

Random Matrix and Potential Theory.

Lecture 1 A random matrix is a matrix whose entries are random.

For example,

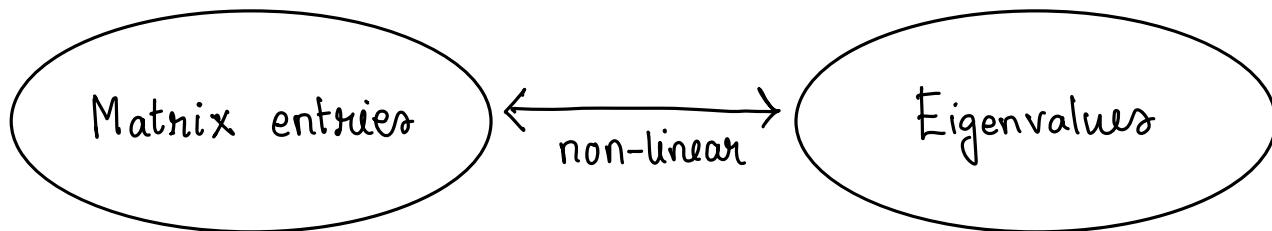
$$X = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \end{pmatrix}_{2 \times 3}$$

is a 2×3 random matrix.

Here, X_{ij} 's are random variables defined on the same probability space.
(X_{ij} maybe real valued or complex valued)

We will be concerned with square random matrices.

The main observable of interest in that case are eigenvalues (also eigenvectors)



Due to this non-linear relation, interesting limit theorems emerge in this area. Note in CLT, we are interested in sum of given random variables; such a relation is linear.

Wigner matrices $\{Z_{i,j}\}_{1 \leq i < j}$ complex-valued iid $E Z_{1,2} = 0 ; E|Z_{1,2}|^2 = 1$.

$\{Y_i\}_{i \geq 1}$ real valued iid. $E Y_i = 0 ; E Y_i^2 < \infty$

$$X_N = \begin{pmatrix} Y_1 & Z_{12} & Z_{13} & \cdots \\ \bar{Z}_{12} & Y_2 & Z_{23} & \\ \bar{Z}_{13} & \bar{Z}_{23} & Y_3 & \end{pmatrix}_{N \times N}$$

\rightarrow Wigner matrix
 \rightarrow random Hermitian matrix

Since X_N is Hermitian, it has real eigenvalues \Rightarrow

$$\lambda_1^N \leq \lambda_2^N \leq \dots \leq \lambda_N^N$$

$$\sum_{i=1}^N (\lambda_i^N)^2 = \text{Tr}(X_N^2) = \sum_{i=1}^N Y_i^2 + 2 \sum_{i < j} |Z_{ij}|^2 = O(N^2).$$

So, we expect $\lambda_i^N = O(N^{1/2})$.

Consider the empirical distribution of eigenvalues \Rightarrow

$$L_N = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i^N / \sqrt{N}} \rightarrow \text{random probability measure.}$$

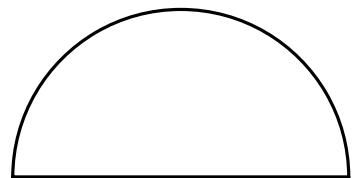
$$L_N(A) = \frac{1}{N} \sum_{i=1}^N \mathbb{1} \left\{ \frac{\lambda_i^N}{\sqrt{N}} \in A \right\} \quad \text{for } A \subseteq \mathbb{R}.$$

$$\int f(x) dL_N(x) = \frac{1}{N} \sum_{i=1}^N f(\lambda_i^N / \sqrt{N})$$

Note if U_1, \dots, U_N are iid from μ , $\tilde{L}_N = \frac{1}{N} \sum_{i=1}^N \delta_{U_i} \rightarrow \mu$
weakly in probability.

Theorem (Wigner) For all bounded continuous function f

$$\int f(x) dL_N(x) \xrightarrow{P} \int f(x) \sigma(x) dx$$



where $\sigma(x) = \frac{1}{2\pi} \sqrt{4 - x^2} \mathbb{1}_{|x| \leq 2}$ is the density of the semicircle distribution.

In other words, L_N converges weakly in probability to the semicircle distribution.

