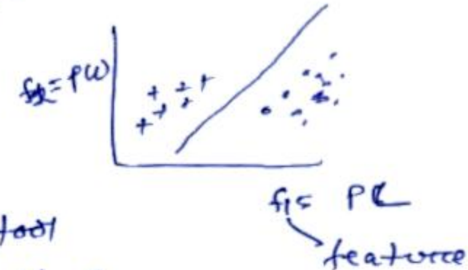
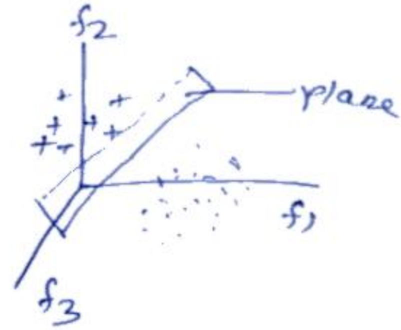
11-1

Linear Algebra



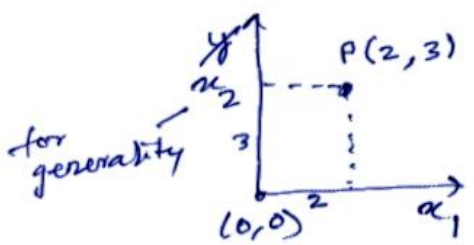


- It gives us mathematical tool to work on a higher dimensional space.

- It also helps to understand and build better intuitions for Machine Learning algorithms.

11.2

Vector / point



point can be

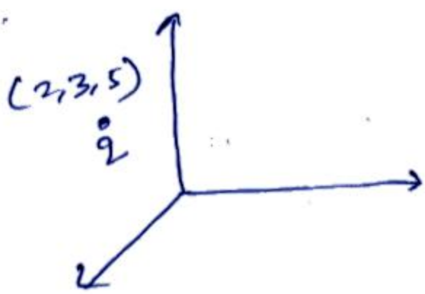
$P = [2, 3]$

x_1 component

x_2 component of the vector.

3D vector/point

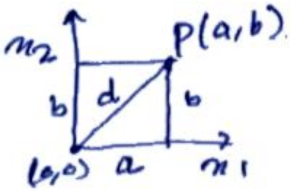
$q = [2, 3, 5]$



How do I represent n-dim point.?

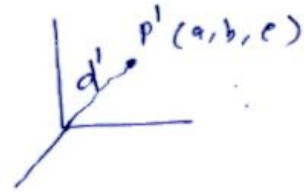
$x = [2, 3, 4, \ldots\ldots]$

distance of a point from origin:



d = dist. betn origin and P

$d = \sqrt{a^2 + b^2}$ by pythagoras theorem



3D

$$d' = \sqrt{a^2+b^2+c^2}$$

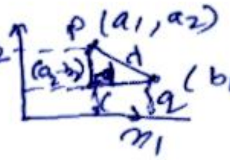
:

nD $d = \sqrt{a_1^2+a_2^2+\dots+a_n^2}$

if $P[a_1, a_2 \dots a_n]$

distance betn two point

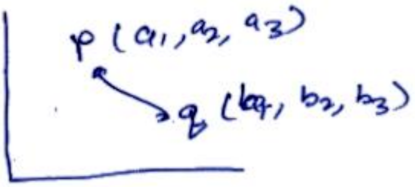
2D



$$\left[d = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2} \right]$$

proof by pythagoras theorem.

3D



$$d = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2}$$

nD $p (a_1, a_2, \dots a_n)$
$q (b_1, b_2 \dots b_n)$

$$\left[d_{pq} = \mathrm{sqrt}\left(\sum_{i=1}^{n} (a_i - b_i)^2 \right) \right]$$

row vector $A = [\overset{1}{a_1}, \overset{2}{a_2}, \dots, \overset{n}{a_n}]_{\underset{\text{row}}{1} \times \underset{\text{col}}{n}}$

column vector

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{\underset{\text{row}}{n} \times \underset{\text{col}}{1}}$$

matrix

$$\begin{array}{c|c} & 1 \quad 2 \quad \dots \quad n \\ \begin{matrix} 1 \\ 2 \\ \vdots \\ m \end{matrix} & \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{m \times n} \end{array}$$

array of array.

