# Random Matrix and Potential Theory.

**Lecture 1** A random matrix is a matrix whose entries are random.

For example,

$$X = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \end{pmatrix}_{2\times 3} \quad \text{is a } 2\times 3 \text{ random matrix.}$$

Here, $X_{ij}$'s are random variables defined on the same probability space.

($X_{ij}$ maybe real valued or complex valued)

We will be concerned with square random matrices.

The main observable of interest in that case are eigenvalues (also eigenvectors)

$$\text{Matrix entries} \xleftrightarrow[\text{non-linear}]{} \text{Eigenvalues}$$

Due to this non-linear relation, interesting limit theorems emerge in this area. Note in CLT, we are interested in sum of given random variables; such a relation is linear.

<u>Wigner matrices</u> $\{Z_{i,j}\}_{1\leq i<j}$ complex-valued iid $\mathbb{E}Z_{1,2}=0$; $\mathbb{E}|Z_{1,2}|^2=1$.

$\{Y_i\}_{i\geq 1}$ real valued iid. $\mathbb{E}Y_1 = 0$; $\mathbb{E}Y_1^2 < \infty$

$$X_N = \begin{pmatrix} Y_1 & Z_{12} & Z_{13} & \cdots \\ \overline{Z}_{12} & Y_2 & Z_{23} & \\ \overline{Z}_{13} & \overline{Z}_{23} & Y_3 & \\ & & & \end{pmatrix}_{N\times N}$$

$\rightarrow$ Wigner matrix

$\rightarrow$ random Hermitian matrix

Since $X_N$ is Hermitian, it has real eigenvalues :→

$$\lambda_1^N \leqslant \lambda_2^N \leqslant \cdots \leqslant \lambda_N^N$$

$$\sum_{i=1}^N (\lambda_i^N)^2 = Tr(X_N^2) = \sum_{i=1}^N Y_i^2 + 2\sum_{i<j} |Z_{ij}|^2 = O(N^2).$$

So, we expect $\lambda_i^N = O(N^{1/2})$.

Consider the empirical distribution of eigenvalues :→

$$L_N = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i^N/\sqrt{N}} \quad \rightarrow \text{random probability measure.}$$

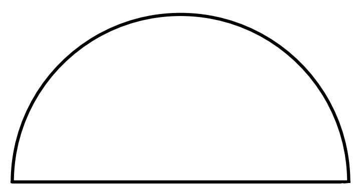
$$L_N(A) = \frac{1}{N}\sum_{i=1}^N \mathbb{1}\left\{\frac{\lambda_i^N}{\sqrt{N}} \in A\right\} \qquad \text{for } A \subseteq \mathbb{R}.$$

$$\int f(x)\, dL_N(x) = \frac{1}{N}\sum_{i=1}^N f\left(\lambda_i^N/\sqrt{N}\right)$$

Note if $U_1, \ldots, U_N$ are iid from $\mu$, $\tilde{L}_N = \frac{1}{N}\sum_{i=1}^N \delta_{U_i} \longrightarrow \mu$ weakly in probability.

<u>Theorem</u> (Wigner) For all bounded continuous function $f$

$$\int f(x)\, dL_N(x) \xrightarrow{p} \int f(x)\, \sigma(x)\, dx$$



where $\sigma(x) = \frac{1}{2\pi}\sqrt{4-x^2}\, \mathbb{1}_{|x|\leqslant 2}$ is the density of the semicircle distribution.

In other words, $L_N$ converges weakly in probability to the semicircle distribution

Wigner's proof is based on moment computations and involves interesting combinatorics. See [AGZ, Section 2]

I will focus on a particular Gaussian class of matrices and prove this theorem using Potential theory. The primary goal is to show you the connections to Potential theory and how it can be used to prove various aspects of $L_N$ and beyond.

## Gaussian ensembles

Gaussian orthogonal ensemble (GOE) $[\beta = 1]$

$$\begin{cases} Z_{i,j} \sim N(0,1) \\ Y_i \sim N(0,2) \end{cases}$$

Gaussian unitary ensemble (GUE) $[\beta = 2]$

$$\begin{cases} Z_{i,j} = (\xi_{i,j} + i\eta_{i,j})/\sqrt{2} \\ \xi_{i,j}, \eta_{i,j}, Y_i \sim N(0,1). \end{cases}$$

Joint density :→ (GUE)

$$C_N \exp\left(-\sum_{i<j} \frac{\xi_{ij}^2}{2} - \sum_{i<j} \frac{\eta_{ij}^2}{2} - \sum_{i=1}^{N} \frac{y_i^2}{2}\right)$$

$$= C_N \exp\left(-\frac{1}{2}\left\{\sum_{i=1}^{N} y_i^2 + 2\sum_{i<j} |z_{ij}|^2\right\}\right)$$

$$= C_N \exp\left(-\frac{1}{2}\{\underbrace{\operatorname{tr} X_N^2}\}\right)$$

↳ depends only on the eigenvalues.

