

# Fuzzy Logic : Introduction

**Debasis Samanta**

IIT Kharagpur

*dsamanta@iitkgp.ac.in*

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# What is Fuzzy logic?

- Fuzzy logic is a mathematical language to **express** something.  
This means it has grammar, syntax, semantic like a language for communication.
- There are some other mathematical languages also known
  - **Relational algebra** (operations on sets)
  - **Boolean algebra** (operations on Boolean variables)
  - **Predicate logic** (operations on well formed formulae (wff), also called predicate propositions)
- **Fuzzy logic deals with Fuzzy set.**

# A brief history of Fuzzy Logic

- First time introduced by **Lotfi Abdelli Zadeh** (1965), University of California, Berkley, USA (1965).



- He is fondly nick-named as **LAZ**

# A brief history of Fuzzy logic



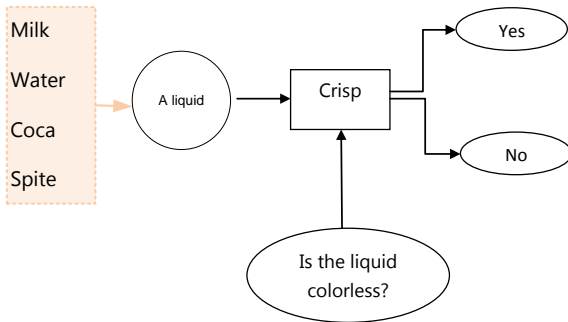
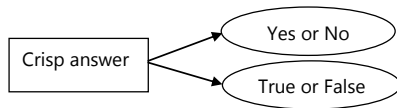
- 1 Dictionary meaning of **fuzzy** is not clear, noisy etc.

Example: Is the picture on this slide is fuzzy?

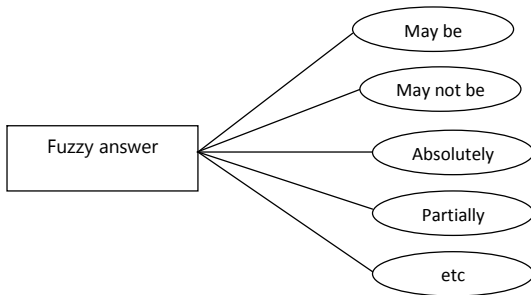
- 2 Antonym of fuzzy is **crisp**

Example: Are the chips crisp?

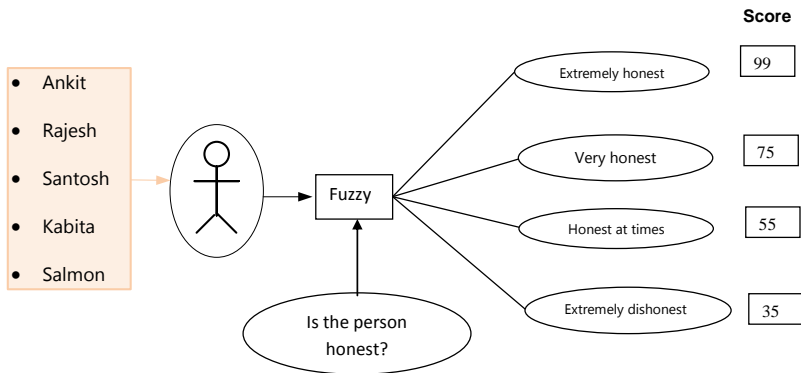
# Example : Fuzzy logic vs. Crisp logic




# Example : Fuzzy logic vs. Crisp logic



# Example : Fuzzy logic vs. Crisp logic



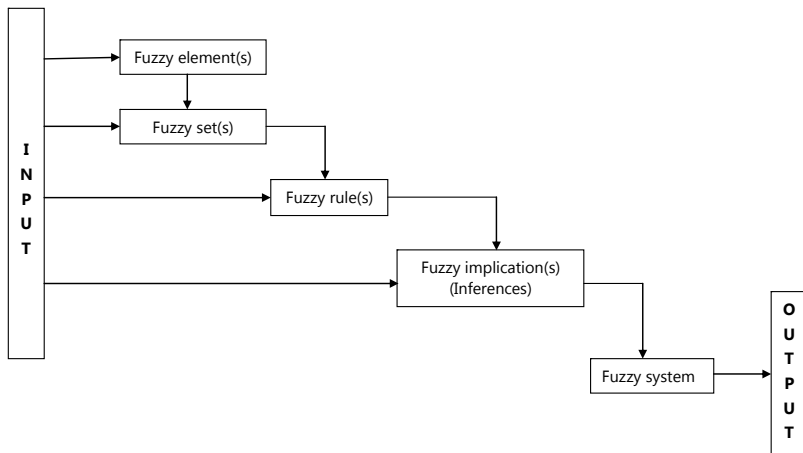
# World is fuzzy!



**Our world is better  
described with  
fuzzily!**



# Concept of fuzzy system



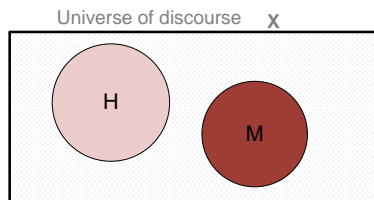
# Concept of fuzzy set

To understand the concept of **fuzzy set** it is better, if we first clear our idea of **crisp set**.

