Computational Intelligence (CI)

Fuzzy Relation

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Motivation

In Fuzzy-set Operation, We have learnt

Given two fuzzy sets, how can another fuzzy set be obtained by different operations.

In **fuzzy relation**

We say that if one element belongs to a fuzzy set, then how is this element related to another fuzzy set?

In other words, it's the relationship between two items that are part of two different fuzzy sets.

Crisp Relation

Suppose, A and B are two (crisp) sets. Then Cartesian product denoted as A x B is a collection of order pairs, such that

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$$

A x B provides a mapping from $a \in A$ to $b \in B$.

This mapping is called as **relation**.

Note :

(1) $A \times B \neq B \times A$

(2) $|A \times B| = |A| \times |B|$

Examples of Crisp Relation

Example 1:

Consider the two crisp sets A and B as given below. A =$\{$ 1, 2, 3, 4$\}$
B = $\{$3, 5, 7 $\}$.

Then, $A \times B = \{(1,3),(1,5),(1,7),(2,3),(2,5),(2,7),(3,3),(3,5),(3,7),(4,3),(4,5),(4,7)\}$

Let us define a relation R as $R = \{(a,b)|b = a+1,(a,b) \in A \times B\}$

Then, $R = \{(2,3),(4,5)\}$ in this case.

We can represent the relation R in a matrix form as follows.

$$R = \begin{array}{c c} & \begin{array}{ccc} 3 & 5 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Fuzzy Relation

Fuzzy relation is a fuzzy set defined on the Cartesian product of crisp sets.

Types:

- Binary Fuzzy Relation
- n-ary Fuzzy Relation

Binary Fuzzy Relation

Let X and Y be two UD (Universe of discourse), then

Binary fuzzy relation R in $X\ x\ Y$ can be defied as

$$R = \left\{ \left((x, y), \mu_R(x, y) \right) \middle| \ (x, y) \epsilon\ X\ x\ Y \right\}$$

This maps each element in $X\ x\ Y$ to a membership grade between 0 and 1.

μ_R is a 2D Membership Function.

Membership values indicate the strength of the relation between the order pair.

n-ary Fuzzy Relation

Fuzzy relation is a fuzzy set defined on the Cartesian product of crisp set $X_1, X_2, \ldots\ldots X_n$

Here, $\boldsymbol{R} = \{(\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots\ldots \boldsymbol{x_n}), \boldsymbol{\mu_R}(\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots\ldots \boldsymbol{x_n}) \mid (\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots\ldots \boldsymbol{x_n}) \in \boldsymbol{X_1} \boldsymbol{x}\, \boldsymbol{X_2}, \ldots \boldsymbol{x}\, \boldsymbol{X_n}\}$

Here, n-tuples $(x_1, x_2, \ldots\ldots x_n)$ may have varying degree of memberships within the relationship.

μ_R is a n-dimensional Membership Function.

Example of Fuzzy Relation

Example :

$A = \{(a_1, 0.2), (a_2, 0.7), (a_3, 0.4)\}$ and $B = \{(b_1, 0.5), (b_2, 0.6)\}$

$$R = A \times B = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

Example of Fuzzy Relation

Let, $X = R^+ = y$ (the positive real line)
and $R = X \times Y$ = "y is much greater than x"

The membership function of $\mu_R(x, y)$ is defined as

$$\mu_R(x,y) = \begin{cases} \frac{(y-x)}{4} & if \quad y > x \\ 0 & if \quad y \leq x \end{cases}$$

Suppose, $X = \{3, 4, 5\}$ and $Y = \{3, 4, 5, 6, 7\}$, then

$$R = \begin{array}{c c} & \begin{array}{ccccc} 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 0.25 & 0.5 & 0.75 & 1.0 \\ 0 & 0 & 0.25 & 0.5 & 0.75 \\ 0 & 0 & 0 & 0.25 & 0.5 \end{bmatrix} \end{array}$$

Operations on Fuzzy Relations

Let R and S be two fuzzy relations on $A \times B$.

