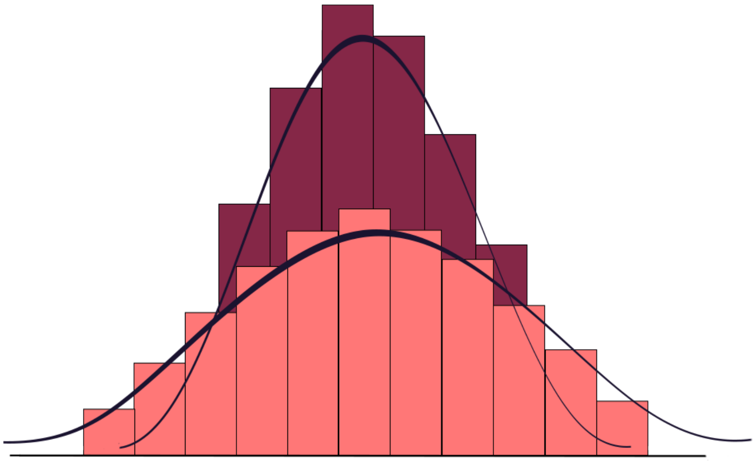
Finding Standard Error of Estimators with Complicated Analytical Form with the Help of Bootstrapping



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Declaration:

I affirm that I have identified all my sources and that no part of my dissertation paper uses unacknowledged materials.

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1 Introduction

In Statistical inference, one of the most important problems is to estimate the value of one or more parameters related to certain distribution. We used to take a sample from that population of interest and combine the observations, that comes under the sample, into a suitable statistic that could estimate the parameter of the interest.

Now, in inference problems, after we have obtained the estimate of the parameter of the distribution, we may look forward -

1. to find out how good the estimator is, in estimating the parameter of interest.

2. to find out the confidence interval of the estimator.

3. to proceed to a test, related to the parameter.

In each of the problems mentioned above, it is very important to to find out the standard error of the estimator. But in the cases where the form of the estimator is very complicated it is not possible to find out the value of standard error of the estimator theoretically. We have to depend on the simulation methods to find out the standard error.

However in real life problems, we are given with a single sample, therefore, we are not allowed to generate as many samples as we wish, like we do in case of simulation. From this single, given sample we can obtain only one value of the statistic. Therefore, in this way we cannot obtain the standard error of the estimator. Also we assume that, we are given with a sample where the large sample properties of the estimator is not applicable.

In such a situation, what will be our tactics to find out the standard error of the estimators of such complicated form? The answer to this question will be discussed in this dissertation. Throughout the dissertation, our objectives will be -

1. Finding a statistical method, by using which we can find out the standard error of such a complicated estimator.

2. To study on the techniques and methodology of the method.

3. To apply the method, for finding the standard error of some estimator of some complicated form, for obtaining it's standard error.

2 Basic Problem

In this section we will be understanding the basic problem that arises when we come to find the variance of an estimator that is very complicated in it's form.

2.1 The Problem

Suppose we are interested in estimation of a parameter θ for a population represented by the distribution function F. Here the functional form of F is known, only the parameter θ is unknown. We can draw a random sample, $\underset{\sim}{x} = (x_1, x_2, ..., x_n)$ of size n, from this population to gather information on θ. In the next step we can combine the components in $\underset{\sim}{x}$ to get a suitable statistic(obtained by any suitable method of estimation for example Maximum Likelyhood Estimation or Method of Moment Estimation etc.) T estimating θ. Next we compute the value of the statistic for the given sample $\underset{\sim}{x} = (x_1, x_2, ..., x_n)$, to get the estimate of the parameter θ.

Now comes the problem of estimating the variance of the estimator, obtained previously. The first thing that we have to keep in mind, is that, the form of the estimator is already complicated. So theoretically we cannot get a function $g(\theta)$ representing the variance of the estimator(Like we have $\frac{\sigma^2}{n}$ as variance of $\bar{x}$, when we draw sample of size n from a $N(\mu, \sigma^2)$) because the expression for variance will be of more complicated form.

Given this situation, the next approach that we can think of, is to perform a simulation. We can go on drawing random samples from the population for a large number of times, Say R times. Then for each of the samples we can get a value of the statistic T. For R samples we can get R values of T. Let us denote these values as $T_1, T_2, ..., T_R$. Then an estimate to the variance of the estimator T can be given by -

$$\frac{1}{R}\sum_{i=1}^{R}(T_i - \bar{T})^2, \ \ where \ \ \bar{T} = \frac{1}{R}\sum_{i=1}^{R} T_i$$

But here the problem is that, we have to draw more than one sample from the population to implement the simulation study. But in a real life problem we are given with a single sample and cannot draw another sample due to the limitation of the resources like time, labour, money etc. In that case we will have only one value of the statistic T and with this single value we cannot empirically obtain the variance in the way described previously.

Also here we assume that the sample size, i.e. n, is not quite large. So the large sample properties of estimator cannot be applied to get a large sample variance of the estimator.

Therefore, in our analysis, the theoretical variance, Simulation technique, the large sample properties of the estimator failed to provide us a figure of estimated variance for the estimator T for θ. Therefore, we have to think about another method to solve this problem.

2.2 A Solution to the Problem

The "Bootstrap Method" may be a solution to the problem discussed earlier. It is basically a modified simulation study, where we generate a number of Bootstrap samples from a given sample, drawn originally from the population. For each of the Bootstrap samples we can calculate the value of the statistic T. The estimate of the variance in this method will be given by the variance of the all values of T computed from the Bootstrap samples. In the next section we will discuss the "Bootstrap Method" elaborately.

3 Bootstrap Method

This section is devoted to the Bootstrap Method. We will learn about the general procedure of Bootstrapping in this part.

3.1 Necessity

In our problem of estimation of θ, from the population distribution represented by the Distribution Function F, we have seen that if the statistic T estimating the parameter θ is of complicated form then it is very difficult to compute the varinace (or the standard error) of the statistic T. Here the theoretical expression of the variance may be tedious to obtain, since the form of estimator is complicated in its form. Also we are given with only one sample, so we cannot make use of the simulation technique. Furthermore we are assuming that the size of the given sample is not large enough for the large sample properties of the estimator to be applied.

These situations give rise to the need of using Bootstrap Method on the sample to compute the estimated variance (or standard error) of the statistic.

3.2 Bootstrap Method

The Bootstrap Method is a technique of estimating the parameters or parametric function related to a population, through averaging the estimates from several small data samples. These small samples, which are known as Bootstrap data, are drawn by simple random sampling with replacement from a given sample.

Suppose we are interested in estimating the parameter θ from a population distribution represented by the distribution function F(x). Here the functional form of F(x) is known, only the parameter θ is unknown. Let here Θ be the parametric space. Now to collect information about θ we draw a random sample $\underset{\sim}{x} = (x_1, x_2, ..., x_n)$ of size n from the population. Suppose $L(\theta, \underset{\sim}{x})$, $\theta \in \Theta$ be the likelyhood function of θ corresponding to the sample $\underset{\sim}{x}$. By maximizing $L(\theta, \underset{\sim}{x})$ for θ over Θ suppose we get the Maximum Likeyhhod Estimator of θ as

$$\hat{\theta}_{MLE} = T(\underset{\sim}{x})$$

Now from the given sample $\underset{\sim}{x} = (x_1, x_2, ..., x_n)$ will draw a bootstrap sample or bootstrap data by SRSWR. Call it $x_1^{*1}, x_2^{*1}, ..., x_n^{*1}$. It is the first bootstrap data.We can calculate the statistic T for is bootstrap data. Lets denote the value of T by T_1.

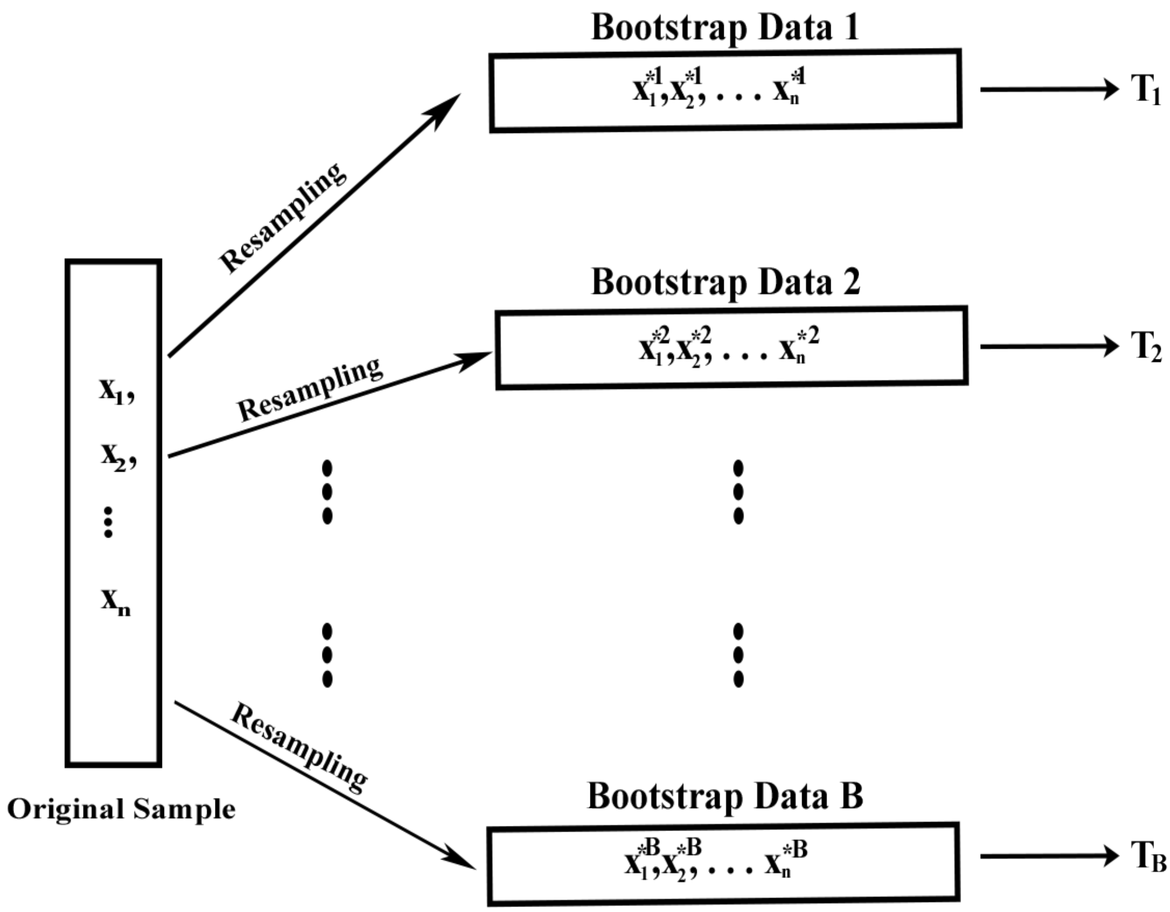

Figure 1: Resampling

In the next step we can again draw the second bootstrap data $x_1^{*2}, x_2^{*2}, ..., x_n^{*2}$ from the initial sample $\underset{\sim}{x} = x_1, x_2, ..., x_n$ and then again can calculated the value of T for this second bootstrap data, and denote it by T_2. In this way we can repeat the procedure for B times to get B values of T, say $T_1, T_2, ..., T_n$, from B bootstrap data.

