Contents

Chapter 1

Sequence and Series

1.1 Introduction

The study of Sequence and Series forms an important part of mathematical analysis and provides the foundation for many advanced topics in calculus and real analysis. A sequence is an ordered list of numbers arranged according to a specific rule, while a series is the sum of the terms of a sequence. Understanding how sequences behave—whether they converge, diverge, or follow certain patterns—helps us analyze real-life phenomena such as population growth, financial investments, and scientific measurements. Similarly, studying series allows us to express complex functions as infinite sums, which is essential in approximation techniques, power series, and Taylor expansions. This chapter introduces the basic concepts, properties, and tests for convergence of sequences and series, preparing learners for deeper mathematical study.

1.2 Real life applications

Here are real-life applications of Sequence and Series categorized according to different B.Tech. courses, written clearly and suitable for academic use:

- **Computer Science & Engineering (CSE):** Algorithm Analysis: Time and space complexity often form arithmetic or geometric sequences (e.g., divide-and-conquer algorithms use recurrence relations and series). Data Compression: Techniques like Huffman coding and wavelet transforms rely on series expansions. Machine Learning: Gradient descent iterations form sequences that converge to optimal solutions. Digital Signal Processing: Fourier series express signals as infinite sums of sine and cosine functions.

- **Electronics & Communication Engineering (ECE):** Signal Processing: Signals are analyzed using Fourier and Laplace series to study frequency components. Communication Systems: Power series help in approximating filter responses and modulation signals. Control Systems: Stability analysis uses sequences representing system states over time.

- **Electrical Engineering (EE):** Circuit Analysis: Transient responses in RC, RL, and RLC circuits are modeled using exponential sequences. Power Systems: Load forecasting uses time-series data (a type of sequence) to predict future consumption. Harmonic Analysis: Series representation is used to study harmonics in AC circuits.

- **Mechanical Engineering (ME):** Vibration Analysis: Periodic motion is represented using Fourier series. Thermodynamics: Infinite series appear in expansion of equations of state and heat transfer calculations. Numerical Methods: Iterative methods (Newton–Raphson, Gauss–Seidel) form sequences that converge to solutions.

- **Civil Engineering:** Structural Analysis: Load distribution along beams or frames is modeled using series approximations. Transportation Engineering: Traffic flow and vehicle count predictions use time-series analysis. Hydrology: Rainfall and flood frequency analysis use probabilistic sequences.

- **Chemical Engineering:** Reaction Kinetics: Concentration profiles follow convergent sequences over time. Process Control: Iterative sequences describe the dynamic behavior of chemical processes. Heat & Mass Transfer: Series solutions are used in solving differential equations for diffusion and conduction.

- **Information Technology (IT):**

  Cryptography: Certain encryption algorithms rely on number sequences and recurrence relations. Image Processing: Pixel intensities are analyzed using discrete Fourier series. Database Indexing: Hash functions and probing techniques use sequences for data organization.

- **Biotechnology:** Genetic Sequencing: DNA and protein sequences are natural examples of ordered data. Population Models: Growth or decay of cells follows geometric or logistic sequences. Enzyme Kinetics: Convergence series help approximate reaction velocity curves.

1.3 Sequences

In this chapter we shall study a special class of functions whose domain is the set $\mathbb{N}$ of natural numbers and range a set of real numbers—the Real Sequences.

1.3.1 Real Sequences

A function whose domain is the set $\mathbf{N}$ of natural numbers and range a set of real numbers is called a **real sequence**. Thus a real sequence is denoted symbolically as $S : \mathbf{N} \to \mathbb{R}$.

Since we shall be dealing with real sequences only, we shall use the term **sequence** to denote a real sequence.

Notation

Since the domain for a sequence is always $\mathbf{N}$, a sequence is specified by the values a_n, $n \in \mathbf{N}$. Thus a sequence may be denoted as

$$\{a_n\}, \quad n \in \mathbf{N} \quad \text{or} \quad \{a_1, a_2, a_3, \ldots, a_{n+1}\}$$

The values $a_1, a_2, a_3, \ldots$ are called the first, second, ... terms of the sequence. The m^{th} and n^{th} terms a_m and a_n for $m \neq n$ are treated as distinct terms even if $a_m = a_n$. Thus the terms of a sequence are arranged in a definite order as first, second, third, ... terms and the terms occurring at different positions are treated as distinct terms even if they have the same value. The number of terms in a sequence is always infinite.

1.3.2 The Range

The **Range** or the **Range Set** is the set consisting of all the distinct elements of a sequence, without repetition and without regard to the position of a term. Thus the range may be a finite or an infinite set, without ever being the null set.

1.3.3 Bounds of a Sequence

Bounded Above Sequences

A sequence $\{a_n\}$ is said to be **bounded above** if there exists a number $K \in \mathbb{R}$ such that

$$a_n \leq K, \quad \forall n \in \mathbb{N}.$$

Bounded Below Sequences

A sequence $\{a_n\}$ is said to be **bounded below** if there exists a number k such that

$$a_n \geq k, \quad \forall n \in \mathbb{N}.$$

1.3.4 Bounded Sequences

A sequence is said to be **bounded** when it is bounded both above and below. K and k are respectively the upper and the lower bounds of the sequence.

Evidently a sequence is bounded if its range is bounded. Also the bounds of the range are the bounds of the sequence.

1.3.5 Monotone Sequences

Definition 1. *A sequence $\{a_n\}$ is:*

- ***Monotonic increasing*** *if* $a_1 \leq a_2 \leq a_3 \leq \cdots$
- ***Monotonic decreasing*** *if* $a_1 \geq a_2 \geq a_3 \geq \cdots$
- ***Monotonic*** *if either increasing or decreasing*

Some sequences are neither increasing nor decreasing (e.g., $\{1, 0, 1, 0, \ldots\}$).

1.3.6 Limit of a Sequence

Definition 2. *A sequence $\{a_n\}$* ***converges*** *to limit l if $\forall \epsilon > 0$, $\exists N$ such that $|a_n - l| < \epsilon$ for all $n > N$. We write $\lim_{n\to\infty} a_n = l$.*

1.3.7 Convergent, Divergent, Oscillating Sequences

Definition 3. *A sequence $\{a_n\}$ is:*

- ***Convergent*** *if* $\lim a_n$ *is finite*
- ***Divergent*** *if* $\lim a_n = \infty (or\ -\infty)$
- ***Oscillating*** *if neither convergent nor divergent*

1.3.8 Examples

Consider the following examples:

(i) $\left\{\frac{1}{3^n}\right\}$ converges to 0

(ii) $\{n^3\}$ diverges to $+\infty$

(iii) $\{-4^n\}$ diverges to $-\infty$

(iv) $\{(-1)^n\}$ oscillates finitely

(v) $\{(-1)^n n\}$ oscillates infinitely

Theorem 1. *Every convergent sequence is bounded.*

Proof. Let $\{a_n\}$ converge to l. For $\varepsilon = 1$, $\exists m \in \mathbb{N}$ such that $|a_n - l| < 1$ for all $n \geq m$. Then:

$$|a_n| \leq \max\{|a_1|, \ldots, |a_{m-1}|, |l| + 1\}$$

Thus $\{a_n\}$ is bounded. $\square$

Theorem 2. *A sequence cannot converge to more than one limit.*

Proof. Suppose $\lim a_n = l$ and $\lim a_n = l'$ with $l \neq l'$. Take $\varepsilon = \frac{|l-l'|}{2}$. For large n, a_n must lie in both $(l - \varepsilon, l + \varepsilon)$ and $(l' - \varepsilon, l' + \varepsilon)$, which are disjoint - a contradiction. $\square$

Theorem 3. *Every convergent sequence is bounded and has a unique limit.*

Proof. Follows immediately from Theorems 1 and 2. $\square$

The unique limit of a convergent sequence is called its **limit point**, denoted as:

$$\lim_{n\to\infty} a_n = l \quad \text{or} \quad a_n \to l \quad \text{as} \quad n \to \infty$$

1.3.9 Examples

Example 1. *Show that the sequence* $\left\{\frac{n+1}{n}\right\}$ *converges to 1.*

Example 2. *Prove that* $\left\{\frac{(-1)^n}{n}\right\}$ *converges to 0.*

1.4 Series

In this section we shall discuss the techniques of testing the behavior of infinite series regarding convergence. The most important technique for series with terms of the same sign (all positive or all negative) is to compare the given series with another suitably chosen series with known behavior. First, we discuss Comparison tests, followed by special convergence tests. Leibniz's test for alternating series will be considered in detail later.

1.4.1 Definition of an Infinite Series

A series is the sum of the terms of a sequence. If $u_1, u_2, u_3, \ldots$ is a sequence, then the sum $u_1 + u_2 + u_3 + \cdots$ of all terms is called an **Infinite Series**, denoted by $\sum_{n=1}^{\infty} u_n$ or simply $\sum u_n$.

Since we cannot add infinitely many terms directly, we associate with each series a sequence (s_n) of partial sums:

$$s_n = u_1 + u_2 + \cdots + u_n$$

1.4.2 Convergence of Series

The sequence (s_n) is called the **Sequence of Partial Sums**. We define convergence as follows:

- If (s_n) converges to a limit L, the series **converges** and L is its sum
- If (s_n) has no limit, the series **does not converge**

An infinite series is said to:

- **Converge** if (s_n) converges
- **Diverge** if (s_n) diverges to $\pm\infty$
- **Oscillate** if (s_n) oscillates finitely or infinitely

1.4.3 A Necessary Condition for Convergence

Theorem 4. *A necessary condition for convergence of an infinite series $\sum u_n$ is that $\lim_{n\to\infty} u_n = 0$.*

Proof. Let $s_n = u_1 + u_2 + \cdots + u_n$ be the partial sums. Since the series converges, $\lim_{n\to\infty} s_n = s$ exists. Then:

$$u_n = s_n - s_{n-1} \Rightarrow \lim_{n\to\infty} u_n = s - s = 0$$

□

Remarks:

(i) $\lim u_n = 0$ does *not* guarantee convergence.

(ii) $\lim u_n \neq 0$ *does* guarantee divergence.

Example 3. *The series* $\sum \frac{n}{n+1}$ *diverges since:*

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$$

1.4.4 Cauchy's General Principle of Convergence

Theorem 5. *A series* $\sum u_n$ *converges iff* $\forall \epsilon > 0$, $\exists m \in \mathbb{N}$ *such that:*

$$|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \epsilon \quad \forall n \geq m, p \geq 1$$

Proof. This follows directly from applying Cauchy's criterion to the sequence of partial sums $\{s_n\}$. □

Example 4. ***The harmonic series*** $\sum \frac{1}{n}$ ***diverges.***

Proof. Assume for contradiction that $\sum \frac{1}{n}$ converges. Then by Cauchy's criterion, for $\epsilon = \frac{1}{2}$ there exists $m \in \mathbb{N}$ such that:

$$\left| \sum_{k=n+1}^{n+p} \frac{1}{k} \right| < \frac{1}{2} \quad \forall n \geq m, p \geq 1$$

Take $n = m$ and $p = m$:

$$\sum_{k=m+1}^{2m} \frac{1}{k} \geq m \cdot \frac{1}{2m} = \frac{1}{2}$$

This contradicts the Cauchy criterion since $\frac{1}{2} \not< \frac{1}{2}$.

Note that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ despite the divergence. □

1.4.5 Positive Term Series

Series with non-negative terms are among the most important types of series. Their key property is that their partial sums form a monotonic increasing sequence.

Let $\sum u_n$ be a series of positive terms with partial sums (s_n), where:

$$s_n = u_1 + u_2 + \cdots + u_n > 0 \quad \forall n$$

Since $s_n - s_{n-1} = u_n > 0$, we have $s_n > s_{n-1}$. Thus (s_n) is strictly increasing.

Theorem 6. *A positive term series can only:*

- *Converge (if partial sums are bounded above)*
- *Diverge to* $+\infty$ *(otherwise)*

It cannot oscillate.

Theorem 7. *A positive term series converges if and only if its partial sums are bounded above.*

Proof. The sequence of partial sums (s_n) is monotonic increasing. By the monotone convergence theorem:

- If bounded above, it converges to its supremum
- If unbounded, it diverges to $+\infty$

□

1.4.6 Geometric Series

The positive term geometric series $\sum_{k=0}^{\infty} r^k$ converges for $r < 1$ and diverges for $r \geq 1$.

Proof. Consider the partial sums $s_n = \sum_{k=0}^{n-1} r^k$:

Case 1: $0 \leq r < 1$

$$s_n = \frac{1 - r^n}{1 - r} \leq \frac{1}{1 - r} \quad \forall n$$

The partial sums are bounded above and monotonic, hence convergent.

Case 2: $r = 1$

$$s_n = n \to \infty \quad \text{as} \quad n \to \infty$$

Case 3: $r > 1$

$$s_n > n \to \infty \quad \text{as} \quad n \to \infty$$

Thus the series converges iff $r < 1$.

1.5 Comparison Series

An important technique for testing convergence is comparison with known series.

Theorem 8. ***The p-series*** $\sum \frac{1}{n^p}$ ***converges if and only if*** $p > 1$**.**

1.6 Comparison Tests for Positive Term Series

1.6.1 First Comparison Test

Theorem 9 (Basic Comparison Test)**.** *Let* $\sum u_n$ *and* $\sum v_n$ *be positive term series. If* $\exists k > 0$ *and* $m \in \mathbb{N}$ *such that* $u_n \leq k v_n \; \forall n > m$*, then:*

- $\sum v_n$ *convergent* $\Rightarrow \sum u_n$ *convergent*
- $\sum u_n$ *divergent* $\Rightarrow \sum v_n$ *divergent*

Theorem 10 (Limit Comparison Test)**.** *If* $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = l$ *where* $0 < l < \infty$*, then* $\sum u_n$ *and* $\sum v_n$ *both converge or both diverge.*

1.6.2 Second Comparison Test

Theorem 11 (Ratio Comparison Test)**.** *For positive term series* $\sum u_n$ *and* $\sum v_n$*, if* $\exists m \in \mathbb{N}$ *such that*

$$\frac{u_{n+1}}{u_n} \leq \frac{v_{n+1}}{v_n} \quad \forall n \geq m$$

then:

- $\sum v_n$ *convergent* $\Rightarrow \sum u_n$ *convergent*
- $\sum u_n$ *divergent* $\Rightarrow \sum v_n$ *divergent*

1.7 Solved Examples

Example 5. *Show that the series* $1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$ *is convergent.*

We have:

$$\frac{1}{2} = \frac{1}{2}$$
$$\frac{1}{3!} < \frac{1}{2^2}$$
$$\frac{1}{4!} < \frac{1}{2^3}$$
$$\vdots$$
$$\frac{1}{n!} < \frac{1}{2^{n-1}}$$

Thus:

$$1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots < 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots$$

Each term after the first is less than the corresponding term of the convergent geometric series $1 + \frac{1}{2} + \frac{1}{2^2} + \cdots$, so by comparison, the given series converges.

Example 6. *Show that $\sum \frac{1}{(\log n)^p}$ diverges for $p > 0$.*

Since $\lim_{n\to\infty} \frac{(\log n)^p}{n} = 0$, we have $(\log n)^p < n$ for sufficiently large all $M \in \mathbb{N}$, hence:

$$\frac{1}{(\log n)^p} > \frac{1}{n} \quad \forall n > M$$

Comparing with the divergent series $\sum \frac{1}{n}$, the given series diverges.

Example 7. *Show that $\sum \left(\frac{2n+1}{(2n+3)^2} + \frac{2n+2}{(2n+4)^2} \right)$ converges.*

Let $u_n = \frac{2n+1}{(2n+3)^2} + \frac{2n+2}{(2n+4)^2}$. Compare with $v_n = \frac{1}{n^2}$:

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(2 + \frac{1}{n}\right)\left(2 + \frac{2}{n}\right) = 1$$

Since $\sum v_n$ converges, $\sum u_n$ converges.

Example 8. *Investigate $\sum \sin \frac{1}{n}$.*

Let $u_n = \sin \frac{1}{n}$ and $v_n = \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1$$

Since $\sum v_n$ diverges, $\sum \sin \frac{1}{n}$ diverges.

Example 9. *Test* $\sum \left((n^3+1)^{1/3} - n\right)$.

Let $u_n = (n^3+1)^{1/3} - n = n\left(\left(1+\frac{1}{n^3}\right)^{1/3} - 1\right) \sim \frac{1}{3n^2}$. *Compare with* $v_n = \frac{1}{n^2}$*:*

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{3}$$

Since $\sum v_n$ *converges,* $\sum u_n$ *converges.*

Example 10. *Test* $\sum \frac{1}{n^{1+1/n}}$.

Let $u_n = \frac{1}{n^{1+1/n}}$ *and* $v_n = \frac{1}{n}$*:*

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1$$

Since $\sum v_n$ *diverges,* $\sum \frac{1}{n^{1+1/n}}$ *diverges.*

Example 11. *Show that* $\sum \frac{1}{1\cdot 3\cdot 5\cdots(2n-1)}$ *converges.*

Proof. Let $u_n = \frac{1}{1\cdot 3\cdot 5\cdots(2n-1)}$. Note that:

$$1\cdot 3\cdot 5\cdots(2n-1) > 1\cdot 2\cdot 2\cdots 2 = 2^{n-1}$$

Thus:

$$u_n < \frac{1}{2^{n-1}} = v_n$$

Since $\sum v_n = \sum \frac{1}{2^{n-1}}$ is a convergent geometric series (ratio $= \frac{1}{2} < 1$), by comparison test, $\sum u_n$ converges. $\square$

Example 12. *Show that* $\sum \frac{2^{n-1}}{(n-1)!}$ *converges.*

Proof. Let $u_n = \frac{2^{n-1}}{(n-1)!}$. For $n > 4$:

$$u_n = \frac{2}{1}\cdot\frac{2}{2}\cdot\frac{2}{3}\cdot\frac{2}{4}\cdots\frac{2}{n-1} < 2\cdot 1\cdot\left(\frac{2}{3}\right)^{n-3}$$

Since $\sum \left(\frac{2}{3}\right)^{n-3}$ is a convergent geometric series, by comparison test, $\sum u_n$ converges. $\square$

Example 13. *Show that* $\sum \frac{2n+4}{(2n-1)(2n+1)(2n+3)}$ *converges.*

Proof. Let $u_n = \frac{2n+4}{(2n-1)(2n+1)(2n+3)}$. Compare with $v_n = \frac{1}{n^2}$:

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^2(2n+4)}{(2n-1)(2n+1)(2n+3)} = \frac{1}{4}$$

Since $\sum v_n = \sum \frac{1}{n^2}$ converges ($p = 2 > 1$), by limit comparison test, $\sum u_n$ converges. □

Example 14. *Show that* $\sum \frac{\log n}{\sqrt{n+1}}$ *diverges.*

Proof. Let $u_n = \frac{\log n}{\sqrt{n+1}}$. Compare with $v_n = \frac{1}{\sqrt{n}}$:

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sqrt{n}\log n}{\sqrt{n+1}} = \infty$$

Since $\sum v_n = \sum \frac{1}{\sqrt{n}}$ diverges ($p = \frac{1}{2} < 1$), by comparison test, $\sum u_n$ diverges. □

Example 15. *Show that* $\sum \frac{\sqrt{n}}{an^{3/2}+b}$ *diverges for* $a > 0$.

Proof. Let $u_n = \frac{\sqrt{n}}{an^{3/2}+b}$. Compare with $v_n = \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n\sqrt{n}}{an^{3/2}+b} = \frac{1}{a}$$

Since $\sum v_n = \sum \frac{1}{n}$ diverges ($p = 1$), by limit comparison test, $\sum u_n$ diverges. □

Example 16. *Show that* $\sum \left[(n^3+1)^{1/3} - n\right]$ *converges.*

Proof. Let $u_n = (n^3+1)^{1/3} - n$. Using binomial expansion:

$$u_n = n\left(1 + \frac{1}{n^3}\right)^{1/3} - n = n\left(1 + \frac{1}{3n^3} - \frac{1}{9n^6} + \cdots\right) - n = \frac{1}{3n^2} + O\left(\frac{1}{n^5}\right)$$

Compare with $v_n = \frac{1}{n^2}$:

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{3}$$

Since $\sum v_n = \sum \frac{1}{n^2}$ converges ($p = 2 > 1$), by limit comparison test, $\sum u_n$ converges. □

Example 17. *Show that* $\sum \frac{n^n}{(n+1)^{n+1}}$ *diverges.*

Proof. Let $u_n = \frac{n^n}{(n+1)^{n+1}}$. Compare with $v_n = \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^{n+1}}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n+1}} = \frac{1}{e}$$

Since $\sum v_n = \sum \frac{1}{n}$ diverges ($p = 1$), by limit comparison test, $\sum u_n$ diverges. □

Example 18. *Show that* $\sum \frac{2n-1}{n!}$ *converges.*

Proof. Let $u_n = \frac{2n-1}{n!}$. For $n \geq 2$:

$$u_n < \frac{2n}{n!} = \frac{2}{(n-1)!} < \frac{2}{2^{n-2}} = \frac{8}{2^n}$$

Since $\sum \frac{8}{2^n}$ converges (geometric series with ratio $\frac{1}{2}$), by comparison test, $\sum u_n$ converges.

□

Exercise

Use the comparison test to determine convergence:

(a) $\sum_{n=1}^{\infty} \frac{n+1}{n^3+5}$

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}}$

(c) $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$

(d) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$

(e) $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n-1}$

Cauchy's Root Test for Infinite Series

Cauchy's Root Test is a fundamental convergence test used to determine whether an infinite series of real or complex numbers converges or diverges. It was developed by Augustin-Louis Cauchy (1789-1857), a pioneering French mathematician. The test is particularly useful when the general term of the series involves exponents or radicals and is expressed in a form suitable for root extraction.

For a given a series $\sum a_n$, define $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.