For GOE :→

Jt density $\propto \exp\left(-\frac{1}{4}\{\operatorname{tr} X_N^2\}\right)$.

## Distribution of eigenvalues :→

$X_N = UDU^*$ where $D$ is diagonal matrix → $n$ free parameters

and $U$ is a unitary/orthogonal matrix.

↳ $N(N-1)$ or $\frac{N(N-1)}{2}$ free parameters.

$$\mathbb{R}^{\beta\frac{N(N-1)}{2}} \times \mathbb{R}^N \xrightarrow{\hat{T}} \mathcal{H}_N^\beta$$

$$z \times \lambda \longrightarrow X_N$$

($\hat{T}$ is "bijective")

Then, Jt density of $(z,\lambda)$ :→ $\propto \exp\left(-\frac{\beta}{4}\sum_{i=1}^{N}\lambda_i^2\right)$ Jacobian$(\hat{T})$

The $\hat{T}$ map :→

$$\underset{\substack{\downarrow \\ \text{unitary/orthogonal}}}{T(U)} = \left(\frac{U_{1,2}}{U_{1,1}}, \dots, \frac{U_{1,N}}{U_{1,1}}, \frac{U_{2,3}}{U_{2,2}}, \dots, \frac{U_{2,N}}{U_{2,2}}, \dots, \frac{U_{N-1,N}}{U_{N-1,N-1}}\right)$$

$T$ is "bijective".
$$\hat{T}(z,\lambda) := T^{-1}(z)\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_N\end{pmatrix}(T^{-1}(z))^*$$

$$\text{Jacobian}(\hat{T}) = g(z)\prod_{i<j}(\lambda_i-\lambda_j)^\beta$$

(can be computed or can be argued that $\lambda_i - \lambda_j$ has to be a factor).

Thus, jt density of eigenvalues :→

$$\boxed{\mathbb{1}_{\lambda_1<\lambda_2<\dots<\lambda_N}\prod_{i<j}(\lambda_i-\lambda_j)^\beta \exp\left(-\frac{\beta}{4}\sum_{i=1}^{N}\lambda_i^2\right)}$$

Caution :→ $T, \hat{T}$ are not bijective. But they turn out to be bijective after discarding some Lebesgue zero measure set and introducing the orders in eigenvalues. For a complete proof, see [AGZ, Section 2.5]

# Lecture 2

Last time we saw jt density of eigenvalues for GUE :→

$$\propto \mathbb{1}_{\lambda_1<\lambda_2<\cdots<\lambda_N} \prod_{i<j}(\lambda_i-\lambda_j)^2 \exp\left(-\frac{1}{2}\sum_{i=1}^{N}\lambda_i^2\right)$$

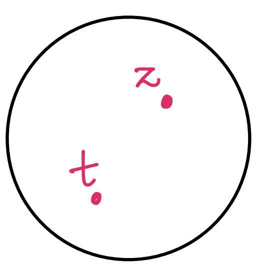
The proportionality constant can be computed but not required.

Let us study this probability density :→

$\prod_{i<j}(\lambda_i-\lambda_j)^2 \longrightarrow$ density is small when $\lambda_i$ and $\lambda_j$ are close. So, two eigenvalues typically cannot be close to each other. So, there is a repulsive force between eigenvalues. They try to stay far apart.

$\exp\left(-\frac{1}{2}\sum_{i=1}^{N}\lambda_i^2\right) \longrightarrow$ density is small when $|\lambda_i|$ is large. Thus the eigenvalues typically can't be too large. Note that this force is opposite of repulsive force.

The two forces balances each other out and produce the semicircle distribution at equilibrium. To explain this, let us introduce the concept of Potential theory where we shall view the eigenvalues as charged particles in a field.




Suppose we distribute one unit charge on a circle according to some measure $\mu$.

Electric potential at $z$ :→ $\propto \underbrace{\log \frac{1}{|z-t|}}$
arising from $t$
$(z \neq t)$

↳ log is due to 2D;
this comes from Gauss Law.

Total electric potential at $z$ :→ $\propto \int\limits_{t \neq z} \log \frac{1}{|z-t|} \, d\mu(t)$

Energy of the system :→ sum of all electric potentials

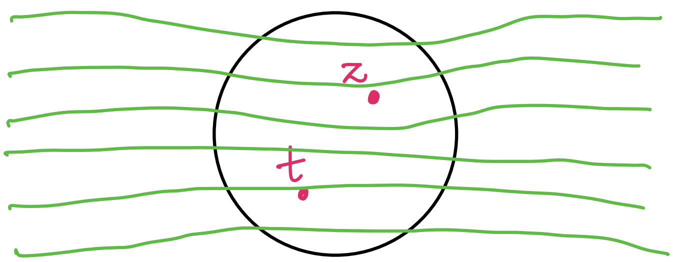
$$\iint\limits_{t \neq z} \log \frac{1}{|z-t|} \, d\mu(t) \, d\mu(z).$$

The system reaches equilibrium when the energy is minimum.

