Computational Intelligence (CI)

T-norm and S-norm Operators

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T-norm / Triangular norm / S-conorm

- It is a generalized intersection operator.

- The intersection of two fuzzy sets A and B is specified in **general** by a function $\boldsymbol{T} : [\mathbf{0}, \mathbf{1}] \times [\mathbf{0}, \mathbf{1}] \to [\mathbf{0}, \mathbf{1}]$, which aggregates two membership grades as follows:

$$\mu_{A \cap B}(x) = T(\mu_A(x), \mu_B(x)) = \mu_A(x) \tilde{*} \mu_B(x)$$

$\tilde{*}$ is a binary operator for the function T.

Note# 2-place real function is **a function which has a two-dimensional domain and one-dimensional range**.

T-norm / S-conorm

T-norm operator is a two-place function $T(.,.)$satisfying following requirements

$$
\begin{array}{ll}
T(0,0) = 0,\ T(a,1) = T(1,a) = a & \text{(boundary)} \\
T(a,b) \leq T(c,d) \text{ if } a \leq c \text{ and } b \leq d & \text{(monotonicity)} \\
T(a,b) = T(b,a) & \text{(commutativity)} \\
T(a,T(b,c)) = T(T(a,b),c) & \text{(associativity).}
\end{array}
$$

Boundary: imposes correct generalization to crisp sets.

Monotonicity: implies that decrease in membership values in A and B can't produce an increase in membership value in $A \cap B$

Commutativity: indicates that the operator is indifferent to the order of the fuzzy sets to be combined.

Associativity: allows intersection of any number of fuzzy sets

T-norm / S-conorm

Four of the most frequently used T-norm operators are

$$
\begin{array}{ll}
\textbf{Minimum:} & T_{min}(a,b) = \min(a,b) = a \wedge b. \\
\textbf{Algebraic product:} & T_{ap}(a,b) = ab. \\
\textbf{Bounded product:} & T_{bp}(a,b) = 0 \vee (a+b-1). \\
\textbf{Drastic product:} & T_{dp}(a,b) = \begin{cases} a, & \text{if } b = 1. \\ b, & \text{if } a = 1. \\ 0, & \text{if } a,b < 1. \end{cases}
\end{array}
$$

Note# from the plot of above T-norm operators it can be observed that

$$T_{dp}(a,b) \leq T_{bp}(a,b) \leq T_{ap}(a,b) \leq T_{min}(a,b).$$

T-norm / S-conorm

Task: Write script to find intersection of A and B using T-norm operators $T_{min}, T_{ap}, T_{bp}, T_{dp}$.

A = .6/a + .3/b + .7/c + .6/d + .5/e + .4/f + .9/g
B = .5/a + .3/b + 1/c + .5/d + .6/e + .4/f + 1/g

Discrete universe of discourse X = {a, b, c, d, e, f, g} can be taken as {1, 2, 3, 4, 5, 6, 7}

T-norm

Task: Consider fuzzy sets A = TRI_MF(x, [1,3,6]) and B = TRI_MF(x, [2,5,7]). Find intersection of A and B using T-norm operators $T_{min}, T_{ap}, T_{bp}, T_{dp}$.

S-norm/T-conorm

- It is a generalized Union operator.

- The Union of two fuzzy sets A and B is specified in **general** by a function $\boldsymbol{S}: [\mathbf{0}, \mathbf{1}] \times [\mathbf{0}, \mathbf{1}] \rightarrow [\mathbf{0}, \mathbf{1}]$, which aggregates two membership grades as follows:

$$\mu_{A \cup B}(x) = S(\mu_A(x), \mu_B(x)) = \mu_A(x) \tilde{+} \mu_B(x)$$

$\tilde{+}$ is a binary operator for the function S.

S-norm/T-conorm

S-norm operator is a two-place function $S(.,.)$satisfying following requirements

$$
\begin{array}{ll}
S(1,1)=1,\ S(0,a)=S(a,0)=a & \text{(boundary)} \\
S(a,b)\leq S(c,d) \text{ if } a\leq c \text{ and } b\leq d & \text{(monotonicity)} \\
S(a,b)=S(b,a) & \text{(commutativity)} \\
S(a,S(b,c))=S(S(a,b),c) & \text{(associativity).}
\end{array}
$$

S-norm/T-conorm

Four of the most frequently used S-norm operators are

Maximum: $S(a,b) = \max(a,b) = a \vee b.$

Algebraic sum: $S(a,b) = a + b - ab.$

Bounded sum: $S(a,b) = 1 \wedge (a+b).$

Drastic sum: $S(a,b) = \begin{cases} a, & \text{if } b = 0. \\ b, & \text{if } a = 0. \\ 1, & \text{if } a,b > 0. \end{cases}$

Note# from the plot of above S-norm operators it can be observed that

$$S_{max}(a,b) \leq S_{as}(a,b) \leq S_{bs}(a,b) \leq S_{ds}$$

S-norm

Task: Consider fuzzy sets A = TRI_MF(x, [1,3,6]) and B = TRI_MF(x, [2,5,7]). Find Union of A and B using S-norm operators $S_{max}, S_{as}, S_{bs}, S_{ds}$.

S-norm/T-conorm

Task: Write script to find Union of A and B using S-norm operators $S_{max}, S_{as}, S_{bs}, S_{ds}$.

A = .6/a + .3/b + .7/c + .6/d + .5/e + .4/f + .9/g
B = .5/a + .3/b + 1/c + .5/d + .6/e + .4/f + 1/g

Generalized De Morgan's Law

De Morgan's Laws

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$$
$$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

Generalized De Morgan's Law: T-norms T(.,.) and T-conorms S(,.,) are duals which support the generalization of DeMorgan's Law.

$$T(a,b) = N(S(N(a), N(b)))$$
$$S(a,b) = N(T(N(a), N(b)))$$

$$a \tilde{*} b = N(N(a) \tilde{+} N(b))$$
$$a \tilde{+} b = N(N(a) \tilde{*} N(b))$$

Where N(.) is the complement operator.

Parameterized T-norm and S-norm

Yager's Class of T-norm Operator

For q > 0

$$T_Y(a,b,q) = 1 - \min\{1, [(1-a)^q + (1-b)^q]^{1/q}\}$$
$$S_Y(a,b,q) = \min\{1, (a^q + b^q)^{1/q}\}.$$

Sugeno's Class of S-norm Operator

For $\lambda \geq -1$,

$$\begin{cases} T_S(a,b,\lambda) = \max\{0, (\lambda+1)(a+b-1) - \lambda ab\}, \\ S_S(a,b,\lambda) = \min\{1, a+b-\lambda ab\}. \end{cases}$$

Thank you