Module 4

1. If $u = \tan^{-1}\frac{x^2+y^2}{x-y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sin 2u$.

2. If $f(u,v) = 3uv^2$, $g(u,v) = u^2 - v^2$, find the Jacobian $\frac{\partial(f,g)}{\partial(u,v)}$.

3. If $z = \sin uv$ where $u = 3x^2$ and $v = \log x$, find $\frac{dz}{dx}$.

4. Verify Euler's theorem for homogeneous function for
$$f(x,y) = x^3 + y^3 + 3x^2y + 3xy^2.$$

5. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u = \frac{3}{x+y+z}$.

6. If $= \cos^{-1}\{\frac{x+y}{\sqrt{x}+\sqrt{y}}\}$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}cot\ u = 0$.

7. If $u = x^2 - 2y,\ v = x + y + z,\ w = x - 2y + 3z,\ find\ \frac{\partial(u,v,w)}{\partial(x,y,z)}$.

8. If $u = f(y - z,\ z - x,\ x - y)$. Prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

9. If $u = f(x^2 + 2yz, y^2 + 2zx)$, show that $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - zy)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0$.

10. If $f(x,y,z,w) = 0$ prove that $\frac{\partial x}{\partial y} \times \frac{\partial y}{\partial z} \times \frac{\partial z}{\partial w} \times \frac{\partial w}{\partial x} = 1$.

11. Find the maxima and minima of the function $f(x,y) = x^3 + y^3 - 63(x + y) + 12xy$. Find also the saddle points.

12. Given the function $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$. Find $f_{xy}(0,0)$.

13. If $u = xf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$ then show that
(i) $xu_x + yu_y = xf\left(\frac{y}{x}\right)$ and
(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$

14. If $u = \cos^{-1}\{\frac{x+y}{\sqrt{x}+\sqrt{y}}\}$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}cotu = 0$
15. If $u = x^2 - 2y,\ v = x + y + z,\ w = x - 2y + 3z$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$
16. Find the maxima and minima of the function $x^3 + y^3 - 3x - 12y + 20$. Also find the saddle point.
17. Evaluate $\int_C (3xy\,dx - y^2dy)$ where C is the arc of the parabola $y = 2x^2$ from $(0,0)$ to $(1,2)$.
18. Evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x + y)dx\,dy$.

19. If $u = f(x^2 + 2yz, y^2 + 2zx)$, Find $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - zy)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0$
20. If $f(v^2 - x^2, v^2 - y^2, v^2 - z^2) = 0$, where v is a function of x, y, z. Show that $\frac{1}{x}\frac{\partial v}{\partial x} + \frac{1}{y}\frac{\partial v}{\partial y} + \frac{1}{z}\frac{\partial v}{\partial z} = \frac{1}{v}$.
21. Find the extreme value, if exists for the following function: $f(x, y) = x^3 + y^3 - 3axy$.
22. If $U = \sin^{-1}\left(\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}\right)$, show that $x^2 U_{xx} + 2xyU_{xy} + y^2 U_{yy} = \frac{\tan U}{144}(13 + \tan^2 U)$.
23. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ show that
    (i) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u = \frac{3}{x+y+z}$
    (ii) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u = -\frac{3}{(x+y+z)^2}$
    (iii) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$
24. Evaluate $\iint dxdy$ over the domain bounded by $y = x^2$ *and* $y^2 = x$
25. If $x = \sqrt{vw}, y = \sqrt{wu}, z = \sqrt{uv}$ and $u = r\sin\theta\cos\varphi, v = r\sin\theta\sin\varphi, w = r\cos\theta$ evaluate $\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}$
26. Evaluate $\int_0^a \int_0^x \int_0^y x^3 y^2 z \, dz \, dy \, dx$.
27. Evaluate $\int_0^{\pi/2} \int_0^{\pi} \sin(x + y) \, dx \, dy$
28. Evaluate $\iint_R \sqrt{4x^2 - y^2}\,dxdy$ where R is the triangular region bounded by the lines $y = 0, x = 1$ and $y = x$.
29. Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2 + y^2) dy \, dx$ by changing to polar coordinates.
30. Evaluate $\iiint z^2 dx \, dy \, dz$ over the region bounded by $z \geq 0$, $x^2 + y^2 + z^2 \leq a^2$.