Union:

$$\mu_{R \cup S}(a, b) = max\{\mu_R(a, b), \mu_S(a, b)\}$$

Intersection:

$$\mu_{R \cap S}(a, b) = min\{\mu_R(a, b), \mu_S(a, b)\}$$

Complement:

$$\mu_{\overline{R}}(a, b) = 1 - \mu_R(a, b)$$

Composition

$$T = R \circ S$$

Union Operation on Fuzzy Relations

Example:

X = {x1, x2, x3},   Y = {y1, y2}, Z = {z1, z2, z3}

Two fuzzy relations R and S defined on XxY and Fuzzy relation T, defined on YxZ.

	y1	y2
x1	0.5	0.1
x2	0.2	0.9
x3	0.8	0.6

R

	y1	y2
x1	0.1	0.2
x2	0.2	0.5
x3	0.6	0.5

S

	z1	z2	z3
y1	0.6	0.5	0.7
y2	0.3	0.1	0.4

T

Find R U S.

	y1	y2
x1	0.5	0.2
x2	0.2	0.9
x3	0.8	0.6

Find R U T.

Not Compatible, we may apply composition operator.

Example of Fuzzy Composition

Let, $R = x$ is relevant to y

and $S = y$ is relevant to z

be two fuzzy relations defined on $X \times Y$ and $Y \times Z$, respectively, where $X = \{1, 2, 3\}$,$Y = \{\alpha, \beta, \gamma, \delta\}$ and $Z = \{a, b\}$.
Assume that R and S can be expressed with the following relation matrices :

$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} \alpha & \beta & \gamma & \delta \end{array} \\ \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.7 \\ 0.4 & 0.2 & 0.8 & 0.9 \\ 0.6 & 0.8 & 0.3 & 0.2 \end{bmatrix} \end{array} \text{ and}$$

$$S = \begin{array}{c} \\ \alpha \\ \beta \\ \gamma \\ \delta \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.3 \\ 0.5 & 0.6 \\ 0.7 & 0.2 \end{bmatrix} \end{array}$$

Example of Fuzzy Composition

Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two fuzzy relations defined on $X \times Y$ and $Y \times Z$, respectively. The **max-min composition** of $\mathcal{R}_1$ and $\mathcal{R}_2$ is a fuzzy set defined by

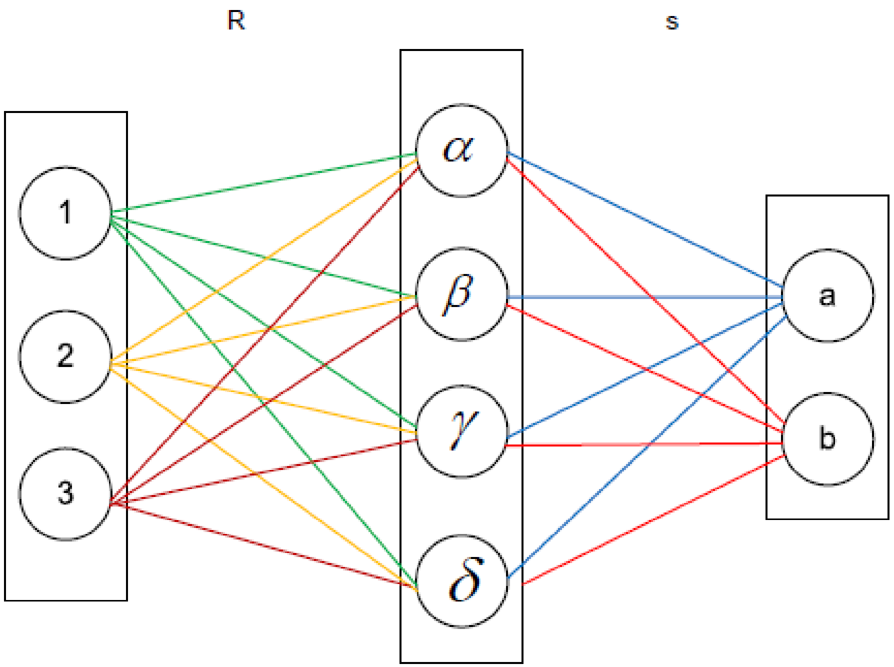
$$\mathcal{R}_1 \circ \mathcal{R}_2 = \{[(x, z), \max_y \min(\mu_{\mathcal{R}_1}(x, y), \mu_{\mathcal{R}_2}(y, z))] | x \in X, y \in Y, z \in Z\}, \quad (3.5)$$