- If $L < 1$, the series converges absolutely.

- If $L > 1$, the series diverges.
- If $L = 1$, the test is inconclusive.

Example 1. Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^n$.

Solution: Let $a_n = \left(\frac{1}{n}\right)^n$. Then,

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{1}{n}\right)^n} = \frac{1}{n}.$$

Now,

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{n} = 0 < 1.$$

Conclusion: Hence, the series converges absolutely.

Example 2. Test the convergence of the series $\sum_{n=1}^{\infty} 2^n$.

Solution: Let $a_n = 2^n$. Then,

$$\sqrt[n]{|a_n|} = \sqrt[n]{2^n} = 2.$$

Now,

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} 2 = 2 > 1.$$

Conclusion: Since $2 > 1$, the series diverges.

Example 3. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n}$.

Solution: Let $a_n = \frac{1}{n}$. Then,

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{1}{n}} = \frac{1}{n^{1/n}}.$$

Now,

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1.$$

Conclusion: $L = 1$, so the root test is inconclusive. (We know from other tests that this series diverges.)

Example 4. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$.

Solution: Let $a_n = \frac{x^n}{n}$. Then,

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left|\frac{x^n}{n}\right|} = \frac{|x|}{n^{1/n}}.$$

As $n \to \infty$, $n^{1/n} \to 1$, so

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{|x|}{n^{1/n}} = |x|.$$

Conclusion: The series converges when $|x| < 1$ and diverges when $|x| > 1$.

D'Alembert's Ratio Test for Infinite Series

D'Alembert's Ratio Test is a popular method for testing the convergence of an infinite series. Consider the series

$$\sum_{n=1}^{\infty} a_n.$$

Define the limit

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

Then:

- If $L < 1$, the series converges absolutely.
- If $L > 1$, or the limit is infinite, the series diverges.
- If $L = 1$, the test is inconclusive.

Example 1. Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n!}.$$

Solution: Let $a_n = \frac{1}{n!}$, hence $a_{n+1} = \frac{1}{(n+1)!}$.

$$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$$
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{n+1} = 0(= L).$$

Conclusion: Since $L = 0 < 1$, the series converges absolutely.

Example 2. Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{2^n}{n}.$$

Solution: Let $a_n = \frac{2^n}{n}$, hence $a_{n+1} = \frac{2^{n+1}}{n+1}$.

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{n+1} \cdot \frac{n}{2^n} = 2 \cdot \frac{n}{n+1}$$
$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} 2 \cdot \frac{n}{n+1} = 2(= L).$$

Conclusion: Since $L = 2 > 1$, the series diverges.

Example 3. Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n}{n+1}.$$

Solution: Let $a_n = \frac{n}{n+1}$, therefore $a_{n+1} = \frac{n+1}{n+2}$.

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n+2} \cdot \frac{n+1}{n} = \frac{(n+1)^2}{n(n+2)}$$
$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{n(n+2)} = 1.$$

Conclusion: Since $L = 1$, the test is inconclusive.

Example 4. Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n!}.$$

Solution: Let $a_n = \frac{x^n}{n!}$, hence $a_{n+1} = \frac{x^{n+1}}{(n+1)!}$.

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1}$$
$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{x}{n+1} = 0.$$

Conclusion: Since $L = 0 < 1$, the series converges absolutely for all real x.

Raabe's Test for Infinite Series

Raabe's Test is an enhancement of D'Alembert's Ratio Test for testing the convergence of positive-term series.

Consider the series:

$$\sum a_n, \quad a_n > 0$$

Define:

$$R_n = n\left(\frac{a_n}{a_{n+1}} - 1\right)$$

Then:

- If $\lim_{n\to\infty} R_n > 1$, the series converges.
- If $\lim_{n\to\infty} R_n < 1$, the series diverges.
- If $\lim_{n\to\infty} R_n = 1$, the test is inconclusive.

Raabe's Test is particularly helpful when the Ratio Test is inconclusive (i.e., limit = 1).

Example 1. Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n \ln n}, \quad n \geq 2.$$

Solution: Let $a_n = \frac{1}{n \ln n}$, hence $a_{n+1} = \frac{1}{(n+1)\ln(n+1)}$.

Use Raabe's Test:

$$\frac{a_n}{a_{n+1}} = \frac{(n+1)\ln(n+1)}{n\ln n}$$

$$\Rightarrow R_n = n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left(\frac{(n+1)\ln(n+1)}{n\ln n} - 1\right)$$

As $n \to \infty$, $R_n \to 0$

Conclusion: Since $R_n < 1$, the series diverges.

Example 2. Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Solution: Let $a_n = \frac{1}{n^2}$, hence $a_{n+1} = \frac{1}{(n+1)^2}$ so:

$$\frac{a_n}{a_{n+1}} = \frac{(n+1)^2}{n^2}$$

$$\Rightarrow R_n = n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left(\frac{(n+1)^2}{n^2} - 1\right) = n\left(1 + \frac{2}{n} + \frac{1}{n^2} - 1\right) = 2 + \frac{1}{n}$$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}\left(2 + \frac{1}{n}\right) = 2 > 1$$

Conclusion: The series converges.

Example 3. Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Solution: Let $a_n = \frac{1}{n}$, hence $a_{n+1} = \frac{1}{n+1}$, so

$$\frac{a_n}{a_{n+1}} = \frac{n+1}{n}$$

$$\Rightarrow R_n = n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left(\frac{n+1}{n} - 1\right) = n\cdot\left(\frac{1}{n}\right) = 1$$

Conclusion: Since $R_n = 1$, the test is inconclusive (We know this is the harmonic series, which diverges).

Example 4. Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}.$$

Solution: Let $a_n = \frac{n!}{n^n}$, hence $a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

We use Raabe's expression:

$$\frac{a_n}{a_{n+1}} = \frac{n!}{n^n} \cdot \frac{(n+1)^{n+1}}{(n+1)!} = \frac{(n+1)^{n+1}}{(n+1)n^n}$$

$$\Rightarrow R_n = n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left(\frac{(n+1)^n}{n^n} - 1\right)$$

Since $\left(\frac{n+1}{n}\right)^n \to e$, as $n \to \infty$, we get:

$$R_n \approx n(e-1) \to \infty$$

Conclusion: Since $R_n \to \infty > 1$, the series converges.

Problems

Test the convergence of the following series.

1.

$$\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{n^2}$$

2.

$$\sum_{n=1}^{\infty} \left(\frac{nx}{n+1}\right)^{n}$$

3.

$$\sum_{n=1}^{\infty} \frac{(n+1)^n x^n}{n+1}$$

4.
$$\sum_{n=1}^{\infty}\left(\frac{1+nx}{n}\right)^{n}$$

5.
$$\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^{2}x^{2}+\left(\frac{4}{5}\right)^{3}x^{3}+\cdots$$

6.
$$\frac{1}{3}+\frac{1\cdot 2}{3\cdot 5}+\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}+\cdots$$

7.
$$u_n=\frac{n^p}{n!}\quad ;\ (p>0)$$

8.
$$1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\cdots$$

9.
$$x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots\quad (x>0)$$

10.
$$\frac{x}{1\cdot 2}+\frac{x^2}{3\cdot 4}+\frac{x^3}{5\cdot 6}+\cdots\quad (x>0)$$

11.
$$\frac{1^2\cdot 2^2}{1!}+\frac{2^2\cdot 3^2}{2!}+\frac{3^2\cdot 4^2}{3!}+\cdots$$

12.
$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\cdots$$

13.
$$\frac{1}{2}+\frac{1\cdot 3}{2\cdot 4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}+\cdots$$

14.
$$\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7}+\cdots$$

MCQ Type questions:

Choose the correct option for each question. Use Cauchy's Root Test wherever applicable.

(i) The series $\sum \left(\frac{1}{n}\right)^n$:

- (a) Diverges
- (b) Converges conditionally
- (c) **Converges absolutely (Correct)**
- (d) Inconclusive

(ii) Cauchy's Root Test is based on the limit:

- (a) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$
- (b) $\lim_{n\to\infty} \frac{1}{a_n}$
- (c) $\limsup_{n\to\infty} \sqrt[n]{|a_n|}$ **(Correct)**
- (d) $\lim_{n\to\infty} \ln a_n$

(iii) If $\limsup \sqrt[n]{|a_n|} = 0.8$, the series $\sum a_n$:

- (a) Diverges
- (b) Is inconclusive
- (c) **Converges absolutely (Correct)**
- (d) Converges conditionally

(iv) If $\limsup \sqrt[n]{|a_n|} = 1.2$, then:

- (a) The series converges

(b) **The series diverges (Correct)**

(c) The series is conditionally convergent

(d) The series is absolutely convergent

(v) The series $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$:

(a) Diverges

(b) **Converges absolutely (Correct)**

(c) Inconclusive

(d) Converges conditionally

(vi) If $\limsup \sqrt[n]{|a_n|} = 1$, then:

(a) The series diverges

(b) The series converges

(c) **The test is inconclusive (Correct)**

(d) The series converges absolutely

(vii) The series $\sum_{n=1}^{\infty} \frac{5^n}{n!}$:

(a) Diverges

(b) Inconclusive

(c) **Converges absolutely (Correct)**

(d) Conditionally convergent

(viii) The series $\sum_{n=1}^{\infty} 2^n$:

(a) **Diverges (Correct)**

(b) Converges conditionally

(c) Converges absolutely

(d) Inconclusive

(ix) Cauchy's Root Test is particularly effective for series involving:

(a) Factorials

(b) Polynomials

(c) **Exponentials and powers (Correct)**

(d) Logarithms

(x) The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$:

(a) Diverges

(b) Inconclusive

(c) **Converges absolutely (Correct)**

(d) Converges conditionally

(xi) The root test gives conclusive results if the limit:

(a) Is equal to 0

(b) Is less than 1

(c) Is greater than 1

(d) **All of the above (Correct)**

(xii) Which of the following series diverges by the root test?

$$\sum_{n=1}^{\infty} 3^n$$

(a) **Yes, it diverges (Correct)**

(b) No, it converges

(c) It is conditionally convergent

(d) Root test is inconclusive

(xiii) The D'Alembert's Ratio Test is applied to a series $\sum a_n$ by evaluating:

(a) $\lim_{n\to\infty} \sqrt[n]{|a_n|}$

(b) $\lim_{n\to\infty} |a_n|$

(c) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ **(Correct)**

(d) $\sum a_n$

(xiv) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, the series:

(a) Diverges

(b) **Converges absolutely, (Correct)**

(c) Converges conditionally

(d) Is oscillating

(xv) If the limit of the ratio is $L > 1$, then the series:

(a) Converges absolutely

(b) Converges conditionally

(c) **Diverges, (Correct)**

(d) Test is inconclusive

(xvi) If the ratio test gives $L = 1$, then:

(a) Series converges

(b) Series diverges

(c) Series is alternating

(d) **The test is inconclusive, (Correct)**

(xvii) For $\sum \frac{2^n}{n!}$, the ratio test yields:

(a) $L = \infty$

(b) $L = 2$

(c) $L = 0$ **, (Correct)**

(d) $L = 1$

(xviii) The ratio test is most effective when the series involves:

(a) Logarithms

(b) Trigonometric terms

(c) **Factorials or exponentials, (Correct)**

(d) Linear terms

(xix) For $\sum \frac{n}{2^n}$, applying the ratio test gives:

(a) $L = 1$

(b) $L = 0$

(c) $L = \frac{1}{2}$**, (Correct)**

(d) $L = n$

(xx) If $a_n = \frac{1}{n}$, then the ratio test gives:

(a) $L = 0$

(b) $L = \infty$

(c) $L = 1$**, (Correct)**

(d) $L = \frac{1}{n}$

(xxi) For the geometric series $\sum ar^n$, the ratio test gives:

(a) $L = a$

(b) $L = n$

(c) $L = 0$

(d) $L = |r|$**, (Correct)**

(xxii) If a series converges by the ratio test, it implies:

(a) The terms are zero

(b) **Absolute convergence, (Correct)**

(c) Alternating series

(d) None of the above

(xxiii) The ratio test fails when:

(a) $L < 1$

(b) $L > 1$

(c) $L = 0$

(d) $L = 1$**, (Correct)**

(xxiv) For $\sum \frac{x^n}{n!}$, applying the ratio test yields:

(a) $L = \infty$

(b) $L = 0$**, (Correct)**

(c) $L = x$

(d) $L = 1$

(xxv) Raabe's Test is used to determine the:

(a) Radius of convergence of a power series

(b) **Convergence or divergence of a positive term series, (Correct)**

(c) Absolute convergence of alternating series

(d) Integral of a rational function

(xxvi) Raabe's Test is especially useful when:

(a) Limit comparison test fails

(b) Integral test fails

(c) **D'Alembert's ratio test is inconclusive, (Correct)**

(d) Root test is undefined

(xxvii) Raabe's Test is based on the limit:

$$R_n = n\left(\frac{a_n}{a_{n+1}} - 1\right)$$

What condition implies convergence?

(a) $\lim R_n < 1$

(b) $\lim R_n = 1$

(c) $\lim R_n > 1$, **(Correct)**

(d) $\lim R_n = 0$

(xxviii) The series $\sum \frac{1}{n^2}$ is:

(a) Divergent by Raabe's Test

(b) **Convergent by Raabe's Test, (Correct)**

(c) Inconclusive by Raabe's Test

(d) Oscillatory

(xxix) For the series $\sum \frac{1}{n}$, Raabe's Test is:

(a) Conclusive

(b) Divergent

(c) **Inconclusive, (Correct)**

(d) Convergent

(xxx) Which of the following satisfies $R_n = 2 + \frac{1}{n}$?

(a) $\sum \frac{1}{n^2}$, **(Correct)**

(b) $\sum \frac{1}{n}$

(c) $\sum \frac{1}{n \ln n}$

(d) $\sum \frac{1}{2^n}$

(xxxi) If $R_n = 0.5$, the series:

(a) Converges

(b) **Diverges, (Correct)**

(c) Is inconclusive

(d) Is alternating

(xxxii) The series $\sum \frac{n!}{n^n}$ is:

(a) Divergent by Raabe's Test

(b) **Convergent by Raabe's Test, (Correct)**

(c) Oscillatory

(d) Convergent by Integral Test

(xxxiii) The value of R_n for the series $\sum \frac{1}{n}$ is:

(a) **1, (Correct)**

(b) 0

(c) ∞

(d) 2

(xxxiv) If $\frac{a_n}{a_{n+1}} = \left(1 + \frac{1}{n}\right)^p$, what is the Raabe's limit?

(a) p, **(Correct)**

(b) $\ln p$

(c) $1/p$

(d) e^p

(xxxv) Raabe's test is valid for:

(a) **Positive term series, (Correct)**

(b) Alternating series

(c) All real series

(d) Complex series

(xxxvi) Which test refines the Ratio Test?

(a) Integral Test

(b) **Raabe's Test, (Correct)**

(c) Root Test

(d) Comparison Test

1.8 Introduction to Alternating Series

An alternating series is a series in which the signs of the terms alternate between positive and negative. Typically, it can be expressed in the form:

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

where $a_n > 0$ for all n, and the signs alternate starting with positive for the first term.

Alternating series appear in many engineering applications, such as signal processing, Fourier series, and approximations in numerical methods.

1.9 Test for Convergence: Alternating Series Test (Leibniz Test)

The Alternating Series Test provides a way to determine if an alternating series converges. For the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ where $a_n > 0$:

The series converges if the following two conditions are satisfied:

(a) The sequence $\{a_n\}$ is decreasing: $a_{n+1} \leq a_n$ for all n sufficiently large.

(b) $\lim_{n\to\infty} a_n = 0$.

If these conditions hold, the series converges. Note that this test gives conditional convergence, not absolute convergence, unless the series of absolute values also converges.

1.9.1 Why Does It Work?

The partial sums of the series oscillate but get closer to the limit because the terms are getting smaller and alternating in sign. The error in approximating the sum with a partial sum is less than the next term's magnitude.

1.10 Examples

1.10.1 Example 1: Alternating Harmonic Series

Consider the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Here, $a_n = \frac{1}{n}$.

- $a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$, so decreasing.

- $\lim_{n\to\infty} \frac{1}{n} = 0$.

Thus, it converges (to $\ln 2$).

1.10.2 Example 2: Non-Converging Series

Consider:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{n}{n+1} = 1 - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \cdots$$

Here, $a_n = \frac{n}{n+1}$.

- a_n is increasing (approaches 1), not decreasing.
- $\lim_{n\to\infty} a_n = 1 \neq 0$.

Thus, it diverges.

1.10.3 Example 3: With Trigonometric Terms

Consider:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{\sin n}{n}$$

Here, $a_n = \frac{|\sin n|}{n}$, but since $\sin n$ oscillates, we adjust to positive terms. However, $|\sin n|/n \to 0$, and decreasing on average, but rigorously, it converges by the test if we consider the envelope.

For simplicity: Assume $a_n = 1/n$, similar to harmonic, it converges.

1.11 Remainder Estimation

For a converging alternating series, the remainder $R_N = S - S_N$ (where S is the sum, S_N the partial sum) satisfies:

$$|R_N| \leq a_{N+1}$$

And the sign of R_N is the same as the first omitted term.

1.11.1 Example of Remainder

For the alternating harmonic series, after 10 terms, $|R_{10}| \leq \frac{1}{11} \approx 0.0909$.

1.12 Main Points to Remember

- An alternating series has terms with alternating signs, like $\sum(-1)^{n+1}a_n$ with $a_n > 0$.
- It converges if a_n is decreasing to 0.
- The test does not apply if a_n does not decrease or does not go to 0.
- Remainder bound: Error less than next term.
- Analogy: Like a zigzag path getting smaller steps, approaching a point without overshooting forever.
- Common example: Alternating harmonic converges, while harmonic diverges.
- In engineering: Used in approximations, e.g., Taylor series for sine or cosine have alternating terms.

1.13 Absolute and Conditional Convergence

1.13.1 Introduction

In infinite series, convergence can be absolute or conditional. These concepts are crucial in engineering mathematics for understanding series behavior in applications like Fourier analysis, signal processing, and error estimation in approximations.

An infinite series $\sum_{n=1}^{\infty} a_n$ may converge or diverge. To distinguish types of convergence, we examine the series of absolute values $\sum_{n=1}^{\infty} |a_n|$.

1.13.2 Absolute Convergence

A series $\sum a_n$ is said to converge **absolutely** if the series of its absolute values $\sum |a_n|$ converges.

If a series converges absolutely, then the original series $\sum a_n$ converges (but not vice versa).

1.13.3 Why It Matters

Absolute convergence implies that rearrangements of terms do not affect the sum (unlike conditional convergence). This is important in engineering for reliable computations.

1.13.4 Tests for Absolute Convergence

Use standard convergence tests on $\sum |a_n|$, such as:

- Ratio Test: If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, absolute convergence.
- Root Test: If $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, absolute convergence.
- Comparison Test, Integral Test, etc., applied to $|a_n|$.

1.13.5 Conditional Convergence

A series $\sum a_n$ is said to converge **conditionally** if $\sum a_n$ converges, but $\sum |a_n|$ diverges.

This often occurs in alternating series where the signs cause cancellation, but absolute values do not converge.

1.13.6 Implications

Conditionally convergent series can have their sums changed by rearranging terms (Riemann's rearrangement theorem). Engineers must be cautious with such series in practical applications to avoid errors.

1.13.7 Examples

1.13.8 Example 1: Absolute Convergence

Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$

The absolute series is $\sum \frac{1}{n^2}$, which is a p-series with p=2 ¿1, so converges.

Thus, the original series converges absolutely (sum is $\frac{\pi^2}{12}$).

1.13.9 Example 2: Conditional Convergence

The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

It converges by the Alternating Series Test.

But the absolute series $\sum \frac{1}{n}$ is the harmonic series, which diverges.

Hence, conditionally convergent (sum is $\ln 2$).

1.13.10 Example 3: Divergence

The series $\sum_{n=1}^{\infty}(-1)^{n+1} = 1 - 1 + 1 - 1 + \cdots$

Neither converges nor absolutely converges (terms don't go to 0).

1.13.11 Example 4: Another Absolute Convergence

Geometric series $\sum_{n=0}^{\infty}(-1/2)^n = 1 - 1/2 + 1/4 - 1/8 + \cdots$

Absolute series $\sum(1/2)^n$ converges (geometric with r=1/2¡1), so absolute convergence (sum=2/3).

1.13.12 Example 5: Conditional Convergence

Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ (the alternating harmonic series).

The absolute series is $\sum \frac{1}{n}$, which is the harmonic series and diverges.

However, the original series satisfies the conditions of the alternating series test: the terms decrease in absolute value ($\frac{1}{n+1} < \frac{1}{n}$) and approach 0 as $n \to \infty$.

Thus, the original series converges conditionally (sum is $\ln 2$).

1.13.13 Example 6: Divergence by Ratio Test

Consider the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

Apply the ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \lim_{n\to\infty} \frac{n+1}{(n+1)^{n+1}} \cdot n^n = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(\frac{1}{1+1/n}\right)^n = \frac{1}{e} < 1$.

Since the limit is less than 1, the series converges absolutely.

Wait, this example was meant for divergence, but it converges. Let me correct with a divergent series.

Consider the series $\sum_{n=1}^{\infty} n \cdot 2^n$.

Apply the ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)\cdot 2^{n+1}}{n\cdot 2^n} = \lim_{n\to\infty} 2 \cdot \frac{n+1}{n} = 2 > 1$.

Since the limit is greater than 1, the series diverges.

1.13.14 Example 7: Convergence by Comparison Test

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^3+n}$.

For large n, $\frac{1}{n^3+n} < \frac{1}{n^3}$, and $\sum \frac{1}{n^3}$ is a p-series with p=3 ¿1, so converges.

By the comparison test, the original series converges.

