Computational Intelligence (CI)

Fuzzy Rules

Dr. Dayal Kumar Behera

School of Computer Engineering
KIIT Deemed to be University, Bhubaneswar, India

Credits:
Soft Computing Applications, Dr. Debasis Samanta, IIT Kharagpur

Fuzzy Extension Principle

It is the principle of extending crisp domain of mathematical expression to fuzzy domain.

Suppose that f is a function from X to Y ($f: X \rightarrow Y$) where X and Y are different universe of discourse.

A is a fuzzy set on X defined as

$$A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \cdots + \mu_A(x_n)/x_n$$

Then the extension principle states that the image of fuzzy set A under the mapping f(.) can be expressed as a fuzzy set B.

$$B = f(A) = \mu_A(x_1)/y_1 + \mu_A(x_2)/y_2 + \cdots + \mu_A(x_n)/y_n$$

where $y_i = f(x_i)$, $i = 1, \ldots, n.$

$$\mu_B(y) = \max_{x = f^{-1}(y)} \mu_A(x)$$

[2]

Example

Let

$$A = 0.1/-2 + 0.4/-1 + 0.8/0 + 0.9/1 + 0.3/2$$

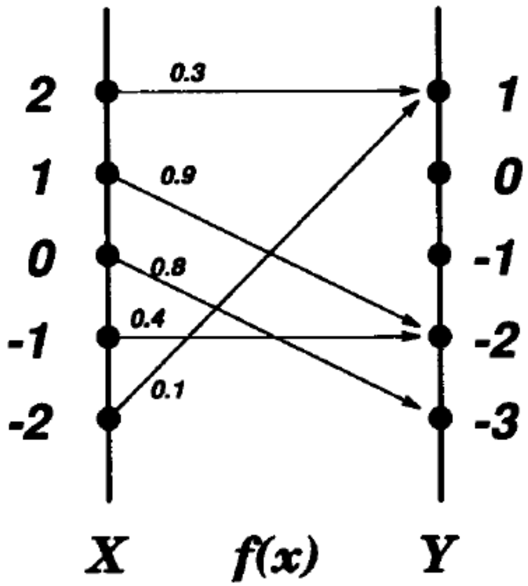
and

$$f(x) = x^2 - 3.$$

Upon applying the extension principle, we have

$$\begin{aligned} B &= 0.1/1 + 0.4/-2 + 0.8/-3 + 0.9/-2 + 0.3/1 \\ &= 0.8/-3 + (0.4 \vee 0.9)/-2 + (0.1 \vee 0.3)/1 \\ &= 0.8/-3 + 0.9/-2 + 0.3/1, \end{aligned}$$

Example



Extension principle on fuzzy sets with discrete universes

Fuzzy Implication/Fuzzy Rule

- A fuzzy implication (also known as fuzzy If-Then rule, fuzzy rule, or fuzzy conditional statement) assumes the form :

  If x **is** A **then** y **is** B

  where, A and B are two linguistic variables defined by fuzzy sets A and B on the universe of discourses X and Y, respectively.

- Often, x **is** A is called the antecedent or premise, while y **is** B is called the consequence or conclusion.

Fuzzy Implication Example

- If pressure is High then temperature is Low
- If mango is Yellow then mango is Sweet else mango is Sour
- If road is Good then driving is Smooth else traffic is High
- The fuzzy implication is denoted as $R : A \rightarrow B$
- In essence, it represents a binary fuzzy relation R on the (Cartesian) product of $A \times B$

Fuzzy Implication Example

- Suppose, P and T are two universes of discourses representing pressure and temperature, respectively as follows.
- $P = \{ 1,2,3,4\}$ and $T = \{ 10, 15, 20, 25, 30, 35, 40, 45, 50 \}$
- Let the linguistic variable High temperature and Low pressure are given as
- $T_{HIGH} =$ $\{(20, 0.2), (25, 0.4), (30, 0.6), (35, 0.6), (40, 0.7), (45, 0.8), (50, 0.8)\}$
- $P_{LOW} = (1, 0.8), (2, 0.8), (3, 0.6), (4, 0.4)$

Fuzzy Implication Example

- Then the fuzzy implication **If temperature is High then pressure is Low** can be defined as

$$R : T_{HIGH} \rightarrow P_{LOW}$$

where, R =

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 20 \\ 25 \\ 30 \\ 35 \\ 40 \\ 45 \\ 50 \end{array} & \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0.6 & 0.6 & 0.6 & 0.4 \\ 0.6 & 0.6 & 0.6 & 0.4 \\ 0.7 & 0.7 & 0.6 & 0.4 \\ 0.8 & 0.8 & 0.6 & 0.4 \\ 0.8 & 0.8 & 0.6 & 0.4 \end{bmatrix} \end{array}$$

Note# Here, the fuzzy rule is represented by a fuzzy relation and cartesian product T x P is calculated to represent the fuzzy relation. We can also use other T-norm operators to interpret the fuzzy rule/relation.