11-3

Dot product

$a = [a_1, a_2, \dots, a_n]$

$b = [b_1, b_2, \dots, b_n]$

· Addition

$a+b = [a_1+b_1, a_2+b_2, \dots, + a_n+b_n]$

multiplication: two types $\begin{cases} \text{dot product} \\ \text{cross product} \end{cases}$

dot product

$$a \cdot b = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

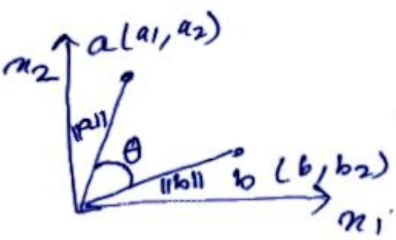
$$= \underset{1\times n}{[a_1, a_2 \dots a_n]} \underset{n\times 1}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}} = a^T b$$

$$a_n = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$ by default vector is treated as column vector.

$$\underset{n\times 1}{a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}} \qquad a^T = [a_1, a_2, \dots \dots a_n]$$

$$\boxed{a \cdot b = a^T b = \sum_{i=1}^{n} a_i b_i}$$ Imp

Geometrically



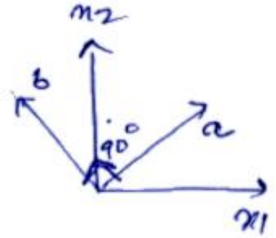
$$\boxed{a \cdot b = \|a\| \, \|b\| \cos\theta}$$ Imp

length of a = dist of a from origin

$$a \cdot b = a_1 b_1 + a_2 b_2 = \|a\| \|b\| \cos\theta$$

$$\theta = \cos^{-1}\left\{\frac{a_1 b_1 + a_2 b_2}{\|a\| \|b\|}\right\}$$

$$\|a\| = \sqrt{a_1^2 + b_1^2}$$



$a \perp b$

$$a \cdot b = \|a\| \|b\| \cos 90$$
$$= \|a\| \|b\| \cdot 0$$
$$= 0$$

Note # if $a \cdot b = 0$ then $a \perp b$.

nD

$a = [\quad]$

$b = [\quad]$

$$a \cdot b = \|a\| \|b\| \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{\sum_{i=1}^{n} a_i b_i}{\|a\| \|b\|}\right)$$

$$a \cdot b = \sum_{i=1}^{n} a_i b_i = 0 \Rightarrow a \perp b.$$

$$\underline{a \cdot a} = a_1 a_1 + a_2 a_2 + a_3 a_3 \cdots + a_n a_n$$
$$= a_1^2 + a_2^2 + \cdots + a_n^2$$
$$= \|a\|^2$$

$\because \|a\| = d = \sqrt{a_1^2 + a_2^2 + \cdots a_n^2}$

$\Rightarrow a_1^2 + a_2^2 + \cdots + a_n^2 = \|a\|^2$

$d = \sqrt{a_1^2 + a_2^2 + \cdots a_n^2}$

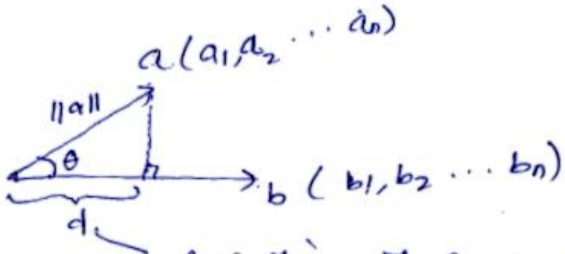
Note If we take both a and b as row vector. so

$a \cdot b = a \cdot b^T$. By default, we assume all vectors to be column vectors unless otherwise stated to avoid confusion. so

$$a \cdot b = a^T \cdot b = b^T \cdot a \quad (\because A^T B = B^T A)$$

11_4

Projection

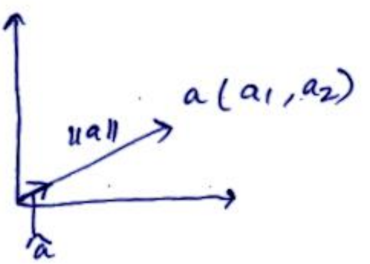


projection of a on b $= \|a\| \cos\theta$ —(1)

$$a \cdot b = \sum_{i=1}^{n} a_i b_i = \|a\| \|b\| \cos\theta$$

$$d = \frac{a \cdot b}{\|b\|} = \frac{\|a\| \|b\| \cos\theta}{\|b\|} = \|a\| \cos\theta.$$

Unit vector



$$\hat{a} = \frac{a}{\|a\|}$$

⇒ is in same direction as vector a.