1.14 Main Points to Remember

- Absolute convergence: $\sum |a_n|$ converges $\implies \sum a_n$ converges.
- Conditional convergence: $\sum a_n$ converges but $\sum |a_n|$ diverges.
- Test absolute convergence by applying convergence tests to the absolute series.
- Alternating series often conditionally converge if absolute terms form a divergent series like harmonic.
- Analogy for absolute: Like a bank account where deposits (positive) and withdrawals (negative) are both limited in total amount, ensuring balance stabilizes.
- Analogy for conditional: Deposits and withdrawals cancel each other out pairwise, but total transactions are infinite; rearrangement could bankrupt you.
- In engineering: Absolute convergence ensures robustness in series expansions; conditional requires careful handling.
- Key example: Alternating harmonic (conditional), alternating $\frac{1}{n^2}$ (absolute).

Miscellaneous Exercise

(a) Examine the convergence of the following series:

   (a) $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

   Answer: Divergent.

   (b) $\sum_{n=1}^{\infty} \frac{2^n}{n}$

   Answer: Divergent.

(c) $\sum_{n=1}^{\infty} \frac{n}{1+2^n}$

Answer: Convergent.

(d) $\sum_{n=1}^{\infty} \frac{x^n}{1+x^n} \quad (x > 0)$

Answer: Convergent if $x \leq 1$, divergent if $x > 1$.

(e) $\sum_{n=1}^{\infty} \frac{nx^n}{n+1} \quad (x > 0)$

Answer: Convergent if $x \leq 1$, divergent if $x > 1$.

(f) $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$

Answer: Divergent.

(g) $\sum_{n=1}^{\infty} \frac{3^2 \cdot 5^2 \cdot 7^2 \cdots (2n+1)^2}{6^2 \cdot 8^2 \cdot 10^2 \cdots (2n+2)^2}$

Answer: Convergent.

(h) $\sum_{n=1}^{\infty} \frac{(n+2)(n+3)}{n(n+1)}$

Answer: Divergent.

(i) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n \quad (x > 0)$

Answer: Convergent if $x < 4$, divergent if $x \geq 4$.

(j) $\sum_{n=1}^{\infty} \frac{x^n}{n^2} \quad (x > 0)$

Answer: Convergent if $x \leq 1$, divergent if $x > 1$.

(k) $\sum_{n=1}^{\infty} \left(\frac{n+2}{n+3}\right)^n x^n \quad (x > 0)$

Answer: Convergent if $x < 1$, divergent if $x \geq 1$.

(l) $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2}$

Answer: Divergent.

(m) $\sum_{n=1}^{\infty} \frac{2^{3n}}{3^{2n}}$

Answer: Convergent.

(n) $\sum_{n=1}^{\infty} \frac{a^n}{1+n^2} \quad (|a| < 1)$

Answer: Convergent.

(o) $1 - \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 3} - \frac{1}{4 \cdot 4} + \cdots$

Answer: Absolutely convergent.

(b) Examine for absolute and conditional convergence:

(a) $\sum_{n=1}^{\infty}(-1)^n \frac{3^{3n}}{3^{2n}}$

Answer: Absolutely convergent.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{2^n}$

Answer: Absolutely convergent.

(c) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \cdots$

Answer: Conditionally convergent.

(d) $\sum_{n=1}^{\infty}(-1)^n \frac{n^2+1}{n^3}$

Answer: Conditionally convergent.

Chapter 2

Calculus of function of several variables

2.1 Introduction

The Calculus of Functions of Several Variables extends the ideas of single-variable calculus to functions that depend on two or more variables. Many real-life phenomena—such as temperature distribution, fluid flow, stress on a beam, or economic models—naturally involve multiple independent variables. This chapter introduces the tools needed to analyze such functions, including partial derivatives, directional derivatives, gradients, multiple integrals, and optimization techniques. These concepts help us describe how multivariable functions change, find maximum and minimum values under constraints, and compute quantities like area, volume, and mass in higher dimensions. Overall, this chapter provides the mathematical foundation for understanding complex systems in science and engineering.

2.2 Real life applications

Computer Science and Engineering (CSE)

- Optimization of machine learning models using multivariable cost functions.
- Image processing techniques using multivariable functions for filtering and enhancement.
- Analysis of algorithms involving multidimensional data.

Electronics and Communication Engineering (ECE)

- Electromagnetic field analysis using multivariable vector calculus.
- Signal processing involving functions of time and frequency.
- Designing antennas and communication systems using gradient and divergence concepts.

Electrical Engineering (EE)

- Power system optimization involving multiple parameters (voltage, current, resistance).
- Electric field and potential distribution analyzed using partial derivatives.
- Control system design based on multivariable transfer functions.

Mechanical Engineering (ME)

- Fluid dynamics involving velocity fields represented as multivariable functions.
- Heat transfer problems using partial differential equations.
- Stress and strain analysis in materials involving multivariable relationships.

Civil Engineering

- Structural analysis of beams, bridges, and buildings involving multiple variables.
- Soil mechanics and pressure distribution using multivariable models.
- Traffic flow and environmental modeling using multivariable calculus.

Chemical Engineering

- Reaction rate modeling with temperature, pressure, and concentration as variables.
- Mass and heat transfer equations involving multivariable functions.
- Optimization of chemical processes using multivariable techniques.

Biotechnology

- Enzyme kinetics involving multiple interacting variables.
- Population modeling using multivariable differential equations.
- Analysis of biological systems with temperature and concentration parameters.

2.3 Basic Definitions

The definition of a function of two variables is very similar to the definition of a function of one variable. The main difference is that, instead of mapping the values of one variable to the values of another variable, we map ordered pairs of variables to another variable.

A function of two variables $z = f(x, y)$ maps each ordered pair (x, y) in a subset D of the real plane $\mathbb{R}^2$ to a unique real number z. The set D is called the domain of the function. The range of f is the set of all real numbers z that has at least one ordered pair $(x, y) \in D$ such that $f(x, y) = z$.

Definition 1.1. (Interior Point). A point $a \in \mathbb{R}^2$ is said to be an interior point if there exists a neighbourhood of a which completely lies inside R.

Definition 1.2. (Boundary Point). A point $a \in \mathbb{R}^2$ is said to be a boundary point if any neighbourhood of a contains infinitely many points that lie inside $\mathbb{R}$ and outside $\mathbb{R}$.

Definition 1.3. (Open Set). A region $R \subseteq \mathbb{R}^2$ is called open if every point in R is an interior point.

Definition 1.4. (Closed Set). A closed set contains its own boundary which contains all its bounded points.

Definition 1.5. (Bounded Region). A region $R \subseteq \mathbb{R}^2$ is called bounded if there exists a disk that contains R.

Definition 1.6. (Connected Region). If any two points in R can be joined by a finite number of line segments all of which lie completely in R.

Definition 1.7. (Level Curves). The set of points on the plane where $f(x, y)$ has a constant value $f(x, y) = C$ is called a level curve of f.

Definition 1.8. (Graph of f). The set of all points $(x, y, f(x, y))$ in the space is called a graph of f. The graph of f is called a surface $w = f(x, y)$.

Definition 1.9. (Limit of a Function of Two Variables). Let $f : (x, y) \to f(x, y)$ be the function defined on some domain $D \subseteq \mathbb{R}^2$. Then the real number L is said to be the limit of the function f as (x, y) approaches (x_0, y_0) if for every $\epsilon > 0$, there exists $\delta > 0$

depending on ϵ and (x_0, y_0) such that for every (x, y) in D, we have

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \implies |f(x, y) - L| < \epsilon$$

(In terms of circular neighbourhood), or

$$0 < |x - x_0| < \delta, \quad 0 < |y - y_0| < \delta \implies |f(x, y) - L| < \epsilon$$

(In terms of square neighbourhood). Symbolically we write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L.$$

Example 1.1. By using the ϵ-δ technique, prove that $\lim_{(x,y)\to(2,3)} xy = 6$.

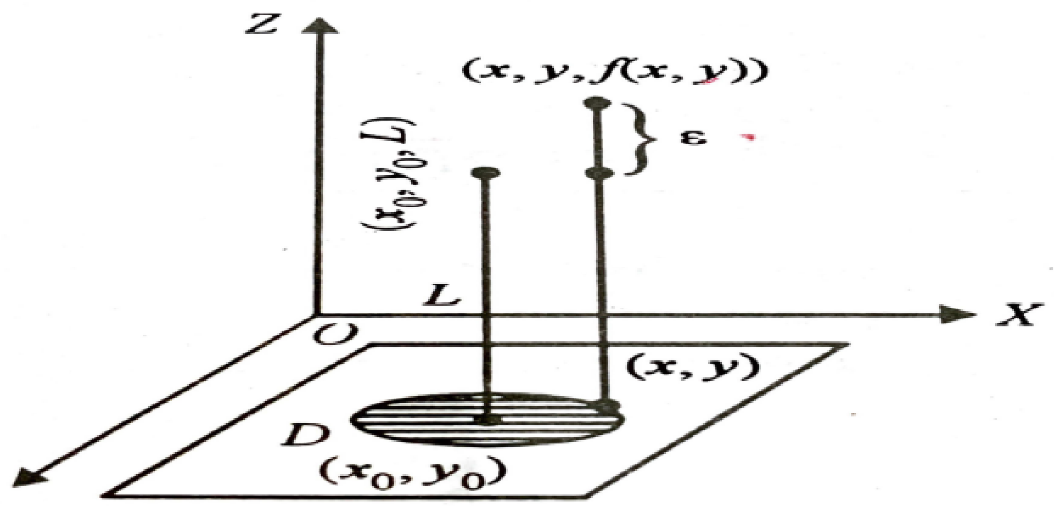


Figure 2.1: Geometrical Interpretation

Solution: We have to show that for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < \sqrt{(x - 2)^2 + (y - 3)^2} < \delta \implies |xy - 6| < \epsilon.$$

Now using the triangle inequality, we have

$$|xy - 6| = |xy - 2 \cdot 3| \leq |x(y - 3) + 3(x - 2)| \leq |x||y - 3| + 3|x - 2|.$$

Since

$$|x-2| \leq \sqrt{(x-2)^2+(y-3)^2}, \quad |y-3| \leq \sqrt{(x-2)^2+(y-3)^2},$$

we get

$$|xy-6| \leq |x|\sqrt{(x-2)^2+(y-3)^2}+3\sqrt{(x-2)^2+(y-3)^2} = (|x|+3)\sqrt{(x-2)^2+(y-3)^2}.$$

Let $\delta = 1$, then we have

$$|x-2| \leq \sqrt{(x-2)^2+(y-3)^2} < 1 \implies -1 < x-2 < 1 \implies 1 < x < 3 \implies |x| < 3.$$

So, we get

$$|xy-6| < (3+3)\sqrt{(x-2)^2+(y-3)^2} = 6\sqrt{(x-2)^2+(y-3)^2}.$$

Now, if we choose $\delta = \min\left\{1, \frac{\epsilon}{6}\right\}$, we then have

$$|xy-6| < 6\delta \leq 6 \cdot \frac{\epsilon}{6} = \epsilon.$$

Thus, for any $\epsilon > 0$, there exists a δ (in this case $\delta = \min\left\{1, \frac{\epsilon}{6}\right\}$) such that

$$|xy-6| < \epsilon \text{ whenever } \sqrt{(x-2)^2+(y-3)^2} < \delta.$$

This implies

$$\lim_{(x,y)\to(2,3)} xy = 6.$$

Example 1.2. By using the ϵ-δ technique, prove that $\lim_{(x,y)\to(1,2)}(3x+2y) = 7$.

Solution: We have to show that for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < \sqrt{(x-1)^2+(y-2)^2} < \delta \implies |3x+2y-7| < \epsilon.$$

Now using triangle inequality, we have

$$|3x+2y-7| = |3x-3+2y-4| = |3(x-1)+2(y-2)| \leq |3(x-1)|+|2(y-2)|$$

$$\leq 3\sqrt{(x-1)^2+(y-2)^2}+2\sqrt{(x-1)^2+(y-2)^2}$$

$$\leq 5\sqrt{(x-1)^2+(y-2)^2}.$$

Now if we chose $\delta = \frac{\varepsilon}{5}$, we then have $|f(x,y)-7| < 5\delta = 5\cdot\frac{\varepsilon}{5} = \varepsilon$. Thus, for any $\varepsilon > 0$, there exists a δ (in this case $\delta = \frac{\varepsilon}{5}$) such that $|3x+2y-7| < \varepsilon$ whenever $\sqrt{(x-1)^2+(y-2)^2} < \delta$. This implies

$$\lim_{(x,y)\to(1,2)} 3x+2y = 7.$$

Example 1.3. By using $\varepsilon - \delta$ technique prove that

$$\lim_{(x,y)\to(1,1)} (x^2+2y) = 3.$$

Solution: We have to show that for given $\varepsilon > 0$, $\exists\delta > 0$ such that

$$0 < \sqrt{(x-1)^2+(y-1)^2} < \delta \Rightarrow |x^2+2y-3| < \varepsilon.$$

Now using triangle inequality, we have

$$|x^2+2y-3| = |x^2-1+2y-2| = |x^2-1+2(y-1)| \leq |x^2-1|+2|y-1|$$

$$\leq |x-1|\cdot|x+1|+2|y-1|.$$

Evidently

$$0 < \sqrt{(x-1)^2+(y-1)^2} < \delta \Rightarrow |x-1| < \delta,\ |y-1| < \delta.$$

Choose $\delta = 1$, then

$$|x-1| < 1 \Rightarrow 0 < x < 2 \Rightarrow 1 < x+1 < 3 \Rightarrow 1 < |x+1| < 3.$$

Therefore,

$$|x^2 + 2y - 3| \leq |x - 1| \cdot |x + 1| + 2|y - 1| < 3|x - 1| + 2|y - 1| < 3\delta + 2\delta = 5\delta.$$

Choosing $\delta = \frac{\varepsilon}{5}$, we get

$$|x^2 + 2y - 3| \leq 5 \cdot \frac{\varepsilon}{5} = \varepsilon \quad \text{i.e.,} \quad |x^2 + 2y - 3| < \varepsilon.$$

Hence, required $\delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$, thus, δ exists. Therefore,

$$\lim_{(x,y)\to(1,1)} (x^2 + 2y) = 3.$$

2.4 Necessary Condition for Existence of the Limit of a Function of Two Variables

Theorem 1. If a function $f(x, y)$ has limit l, finite or infinite as $(x, y) \to (x_0, y_0)$ then $f(x, y)$ tends to l as $(x, y) \to (x_0, y_0)$ along any path.

Theorem 2. If a function has distinct limits as $(x, y) \to (x_0, y_0)$ along two distinct paths, then

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y)$$

does not exist.

Example 2.1. Find whether

$$\lim_{(x,y)\to(0,0)} \frac{x \sin(x^2 + y^2)}{x^2 + y^2}$$

exists or not, and if it exists, find the value.

Solution: We consider the path along x-axis i.e. $y = 0$, we get

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{x\to 0} f(x, 0) = \lim_{x\to 0} \frac{x \sin x^2}{x^2} = 0.$$

Now, we consider the path along y-axis i.e. $x = 0$, we get

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0} f(0,y) = 0.$$

Now, we consider the path along the straight line $y = mx$, we get

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{x\sin(x^2+m^2x^2)}{x^2+m^2x^2} = 0.$$

Hence the limit exists and is equal to zero.

Example 2.2. Find

1.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}.$$

2.

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}.$$

Solution: 1. Let us consider the path $y = mx$. As $(x,y)\to(0,0)$, we get $x\to 0$. Therefore,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{mx^2}{(1+m^2)x^2} = \frac{m}{1+m^2}$$

which depends on m. Hence, for different values of m, we get different limits. Thus, the limit does not exist.

Alternative way: Let $x = r\cos\theta$ and $y = r\sin\theta$. Then

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{r\to 0} \frac{r^2\sin\theta\cos\theta}{r^2} = \sin\theta\cos\theta$$

which depends on θ. So, for different values of θ, we get different limits. Hence, the limit does not exist.

Solution: 2. Let us consider the path $y^3 = mx$. As $(x,y)\to(0,0)$, we get $x\to 0$. Therefore,

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} = \lim_{(x,y)\to(0,0)} \frac{mx^2}{(1+m^2)x^2} = \frac{m}{1+m^2}$$

which depends on m. Hence, for different values of m, we get different limits. Thus, the limit does not exist.

Example 2.3. Show that

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0.$$

Solution: We have

$$|f(x,y) - 0| = \left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| = \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \frac{|xy|}{\sqrt{x^2+y^2}}$$

$$\leq \frac{1}{2}(x^2+y^2)\frac{1}{\sqrt{x^2+y^2}} \quad (\because |pq| \leq \frac{p^2+q^2}{2})$$

$$\leq \frac{1}{2}\sqrt{x^2+y^2}.$$

Now, if we chose $\delta = \frac{\varepsilon}{2}$, we then have $|f(x,y) - 0| < 2\delta = 2\frac{\varepsilon}{2} = \varepsilon$. Thus, for any $\varepsilon > 0$, there exists a δ (in this case $\delta = \frac{\varepsilon}{2}$) such that

$|f(x,y) - 0| < \varepsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$.

$\therefore$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0.$$

2.5 Iterated Limit

Let the function $z = f(x,y)$ be defined in some neighbourhood of (x_0, y_0). Then $\lim_{y\to y_0} f(x,y)$ is a function of x, say $\phi(x)$ provided it exists. If then $\lim_{x\to x_0} \phi(x)$ exists and is

equal to λ, a finite value of infinite one with a fixed sign, then we write

$$\lim_{x\to x_0} \lim_{y\to y_0} f(x,y) = \lambda,$$

which is called the iterated limit of $f(x,y)$ at point (x_0, y_0), where the limit for $y \to y_0$ is taken first and then afterwards for $x \to x_0$. Similarly, if we change the order of taking the limits, we get other iterated limit

$$\lim_{y\to y_0}\lim_{x\to x_0} f(x,y) = \mu(\text{say}).$$

Theorem 3. If the limit

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$$

exists and equal to l, for each fixed value of y,

$\lim_{x\to x_0} f(x,y)$ exists in neighbourhood of y_0 and also for each fixed value of x, $\lim_{y\to y_0} f(x,y)$

exists in a neighbourhood of x_0, then

$$\lim_{x\to x_0}\lim_{y\to y_0} f(x,y) = \lim_{y\to y_0}\lim_{x\to x_0} f(x,y) = l.$$

Example 3.1. Show that the iterated limit

$$\lim_{(x,y)\to(0,0)} x\sin(1/y)$$

exists and is equal to 0 but the single limit $\lim_{x\to 0} x_1 \sin(1/y)$ $(x_1 \neq 0)$ does not exists.

2.6 Continuity of a Function of Two Variables

Let $f(x,y)$ be a function of two variables defined in the open region D of $\mathbb{R}^2$. Let $P(x_0,y_0)$ be the point of D. Then f is said to be continuous at a point $P(x_0,y_0)$ if and only if

(i) f is defined at (x_0,y_0),

(ii)

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$$

exists, and

(iii)

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).$$

Example 4.1. Test the continuity of $f(x,y)$ at origin if

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Solution: Let $x = r\cos\theta$ and $y = r\sin\theta$. Then $r = \sqrt{x^2+y^2}$. So, we have

$$|f(x,y) - f(0,0)| = \left|\frac{2xx^2 - y^2}{x^2+y^2}\right| = \left|\frac{2r^3(\cos^2\theta - \sin^2\theta)\cos\theta}{r^2(\cos^2\theta + \sin^2\theta)}\right|$$

$$= |2r\cos 2\theta\cos\theta| \leq 2r \leq 2\sqrt{x^2+y^2}.$$

Now, if we chose $\delta = \frac{\varepsilon}{2}$, then we have $|f(x,y) - f(0,0)| < 2\delta = 2\frac{\varepsilon}{2} = \varepsilon$. Thus, for any $\varepsilon > 0$, there exists a δ (in this case $\delta = \frac{\varepsilon}{2}$) such that

$|f(x,y) - f(0,0)| < \varepsilon$ whenever $\sqrt{x^2+y^2} < \delta$.

$\therefore$ $f(x,y)$ is continuous at origin.

Example 4.2. Prove that the function $f(x,y) = xy$ is continuous everywhere.

Solution: Let (x_0, y_0) be any point of $\mathbb{R}^2$. Then for a given $\varepsilon > 0$, we have to find $\delta > 0$ such that

$d((x,y),(x_0,y_0)) < \delta \Rightarrow |xy - x_0y_0| < \varepsilon$.

Now

$|xy - x_0y_0| = |x(y-y_0) + y_0(x-x_0)| \leq |x|\cdot|y-y_0| + |y_0|\cdot|x-x_0|$.

We have

$d((x,y),(x_0,y_0)) = \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta \Rightarrow |x-x_0| < \delta$.

Choose first $\delta = 1$. So

$$|x - x_0| < 1 \implies x_0 - 1 < x < x_0 + 1 \implies |x| < |x_0 - 1|.$$

If we choose,

$$|y - y_0| < \frac{\epsilon}{2|x_0+1|} \quad \text{and} \quad |x - x_0| < \frac{\epsilon}{2|y_0|}.$$

Then

$$|xy - x_0y_0| < |x_0+1|\cdot\frac{\epsilon}{2|x_0+1|} + |y_0|\cdot\frac{\epsilon}{2|y_0|} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, if we choose $\delta = \min\left\{1, \frac{\epsilon}{2|x_0+1|}, \frac{\epsilon}{2|y_0|}\right\}$, we get

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta \implies |xy - x_0y_0| < \epsilon.$$

Hence δ exists. Therefore function $f(x,y)$ is continuous everywhere in $\mathbb{R}^2$.

Example 4.3. Show that the given function is continuous at the origin.

$$f(x,y) = \begin{cases} (x+y)\sin\frac{1}{x+y} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}.$$

Solution:

$$|f(x,y) - f(0,0)| = \left|(x+y)\sin\frac{1}{x+y}\right| \leq |x+y| \leq |x|+|y| \leq 2\sqrt{x^2+y^2} \quad \Big(\because |x| \leq \sqrt{x^2+y^2} \text{ and}$$

So, choose $\delta = \frac{\epsilon}{2}$.