After obtaining B such values of T we can take the estimate of θ as-

$$\hat{\theta}_{bootstrap} = \frac{1}{B}\sum_{i=1}^{B} T_i$$

This is the general procedure for Bootstrapping for estimating a parameter by generating multiple samples with replacement from a single given sample. It is an extreamly powerful method for our purpose. We will see in the next section how this method will serve our purpose to get the variance of a complicated estimator from a single given sample.

3.3 General Method to find the variance of a complicated estimator using Bootstrap Method

We have seen how the bootstrap method works. How lets see how we can make use of this method to find out the variance of an estimator of complicated mathematical form.

Suppose like before, we are interested in the parameter θ from the distribution with Distribution Function F(x), here only the θ is unknown and the form of the distribution function is known. Here, like we did previously, we can obtain an estimator of the parameter θ, through some method of estimation(like: Maximum Likelyhood Estimation, Method of Moment Estimation). Say the estimator being $T(\underset{\sim}{x})$, where $\underset{\sim}{x} = (x_1, x_2, ..., x_n)$ is the given sample. We assume that here the form of $T(\underset{\sim}{x})$ is very complicated.

Unlike the example given perviously, here we are interested in estimating the variance of the estimator $T(\underset{\sim}{x})$. Here also, we have to generate bootstrap data by SRSWR from $\underset{\sim}{x} = (x_1, x_2, ..., x_n)$. Say we create B such bootstrap data namely-

$$\{(x_1^{*1}, x_2^{*1}, ...x_n^{*1}), (x_1^{*2}, x_2^{*2}, ...x_n^{*2}), ..., (x_1^{*B}, x_2^{*B}, ...x_n^{*B})\}$$

Now for each of the B bootstrap data, we can calculate the estimator T. Let's say $T_1, T_2, ...T_B$ are the values of the statistic T for respectively, $1^{st}, 2^{nd}, ..., B^{th}$ bootstrap data. Now our target is to get an estimate of the variance of T.

Now, here a small thing that we have to remember that T is a statistic, which is calculated based on the values that comes under the chosen sample. Now the observations that comes into a sample changes every time we choose a new sample. Which is why the value of T is supposed to vary over sample to sample. This fluctuations in the values of T is known as "Sampling Fluctuations". The standard error of T measures this variability of T over different samples, which we want to get estimated.

Here we have calculated B different values of T for B bootstrap data generated from a single sample. The bootstrap data are chosen randomly from the give sample. Therefore, the fluctuations in the values of T for different bootstrap data are due to chance causes of variation. Hence the variability in the values of T over different bootstrap data gives the idea about the standard error of T. Hence we can estimate the standard error of T as-

$$\hat{SE}(T) = \sqrt{\frac{1}{B}\sum_{i=1}^{B}(T_i - \frac{1}{B}\sum_{j=1}^{B}T_j)^2}$$

This formulae will serve our purpose in finding variances to the complicated esimators. We will illustrate the technique described above through some examples covering estimator of the parameters of several population distributions.

4 Use of Bootstrap in estimation of parameters with simple estimator

We have learned the method of Bootstrapping and its working principal, now we will apply it in estimation of parameters from a population that have a complicated estimator, Like we have planned. But before proceeding further let us see how this method works for the parameters whose estimator is not so complicated so that that we can check how close the bootstrap obtained variance are, as compared to the original variance obtained from the theory. Also we will see how the the result will have been if we used simulation technique instead of Bootstrapping.

4.1 Variance of estimator of μ for Normal Distribution $(N(\mu,1))$

Here, the pdf(Probability Density Function) of the $N(\mu,1)$ is given by -

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-\mu)^2}, -\infty < x < \infty$$

where μ is unknown parameter such that $-\infty < \mu < \infty$.

We are to estimate the parameter μ. Suppose a random sample $\underset{\sim}{x} = (x_1, x_2, ..., x_n)$ is chosen from the given population. The log-likelyhood function is given by-

$$l(\mu;\underset{\sim}{x}) = ln(L(\mu;\underset{\sim}{x})) = -\frac{n}{2}ln(2\pi) - \frac{1}{2}\sum_{i=1}^{n}(x_i - \mu)^2, -\infty < \mu < \infty$$

now, taking the partial derivative of the log-likelyhood $l(\mu;\underset{\sim}{x})$ w.r.t μ we get-

$$\begin{aligned}
\frac{\partial l}{\partial \mu} &= \sum_{i=1}^{n}(x_i - \mu) \\
&= \sum_{i=1}^{n} x_i - n\mu \\
&= n(\bar{x} - \mu) \quad where, \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \\
&= \begin{cases} > 0 \quad if \ \ \mu < \bar{x} \\ < 0 \quad if \ \ \mu > \bar{x} \end{cases}
\end{aligned}$$

Therefore, at $\mu = \bar{x}$ the log-likelyhood $l(\mu;\underset{\sim}{x})$ is maximized. Hence, the MLE of μ is given by, $\hat{\mu} = \bar{x}$. We here will draw B bootstrap data from the given sample $\underset{\sim}{x}$, namely $\underset{\sim}{x}_1, \underset{\sim}{x}_2, ..., \underset{\sim}{x}_B$. From each of the $\underset{\sim}{x}_i (i = 1(|)B)$ we calculate $\hat{\mu}$. Denote these $\hat{\mu}'s$ as $\hat{\mu}_1, \hat{\mu}_2, ..., \hat{\mu}_B$.

Now, we will calculate the $\hat{\mu}$ and $\hat{SE}(\hat{\mu})$ as follows,

$$\hat{\mu}_{bootstrap} = \frac{1}{B}\sum_{i=1}^{B}\hat{\mu}_i$$

$$\hat{SE}(\hat{\mu})_{bootstrap} = \sqrt{\frac{1}{B}\sum_{i=1}^{B}(\hat{\mu}_i - \frac{1}{B}\sum_{j=1}^{B}\hat{\mu}_j)^2}$$

Again for simulation, we will obtain R samples from the the distribution $N(\mu, 1)$. Then for each samples we calculate R values of $\hat{\mu}$ namely $\hat{\mu}_1, \hat{\mu}_2, ..., \hat{\mu}_R$. Then we obtain the $\hat{\mu}$ and $\hat{SE}(\hat{\mu})$ as follows,

$$\hat{\mu}_{simulation} = \frac{1}{R}\sum_{i=1}^{R}\hat{\mu}_i$$

$$\hat{SE}(\hat{\mu})_{simulation} = \sqrt{\frac{1}{R}\sum_{i=1}^{R}(\hat{\mu}_i - \frac{1}{R}\sum_{j=1}^{R}\hat{\mu}_j)^2}$$

Obtained Result:

Here we have generated a random sample from $N(\mu,1)$. For the purpose of generation we have take μ=50. Also we have taken n=20, B=100, R=1000. Here while calculation we have repeated the bootstrap method for R=1000 times just to see how the bootstrap obtained results are varying. The results are given as follows-

Quantity	Value by Bootstrap	Value by Simulation
$\hat{\mu}$	50.001	50.012
$\hat{SE}(\hat{\mu})$	0.211	0.221

Table 1: Results for estimation of μ in $N(\mu,1)$

Observe that, here the original $\hat{SE}(\hat{\mu}) = \frac{1}{\sqrt{n}} = 0.224$, which is very close to the results that we obtained.

Now to get an idea about the variability around the original values we have plotted the histogram of bootstrap obtained estimates and the simulation obtained estimates. Observe that here the bootstrap

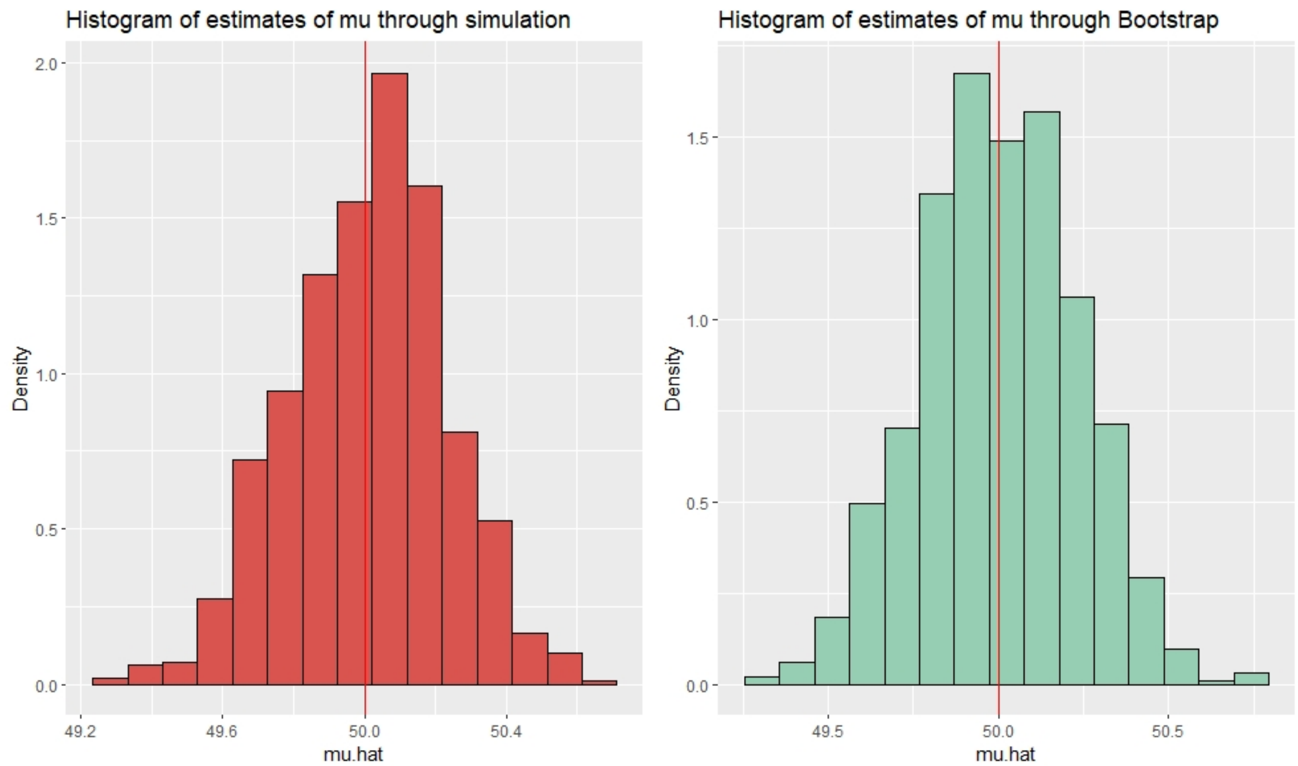

Figure 2: Histogram of the Estimates of μ

obtained estimates(Histogram in green) are clustering to the original parameter value(marked in red line).

