MATHEMATICAL TREATMENT OF THE THEORY OF CONSUMER'S CHOICE

In this appendix we will study the mathematical derivation of consumer's equilibrium, the demand function and Slutsky equation which describes decomposition of price effect into substitution effect and income effect. We will also solve some numerical problems based on these concepts.

Mathematical Derivation of Conditions for Consumer's Equilibrium : Lagrangian Method

Those who know mathematics can understand the condition for consumer's equilibrium easily with the Lagrangian multiplier method which seeks to solve the maximisation and minimisation problems subject to some constraints.

Consider a consumer who intends to spends his entire given income (I) on the two goods, X and Y. P_x and P_y are the prices of two goods respectively which the consumer takes as given and constant for him. Consumer's budget constraint can be written by means of the following budget equation.

$$I = P_x \cdot X + P_y \cdot Y \qquad \text{...(1)}$$

where X and Y are the quantities of the two goods X and Y. The budget equation (1) describes that the expenditure on the two goods $P_x.X + P_y \cdot Y$ equals (more generally speaking cannot exceed) his given income I.

The general ordinal utility function visualised by indifference curve analysis is given by

$$U = f(X, Y) \qquad \text{...(2)}$$

Equation (2) postulates that total utility (U) derived from consumption depends on the quantities X and Y of the two goods X and Y respectively. Note that in the indifference curve ordinal utility approach, total utility is not the *sum* of independent utilities derived separately from the two goods X and Y.

The aim of the consumer is to purchase such quantities of the two goods which maximise his total utility as given by the equation $U = f(x, y)$ subject to the budget constraint as given by equation (1). So this is a *constrained maximisation problem* which can be solved through a mathematical technique called Lagrangian method which is explained below.

First Step in this method is to form Lagrangian function from the above two equations which is as under :

$$L = f(X, Y) + \lambda(I - P_x X - P_y Y) \qquad \text{...(3)}$$

where λ is Lagrangian multiplier. For maximisation of utility it is necessary that partial derivative

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of L with respect to x, y and λ of the above Lagraugian function be zero. Thus, for maximisation of utility.

$$\frac{\partial L}{\partial X} = \frac{\partial f}{\partial X} - \lambda P_x = 0 \quad ...(4)$$

$$\frac{\partial L}{\partial Y} = \frac{\partial f}{\partial Y} - \lambda P_y = 0 \quad ...(5)$$

$$\frac{\partial L}{\partial \lambda} = I - P_x X - P_y Y = 0 \quad ...(6)$$

(Note that 'f' in the above equation represents the utility function).

It will be noticed that equation (6) coincides with the budget equation (1). Knowing that $\frac{\partial f}{\partial x} = MU_x$ and $\frac{\partial f}{\partial y} = MU_y$ the above equations can be rewritten as

$$MU_x - \lambda P_x = 0 \quad ...(7)$$

$$MU_y - \lambda P_y = 0 \quad ...(8)$$

$$I - P_x X - P_x Y = 0 \quad ...(9)$$

Rewritting the above equations we have

$$MU_x = \lambda P_x \quad ...(7)$$

$$MU_y = \lambda P_y \quad ...(8)$$

$$I = P_x . X + P_y . Y \quad ...(9)$$

The above three equations can be solved simultaneously to obtain the values of X and Y. In order to do so we divide (7) by (8) we have

$$\frac{MU_x}{MU_y} = MRS_{xy} = \frac{\lambda P_x}{\lambda P_y}$$

or

$$MRS_{xy} = \frac{P_x}{P_y} \quad ....(10)$$

In order to maximise satisfaction subject to the budget constraint, the equation (10) must be met along with the budget constraint equation (9).

It will be recalled that equality of MRS_{xy} with the price ratio is the necessary condition of consumer's equilibrium.

In order to obtain the values of X and Y we find out the value of X or Y from equation (10) and then substitute its value in the budget equation (9) to obtain the value of the other. This will become clear from the numerical problems solved below. Note that Lagrangean multiplier (λ) is equal to the marginal utility or satisfaction obtained from the last rupee spent on each commodity.

Numerical Problem 1

Utility function of an individual is given by $U = f(x, y) = x^{3/4} y^{1/4}$. Find out the optimal quantities of the two goods using Lagrangian method, if it is given that price of good x is Rs. 6 per unit, price of good y is Rs. 3 per unit and income of the individual (I) is equal to Rs. 120.