$X$  = The entire population of India.

$H$  = All Hindu population =  $\{ h_1, h_2, h_3, \dots, h_L \}$

$M$  = All Muslim population =  $\{ m_1, m_2, m_3, \dots, m_N \}$



Here, All are the sets of finite numbers of individuals.

Such a set is called **crisp set**.

# Example of fuzzy set

Let us discuss about fuzzy set.

$X$  = All students in IT60108.

$S$  = All **Good students**.

$S = \{ (s, g) \mid s \in X \}$  and  $g(s)$  is a measurement of goodness of the student  $s$ .

**Example:**

$S = \{ (\text{Rajat}, 0.8), (\text{Kabita}, 0.7), (\text{Salman}, 0.1), (\text{Ankit}, 0.9) \}$  etc.

# Fuzzy set vs. Crisp set

Crisp Set	Fuzzy Set
1. $S = \{ s \mid s \in X \}$	1. $F = (s, \mu) \mid s \in X \text{ and } \mu(s) \text{ is the degree of } s.$
2. It is a collection of elements.	2. It is collection of ordered pairs.
3. Inclusion of an element $s \in X$ into $S$ is crisp, that is, has strict boundary <b>yes</b> or <b>no</b> .	3. Inclusion of an element $s \in X$ into $F$ is fuzzy, that is, if present, then with a degree of <b>membership</b> .

# Fuzzy set vs. Crisp set

**Note:** A crisp set is a fuzzy set, but, a fuzzy set is not necessarily a crisp set.

Example:

$$H = \{ (h_1, 1), (h_2, 1), \dots, (h_L, 1) \}$$

$$\text{Person} = \{ (p_1, 1), (p_2, 0), \dots, (p_N, 1) \}$$

In case of a crisp set, the elements are with extreme values of degree of membership namely either 1 or 0.

How to decide the degree of memberships of elements in a fuzzy set?

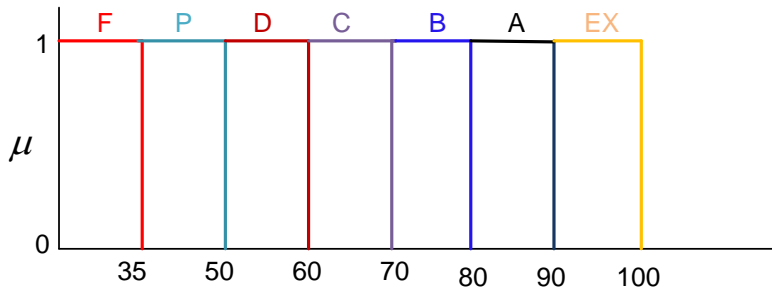
City	Bangalore	Bombay	Hyderabad	Kharagpur	Madras	Delhi
DoM	0.95	0.90	0.80	0.01	0.65	0.75

How the cities of **comfort** can be judged?

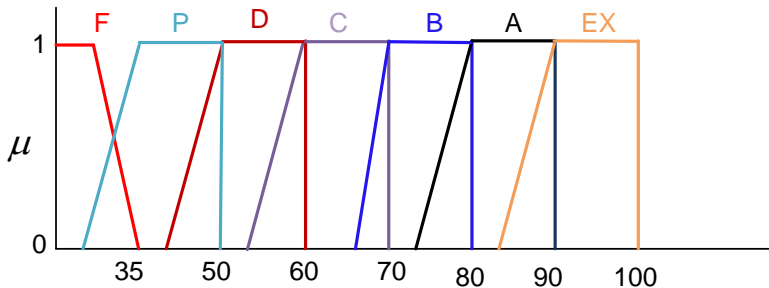
# Example: Course evaluation in a crisp way

- 1  $EX = \text{Marks} \geq 90$
- 2  $A = 80 \leq \text{Marks} < 90$
- 3  $B = 70 \leq \text{Marks} < 80$
- 4  $C = 60 \leq \text{Marks} < 70$
- 5  $D = 50 \leq \text{Marks} < 60$
- 6  $P = 35 \leq \text{Marks} < 50$
- 7  $F = \text{Marks} < 35$

# Example: Course evaluation in a crisp way



# Example: Course evaluation in a fuzzy way





# Few examples of fuzzy set

- High Temperature
- Low Pressure
- Color of Apple
- Sweetness of Orange
- Weight of Mango

Note: Degree of membership values lie in the range  $[0...1]$ .

# Some basic terminologies and notations

## Definition 1: Membership function (and Fuzzy set)

If  $X$  is a universe of discourse and  $x \in X$ , then a fuzzy set  $A$  in  $X$  is defined as a set of ordered pairs, that is

$A = \{(x, \mu_A(x)) | x \in X\}$  where  $\mu_A(x)$  is called the **membership function** for the fuzzy set  $A$ .

### Note:

$\mu_A(x)$  map each element of  $X$  onto a membership grade (or membership value) between 0 and 1 (both inclusive).

### Question:

How (and who) decides  $\mu_A(x)$  for a Fuzzy set  $A$  in  $X$ ?