Wigner's proof is based on moment computations and involves interesting combinatorics. See [AGZ, Section 2]

I will focus on a particular Gaussian class of matrices and prove this theorem using Potential theory. The primary goal is to show you the connections to Potential theory and how it can be used to prove various aspects of L_N and beyond.

Gaussian ensembles

Gaussian orthogonal ensemble
 $(GOE) [\beta=1]$

$$\begin{cases} z_{i,j} \sim N(0, 1) \\ y_i \sim N(0, 2) \end{cases}$$

Gaussian unitary ensemble
 $(GUE) [\beta=2]$

$$\begin{cases} z_{i,j} = (\varepsilon_{i,j} + i\eta_{i,j})/\sqrt{2} \\ \varepsilon_{i,j}, \eta_{i,j}, y_i \sim N(0, 1). \end{cases}$$

Joint density $\Rightarrow (GUE)$

$$C_N \exp \left(- \sum_{i < j} \frac{\varepsilon_{ij}^2}{2} - \sum_{i < j} \frac{\eta_{ij}^2}{2} - \sum_{i=1}^N \frac{y_i^2}{2} \right)$$

$$= C_N \exp \left(- \frac{1}{2} \left\{ \sum_{i=1}^N y_i^2 + 2 \sum_{i < j} |z_{ij}|^2 \right\} \right)$$

$$= C_N \exp \left(- \underbrace{\frac{1}{2} \left\{ \text{tr } X_N^2 \right\}}_{\text{depends only on the eigenvalues.}} \right)$$

For GOE \Rightarrow

$$\text{JT density} \propto \exp \left(- \frac{1}{4} \left\{ \text{tr } X_N^2 \right\} \right).$$

Distribution of eigenvalues \Rightarrow

$X_N = UDU^*$ where D is diagonal matrix $\rightarrow n$ free parameters

and U is a unitary/orthogonal matrix.

$\hookrightarrow N(N-1)$ on $\frac{N(N-1)}{2}$ free parameters.

$$\begin{array}{ccc} \mathbb{R}^{\frac{\beta N(N-1)}{2}} \times \mathbb{R}^N & \xrightarrow{\hat{T}} & \mathcal{H}_N^\beta \\ z \times \lambda & \longrightarrow & X_N \end{array} \quad (\hat{T} \text{ is "bijective"})$$

Then, jt density of $(z, \lambda) \Rightarrow \propto \exp\left(-\frac{\beta}{4} \sum_{i=1}^N \lambda_i^2\right) \text{ Jacobian}(\hat{T})$

The \hat{T} map \Rightarrow

$$T(U) = \left(\frac{U_{1,2}}{U_{1,1}}, \dots, \frac{U_{1,N}}{U_{1,1}}, \frac{U_{2,3}}{U_{2,2}}, \dots, \frac{U_{2,N}}{U_{2,2}}, \dots, \frac{U_{N-1,N}}{U_{N-1,N-1}} \right)$$

\downarrow
unitary/orthogonal

$$T \text{ is "bijective."} \quad \hat{T}(z, \lambda) := T^{-1}(z) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{pmatrix} (\bar{T}'(z))^*$$

$$\text{Jacobian}(\hat{T}) = g(z) \prod_{i < j} (\lambda_i - \lambda_j)^\beta \quad (\text{can be computed or can be argued that } \lambda_i - \lambda_j \text{ has to be a factor}).$$

Thus, jt density of eigenvalues \Rightarrow

$$\mathbf{1}_{\lambda_1 < \lambda_2 < \dots < \lambda_N} \prod_{i < j} (\lambda_i - \lambda_j)^\beta \exp\left(-\frac{\beta}{4} \sum_{i=1}^N \lambda_i^2\right)$$

Caution $\Rightarrow T, \hat{T}$ are not bijective. But they turn out to be bijective after discarding some Lebesgue zero measure set and introducing the orders in eigenvalues. For a complete proof, see [AGZ, Section 2.5]

Lecture 2 Last time we saw jf density of eigenvalues for GUE :→

$$\propto \prod_{\lambda_1 < \lambda_2 < \dots < \lambda_N} \prod_{i < j} (\lambda_i - \lambda_j)^2 \exp\left(-\frac{1}{2} \sum_{i=1}^N \lambda_i^2\right)$$

The proportionality constant can be computed but not required.