or, equivalently,

$$\begin{aligned} \mu_{\mathcal{R}_1 \circ \mathcal{R}_2}(x, z) &= \max_y \min[\mu_{\mathcal{R}_1}(x, y), \mu_{\mathcal{R}_2}(y, z)] \\ &= \vee_y [\mu_{\mathcal{R}_1}(x, y) \wedge \mu_{\mathcal{R}_2}(y, z)], \end{aligned} \quad (3.6)$$

with the understanding that $\vee$ and $\wedge$ represent max and min, respectively.

$$\begin{aligned} \mu (2, a) &= \max(0.4 \wedge 0.9,\ 0.2 \wedge 0.2,\ 0.8 \wedge 0.5,\ 0.9 \wedge 0.7) \\ &= \max(0.4,\ 0.2,\ 0.5,\ 0.7) \\ &= 0.7 \text{ (by max-min composition).} \end{aligned}$$


R
s
1
2
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b

Example of Fuzzy Composition

Definition 3.4 *Max-product composition*

Assuming the same notation as used in the definition of max-min composition, we can define **max-product composition** as follows:

$$\mu_{\mathcal{R}_1 \circ \mathcal{R}_2}(x, z) = \max_y [\mu_{\mathcal{R}_1}(x, y)\mu_{\mathcal{R}_2}(y, z)]. \tag{3.8}$$

$$\begin{aligned} \mu(2, a) &= \max(0.4 \times 0.9,\ 0.2 \times 0.2,\ 0.8 \times 0.5,\ 0.9 \times 0.7) \\ &= \max(0.36,\ 0.04,\ 0.40,\ 0.63) \\ &= 0.63 \text{ (by max-product composition).} \end{aligned}$$

Another Example of Fuzzy Composition

Consider the following two sets P and D, which represent a set of paddy plants and a set of plant diseases. More precisely

$P = \{P_1, P_2, P_3, P_4\}$ a set of four varieties of paddy plants
$D = \{D_1, D_2, D_3, D_4\}$ of the four various diseases affecting the plants

In addition to these, also consider another set $S = \{S_1, S_2, S_3, S_4\}$ be the common symptoms of the diseases.

Let, R be a relation on $P \times D$, representing which plant is susceptible to which diseases, then R can be stated as

$$R = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{bmatrix} 0.6 & 0.6 & 0.9 & 0.8 \\ 0.1 & 0.2 & 0.9 & 0.8 \\ 0.9 & 0.3 & 0.4 & 0.8 \\ 0.9 & 0.8 & 0.4 & 0.2 \end{bmatrix} \end{array} \quad T = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.1 & 0.2 & 0.7 & 0.9 \\ 1.0 & 1.0 & 0.4 & 0.6 \\ 0.0 & 0.0 & 0.5 & 0.9 \\ 0.9 & 1.0 & 0.8 & 0.2 \end{bmatrix} \end{array}$$

Examples of Fuzzy Composition

Also, consider T be the another relation on $D \times S$, which is given by

$$T = \begin{array}{c c} & \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{array}{c} D_1 \\ D_2 \\ D_3 \\ D_4 \end{array} & \begin{bmatrix} 0.1 & 0.2 & 0.7 & 0.9 \\ 1.0 & 1.0 & 0.4 & 0.6 \\ 0.0 & 0.0 & 0.5 & 0.9 \\ 0.9 & 1.0 & 0.8 & 0.2 \end{bmatrix} \end{array}$$

Obtain the association of plants with the different symptoms of the disease using **max-min composition**.

Hint: Find $R \circ T$, and verify that

$$R \circ T = \begin{array}{c c} & \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{array}{c} P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} & \begin{bmatrix} 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.7 & 0.9 \end{bmatrix} \end{array}$$

Thank you