Hence, for any $\epsilon > 0$, there exists a $\delta = \frac{\epsilon}{2}$ such that $|f(x,y) - f(0,0)| < \epsilon$ whenever $\sqrt{x^2+y^2} < \delta$. Hence, $f(x,y)$ is continuous at the origin.

Example 4.4. Show that the given function is continuous at the origin.

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}.$$

Solution: Let $x = r\cos\theta$, $y = r\sin\theta$.

$$\therefore \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = r|\cos\theta\sin\theta| \leq r = \sqrt{x^2+y^2} < \epsilon,$$

If

$$x^2 < \frac{\epsilon^2}{2}, \quad y^2 < \frac{\epsilon^2}{2},$$

or, if

$$|x| < \frac{\epsilon}{\sqrt{2}}, \quad |y| < \frac{\epsilon}{\sqrt{2}}.$$

Thus

$$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| < \epsilon, \quad \text{when } |x| < \frac{\epsilon}{\sqrt{2}}, \quad |y| < \frac{\epsilon}{\sqrt{2}}.$$

$$\implies \lim_{(x,y)\to(x_0,y_0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$$

$$\therefore \lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(0,0).$$

Hence, f is continuous at $(0,0)$.

5. Partial Derivatives

The ordinary derivative of a function of several variables with respect to one of the independent variables, keeping all other independent variables constant is called the partial derivative of the function with respect to the variable. Partial derivative of $f(x,y)$ with respect to x generally denoted by $\frac{\partial f}{\partial x}$ or $f_x(x,y)$, while those with respect to y are denoted by $\frac{\partial f}{\partial y}$ or $f_y(x,y)$.

$$\therefore \frac{\partial f}{\partial x} = \lim_{\delta x\to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$$

and

$$\frac{\partial f}{\partial y} = \lim_{\delta y\to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$$

when these limits exist.

The partial derivatives at a particular point (a,b) are often denoted by

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)}, \quad \frac{\partial f(a,b)}{\partial x} \quad \text{or} \quad f_x(a,b)$$

and

$$\left.\frac{\partial f}{\partial y}\right|_{(a,b)}, \quad \frac{\partial f(a,b)}{\partial y} \quad \text{or} \quad f_y(a,b)$$

$$\therefore f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h},$$

$$f_y(a,b) = \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k}.$$

Example 5.1. If

$$f(x,y)=\begin{cases}\frac{xy}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0).\end{cases}$$

Show that both the partial derivatives exist at $(0,0)$ but the function is not continuous there.

Solution: Putting $y=mx$, we see that

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{x\to 0} f(x,mx)=\frac{m}{1+m^2},$$

so that the limit depends on the value of m, i.e., on the path of approach, and is different for the different paths followed and therefore does not exist. Hence the function $f(x,y)$ is not continuous.

Again

$$f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0}{h}=0,$$
$$f_y(0,0)=\lim_{k\to 0}\frac{f(0,0+k)-f(0,0)}{k}=\lim_{k\to 0}\frac{0}{k}=0.$$

Hence, $f_x(0,0)=f_y(0,0)$.

Example 5.2. The function f is defined by $f(x,y)=\frac{2xy}{x^2+y^2}$ at $(x,y)\neq(0,0)$ and $f(0,0)=0$ is not continuous at $(0,0)$, however, the partial derivatives exist at $(0,0)$.

Solution: See the previous approach to the question.

Example 5.3. If the function $f(x,y)=\frac{x^2-y^2}{x^2+y^2+1}$ then by definition find the values of $f_y(0,0)$ and $f_x(0,0)$.

Solution: Here

$$f_y(0,0)=\lim_{k\to 0}\frac{f(0,0+k)-f(0,0)}{k}=\lim_{k\to 0}\frac{\frac{0-(0+k)^2}{0+(0+k)^2+1}-0}{k}=0.$$

$$f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{\frac{(0+h)^2-0}{(0+h)^2+0+1}-0}{h}=0.$$

6. Partial Derivatives of Higher Order

If a function $f(x,y)$ has partial derivatives of the first order at each point (x,y) of a certain region $D\subset\mathbb{R}^2$, then f_x, f_y are themselves functions of x,y and may also possess partial derivatives. These are called second-order partial derivatives of f and are denoted by

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}, \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy},$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}, \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}.$$

The second-order partial derivatives at a particular point (a, b) are often denoted by

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(a,b)}, \quad \frac{\partial^2 f(a,b)}{\partial x^2} \quad \text{or} \quad f_{xx}(a,b);$$

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(a,b)}, \quad \frac{\partial^2 f(a,b)}{\partial x \partial y} \quad \text{or} \quad f_{xy}(a,b)$$

and so on.

Thus

$$f_{xx}(a,b) = \lim_{h\to 0} \frac{f_x(a+h,b) - f_x(a,b)}{h} = \lim_{h\to 0} \frac{f(a+2h,b) - 2f(a+h,b) + f(a,b)}{h^2},$$

$$f_{yy}(a,b) = \lim_{k\to 0} \frac{f_y(a,b+k) - f_y(a,b)}{k} = \lim_{k\to 0} \frac{f(a,b+2k) - 2f(a,b+k) + f(a,b)}{k^2},$$

$$f_{xy}(a,b) = \lim_{h\to 0} \frac{f_y(a+h,b) - f_y(a,b)}{h} = \lim_{h\to 0}\lim_{k\to 0} \frac{f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b)}{hk},$$

$$f_{yx}(a,b) = \lim_{k\to 0} \frac{f_x(a,b+k) - f_x(a,b)}{k} = \lim_{k\to 0}\lim_{h\to 0} \frac{f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b)}{kh}$$

in case the limit exists.

Note: Here we have used $f_{xy} = \frac{\partial f_y}{\partial x}$. However, some authors also define $f_{xy} = \frac{\partial f_x}{\partial y}$. Similar changes can be made in defining f_{yx}.

Example 6.1. Let

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Solution: Using the definitions of partial derivatives, we have

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0,$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0,$$

$$f_x(h,0) = \lim_{h\to 0} \frac{f(h+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(2h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0,$$

$$f_y(h,0) = \lim_{k\to 0} \frac{f(h,0+k) - f(0,0)}{k} = \lim_{k\to 0} \frac{f(h,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{\frac{hk(h^2-k^2)}{h^2+k^2}}{k} = \lim_{k\to 0} \frac{h(1-(k/h)^2)}{1+(k/h)^2} =$$

$$f_y(0,k) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = 0.$$

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{hk(h^2-k^2)}{h^2+k^2}}{h} = \lim_{h\to 0} \frac{k((h/k)^2-1)}{(h/k)^2+1} = -k.$$

Therefore,

$$f_{xx}(0,0) = \lim_{h\to 0} \frac{f_x(0+h,0) - f_x(0,0)}{h} = \lim_{h\to 0} \frac{f_x(h,0) - f_x(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0,$$

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,0+k) - f_x(0,0)}{k} = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} = -1,$$

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1,$$

and

$$f_{yy}(0,0) = \lim_{k\to 0} \frac{f_y(0,0+k) - f_y(0,0)}{k} = \lim_{k\to 0} \frac{f_y(0,k) - f_y(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0.$$

Hence,

$$f_{xy}(0,0) \neq f_{yx}(0,0).$$

Solution: Let $f(x,y) = x^3y^2 - xy^5$. Then

$$f_x = \frac{\partial f}{\partial x} = 3x^2y^2 - y^5,$$

$$f_y = \frac{\partial f}{\partial y} = 2x^3y - 5xy^4,$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(f_x) = 6xy^2,$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(f_y) = 6x^2y - 5y^4,$$

$$f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(f_x) = 6x^2y - 5y^4,$$
$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(f_y) = 2x^3 - 20xy^3.$$

Example 6.3. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, then show that

(i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$.

(ii) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$.

Solution: Here

$$\begin{aligned} u &= \log(x^3 + y^3 + z^3 - 3xyz), \\ \frac{\partial u}{\partial x} &= \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}, \\ \frac{\partial u}{\partial y} &= \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz}, \\ \frac{\partial u}{\partial z} &= \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}. \end{aligned}$$

Hence,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)} = \frac{3}{x+y+z}. \tag{1}$$

$$\begin{aligned} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u &= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right) \\ &= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} = -\frac{9}{(x+y+z)^2}. \end{aligned} \tag{2}$$

2.7 Homogeneous Function

A function f of two independent variables x, y is said to be a **homogeneous function of degree** n if it can be written as:

$$f(x, y) = x^n \phi\left(\frac{y}{x}\right)$$

where ϕ denotes a function of $\frac{y}{x}$.

Alternatively:

$$f(tx, ty) = t^n f(x, y) \quad \text{for any } t > 0.$$

Similarly, a function f of three independent variables x, y, z is homogeneous of degree n if:

$$f(x, y, z) = x^n \phi\left(\frac{y}{x}, \frac{z}{x}\right) \quad \text{or} \quad f(tx, ty, tz) = t^n f(x, y, z).$$

Example 1: $f(x, y) = 4x^3 + 4x^2y + 4xy^2 + y^3$ is homogeneous of degree 3. Here

$$f(x, y) = 4x^3 + 4x^2y + 4xy^2 + y^3$$

Now, substitute $x \to tx$ and $y \to ty$:

$$\begin{aligned} f(tx, ty) &= 4(tx)^3 + 4(tx)^2(ty) + 4(tx)(ty)^2 + (ty)^3 \\ &= 4t^3x^3 + 4t^3x^2y + 4t^3xy^2 + t^3y^3 \\ &= t^3(4x^3 + 4x^2y + 4xy^2 + y^3) \\ &= t^3 f(x, y) \end{aligned}$$

Therefore, $f(x, y)$ is homogeneous of degree 3.

Example 2: $f(x, y, z) = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$ is homogeneous of degree 2.

We are given:

$$f(x, y, z) = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$$

Substitute $x \to tx$, $y \to ty$, and $z \to tz$:

$$\begin{aligned} f(tx, ty, tz) &= (tx)^2 + (ty)^2 + (tz)^2 + 2(tx)(ty) + 2(ty)(tz) + 2(tz)(tx) \\ &= t^2x^2 + t^2y^2 + t^2z^2 + 2t^2xy + 2t^2yz + 2t^2zx \\ &= t^2(x^2 + y^2 + z^2 + 2xy + 2yz + 2zx) \\ &= t^2 f(x, y, z) \end{aligned}$$

Therefore, $f(x, y, z)$ is homogeneous of degree 2.

2.8 Euler's Theorem on Homogeneous Function of Two Variables

Theorem 12. *If u is a homogeneous function of degree n in x, y, then:*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Proof. Let $u = f(x, y)$ is a homogeneous function of degree n in x,y, then

$$u = x^n\phi(\frac{y}{x}) \tag{2.1}$$

Now we calculate

$$\frac{\partial u}{\partial x} = nx^{n-1}\phi(\frac{y}{x}) + x^n\phi'(\frac{y}{x}) \cdot (-\frac{y}{x^2})$$

and

$$\frac{\partial u}{\partial y} = x^n\phi'(\frac{y}{x}) \cdot (\frac{1}{x})$$

Now

$$\begin{aligned} x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} &= x\left[nx^{n-1}\phi(\frac{y}{x}) + x^n\phi'(\frac{y}{x}) \cdot (-\frac{y}{x^2})\right] + y \cdot x^n\phi'(\frac{y}{x}) \cdot (\frac{1}{x}) \\ &= nx^n\phi(\frac{y}{x}) - x^{n-1}y\phi'(\frac{y}{x}) + x^{n-1}y\phi'(\frac{y}{x}) \\ &= nx^n\phi(\frac{y}{x}) \\ &= nu \end{aligned}$$

Hence proved. □

2.9 Euler's Theorem on Homogeneous Function of Three Variables

Theorem 13. *If u is a homogeneous function of degree n in x, y, z, then:*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

Proof. Let $u = f(x, y, z)$ is a homogeneous function of degree n in x,y,z, then

$$u = x^n f(\frac{y}{x}, \frac{z}{x})$$

Let $v = \frac{y}{x}$ and $w = \frac{z}{x}$, then $u = x^n f(v, w)$. Now same way as proof of Theorem 1, we prove this. □

Let u(x,y) be a homogeneous function of degree n, then

(i) $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = n(n-1)u$

(ii) $\dfrac{\partial}{\partial x}\left(x \dfrac{\partial u}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(y \dfrac{\partial u}{\partial y}\right) = nu + (n-1)\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}\right)$

2.10 Examples

Example 19. *Verify Euler's theorem for* $u = \frac{x^3+y^3}{x+y}$.

Step 1: Check if u is homogeneous.

We write:

$$u = \frac{x^3 + y^3}{x + y}$$

Now substitute $x \to tx$, $y \to ty$:

$$u(tx, ty) = \frac{(tx)^3 + (ty)^3}{tx + ty} = \frac{t^3(x^3 + y^3)}{t(x + y)} = t^2 \cdot \frac{x^3 + y^3}{x + y} = t^2 u(x, y)$$

So, u is homogeneous of degree 2.

Step 2: Verify Euler's Theorem: From Euler's theorem, we get

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u$$

We are to verify the above.

Here

$$u = \frac{x^3 + y^3}{x + y}$$

Now

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{(x+y)(3x^2)-(x^3+y^3)(1)}{(x+y)^2} \\ &= \frac{3x^2(x+y)-(x^3+y^3)}{(x+y)^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{(x+y)(3y^2)-(x^3+y^3)(1)}{(x+y)^2} \\ &= \frac{3y^2(x+y)-(x^3+y^3)}{(x+y)^2}\end{aligned}$$

Now compute:

$$\begin{aligned}x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} &= \frac{x\left[3x^2(x+y)-(x^3+y^3)\right]}{(x+y)^2}+\frac{y\left[3y^2(x+y)-(x^3+y^3)\right]}{(x+y)^2} \\ &= \frac{3x^3(x+y)-x\cdot(x^3+y^3)+3y^3(x+y)-y\cdot(x^3+y^3)}{(x+y)^2} \\ &= \frac{3x^3(x+y)+3y^3(x+y)-(x^3+y^3)(x+y)}{(x+y)^2} \\ &= \frac{(x+y)\left[3x^3+3y^3-x^3-y^3\right]}{(x+y)^2} \\ &= \frac{(x+y)(2x^3+2y^3)}{(x+y)^2}\end{aligned}$$

So:

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{(x+y)(2x^3+2y^3)}{(x+y)^2} = \frac{2(x^3+y^3)}{x+y} = 2u$$

Hence, Euler's theorem is verified.

Example 20. *If* $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, *then prove that* $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\cot u = 0$

Let $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right) \Rightarrow \cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$.

Define:

$$v(x,y) = \cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

Let us now prove that $v(x,y)$ is a homogeneous function and then apply Euler's theorem.

Step 1: Check homogeneity

Replace x with tx, and y with ty:

$$v(tx,ty)=\frac{tx+ty}{\sqrt{tx}+\sqrt{ty}}=\frac{t(x+y)}{\sqrt{t}\left(\sqrt{x}+\sqrt{y}\right)}=\sqrt{t}\cdot\frac{x+y}{\sqrt{x}+\sqrt{y}}=t^{\frac{1}{2}}\cdot v(x,y)$$

Hence, v is a homogeneous function of degree $\frac{1}{2}$.

Step 2: Apply Euler's Theorem to $v=\cos u$

Euler's theorem for homogeneous functions $v(x,y)$ of degree $\frac{1}{2}$ gives:

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=\frac{1}{2}v \tag{2.2}$$

Using chain we calculate partial derivative of v with respect to x and y,

$$\frac{\partial v}{\partial x}=-\sin u\cdot\frac{\partial u}{\partial x},\quad \frac{\partial v}{\partial y}=-\sin u\cdot\frac{\partial u}{\partial y}$$

Put $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ value in (2.2), we get

$$\begin{aligned}
x\cdot(-\sin u)\cdot\frac{\partial u}{\partial x}+y\cdot(-\sin u)\cdot\frac{\partial u}{\partial y}&=\frac{1}{2}\cdot\cos u\\
\Rightarrow-\sin u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)&=\frac{1}{2}\cos u\\
\Rightarrow-\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)&=\frac{1}{2}\cot u\\
\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\cot u&=0
\end{aligned}$$

Problems:

(i) If $u(x,y)=x\,f\left(\frac{y}{x}\right)+g\left(\frac{y}{x}\right)$, show that (i) $x\,\frac{\partial u}{\partial x}+y\,\frac{\partial u}{\partial y}=x\,f\left(\frac{y}{x}\right)$
(ii)$x^2\,\frac{\partial^2 u}{\partial x^2}+2xy\,\frac{\partial^2 u}{\partial x\,\partial y}+y^2\,\frac{\partial^2 u}{\partial y^2}=0.$

(ii) If $u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, prove that $x\,\frac{\partial u}{\partial x}+y\,\frac{\partial u}{\partial y}=\frac{1}{2}\sin 2u.$

(iii) If $z=x^2y^3$, verify that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=5z.$

Chain Rule

Introduction

The chain rule explains how a function depending on intermediate variables changes when those variables depend on others. It links derivatives across layers of dependence and is expressed as the product of Jacobians.

Engineering Application

In engineering, it is used to trace how small changes in inputs (like load, temperature, voltage, pressure) affect outputs (like stress, displacement, current, reaction rate). Applications appear in mechanical, electrical, civil, chemical and computer engineering, including optimization and machine learning.

Chain Rule for computing Derivatives of $y = f(u(x))$

Let y be a differentiable function of u and u be a differentiable function of x say, $y = f(u(x))$, then the derivative of y with respect to x is given by the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Example 1: Let $y = \sin(u^2)$ where $u = x - 2$, find $\frac{dy}{dx}$.
Solution: we have

$$\frac{dy}{du} = \cos(u^2) \cdot 2u, \qquad \frac{du}{dx} = 1.$$

Therefore,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2u\cos(u^2) \cdot 1 = 2(x-2)\cos(x-2)^2$$

Chain Rule for computing Derivatives of $y = f(u(x), v(x))$

Suppose y is a function of two variables u and v with continuous partial derivative $\dfrac{\partial y}{\partial u}$ and $\dfrac{\partial y}{\partial v}$, and that u and v are differentiable functions of x. Then y is a differentiable function of x, i.e., $y = f(u(x), v(x))$, and

$$\frac{dy}{dx} = \frac{\partial y}{\partial u}\frac{du}{dx} + \frac{\partial y}{\partial v}\frac{dv}{dx}.$$

Example 1: Let $y = u^3 - u^2 v$ where $u = x - 2$, and $v = x^2$, find $\frac{dy}{dx}$.

Solution: Here $y = u^3 - u^2 v$ where $u = x - 2$, and $v = x^2$

Now we calculate

$$\frac{\partial y}{\partial u} = 3u^2 - 2uv, \qquad \frac{\partial y}{\partial v} = -u^2, \qquad \frac{du}{dx} = 1, \qquad \frac{dv}{dx} = 2x.$$

Therefore,

$$\begin{aligned} \frac{dy}{dx} &= \frac{\partial y}{\partial u}\frac{du}{dx} + \frac{\partial y}{\partial v}\frac{dv}{dx} \\ &= (3u^2 - 2uv)(1) + (-u^2)(2x) \\ &= 3u^2 - 2uv - 2u^2 x. \end{aligned}$$

Substituting for u and v now gives $\dfrac{dy}{dx}$ as a function of the independent variable x:

$$\begin{aligned} \frac{dy}{dx} &= 3(x-2)^2 - 2(x-2)x^2 - 2(x-2)^2 x \\ &= -4x^3 + 15x^2 - 20x + 12 \end{aligned}$$

Chain Rule for computing Partial Derivatives of $z = f(x(u, v), y(u, v))$

Suppose that z is a function of x and y, and each of these is in turn a function of u and v. (In symbols, $z = f(x(u, v), y(u, v))$.) Assuming that all partial derivatives exist and are continuous, obtain formulas for

$$\frac{\partial z}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v}.$$

To compute $\dfrac{\partial z}{\partial u}$, notice that there are two paths from z to u: one via x and the other via y. This gives

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}.$$

Similarly, following the two paths from z to v gives

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

Example 1: Let $z = \sin(xy)$, and $x = u + v, y = u^2 + v$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial u}$

Solution: We are given

$$z = \sin(xy), \quad x = u + v, \quad y = u^2 + v.$$

We want to compute $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

Step 1: Chain rule

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

Step 2: Partial derivatives of z with respect to x and y

Since $z = \sin(xy)$,

$$\frac{\partial z}{\partial x} = \cos(xy) \cdot y, \qquad \frac{\partial z}{\partial y} = \cos(xy) \cdot x.$$

Step 3: Partial derivatives of x and y with respect to u, v

$$\frac{\partial x}{\partial u} = 1, \qquad \frac{\partial x}{\partial v} = 1,$$
$$\frac{\partial y}{\partial u} = 2u, \qquad \frac{\partial y}{\partial v} = 1.$$

Step 4: Substitute into the chain rule

$$\frac{\partial z}{\partial u} = \cos(xy) \cdot y \cdot (1) + \cos(xy) \cdot x \cdot (2u),$$
$$\frac{\partial z}{\partial u} = \cos(xy)\,(y + 2ux).$$

$$\frac{\partial z}{\partial v} = \cos(xy) \cdot y \cdot (1) + \cos(xy) \cdot x \cdot (1),$$

$$\frac{\partial z}{\partial v} = \cos(xy)\,(x+y).$$

Step 5: Express in terms of u, v only

Since $x = u + v$ and $y = u^2 + v$, we have

$$\frac{\partial z}{\partial u} = \cos\big((u+v)(u^2+v)\big)\,\Big((u^2+v) + 2u(u+v)\Big),$$

$$\frac{\partial z}{\partial v} = \cos\big((u+v)(u^2+v)\big)\,\Big((u+v) + (u^2+v)\Big).$$

Example 21. *If $u = f(x^2+2yz, y^2+2zx)$, show that $(y^2-zx)\frac{\partial u}{\partial x} + (x^2-zy)\frac{\partial u}{\partial y} + (z^2-xy)\frac{\partial u}{\partial z} = 0$*

Let $p = x^2 + 2yz$ and $\mathrm{q} = \mathrm{y}^2 + 2zx$, then $u = f\big(p, q\big)$

where $u = f$ is a differentiable function of two variables p, q.