Let us study this probability density :→

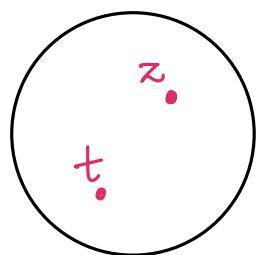
$$\prod_{i < j} (\lambda_i - \lambda_j)^2 \rightarrow \text{density is small when } \lambda_i \text{ and } \lambda_j \text{ are close.}$$

So, two eigenvalues typically cannot be close to each other. So, there is a repulsive force between eigenvalues. They try to stay far apart.

$$\exp\left(-\frac{1}{2} \sum_{i=1}^N \lambda_i^2\right) \rightarrow \text{density is small when } |\lambda_i| \text{ is large.}$$

Thus the eigenvalues typically can't be too large. Note that this force is opposite of repulsive force.

The two forces balances each other out and produce the **semicircle distribution** at **equilibrium**. To explain this, let us introduce the concept of **Potential theory** where we shall view the eigenvalues as **charged particles in a field**.



Suppose we distribute one unit charge on a circle according to some measure μ .

Electric potential at $z \Rightarrow \propto \underbrace{\log \frac{1}{|z-t|}}_{\text{arising from } t} \quad (z \neq t)$
 This comes from Gauss Law.
 ↳ log is due to 2D;

Total electric potential at $z \Rightarrow \propto \int_{t \neq z} \log \frac{1}{|z-t|} d\mu(t)$

Energy of the system \Rightarrow sum of all electric potentials

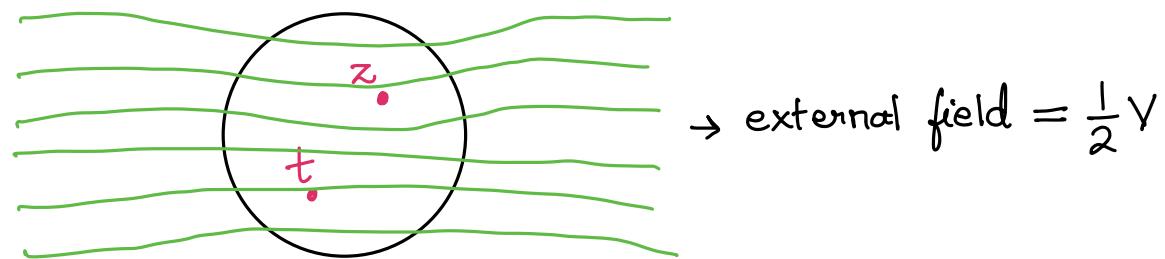
$$\iint_{t \neq z} \log \frac{1}{|z-t|} d\mu(t) d\mu(z).$$

The system reaches equilibrium when the energy is minimum.

The fundamental problem in electrostatics is

$$\text{minimize } \iint_{t \neq z} \log \frac{1}{|z-t|} d\mu(t) d\mu(z) \text{ over measures } \mu.$$

The situation becomes a bit more complicated in presence of an external field.



Electric potential at $z \Rightarrow \log \frac{1}{|z-t|} + \frac{1}{2}V(z) + \frac{1}{2}V(t)$
 arising from t

Then equilibrium is attained by minimizing (assume $\mu(x \neq y) = 0$)

$$\left. \begin{aligned} & \iint \left[\log \frac{1}{|z-t|} + \frac{1}{2}V(z) + \frac{1}{2}V(t) \right] d\mu(z) d\mu(t) \\ &= - \iint \log |z-t| d\mu(z) d\mu(t) + \int V(z) d\mu(z). \end{aligned} \right\} (*)$$

The minimizer of $(*)$ is called equilibrium measure.

How is it related to GOE/GUE eigenvalues?

