ADVANCED ENGINEERING PHYSICS

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WBUT

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- PH301 – ME, PE, Civil, AUE, CSE, IT
- PH401 – ECE

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Advanced Engineering Physics

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Preface

Engineering and technology, in essence, are applied aspects of science. Science provides concepts, theories as well as formulae to engineering and technology and helps in their progress. Any technological development depends on better scientific understanding of the working of nature. New inventions are the results of varied scientific research and experimentation. Physical science and technology are very closely interlinked. Some modern technologies like biotechnology, food technology, etc., are very much dependent on bioscience along with physics. Technologists, today, require sound knowledge of physics to be able to face current and new challenges in this field at present and in the future.

Aim of this Book

The West Bengal University of Technology recently reorganized the physics syllabus for its BTech program. It has divided the entire syllabus into two parts. Topics to be studied by the students of both physics-based and chemistry-based streams of engineering have been compiled as the first part and have been made compulsory for all first year students.

Some other topics have been put under the second part, which are to be studied as part of the compulsory curriculum by the second year students of physics-based streams only. We have compiled the following topics of the second part of the BTech syllabus in this book: (a) vector analysis, (b) electrostatics, (c) dielectrics, (d) magnetostatics, (e) electromagnetic field theory, (f) classical mechanics, (g) quantum mechanics and (h) statistical mechanics. The book is hence appropriately titled, *Advanced Engineering Physics*.

Scope of this Book

This book is designed as a first reader for second-year students of Mechanical Engineering, Civil Engineering, Production Engineering, Automobile Engineering, Computer Science Engineering, Information Technology, and Electronics and Communication Engineering studying the course, *Physics-II (PH301, PH401)* at West Bengal University of Technology. It will also be useful for students of other universities pursuing BSc (Honours) and BSc (General) courses.

Salient Features of the Book

Though many books covering the aforesaid topics are available in the market, none of them entirely cover the new syllabus and a majority of them discuss topics either very briefly or in a highly exaggerated manner. For this reason, these books are not completely suitable for students pursuing second year of BTech at WBUT, for understanding their physics course. Keeping all these points in mind, we decided to pen this physics textbook.

CHAPTER

1

Vector Analysis

1.1 INTRODUCTION

The physical quantities which we usually come across in physics and also our daily lives are broadly classified into two types: (i) scalars, and (ii) vectors. A scalar is a quantity which is completely specified by its magnitude and has no reference to the idea of direction. For example—mass, length, time, temperature, work, speed, electric charge, etc., are scalars and are added according to the ordinary rules of algebra. The physical quantities which have both magnitude and direction and which can be added according to the triangle rule are called vector quantities. Acceleration, momentum, velocity, weight, electric field intensity, magnetic field intensity etc., are vector quantities as each involve magnitude and direction and follow the triangle rule. If any physical quantity has both magnitude and direction but does not add up according to the triangle rule, it will not be called a vector quantity. Electric current in a wire has magnitude as well as direction but there is no meaning of triangle rule there. Thus, electric current is not a vector quantity.

1.2 REPRESENTATION OF A VECTOR

A vector is represented analytically by bold-faced letter such as **A** or putting an arrow on the top of it such as $\vec{A}$ in Fig. 1.1.

Fig. 1.1 **Vector representation.**

1.3 SOME IMPORTANT DEFINITIONS ABOUT VECTORS

Equality of vectors Two vectors $\vec{A}$ and $\vec{B}$ are said to be equal if they have the same magnitude and direction regardless of their initial points [Fig 1.2].

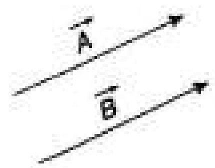


Fig. 1.2 **Equality of two vectors.**

Negative vector A vector having the same magnitude but direction opposite to that of the given vector is called a negative vector relative to that vector. In Fig. 1.3, $\vec{A}$ and $\vec{B}$ are opposite vectors.

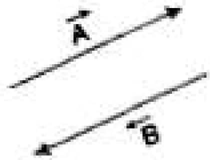


Fig. 1.3 Opposite vectors.

Unit vector A vector of unit magnitude is called a unit vector. If any vector $\vec{A}$ is of magnitude $|\vec{A}|$ then the unit vector is represented by $\frac{\vec{A}}{|\vec{A}|}$.

Null or zero vector It is a vector having zero magnitude. It has no definite direction. It is denoted by $\vec{O}$.

Coplanar vectors If a system of vectors lie in the same plane then they are called coplanar vectors. In Fig. 1.4, $\vec{A}$, $\vec{B}$ and $\vec{C}$ are coplanar vectors.

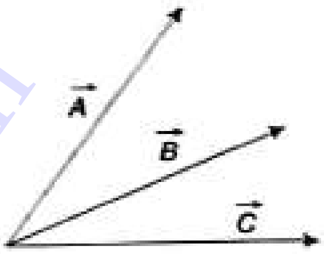


Fig. 1.4 Coplanar vectors.

Collinear or parallel vectors Vectors which have the same line of action or having lines of action parallel to the same line are called collinear vectors.

Reciprocal vectors A vector $\vec{A}^{-1}$ having the same direction as that of a given vector $\vec{A}$ but magnitude as the reciprocal of $\vec{A}$ is known as the reciprocal vector of $\vec{A}$. Thus $\vec{A}^{-1} = \frac{1}{|\vec{A}|}\hat{A}$.

Polar vectors A vector which has a linear motion in a particular direction and changes its sign under inversion or reflection is called a polar vector. Linear velocity, linear momentum, force, etc., are examples of polar vectors.

Axial vector A vector corresponding to the rotation about a certain axis is called an axial vector. Angular velocity, angular momentum, torque, etc., are axial vectors.

1.4 RESOLUTION OF A VECTOR INTO COMPONENTS

Any vector can be resolved into component vectors along three axes of the cartesian coordinate system. Here, the three axes OX, OY and OZ are mutually perpendicular to each other.

If $\hat{i}$, $\hat{j}$ and $\hat{k}$ be the unit vectors along the x, y and z axes and A_x, A_y and A_z be the vector intercepts of A along the x, y and z axes respectively, then we may write $\vec{A} = \hat{i}\,A_x + \hat{j}\,A_y + \hat{k}\,A_z$.

By geometry, the magnitude of the vector $\vec{A}$ is

$$A = |\vec{A}| = \sqrt{OP^2 + OQ^2 + OR^2}$$

or,

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \qquad \ldots(1.1)$$

The unit vector along $\vec{A}$ is given by

$$\hat{A} = \frac{\vec{A}}{A} = \frac{A_x}{A}\hat{i} + \frac{A_y}{A}\hat{j} + \frac{A_z}{A}\hat{k} \qquad \ldots(1.2)$$

Here,

$$\frac{A_x}{A} = \cos\alpha, \frac{A_y}{A} = \cos\beta, \frac{A_z}{A} = \cos\gamma \qquad \ldots(1.3)$$

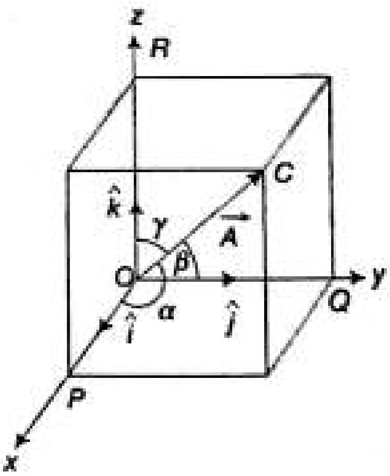


Fig. 1.5 Resolution of a vector into components.

are called the direction cosines of vector $\vec{A}$. Generally they are represented by the letters l, m and n respectively and α, β, γ are the angles as shown in Fig. 1.5.