4.2 Variance of estimator of p for Bernoulli Distribution (Bernoulli(p))

Here the Probability Mass Function of $Bernoulli(p)$ is given by-

$$f(x) = p^x(1-p)^{1-x}, x = 0, 1$$

here p, such that $0 < p < 1$ is the unknown parameter and it is to be estimated. So we choose a random sample $\underset{\sim}{x} = (x_1, x_2, ..., x_n)$. The log-likelyhood function is given by-

$$l(p;\underset{\sim}{x}) = ln(L(p;\underset{\sim}{x})) = (\sum_{i=1}^{n} x_i)ln(p) + (n - \sum_{i=1}^{n} x_i)ln(1-p), 0 < p < 1$$

Therefore, taking the partial derivative on the log-likelyhood $l(p,\underset{\sim}{x})$ w.r.t p we get-

$$\begin{aligned}
\frac{\partial l}{\partial p} &= \frac{\sum_{i=1}^{n} x_i}{p} - \frac{n - \sum_{i=1}^{n} x_i}{1-p} \\
&= \frac{n\bar{x}}{p} - \frac{n - n\bar{x}}{1-p} \quad where, \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \\
&= n(\frac{\bar{x}}{p} - \frac{1-\bar{x}}{1-p}) \\
&= n(\frac{\bar{x}(1-p) - (1-\bar{x})p}{p(1-p)}) \\
&= \frac{n}{p(1-p)}(\bar{x} - p) \\
&= \begin{cases} > 0 \quad if \;\; p < \bar{x} \\ < 0 \quad if \;\; p > \bar{x} \end{cases}
\end{aligned}$$

Therefore at $p = \bar{x}$ the log-likelyhood $l(p;\underset{\sim}{x})$ is maximized. Hence we get the MLE of p, which is given by, $\hat{p} = \bar{x}$.

Now, for bootstrap we will draw B bootstrap data from the sample $\underset{\sim}{x}$ and for each bootstrap data we compute $\hat{p}$ and denote the values as $\hat{p}_1, \hat{p}_2, ..., \hat{p}_B$ and obtain the bootstrap estimate of p and $\hat{SE}(\hat{p})$ as follows-

$$\hat{p}_{bootstrap} = \frac{1}{B}\sum_{i=1}^{n} \hat{p}_i$$

$$\hat{SE}(\hat{p})_{bootstrap} = \sqrt{\frac{1}{B}\sum_{i=1}^{n}(\hat{p}_i - \frac{1}{B}\sum_{j=1}^{n} \hat{p}_j)^2}$$

For simulation, like we did in the previous example of Normal mean, we will generate R samples from the distribution and calculate the value of $\hat{p}$ for each of them, say, the values are $\hat{p}_1, \hat{p}_2, ..., \hat{p}_R$. Then the $\hat{p}$ and $\hat{SE}(\hat{p})$ is given by-

$$\hat{p}_{simulation} = \frac{1}{R}\sum_{i=1}^{n}\hat{p}_i$$

$$\hat{SE}(\hat{p})_{simulation} = \sqrt{\frac{1}{R}\sum_{i=1}^{n}(\hat{p}_i - \frac{1}{R}\sum_{j=1}^{n}\hat{p}_j)^2}$$

Obtained Result:

Here we have generated a random sample from $Bernoulli(p)$. For the purpose of generation we have take p=0.3. Also we have taken n=20, B=100, R=1000. Here while calculation we have repeated the bootstrap method for R=1000 times just to see how the bootstrap obtained results are varying. The results are given as follows- Also the theoretical $\hat{SE}(\hat{p}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.1024$, close to both the bootstrap

Quantity	Value by Bootstrap	Value by Simulation
$\hat{p}$	0.2991	0.3042
$\hat{SE}(\hat{p})$	0.0985	0.1002

Table 2: Results for estimation of p in $Bernoulli(p)$

obtained and the simulation obtained result.

We have also plotted the histogram of the estimates of the simulation study and also the bootstrap

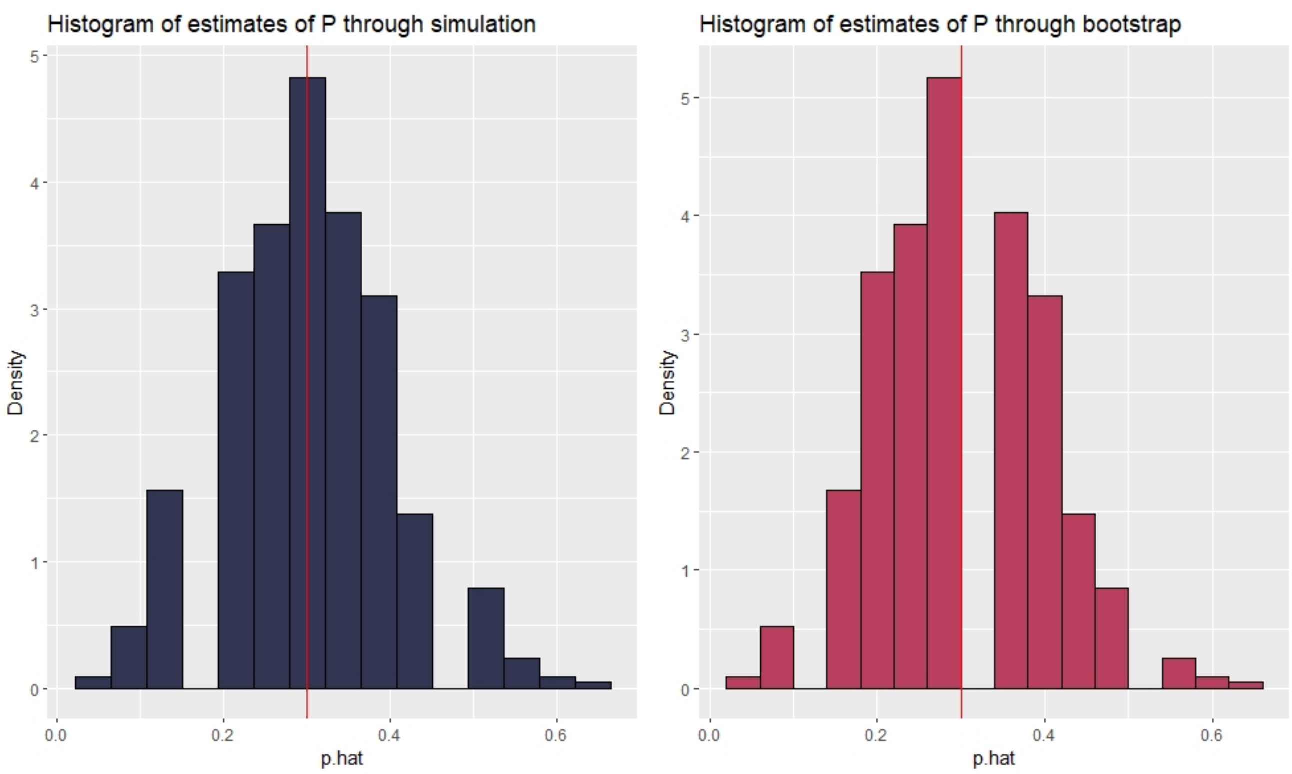

Figure 3: Histogram of the Estimates of p

obtained estimates replicated for R times, seperately; to get a visual idea about the variability of the estimators. Here the bootstrap obtained estimates of p are clustered around p.

5 Use of Bootstrap in estimation of parameters with complicated estimator

Now that we get an idea that the bootstrap method works good for estimators of the parameter with simple form, we will apply the method for some distributions with some complicated form of estimator. Here we will not going for simulation. We will only calculate the estimator T, estimating θ in certain distribution, for B bootstrap data and denote it as $T_1, T_2, ..., T_B$. Now we will calculate $\hat{\theta}$ and $\hat{SE}(\hat{\theta})$ as follows-

$$\hat{\theta} = \frac{1}{B}\sum_{i=1}^{B} T_i$$

$$\hat{SE}(\hat{\theta}) = \sqrt{\frac{1}{B}\sum_{i=1}^{B}(T_i - \frac{1}{B}\sum_{j=1}^{B} T_j)^2}$$

let's try it for some distributions.

5.1 Variance of estimator of μ and σ for $DE(\mu, \sigma^2)$

Here the Probability Density Function is given by,

$$f(x) = \frac{1}{2\sigma} e^{-|\frac{(x-\mu)}{\sigma}|}, -\infty < x < \infty$$

where μ and σ are the two parameters, such that $-\infty < \mu < \infty$ and $\sigma > 0$, that we are interested.

we choose a random sample, $\underset{\sim}{x} = (x_1, x_2, ..., x_3)$, from the population. Here the log-likelyhood function is given by-

$$l(\mu, \sigma; \underset{\sim}{x}) = ln(L(\mu, \sigma; \underset{\sim}{x})) = -ln(2^n) - nln(\sigma) - \sum_{i=1}^{n} |\frac{x_i - \mu}{\sigma}|; -\infty < \mu < \infty, \sigma > 0$$

For the maximisation of $l(\mu, \sigma; \underset{\sim}{x})$ we will make use of the following result-

Result:

Given a data set $(x_1, x_2, ..., x_n)$ and for any arbitrary real number A, the expression $\sum_{i=1}^{n} |x_i - A|$ is minimized if $A = median(\underset{\sim}{x})$.