Step 1: Compute the partial derivatives of u using the chain rule.

Let

$$f_p = \frac{\partial f}{\partial p}, \qquad f_q = \frac{\partial f}{\partial q}.$$

Then by the chain rule

$$\frac{\partial u}{\partial x} = f_p\,\frac{\partial p}{\partial x} + f_q\,\frac{\partial q}{\partial x}, \qquad \frac{\partial u}{\partial y} = f_p\,\frac{\partial p}{\partial y} + f_q\,\frac{\partial q}{\partial y},$$

$$\frac{\partial u}{\partial z} = f_p\,\frac{\partial p}{\partial z} + f_q\,\frac{\partial q}{\partial z}.$$

Since $p = x^2 + 2yz$ and $q = y^2 + 2zx$, their partial derivatives are

$$\frac{\partial p}{\partial x} = 2x, \quad \frac{\partial p}{\partial y} = 2z, \quad \frac{\partial p}{\partial z} = 2y,$$

$$\frac{\partial q}{\partial x} = 2z, \quad \frac{\partial q}{\partial y} = 2y, \quad \frac{\partial q}{\partial z} = 2x.$$

Hence

$$\frac{\partial u}{\partial x} = f_p(2x) + f_q(2z),$$
$$\frac{\partial u}{\partial y} = f_p(2z) + f_q(2y),$$
$$\frac{\partial u}{\partial z} = f_p(2y) + f_q(2x).$$

Step 2: Form the combination and simplify.

Substituting the expressions of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ in $(y^2 - zx)\,\frac{\partial u}{\partial x} + (x^2 - zy)\,\frac{\partial u}{\partial y} + (z^2 - xy)\,\frac{\partial u}{\partial z}$. we get

$$\begin{aligned}&(y^2 - zx)\big(f_p 2x + f_q 2z\big) + (x^2 - zy)\big(f_p 2z + f_q 2y\big) + (z^2 - xy)\big(f_p 2y + f_q 2x\big)\\ =&2f_p\Big[x(y^2 - zx) + z(x^2 - zy) + y(z^2 - xy)\Big] + 2f_q\Big[z(y^2 - zx) + y(x^2 - zy) + x(z^2 - xy)\Big].\end{aligned}$$

We simplify the two bracketed expressions separately.

For the first bracket:

$$\begin{aligned}x(y^2 - zx) + z(x^2 - zy) + y(z^2 - xy) &= xy^2 - x^2z + zx^2 - z^2y + yz^2 - xy^2\\ &= \big(xy^2 - xy^2\big) + \big(-x^2z + zx^2\big) + \big(-z^2y + yz^2\big)\\ &= 0.\end{aligned}$$

For the second bracket:

$$\begin{aligned}z(y^2 - zx) + y(x^2 - zy) + x(z^2 - xy) &= zy^2 - z^2x + yx^2 - y^2z + xz^2 - x^2y\\ &= \big(zy^2 - y^2z\big) + \big(yx^2 - x^2y\big) + \big(xz^2 - z^2x\big)\\ &= 0.\end{aligned}$$

Therefore both brackets vanish, so $(y^2 - zx)\,\frac{\partial u}{\partial x} + (x^2 - zy)\,\frac{\partial u}{\partial y} + (z^2 - xy)\,\frac{\partial u}{\partial z} = 0$.

Problems:

(i) Suppose $z = x^2 + y^2$, where $y = e^{3x}$. Compute $\dfrac{dz}{dx}$.

(ii) If $z = f(x, y)$, where $x = e^u \cos v$ and $y = e^u \sin v$, then show that $y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = e^2 u \frac{\partial z}{\partial y}$

(iii) If $f(x, y, z, w) = 0$ prove that $\frac{\partial x}{\partial y} \times \frac{\partial y}{\partial z} \times \frac{\partial z}{\partial w} \times \frac{\partial w}{\partial x} = 1$.

(iv) If $u = f(x + y,\ y + z,\ z + x)$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

(v) Let $z = \ln(u^2 + v^2), \quad u = x^2 - y^2,\ v = 2xy$. Show that $\frac{\partial z}{\partial x} = \frac{2x}{x^2+y^2}, \frac{\partial z}{\partial y} = \frac{2y}{x^2+y^2}$.

(vi) Let $w = e^{x^2+y^2+z^2}, \quad x = r\cos\theta,\ y = r\sin\theta,\ z = r$. Find $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial \theta}$.

2.10.1 Implicit Function

Implicit function is a function defined for differentiation of functions containing the variables, which cannot be easily expressed in the form of $y = f(x)$.

An expression of an implicit function contains two or more than two variables, written and expressed in the form of an equation f(x, y) = 0 or g(x, y, z) = 0, with the expression on the left-hand side of the equation containing the variables, constants, coefficients, and equalized to zero.

The function of the form $g(x, y) = 0$ or an equation, $x^2 + y^2 + 4xy + 25 = 0$ is an example of implicit function, where the dependent variable "y" and the independent variable "x" cannot be easily segregated to represent it as a function of the form $y = f(x)$.

Examples

(i) $f(x, y) = xy^2 - 3y - e^x$.

(ii) $f(x,y) = y^5 - 5xy + 4x^2$.

(iii) $f(x,y) = e^y + x - y + logy$.

Properties of Implicit Function

The following are some of the important properties of implicit function, which are helpful in a better understanding of this function.

- The implicit function cannot be expressed in the form of $y = f(x)$.

- The implicit function is always represented as a combination of variables as $f(x,y) = 0$.

- The implicit function is a non-linear function with many variables.

- The implicit function is written both in terms of the dependent variable and independent variable.

- The vertical line draw through the graph of an implicit function cuts it across more than one point.

(Two variables, one equation) If $f(x,y) = 0$ gives y as an implicit function of x in a region on $x - y$ plane then $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$ provided $\frac{\partial f}{\partial y} \neq 0$.

(Three variables, one equation) If $f(x,y,z) = 0$ gives z as an implicit function of x and y in a region on $x - y$ plane then $\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}$ and $\frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}$ provided $\frac{\partial f}{\partial z} \neq 0$.

Illustrative Examples

For each of the following implicit relationships,

(i) Find $\frac{dy}{dx}$, if $(cosx)^y = (siny)^x$.

Let $f(x, y) = (cosx)^y - (siny)^x$.

So,

$$\frac{\partial f}{\partial x} = y(cosx)^{y-1}(-sinx) - (siny)^x logsiny.$$

and

$$\frac{\partial f}{\partial y} = (cosx)^y logcosx - x(siny)^{x-1}cosy.$$

By Theorem-2.10.1,

$$\begin{aligned} \frac{dy}{dx} &= -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \\ &= -\frac{-y(cosx)^{y-1}sinx - (siny)^x logsiny}{(cosx)^y logcosx - x(siny)^{x-1}cosy} \\ &= \frac{y(cosx)^{y-1}sinx + (cosx)^y logsiny}{(cosx)^y logcosx - x\frac{cosx}{siny}cosy}, \qquad \text{[from the given relation]} \\ &= \frac{ytanx + logsiny}{logcosx - xcoty}. \end{aligned}$$

(ii) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, if $x^3 + y^3 - 3xy = 0$.

Here, $f(x, y) = x^3 + y^3 - 3xy$

So,

$$\frac{\partial f}{\partial x} = 3x^2 - 3y.$$

and

$$\frac{\partial f}{\partial y} = 3y^2 - 3x.$$

By Theorem-2.10.1,

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{3x^2 - 3y}{3y^2 - 3x} = \frac{y - x^2}{y^2 - x}.$$

and

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{y - x^2}{y^2 - x}\right) \\
&= \frac{(y^2 - x)\frac{d}{dx}(y - x^2) - (y - x^2)\frac{d}{dx}(y^2 - x)}{(y^2 - x)^2} \\
&= \frac{(y^2 - x)(\frac{dy}{dx} - 2x) - (y - x^2)(2y\frac{dy}{dx} - 1)}{(y^2 - x)^2} \\
&= \frac{-2xy}{(y^2 - x)^3} \qquad \text{[Using Equation2.10.1].}
\end{aligned}$$

(iii) Find $\frac{\partial x}{\partial y}$ and $\frac{\partial x}{\partial z}$, if $2x^2 - yz + xz^2 = 4$.

Let $f(x, y, z) = 2x^2 - yz + xz^2 - 4$,

Now,

$$\begin{aligned}
\frac{\partial x}{\partial y} &= -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}} = -\frac{\frac{\partial}{\partial y}(2x^2 - yz + xz^2 - 4)}{\frac{\partial}{\partial x}(2x^2 - yz + xz^2 - 4)} \\
&= \frac{z}{4x + z^2}.
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial x}{\partial z} &= -\frac{\frac{\partial f}{\partial z}}{\frac{\partial f}{\partial x}} = -\frac{\frac{\partial}{\partial z}(2x^2 - yz + xz^2 - 4)}{\frac{\partial}{\partial x}(2x^2 - yz + xz^2 - 4)} \\
&= \frac{y - 2xz}{4x + z^2}.
\end{aligned}$$

Related Problems

1. If $x^3 + 3x^2y + 6xy^2 + y^3 = 1$, Find $\frac{dy}{dx}$ using differential formula of implicit functions.

2. If $x^n + y^n = a^n$, Find $\frac{dy}{dx^2}$

2.10.2 Total Differentials

For function $z = f(x, y)$ whose partial derivatives exists, total differential of z is

$$dz = \frac{\partial f}{\partial x}(x, y)\hat{A}dx + \frac{\partial f}{\partial y}(x, y).dy, \tag{2.3}$$

where dz is sometimes written df. On the one hand, the exact value of function is $f(x + \Delta x, y + \Delta y) = f(x, y) + \Delta z$.

On the other hand, if differentials dx and dy are small, then $dz \approx \Delta z$, and so the value of the function could be (linearly) approximated by,

$$f(x + \Delta x, +\Delta y) \approx f(x, y) + dz = f(x, y) + \frac{\partial f}{\partial x}(x, y)\hat{A}dx + \frac{\partial f}{\partial y}(x, y)\hat{A}dy.$$

Total differentials can be generalized. For a function $f = f(x, y, z)$ whose partial derivatives exists, the total differential of f is given by

$$df = \frac{\partial f}{\partial x}(x, y, z)\hat{A}dx + \frac{\partial f}{\partial y}(x, y, z)\hat{A}dy + \frac{\partial f}{\partial z}(x, y, z)\hat{A}dz.$$

Second Order Differential

Let $z = f(x, y)$ be a function of two independent variables x and y. The second order differential of $z = f(x, y)$ is defined by $d^2z = d(dz)$.

Illustrative Examples

(i) Let $z = x^4e^{3y}$. Find dz

We compute the partial derivatives:
$\frac{\partial f}{\partial x} = 4x^3e^{3y}$ and $\frac{\partial f}{\partial y} = 3x^4e^{3y}$.

Following equation 2.3, we have

$$dz = 4x^3e^{3y}dx + 3x^4e^{3y}dy.$$

(ii) Let $z = x^2y - 3y$. Find dz when $x = 4$, $y = 3$, $\Delta x = -0.01$, $\Delta y = 0.02$.

We compute the partial derivatives:
$\frac{\partial f}{\partial x} = 2xy$ and $\frac{\partial f}{\partial y} = 3x^4e^{3y}$.

Following equation 2.3, we have

$$dz = 2xydx + (x^2 - 3)dy = 2 \times 4 \times 3 \times (-0.01) + (4^2 - 3) \times 0.02 = 0.02.$$

(iii) If $z = x^2y - 3y$, determine d^2z.

$$\begin{aligned} d^2z &= \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}.dy\right)^2 z \\ &= \frac{\partial^2 z}{\partial x^2}(dx)^2 + 2\frac{\partial^2 z}{\partial x \partial y}dxdy + \frac{\partial^2 z}{\partial y^2}(dy)^2. \end{aligned}$$

Here,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(2xy) = 2y$$

and

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial x}(x^2 - 3) = 0$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial y}) = \frac{\partial}{\partial x}(x^2 - 3) = 2x.$$

$$\begin{aligned} d^2z &= 2y(dx)^2 + 4xdxdy + 0.(dy)^2 \\ &= 2y(dx)^2 + 4xdxdy. \end{aligned}$$

2.10.3 Jacobians

Let $f_1, f_2, ..., f_n$ be n real-valued functions of n variables $x_1, x_2, ..., x_n$ having first order partial derivatives at a point $a = (a_1, a_2, ..., a_n)$. Then the matrix

$$\begin{bmatrix} \frac{\partial f_1(a)}{\partial x_1} & \frac{\partial f_1(a)}{\partial x_2} & \frac{\partial f_1(a)}{\partial x_3} & \cdots & \frac{\partial f_1(a)}{\partial x_n} \\ \frac{\partial f_2(a)}{\partial x_1} & \frac{\partial f_2(a)}{\partial x_2} & \frac{\partial f_2(a)}{\partial x_3} & \cdots & \frac{\partial f_2(a)}{\partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n(a)}{\partial x_1} & \frac{\partial f_n(a)}{\partial x_2} & \frac{\partial f_n(a)}{\partial x_3} & \cdots & \frac{\partial f_n(a)}{\partial x_n} \end{bmatrix}$$

is called the Jacobian matrix of the functions $f_1, f_2, ..., f_n$ at $(a_1, a_2, ...a_n)$. The determinant of this matrix is called the Jacobian of the functions at $(a_1, a_2, ...a_n)$. It is denoted by $\frac{\partial(f_1, f_2, ..., f_n)}{\partial(x_1, x_2, ..., x_n)}$ or $J(f_1, f_2, ..., f_n)$.

If $u = f(x, y)$ and $v = g(x, y)$ are differentiable functions of two independent variables x and y in a region, then the determinant

$$\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}$$

i.e.,

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is called the Jacobian of u, v with respect to x, y. It is also called the Jacobian of transformation $u = f(x, y)$ and $v = g(x, y)$ and is written as $\frac{\partial(u,v)}{\partial(x,y)}$.

Numerical Illustration

(i) If $u = x^2$, $v = y^2$, then find $\frac{\partial(u,v)}{\partial(x,y)}$

Here, $u = x^2$ and $v = y^2$

So, $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 0$

and $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 2y$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 0 \\ 0 & 2y \end{vmatrix} = 4xy$$

Related Problems

1. If $x = rcos\theta$, $y = rsin\theta$, $z = z$, then find $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$

2. If $u = xyz$, $v = xy + yz + zx$, $w = x + y + z$, then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Theorem 1: Necessary Conditions of Extrema

If a function $z = f(x, y)$ has a maximum or minimum point at (a, b), then

$$f_x(a, b) = f_y(a, b) = 0,$$

provided these partial derivatives exist.

Notes:

(i) The converse of the above theorem is not true in general. Let us consider the function

$$f(x, y) = x^2y^3.$$

Here,

$$f_x = 2xy^3, \quad f_y = 3x^2y^2.$$

So $f_x(0, 0) = f_y(0, 0) = 0$, but $(0, 0)$ is not an extreme value. This is because there exist no $h > 0$, $k > 0$, such that $f(0 \pm h, 0 \pm k)$ changes sign. For instance,

$$f(h, h) = h^5 > 0, \quad f(-h, -h) = -h^5 < 0 \text{ for } h > 0.$$

But

$$f(\epsilon, -\epsilon) = -\epsilon^5 < 0 \text{ for small } \epsilon > 0.$$

(ii) A function $f(x,y)$ may have an extreme value at (a,b) even if $f_x(a,b)$ and $f_y(a,b)$ do not exist.

For example, let

$$f(x,y) = |x| + |y|.$$

Then:

$$\lim_{h\to 0^+} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0^+} \frac{|h|-0}{h} = 1,$$
$$\lim_{h\to 0^-} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0^-} \frac{|h|-0}{h} = -1.$$

So $f_x(0,0)$ does not exist. Similarly, $f_y(0,0)$ does not exist. However, $f(0,0)=0$ is a minimum of $f(x,y)$.

(iii) For extrema, $df = f_x\,dx + f_y\,dy = 0$. Therefore, $df = 0$ may be considered as a necessary condition for extrema of a function.

Theorem 2: Conditions of Extrema

Let $z = f(x,y)$ be a continuous function having second-order partial derivatives and let (a,b) be a point satisfying:

$$f_x(a,b) = f_y(a,b) = 0.$$

Let

$$H(x,y) = f_{xx}(x,y)f_{yy}(x,y) - [f_{xy}(x,y)]^2 .$$

Then:

(i) $f(x,y)$ has a local maximum at (a,b) if $H(a,b) > 0$ and $f_{xx}(a,b) < 0$ (or $f_{yy}(a,b) < 0$).

(ii) $f(x,y)$ has a local minimum at (a,b) if $H(a,b) > 0$ and $f_{xx}(a,b) > 0$ (or $f_{yy}(a,b) > 0$).

(iii) $f(x,y)$ has neither a maximum nor a minimum at (a,b) if $H(a,b) < 0$.

(iv) The case is doubtful and needs further investigation if $H(a,b) = 0$.

Saddle Point

A point (a, b) is said to be a saddle point of $f(x, y)$ if it has neither a maximum nor a minimum value at (a, b), though:

$$f_x(a, b) = f_y(a, b) = 0.$$

Note: Clearly a point (a, b) will be a saddle point of $f(x, y)$ if condition (iii) of Theorem 2, i.e.,

$$f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 < 0$$

is satisfied.

2.11 Examples

Problem 1: Find the maxima and minima for the function

$$f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$$

Solution: Let

$$f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$$

Then,

$$f_x = \frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x + 72, \quad f_y = \frac{\partial f}{\partial y} = 6xy - 30y$$

The critical points are obtained by solving:

$$f_x = 0 \quad \text{and} \quad f_y = 0$$

$$3x^2 + 3y^2 - 30x + 72 = 0 \tag{1}$$

$$6xy - 30y = 0 \tag{2}$$

From equation (2):

$$y(6x - 30) = 0 \Rightarrow y = 0 \quad \text{or} \quad x = 5$$

Case 1: If $y = 0$, then equation (1) becomes:

$$3x^2 - 30x + 72 = 0 \Rightarrow x^2 - 10x + 24 = 0 \Rightarrow x = 4, 6$$

So, critical points are $(4, 0)$ and $(6, 0)$

Case 2: If $x = 5$, then equation (1) becomes:

$$3(25) + 3y^2 - 150 + 72 = 0 \Rightarrow 75 + 3y^2 - 150 + 72 = 0 \Rightarrow 3y^2 - 3 = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$$

So, critical points are $(5, 1)$ and $(5, -1)$

Hence, the critical points are:

$$(4, 0),\ (6, 0),\ (5, 1),\ (5, -1)$$

Now, we compute the second-order partial derivatives:

$$f_{xx} = 6x - 30, \quad f_{xy} = 6y, \quad f_{yy} = 6x - 30$$

The Hessian determinant is:

$$H(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = (6x - 30)^2 - 36y^2$$

At $(4, 0)$:

$$H = (6 \cdot 4 - 30)^2 - 36 \cdot 0^2 = (-6)^2 = 36 > 0, \quad f_{xx} = -6 < 0 \Rightarrow \text{Local Maximum at } (4, 0)$$

$$f(4, 0) = 4^3 + 3 \cdot 4 \cdot 0^2 - 15 \cdot 4^2 - 15 \cdot 0^2 + 72 \cdot 4 = 64 - 240 + 288 = 112$$

At $(6, 0)$:

$$H = (6 \cdot 6 - 30)^2 - 36 \cdot 0^2 = (6)^2 = 36 > 0, \quad f_{xx} = 6 > 0 \Rightarrow \text{Local Minimum at } (6, 0)$$

$$f(6, 0) = 6^3 + 3 \cdot 6 \cdot 0^2 - 15 \cdot 6^2 - 15 \cdot 0^2 + 72 \cdot 6 = 216 - 540 + 432 = 108$$

At $(5, 1)$ and $(5, -1)$:

$$H = (6 \cdot 5 - 30)^2 - 36 \cdot 1^2 = 0 - 36 = -36 < 0 \Rightarrow \text{Saddle points at } (5, 1) \text{ and } (5, -1)$$

Conclusion:

- Local maximum at $(4, 0)$, value $= 112$
- Local minimum at $(6, 0)$, value $= 108$
- Saddle points at $(5, 1)$ and $(5, -1)$

Problem 2: Examine the extreme values for the function

$$f(x, y) = x^3 + y^3 - 3axy$$

Solution:

Given:

$$f(x, y) = x^3 + y^3 - 3axy$$

First-order partial derivatives:

$$f_x = \frac{\partial f}{\partial x} = 3x^2 - 3ay, \quad f_y = \frac{\partial f}{\partial y} = 3y^2 - 3ax$$

Set $f_x = 0$ and $f_y = 0$ for critical points:

$$x^2 - ay = 0 \tag{1}$$

$$y^2 - ax = 0 \tag{2}$$

Subtracting (1) from (2):

$$y^2 - x^2 - ax + ay = 0 \Rightarrow (y - x)(y + x + a) = 0$$

So, either $y = x$ or $x + y + a = 0$

Case 1: $y = x$

Substitute into (1):

$$x^2 - ax = 0 \Rightarrow x(x - a) = 0 \Rightarrow x = 0, a$$

So, critical points: $(0, 0)$, (a, a)

Case 2: $y = -(x + a)$

Substitute into (1):

$$x^2 + a(x+a) = 0 \Rightarrow x^2 + ax + a^2 = 0 \Rightarrow x = \frac{-a \pm \sqrt{a^2 - 4a^2}}{2} = \frac{-a \pm \sqrt{-3a^2}}{2}$$

Since the discriminant is negative, the roots are complex unless $a = 0$. Hence, no additional real solutions.