If we put the values of A_x, A_y and A_z from Eq. (1.3) in Eq. (1.1), we get

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \qquad \text{...(1.4)}$$

or,
$$l^2 + m^2 + n^2 = 1 \qquad \text{...(1.5)}$$

Thus, the sum of squares of the direction cosines is equal to unity.

1.5 PRODUCT OF TWO VECTORS

The product of two vectors is classified as (i) scalar product or dot product, and (ii) vector product or cross product.

1.5.1 Scalar or Dot Product

The scalar product of two vectors $\vec{A}$ and $\vec{B}$ is equal to the product of the magnitudes of these vectors multiplied by the cosine of the angle between them as shown in Fig. 1.6.

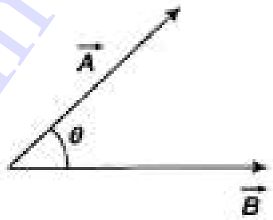


Fig. 1.6 Scalar product of two vectors $\vec{A}$ and $\vec{B}$.

Hence, $\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos \theta$

where θ is the angle between directions of A and B.

Note that $\vec{A} \cdot \vec{B}$ is a scalar and not a vector.

1.5.2 Properties and Other Results of Scalar Product

(i) Scalar product of two vectors obeys *commutative* law

i.e., $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

(ii) If two vectors $\vec{A}$ and $\vec{B}$ are mutually perpendicular, $\cos \theta = 0$, $\vec{A} \cdot \vec{B} = 0$. This condition is known as the *orthogonality* condition of two vectors. If this property is applied to a unit vector, then

$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

(iii) Scalar product of two vectors is *distributive*

i.e., $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$

(iv) Scalar multiplication is *associative*. If m, n are two scalars and A, B are two vectors, then

$$m\vec{A} \cdot n\vec{B} = mn\,(\vec{A} \cdot \vec{B}) = (n\vec{A}) \cdot (m\vec{B}) = \vec{A} \cdot mn\,\vec{B}$$

1.5.3 Physical Applications of Scalar Product

(i) Work done Work done by a force $\vec{F}$ causing a displacement $\vec{d}$ is given by Fig. 1.7.

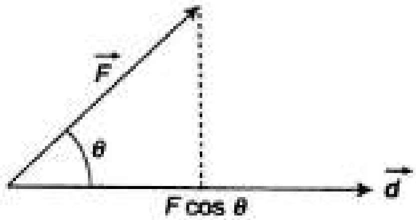


Fig. 1.7 Work done by a force.

$$W = F \cos \theta\, d = \vec{F} \cdot \vec{d}$$

(ii) Power The rate of doing work is power. So, power is

$$p = \frac{dW}{dt} = \frac{d}{dt}(\vec{F} \cdot \vec{d}) = \vec{F} \cdot \vec{v}$$

where $\vec{v}$ is the velocity of the body.

(iii) Electric flux Let us consider an elementary area $\vec{ds}$ in an electric field $\vec{E}$ [Fig. 1.8]. The normal electric flux coming out of the area $= \vec{E}.\hat{n}\,ds$ where $\hat{n}$ is the outward unit normal to the surface.

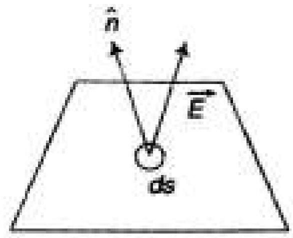


Fig. 1.8 Normal component of electric flux.

(iv) Magnetic flux (φ) The magnetic flux of a magnetic field of flux density $\vec{B}$ passing normally through an area $\vec{S}$ is

$$\varphi = \vec{B}.\vec{S}$$

1.5.4 Vector or Cross Product

The vector product of two vectors $\vec{A}$ and $\vec{B}$ is defined as a vector whose magnitude is equal to the product of the magnitudes of the vectors and the sine of the angle between their directions and its direction is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$. The vector product is represented as

$$\vec{C} = \vec{A} \times \vec{B} = |\vec{A}|\,|\vec{B}| \sin\theta\, \hat{n}$$

The direction of $\vec{C}$ is determined by the screw rule [Fig. 1.9] or any right-hand rule.

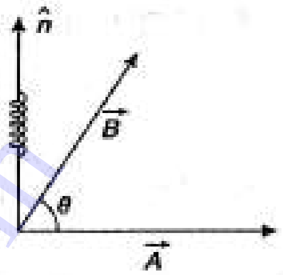


Fig. 1.9 **Representation of cross product of two vectors $\vec{A}$ and $\vec{B}$.**

1.5.5 Properties and Other Results of Vector Product

(i) Vector product does not obey *commutative* law.

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

(ii) The *distributive law* for cross products holds good.

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

(iii) The vector product is *associative*.

$$\vec{A} \times (m\vec{B}) = (m\vec{A}) \times \vec{B}$$

(iv) For the orthonormal vector triad $\hat{i}, \hat{j}, \hat{k}$,

$$\hat{i} \times \hat{j} = -\hat{j} \times \hat{i} = \hat{k};\ \hat{j} \times \hat{k} = -\hat{k} \times \hat{j} = \hat{i} \text{ and } \hat{k} \times \hat{i} = -\hat{i} \times \hat{k} = \hat{j}$$

and $$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

(v) If $\vec{A} = A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}$ and $\vec{B} = B_x\,\hat{i} + B_y\,\hat{j} + B_z\,\hat{k}$, then

$$A \times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

1.5.6 Physical Application of Vector Product

(i) Moment of a force or torque ($\vec{\tau}$) The moment of a force (or torque) about a fixed point is the vector $\vec{\tau} = \vec{r} \times \vec{F}$, where $\vec{r}$ is the position vector of the particle and $\vec{F}$ is the applied force.

(ii) Angular momentum ($\vec{L}$) Angular momentum of a particle is defined as the moment of linear momentum, i.e., $\vec{L} = \vec{r} \times \vec{p}$ where p is the momentum of the particle.

(iii) Force on a moving charge in a magnetic field When a charged particle moves in a magnetic field, a force acts on it. If a charge q moves with a velocity $\vec{v}$ in a uniform magnetic field $\vec{B}$, then the force experienced by the charge is

$$\vec{F} = q\,(\vec{v} \times \vec{B}) \qquad \ldots(1.6)$$

1.6 TRIPLE PRODUCT

Suppose we have three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$. If the vector product of two vectors $\vec{B}$ and $\vec{C}$ is a vector, this may be multiplied scalarly or vectorially with the first vector $\vec{A}$. This is known as triple product. There are two types of triple product

(i) $\vec{A}.(\vec{B}\times\vec{C})$ is known as scalar triple product.

(ii) $\vec{A}\times(\vec{B}\times\vec{C})$ is known as vector triple product.

1.6.1 Scalar Triple Product

The scalar triple product of three vectors is a scalar and is represented as

$$[\vec{A}\ \vec{B}\ \vec{C}] = \vec{A}.(\vec{B}\times\vec{C})$$
$$= \vec{B}.(\vec{C}\times\vec{A}) = \vec{C}.(\vec{A}\times\vec{B})$$

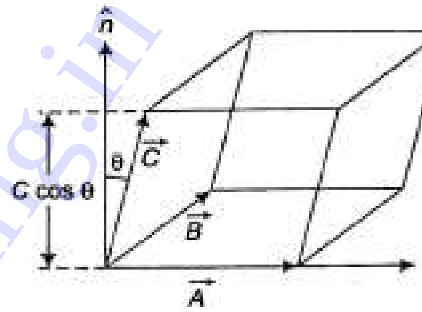


Fig. 1.10 Volume of a parallelepiped.