Now, $l(\mu, \sigma; \underset{\sim}{x})$ is a decreasing function of $\sum_{i=1}^{n} |x_i - \mu|$. Hence, $l(\mu, \sigma; \underset{\sim}{x})$ is maximized if $\sum_{i=1}^{n} |x_i - \mu|$ is minimized, i.e. $\mu = median(\underset{\sim}{x})$. Now if we take partial derivative of $l(\mu, \sigma; \underset{\sim}{x})$ w.r.t σ, we get-

$$
\begin{aligned}
\frac{\partial l}{\partial \sigma} &= -\frac{n}{\sigma} + \frac{\sum_{i=1}^{n} |x_i - median(\underset{\sim}{x})|}{\sigma^2} \qquad putting, \mu = median(\underset{\sim}{x}) \\
&= \frac{n}{\sigma^2}\Big(\frac{1}{n}\sum_{i=1}^{n} |x_i - median(\underset{\sim}{x})| - \sigma\Big) \\
&= \begin{cases} > 0 & if \;\; \sigma < \frac{1}{n}\sum_{i=1}^{n} |x_i - median(\underset{\sim}{x})| \\ < 0 & if \;\; \sigma > \frac{1}{n}\sum_{i=1}^{n} |x_i - median(\underset{\sim}{x})| \end{cases}
\end{aligned}
$$

Therefore, at $\mu = median(\underset{\sim}{x})$ and $\sigma = \frac{1}{n}\sum_{i=1}^{n} |x_i - median(\underset{\sim}{x})|$ the log-likelyhood $l(\mu, \sigma; \underset{\sim}{x})$ is maximized. Hence we have the MLE's of μ and σ-

$$\hat{\mu} = median(\underset{\sim}{x}), \quad \hat{\sigma} = \frac{1}{n}\sum_{i=1}^{n} |x_i - median(\underset{\sim}{x})|$$

After applying the bootstrap method we get the following results-

Obtained Result:

Here we have generated the initial sample by setting (μ, σ)=(10,2). The sample size has been taken as n=15 and the number of bootstrap data generated is B=1000. The estimates are given in the following table-

Quantity	Value by Bootstrap
$\hat{\mu}$	10.224
$\hat{SE}(\hat{\mu})$	0.556
$\hat{\sigma}$	1.715
$\hat{SE}(\hat{\sigma})$	0.451

Table 3: Results for estimation of μ and σ in $DE(\mu, \sigma)$

The Histograms of the B estimates of μ and σ obtained from B bootstrap data are given below, to get a visual idea about the variability of the estimator.

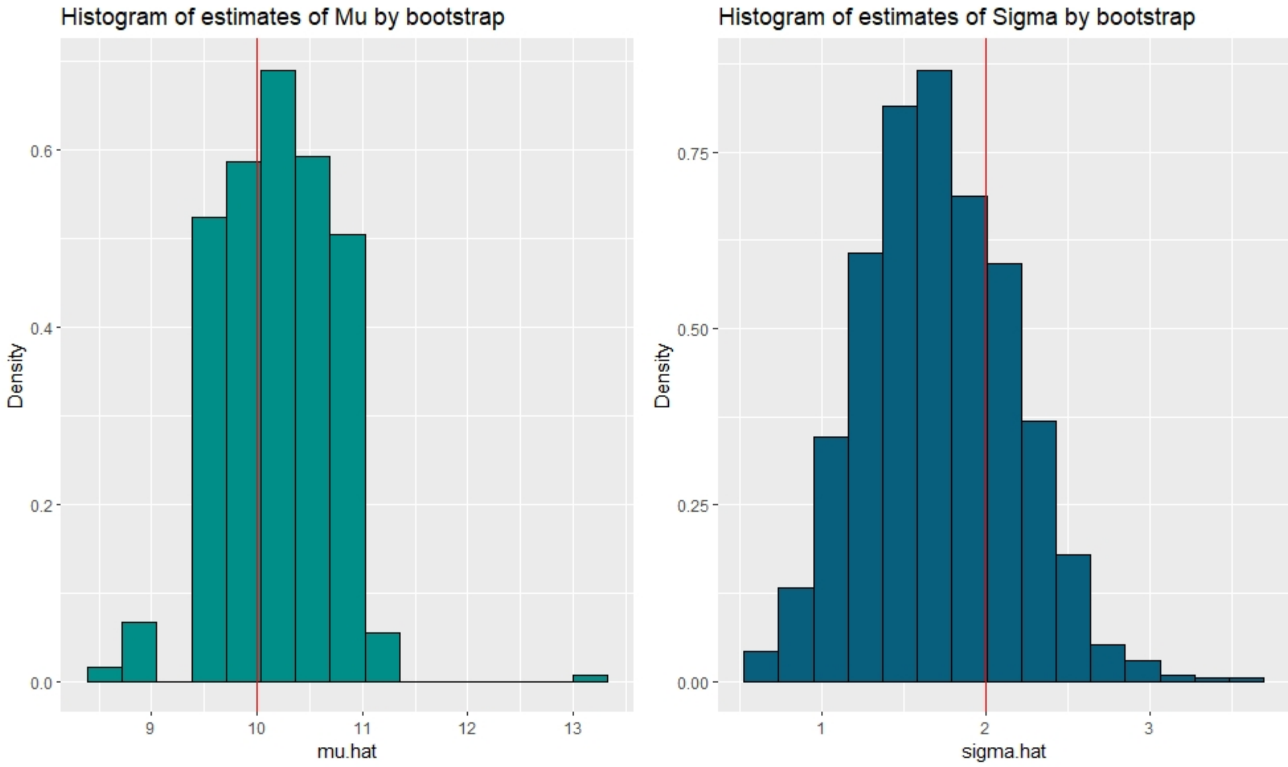

Figure 4: Histogram of estimates of μ and σ

5.2 Variance of estimator of α for $Beta(\alpha, \alpha)$

Here the Probability Density Function is given by-

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\alpha-1}}{B(\alpha,\alpha)}, 0 < x < 1$$

where $B(\alpha,\alpha) = \int_0^1 x^{\alpha-1}(1-x)^{\alpha-1}dx$ for $\alpha > 0$, is the Beta integral of parameter (α,α). We are interested in the estimation of α.

Here we choose a random sample $\underset{\sim}{x} = (x_1, x_2, ..., x_n)$ from the population. Let us calculate the Method of Moments estimator of α. Here, we know that, in general for a $X\tilde{}B(m,n)$, the $\mu_2 = V(X) = \frac{mn}{(m+n)^2(m+n+1)}$. Hence here in case of $B(\alpha,\alpha)$

$$\mu_2 = \frac{1}{4(2\alpha+1)}$$

and

$$m_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2, where \ \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

according to the method of moment we equate μ_2 and m_2, i.e.

$$\begin{aligned} m_2 &= \mu_2 = \frac{1}{4(2\alpha+1)} \\ i.e.\ \ 4(2\alpha+1) &= \frac{1}{m_2} \\ i.e.\ \ \alpha &= \frac{1-4m_2}{8m_2} \end{aligned}$$

therefore, by the Method of Moment we get the estimate of α as $\hat{\alpha} = \frac{1-4m_2}{8m_2}$.

We apply the bootstrap method to estimate α and the estimate of $SE(\hat{\alpha})$. The results-

Obtained Result:

To generated initial sample we set α=3, n=15 and B=1000. The estimates-

Quantity	Value by Bootstrap
$\hat{\alpha}$	3.744
$\hat{SE}(\hat{\alpha})$	1.4483

Table 4: Results for estimation of α in $B(\alpha,\alpha)$

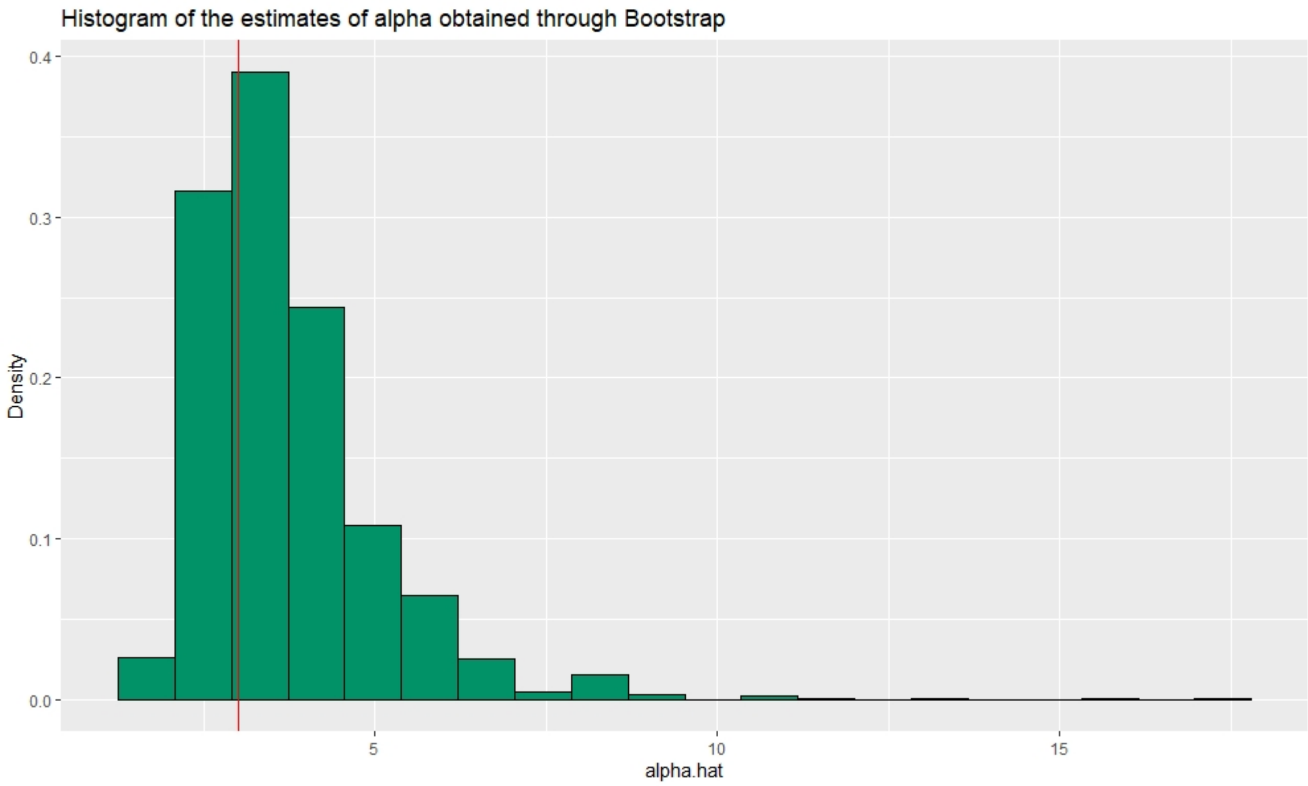

Figure 5: Histogram of estimates of α

The given histogram of the B estimates of the α obtained from bootstrap give the visual idea about the variability of the estimator $\hat{\alpha}$.

5.3 Variance of estimator of θ for $N(\theta, \theta^2)$

Here the Probability Density function is given by-

$$f(x) = \frac{1}{\theta\sqrt{(2\pi)}} e^{-\frac{1}{2}(\frac{x-\theta}{\theta})^2}, -\infty < x < \infty$$

where $\theta > 0$ is the parameter of interest.