# Some basic terminologies and notations

## Example:

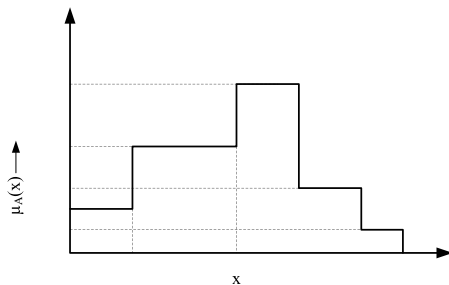
$X$  = All cities in India

$A$  = City of comfort

$A = \{(\text{New Delhi}, 0.7), (\text{Bangalore}, 0.9), (\text{Chennai}, 0.8), (\text{Hyderabad}, 0.6), (\text{Kolkata}, 0.3), (\text{Kharagpur}, 0)\}$

# Membership function with discrete membership values

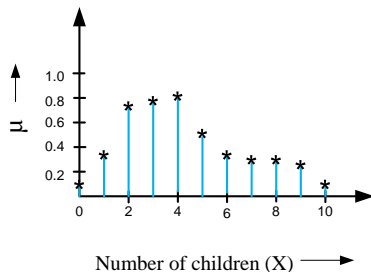
The membership values may be of discrete values.



A fuzzy set with discrete values of  $\mu$

# Membership function with discrete membership values

Either elements or their membership values (or both) also may be of discrete values.



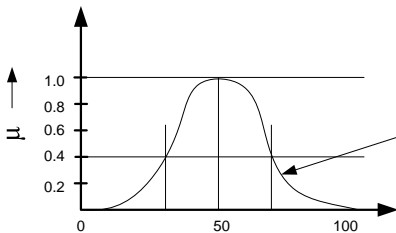
$$A = \{(0,0.1), (1,0.30), (2,0.78), \dots, (10,0.1)\}$$

Note : X = discrete value

How you measure happiness ??

A = "Happy family"

# Membership function with continuous membership values



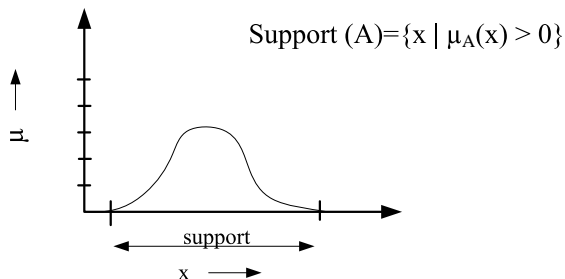
$$\mu_B(x) = \frac{1}{1 + \left(\frac{x-50}{10}\right)^4}$$

$$B = \{(x, \mu_B(x))\}$$

Note :  $x = \text{real value}$   
 $= \mathbb{R}^+$

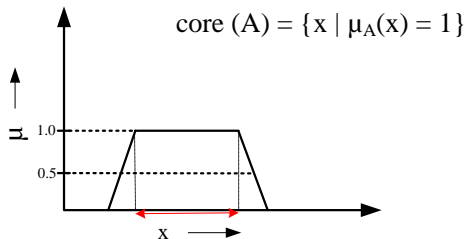
# Fuzzy terminologies: Support

**Support:** The support of a fuzzy set  $A$  is the set of all points  $x \in X$  such that  $\mu_A(x) > 0$



# Fuzzy terminologies: Core

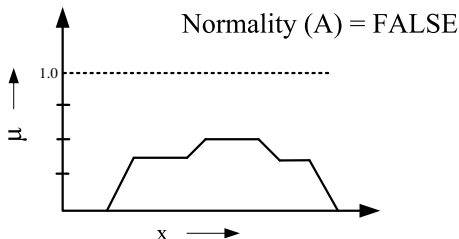
**Core:** The core of a fuzzy set  $A$  is the set of all points  $x$  in  $X$  such that  $\mu_A(x) = 1$





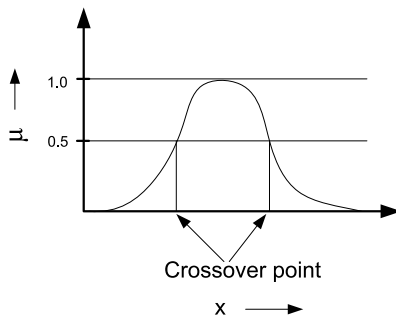
# Fuzzy terminologies: Normality

**Normality** : A fuzzy set  $A$  is a normal if its core is non-empty. In other words, we can always find a point  $x \in X$  such that  $\mu_A(x) = 1$ .



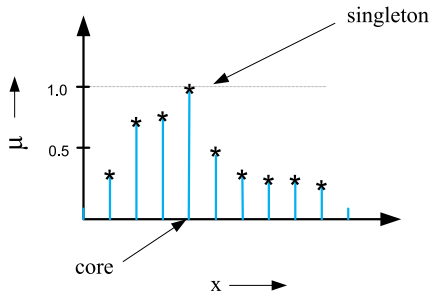
# Fuzzy terminologies: Crossover points

**Crossover point** : A crossover point of a fuzzy set  $A$  is a point  $x \in X$  at which  $\mu_A(x) = 0.5$ . That is  
 $\text{Crossover}(A) = \{x | \mu_A(x) = 0.5\}$ .