Properties of scalar triple product The scalar triple product of three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ represents the volume of a parallelepiped whose three adjacent sides are $\vec{A}$, $\vec{B}$ and $\vec{C}$.

The volume of the parallelepiped with sides $\vec{A}$, $\vec{B}$ and $\vec{C}$ [Fig 1.10]

$$= \text{(Height of the parallelepiped)} \times \text{Area of the base}$$
$$= C\cos\theta\,|(\vec{A}\times\vec{B})|$$
$$= \vec{C}.(\vec{A}\times\vec{B}) = (\vec{A}\times\vec{B}).\vec{C}$$

The sign of $\vec{C}.(\vec{A}\times\vec{B})$ can be either positive or negative according as $\vec{C}$, $\vec{A}$ and $\vec{B}$ do or do not form a right-handed system. The scalar quantity $(\vec{A}\times\vec{B}).\vec{C}$ denotes the volume of the parallelepiped.

Any face of the parallelepiped may be taken as base. Hence, its volume can be represented by any of the three expressions

$$\vec{A}.(\vec{B}\times\vec{C}),\ \vec{B}.(\vec{C}\times\vec{A}),\ \vec{C}.(\vec{A}\times\vec{B})$$

Scalar triple product of $\vec{A}$, $\vec{B}$ and $\vec{C}$ is given by $\vec{A}.(\vec{B}\times\vec{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$

1.6.2 Vector Triple Product

The vector triple product of three vectors is a vector product of one vector with the vector product of the other two vectors. The vector triple product of three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ is $\vec{A}\times(\vec{B}\times\vec{C})$.

Vector triple product can be written as the sum of two terms

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}\,(\vec{A}.\vec{C}) - \vec{C}\,(\vec{A}.\vec{B}) \qquad \ldots(1.7)$$

In general,

$$\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}$$

The vector $\vec{A}\times(\vec{B}\times\vec{C})$ represents a vector coplanar with $\vec{B}$ and $\vec{C}$, but $(\vec{A}\times\vec{B})\times\vec{C}$ represents a vector coplanar with $\vec{A}$ and $\vec{B}$. So, the product does not represent the same vector.

1.7 SCALAR AND VECTOR FIELDS

A mathematical function, or a graphical sketch, constructed so as to describe the variation of a quantity in a given region is said to represent the *field* of that quantity associated with that region. According to the nature of the physical quantity, there are two main kinds of fields.

(a) Scalar fields (b) Vector fields

A field is a spatial distribution of quantity, which may or may not be a function of time. The *scalar field* is for a scalar quantity and *vector field* for a vector quantity. The distribution of temperature in a medium, distribution of electrostatics or gravitational potential, etc., are examples of scalar fields. A scalar field is represented by a function in space as $\psi(x, y, z)$. If the scalar quantity is also varying with time then the function is $\psi(x, y, z, t)$. The distribution of magnetic and electric field intensity, the distribution of velocity in a fluid, the force field are examples of vector fields. The vector field $\vec{F}$ in the cartesian coordinate system can be written as $\vec{F}(x, y, z, t) = F_x(x, y, z, t)\,\hat{i} + F_y(x, y, z, t)\,\hat{j} + F_z(x, y, z, t)\,\hat{k}$

1.8 GRADIENT OF SCALAR FIELD

If $\psi(x, y, z)$ be a scalar function of position of coordinates (x, y, z) in space, then ψ can be differentiated with respect to x keeping the other two coordinates y and z constant. This type of differentiation is known as partial derivatives. The partial derivatives along the three orthogonal axes are $\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}$ and $\frac{\partial \psi}{\partial z}$.

The gradient of a scalar point function $\psi(x, y, z)$ is defined as $\vec{\nabla}\psi$ and given by

$$\text{grad } \psi = \vec{\nabla}\psi = \hat{i}\frac{\partial \psi}{\partial x} + \hat{j}\frac{\partial \psi}{\partial y} + \hat{k}\frac{\partial \psi}{\partial t} \qquad \ldots(1.8)$$

where vector differential operator ∇ (del) is defined as

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

The 'del' operator is a vector operator, when it operates on a scalar point function; it converts the scalar function into a vector function, grad $\psi(\vec{\nabla}\psi)$ is a vector quantity.

According to the theory of partial derivatives,

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy + \frac{\partial \psi}{\partial z}dz \qquad \ldots(1.9)$$

This shows how ψ varies as we go a small distance (dx, dy, dz), away from the point (x, y, z). The above relation can be written as

$$d\psi = \left(\hat{i}\frac{\partial \psi}{\partial x} + \hat{j}\frac{\partial \psi}{\partial y} + \hat{k}\frac{\partial \psi}{\partial z}\right)\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \vec{\nabla}\psi . d\vec{r} \qquad \ldots(1.10)$$

where, $d\vec{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$ is the infinitesimal displacement vector.

1.8.1 Geometrical Interpretation of the Gradient

Like an ordinary vector, a gradient has both magnitude and direction. From Eq. (1.10), we have

$$d\psi = \vec{\nabla}\psi . d\vec{r} = |\vec{\nabla}\psi| |d\vec{r}| \cos\theta \qquad \ldots(1.11)$$

where θ is the angle between the vector $\vec{\nabla}\psi$ and $d\vec{r}$. For a fixed value of $|d\vec{r}|$, the maximum charge of ψ occurs when $\theta = 0$. So, for fixed distance $|d\vec{r}|$, $d\psi$ is greatest when one moves in the same direction as $\vec{\nabla}\psi$.

Putting $\cos\theta = 1$ in Eq. (1.11) we get maximum rate of increase of the function,

$$\left.\frac{d\psi}{dr}\right|_{\max} = |\vec{\nabla}\psi| \quad \text{...(1.12)}$$

The gradient of a scalar field is a vector whose magnitude is equal to the maximum rate of change of scalar field and direction is along that change.

Example: The electric field intensity $\vec{E} = -\vec{\nabla}V$ where V is the electric potential.

1.8.2 Directional Derivative

Suppose $\hat{e}$ represents a unit vector along a specific direction. The component of $\vec{\nabla}\psi$ along $\hat{e}$ is $\vec{\nabla}\psi.\hat{e}$. This is known as the directional derivative of ψ along $\hat{e}$. The directional derivative of ψ in the direction $\hat{e}$ is the component of $\vec{\nabla}\psi$ along $\hat{e}$. Since $\vec{\nabla}\psi.\hat{e} \le |\vec{\nabla}\psi|$, $\vec{\nabla}\psi$ is equal to the largest directional derivative of ψ.

1.9 DIVERGENCE OF VECTOR FIELD

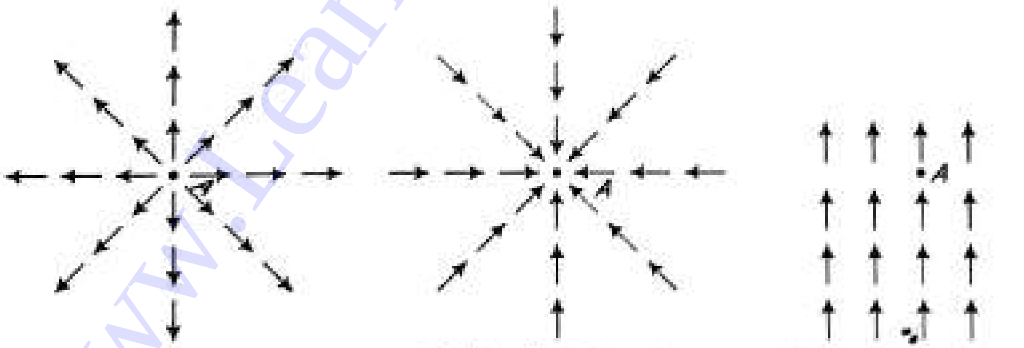
The divergence of a vector field at any point is defined as the net outflow or flux of that field per unit volume.