Here we draw a random sample $\underset{\sim}{x} = (x_1, x_2, ..., x_n)$ of size n from the population. The log-likelyhood function $l(\theta; \underset{\sim}{x})$ is given by-

$$l(\theta;\underset{\sim}{x}) = ln(L(\theta;\underset{\sim}{x})) = -nln(\sqrt{2\pi}) - nln(\theta) - \frac{1}{2}\sum_{i=1}^{n}(\frac{x_i-\theta}{\theta})^2, \theta > 0$$

By partially differentiating both side w.r.t θ we get-

$$\begin{aligned}
\frac{\partial l}{\partial \theta} &= -\frac{n}{\theta} - \frac{1}{2}((-2)\frac{\sum_{i=1}^{n} x_i^2}{\theta^3} - (-1)\frac{2\sum_{i=1}^{n} x_i}{\theta^2}) \\
&= -\frac{n}{\theta} + \frac{\sum_{i=1}^{n} x_i^2}{\theta^3} - \frac{\sum_{i=1}^{n} x_i}{\theta^2} \\
&= -\frac{n}{\theta^3}(\theta^2 + m_1'\theta - m_2') \quad where, m_r' = \frac{1}{n}\sum_{i=1}^{n} x_i^r, r = 1,2
\end{aligned}$$

now equating $\frac{\partial l}{\partial \theta}$ with 0 we get,

$$\begin{aligned}
\frac{\partial l}{\partial \theta} &= 0 \\
i.e.\ -\frac{n}{\theta^3}(\theta^2 + m_1'\theta - m_2') &= 0 \\
i.e.\ (\theta^2 + m_1'\theta - m_2') &= 0 \\
i.e.\ \theta &= \frac{-m_1' \pm \sqrt{m_1'^2 + 4m_2'}}{2}
\end{aligned}$$

we will take only $\theta = \frac{-m_1' + \sqrt{m_1'^2 + 4m_2'}}{2}$ since $\theta > 0$. Therefore, we get the MLE of θ as-

$$\hat{\theta} = \frac{-m_1' + \sqrt{m_1'^2 + 4m_2'}}{2}$$

On applying bootstrap using the estimator we get the following result-

Obtained Results:

Here we have used B=1000 as bootstrap number and has taken the sample size n=15. Also we have taken $\theta = 3$ for generating the sample. The estimates are given in the following table-

Quantity	Value by Bootstrap
$\hat{\theta}$	2.6578
$\hat{SE}(\hat{\theta})$	0.3487

Table 5: Results for estimetion of θ in N(θ,θ^2)

The following histogram will give the visual idea about the clustering of the B esimates obtained through bootstrap.

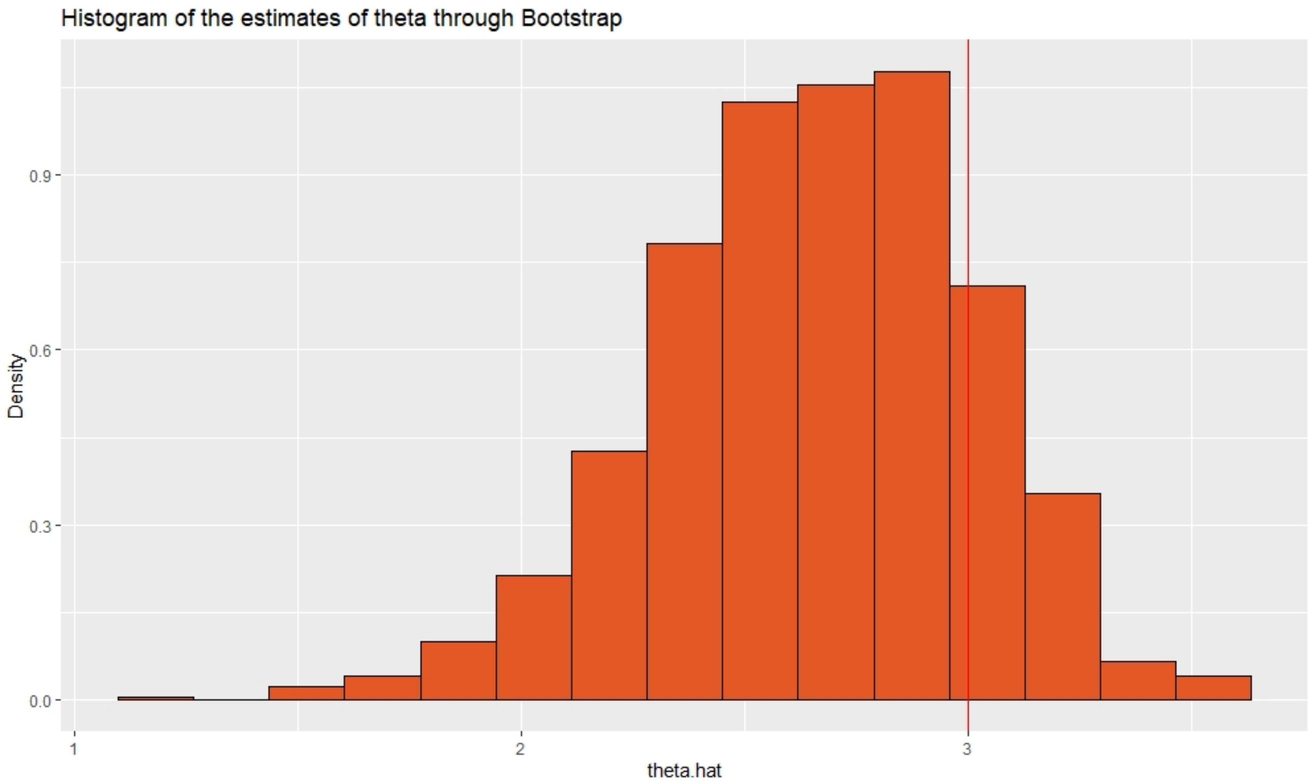

Figure 6: Histogram of the estimates of θ

5.4 Variance of estimator of α and β for $Beta(\alpha,\beta)$

In case of this distribution the Probability Density Function is given by-

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, 0 < x < 1$$

where $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$ is the Beta function with parameters $\alpha, \beta(\alpha, \beta > 0)$. Here $\alpha > 0$ and $\beta > 0$ are the parameters of interest.

Suppose a random sample $\underset{\sim}{x} = (x_1, x_2, ..., x_n)$ of size n is chosen from the distribution. Again here we will calculate the Method of Moment estimators of α and β. We know that, for $X\tilde{}B(m,n)$, the $\mu_1' = E(X) = \frac{m}{m+n}$ and $\mu_2 = V(X) = \frac{mn}{(m+n)^2(m+n+1)}$. Therefore, in case of $B(\alpha,\beta)$ distribution-

$$\mu_1' = \frac{\alpha}{\alpha+\beta}$$

$$\mu_2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Now by the Method of Moment estimation technique we equate the μ_1' and μ_2 to respectively $m_1' = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $m_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$-

$$\begin{aligned} \mu_1' &= m_1' \\ i.e.\ m_1' &= \frac{\alpha}{\alpha+\beta} \end{aligned} \tag{1}$$

and

$$\begin{aligned} \mu_2 &= m_2 \\ i.e.\ m_2 &= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \\ i.e.\ m_2 &= (\frac{\alpha}{\alpha+\beta})(1-\frac{\alpha}{\alpha+\beta})\frac{1}{(\alpha+\beta+1)} \\ i.e.\ m_2 &= \frac{m_1'(1-m_1')}{(\alpha+\beta+1)} \quad (from\ (1)) \\ i.e.\ \alpha+\beta+1 &= \frac{m_1'(1-m_1')}{m_2} \\ i.e.\ \frac{\alpha}{m_1'} &= \frac{m_1'(1-m_1')}{m_2} - 1 \quad (from\ (1)) \\ i.e.\ \alpha &= m_1'(\frac{m_1'(1-m_1')}{m_2} - 1) \end{aligned}$$

Therefore, $\hat{\alpha} = m_1'(\frac{m_1'(1-m_1')}{m_2} - 1)$ putting $\alpha = \hat{\alpha}$ in (1) we get

$$\begin{aligned}
i.e.\ m_1' &= \frac{\hat{\alpha}}{\hat{\alpha}+\beta} \\
i.e.\ \frac{1}{m_1'} &= \frac{\hat{\alpha}+\beta}{\hat{\alpha}} \\
i.e.\ \frac{1}{m_1'} &= 1 + \frac{\beta}{\hat{\alpha}} \\
i.e.\ \beta &= \hat{\alpha}(\frac{1}{m_1'} - 1) \\
i.e.\ \beta &= m_1'(\frac{m_1'(1-m_1')}{m_2} - 1)(\frac{1-m_1'}{m_1'}) \quad (putting\ \hat{\alpha}) \\
i.e.\ \beta &= (1-m_1')(\frac{m_1'(1-m_1')}{m_2} - 1)
\end{aligned}$$

Therefore, the method of moment estimator of α and β is given by-

$$\hat{\alpha} = m_1'(\frac{m_1'(1-m_1')}{m_2} - 1)$$

$$\hat{\beta} = (1-m_1')(\frac{m_1'(1-m_1')}{m_2} - 1)$$

On applying the bootstrap we get the following results-

Obtained Results:

Here we have used $(\alpha, \beta) = (4, 5)$ for generating the sample. The bootstrap iteration B is taken to be B=1000 and have taken a sample of size n=15. The estimates are given in the table-

Quantity	Value by Bootstrap
$\hat{\alpha}$	4.0945
$\hat{SE}(\hat{\alpha})$	1.3155
$\hat{\beta}$	6.6471
$\hat{SE}(\hat{\beta})$	2.7383

Table 6: Results of estimation of α and β in $B(\alpha, \beta)$

Histogram of the bootstrap estimates give the image of variability of the estimators of α and β. These are shown below-