# Fuzzy terminologies: Fuzzy Singleton

**Fuzzy Singleton** : A fuzzy set whose support is a single point in  $X$  with  $\mu_A(x) = 1$  is called a fuzzy singleton. That is  $|A| = |\{x \mid \mu_A(x) = 1\}| = 1$ . Following fuzzy set is not a fuzzy singleton.



# Fuzzy terminologies: $\alpha$ -cut and strong $\alpha$ -cut

**$\alpha$ -cut and strong  $\alpha$ -cut :**

The  $\alpha$ -cut of a fuzzy set  $A$  is a crisp set defined by

$$A_{\alpha} = \{x \mid \mu_A(x) \geq \alpha \}$$

Strong  $\alpha$ -cut is defined similarly :

$$A_{\alpha}' = \{x \mid \mu_A(x) > \alpha \}$$

**Note :**  $\text{Support}(A) = A_0'$  and  $\text{Core}(A) = A_1$ .

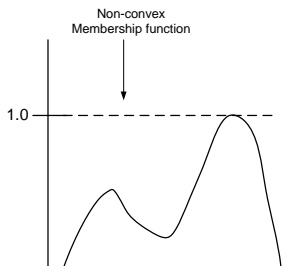
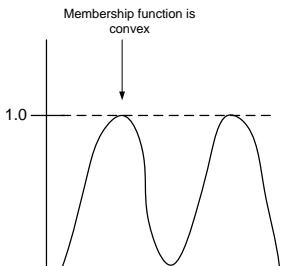
# Fuzzy terminologies: Convexity

**Convexity** : A fuzzy set  $A$  is convex if and only if for any  $x_1$  and  $x_2 \in X$  and any  $\lambda \in [0, 1]$

$$\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$

**Note :**

- $A$  is convex if all its  $\alpha$ - level sets are convex.
- Convexity ( $A_\alpha$ )  $\implies A_\alpha$  is composed of a single line segment only.



## Bandwidth :

For a normal and convex fuzzy set, the bandwidth (or width) is defined as the distance the two unique crossover points:

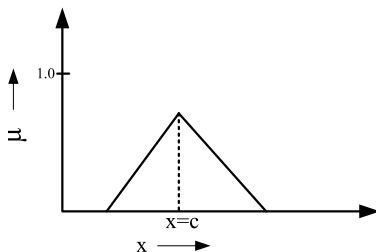
$$\text{Bandwidth}(A) = |x_1 - x_2|$$

where  $\mu_A(x_1) = \mu_A(x_2) = 0.5$

# Fuzzy terminologies: Symmetry

## Symmetry :

A fuzzy set  $A$  is symmetric if its membership function around a certain point  $x = c$ , namely  $\mu_A(x + c) = \mu_A(x - c)$  for all  $x \in X$ .



# Fuzzy terminologies: Open and Closed

A fuzzy set  $A$  is

## Open left

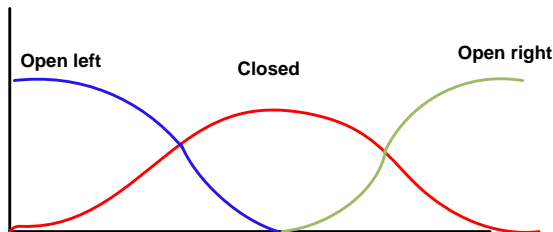
If  $\lim_{x \rightarrow -\infty} \mu_A(x) = 1$  and  $\lim_{x \rightarrow +\infty} \mu_A(x) = 0$

## Open right:

If  $\lim_{x \rightarrow -\infty} \mu_A(x) = 0$  and  $\lim_{x \rightarrow +\infty} \mu_A(x) = 1$

## Closed

If :  $\lim_{x \rightarrow -\infty} \mu_A(x) = \lim_{x \rightarrow +\infty} \mu_A(x) = 0$





# Fuzzy vs. Probability

**Fuzzy** : When we say about certainty of a thing

Example: A patient come to the doctor and he has to diagnose so that medicine can be prescribed.

Doctor prescribed a medicine with certainty 60% that the patient is suffering from flue. So, the disease will be cured with certainty of 60% and uncertainty 40%. Here, in stead of flue, other diseases with some other certainties may be.

**Probability**: When we say about the chance of an event to occur

Example: India will win the T20 tournament with a chance 60% means that out of 100 matches, India own 60 matches.

# Prediction vs. Forecasting

The Fuzzy vs. Probability is analogical to Prediction vs. Forecasting

**Prediction** : When you start guessing about things.

**Forecasting** : When you take the information from the past job and apply it to new job.

**The main difference:**

**Prediction** is based on the best guess from experiences.

**Forecasting** is based on data you have actually recorded and packed from previous job.

# Fuzzy Membership Functions

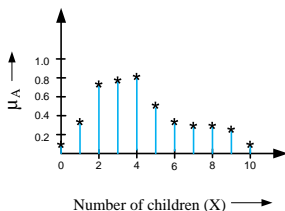
# Fuzzy membership functions

A fuzzy set is completely characterized by its membership function (sometimes abbreviated as *MF* and denoted as  $\mu$ ). So, it would be important to learn how a membership function can be expressed (mathematically or otherwise).