The divergence of a vector point function $\vec{A}$ is denoted by div $\vec{A}$ and can be written as

$$\text{div}\,\vec{A} = \vec{\nabla}.\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(\hat{i}A_x + \hat{j}A_y + \hat{k}A_z)$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad \text{...(1.13)}$$

Physical meaning $\vec{\nabla}.\vec{A}$ is the measure of how much the vector $\vec{A}$ spreads out (i.e., diverges) from the point in question.



(a) Positive divergence. (b) Negative divergence. (c) Zero divergence.

Fig. 1.11 Divergence of a vector field.

For example, the vector function in Fig. 1.11(a) has a large positive divergence at the point A. It indicates a net outflow, while a negative value of divergence [Fig. 1.11(b)] represents a net inflow and the function in Fig. 1.11(c) has zero divergence at A; it is not spreading out at all. In this case, $\vec{\nabla}.\vec{A} = 0$ and it implies that there is no inflow or outflow.

A vector $\vec{A}$, which satisfies the condition div $\vec{A} = 0$ is called a *solenoidal vector*. For example, the magnetic field vector $\vec{B}$ is a solenoidal vector.

The points at which the divergence of $\vec{A}$ is greater than zero are called *sources* and the points at which divergence of $\vec{A}$ is less than zero are called sinks. The divergence of a vector field is a scalar quantity. The divergence of a vector $\vec{A}$ may be written as

$$\vec{\nabla}.\vec{A} = \underset{\Delta V \to 0}{\text{Lt}} \frac{\iint_S \vec{A}.\vec{dS}}{\Delta V} \qquad \ldots(1.14)$$

where dS is the surface enclosing the volume ΔV.

1.9.1 Divergence of a Fluid

Let us consider an elementary parallelepiped of volume $\Delta x\ \Delta y\ \Delta z$ within a fluid as shown in Fig. 1.12, where Δx, Δy and Δz are length, breadth and height of the parallelepiped. Suppose $\vec{v}$ represents a vector point at the centre point Q of the parallelepiped.

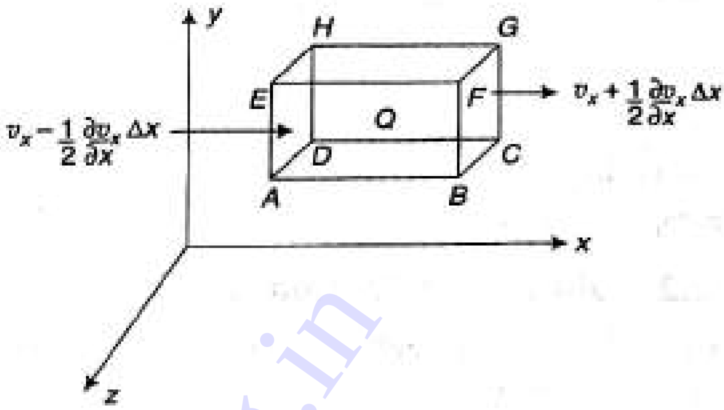


Fig. 1.12 Representation of the divergence of $\vec{v}$ in cartesian coordinates.

Here $\quad \vec{v} = \hat{i}v_x + \hat{j}\,v_y + \hat{k}\,v_z$

The x component of $\vec{v}$ at the face $ADHE = v_x - \frac{1}{2}\frac{\partial v_x}{\partial x} \times \Delta x$

The x component of $\vec{v}$ at the face $BCGF = v_x + \frac{1}{2}\frac{\partial v_x}{\partial x}\Delta x$

Per unit time, the volume of the fluid crossing $ADHE$

$$= \left(v_x - \frac{1}{2}\frac{\partial v_x}{\partial x}\Delta x\right)\Delta y\ \Delta z \qquad \ldots(1.15)$$

Per unit time, the volume of the fluid crossing $BCGF$

$$= \left(v_x + \frac{1}{2}\frac{\partial v_x}{\partial x}\Delta x\right)\Delta y\ \Delta z \qquad \ldots(1.16)$$

Hence in the x direction, the gain in volume of the fluid per unit time

$$= \left[\left(v_x + \frac{1}{2}\frac{\partial v_x}{\partial x}\Delta x\right) - \left(v_x - \frac{1}{2}\frac{\partial v_x}{\partial x}\Delta x\right)\right]\Delta y\ \Delta z$$

$$= \frac{\partial v_x}{\partial x}\Delta x\ \Delta y\ \Delta z \qquad \ldots(1.17)$$

Similarly, the gain in volume of the fluid per unit time along the y direction

$$= \frac{\partial v_y}{\partial y}\Delta x\ \Delta y\ \Delta z \qquad \ldots(1.18)$$

and along the z direction

$$= \frac{\partial v_z}{\partial z}\Delta x\ \Delta y\ \Delta z \qquad \ldots(1.19)$$

So, the total flux or gain of fluid per unit volume per unit time

$$= \frac{\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\Delta x\ \Delta y\ \Delta z}{\Delta x\ \Delta y\ \Delta z}$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (\hat{i}\, v_x + \hat{j}\, v_y + \hat{k}\, v_z)$$

$$= \vec{\nabla} \cdot \vec{v} \qquad \text{...(1.20)}$$

If there is no gain of fluid anywhere, then $\vec{\nabla} \cdot \vec{v} = 0$. This is called the continuity equation for an incompressible fluid.

1.10 CURL OF A VECTOR FIELD

The curl of a vector field at any point measures the rate of rotation of that vector. The curl of a vector field is also known as circulation or rotation. The curl of a continuously differentiable vector point function $\vec{A}\,(x, y, z)$ is defined by the equation

$$\text{Curl } \vec{A} = \vec{\nabla} \times \vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (\hat{i}\, A_x + \hat{j}\, A_y + \hat{k}\, A_z)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{j}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \qquad \text{...(1.21)}$$

1.10.1 Physical Meaning

Curl is a measure of how much the vector '*curl around*' the point in question. For example, the existence of curl of $\vec{v}$, the velocity at a point in a space indicates **circulation *or* vorticity** at that point of the liquid flow. If curl $\vec{v} = 0$, it means that if a wheel is placed in the liquid, it will not rotate. But if curl $\vec{v} \neq 0$, the wheel will rotate.

If a free magnetic pole is placed near a current-carrying conductor, the pole rotates around the conductor, which means $\oint_c \vec{H} \cdot \vec{dl} \neq 0$, so curl $\vec{H} \neq 0$. But in the case of an electrostatic field, $\oint_c \vec{E} \cdot \vec{dl} = 0$, so curl $\vec{E} = 0$.

The curl of a vector field at a point is defined as the amount of maximum line integral at any point in a vector field per unit area around a closed curve and is directed along the normal to the plane of the area.

Thus,
$$\text{curl } \vec{A} = \vec{\nabla} \times \vec{A} = \underset{\Delta s \to 0}{\text{Lt}} \frac{\left[\oint_c \vec{A} \cdot \vec{dl}\right]_{\max}}{\Delta s}\, \hat{n} \qquad \text{...(1.22)}$$

Irrotational vector If the curl of a vector field $\vec{A}$ is zero then the vector field $\vec{A}$ is called an *irrotational vector*. Gravitational field, electrostatic fields, etc., are irrotational fields.