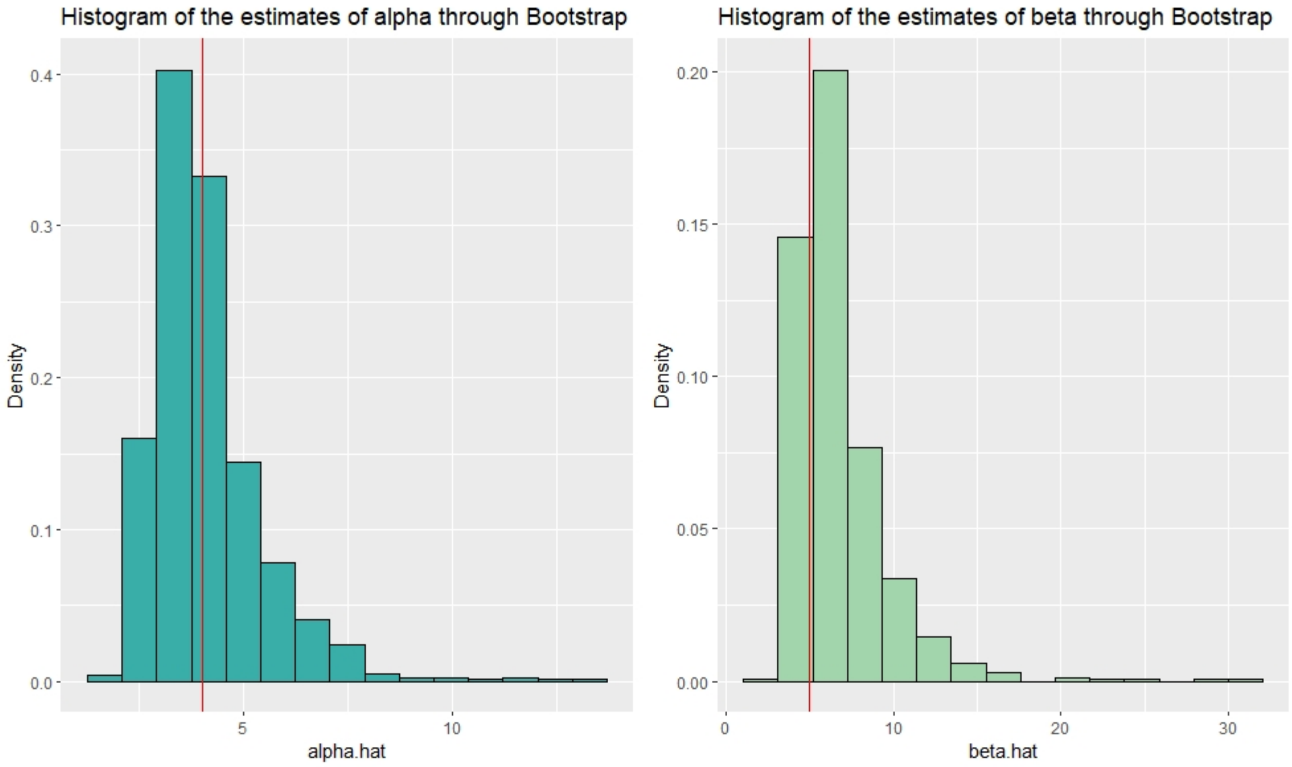

Figure 7: Histogram of estimates of α and β

5.5 Variance of estimator of π and λ for $ZIP(\pi, \lambda)$

Here the Probability Mass Function is given by-

$$f(x) = \begin{cases} \pi + (1-\pi)e^{-\lambda} & if \;\; x = 0 \\ (1-\pi)\frac{e^{-\lambda}\lambda^x}{x!} & if \;\; x = 1, 2, 3, \ldots \end{cases}$$

Here $0 \leq \pi < 1$ and $\lambda > 0$ are the two parameters of interests. Let us draw a random sample $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$ from this distribution. Here also we will find the estimators of π and λ by the Method of Moment estimation. Before that, we would like to state the following result relating to the mean and variance of $ZIP(\pi, \lambda)$, without proof.

Result:

If a random variable $X \tilde{} ZIP(\pi, \lambda)$; where $0 \leq \pi < 1$, $\lambda > 0$; then-

$$E(X) = \lambda(1-\pi)$$

$$V(X) = \lambda(1-\pi)(1+\pi\lambda)$$

Therefore, for $ZIP(\pi,\lambda)$-

$$\mu_1' = \lambda(1-\pi)$$

$$\mu_2 = \lambda(1-\pi)(1+\pi\lambda)$$

Equating this population moments to the corresponding sample moments, i.e. $m_1' = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $m_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$, we get-

$$\begin{aligned} m_1' &= \mu_1' \\ i.e.\ m_1' &= \lambda(1-\pi) \end{aligned} \tag{2}$$

and

$$\begin{aligned} m_2 &= \mu_2 \\ i.e.\ m_2 &= \lambda(1-\pi)(1+\pi\lambda) \\ i.e.\ m_2 &= m_1'(1+\pi\lambda) \quad (from\ (2)) \\ i.e.\ \frac{m_2}{m_1'} &= (1+\pi\lambda) \\ i.e.\ \pi\lambda &= \frac{m_2}{m_1'} - 1 \end{aligned} \tag{3}$$

Now from (2) we get-

$$\begin{aligned} m_1' &= \lambda(1-\pi) \\ i.e.\ m_1' &= \lambda - \pi\lambda \\ i.e.\ m_1' &= \lambda - \frac{m_2}{m_1'} + 1 \\ i.e.\ \lambda &= m_1' + \frac{m_2}{m_1'} - 1 \end{aligned}$$

Therefore, we get $\hat{\lambda} = m_1' + \frac{m_2}{m_1'} - 1$. Now putting this $\hat{\lambda}$ in (3) we get-

$$\begin{aligned} \pi(m_1' + \frac{m_2}{m_1'} - 1) &= \frac{m_2}{m_1'} - 1 \\ i.e.\ \pi &= \frac{\frac{m_2}{m_1'} - 1}{m_1' + \frac{m_2}{m_1'} - 1} \\ i.e.\ \pi &= \frac{m_2 - m_1'}{m_1'^2 + m_2 - m_1'} \end{aligned}$$

Hence, the estimates of the parameters π and λ is given by-

$$\hat{\pi} = \frac{m_2 - m_1'}{m_1'^2 + m_2 - m_1'}$$

$$\hat{\lambda} = m_1' + \frac{m_2}{m_1'} - 1$$

We apply bootstrap on these estimates. The results are given below-

Obtained Results:

Here we used (π, λ)=(0.5,2) to generate random sample. The bootstrap number has been taken as B=1000 and the sample size, is taken to be n=15. The estimates are given in the following table.

Quantity	Value by Bootstrap
$\hat{\pi}$	0.2827
$\hat{SE}(\hat{\pi})$	0.2110
$\hat{\lambda}$	1.7110
$\hat{SE}(\hat{\lambda})$	0.3797

Table 7: Results of estimation of (π, λ) in $ZIP(\pi, \lambda)$

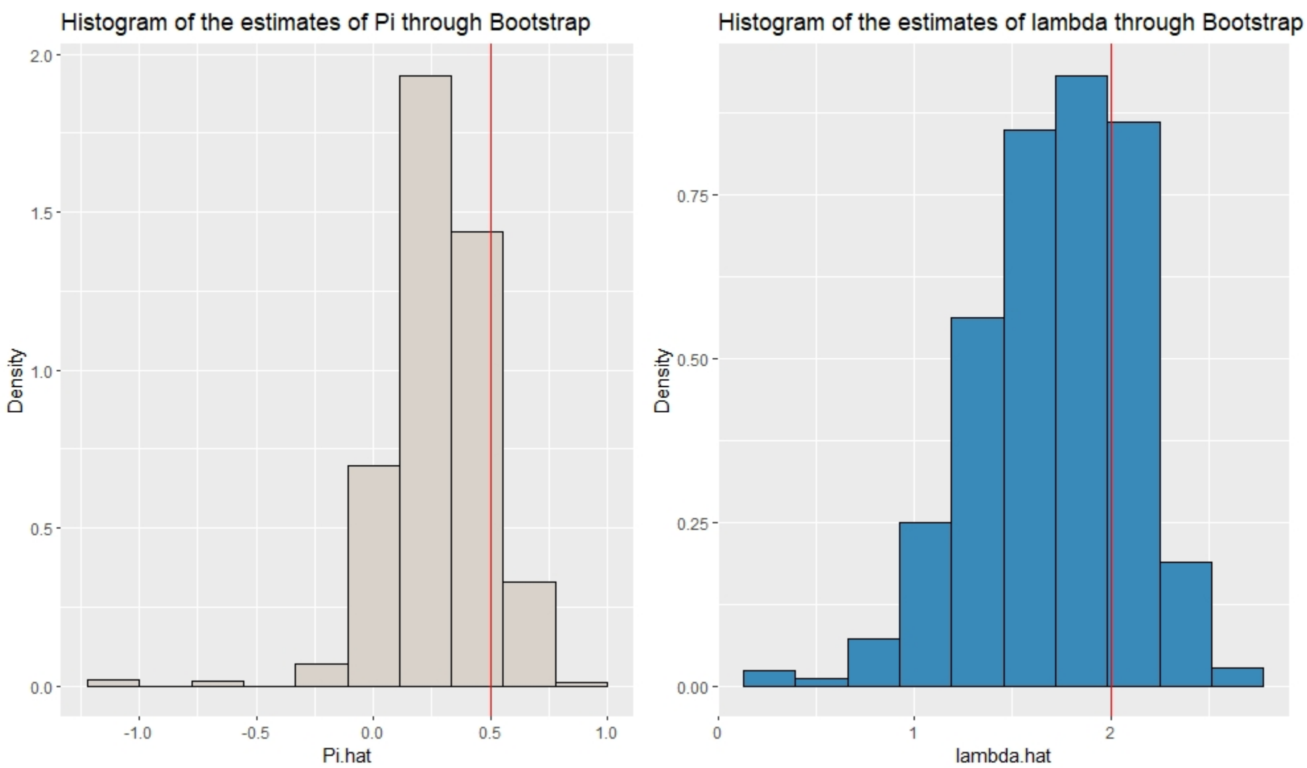

Figure 8: Histogram of extimates of π and λ

The graph of the estimates of π and λ is given to get the idea of the variabilty of the estimators of π and λ.