Therefore, the only real critical points are $(0,0)$ and (a,a)

Second-order partial derivatives:

$$f_{xx} = 6x, \quad f_{xy} = -3a, \quad f_{yy} = 6y$$

Let:

$$A = f_{xx}, \quad B = f_{xy}, \quad C = f_{yy}, \quad H = AC - B^2$$

At $(0,0)$:

$$A = 0, \quad B = -3a, \quad C = 0, \quad H = (0)(0) - (-3a)^2 = -9a^2 < 0 \Rightarrow \text{Saddle point at } (0,0)$$

At (a,a):

$$A = 6a, \quad B = -3a, \quad C = 6a, \quad H = 36a^2 - 9a^2 = 27a^2 > 0$$

Since $H > 0$, we examine the sign of A:

- If $a > 0 \Rightarrow A = 6a > 0 \Rightarrow$ Local minimum at (a,a)
- If $a < 0 \Rightarrow A = 6a < 0 \Rightarrow$ Local maximum at (a,a)

Value at (a,a):

$$f(a,a) = a^3 + a^3 - 3a \cdot a \cdot a = 2a^3 - 3a^3 = -a^3$$

Conclusion:

- Saddle point at $(0,0)$
- Local minimum at (a,a) if $a > 0$, value $= -a^3$

- Local maximum at (a, a) if $a < 0$, value $= -a^3$

Problem 3: A rectangular box, open at the top, is to have a given capacity V. Find the dimensions of the box requiring the least material for its construction.

Solution:

Let the dimensions of the box be:

$$\text{Length} = x, \quad \text{Breadth} = y, \quad \text{Height} = z$$

1. Volume constraint:

Since the volume of the box is given by

$$V = xyz \tag{1}$$

2. Surface area to minimize:

As the box is open at the top, the surface area A to be minimized (material used) is:

$$A = \text{Bottom area} + \text{Lateral areas} = xy + 2yz + 2xz \tag{2}$$

Step 1: Use the volume constraint (1) to express z in terms of x and y:

$$z = \frac{V}{xy}$$

Step 2: Substitute z into equation (2):

$$\begin{aligned} A &= xy + 2y \cdot \frac{V}{xy} + 2x \cdot \frac{V}{xy} \\ &= xy + \frac{2V}{x} + \frac{2V}{y} \end{aligned} \tag{3}$$

Now, minimize A with respect to x and y. Take partial derivatives:

$$\frac{\partial A}{\partial x} = y - \frac{2V}{x^2} = 0 \tag{4}$$

$$\frac{\partial A}{\partial y} = x - \frac{2V}{y^2} = 0 \tag{5}$$

From equation (4):

$$y = \frac{2V}{x^2}$$

Substitute into equation (5):

$$x = \frac{2V}{\left(\frac{2V}{x^2}\right)^2} = \frac{2Vx^4}{4V^2} = \frac{x^4}{2V} \Rightarrow x^5 = 2V^2 \Rightarrow x = \sqrt[5]{2V^2}$$

Now substitute back:

$$y = \frac{2V}{x^2} = \frac{2V}{\left(\sqrt[5]{2V^2}\right)^2}$$

Finally, from the volume equation:

$$z = \frac{V}{xy}$$

Thus, the optimal dimensions that minimize the surface area are:

$$x = \sqrt[5]{2V^2}, \quad y = \frac{2V}{x^2}, \quad z = \frac{V}{xy}$$

Multiple Choice Questions

(i) The necessary conditions for $f(x, y) = 0$ to have extremum are:

(a) $f_x = 0, f_y = 0$ and $f_{xx} > 0, f_{yy} > 0$

(b) $f_{xy} = 0, f_{yx} = 0$

(c) $f_{xx} = 0, f_{yy} = 0$

(d) $f_x = 0, f_y = 0$

(ii) The function $f(x, y)$ is to be maximum if:

(a) $f_x = f_y = 0, f_{xx}f_{yy} - (f_{xy})^2 < 0, f_{xx} > 0$

(b) $f_x = f_y = 0, f_{xx}f_{yy} - (f_{xy})^2 > 0, f_{xx} > 0$

(c) $f_x = f_y = 0, f_{xx}f_{yy} - (f_{xy})^2 > 0, f_{xx} < 0$

(d) None of these

(iii) The minimum value of $x^2 + y^2 + 6x + 15$ is:

(a) 3

(b) 6

(c) 5

(d) 7

(iv) If a function $f(x, y)$ has maximum or minimum value at the point $(1, 2)$, then $f_x(1, 2) =$:

(a) 0

(b) 1

(c) any non-zero value

(d) none of these

(v) If $f_x(2, 3) = f_y(2, 3) = 0$, then $f(x, y)$ must have extreme value at $(2, 3)$:

(a) Yes

(b) No

(vi) If $f(x, y) = x^2 - y^3$, then $f(x, y) - f(0, 0)$ keeps the same sign in the neighbourhood of $(0, 0)$. This statement is:

(a) True

(b) False

(vii) If at (a, b), $f_x = f_y = 0, f_{xx}f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} > 0$, then at (a, b), $f(x, y)$ has a:

(a) minimum

(b) saddle point

(c) maximum

(d) none of these

(viii) If $f(x, y) = xy$, then at $(0, 0)$, $f(x, y)$ has a:

(a) minimum

(b) maximum

(c) saddle

(d) none of these

(ix) If $f_x(a, b) = f_y(a, b) = 0$, then (a, b) is known as:

(a) saddle point

(b) critical point

(c) none of these

(x) The critical (or stationary) point of the function $f(x, y) = 1 + x^2 + y^2$ is:

(a) $(0, 0)$

(b) $(0, 1)$

(c) $(1, 0)$

(d) $(1, 1)$

(xi) $f(x, y) = x^2 + y^2$ has a critical (or stationary) point at $(0, 0)$ and at this point it:

(a) is maximum

(b) has a saddle point

(c) is minimum

(d) needs further investigation for extrema

(xii) If $f_x(a,b) = f_y(a,b) = 0$, then (a,b) is a saddle point of $f(x,y)$ if:

(a) $f_{xx}(a,b) < 0$

(b) $f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2 < 0$

(c) $f_{yy}(a,b) < 0$

(d) $f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2 > 0$

Answers:

1.(d)	2.(c)	3.(b)	4.(a)	5.(b)	6.(b)
7.(a)	8.(c)	9.(b)	10.(a)	11.(c)	12.(b)

Problems

(i) Find the extrema (i.e., maxima and minima) of the following functions:

(a) $x^3 + 3xy^2 - 3x^2 - 3x^2 + 4$

(b) $x^3 + y^3 + 3xy$

(c) $(x^2 + y^2)e^{6x^2x^2}$

(d) $\sin x + \sin y + \sin(x + y)$

(e) $x^3 + y^3 - 3x - 12y + 20$

(f) $x^3 + y^3 - 63(x + y) + 12xy$

(ii) Find the saddle points of the following functions:

(a) $3x^3y - 15x^2 - 15y^2 + 72x$

(b) $x^3 + y^3 - 3xy$

(c) $x^3 + 3xy^2 - 3x^2 + 4$

(d) $x^3 + y^3 - 3x - 12y + 20$

(e) $x^3 + y^3 - 63(x + y) + 12xy$

(iii) A rectangular box, open at the top, is to have a volume of 32 cft. Find the dimensions of the box requiring least material for its construction.

(iv) Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

Hint: Take $2x, 2y, 2z$ as the length, breadth, and height of the rectangular solid, where (x, y, z) lies on a sphere:

$$x^2 + y^2 + z^2 = R^2$$

(v) Using Lagrange's multiplier method, find the maximum and minimum values of the following functions subject to the given constraints $x^2 + y^2$ subject to $3x + 2y = 6$ xy^2 subject to $x + y = 1$.

(vi) Using the method of Lagrange's multipliers, find the largest product of the numbers x, y and z when $x + y + z^2 = 16$.

(vii) Using the Lagrange's method of multipliers, find the shortest distance from the point $(1, 2, 2)$ to the sphere $x^2 + y^2 + z^2 = 36$.

(viii) If $u = \dfrac{a^3}{x^2} + \dfrac{b^3}{y^2} + \dfrac{c^3}{z^2}$, where $x + y + z = 1$, prove using Lagrange's method of multipliers, that the critical point of u is given by:

$$x = \frac{a}{a + b + c}, \quad y = \frac{b}{a + b + c}, \quad z = \frac{c}{a + b + c}$$

Answers to Problems

(i) (a) Maximum at (0, 0) and maximum value is 4, minimum at (2, 0) and minimum value is 0.

(b) Maximum at (–1, –1) and maximum value is 1.

(c) Minimum at (0, 0), (–1, 0), $\left(-\frac{1}{2}, 0\right)$, and the minimum values are 0, e^{-4}, $\frac{1}{4}e^{-5/2}$ respectively.

(d) Maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, and the maximum value is $\frac{3\sqrt{3}}{2}$

(e) Maximum at (–1, –2), maximum value is 38; minimum at (1, 2), minimum value is 2.

(f) Maximum at (–7, –7), maximum value is 784; minimum at (3, 3), minimum value is –216.

(ii) (a) (5, 1) and (5, –1)

(b) (0, 0)

(c) (1, 1) and (–1, –1)

(d) (–1, –2) and (1, –2)

(e) (–1, 5) and (5, –1)

(iii) Length = 4 ft., breadth = 4 ft., height = 2 ft.

(iv) The solid of maximum volume that can be inscribed in a sphere is a cube.

(v) (a) Minimum at $\left(\frac{18}{13}, \frac{12}{13}\right)$; minimum value $= \frac{36}{13}$

(b) Maximum at $\left(\frac{3}{5}, \frac{2}{5}\right)$; maximum value $= \frac{108}{3125}$

(vi) $\frac{4096}{25\sqrt{5}}$

(vii) 3

2.12 Lagrange multiplier method

So far, we have mastered the art of finding the maximum and minimum values (extrema) of functions by setting derivatives to zero. But what if our optimization problem has a restriction? For example, what is the maximum profit we can achieve *given* a fixed budget? Or, what is the rectangle with the maximum area *given* a fixed perimeter? These "given" statements are **constraints**. This chapter introduces a powerful and elegant method for solving such problems: the method of Lagrange Multipliers.

2.12.1 The Problem We Aim to Solve

We want to optimize a function of two or more variables, subject to a constraint.

- **Objective Function:** The function we want to maximize or minimize. Denoted as $f(x, y)$.
- **Constraint:** A condition that the variables must satisfy. Denoted as $g(x, y) = c$.

In formal terms: Find the extreme values of $f(x, y)$ subject to the constraint $g(x, y) = c$.

2.12.2 The Geometric Insight: The Delivery Driver Analogy

Imagine a city where the temperature at every location (x, y) is given by $T = f(x, y)$. We can draw **isotherms** on a map—lines of equal temperature (level curves of f).

You are a delivery driver. Your route is strictly defined by the equation $g(x, y) = c$. You want to find the hottest point on your assigned route.

As you drive, you cross the city's isotherms. The temperature changes. At the **hottest point on your route**, your road just touches (is tangent to) a specific isotherm. If the road crossed the isotherm, you would be moving to a warmer or cooler area, meaning you hadn't yet reached the extreme point.

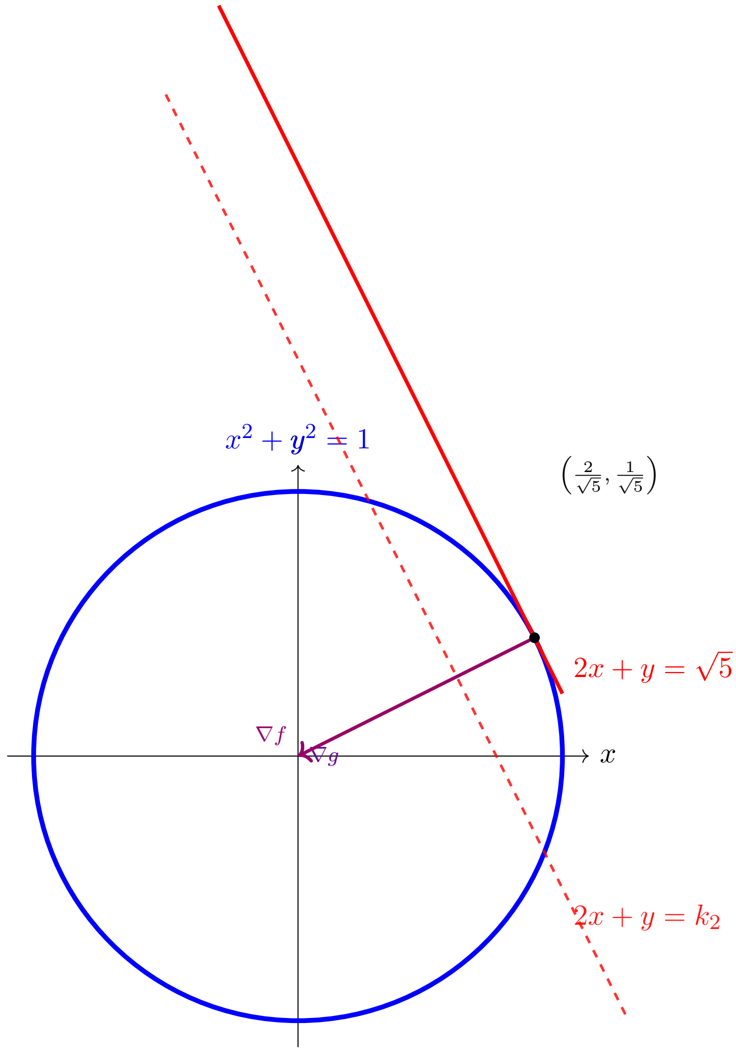


Figure 2.2: Accurate geometric interpretation of the Lagrange multiplier method for the objective function $f(x, y) = 2x + y$ subject to the constraint $x^2 + y^2 = 1$. The maximum value of $\sqrt{5}$ occurs at the point of tangency shown, where the gradients ∇f and ∇g are parallel.

2.12.3 The Connection to Gradients

Recall: The **gradient** of a function, ∇f, is a vector that points in the direction of the steepest increase. It is always **perpendicular** to the level curve (isotherm) at any point. The gradient of the constraint, ∇g, is always **perpendicular** to the constraint curve (your road). Since at the optimum point the road and the isotherm are tangent, their perpendicular vectors must be parallel! Therefore, at the point (x_0, y_0) that solves our problem:

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

Here, λ is a scalar called the **Lagrange multiplier**.

2.13 The Method: A Step-by-Step

To find the maximum and minimum values of $f(x, y)$ subject to $g(x, y) = c$:

(i) **Form the Lagrangian function:**

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c)$$

(ii) **Compute partial derivatives (for each variable and for λ):** $\frac{\partial \mathcal{L}}{\partial x}, \frac{\partial \mathcal{L}}{\partial y}, \frac{\partial \mathcal{L}}{\partial \lambda}$

(iii) **Solve the System of Equations:**

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} = 0 \quad &\Rightarrow \quad f_x - \lambda g_x = 0 \\ \frac{\partial \mathcal{L}}{\partial y} = 0 \quad &\Rightarrow \quad f_y - \lambda g_y = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = 0 \quad &\Rightarrow \quad g(x, y) = c \end{aligned}$$

(iv) **Plug solutions back into $f(x, y)$ to get maximum/minimum values.**

2.14 Worked-out Examples

2.14.1 Optimization With One Constraint

Example 1: Find the extreme values of $f(x, y) = 2x + y$ on the circle $x^2 + y^2 = 1$.
Solution:

(i) Identify $f(x, y) = 2x + y$ and constraint $g(x, y) = x^2 + y^2 = 1$.

(ii) Form the Lagrangian:

$$\mathcal{L}(x, y, \lambda) = (2x + y) - \lambda(x^2 + y^2 - 1)$$

(iii) Find the partial derivatives and and the system of equations:

$$\frac{\partial \mathcal{L}}{\partial x} = 2 - 2\lambda x = 0 \tag{2.4}$$
$$\frac{\partial \mathcal{L}}{\partial y} = 1 - 2\lambda y = 0 \tag{2.5}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(x^2 + y^2 - 1) = 0 \tag{2.6}$$

(iv) Solve the system: From the first two equations:

$$\lambda = \frac{1}{x} \quad \text{(from Eq. 2.22)}$$
$$\lambda = \frac{1}{2y} \quad \text{(from Eq. 2.5)}$$
$$\text{So, } \frac{1}{x} = \frac{1}{2y} \implies x = 2y$$

Substitute $x = 2y$ into the constraint (Eq. 2.6):

$$(2y)^2 + y^2 = 1$$
$$4y^2 + y^2 = 1$$
$$5y^2 = 1$$
$$y = \pm\frac{1}{\sqrt{5}}$$
$$x = 2y = \pm\frac{2}{\sqrt{5}}$$

Points: $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ and $\left(-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)$.

(v) Evaluate f at the points:

$$f\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) = 2 \cdot \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{5}} = \frac{5}{\sqrt{5}} = \sqrt{5}$$
$$f\left(-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right) = 2 \cdot \left(-\frac{2}{\sqrt{5}}\right) + \left(-\frac{1}{\sqrt{5}}\right) = -\frac{5}{\sqrt{5}} = -\sqrt{5}$$

The maximum value is $\sqrt{5}$ and the minimum value is $-\sqrt{5}$.

Example 2: Find the point on the plane $2x + 3y - z = 4$ that is closest to the origin.
Solution:

(i) **Formulate the problem:** We want to minimize the distance from a point (x, y, z) to the origin $(0, 0, 0)$. To make calculus easier, we minimize the *square* of the distance, which has the same critical points.

$$\begin{aligned} \text{Objective:} \quad & f(x, y, z) = x^2 + y^2 + z^2 \\ \text{Constraint:} \quad & g(x, y, z) = 2x + 3y - z = 4 \end{aligned}$$

(ii) **Form the Lagrangian:**

$$\mathcal{L}(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(2x + 3y - z - 4)$$

(iii) **Find the partial derivatives and and the system of equations:**

$$\frac{\partial \mathcal{L}}{\partial x} = 2x - 2\lambda = 0 \tag{2.7}$$

$$\frac{\partial \mathcal{L}}{\partial y} = 2y - 3\lambda = 0 \tag{2.8}$$

$$\frac{\partial \mathcal{L}}{\partial z} = 2z + \lambda = 0 \tag{2.9}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(2x + 3y - z - 4) = 0 \tag{2.10}$$

(iv) **Solve the system:** From 2.7: $x = \lambda$

From 2.8: $y = \frac{3}{2}\lambda$

From 2.9: $z = -\frac{1}{2}\lambda$

Substitute x, y, z into 2.10:

$$2(\lambda) + 3\left(\frac{3}{2}\lambda\right) - \left(-\frac{1}{2}\lambda\right) - 4 = 0$$

$$2\lambda + \frac{9}{2}\lambda + \frac{1}{2}\lambda - 4 = 0$$

$$\frac{12}{2}\lambda - 4 = 0 \implies 6\lambda - 4 = 0 \implies \lambda = \frac{2}{3}$$

Now find the point:

$$x = \frac{2}{3}, \quad y = \frac{3}{2} \cdot \frac{2}{3} = 1, \quad z = -\frac{1}{2} \cdot \frac{2}{3} = -\frac{1}{3}$$

The point on the plane closest to the origin is $\left(\frac{2}{3}, 1, -\frac{1}{3}\right)$. The minimum distance is:

$$d = \sqrt{f(x,y,z)} = \sqrt{\left(\frac{2}{3}\right)^2 + (1)^2 + \left(-\frac{1}{3}\right)^2} = \sqrt{\frac{4}{9} + 1 + \frac{1}{9}} = \sqrt{\frac{14}{9}} = \frac{\sqrt{14}}{3}$$

Example 3: Maximize $f(x,y,z) = xyz$ subject to $x + y + z = 6$.

(i) **Objective and Constraint:**

$$f(x,y,z) = xyz, \qquad g(x,y,z) = x + y + z - 6 = 0.$$

(ii) **Lagrangian:**

$$\mathcal{L}(x,y,z,\lambda) = xyz + \lambda(x + y + z - 6).$$

(iii) **Find the partial derivatives and and the system of equations:**

$$\frac{\partial \mathcal{L}}{\partial x} = yz + \lambda = 0 \tag{2.11}$$
$$\frac{\partial \mathcal{L}}{\partial y} = xz + \lambda = 0 \tag{2.12}$$
$$\frac{\partial \mathcal{L}}{\partial z} = xy + \lambda = 0 \tag{2.13}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = x + y + z - 6 = 0 \quad . \tag{2.14}$$

(iv) **Solve the system** From 2.11 and 2.12:

$$yz = xz \quad \Rightarrow \quad y = x \quad (\text{if } z \neq 0).$$

From 2.11 and 2.13:

$$yz = xy \quad \Rightarrow \quad z = x \quad (\text{if } y \neq 0).$$

Thus,

$$x = y = z.$$

Using 2.14:

$$x + y + z = 6 \quad \Rightarrow \quad 3x = 6 \quad \Rightarrow \quad x = 2.$$

So,

$$(x, y, z) = (2, 2, 2).$$

(v) **Evaluate**

$$f(2, 2, 2) = 2 \cdot 2 \cdot 2 = 8.$$

$$\text{Maximum value } = 8 \text{ at } (2, 2, 2).$$

Note:

But how do we know this is a maximum, not a minimum? But how do we know this is a maximum, not a minimum?