Interpretation of Fuzzy Rule

In general, there are two ways to interpret the fuzzy rule $A \rightarrow B$ as

- A coupled with B
- A entails B

A coupled with B

$R : A \to B = A \times B = \int_{X \times Y} \mu_A(x) * \mu_B(y)|_{(x,y)}$; where $*$ is called a T-norm operator.

T-norm operator

The most frequently used T-norm operators are:

Minimum : $T_{min}(a, b) = min(a, b) = a \wedge b$

Algebric product : $T_{ap}(a, b) = ab$

Bounded product : $T_{bp}(a, b) = 0 \vee (a + b - 1)$

Drastic product : $T_{dp} = \begin{cases} a & if \quad b = 1 \\ b & if \quad a = 1 \\ 0 & if \quad a, b < 1 \end{cases}$

Here, a = $\mu_A(x)$ and b = $\mu_B(y)$. T_* is called the function of T-norm operator.

A coupled with B

In the following, few implications of $R : A \to B$

Min operator:

$R_m = A \times B = \int_{X \times Y} \mu_A(x) \wedge \mu_B(y)|_{(x,y)}$ or $f_{min}(a,b) = a \wedge b$

[Mamdani rule]

Algebric product operator

$R_{ap} = A \times B = \int_{X \times Y} \mu_A(x).\mu_B(y)|_{(x,y)}$ or $f_{ap}(a,b) = ab$

[Larsen rule]

A entails B

There are three main ways to interpret such implication:

Material implication :

$$R : A \to B = \bar{A} \cup B$$

Propositional calculus :

$$R : A \to B = \bar{A} \cup (A \cap B)$$

Extended propositional calculus :

$$R : A \to B = (\bar{A} \cap \bar{B}) \cup B$$

A entails B

Classical operator to interpret Fuzzy Rule is **Zadeh's max-min rule**.

Fuzzy rule $IF\ x\ is\ A\ THEN\ y\ is\ B$ can be interpreted as

$$R : A \to B = \bar{A} \cup (A \cap B)$$

Which is equivalent to

$$(A \times B) \cup (\bar{A} \times Y)$$

Note# This is the implication relation matrix.

Fuzzy Rule: Zadeh's Max-Min Operator

$IF\ x\ is\ A\ THEN\ y\ is\ B$

$$(A \times B) \cup (\bar{A} \times Y)$$

$IF\ x\ is\ A\ THEN\ y\ is\ B\ ELSE\ y\ is\ C$

$$(A \times B) \cup (\bar{A} \times C)$$

Example

Let X = {a, b, c, d}, Y = {1, 2, 3, 4}. Fuzzy set A and B are defined on X and Y, respectively. Determine the implication relation of the fuzzy rule $IF\ x\ is\ A\ THEN\ y\ is\ B$.

$$A = \frac{0.8}{b} + \frac{0.6}{c} + \frac{1.0}{d} \qquad B = \frac{0.2}{1} + \frac{1.0}{2} + \frac{0.8}{3}$$

Example

$$A \times B = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array}$$

$$\bar{A} \times Y = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$(A \times B) \cup (\bar{A} \times Y) = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.4 & 0.6 & 0.6 & 0.4 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array}$$

Example 2

Let X = {a, b, c, d}, Y = {1, 2, 3, 4}. Fuzzy set A is defined on X. Fuzzy set B, C are defined on Y. Determine the implication relation of the fuzzy rule $IF\ x\ is\ A\ THEN\ y\ is\ B\ ELSE\ y\ is\ C$.

$$A = \frac{0.8}{b} + \frac{0.6}{c} + \frac{1.0}{d}$$

$$A = \frac{0.2}{1} + \frac{1.0}{2} + \frac{0.8}{3}$$

$$C = \frac{0.4}{2} + \frac{1.0}{3} + \frac{0.8}{4}$$

Example 2

$$A \times B = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array}$$

$$\bar{A} \times C = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0.4 & 1.0 & 0.8 \\ 0 & 0.2 & 0.2 & 0.2 \\ 0 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$(A \times B) \cup (\bar{A} \times C) \quad \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0.4 & 1.0 & 0.8 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.2 & 0.6 & 0.6 & 0.4 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array}$$

Thank you