1.11 CURL IN THE CONTEXT OF ROTATIONAL MOTION

Consider a rigid body R rotating about an axis passing through O [Fig 1.13] with an angular velocity $\vec{\omega}$. If $\vec{r}$ be the position vector of a point P on the rigid body, then its linear velocity

$$\vec{v} = \vec{\omega} \times \vec{r} \qquad \text{...(1.23)}$$

$$= (\hat{i}\omega_1 + \hat{j}\omega_2 + \hat{k}\omega_3) \times (\hat{i}x + \hat{j}y + \hat{k}z)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \hat{i}(\omega_2 z - \omega_3 y) - \hat{j}(\omega_1 z - \omega_3 x) + \hat{k}(\omega_1 y - \omega_2 x)$$

So, $\quad \text{curl } \vec{v} = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (\omega_2 z - \omega_3 y) & (\omega_3 x - \omega_1 z) & (\omega_1 y - \omega_2 x) \end{vmatrix}$

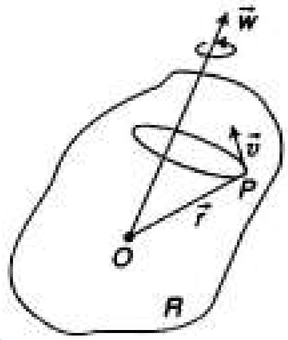


Fig. 1.13 Rotational motion of a rigid body.

$$= \hat{i}(\omega_1 + \omega_1) + \hat{j}(\omega_2 + \omega_2) + \hat{k}(\omega_3 + \omega_3)$$

$$= 2\vec{\omega} \qquad \text{...(1.24)}$$

Thus, we see that the curl of a vector field $\vec{v}$ is associated with the rotational properties of the vector field and shows that the angular velocity of a uniformly rotating body is one half the curl of the linear velocity.

1.11.1 Some Important Rules on Gradient, Divergence and Curl

(i) $\vec{\nabla}(\varphi + \psi) = \vec{\nabla}\varphi + \vec{\nabla}\psi$

(ii) $\vec{\nabla} \cdot (\vec{P} + \vec{Q}) = \vec{\nabla} \cdot \vec{P} + \vec{\nabla} \cdot \vec{Q}$

(iii) $\vec{\nabla} \times (\vec{P} + \vec{Q}) = \vec{\nabla} \times \vec{P} + \vec{\nabla} \times \vec{Q}$

(iv) $\vec{\nabla} \cdot (\psi \vec{P}) = \vec{\nabla}\psi \cdot \vec{P} + \psi \vec{\nabla} \cdot \vec{P}$

(v) $\vec{\nabla} \times (\psi \vec{P}) = \vec{\nabla}\psi \times \vec{P} + \psi \vec{\nabla} \times \vec{P}$

(vi) $\vec{\nabla} \cdot (\vec{P} \times \vec{Q}) = \vec{Q} \cdot (\vec{\nabla} \times \vec{P}) - \vec{P} \cdot (\vec{\nabla} \times \vec{Q})$

(vii) $\vec{\nabla} \times (\vec{P} \times \vec{Q}) = (\vec{Q} \cdot \vec{\nabla})\vec{P} - \vec{Q}(\vec{\nabla} \cdot \vec{P}) - (\vec{P} \cdot \vec{\nabla})\vec{Q} + \vec{P}(\vec{\nabla} \cdot \vec{Q})$

(viii) $\vec{\nabla}(\vec{P} \cdot \vec{Q}) = (\vec{Q} \cdot \vec{\nabla})\vec{P} + (\vec{P} \cdot \vec{\nabla})\vec{Q} + \vec{Q} \times (\vec{\nabla} \times \vec{P}) + \vec{P} \times (\vec{\nabla} \times \vec{Q})$

(ix) $\vec{\nabla} \cdot (\vec{\nabla}\psi) = \nabla^2 \psi = \dfrac{\partial^2 \psi}{\partial x^2} + \dfrac{\partial^2 \psi}{\partial y^2} + \dfrac{\partial^2 \psi}{\partial z^2}$

(x) $\vec{\nabla} \times (\vec{\nabla}\psi) = 0$

(xi) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{P}) = 0$

(xii) $\vec{\nabla} \times (\vec{\nabla} \times \vec{P}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{P}) - \nabla^2 \vec{P}$

Here $\vec{P}$ and $\vec{Q}$ are differentiable vector functions and φ and ψ are differentiable scalar functions.

1.12 VECTOR INTEGRATION

In vector analysis, the integrals that generally come are the line integral, the surface integral and the volume integral.

1.12.1 Line Integration

Let $\vec{dl}$ be an element of length on a smooth curve PQ and $\vec{A}$ be continuous vector point function. The scalar product of $\vec{A}$ with the line element $\vec{dl}$ is called the line integral of the vector $\vec{A}$ and for an extended path it will be equal to the integral

$$\int_P^Q \vec{A}.\vec{dl} = \int_P^Q A\, dl \cos\theta \qquad \ldots(1.25)$$

It is defined as the line integral of the vector $\vec{A}$ along the curve PQ [Fig. 1.14], where θ is the angle between $\vec{A}$ and elementary length $\vec{dl}$. In terms of the components of $\vec{A}$ along three cartesian coordinates, we have

$$\int_P^Q \vec{A}.\vec{dl} = \int_P^Q (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z).(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int_P^Q (A_x\, dx + A_y\, dy + A_z\, dz) \qquad \ldots(1.26)$$

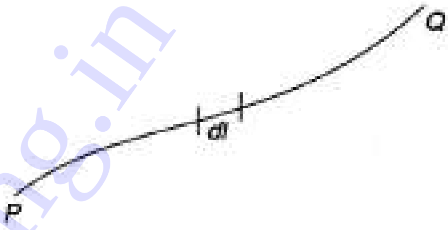


Fig. 1.14 Elementary length *dl* along the curve *PQ*.

If the path of integration is a closed curve, then we write $\oint$ instead of $\int$.

If the value of line integral depends only on the initial and final points in the vector field and independence of the path then the vector field is called ***conservative field***. All central force fields such as gravitational field and electrostatic field are conservative fields. For a conservative force field,

$$\oint_C \vec{A}\cdot\vec{dl} = 0 \qquad \ldots(1.27)$$

If the line integral over a closed path in a vector field $\vec{A}$ is zero, then $\vec{A}$ will be the gradient of a scalar function i.e, $\vec{A} = \vec{\nabla}\psi$, where ψ is the scalar point function. If $\vec{A}$ is conservative then $\vec{\nabla} \times \vec{A}$ will be zero.

1.12.2 Surface Integral

The surface integral of a vector field $\vec{F}$ over a piecewise smooth surface S in space is defined as the integral of the normal component of $\vec{F}$ across the surface and can be written as

$$\iint_S \vec{F}.\vec{dS} \quad \text{or} \quad \iint_S \vec{F}\cdot\hat{n}\, dS$$

where ds is the elementary surface on S and $\hat{n}$ is a unit vector along the outward drawn normal to the surface.

If $\qquad \vec{F} = \hat{i}F_x + \hat{j}F_y + \hat{k}F_z$ and $\vec{ds} = \hat{i}dy\,dz + \hat{j}dx\,dz + \hat{k}dx\,dy$

Then $\qquad \iint_S \vec{F}.\vec{dS} = \iint_S (F_x dy\,dz + F_y dx\,dz + F_z dx\,dy)$

The surface integral can also be written as

$$\int_S \vec{F}.\vec{dS} = \int_S \vec{F}\cdot\hat{n}\,dS$$

The notation $\oiint_S$ is used for a closed surface S. Sometimes $\oint_S$ may also be used.

