Example

Find Cartesian product and co-product

Let us consider the reference sets $X = \{m, n\}$ and $Y = \{p, q, r\}$ and the fuzzy sets A and B defined on them.

$$A = \frac{0.3}{m} + \frac{0.7}{n}, \qquad B = \frac{0.5}{p} + \frac{0.1}{q} + \frac{0.8}{r}$$

Proper subset of a fuzzy set

Let us consider two fuzzy sets: $A(x)$ and $B(x)$, such that all $x \in X$. The fuzzy set $A(x)$ is called the proper subset of $B(x)$, if $\mu_A(x) < \mu_B(x)$.

It can be represented as follows:

$$A(x) \subset B(x), \text{if } \mu_A(x) < \mu_B(x).$$

Example:

$A(x) = \{(x1, 0.1), (x2, 0.2), (x3, 0.3), (x4, 0.4)\}$
$B(x) = \{(x1, 0.5), (x2, 0.7), (x3, 0.8), (x4, 0.9)\}$

Is A subset of B?

As $\mu_A(x) < \mu_B(x)$ for all $x \in X$, $A(x) \subset B(x)$.

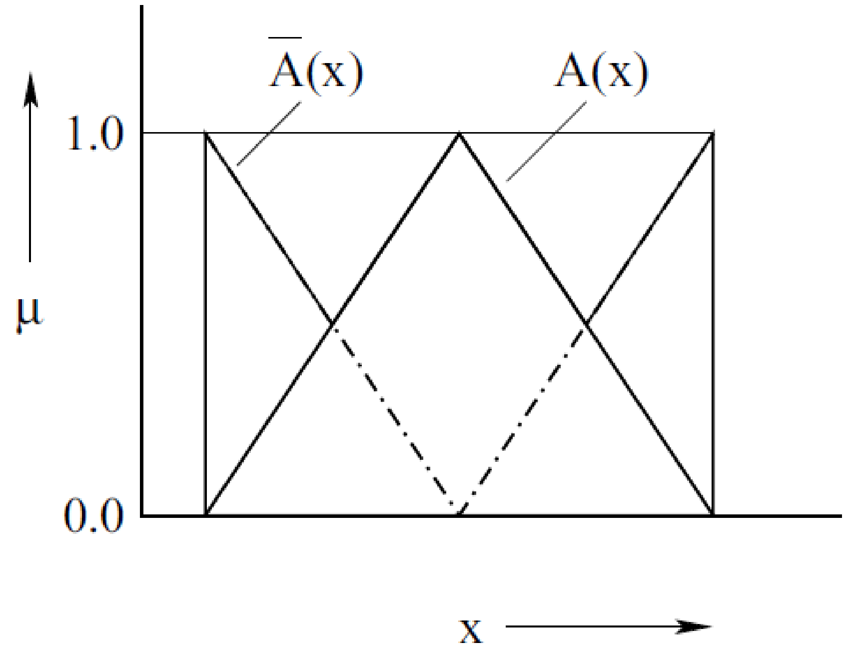
Few Properties of Fuzzy Sets

Fuzzy sets follow the many properties of crisp sets, except the following:

Law of excluded middle:

In crisp set,

$$A \cup \overline{A} = X$$

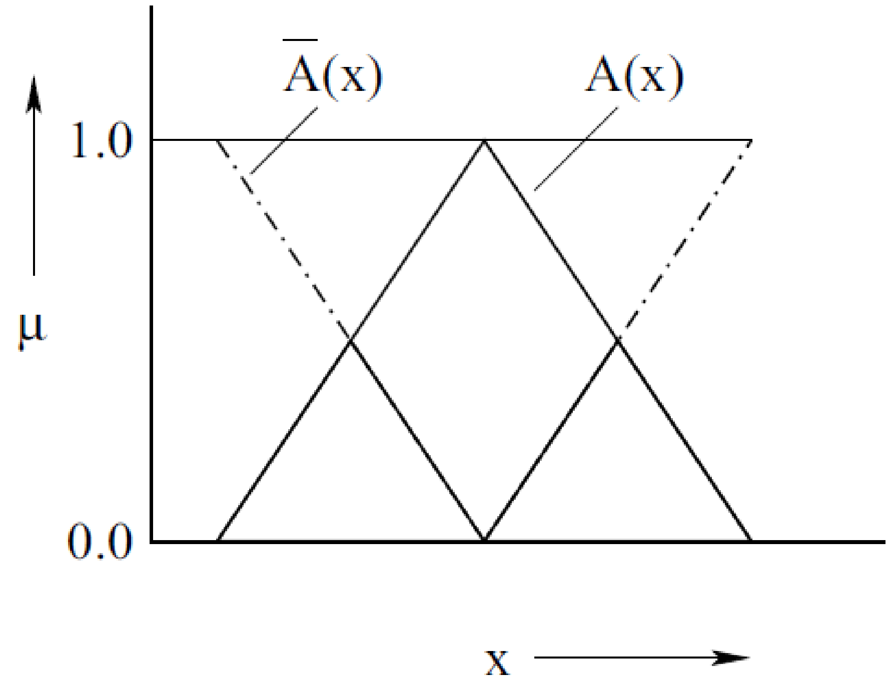


In fuzzy set,

$$A \cup \overline{A} \neq X$$

Few Properties of Fuzzy Sets

Law of contradiction:



In crisp set,

$$A \cap \overline{A} = \emptyset$$

In fuzzy set,

$$A \cap \overline{A} \neq \emptyset$$

Fuzziness of Fuzzy Sets

The fuzziness of a fuzzy set can be measured using the concept of **entropy** (DeLuca and Termini[1]).

Let us consider *X* = *{x*1*, x*2*, . . . , xn}* be a discrete universe of discourse.

The entropy of a fuzzy set *A*(*x*) could be determined as follows:

$$H(A) = -\frac{1}{n}\sum_{i=1}^{n}[\mu_A(x_i)log\{\mu_A(x_i)\} + \{1-\mu_A(x_i)\}log\{1-\mu_A(x_i)\}]$$

Note# Entropy measures the level of disorder or uncertainty.

[1] A. DeLuca, S. Termini, "A definition of non-probabilistic entropy in the setting of fuzzy set theory," *Information and Control*, vol. 20, pp. 301–312, 1971.

Example

Let $A(x)$ be a fuzzy set in a discrete universe of discourse as given below.

$A(x) = \{(x1, 0.1), (x2, 0.3), (x3, 0.4), (x4, 0.5)\}$.

Calculate its entropy value.

The value of entropy, $H(A) = -\frac{1}{4}[\{0.1 \times log(0.1) + 0.9 \times log(0.9)\} + \{0.3 \times log(0.3) + 0.7 \times log(0.7)\} + \{0.4 \times log(0.4) + 0.6 \times log(0.6)\} + \{0.5 \times log(0.5) + 0.5 \times log(0.5)\}] = 0.2499$

Inaccuracy of Fuzzy Sets

The inaccuracy of the fuzzy set *B*(*x*) can be measured with respect to the fuzzy set *A*(*x*) as follows[2].

$$I(A;B) = -\frac{1}{n}\sum_{i=1}^{n}[\mu_A(x_i)log(\mu_B(x_i)) + (1-\mu_A(x_i))log(1-\mu_B(x_i))]$$

[2] R. Verma, B.D. Sharma, ”A measure of inaccuracy between two fuzzy sets,” *Cybernetics and Information Technologies (Bulgarian Academy of Sciences)*, vol. 11, no. 2, pp. 13–23, 2011

Example

Let us consider the following two fuzzy sets in the same discrete universe of discourse:

$A(x) = \{(x1, 0.1), (x2, 0.2), (x3, 0.3), (x4, 0.4)\}$
$B(x) = \{(x1, 0.5), (x2, 0.7), (x3, 0.8), (x4, 0.9)\}$

Determine inaccuracy of the fuzzy set $B(x)$ with respect to the fuzzy set $A(x)$.

The inaccuracy of $B(x)$ with respect to $A(x)$, $I(A;B) = -\frac{1}{4}[\{0.1 \times log(0.5) + 0.9 \times log(0.5)\} + \{0.2 \times log(0.7) + 0.8 \times log(0.3)\} + \{0.3 \times log(0.8) + 0.7 \times log(0.2)\} + \{0.4 \times log(0.9) + 0.6 \times log(0.1)\}] = 0.4717$

Thank you