The fundamental problem in electrostatics is

minimize $\iint\limits_{t \neq z} \log \frac{1}{|z-t|} \, d\mu(t) \, d\mu(z)$ over measures $\mu$.

The situation becomes a bit more complicated in presence of an external field.




→ external field $= \frac{1}{2} V$

Electric potential at $z$ :→ $\log \frac{1}{|z-t|} + \frac{1}{2} V(z) + \frac{1}{2} V(t)$
arising from $t$

Then equilibrium is attained by minimizing (assume $\mu(x \neq y) = 0$)

$$\left.\begin{aligned} &\iint \left[ \log \frac{1}{|z-t|} + \frac{1}{2} V(z) + \frac{1}{2} V(t) \right] d\mu(z) \, d\mu(t) \\ &= -\iint \log |z-t| \, d\mu(z) \, d\mu(t) + \int V(z) \, d\mu(z) \end{aligned}\right\} (*)$$

The minimizer of $(*)$ is called equilibrium measure.

How is it related to GOE/GUE eigenvalues?

Recall the jt density of GUE (ignore the proportionality const) :→

$$\prod_{i<j} (\lambda_i - \lambda_j)^2 \exp\left(-\frac{1}{2}\sum_{i=1}^{N} \lambda_i^2\right)$$

$$= \exp\left(2\sum_{i<j} \log|\lambda_i - \lambda_j| - \frac{1}{2}\sum_{i=1}^{N} \lambda_i^2\right)$$

$$\propto \exp\left( \sum_{i\neq j} \log\left|\frac{\lambda_i}{\sqrt{N}} - \frac{\lambda_j}{\sqrt{N}}\right| - \frac{1}{2}\sum_{i=1}^{N} \lambda_i^2\right)$$

$$= \exp\left(-N^2\left\{-\frac{1}{N^2}\sum_{i\neq j} \log\left|\frac{\lambda_i}{\sqrt{N}} - \frac{\lambda_j}{\sqrt{N}}\right| - \frac{1}{N}\sum_{i=1}^{N} \frac{(\lambda_i/\sqrt{N})^2}{2}\right\}\right)$$

$$= \exp\left(-N^2 I_V(L_N)\right).$$

where $L_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i/\sqrt{N}}$ , $V(x) = \frac{x^2}{2}$

$$I_V(\mu) := \iint_{z\neq t} -\log|z-t|\, d\mu(t)\, d\mu(z) + \int V(z)\, d\mu(z)$$

$\mu$ is now a prob. measure on $\mathbb{R}$.

Now it is intuitively clear, $L_N \longrightarrow \mu_{eq}$ (equilibrium measure)

($\mu_{eq}$ will be semicircle in this case)

Furthermore, this formulation also tells you

$$\mathbb{P}(L_N \approx \nu) \approx \exp\left(N^2[I_V(\mu) - I_V(\nu)]\right)$$

This is related to large deviations of empirical distribution.

Historical notes :→ The variational problem in (*) was first studied by Gauss and its often known as Gauss variational problem. Frostman studied this problem in 1920s and showed uniqueness and existence of equilibrium measure. Then, the Polish school led by Franciszek Leja made significant contributions in 1935–1960. Then, in the late $20^{th}$ century, it gathered renewed interest as they found to be useful in studying orthogonal polynomials and as we see in random matrix theory as well.

Theorem :→ Suppose $V$ is a continuous potential with

$$V(x) \geq 2\log(1+x^2) \quad \text{for large } x.$$

Set :→ $F_V := \inf\{ I_V(\mu) \mid \mu \text{ probability measure on } \mathbb{R} \}$

Then,

(a) $F_V$ is finite

(b) there exist a unique probability measure $\mu_{eq}$ such that

$$I_V(\mu_{eq}) = F_V$$

(c) $\mu_{eq}$ has a compact support say $S$.

(d) $$\int \left[ -\log|z-t| + \tfrac{1}{2}V(z) + \tfrac{1}{2}V(t) \right] d\mu_{eq}(t) = F_V \quad \text{for } z \in S$$

$$\int \left[ -\log|z-t| + \tfrac{1}{2}V(z) + \tfrac{1}{2}V(t) \right] d\mu_{eq}(t) \geq F_V \quad \text{for } z \notin S$$

(e) (d) determines $\mu_{eq}$ uniquely.