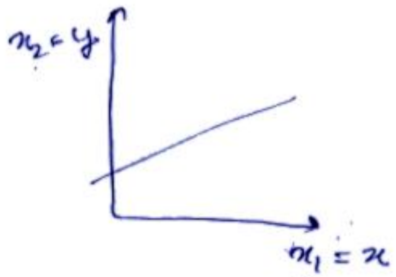
⇒ $\|\hat{a}\| = 1$

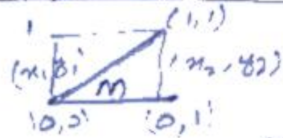
$d = \frac{1 \cos\theta}{\cancel{\cos\theta}}$

$d = 1 \cos 90 = 0$

(9)

11.5 Equation of a Line (2D), Plane (3D), Hyperplane (n-D)





$m = \frac{y_2 - y_1}{x_2 - x_1}$

$y = mx + c$

$ax + by + c = 0$ — general form of a line

$y = -\frac{c}{b} - \frac{a}{b}x$

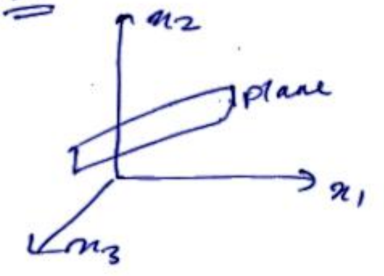
$= c + mx$

2D

$ax_1 + bx_2 + c = 0$

$\boxed{w_1x_1 + w_2x_2 + w_0 = 0}$ — eqn of line in 2D

3D



$ax + by + cz + d = 0$

$w_1x_1 + w_2x_2 + w_3x_3 + w_0 = 0$

— eqn of plane in 3D

n-D (hyperplane)

$w_0 + w_1x_1 + w_2x_2 + \cdots\cdots + w_nx_n = 0$

summation notation $\Rightarrow$ $w_0 + \sum_{i=1}^{n} w_ix_i = 0$

vector notation $\Rightarrow$ $w_0 + \underbrace{[w_1 \; w_2 \; \cdots\cdots \; w_n]}_{w_{1\times n}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x_{n\times 1}} = 0$

$w_0 + w_1x_1 + w_2x_2 + \cdots\cdots + w_nx_n = 0.$

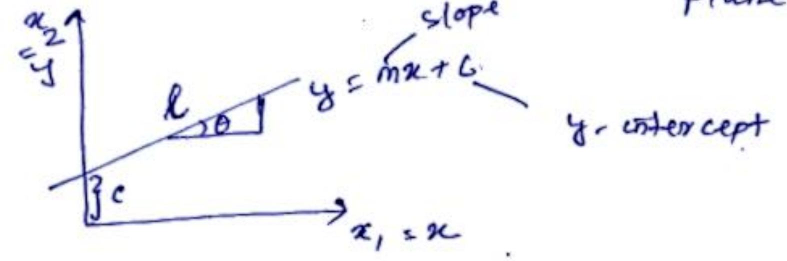
$\Rightarrow$

$$w_0 + [w_1 w_2 \cdots w_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0 \Rightarrow w_0 + w^T x = 0$$

$\Rightarrow$

$$W_{n\times 1} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}_{n\times 1} \qquad X_{n\times 1} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$

$\underset{\text{plane}}{\Pi} : w_0 + w^T x = 0$



2D: $w_1 x_1 + w_2 x_2 + w_0 = 0$

$$x_2 = \frac{-w_0}{w_2} - \frac{w_1}{w_2} x_1$$

$$y = c + mx$$

l is passing through origin, then $c = 0$

if $c = 0$, $\frac{-w_0}{w_2} = 0 \Rightarrow w_0 = 0$

l passes through origin

2D : $w_1 x_1 + w_2 x_2 = 0$

3D : $w_1 x_1 + w_2 x_2 + w_3 x_3 = 0$

$\vdots$

nD : $w_1 x_1 + w_2 x_2 + w_3 x_3 + \cdots + w_n x_n = 0$

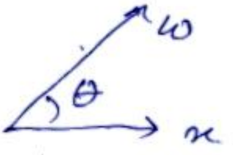
$\Rightarrow \boxed{w^T x = 0}$ Imp

eq^n of plane passes through origin.

Imp $\boxed{w^T x + w_0 = 0}$ eq^n of hyperplane not passing through origin.