For non-negative x, y, z with a fixed sum, the Arithmetic Mean–Geometric Mean inequality gives:

$$\frac{x + y + z}{3} \geq \sqrt[3]{xyz}.$$

Here,

$$\frac{x + y + z}{3} = \frac{6}{3} = 2.$$

So,

$$\sqrt[3]{xyz} \leq 2 \quad \Longrightarrow \quad xyz \leq 2^3 = 8.$$

Equality holds only when

$$x = y = z = 2.$$

Therefore, the product xyz is **maximized at 8** when $(x, y, z) = (2, 2, 2)$. There is no finite minimum since the product can be zero (if one variable is zero) or negative (if one variable is negative).

Rule for Lagrange Problems

- If the constraint defines a **bounded region** for example, a circle

$$x^2 + y^2 = 1,$$

  then the objective function usually has both a maximum and a minimum value.

- If the constraint defines an **unbounded region** for example, a plane

$$x + y + z = 6,$$

  then often you obtain only a maximum (or only a minimum), but not both.

2.14.2 Optimization with Two Constraints

Problem: Find the maximum and minimum values of $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the sphere $x^2 + y^2 + z^2 = 1$. **Solution:**

(i) **Identify the functions:** We have one objective function and two constraints.

$$\begin{aligned} \text{Objective:} \quad & f(x, y, z) = x + 2y + 3z \\ \text{Constraint 1:} \quad & g(x, y, z) = x - y + z = 1 \\ \text{Constraint 2:} \quad & h(x, y, z) = x^2 + y^2 + z^2 = 1 \end{aligned}$$

(ii) **Form the Lagrangian (with two multipliers, λ and μ):**

$$\mathcal{L}(x, y, z, \lambda, \mu) = x + 2y + 3z - \lambda(x - y + z - 1) - \mu(x^2 + y^2 + z^2 - 1)$$

(iii) **Find the partial derivatives and and the system of equations:**

$$\frac{\partial \mathcal{L}}{\partial x} = 1 - \lambda - 2\mu x = 0 \tag{2.15}$$

$$\frac{\partial \mathcal{L}}{\partial y} = 2 + \lambda - 2\mu y = 0 \tag{2.16}$$

$$\frac{\partial \mathcal{L}}{\partial z} = 3 - \lambda - 2\mu z = 0 \tag{2.17}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(x - y + z - 1) = 0 \tag{2.18}$$

$$\frac{\partial \mathcal{L}}{\partial \mu} = -(x^2 + y^2 + z^2 - 1) = 0 \tag{2.19}$$

(iv) **Solve the system:** Use equations 2.15, 2.16, and 2.17 to express x, y, z in terms of λ and μ.

$$2\mu x = 1 - \lambda \implies x = \frac{1-\lambda}{2\mu}$$
$$2\mu y = 2 + \lambda \implies y = \frac{2+\lambda}{2\mu}$$
$$2\mu z = 3 - \lambda \implies z = \frac{3-\lambda}{2\mu}$$

Substitute x, y, z into constraint 2.18:

$$\left(\frac{1-\lambda}{2\mu}\right) - \left(\frac{2+\lambda}{2\mu}\right) + \left(\frac{3-\lambda}{2\mu}\right) = 1$$
$$\frac{1-\lambda-2-\lambda+3-\lambda}{2\mu} = 1$$

$$\frac{2-3\lambda}{2\mu} = 1 \qquad \implies 2 - 3\lambda = 2\mu \tag{2.20}$$

Now substitute x, y, z into constraint 2.19:

$$\left(\frac{1-\lambda}{2\mu}\right)^2 + \left(\frac{2+\lambda}{2\mu}\right)^2 + \left(\frac{3-\lambda}{2\mu}\right)^2 = 1$$

$$\frac{(1-\lambda)^2 + (2+\lambda)^2 + (3-\lambda)^2}{4\mu^2} = 1 \tag{2.21}$$

Calculate the numerator:

$$\begin{aligned}
&(1 - 2\lambda + \lambda^2) + (4 + 4\lambda + \lambda^2) + (9 - 6\lambda + \lambda^2)\\
&= (1 + 4 + 9) + (-2\lambda + 4\lambda - 6\lambda) + (\lambda^2 + \lambda^2 + \lambda^2)\\
&= 14 - 4\lambda + 3\lambda^2
\end{aligned}$$

From 2.20, $4\mu^2 = (2 - 3\lambda)^2$, substitute into 2.21:

$$\frac{14 - 4\lambda + 3\lambda^2}{(2-3\lambda)^2} = 1$$

$$14 - 4\lambda + 3\lambda^2 = 4 - 12\lambda + 9\lambda^2$$

$$10 + 8\lambda - 6\lambda^2 = 0 \implies 3\lambda^2 - 4\lambda - 5 = 0$$

Solve the quadratic equation:

$$\lambda = \frac{4 \pm \sqrt{(-4)^2 - 4(3)(-5)}}{2(3)} = \frac{4 \pm \sqrt{16+60}}{6} = \frac{4 \pm \sqrt{76}}{6} = \frac{4 \pm 2\sqrt{19}}{6} = \frac{2 \pm \sqrt{19}}{3}$$

Let's define the two cases:

$$\lambda = \frac{2+\sqrt{19}}{3}, \quad \lambda = \frac{2-\sqrt{19}}{3}$$

(v) **Find the points and evaluate f:**

Case 1: $\lambda = \frac{2+\sqrt{19}}{3}$

From 2.20: $\mu = \frac{2-3\lambda}{2} = \frac{2-(2+\sqrt{19})}{2} = -\frac{\sqrt{19}}{2}$

Now find the coordinates:

$$x = \frac{1-\lambda}{2\mu} = \frac{1 - \frac{2+\sqrt{19}}{3}}{2 \cdot (-\frac{\sqrt{19}}{2})} = \frac{\frac{1-\sqrt{19}}{3}}{-\sqrt{19}} = \frac{\sqrt{19}-1}{3\sqrt{19}}$$

$$y = \frac{2+\lambda}{2\mu} = \frac{2 + \frac{2+\sqrt{19}}{3}}{2 \cdot (-\frac{\sqrt{19}}{2})} = \frac{\frac{8+\sqrt{19}}{3}}{-\sqrt{19}} = -\frac{8+\sqrt{19}}{3\sqrt{19}}$$

$$z = \frac{3-\lambda}{2\mu} = \frac{3 - \frac{2+\sqrt{19}}{3}}{2 \cdot (-\frac{\sqrt{19}}{2})} = \frac{\frac{7-\sqrt{19}}{3}}{-\sqrt{19}} = \frac{\sqrt{19}-7}{3\sqrt{19}}$$

Now evaluate f at this point:

$$\begin{aligned}
f_1 &= x + 2y + 3z \\
&= \frac{\sqrt{19}-1}{3\sqrt{19}} + 2\left(-\frac{8+\sqrt{19}}{3\sqrt{19}}\right) + 3\left(\frac{\sqrt{19}-7}{3\sqrt{19}}\right) \\
&= \frac{\sqrt{19} - 1 - 16 - 2\sqrt{19} + 3\sqrt{19} - 21}{3\sqrt{19}} \\
&= \frac{(1-2+3)\sqrt{19} + (-1-16-21)}{3\sqrt{19}} = \frac{2\sqrt{19}-38}{3\sqrt{19}} = \frac{2}{3} - \frac{38}{3\sqrt{19}}
\end{aligned}$$

Case 2: $\lambda = \frac{2-\sqrt{19}}{3}$

From 2.20: $\mu = \frac{2-3\lambda}{2} = \frac{2-(2-\sqrt{19})}{2} = \frac{\sqrt{19}}{2}$

Now find the coordinates:

$$\begin{aligned}
x &= \frac{1-\lambda}{2\mu} = \frac{1-\frac{2-\sqrt{19}}{3}}{2\cdot(\frac{\sqrt{19}}{2})} = \frac{\frac{1+\sqrt{19}}{3}}{\sqrt{19}} = \frac{1+\sqrt{19}}{3\sqrt{19}} \\
y &= \frac{2+\lambda}{2\mu} = \frac{2+\frac{2-\sqrt{19}}{3}}{2\cdot(\frac{\sqrt{19}}{2})} = \frac{\frac{8-\sqrt{19}}{3}}{\sqrt{19}} = \frac{8-\sqrt{19}}{3\sqrt{19}} \\
z &= \frac{3-\lambda}{2\mu} = \frac{3-\frac{2-\sqrt{19}}{3}}{2\cdot(\frac{\sqrt{19}}{2})} = \frac{\frac{7+\sqrt{19}}{3}}{\sqrt{19}} = \frac{7+\sqrt{19}}{3\sqrt{19}}
\end{aligned}$$

Now evaluate f at this point:

$$\begin{aligned}
f_2 &= x + 2y + 3z \\
&= \frac{1+\sqrt{19}}{3\sqrt{19}} + 2\left(\frac{8-\sqrt{19}}{3\sqrt{19}}\right) + 3\left(\frac{7+\sqrt{19}}{3\sqrt{19}}\right) \\
&= \frac{1+\sqrt{19}+16-2\sqrt{19}+21+3\sqrt{19}}{3\sqrt{19}} \\
&= \frac{(1-2+3)\sqrt{19}+(1+16+21)}{3\sqrt{19}} = \frac{2\sqrt{19}+38}{3\sqrt{19}} = \frac{2}{3} + \frac{38}{3\sqrt{19}}
\end{aligned}$$

$$\begin{aligned}
\text{Maximum value:} \quad & \frac{2}{3} + \frac{38}{3\sqrt{19}} \\
\text{Minimum value:} \quad & \frac{2}{3} - \frac{38}{3\sqrt{19}}
\end{aligned}$$

2.15 Exercise

(i) Find the maximum and minimum values of

$$f(x,y) = 81x^2 + y^2$$

subject to the constraint

$$4x^2 + y^2 = 9.$$

(ii) Find the maximum and minimum values of

$$f(x,y) = 8x^2 - 2y$$

subject to the constraint

$$x^2 + y^2 = 1.$$

(iii) Find the maximum and minimum values of

$$f(x, y, z) = y^2 - 10z$$

subject to the constraint

$$x^2 + y^2 + z^2 = 36.$$

(iv) Find the maximum and minimum values of

$$f(x, y, z) = xyz$$

subject to the constraint

$$x + 9y^2 + z^2 = 4.$$

Assume that $x \geq 0$ for this problem. Why is this assumption needed?

(v) Find the maximum and minimum values of

$$f(x, y, z) = 3x^2 + y$$

subject to the constraints

$$4x - 3y = 9, \quad x^2 + z^2 = 9.$$

2.16 Line integral

Let $z = f(x, y)$ be a continuous function at every point on a plane curve in the xy-plane whose parametric equations are

$$x = \Phi(t), \quad y = \psi(t), \tag{2.22}$$

for some real values of t.

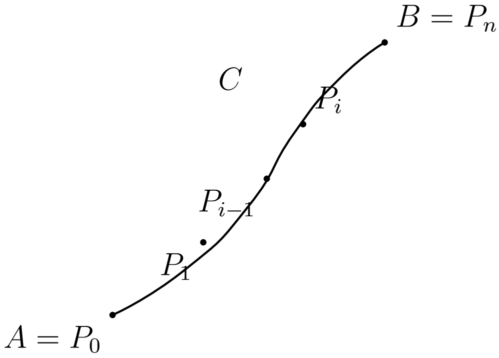
Consider any arc C of the curve (2.22) enclosed between two points A and B. Let a, b be the values of the parameter t corresponding to these points. Divide the arc C into n parts by the points

$$A = P_0,\ P_1,\ \ldots,\ P_{n-1},\ P_n = B,$$

and let

$$t_0 = a,\ t_1,\ t_2,\ \ldots,\ t_n = b$$

be the parameter values at these points respectively.



Also let

$$x_i = \Phi(t_i), \quad y_i = \psi(t_i), \quad i = 1, 2, \ldots, n,$$

and ξ_i be any point in $[t_{i-1}, t_i],\ i = 1, 2, \ldots, n.$

Consider the sum

$$\sum_{i=1}^{n} f(x_i, y_i)\,\Delta x_i, \quad \text{where } \Delta x_i = x_i - x_{i-1}.$$

Now let the number of partitions $n \to \infty$ in such a way that $\Delta x_i \to 0$. Then the sum

$$\sum_{i=1}^{n} f(x_i, y_i)\,\Delta x_i$$

varies, and if

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta x_i$$

exists, the function $f(x, y)$ is line integrable along C and is denoted by

$$\int_C f(x, y)\, dx \quad \text{or by} \quad \int_{AB} f(x, y)\, dx.$$

Similarly, we can define the line integral

$$\int_C f(x, y)\, dy, \quad \text{i.e.} \int_{AB} f(x, y)\, dy.$$

Note

(i) Using (1), the above line integrals can be written as

$$\int_C f(x, y)\, dx = \int_{t_0}^{t_n} f\big(\Phi(t), \psi(t)\big)\, \Phi'(t)\, dt \tag{2}$$

and

$$\int_C f(x, y)\, dy = \int_{t_0}^{t_n} f\big(\Phi(t), \psi(t)\big)\, \psi'(t)\, dt \tag{3}$$

(ii) When the equation of the curve is of the form $y = \Phi(x),\ x_0 \leq x \leq x_n$, then

$$\int_C f(x, y)\, dx = \int_{x_0}^{x_n} f(x, \Phi(x))\, dx \tag{4}$$

(iii) If f_1, f_2 are line integrable along C, then $f_1 \pm f_2$ are also line integrable and

$$\int_C (f_1 \pm f_2)\, dx = \int_C f_1\, dx \pm \int_C f_2\, dx \tag{5}$$

(iv) If arc $AB =$ arc $AC+$ arc CB, then

$$\int_{AB} f\, dx = \int_{AC} f\, dx + \int_{CB} f\, dx \tag{6}$$

(v) If $f(x, y)\, dx + \phi(x, y)\, dy$ can be expressed as

$$f(x, y)\, dx + \phi(x, y)\, dy = dU(x, y),$$

then

$$\int_C f(x, y)\, dx + \phi(x, y)\, dy = 0, \quad \text{if } C \text{ is a closed curve.}$$

Illustrative Examples

Example 1. Evaluate $\int_C f(x,y)\,dx$ where $f(x,y) = x^2 + y^3$ and the curve C is the arc of the parabola $y = x^2$ in the xy-plane from $(0,0)$ to $(1,1)$.

Solution:

Method 1

$$\begin{aligned}\int_C f(x,y)\,dx = \int_C (x^2+y^3)\,dx &= \int_0^1 \left(x^2 + (x^2)^3\right)\,dx \quad (\text{since } y = x^2)\\ &= \int_0^1 \left(x^2 + x^6\right)\,dx\\ &= \left[\frac{x^3}{3} + \frac{x^7}{7}\right]_0^1 = \frac{1}{3} + \frac{1}{7} = \frac{10}{21}.\end{aligned}$$

Method 2

The parametric form of the parabola $y = x^2$ is $x = t,\ y = t^2$.

$$\begin{aligned}\int_C f(x,y)\,dx = \int_C (x^2+y^3)\,dx &= \int_0^1 \left(t^2 + t^6\right)\,dt \quad (\text{at } (0,0) : t = 0,\ (1,1) : t = 1)\\ &= \frac{1}{3} + \frac{1}{7} = \frac{10}{21}.\end{aligned}$$

Example 2. Evaluate

$$\int_C 2xy\,dx + (x^2 - y^2)\,dy$$

where C is the line segment AB from $A(0,0)$ to $B(2,1)$.

Solution:

Method 1

The equation of the line segment AB is

$$\frac{x-0}{2} = \frac{y-0}{1} \quad \Rightarrow \quad y = \tfrac{1}{2}x.$$

$$\begin{aligned}\int_C 2xy\,dx + (x^2 - y^2)\,dy &= \int_0^2 \left\{2x \cdot \tfrac{1}{2}x\,dx + \left(x^2 - \tfrac{1}{4}x^2\right) \cdot \tfrac{1}{2}\,dx\right\} \\ &= \int_0^2 \left(x^2 + \frac{3x^2}{8}\right) dx \\ &= \frac{11}{8}\int_0^2 x^2\,dx = \frac{11}{8}\left[\frac{x^3}{3}\right]_0^2 = \frac{11}{8} \cdot \frac{8}{3} = \frac{11}{3}\end{aligned}$$

Method 2

The parametric equation of AB is

$$x = 2t, \quad y = t, \quad (0 \le t \le 1).$$

Then

$$\begin{aligned}\int_C 2xy\,dx + (x^2 - y^2)\,dy &= \int_0^1 \left(2 \cdot 2t \cdot t \cdot d(2t) + \left((2t)^2 - t^2\right) dt\right) \\ &= \int_0^1 \left(8t^2\,dt + 3t^2\,dt\right) = 11\int_0^1 t^2\,dt = 11\left[\frac{t^3}{3}\right]_0^1 = \frac{11}{3}.\end{aligned}$$

Example 3. Evaluate

$$\int_C (x^2 + y^2)\,dx - 2xy\,dy$$

where curve C is the rectangle in the xy-plane bounded by $x = 0$, $x = a$, $y = 0$, $y = b$.

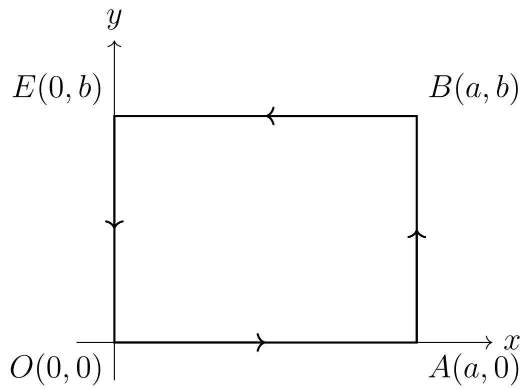
The curve C consists of the straight lines OA, AB, BE, and EO.

$$O(0,0) \to A(a,0) \to B(a,b) \to E(0,b) \to O(0,0).$$

Solution:

We evaluate the integral over each side separately.

On OA**:** $(0,0) \to (a,0)$



Here $y = 0$, $dy = 0$, $x : 0 \to a$.

$$\int_{OA} (x^2 + y^2)\, dx - 2xy\, dy = \int_0^a x^2\, dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}.$$

On AB**:** $(a, 0) \to (a, b)$

Here $x = a$, $dx = 0$, $y : 0 \to b$.

$$\int_{AB} (x^2 + y^2)\, dx - 2xy\, dy = \int_0^b -2ay\, dy = -a\,[y^2]_0^b = -ab^2.$$

On BE**:** $(a, b) \to (0, b)$

Here $y = b$, $dy = 0$, $x : a \to 0$.

$$\int_{BE} (x^2 + y^2)\, dx - 2xy\, dy = \int_a^0 (x^2 + b^2)\, dx = -\int_0^a (x^2 + b^2)\, dx$$
$$= -\left[\frac{x^3}{3} + b^2 x\right]_0^a = -\left(\frac{a^3}{3} + ab^2\right).$$

On EO**:** $(0, b) \to (0, 0)$

Here $x = 0$, $dx = 0$, $y : b \to 0$.

$$\int_{EO} (x^2 + y^2)\, dx - 2xy\, dy = \int_b^0 0\, dy = 0.$$

Adding all parts:

$$\int_C (x^2 + y^2)\, dx - 2xy\, dy = \frac{a^3}{3} - ab^2 - \left(\frac{a^3}{3} + ab^2\right) + 0 = -2ab^2.$$

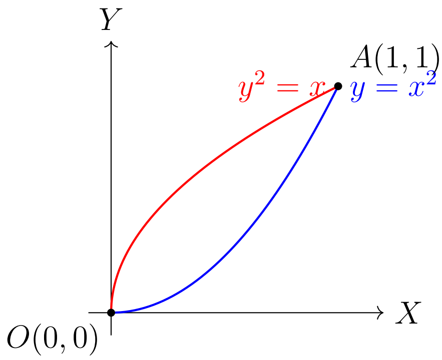
Example 4. Evaluate

$$\int_C (xy - x^2)\, dx + y\, dy$$

where C is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$.

Solution:

The two curves $y = x^2$ and $y^2 = x$ intersect at origin $O(0,0)$ and $A(1,1)$.



$$\begin{aligned}
\int_C (xy - x^2)\,dx + y\,dy &= \int_{OA} (xy - x^2)\,dx + y\,dy + \int_{AO} (xy - x^2)\,dx + y\,dy \\
&= \int_0^1 \left(x \cdot x^2 - x^2\right) dx + x^2\,d(x^2) \quad \left[\text{on } OA,\ y = x^2\right] \\
&\quad + \int_1^0 \left(y^2 \cdot y - y^4\right) d(y^2) + y\,dy \quad \left[\text{on } AO,\ x = y^2\right] \\
&= \int_0^1 \left(x^3 - x^2 + 2x^3\right) dx + \int_1^0 \left(2y^4 - 2y^5 + y\right) dy \\
&= \left[\frac{3x^4}{4} - \frac{x^3}{3}\right]_0^1 + \left[\frac{2y^5}{5} - \frac{2y^6}{6} + \frac{y^2}{2}\right]_1^0 \\
&= \left(\frac{3}{4} - \frac{1}{3}\right) + \left(\frac{2}{5} - \frac{1}{3} + \frac{1}{2}\right) = \frac{5}{12} - \frac{17}{30} = -\frac{3}{20}
\end{aligned}$$

Exercise

(i) Evaluate

$$\int_C xy\,dx + (x^2 + y^2)\,dy$$

where C consists of a part of the x-axis from $x = 2$ to $x = 4$ and a portion of the line $x = 4$ from $y = 0$ to $y = 12$.

(ii) Evaluate the line integral

$$\int_C \left[(x^2 + xy)\,dx + (x^2 + y^2)\,dy\right]$$

where C is the square formed by the lines $x = \pm 1$, $y = \pm 1$.

(iii) Show that

$$\int_C \big(\cos x \sin y - xy\big)\, dx + \sin x \cos y\, dy = 0$$

where C is the circle $x^2 + y^2 = 1$ in the xy-plane described in the positive sense.