Recall the jf density of GUE (ignore the proportionality const) \Rightarrow

$$\begin{aligned}
 & \prod_{i < j} (\lambda_i - \lambda_j)^2 \exp\left(-\frac{1}{2} \sum_{i=1}^N \lambda_i^2\right) \\
 &= \exp\left(2 \sum_{i < j} \log |\lambda_i - \lambda_j| - \frac{1}{2} \sum_{i=1}^N \lambda_i^2\right) \\
 &\propto \exp\left(\sum_{i \neq j} \log \left|\frac{\lambda_i}{\sqrt{N}} - \frac{\lambda_j}{\sqrt{N}}\right| - \frac{1}{2} \sum_{i=1}^N \lambda_i^2\right) \\
 &= \exp\left(-N^2 \left\{-\frac{1}{N^2} \sum_{i \neq j} \log \left|\frac{\lambda_i}{\sqrt{N}} - \frac{\lambda_j}{\sqrt{N}}\right| - \frac{1}{N} \sum_{i=1}^N \frac{(\lambda_i/\sqrt{N})^2}{2}\right\}\right) \\
 &= \exp(-N^2 I_V(L_N)).
 \end{aligned}$$

where $L_N = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i/\sqrt{N}}$, $V(x) = \frac{x^2}{2}$

$$I_V(\mu) := \iint_{z \neq t} -\log |z-t| d\mu(t) d\mu(z) + \int V(z) d\mu(z)$$

μ is now a prob. measure on \mathbb{R} .

Now it is intuitively clear, $L_N \rightarrow \mu_{eq}$ (equilibrium measure)
 $(\mu_{eq}$ will be semicircle in this case)

Furthermore, this formulation also tells you

$$P(L_N \approx v) \approx \exp(N^2 [I_V(\mu) - I_V(v)])$$

This is related to large deviations of empirical distribution.

Historical notes \Rightarrow The variational problem in (*) was first studied by Gauss and its often known as Gauss variational problem. Frostman studied this problem in 1920s and showed uniqueness and existence of equilibrium measure. Then, the Polish school led by Franciszek Leja made significant contributions in 1935–1960. Then, in the late 20th century, it gathered renewed interest as they found to be useful in studying orthogonal polynomials and as we see in random matrix theory as well.

Theorem \Rightarrow Suppose V is a continuous potential with

$$V(x) \geq 2 \log(1+x^2) \text{ for large } x.$$

$$\text{Set} : F_V := \inf \left\{ I_V(\mu) \mid \mu \text{ probability measure on } \mathbb{R} \right\}$$

Then,

(a) F_V is finite

(b) there exist a unique probability measure μ_{eq} such that

$$I_V(\mu_{eq}) = F_V$$

(c) μ_{eq} has a compact support say S .

$$(d) \int \left[-\log|z-t| + \frac{1}{2}V(z) + \frac{1}{2}V(t) \right] d\mu_{eq}(t) = F_V \quad \text{for } z \in S$$

$$\int \left[-\log|z-t| + \frac{1}{2}V(z) + \frac{1}{2}V(t) \right] d\mu_{eq}(t) \geq F_V \quad \text{for } z \notin S$$

(e) (d) determines μ_{eq} uniquely.

Lecture 3 In fact μ_{eq} will have a density. I will prove (d).

$$\text{Set } U_V(z) := \int_{t \neq z} \log \frac{1}{|z-t|} d\mu_{eq}(t) + \frac{1}{2} V(z)$$

$$\text{To show } \Rightarrow U_V(z) \geq F_V - \frac{1}{2} \int V(t) d\mu_{eq}(t) = \tilde{F}_V \quad \text{for all } z$$

$A = \{z \mid U_V(z) < \tilde{F}_V\}$ has +ve Leb measure for the sake of contradiction.

Then $A_0 = \{z \mid U_V(z) < \tilde{F}_V - \frac{1}{n_0}\}$ has +ve Leb measure for n_0 large enough.

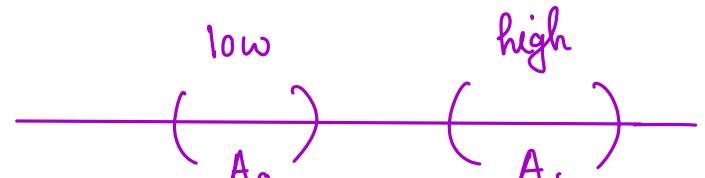
$$\text{Now } \int U_V(z) d\mu_{eq}(z) = \tilde{F}_V$$

So, there must be a compact set A_1 s.t. $U_V(z) > \tilde{F}_V - \frac{1}{2n_0} \quad z \in A_1$

and A_1 has +ve Leb measure. Let $m = \mu_{eq}(A_1) > 0$.