1.12.3 Volume Integral

Let $\vec{F}$ be a single-valued continuous vector function in volume V. Suppose the volume is divided into a large number of small volume elements (dV). The volume integral of $\vec{F}$ is the sum of the products of values of vector fields and the volume for all elements and for infinite large volume elements, volume integral

$$= \iiint_V \vec{F}\, dV$$

If $\vec{F} = \hat{i}F_x + \hat{j}F_y + \hat{k}F_z$ then volume integral

$$= \iiint_V (\hat{i}F_x + \hat{j}F_y + \hat{k}F_z)\, dV$$

$$= \hat{i} \iiint_V F_x\, dV + \hat{j}\iiint_V F_y\, dV + \hat{k}\iiint_V F_z\, dV$$

For scalar point function $\psi(x, y, z)$, volume integration will be

$$= \iiint_V \psi\, dV = \iiint_V \psi\, dxdydz$$

It is a scalar quantity.

The notations $\int_V \vec{F}dV$ or $\int_V \psi dV$ are also used to indicate volume integration for the respective vector and scalar fields.

1.13 INTEGRAL THEOREMS

There are mainly three fundament integral theorems in relation to the integration of vector fields: (i) Gauss' Divergence Theorem (ii) Stoke's Theorem, and (iii) Theorem for Gradient. The first one is a correlation between a closed surface and its enclosed volume. The second one is the theorem for curls, which correlates a closed line to its enclosed surface. The third one is the theorem for gradient, which correlates closed points to a line.

1.13.1 Gauss' Divergence Theorem

Statement **The theorem states that the surface integral of the normal component of a vector field taken around a closed surface is equal to the integral of the divergence of the vector taken over the volume enclosed by the surface.**

Let us consider a volume V enclosed by a closed surface S. If a vector function $\vec{F}$ is continuously differentiable throughout V then from Gauss' divergence theorem.

$$\oiint_S \vec{F}.\vec{dS} = \iiint_V (\vec{\nabla}\cdot\vec{F})\, dV$$

or,

$$\oiint_S \vec{F}.\hat{n}dS = \iiint_V (\vec{\nabla}\cdot\vec{F})\, dV \qquad \ldots(1.28)$$

where $\hat{n}$ is the outward drawn unit normal to the surface.

Significance If $\vec{F}$ represents the velocity of a fluid, then the flux of $\vec{F}$ is the total amount of fluid passing through the surface per unit time. A point of positive divergence acts as a surface of the fluid measuring the amount of fluid it is producing but a point of negative divergence measures the amount of fluid it is absorbing. The divergence theorem is basically a statement of incompressibility of fluid. For incompressible fluid, the volume integral of the divergence measures the net amount of fluid that is produced inside the region, and the surface integral of $\vec{F}$ (velocity flux) over the closed surface S gives the net amount of fluid flowing out through the surface per unit time. Hence, both are equal from the law of conservation of mass. So, from divergence theorem

Amount of fluid produced inside the volume = Net amount of fluid flowing out through the surface

1.13.2 Stoke's Theorem

Statement **The line integral of the tangential component of a vector taken around a closed path is equal to the surface integral of the normal component of the curl of the vector taken over any surface having the path as the boundary.**

Mathematically, the theorem states that if $\vec{F}$ is a continuously differentiable vector point function in a region of space and S is an open two-sided surface bounded by a closed, non-intersecting curve C, then

$$\oint_c \vec{F}.\vec{dr} = \iint_s (\vec{\nabla} \times \vec{F}) \cdot \vec{dS}$$

or,

$$\oint_c \vec{F}.\vec{dr} = \iint_s (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, dS \qquad \text{...(1.29)}$$

where $\hat{n}$ is the outward drawn unit normal to the surface S. Stoke's theorem relates surface integral with line integral.

Significance Curl measures the amount of twist or rotation of the vector $\vec{F}$ at each point, the integral of the curl through a surface or flux of the curl can be determined if we go all around the edge and find how much of the fluid is flowing out of the boundary. There is no restriction on the shape of the surface S. The only condition is that the boundary of the surface S must coincide with the curve C.

1.14 COORDINATE SYSTEM

For solving three-dimensional problems, we require a coordinate system. There are mainly three types of coordinate systems (i) cartesian coordinate system, (ii) cylindrical coordinate system, and (iii) spherical coordinate system.

1.14.1 Cartesian Coordinate System (x, y, z)

In the cartesian coordinate system, let P be the position vector with respect to origin O given by [Fig 1.15] by $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$, where $\hat{i}, \hat{j}, \hat{k}$, are unit vectors along the X, Y and Z respectively. Here unit vectors are not the functions of the coordinates.

Now elementary displacement

$$\vec{dr} = \hat{i}dx + \hat{j}dy + \hat{k}dz \qquad \text{...(1.30)}$$

Elementary surface area in XY plane is

$$dS = dx\,dy$$

and the corresponding volume element

$$dV = dx\,dy\,dz$$

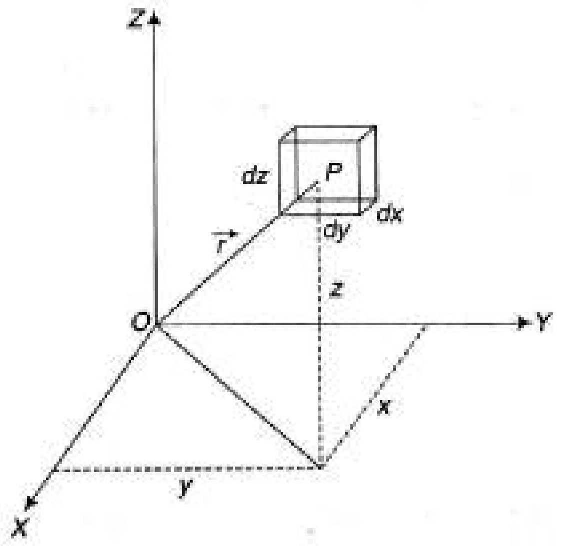


Fig. 1.15 The elementary volume element dV in the cartesian coordinate system.

1.14.2 Cylindrical Polar Coordinate System (ρ, φ, z)

In the cylindrical coordinate system, the position of the point P is $P(\rho, \varphi, z)$ [Fig. 1.16(a,b)]. The unit vector $\hat{\rho}$ at P is directed radially outward from the z axis. The unit vector $\hat{\varphi}$ is directed along the direction of increasing of φ and the unit vector $\hat{k}$ is along the direction of the z axis.