5.6 Variance of estimator of θ for $N(\theta,\theta)$

In case of this distribution the Probability Density Function is given by-

$$f(x) = \frac{1}{\sqrt{\theta 2\pi}} e^{-\frac{1}{2}\frac{(x-\theta)^2}{\theta}}, -\infty < x < \infty$$

Here $\theta > 0$ is the parameter of interest. We will use the MLE to estimate θ. Suppose a random sample $\underset{\sim}{x} = (x_1, x_2, ..., x_n)$ is chosen from the distribution. The log-liklyhood function $l(\theta;\underset{\sim}{x})$ is given by-

$$\begin{aligned}
l(\theta;\underset{\sim}{x}) &= ln(L(\theta;\underset{\sim}{x})) = -\frac{n}{2}ln(\theta) - \frac{1}{2}\sum_{i=1}^{n}\frac{(x_i-\theta)^2}{\theta} \\
i.e.\ l(\theta;\underset{\sim}{x}) &= -\frac{n}{2}ln(\theta) - \frac{1}{2}\frac{\sum_{i=1}^{n}x_i^2 - 2\theta\sum_{i=1}^{n}x_i + n\theta^2}{\theta} \\
i.e.\ l(\theta;\underset{\sim}{x}) &= -\frac{n}{2}ln(\theta) - \frac{1}{2}(\frac{\sum_{i=1}^{n}x_i^2}{\theta} - 2\sum_{i=1}^{n}x_i + n\theta)
\end{aligned}$$

Therefore, differentiating the log-likelyhood $l(\theta;\underset{\sim}{x})$ w.r.t θ we have-

$$\begin{aligned}
\frac{\partial l}{\partial \theta} &= -\frac{n}{2\theta} - \frac{1}{2}((-1)\frac{\sum_{i=1}^{n}x_i^2}{\theta^2} + n) \\
&= -\frac{n}{2\theta} + \frac{\sum_{i=1}^{n}x_i^2}{2\theta^2} - \frac{n}{2} \\
&= -\frac{n}{2\theta^2}(\theta^2 + \theta - \frac{1}{n}\sum_{i=1}^{n}x_i^2) \\
&= -\frac{n}{2\theta^2}(\theta^2 + \theta - m_2') \quad where\ \ m_2' = \frac{1}{n}\sum_{i=1}^{n}x_i^2
\end{aligned}$$

Now equating $\frac{\partial l}{\partial \theta}$ with 0 we get-

$$\begin{aligned}
\frac{\partial l}{\partial \theta} &= 0 \\
i.e.\ -\frac{n}{2\theta^2}(\theta^2 + \theta - m_2') &= 0 \\
i.e.\ (\theta^2 + \theta - m_2') &= 0 \\
i.e.\ \theta &= \frac{-1 \pm \sqrt{1 + 4m_2'}}{2}
\end{aligned}$$

Since $\theta > 0$ we write the MLE of θ as-

$$\hat{\theta} = \frac{-1+\sqrt{1+4m_2'}}{2}$$

We have the done the estimation of θ, using this estimator, by boostrap. Here are the results of the estimation-

Obtained Result:

Like before, we here used initial value of $\theta = 3$ for the purpose of generating a sample. of size n=15. There are B=1000 bootstrap data from this initial sample. The estimates of θ and $SE(\hat{\theta})$ is given below-

Quantity	Value by Bootstrap
$\hat{\theta}$	2.6417
$\hat{SE}(\hat{\theta})$	0.3949

Table 8: Result of estimetion of θ in $N(\theta, \theta)$

From the following histogram of the estimates of θ we get a visual idea about the variability of the estimator $\hat{\theta}$-

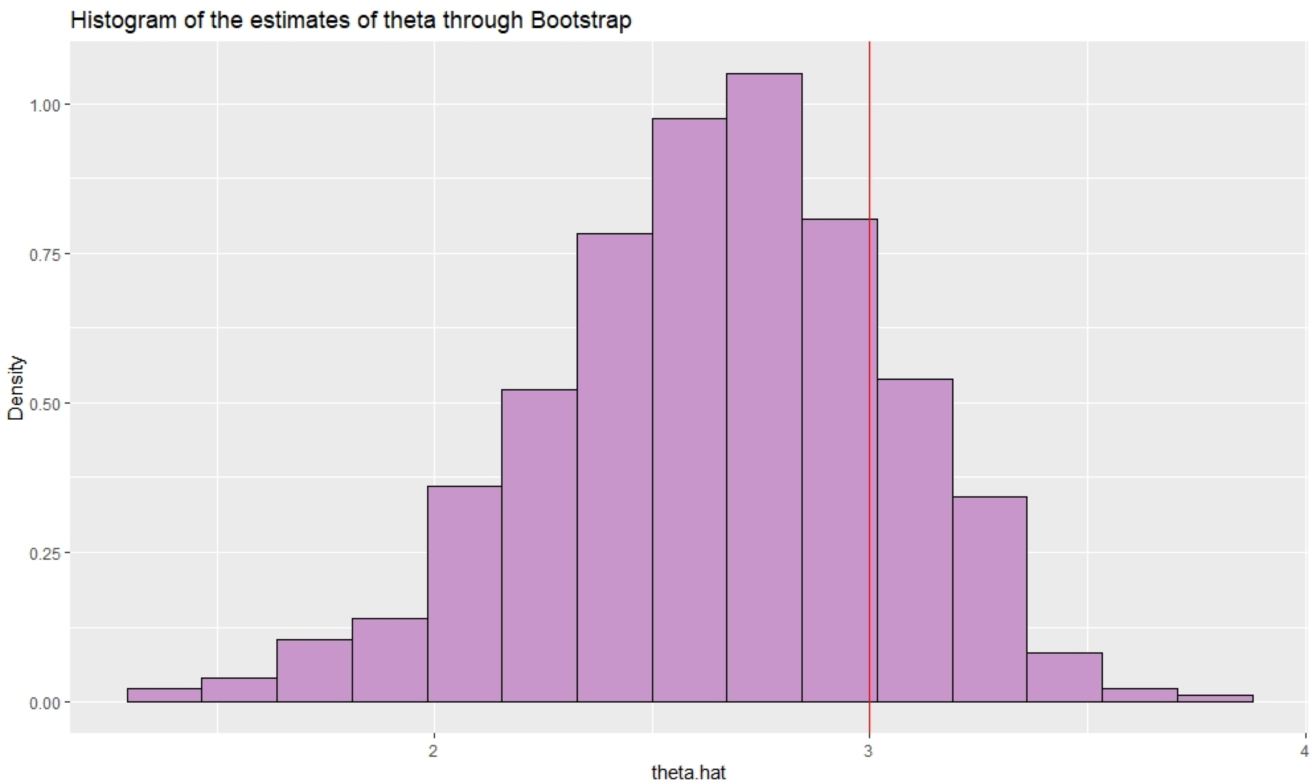

Figure 9: Histogram of extimates of θ

6 Conclusion

Throughout the dissertation our main target was to find a method of estimation of the Standard Error of an estimator, which has a complicated expression. Also we should remember that here we are unable to use the method of simulation or the large sample behaviours of an estimator, because we are given with a single small size sample.

Given this conditions, we have developed an idea of resampling from a given sample, to generate more and more data points. This idea has been implemented into the "Boostrap Method", which is used to estimated the standard error of the estimator. Throughout the dissertation we have learned-

- The general idea and procedure of Bootstrap Method
- Under which circumstances it can be applied
- How it can be used in estimations of the standard error of a complicated form estimator

We have used this method to calculate the standard error of estimators to estimated several parameters that are coming from diffrent distributions. Not only parameters we can apply the method on several parametric functions also. However, the goodness of the estimation will always depend on the efficiency of the chosen estimator. We need to study about the estimator that we have chosen, before applying bootstrap on it.

Once we have obtained the standard deviation of the required statistic, it will facilitate us to

1. get idea of variability of the statistic.
2. perform testing involving the statistic.
3. get confidence intervals of the related parameter.

7 Acknowledgement

First and foremost, I would like to thank to my superviser, Prof. Surupa Chakraborty for her guidence through each steps of my dissertation. Without her assistance I could never finish this dissertation within the stipulated time period.

Besides, I would like to thank all the professors in my department, for their help, by providing me necessary advice and additional equipment that helped me a lot in my dissertation.

Also a special thanks to my parents and friends, for helping me to finalise my dissertaion.

8 References

An Introduction to Statistical Learning with Applications in R - By Gareth James • Daniela Witten • Trevor Hastie Robert Tibshirani (Page-187, ”The Bootstrap”).

9 APPENDIX: R Codes

Here we have given all the R codes that is used while estimating different parameters of different population distribution.

List of Codes

Code 1: Code Related to Section 4.1

```
rm(list=ls())
library(ggplot2)
library(patchwork)
library(dplyr)
set.seed(seed=987654321)
mu=50 #parameter
R=1000 #simulation number
B=100 #bootstrap number
n=20 #sample size
#simulation====================================
```

```
data=data.frame(x.bar.simu=replicate(R,mean(rnorm(n,mu,1)))) #genearting
    sample means
mu.est=mean(data[,1]);mu.est
se.x.bar.simu=sqrt((R-1)/R)*sd(data[,1]);se.x.bar.simu #calculating the
   SE of sample mean
g1=ggplot(data,aes(x=x.bar.simu))+
        geom_histogram(aes(y=..density..),fill="#D9534F",
                 colour="black",bins=15)+
        geom_vline(xintercept=50,color="red")+
        labs(x="mu.hat",y="Density",
                 title="Histogram of estimates of mu through simulation"
                    ) #histogram of x-bar in simulation
#bootstrap========================================
x.bar.boot=se.x.bar.star=array(0)
for(i in 1:R){ #simulation loop
  x.bar.star=array(0) #bootstrap mean initialization
  x=rnorm(n,mu,1) #original sample
  for(j in 1:B){ #bootstrap loop
    x.star=sample(x,n,replace=T) #bootstrap data
    x.bar.star[j]=mean(x.star)} #bootstrap mean generation
  se.x.bar.star[i]=sqrt((B-1)/B)*sd(x.bar.star) #SE of sample bootstrap
     means
  x.bar.boot[i]=mean(x.bar.star)} #estimate of sample mean
mu.est=mean(x.bar.boot);mu.est
se.x.bar.boot=mean(se.x.bar.star);se.x.bar.boot #estimate of SE of
   sample mean
data=data%>%
  mutate(x.bar.boot=x.bar.boot)
g2=ggplot(data,aes(x=x.bar.boot))+
        geom_histogram(aes(y=..density..),fill="#96CEB4",
                 colour="black",bins=15)+
        geom_vline(xintercept=50,color="red")+
        labs(x="mu.hat",y="Density",
```

```
                  title="Histogram of estimates of mu through Bootstrap") #histogram of x-bar in bootstrapping
g1+g2
#theoretical SE of x-bar===========================
se.x.bar.orig=1/sqrt(n);se.x.bar.orig
```