Get μ_0 supported on A_0

$$\text{s.t. } \mu_0(A_0) = m, \quad I_V(\mu_0) < \infty,$$



$$\mu_r^* = \begin{cases} \mu_{eq} + r\mu_0 & \text{on } A_0 \\ (1-r)\mu_{eq} & \text{on } A_1 \\ \mu_{eq} & \text{otherwise} \end{cases}$$

One can check $I_V(\mu_r^*) < I_V(\mu_{eq})$ for small enough $r > 0$.

This is a contradiction.

$$B_\varepsilon = \{z \mid U_V(z) > \tilde{F}_V + \varepsilon\} \cap \text{Supp}(\mu_{eq})$$

$$\begin{aligned}
\int_{B_\varepsilon} u_v(z) d\mu_{eq}(t) &\geq \int_{B_\varepsilon} u_v(z) d\mu_{eq}(t) + \int_{\text{Supp}(\mu_{eq}) \setminus B_\varepsilon} u_v(t) d\mu_{eq}(t) \\
&\geq \mu_{eq}(B_\varepsilon) (\tilde{F}_v + \varepsilon) = \tilde{F}_v + \varepsilon \mu_{eq}(B_\varepsilon) \\
&\quad + (1 - \mu_{eq}(B_\varepsilon)) \tilde{F}_v
\end{aligned}$$

So, $\mu_{eq}(B_\varepsilon) = 0 \quad \forall \varepsilon > 0$. So, this proves (d).

Towards the semicircle $\Rightarrow \mu_{eq}$ has a density say h

For $V(x) = \frac{x^2}{2}$, it's reasonable to assume h is supported on a symmetric interval. $\Rightarrow [-a, a]$

$$\int_{-a}^a -\log|z-t| h(t) dt + \frac{z^2}{4} \text{ is const on } [-a, a].$$

So,

$$\underbrace{\int_{-a}^a \frac{h(t) dt}{t-z}}_{\text{the integrand}} + \frac{z^2}{4} = 0 \quad \forall z \in [-a, a].$$

\hookrightarrow the integrand is not integrable as $t \rightarrow z$

So, this integral is interpreted as Cauchy principal value.

$$\int_{-a}^{*a} \frac{h(t) dt}{t-z} := \lim_{\varepsilon \downarrow 0} \left\{ \int_{-a}^{z-\varepsilon} \frac{h(t) dt}{t-z} + \int_{z+\varepsilon}^a \frac{h(t) dt}{t-z} \right\}$$

After a change of variable \Rightarrow

$$A[\phi](z) - \frac{1}{\pi} \int_1^{*1} \frac{\phi(u) du}{u-z} = f(z) = -\frac{az}{2\pi} \quad \text{for } u \in [-1, 1]$$

where $\phi(u) = h(au)$.

This is an integral equation known as **airfoil equation**.

which appears in aerodynamics. Solutions are given by

$$\phi(x) = -\frac{1}{\sqrt{1-x^2}} A[\sqrt{1-y^2} f(y)](x) + \frac{C}{\sqrt{1-x^2}} \quad \left(\begin{array}{l} \text{see [Tri, Sec 4.3]} \\ \text{for details} \end{array} \right)$$

$$= \frac{1}{\sqrt{1-x^2}} \int_1^{*1} \underbrace{\frac{a}{2\pi} \frac{y\sqrt{1-y^2}}{y-x} dy}_{\text{has an antiderivative}} + \frac{C}{\sqrt{1-x^2}} \quad x \in [-1, 1]$$

$$h(x) = \phi\left(\frac{x}{a}\right) = \frac{1}{2\pi} \sqrt{a^2 - x^2} + \frac{\tilde{C}}{\sqrt{a^2 - x^2}} \quad x \in [-a, a]$$

$$\int_{-a}^a h(x) dx = \frac{1}{2\pi} \cdot \frac{a^2 \pi}{2} + \tilde{C} \pi = 1 \Rightarrow \tilde{C} = \left(1 - \frac{a^2}{4}\right)/\pi$$

$$\text{Thus } h(x) = \frac{1}{2\pi} \sqrt{a^2 - x^2} + \frac{1 - \frac{a^2}{4}}{\pi \sqrt{a^2 - x^2}} \rightarrow \mu_a$$

$\tilde{C} \geq 0$ as $h \geq 0$ so $a \leq 2$.

Now, compute $I_V(\mu_a)$ to see that $a=2$ is minimizer.