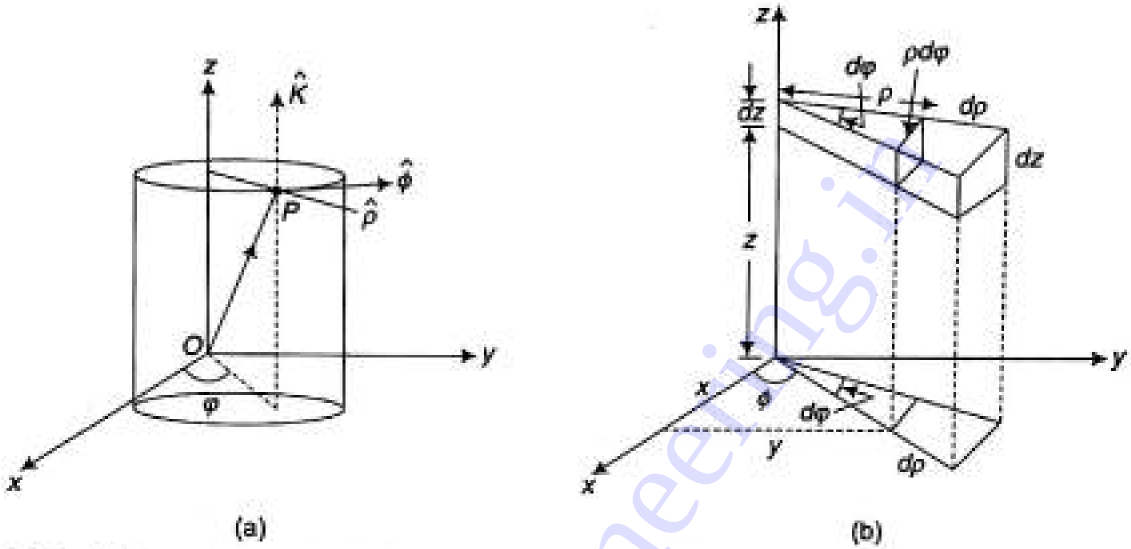


Fig. 1.16 (a) A point in cylindrical coordinate system. (b) The elementary volume element dV in cylindrical coordinate system.

From Fig. 1.16(b)

$$x = \rho \cos \varphi,\ y = \rho \sin \varphi \text{ and } z = z$$

Again in the cartesian coordinates, the position vector $\vec{r}$ can be written as

$$\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$$

$$= \hat{i}\rho \cos \varphi + \hat{j}\rho \sin \varphi + \hat{k}z \quad \ldots(1.31)$$

The elementary curve length $(dr)^2 = (dx)^2 + (dy)^2 + (dz)^2 = (d\rho)^2 + \rho^2(d\varphi)^2 + (dz)^2$ ...(1.32)

and
$$d\vec{r} = \hat{\rho}d\rho + \hat{\varphi}(\rho d\varphi) + \hat{k}\,dz \quad \ldots(1.33)$$

The volume element in cylindrical coordinate can be expressed as [from Fig 1.16(b)]

$$dV = d\rho \times \rho d\varphi \times dz = \rho d\rho\, d\varphi\, dz \quad \ldots(1.34)$$

The unit vectors in the cylindrical coordinates system are:

$$\hat{\rho} = \frac{\partial \vec{r}}{\partial \rho}\bigg/\left|\frac{\partial \vec{r}}{\partial \rho}\right| = \hat{i}\cos \varphi + \hat{j}\sin \varphi$$

$$\hat{\varphi} = \frac{\partial \vec{r}}{\partial \varphi}\bigg/\left|\frac{\partial \vec{r}}{\partial \varphi}\right| = -\hat{i}\sin \varphi + \hat{j}\cos \varphi$$

$$\hat{k} = \frac{\partial \vec{r}}{\partial z}\bigg/\left|\frac{\partial \vec{r}}{\partial z}\right| = \hat{k}$$

1.14.3 Spherical Polar Coordinate System (r, θ, φ)

In the spherical coordinate system, the position of the point A is A (r, θ, φ) [Fig 1.17]. The unit vector $\hat{r}$ is directed radially outward. The unit vector $\hat{\theta}$ is normal to the conical surface and it is directed to the directional in which θ increases.

The unit vector $\hat{\varphi}$ is along the direction in which φ increases. The relation between cartesian and spherical polar coordinates is

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi \text{ and } z = r\cos\theta$$

So, the position vector

$$\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$$
$$= \hat{i}r\sin\theta\cos\varphi + \hat{j}r\sin\theta\sin\varphi + \hat{k}r\cos\theta \qquad \ldots(1.35)$$

If dl is the elementary length then in spherical coordinates

$$\vec{dl} = \hat{r}dr + \hat{\theta}rd\theta + \hat{\varphi}r\sin\theta d\varphi \qquad \ldots(1.36)$$

and $dl^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\varphi)^2 \qquad \ldots(1.37)$

The surface element ds in spherical coordinates is obtained from Fig. 1.17.

$$dS = AB \times AC = r\sin\theta d\varphi \times rd\theta$$
$$= r^2\sin\theta\, d\theta\, d\varphi \qquad \ldots(1.38)$$

and volume element $dV = dS \times \text{height}(dr)$

$$= r^2\sin\theta\, d\theta\, d\varphi\, dr \qquad \ldots(1.39)$$

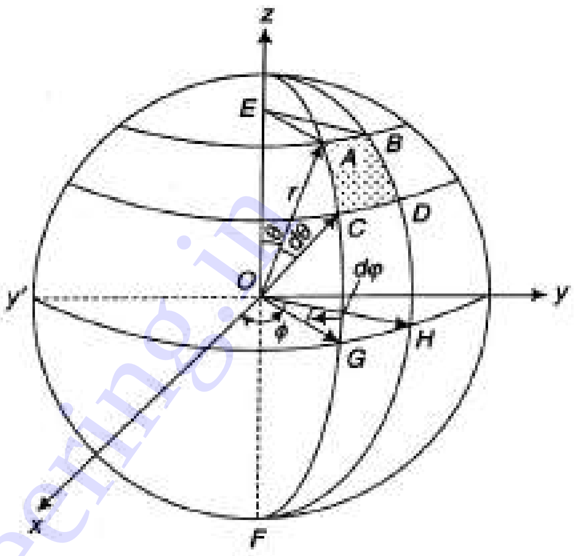


Fig. 1.17 Surface element in spherical polar coordinate system.

The unit vectors in the spherical polar coordinate system are

$$\hat{r} = \frac{\partial\vec{r}}{\partial r}\bigg/\left|\frac{\partial\vec{r}}{\partial r}\right| = \hat{i}\sin\theta\cos\varphi + \hat{j}\sin\theta\sin\varphi + \hat{k}\cos\theta$$

$$\hat{\theta} = \frac{\partial\vec{r}}{\partial\theta}\bigg/\left|\frac{\partial\vec{r}}{\partial\theta}\right| = \hat{i}\cos\theta\cos\varphi + \hat{j}\cos\theta\sin\varphi - \hat{k}\sin\theta$$

and

$$\hat{\varphi} = \frac{\partial\vec{r}}{\partial\varphi}\bigg/\left|\frac{\partial\vec{r}}{\partial\varphi}\right| = -\hat{i}\sin\varphi + \hat{j}\cos\varphi$$

1.14.4 Gradient, Divergence, Curl and Laplacian in Cartesian, Cylindrical and Spherical Coordinate System

(i) In cartesian coordinate system

(a) Gradient of a scalar function $\psi(x, y, z)$

$$\vec{\nabla}\psi = \hat{i}\frac{\partial\psi}{\partial x} + \hat{j}\frac{\partial\psi}{\partial y} + \hat{k}\frac{\partial\psi}{\partial z}$$

(b) Divergence of vector field $\vec{A}(x, y, z)$

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

(c) Curl of vector field $\vec{A}(x, y, z)$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

(d) Laplacian of $\psi(x, y, z)$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}$$

(ii) In cylindrical coordinate system

(a) Gradient of scalar function $\psi(\rho, \varphi, z)$

$$\vec{\nabla}\psi = \hat{\rho}\frac{\partial \psi}{\partial \rho} + \hat{\varphi}\frac{1}{\rho}\frac{\partial \psi}{\partial \varphi} + \hat{k}\frac{\partial \psi}{\partial z}$$