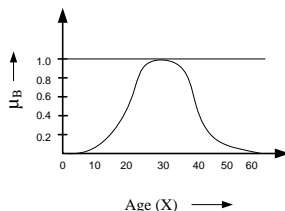
**Note:** A membership function can be on

- (a) a discrete universe of discourse and
- (b) a continuous universe of discourse.

**Example:**



A = Fuzzy set of "Happy family"

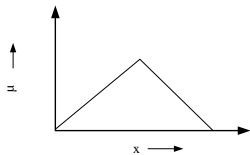


B = "Young age"

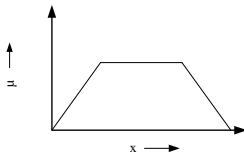
# Fuzzy membership functions

So, membership function on a discrete universe of course is trivial. However, a membership function on a continuous universe of discourse needs a special attention.

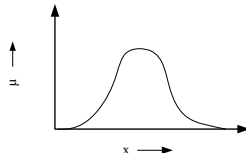
Following figures shows a typical examples of membership functions.



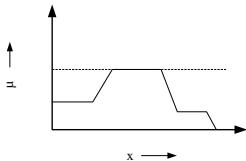
< triangular >



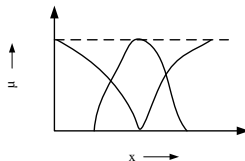
< trapezoidal >



< curve >



< non-uniform >



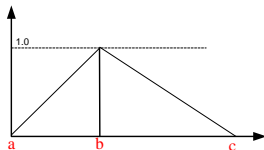
< non-uniform >

# Fuzzy MFs : Formulation and parameterization

In the following, we try to parameterize the different MFs on a continuous universe of discourse.

**Triangular MFs** : A triangular MF is specified by three parameters  $\{a, b, c\}$  and can be formulated as follows.

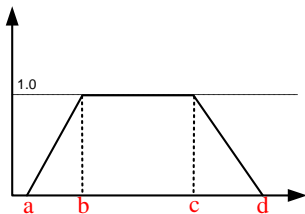
$$\text{triangle}(x; a, b, c) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ \frac{c-x}{c-b} & \text{if } b \leq x \leq c \\ 0 & \text{if } c \leq x \end{cases} \quad (1)$$



# Fuzzy MFs: Trapezoidal

A trapezoidal MF is specified by four parameters  $\{a, b, c, d\}$  and can be defined as follows:

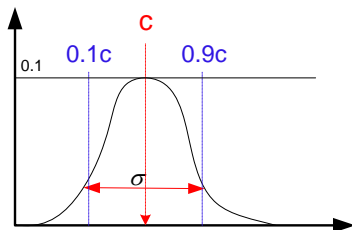
$$\text{trapeziod}(x; a, b, c, d) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } b \leq x \leq c \\ \frac{d-x}{d-c} & \text{if } c \leq x \leq d \\ 0 & \text{if } d \leq x \end{cases} \quad (2)$$



# Fuzzy MFs: Gaussian

A Gaussian MF is specified by two parameters  $\{c, \sigma\}$  and can be defined as below:

$$\text{gaussian}(x; c, \sigma) = e^{-\frac{1}{2}(\frac{x-c}{\sigma})^2}.$$

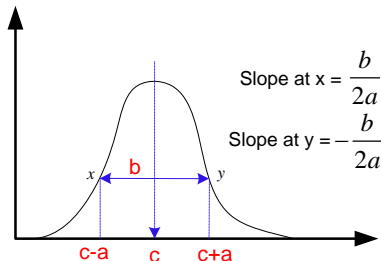




# Fuzzy MFs: Generalized bell

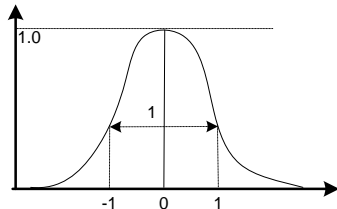
It is also called **Cauchy MF**. A generalized bell MF is specified by three parameters  $\{a, b, c\}$  and is defined as:

$$\text{bell}(x; a, b, c) = \frac{1}{1 + \left| \frac{x-c}{a} \right|^{2b}}$$

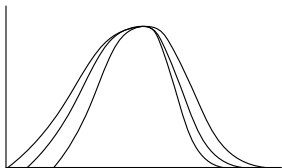


# Example: Generalized bell MFs

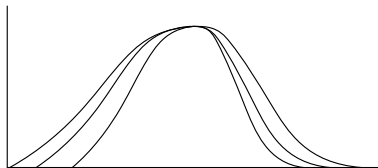
Example:  $\mu(x) = \frac{1}{1+x^2}$  ;  
 $a = b = 1$  and  $c = 0$ ;



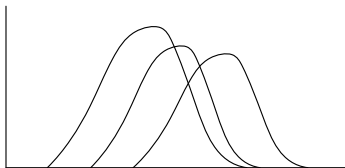
# Generalized bell MFs: Different shapes