(iv) Evaluate

$$\int_C (2x + y^2)\, dx + (3y - 4x)\, dy$$

where C is the triangle with vertices $(0,0), (2,0), (2,1)$.

(v) Show that the line integral

$$\int_C \big(2xy - x^2\big)\, dx + \big(x + y^2\big)\, dy$$

where C is the closed curve of the region bounded by $y = x^2$, $y^2 = x$, is $\dfrac{1}{30}$.

(vi) Show that the line integral

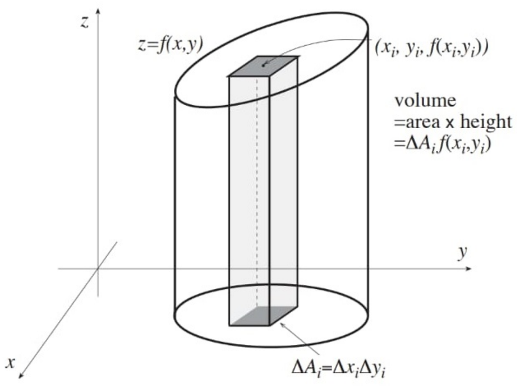
$$\int_{(1,0)}^{(2,1)} (2xy - y^4 + 3)\, dx + (x^2 - 4xy^3)\, dy$$

is independent of the path joining $(1,0)$ and $(2,1)$ and the value is 5.

2.17 Double Integral

Question: What is the volume above a region R and below the graph of $z = f(x,y)$?

Answer: It is a double integral of $f(x,y)$ over R. 2 Consider a single valued bounded function $z = f(x,y)$ of two independent variables x and y, defined on a region R. Subdivide the region R into n smaller rectangles. These rectangles form a partition of the region R. Let the areas of the smaller rectangular pieces are $\Delta A_i = \Delta x_i \Delta y_i,\ i = 1, 2, \cdots, n$. Let (x_i, y_i) be a point belongs to the smaller rectangular region whose area is given by ΔA_i.



Consider the sum

$$f(x_1, y_1)\Delta A_1 + f(x_2, y_2)\Delta A_2 + \cdots + f(x_n, y_n)\Delta A_n = \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i. \tag{2.23}$$

Now let the number of partition $n \to \infty$ in such a way that $\Delta A_i \to 0$. Then if the following limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i \tag{2.24}$$

exists then we say that the function $f(x, y)$ is integrable on R with respect to x and y. The value of the limit is called the double integral and is given by

$$\iint\limits_{R} f(x, y)dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i \tag{2.25}$$

Note: The double integral $\iint f(x, y)dy\ dx$ starts with $\int f(x, y)dy$. For each fixed x we integrate with respect to y. The answer depends on x. Now integrate again, this time with respect to x.

2.17.1 Properties of Double Integral

(i) Linearity: $\iint\limits_{R}(f + g)dA = \iint\limits_{R} f\, dA + \iint\limits_{R} g\, dA$.

(ii) Constant comes outside: $\iint\limits_{R} cf(x, y)\, dA = c\iint\limits_{R} f(x, y)\, dA$, where c is a constant.

(iii) R splits into R_1 and R_2(non-overlapping): $\iint\limits_{R} f\, dA = \iint\limits_{R_1} f\, dA + \iint\limits_{R_2} f\, dA$.

(iv) $\iint\limits_{R} f\, dA \geq \iint\limits_{R} g\, dA$, if $f \geq g$ at all points of R.

2.17.2 Interpretations of Double Integrals

(i) If $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous and $f(x,y) \geq 0$, then $\iint\limits_R f(x,y)dA$ is the volume of the solid lying over the region R in the xy-plane and below the graph of $z = f(x,y)$.

(ii) If $f(x,y) = 1$ for all $x,\ y$, then $\iint\limits_R dA$ gives the area of the region R.

(iii) Integral of density is the total mass.

(iv) Integral of charge density is the total charge.

In the next example we see how double integral is used to find the area.

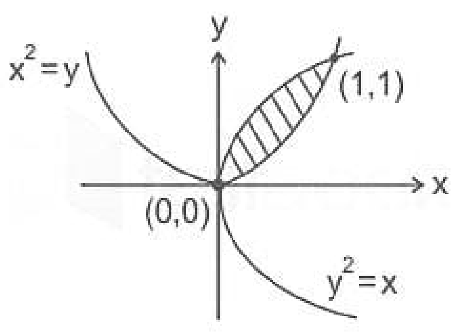
Example:

Find the area of the region bounded by the curves $y = x^2$ and $y^2 = x$.

Solution:

2 The curves $y = x^2$ and $y^2 = x$ meet at the points (1,1) and (0,0). The region of integration is shaded. The boundary is composed of two curves $y = x^2$ and $y = \sqrt{x}$ defined on $0 \leq x \leq 1$. Therefore

$$\iint\limits_R dx\ dy = \int_0^1 \int_{x^2}^{\sqrt{x}} dy\ dx = \int_0^1 [y]_{x^2}^{\sqrt{x}} dx$$
$$= \int_0^1 (\sqrt{x} - x^2)dx = \frac{1}{3}$$



2.17.3 Double Integral over Rectangular Region

Theorem 14 (Fubini's Theorem)**.** *Let $f(x,y)$ be continuous throughout the rectangular region $R:\ a \le x \le b,\ c \le y \le d$. Then*

$$\iint\limits_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dx\,dy = \int_a^b \int_c^d f(x,y)dy\,dx$$

Example:

Evaluate $\iint\limits_R \frac{x-y}{x+y}dx\,dy$, where $R = \{0 \le x \le 1,\ 0 \le y \le 1\}$.

Solution:

$$\begin{aligned}
\iint\limits_R \frac{x-y}{x+y}dx\,dy &= \int_0^1 \int_0^1 \frac{x-y}{x+y}dy\,dx = \int_0^1 \int_0^1 \left(\frac{2x}{x+y} - 1\right) dy\,dx \\
&= \int_0^1 \left[2x\log(x+y) - y\right]_0^1 dx \\
&= \int_0^1 [2x\{\log(x+1) - \log x\} - 1]dx \\
&= \int_0^1 2x\log(x+1)dx - \int_0^1 2x\log x dx - \int_0^1 dx \\
&= 0, \text{ as } \int_0^1 x\log(x+1)dx = \frac{1}{4} \text{ and } \int_0^1 x\log(x)dx = -\frac{1}{4}
\end{aligned}$$

Example:

Evaluate $\int_0^{\frac{\pi}{2}} \int_0^{\pi} \sin(x+y)\,dx\,dy$.

Solution:

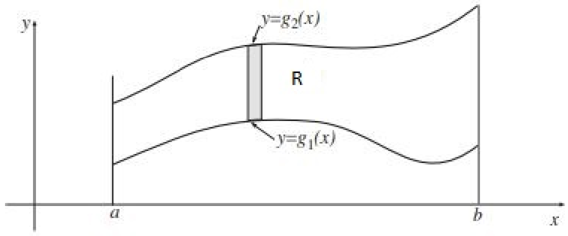
Here the region R is the rectangle formed by the straight lines $x = 0, x = \frac{\pi}{2}$ and $y = 0, y = \pi$.

$$\begin{aligned}
\int_0^{\frac{\pi}{2}}\int_0^{\pi} \sin(x+y)\,dx\,dy &= \int_0^{\frac{\pi}{2}}\left[\int_0^{\pi}\sin(x+y)\,dx\right]dy \\
&= \int_0^{\frac{\pi}{2}}\left[\int_0^{\pi}\sin(x+y)\,dy\right]dx \\
&= \int_0^{\frac{\pi}{2}}\left[-\cos(x+y)\right]_0^{\pi}dx \\
&= \int_0^{\frac{\pi}{2}}\left[-\cos(\pi+x)+\cos(x)\right]dx \\
&= \int_0^{\frac{\pi}{2}}\left[-(-\cos x)+\cos x\right]dx \\
&= 2\int_0^{\frac{\pi}{2}}\cos x\,dx \\
&= 2(1-0) \\
&= 2.
\end{aligned}$$

2.17.4 Double Integrals over Non-rectangular Regions

Theorem 15 (Fubini's Theorem Stronger Form). *Let $f(x,y)$ be continuous throughout a region R. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous functions of x on the interval $[a,b]$, then*

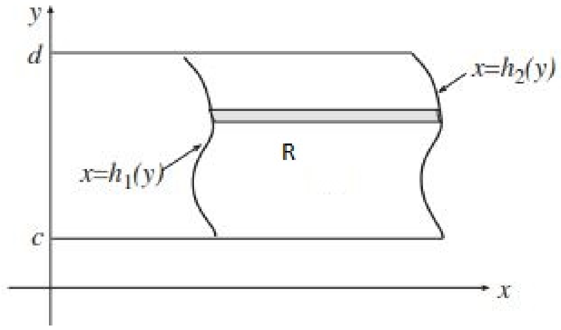
$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dy\ dx$$



Theorem 16 (Fubini's Theorem Stronger Form). *Let $f(x,y)$ be continuous throughout a region R. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are*

continuous functions of y on the interval $[c, d]$. then

$$\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dx\ dy$$

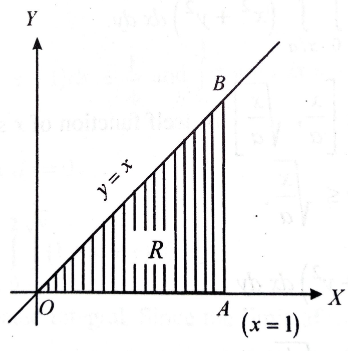


Example:

Evaluate $\iint\limits_R \sqrt{4x^2 - y^2}dx\,dy$ over the triangle formed by the straight lines $y = 0$, $x = 1$ and $y = x$.

Solution:

The region R is shown in the figure by shade. Here the boundary of R can be decomposed]into two lines $y = 0$ (line OA) and $y = x$ (line OB) defined on the interval $0 \leq x \leq 1$.



So the required integral

$$\iint\limits_R \sqrt{4x^2 - y^2}dx\,dy = \int_0^1 \int_0^x \sqrt{4x^2 - y^2}dy\,dx$$

Hence

$$\iint\limits_R \sqrt{4x^2 - y^2}dx\,dy = \int_0^1 \left[\frac{y}{2}\sqrt{4x^2 - y^2} + \frac{4x^2}{2}\sin^{-1}\frac{y}{2x}\right]_{y=0}^{x} dx$$
$$= \int_0^1 \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right) x^2\,dx$$
$$= \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right)\left[\frac{x^3}{3}\right]_0^1$$
$$= \frac{3\sqrt{3} + 2\pi}{18}$$

2.17.5 Exercise

(i) Evaluate $\int_0^1 \int_0^x xy\,dy\,dx$

(ii) Evaluate $\int_1^2 \int_y^{y^2} dx\,dy$

(iii) Evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$

(iv) Evaluate $\iint\limits_R \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy$, where $R = \{|x| \le 1, |y| \le 1\}$.

(v) Determine $\iint\limits_R (x^2 + y^2)dx\,dy$ where R is the region bounded by $y = x^2$, $x = 2$, $y = 1$.

(vi) Evaluate $\iint\limits_R (x^2 + y^2)dx\,dy$ where R is the region bounded by $xy = 1$, $y = 0$, $y = x$ and $x = 2$.

(vii) Evaluate $\iint xy(x+y)\,dx\,dy$ over the region enclosed by the cures $y = x^2$ and $y = x$.

(viii) Evaluate $\iint \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy$ over the positive quadrant of the circle $x^2 + y^2 = 1$.

2.17.6 Change of Variables

The concept of change of variables had evolved to facilitate the evaluation of some typical integrals. Let the variables x and y be transformed to the variables u and v with the help of the transformation equations $x = f(u, v)$ and $y = g(u, v)$. Then the region R in the xy plane of the integral $\iint_R F(x, y)\,dx\,dy$ is mapped to the region R' in the uv plane. We then have

$$\iint_R F(x, y)\,dx\,dy = \iint_{R'} F(f(u, v), g(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv,$$

where $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ is the Jacobian.

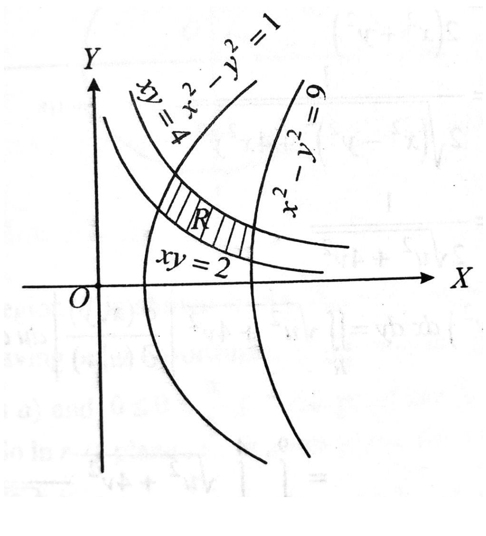
Note: In evaluating a double integral over a region R, it is often convenient to change the variable into a new plane.

Example:

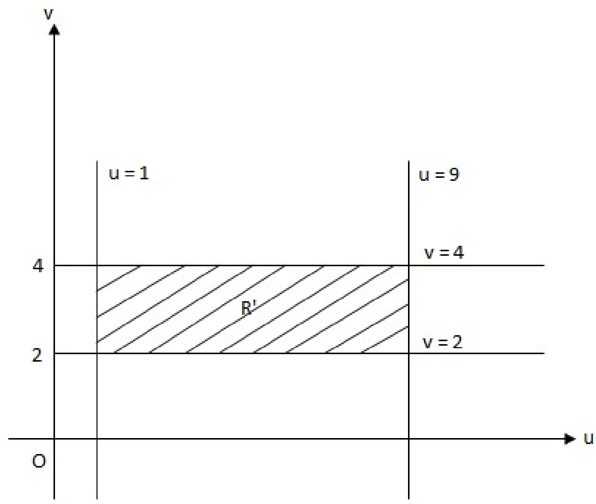
Evaluate $\iint_R (x^2 + y^2)dx\,dy$ where R is bounded by $x^2 - y^2 = 1$, $x^2 - y^2 = 9$, $xy = 2$, $xy = 4$.

Solution:

First look at the region R in the xy plane given in the problem.



If we give the transformation $x^2 - y^2 = u$ and $xy = v$, then the curves $x^2 - y^2 = 1$, $x^2 - y^2 = 9$ in xy plane transform to $u = 1$, $u = 9$ in the uv-plane; $xy = 2$, $xy = 4$ transform to $v = 2$, $v = 4$ in the uv-plane. Now look at the uv-plane in the following figure.



Now,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2(x^2 + y^2)$$

Therefore,

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2(x^2+y^2)} = \frac{1}{2\sqrt{(x^2-y^2)^2 + 4x^2y^2}} = \frac{1}{2\sqrt{u^2+4v^2}}$$

Hence,

$$\begin{aligned} \iint_R (x^2+y^2)\,dx\,dy &= \iint_{R'} \sqrt{u^2+4v^2}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \\ &= \int\limits_{u=1}^{9}\int\limits_{v=2}^{4} \sqrt{u^2+4v^2}\frac{1}{2\sqrt{u^2+4v^2}} du\,dv \\ &= \frac{1}{2}\int\limits_{u=1}^{9}\int\limits_{v=2}^{4} du\,dv \\ &= \frac{1}{2}\int\limits_{u=1}^{9} (4-2)du = 8 \end{aligned}$$

2.17.7 Exercise

(i) Evaluate $\int\limits_0^1 dx \int\limits_0^{1-x} e^{\frac{y}{x+y}} dy$ using the transformation $u = x + y$, $uv = y$.

(ii) Using transformation $x + y = u, \ y = uv$, show that

$$\int_0^1 \int_0^{1-x} \left(\frac{y}{x+y} \right) dx\, dy = \frac{1}{2}(e-1).$$

(iii) Evaluate $\iint\limits_R (x+y)^2\, dx\, dy$ where R is the parallelogram in the xy plane with vertices $(1,0), \ (3,1), \ (2,2), \ (0,1)$ using the transformation $u = x + y$, $v = x - 2y$.

2.18 Triple Integral

The triple integral is defined in a manner entirely analogous to the definition of the double integral. Let $w = f(x, y, z)$ be a function of three independent variables x, y and z defined on a three dimensional region V in the space. Subdivide V into n number of elementary volumes given by $\Delta v_i = \Delta x_i \Delta y_i \Delta z_i, \ i = 1, 2, \cdots, n$. Let $(x_i, y_i, z_i), i = 1, 2, \cdots, n$ be any point inside the ith sub-division. Consider the sum

$$\sum_{i=1}^{n} f(x_i, y_i, z_i) \Delta v_i$$

Now let the number of partitions $n \to \infty$ in such a way that each volume $\Delta v_i \to 0$. Then if the following limit

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i, z_i) \Delta v_i$$

exits then we say that the function $f(x, y, z)$ in integrable on V. The value of the limit is called the triple integral of $f(x, y, z)$ on V and denoted by

$$\iiint\limits_V f(x, y, z)\, dV = \iiint\limits_V f(x, y, z)\, dx\, dy\, dz. \tag{2.26}$$

Note: If $f(x, y, z) = 1$ then the triple integral $\iiint\limits_V dV$ gives the volume of V.

Theorem 17. *If $f(x, y, z)$ is continuous on the region V then its triple integral exists on V.*

Just as a double integral can be evaluated by two single integrations, a triple integral can be evaluated by three single integrations. If V in $\mathbb{R}^3$ is defined by

$$a \leq x \leq b,\ c \leq y \leq d,\ k \leq z \leq l \tag{2.27}$$

and $f(x, y, z)$ is continuous on the region, then

$$\begin{aligned}\iiint\limits_V f(x,y,z)dV &= \int_a^b \int_c^d \int_k^l f(x,y,z)\,dz\,dy\,dx \\ &= \int_c^d \int_k^l \int_a^b f(x,y,z)\,dx\,dy\,dz\end{aligned}$$

and so on. If the limits z_1 and z_2 of z be the functions of x, y; y_1 and y_2 of y be the functions of x and the limits of x be constants, then in such case we can express the triple integral as an iterated integral of the form

$$\int\limits_{x=a}^{b} \int\limits_{y=g_1(x)}^{g_2(x)} \int\limits_{z=f_1(x,y)}^{f_2(x,y)} F(x,y,z)dz\,dy\,dx = \int\limits_{x=a}^{b} \left[\int\limits_{y=g_1(x)}^{g_2(x)} \left\{ \int\limits_{z=f_1(x,y)}^{f_2(x,y)} F(x,y,z)dz \right\} dy \right] dx$$

where the innermost integral is to be evaluated first.

Example:

Evaluate $\int\limits_0^a \int\limits_0^x \int\limits_0^y x^3 y^2 z\,dz\,dy\,dx$.

Solution

$$\int_0^a \int_0^x \int_0^y x^3y^2z\,dz\,dy\,dx = \int_{x=0}^{a}\left[\int_{y=0}^{x}\left\{\int_{z=0}^{y} x^3y^2z\,dz\right\}dy\right]dx$$
$$= \int_{x=0}^{a}\left[\int_{y=0}^{x}\left[\frac{x^3y^2z^2}{2}\right]_{z=0}^{y}dy\right]dx$$
$$= \int_{x=0}^{a}\left[\int_{y=0}^{x}\frac{1}{2}x^3y^4dy\right]dx$$
$$= \int_{x=0}^{a}\left[\frac{1}{2}x^3\frac{y^5}{5}\right]_{y=0}^{x}dx$$
$$= \frac{1}{10}\int_{x=0}^{a} x^3\cdot x^5\,dx$$
$$= \frac{1}{10}\int_{x=0}^{a} x^8\,dx = \frac{a^9}{90}$$

Example:

Evaluate $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$.

Solution:

On integrating first with respect to z, keeping x and y constants, we get

$$I = \int_0^a\int_0^x \left[e^{(x+y)+z}\right]_0^{x+y}dy\,dx, \quad \text{[Here } (x+y) \text{ treated as some constant]}$$
$$= \int_0^a\int_0^x \left[e^{(x+y)+(x+y)} - e^{(x+y)+0}\right]dy\,dx$$
$$= \int_0^a\int_0^x \left[e^{2(x+y)} - e^{(x+y)}\right]dy\,dx$$
$$= \int_0^a \left[\frac{e^{2x+2y}}{2} - \frac{e^{x+y}}{1}\right]_0^x dx, \quad \text{(Integrating with respect to } y,\ \text{keeping } x \text{ constant)}$$
$$= \int_0^a \left[\left(\frac{e^{4x}}{2} - e^{2x}\right) - \left(\frac{e^{2x}}{2} - e^x\right)\right]dx$$

On integrating with respect to x,

$$
\begin{aligned}
I &= \left[-\frac{e^{4x}}{8} + \frac{e^{2x}}{2} - \frac{e^{2x}}{4} + e^{x}\right]_0^a \\
&= \left(\frac{e^{4a}}{8} - \frac{e^{2a}}{2} + \frac{e^{2a}}{4} + e^{a}\right) - \left(\frac{1}{8} - \frac{1}{2} + \frac{1}{4} + 1\right) \\
&= \left(\frac{e^{4a}}{8} - \frac{3}{4}e^{2a} + e^{a} - \frac{3}{8}\right).
\end{aligned}
$$

2.18.1 Exercise

(i) Evaluate $\iiint (x+y+z+1)^4 dx\,dy\,dz$ over the region defined by $x \geq 0$, $y \geq 0$, $z \geq 0$, $x+y+z \leq 1$.

(ii) Evaluate $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\,dx\,dy\,dz$

(iii) Evaluate $\int_0^a \int_0^x \int_0^y xyz\,dx\,dy\,dz$

(iv) Evaluate $\int_0^1 \int_0^{1-x} \int_0^{(x+y)^2} x\,dx\,dy\,dz$