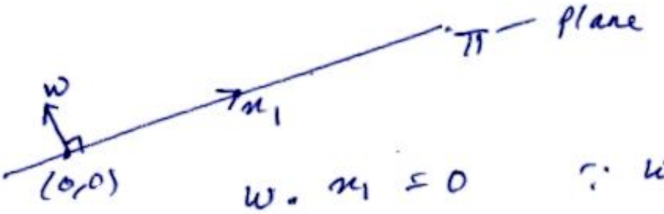
$\pi_n : w^T x = 0$

$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$



$w \cdot x = w^T x = \|w\| \|x\| \cos\theta$

if $w \perp x \Rightarrow \theta_{w,x} = 90^\circ$

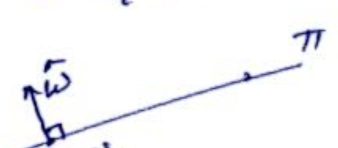


$w \cdot x_1 = 0 \quad \because w \perp x_1$

If $w \perp \pi$ then

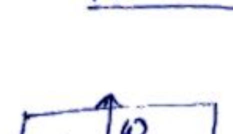
$w \cdot x_i = 0 \quad \forall \; x_i \in \pi$

$w^T x = 0$



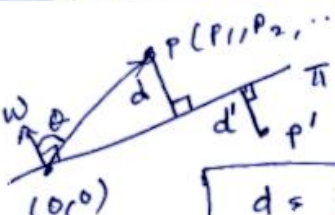
w geometrically represents a vector perpendicular to the plane passes through origin.

$\hat{w} = \frac{w}{\|w\|}$



11-6

Distance of a point from a plane



$w^T x = 0$

$w_{n\times1}$ $\quad$ $P_{n\times1}$

$$\boxed{d = \frac{w^T p}{\|w\|}}$$

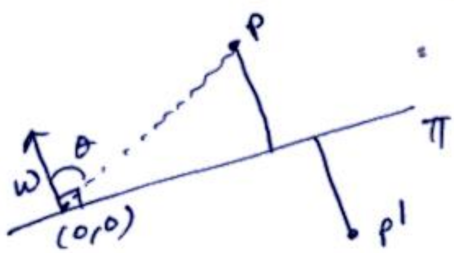
$P'(P_1', P_2', \cdots P_n')$

$d = \frac{w \cdot p}{\|w\|}$ $\quad$ if $\|w\| = 1$ unit vector.

$d = w \cdot p$

half-spaces — line/plane ~~half~~ divide the space into two region.
(upper part and below part)

line : 2D



P is in one half space and P' is in another half space

$d = \frac{w \cdot p}{\|w\|} = +ve$

$d' = \frac{w \cdot p'}{\|w\|} = -ve$

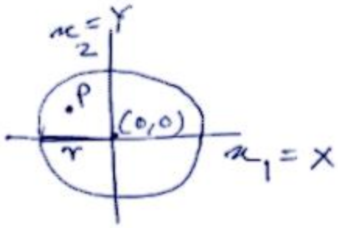
→ dot product is the work force for linear algebra.

Note if $\frac{w \cdot p}{\|w\|}$ is +ve, then P lies in the same direction of w.

if $\frac{w \cdot p}{\|w\|}$ is -ve, then P lies in the opposite direction of w.

11x

circle



$x^2+y^2=r^2$ if center is at (0,0)

$(x-h)^2+(y-k)^2=r^2$ if center is at (h,k)

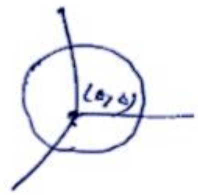
for a point $P(x_1,x_2)$.

$x_1^2+x_2^2 < r^2 \Rightarrow$ P lies inside the circle

$x_1^2+x_2^2 > r^2 \Rightarrow$ P lies outside

$x_1^2+x_2^2 = r^2 \Rightarrow$ P lies on circle

3D x_1, x_2, x_3 are dimension



Sphere $x_1^2+x_2^2+x_3^2=r^2$

⋮

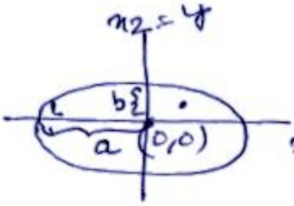
nD $x_1, x_2, x_3 \dots$

hyper-sphere $x_1^2+x_2^2+x_3^2+\dots+x_n^2 = r^2$

Equation $= \sum_{i=1} x_i^2 = r^2$

11-8

Equation of ellipse



Equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

for a point $P(x_1, y_1)$ if $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} < 1$, then P lies inside the ellipse

if $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} > 1$ then P lies outside

3D equation of 3D-ellipse (ellipsoid)

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$$

nD (hyperellipsoid)

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2} = 1 \quad \text{lies on ellipsoid}$$

> 1, outside

< 1, inside