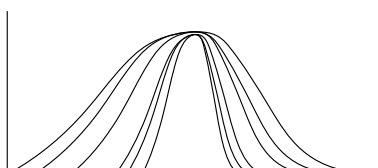
Changing a



Changing b



Changing a

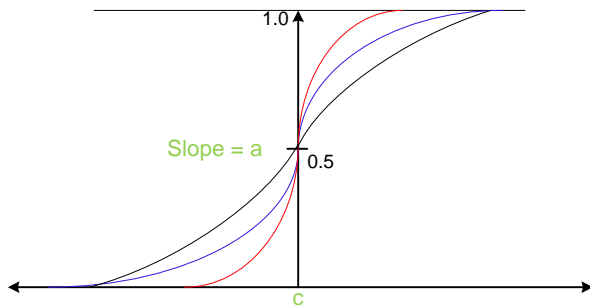


Changing a and b

# Fuzzy MFs: Sigmoidal MFs

Parameters:  $\{a, c\}$  ; where  $c$  = crossover point and  $a$  = slope at  $c$ ;

$$\text{Sigmoid}(x;a,c) = \frac{1}{1 + e^{-[\frac{a}{x-c}]}}$$



# Fuzzy MFs : Example

Example : Consider the following grading system for a course.

Excellent = Marks  $\leq 90$

Very good =  $75 \leq \text{Marks} \leq 90$

Good =  $60 \leq \text{Marks} \leq 75$

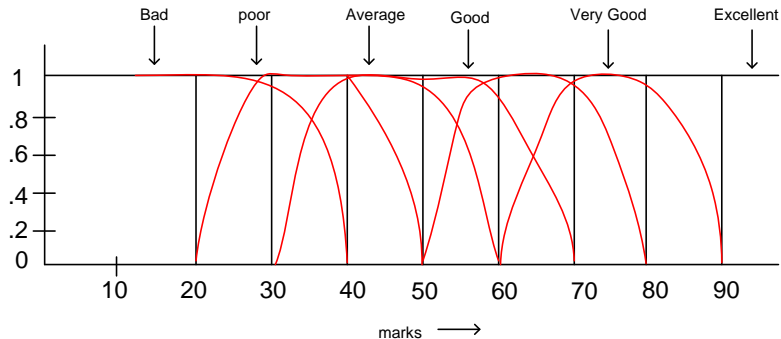
Average =  $50 \leq \text{Marks} \leq 60$

Poor =  $35 \leq \text{Marks} \leq 50$

Bad = Marks  $\leq 35$

# Grading System

A fuzzy implementation will look like the following.



You can decide a standard fuzzy MF for each of the **fuzzy grade**.

# Operations on Fuzzy Sets

# Basic fuzzy set operations: Union

**Union ( $A \cup B$ ):**

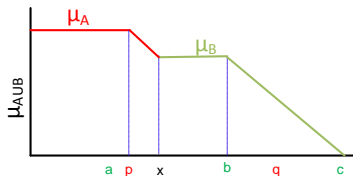
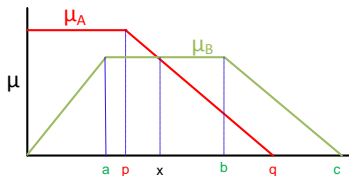
$$\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

Example:

$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$  and

$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\};$

$C = A \cup B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 0.5)\}$





# Basic fuzzy set operations: Intersection

## Intersection ( $A \cap B$ ):

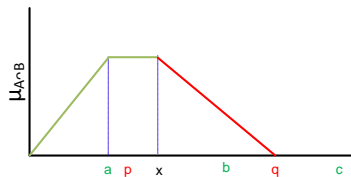
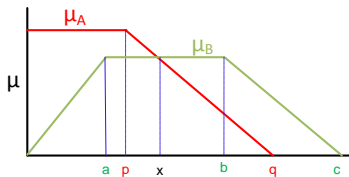
$$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

Example:

$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$  and

$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\};$

$C = A \cap B = \{(x_1, 0.2), (x_2, 0.1), (x_3, 0.4)\}$



# Basic fuzzy set operations: Complement

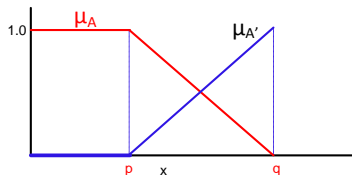
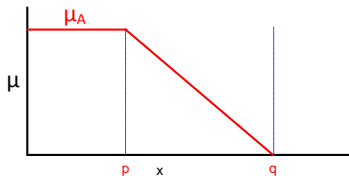
## Complement ( $A^C$ ):

$$\mu_{A^C}(x) = 1 - \mu_A(x)$$

Example:

$$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$$

$$C = A^C = \{(x_1, 0.5), (x_2, 0.9), (x_3, 0.6)\}$$



# Basic fuzzy set operations: Products

**Algebraic product or Vector product ( $A \bullet B$ ):**

$$\mu_{A \bullet B}(X) = \mu_A(X) \bullet \mu_B(X)$$

**Scalar product ( $\alpha \times A$ ):**

$$\mu_{\alpha A}(X) = \alpha \cdot \mu_A(X)$$

# Basic fuzzy set operations: Sum and Difference

**Sum ( $A + B$ ):**

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$$

**Difference ( $A - B = A \cap B^C$ ):**

$$\mu_{A-B}(x) = \mu_{A \cap B^C}(x)$$

**Disjunctive sum:  $A \oplus B = (A^C \cap B) \cup (A \cap B^C)$**

**Bounded Sum:  $|A(x) \oplus B(x)|$**

$$\mu_{|A(x) \oplus B(x)|} = \min\{1, \mu_A(x) + \mu_B(x)\}$$

**Bounded Difference:  $|A(x) \ominus B(x)|$**

$$\mu_{|A(x) \ominus B(x)|} = \max\{0, \mu_A(x) + \mu_B(x) - 1\}$$

# Basic fuzzy set operations: Equality and Power

**Equality ( $A = B$ ):**

$$\mu_A(X) = \mu_B(X)$$

**Power of a fuzzy set  $A^\alpha$ :**

$$\mu_{A^\alpha}(X) = \{\mu_A(X)\}^\alpha$$

- If  $\alpha < 1$ , then it is called *dilation*
- If  $\alpha > 1$ , then it is called *concentration*

# Basic fuzzy set operations: Cartesian product

## Cartesian Product ( $A \times B$ ):

$$\mu_{A \times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\}$$

### Example 3:

$$A(x) = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5), (x_4, 0.6)\}$$

$$B(y) = \{(y_1, 0.8), (y_2, 0.6), (y_3, 0.3)\}$$

$$A \times B = \min\{\mu_A(x), \mu_B(y)\} =$$

	$y_1$	$y_2$	$y_3$
$x_1$	0.2	0.2	0.2
$x_2$	0.3	0.3	0.3
$x_3$	0.5	0.5	0.3
$x_4$	0.6	0.6	0.3

# Properties of fuzzy sets

## Commutativity :

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

## Associativity :

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

## Distributivity :

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

# Properties of fuzzy sets

## Idempotence :

$$A \cup A = A$$

$$A \cap A = \emptyset$$

$$A \cup \emptyset = A$$

$$A \cap \emptyset = \emptyset$$

## Transitivity :

If  $A \subseteq B, B \subseteq C$  then  $A \subseteq C$

## Involution :

$$(A^c)^c = A$$

## De Morgan's law :

$$(A \cap B)^c = A^c \cup B^c$$

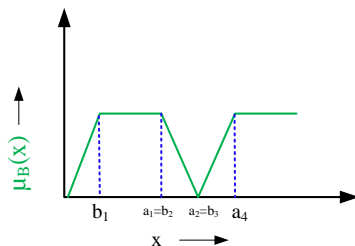
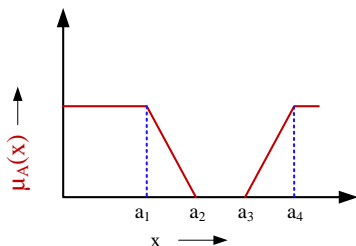
$$(A \cup B)^c = A^c \cap B^c$$



# Few Illustrations on Fuzzy Sets

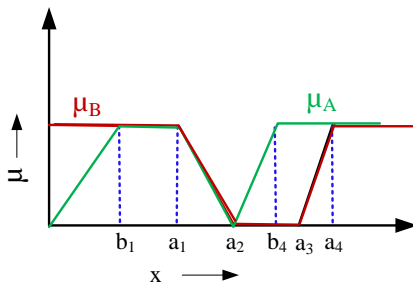
# Example 1: Fuzzy Set Operations

Let A and B are two fuzzy sets defined over a universe of discourse X with membership functions  $\mu_A(x)$  and  $\mu_B(x)$ , respectively. Two MFs  $\mu_A(x)$  and  $\mu_B(x)$  are shown graphically.



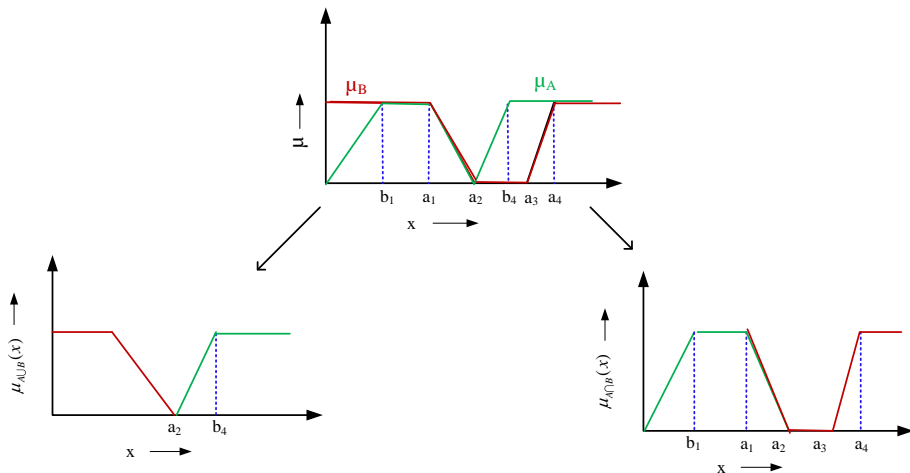
# Example 1: Plotting two sets on the same graph

Let's plot the two membership functions on the same graph



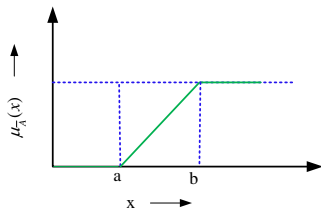
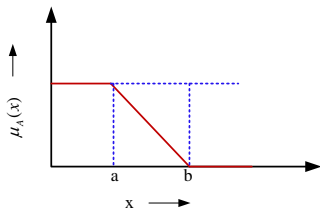
# Example 1: Union and Intersection

The plots of union  $A \cup B$  and intersection  $A \cap B$  are shown in the following.