This gives the **semicircle**.

Summary and Beyond

We saw how potential theory can be used to derive limits of empirical distribution in random matrix theory.

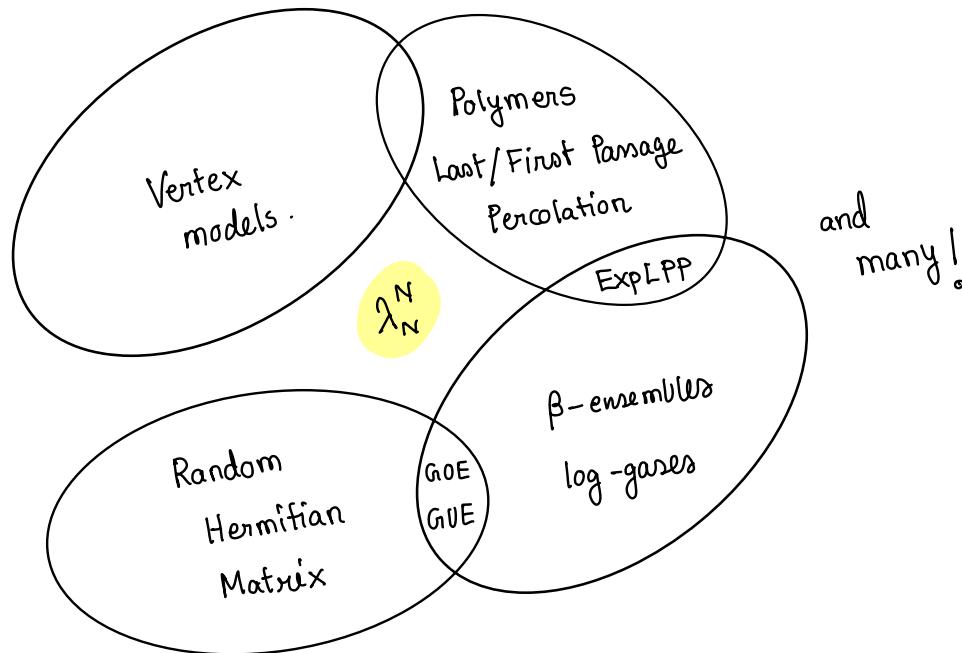
In general, potential theory is useful in

- a) Law of Large numbers for empirical distribution
- b) Large deviations for empirical distribution and largest eigenvalue
- c) Fluctuations $\Rightarrow \sum_{i=1}^N \frac{\lambda_i}{\sqrt{N}} \longrightarrow N(0, \sigma^2)$ (for GUE)

The theory works well even for beyond random matrix theory, for models where similar structures are present.

Often in RMT, the quantity of interest is largest eigenvalue λ_N^N

KPZ universality class.



and many!

λ_N^N and their analogues
show same
limiting distribution

↓
Tracy-Widom distribution

References :-

AGZ :- An introduction to random matrices - Anderson-Guionnet-Zeitouni

ST :- Logarithmic Potentials with external fields - Saff-Totik

Tri :- Integral equations - Tricomi

F :- Log-gases and Random matrices - Forrester.

Jo1 :- On Fluctuations of eigenvalues of random Hermitian matrices - Johansson

Jo2 :- Shape fluctuations and random matrices - Johansson

Jo3 :- Discrete orthogonal polynomial ensembles and the Plancharel measure - Johansson.

Extra :-

$$A[\phi](x) = \frac{1}{\pi} \int_{-1}^{*1} \frac{\phi(y) dy}{y-x} \quad \rightarrow \text{finite Hilbert transformation.}$$

$$\text{Hilbert transformation} \rightarrow H[\phi](x) = \frac{1}{\pi} \int_{-\infty}^{*\infty} \frac{\phi(y)}{y-x} dx.$$

$$\text{Fact} \rightarrow H[H[\phi]] = -\phi \quad (\text{Reciprocity theorem})$$

Thus the equation $H[\phi](x) = f(x)$ has a unique solⁿ for ϕ .

However, $A[\phi]$ does not have such nice reciprocity result.

In fact, the equation, $A[\phi](x) = f(x)$ —— (*) has many solutions.

$$\phi(x) = \frac{1}{\sqrt{1-x^2}} \quad A[\phi] = 0$$