$120 = 6x + 3y$

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$(120 - 6x - 3Y = 0$

Solution:

Given : $U = x^{3/4} y^{1/4}$

$P_x = 6$, $P_y = 3$ and I = Rs. 120

Lagrangian expression for the above problem is

$$L = x^{3/4} y^{1/4} + \lambda(120 - 6x - 3y)$$

Differentiating the Legrangian function (L) with respect to x, y, λ and setting them equal to zero we have

$$\frac{\partial L}{\partial x} = \frac{3}{4} x^{-1/4} y^{1/4} - 6\lambda = 0 \qquad \text{...(1)}$$

$$\frac{\partial L}{\partial y} = \frac{1}{4} x^{3/4} y^{-3/4} - 3\lambda = 0 \qquad \text{...(2)}$$

$$\frac{\partial L}{\partial \lambda} = 120 - 6x - 3y = 0 \qquad \text{...(3)}$$

Through rearrangement we have

$$\frac{3}{4} x^{-1/4} y^{1/4} = 6\lambda \qquad \text{...(4)}$$

$$\frac{1}{4} x^{3/4} y^{-3/4} = 3\lambda \qquad \text{...(5)}$$

$$6x + 3Y = 120 \qquad \text{...(6)}$$

To solve for x we divide the equation (4) by equation (5). Thus

$$\frac{\frac{3}{4} x^{-1/4} y^{1/4}}{\frac{1}{4} x^{3/4} y^{-\frac{3}{4}}} = \frac{6\lambda}{3\lambda}$$

$$3x^{-1} \cdot y = \frac{6}{3}$$

or $$\frac{3y}{x} = 2 \text{ or } x = \frac{3y}{2}$$

Substituting the value of $x = \frac{3y}{2}$ in the budget equation (6) we have

$$6\frac{3y}{2} + 3y = 120$$

$$12y = 120$$

$$y = 10$$

Now, substituting the value of y = 10 in the budget equation (6) we have

$$6x + 30 = 120$$

$$x = 15$$

Thus the optimal quantities of x and y which maxise utility are $x = 15$ and $y = 10$.

Note. We can also find out the values of λ by substituting the values of x and y either in equation (4) or equation (5).

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Numerical Problem 2

There are two commodities X_1 and X_2 on which a consumer spends his entire income in a day. He has utility function $U = \sqrt{X_1 X_2}$. Find out the optimal quantities of X_1 and X_2 if prices of X_1 and X_2 are Rs. 5 and Rs. 2 respectively and his daily income equals Rs. 500.

Solution

$$I = P_X X + P_Y Y$$

$$\frac{MU_X}{MU_Y} = \frac{P_X}{P_Y}$$

Given utility function : $TU = \sqrt{X_1 X_2}$

or $U = X_1^{1/2} X_2^{1/2}$

and with $P_{x1} = 5$, $P_{x2} = 2$ and I = Rs 500, budget equation is

$$500 = 5X_1 + 2X_2$$

Defferentiating the given utility function with respect ot X_1 and X_2 we have

$$MU = \frac{dTU}{dQ}$$

$$MU_{x1} = \frac{\partial U}{\partial X_1} = \frac{1}{2} X_1^{-1/2} X_2^{1/2} \quad \text{...(1)}$$

$$MU_{x2} = \frac{\partial U}{\partial X_2} = \frac{1}{2} X_1^{1/2} X_2^{-1/2} \quad \text{...(2)}$$

Dividing equations (1) by equation (2)

$$MRS_{xy} = \frac{MU_{x1}}{MU_{x2}} = \frac{\frac{1}{2} X_1^{-1/2} \cdot X_2^{1/2}}{\frac{1}{2} X_1^{1/2} \cdot X_2^{-1/2}} = X_1^{-1} X_2 = \frac{X_2}{X_1}$$

For maximisation of utility it is required that

$$MRS_{xy} = \frac{P_{x1}}{P_{x2}}$$

or

$$\frac{X_2}{X_1} = \frac{5}{2}$$

$$X_2 = \frac{5X_1}{2}$$

Substituting the value of X_2 in the budget equation, we have

$$I = X_1 P_{x1} + X_2 P_{x2} \text{ (Budget Equation)}$$

$$500 = X_1\, 5 + \frac{5X_1 .2}{2}$$

$$10\, X_1 = 500$$

$$X_1 = 50$$

Now substituting the value of X_1 in the budget equation

$$500 = 50 \times 5 + 2X_2$$

$$500 = 250 + 2X_2$$

$$2X_2 = 250$$

$$X_2 = 125$$

Thus the optimal quantities of X_1 and X_2 are 50 and 125 respectively.

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