Code 2: Code Related to Section 4.2

```
rm(list=ls())
library(ggplot2)
library(patchwork)
library(dplyr)
set.seed(seed=987654321)
P=0.3 #parameter
R=1000 #simulation number
B=100#bootstrap number
n=20 #sample size
 #simulation==============================================
data=data.frame(p=replicate(R,mean(rbinom(n,1,P))))#generating sample proportions
p.est.simu=mean(data[,1]);p.est.simu
se.p.simu=sqrt((R-1)/R)*sd(data[,1]);se.p.simu #calculate SE of the sample proportion
g1=ggplot(data,aes(x=p))+
        geom_histogram(aes(y=..density..),bins=15,
                       fill="#313552",colour="black")+
        geom_vline(xintercept=P,colour="red")+
        labs(x="p.hat",y="Density",
            title="Histogram of estimates of P through simulation") #histogram of p in simulation
#bootstrap===============================================
p.boot=se.p.star=array(0)
for(i in 1:R){ #simulation loop
        p.star=array(0) #bootstrap p initialization
```

```
        x=rbinom(n,1,P) #original sample
        for(j in 1:B){ #bootstrap loop
                x.star=sample(x,n,replace=T) #bootstrap data
                p.star[j]=mean(x.star)} #bootstrap p
        se.p.star[i]=sqrt((B-1)/B)*sd(p.star) #SE of the SE of p
        p.boot[i]=mean(p.star)} #estimates of p
p.est.boot=mean(p.boot);p.est.boot
se.p.boot=mean(se.p.star);se.p.boot #estimate of SE of p
data=data%>%
        mutate(p.boot=p.boot)
g2=ggplot(data,aes(x=p))+
        geom_histogram(aes(y=..density..),fill="#B8405E",
                        bins=16,colour="black")+
        geom_vline(xintercept=0.3,colour="red")+
        labs(x="p.hat",y="Density",
             title="Histogram of estimates of P through bootstrap") #
                histogram of x-bar in bootstrapping
g1+g2
#theoretical SE of p==================================
se.p.orig=sqrt(p.est.boot*(1-p.est.boot)/n);se.p.orig
```

Code 3: Code Related to Section 5.1

```
rm(list=ls())
library(nimble)
library(ggplot2)
library(patchwork)
set.seed(seed=123456789)
n=15 #sample size
B=1000 #bootstrap number
mu=10 #initial Value Of the parameter to generate the data
sigma=2
x=rdexp(n,location=mu,scale=sigma) #given data
#bootstrap========================
```

```
mu.hat=sigma.hat=array(0)
for(i in 1:B){
  x.star=sample(x,n,replace=T) #bootstrap data
  mu.hat[i]=median(x.star) #MLE for mu
  sigma.hat[i]=mean(abs(x.star-median(x.star)))} #MLE for sigma
mu.est=mean(mu.hat);mu.est #estimate of the mu by bootstrap
sigma.est=mean(sigma.hat);sigma.est #estimate of the sigma by bootstrap
SE.mu.hat=sqrt((B-1)/B)*sd(mu.hat);SE.mu.hat #se of mu
SE.sigma.hat=sqrt((B-1)/B)*sd(sigma.hat);SE.sigma.hat #se of sigma
#graphs============================
data=data.frame(mu.hat=mu.hat,sigma.hat=sigma.hat)
g1=ggplot(data,aes(x=mu.hat))+
    geom_histogram(aes(y=..density..),bins=15,
                   fill="#008E89",colour="black")+
    geom_vline(xintercept=mu,colour="red")+
    labs(x="mu.hat",y="Density",
         title="Histogram of estimates of Mu by bootstrap")
g2=ggplot(data,aes(x=sigma.hat))+
    geom_histogram(aes(y=..density..),bins=15,
                   fill="#085E7D",colour="black")+
    geom_vline(xintercept=sigma,colour="red")+
    labs(x="sigma.hat",y="Density",
         title="Histogram of estimates of Sigma by bootstrap")
g1+g2
```

Code 4: Code Related to Section 5.2

```
rm(list=ls())
library(ggplot2)
set.seed(seed=123456789)
n=15
B=1000
alpha=3 #initial parameter value
alpha.hat=array(0)
```

```
x=rbeta(n,alpha,alpha)
#bootstrap========================
for(i in 1:B){ #loops for bootstrapping
  x.star=sample(x,n,replace=T) #bootstrap data
  m2=((n-1)/n)*var(x.star)
  alpha.hat[i]=(1-4*m2)/(8*m2)}
alpha.est=mean(alpha.hat);alpha.est #estimate of alpha
SE.alpha.hat=sqrt((B-1)/B)*sd(alpha.hat);SE.alpha.hat #se of alpha
#graphs===========================
data=data.frame(alpha.hat=alpha.hat)
ggplot(data,aes(x=alpha.hat))+
  geom_histogram(aes(y=..density..),fill="#019267",
               colour="black",bins=20)+
  geom_vline(xintercept=alpha,colour="red")+
  labs(x="alpha.hat",
       y="Density",
       title="Histogram of the estimates of alpha obtained through
          Bootstrap")
```

Code 5: Code Related to Section 5.3

```
rm(list=ls())
library(ggplot2)
set.seed(seed=123456789)
n=15
B=1000
theta=3 #initial parameter value
theta.hat=array(0)
x=rnorm(n,theta,theta)
#bootstrap========================
for(i in 1:B){ #loops for bootstrapping
  x.star=sample(x,n,replace=T) #bootstrap data
  m1.dash=mean(x.star)
  m2.dash=mean(x.star^2)
```

```
  k=sqrt(m1.dash^2+4*m2.dash)
  theta.hat[i]=(k-m1.dash)/2}
theta.est=mean(theta.hat);theta.est #estimate of theta
SE.theta.hat=sqrt((B-1)/B)*sd(theta.hat);SE.theta.hat #se of theta
#graphs===========================
data=data.frame(theta.hat=theta.hat)
ggplot(data,aes(x=theta.hat))+
  geom_histogram(aes(y=..density..),bins=15,
                 fill="#E45826",colour="black")+
  geom_vline(xintercept=theta,colour="red")+
  labs(x="theta.hat",
       y="Density",
       title="Histogram of the estimates of theta through Bootstrap")
```

Code 6: Code Related to Section 5.4

```
rm(list=ls())
library(ggplot2)
library(patchwork)
set.seed(seed=123456789)
n=15 #sample size
B=1000 #bootstrap number
alpha=4 #initial parameter value
beta=5
x=rbeta(n,alpha,beta) #given data
alpha.hat=beta.hat=array(0)
#bootstrap========================
for(i in 1:B){ #bootstrap loop
  x.star=sample(x,n,replace=T) #bootstrap data
  m1.dash=mean(x.star)
  m2=((n-1)/n)*var(x.star)
  alpha.hat[i]=m1.dash*((m1.dash*(1-m1.dash)/m2)-1) #MME for alpha
  beta.hat[i]=(1-m1.dash)*((m1.dash*(1-m1.dash)/m2)-1)} #MME for beta
alpha.est=mean(alpha.hat);alpha.est #estimate of the alpha by bootstrap
```

```
beta.est=mean(beta.hat);beta.est #estimate of the beta by bootstrap
SE.alpha.hat=sqrt((B-1)/B)*sd(alpha.hat);SE.alpha.hat #se of alpha
SE.beta.hat=sqrt((B-1)/B)*sd(beta.hat);SE.beta.hat #se of alpha
#graphs============================
data=data.frame(alpha.hat=alpha.hat,beta.hat=beta.hat)
g1=ggplot(data,aes(x=alpha.hat))+
     geom_histogram(aes(y=..density..),fill="#39AEA9",
                   colour="black",bins=15)+
     geom_vline(xintercept=alpha,colour="red")+
     labs(x="alpha.hat",
          y="Density",
          title="Histogram of the estimates of alpha through Bootstrap")
g2=ggplot(data,aes(x=beta.hat))+
      geom_histogram(aes(y=..density..),fill="#A2D5AB",
                    colour="black",bins=15)+
      geom_vline(xintercept=beta,colour="red")+
      labs(x="beta.hat",
           y="Density",
           title="Histogram of the estimates of beta through Bootstrap")
g1|g2
```

Code 7: Code Related to Section 5.5

```
rm(list=ls())
library(VGAM)
library(ggplot2)
library(patchwork)
set.seed(seed=123456789)
n=15
B=1000
Pi = 0.5
lambda = 2  #initial parameter value
Pi.hat = array(0)
lambda.hat = array(0)
```

```
x=rzipois(n, lambda=lambda, pstr0=Pi)
#bootstrap=========================
for(i in 1:B){ #loops for bootstrapping
  x.star=sample(x,n,replace=T) #bootstrap data
  m1.dash=mean(x.star)
  m2=((n-1)/n)*var(x.star)
  lambda.hat[i]=m1.dash+(m2/m1.dash)-1
  Pi.hat[i]=(m2-m1.dash)/(m1.dash^2+m2-m1.dash)}
Pi.hat[!is.finite(Pi.hat)]=NA
lambda.hat[!is.finite(lambda.hat)]=NA
lambda.est=mean(lambda.hat,na.rm=T);lambda.est #estimate of lambda
Pi.est=mean(Pi.hat,na.rm=T);Pi.est  #estimate of Pi
SE.lambda.hat=sqrt((B-1)/B)*sd(lambda.hat, na.rm=T);SE.lambda.hat #se of
    lambda
SE.Pi.hat=sqrt((B-1)/B)*sd(Pi.hat, na.rm=T);SE.Pi.hat #se of Pi
#graphs============================
data=data.frame(Pi.hat=Pi.hat,lambda.hat=lambda.hat)
g1=ggplot(data,aes(x=Pi.hat))+
      geom_histogram(aes(y=..density..),fill="#D8D2CB",
                     colour="black",bins=10)+
      geom_vline(xintercept=Pi,colour="red")+
      labs(x="Pi.hat",
           y="Density",
           title="Histogram of the estimates of Pi through Bootstrap")
g2=ggplot(data,aes(x=lambda.hat))+
      geom_histogram(aes(y=..density..),fill="#398AB9",
                     colour="black",bins=10)+
      geom_vline(xintercept=lambda,colour="red")+
      labs(x="lambda.hat",
           y="Density",
           title="Histogram of the estimates of lambda through Bootstrap
              ")
g1|g2
```

Code 8: Code Related to Section 5.6

```
rm(list=ls())
library(ggplot2)
set.seed(seed=123456789)
n=15
B=1000
theta=3 #initial parameter value
theta.hat=array(0)
x=rnorm(n,theta,sqrt(theta))
#bootstrap========================
for(i in 1:B){ #loops for bootstrapping
  x.star=sample(x,n,replace=T) #bootstrap data
  m2.dash=mean(x.star^2)
  k=sqrt(1+4*m2.dash)
  theta.hat[i]=(k-1)/2}
theta.est=mean(theta.hat);theta.est #estimate of theta
SE.theta.hat=sqrt((B-1)/B)*sd(theta.hat);SE.theta.hat #se of theta
#graphs============================
data=data.frame(theta.hat=theta.hat)
ggplot(data,aes(x=theta.hat))+
  geom_histogram(aes(y=..density..),fill="#C996CC",
                 colour="black",bins=15)+
  geom_vline(xintercept=theta,colour="red")+
  labs(x="theta.hat",
       y="Density",
       title="Histogram of the estimates of theta through Bootstrap")
```