# Example 1: Intersection

The plots of union  $\mu_{\bar{A}}(x)$  of the fuzzy set  $A$  is shown in the following.



# Fuzzy set operations: Practice

Consider the following two fuzzy sets  $A$  and  $B$  defined over a universe of discourse  $[0,5]$  of real numbers with their membership functions

$$\mu_A(x) = \frac{x}{1+x} \text{ and } \mu_B(x) = 2^{-x}$$

Determine the membership functions of the following and draw them graphically.

i.  $\bar{A}, \bar{B}$

ii.  $A \cup B$

iii.  $A \cap B$

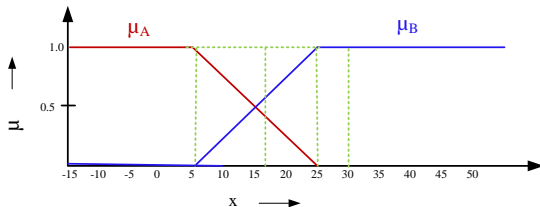
iv.  $(A \cup B)^c$  [Hint: Use De' Morgan law]

## Example 2: A real-life example

Two fuzzy sets  $A$  and  $B$  with membership functions  $\mu_A(x)$  and  $\mu_B(x)$ , respectively defined as below.

$A = \text{Cold climate}$  with  $\mu_A(x)$  as the MF.

$B = \text{Hot climate}$  with  $\mu_B(x)$  as the M.F.



Here,  $X$  being the universe of discourse representing entire range of temperatures.

## Example 2: A real-life example

What are the fuzzy sets representing the following?

- 1 **Not cold climate**
- 2 **Not hot climate**
- 3 **Extreme climate**
- 4 **Pleasant climate**

Note: Note that "Not cold climate"  $\neq$  "Hot climate" and vice-versa.



## Example 2 : A real-life example

Answer would be the following.

❶ **Not cold climate**

$\bar{A}$  with  $1 - \mu_A(x)$  as the MF.

❷ **Not hot climate**

$\bar{B}$  with  $1 - \mu_B(x)$  as the MF.

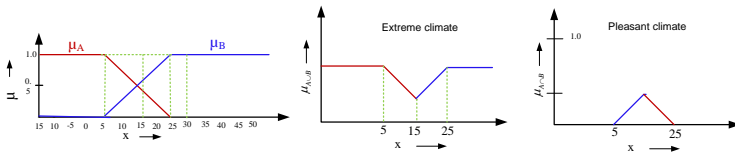
❸ **Extreme climate**

$A \cup B$  with  $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$  as the MF.

❹ **Pleasant climate**

$A \cap B$  with  $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$  as the MF.

The plot of the MFs of  $A \cup B$  and  $A \cap B$  are shown in the following.



# Few More on Membership Functions

# Generation of MFs

Given a membership function of a fuzzy set representing a **linguistic hedge**, we can derive many more MFs representing several other linguistic hedges using the concept of **Concentration** and **Dilation**.

- **Concentration:**

$$A^k = [\mu_A(x)]^k ; k > 1$$

- **Dilation:**

$$A^k = [\mu_A(x)]^k ; k < 1$$

Example : Age = { Young, Middle-aged, Old }

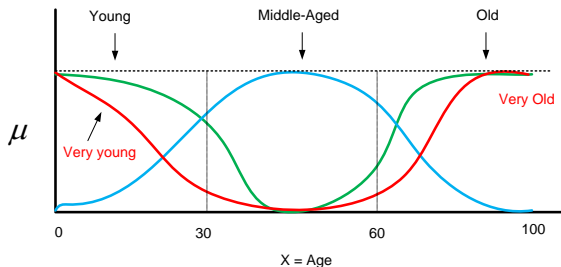
Thus, corresponding to Young, we have : Not young, Very young, Not very young and so on.

Similarly, with Old we can have : old, very old, very very old, extremely old etc.

Thus, **Extremely old** =  $((old)^2)^2$  and so on

Or, **More or less old** =  $A^{0.5} = (old)^{0.5}$

# Linguistic variables and values



$$\mu_{\text{young}}(x) = \text{bell}(x, 20, 2, 0) = \frac{1}{1 + (\frac{x}{20})^4}$$

$$\mu_{\text{old}}(x) = \text{bell}(x, 30, 3, 100) = \frac{1}{1 + (\frac{x-100}{30})^6}$$

$$\mu_{\text{middle-aged}} = \text{bell}(x, 30, 60, 50)$$

$$\text{Not young} = \overline{\mu_{\text{young}}(x)} = 1 - \mu_{\text{young}}(x)$$

$$\text{Young but not too young} = \mu_{\text{young}}(x) \cap \overline{\mu_{\text{young}}(x)}$$

# Any questions??