(b) Divergence of vector field $\vec{A}$ (ρ, φ, z)

$$\vec{\nabla}\cdot\vec{A} = \left(\hat{\rho}\frac{\partial}{\partial \rho} + \hat{\varphi}\frac{1}{\rho}\frac{\partial}{\partial \varphi} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(\hat{\rho}A_\rho + \hat{\varphi}A_\varphi + \hat{k}A_z)$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

(c) Curl of vector field $\vec{A}(\rho, \varphi, z)$

$$\vec{\nabla}\times\vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\varphi} & \hat{k} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ A_\rho & A_\varphi & A_z \end{vmatrix}$$

(d) Laplacian of $\psi(\rho, \varphi, z)$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \varphi^2} + \frac{\partial^2 \psi}{\partial z^2}.$$

(iii) In spherical coordinate system

(a) Gradient of a scalar function $\psi(r, \theta, \varphi)$

$$\vec{\nabla}\psi = \hat{r}\frac{\partial \psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial \psi}{\partial \theta} + \hat{\theta}\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \varphi}$$

(b) Divergence of vector field $\vec{A}$ (r, θ, φ)

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}$$

(c) Curl of vector field $\vec{A}(r, \theta, \varphi)$

$$\vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ A_r & rA_\theta & r\sin\theta A_\varphi \end{vmatrix}$$

(d) Laplacian of $\psi(r, \theta, \varphi)$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \psi}{\partial \varphi^2}.$$

Worked Out Problems

Example 1.1 If $\vec{A} = 4\hat{i} + 3\hat{j} + \hat{k}$, $\vec{B} = 2\hat{i} - \hat{j} + 2\hat{k}$ find a unit vector perpendicular to vectors $\vec{A}$ and $\vec{B}$. Find also the angle between the vector $\vec{A}$ and $\vec{B}$.

Sol. $$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 3 & 1 \\ 2 & -1 & 2 \end{vmatrix} = 7\hat{i} - 6\hat{j} - 10\hat{k}$$

and $|\vec{A} \times \vec{B}| = \sqrt{7^2 + (-6)^2 + (-10)^2} = \sqrt{185}$

The unit normal $\hat{n} = \dfrac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} = \dfrac{7\hat{i} - 6\hat{j} - 10\hat{k}}{\sqrt{185}}$

If θ is the angle between $\vec{A}$ and $\vec{B}$, then

$|\vec{A} \times \vec{B}| = |\vec{A}|\,|\vec{B}| \sin\theta$, here $|\vec{A}| = \sqrt{4^2 + 3^2 + 1^2} = \sqrt{26}$

$|\vec{B}| = \sqrt{2^2 + 1^2 + 2^2} = 3$

$$\therefore \quad \sin\theta = \frac{|\vec{A} \times \vec{B}|}{|\vec{A}| \times |\vec{B}|} = \frac{\sqrt{185}}{3\sqrt{26}}$$

$$\therefore \quad \theta = \sin^{-1}\left(\frac{\sqrt{185}}{3\sqrt{26}}\right) = 62.77^\circ$$

Example 1.2 Show that the vectors $\vec{A} = 3\hat{i} - 2\hat{j} + \hat{k}$, $\vec{B} = \hat{i} - 3\hat{j} + 5\hat{k}$ and $\vec{C} = 2\hat{i} + \hat{j} - 4\hat{k}$ from the sides of a right-angled triangle.

Sol. In our problem we find that

$$\vec{B} + \vec{C} = (\hat{i} - 3\hat{j} + 5\hat{k}) + (2\hat{i} + \hat{j} - 4\hat{k})$$
$$= 3\hat{i} - 2\hat{j} + \hat{k} = \vec{A}$$

$\therefore$ The given vectors form a triangle.

Again $|\vec{A}| = \sqrt{9 + 4 + 1} = \sqrt{14}$

$|\vec{B}| = \sqrt{1 + 9 + 25} = \sqrt{35}$

$|\vec{C}| = \sqrt{4 + 1 + 16} = \sqrt{21}$

$|\vec{A}|^2 + |\vec{C}|^2 = 14 + 21 = 35 = |\vec{B}|^2$.

$\therefore$ $\vec{A}, \vec{B}, \vec{C}$ form a right-angled triangle.

Example 1.3 If $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$, then show that $\vec{A}$ and $\vec{B}$ are perpendicular.

Sol. $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$

or, $|\vec{A} + \vec{B}|^2 = |\vec{A} - \vec{B}|^2$

or, $2\vec{A} \cdot \vec{B} = -2\vec{A} \cdot \vec{B}$

or, $4\vec{A} \cdot \vec{B} = 0$, So $\vec{A}$ and $\vec{B}$ are perpendicular to each other.

Example 1.4 The three adjacent sides of a parallelepiped are represented by $\hat{i}+2\hat{j}$, $4\hat{j}$ and $\hat{j}+3\hat{k}$. Calculate its volume.

Sol. Let $\vec{A}=\hat{i}+2\hat{j}$, $\vec{B}=4\hat{j}$ and $\vec{C}=\hat{j}+3\hat{k}$

The volume of the parallelepiped is $V=\vec{A}\cdot(\vec{B}\times\vec{C})$

Here $$\vec{B}\times\vec{C}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 4 & 0\\ 0 & 1 & 3\end{vmatrix}=12\hat{i}$$

$$\therefore \quad \vec{A}\cdot(\vec{B}\times\vec{C})=(\hat{i}+2\hat{j})\cdot 12\hat{i}=12$$

So, volume $V = 12$ units

Example 1.5 For what value of m are the following three vectors coplanar?

$$\vec{A}=3\hat{i}+2\hat{j}+\hat{k},\ \vec{B}=3\hat{i}+4\hat{j}+5\hat{k} \text{ and } \vec{C}=\hat{i}+\hat{j}-m\hat{k}$$

Sol. Three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ will be coplanar if their scalar triple product is zero, i.e., $\vec{A}\cdot(\vec{B}\times\vec{C})=0$

So, $$\begin{vmatrix}3 & 2 & 1\\ 3 & 4 & 5\\ 1 & 1 & -m\end{vmatrix}=0$$

or, $3(-4m-5)+2(5+3m)+1(3-4)=0$

or, $m=-1$

Example 1.6 Find the area of the parallelogram determined by the vectors $\vec{A}=3\hat{i}+2\hat{j}$ and $\vec{B}=2\hat{j}-4\hat{k}$.

Sol. The area of the parallelogram is

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 2 & 0\\ 0 & 2 & -4\end{vmatrix}=-8\hat{i}+12\hat{j}+6\hat{k}$$

The magnitude of the area of the parallelogram is $\sqrt{64+144+36}=2\sqrt{61}$ units.

Example 1.7 If $\hat{a}$ and $\hat{b}$ are unit vectors and θ is the angle between them, show that $2\sin\frac{\theta}{2}=|\hat{a}-\hat{b}|$.

[WBUT 2012]

Sol.
$$|\hat{a}-\hat{b}|^2=(\hat{a}-\hat{b})\cdot(\hat{a}-\hat{b})$$
$$=|\hat{a}|^2+|\hat{b}|^2-2\hat{a}\cdot\hat{b} \qquad |\hat{a}|=1$$
$$=1+1-|\hat{a}|\,|\hat{b}|\cos\theta \qquad |\hat{b}|=1$$
$$=2(1-\cos\theta)$$
$$=4\sin^2\frac{\theta}{2}$$
$$\therefore \quad |\hat{a}-\hat{b}|=2\sin\frac{\theta}{2}$$

Example 1.8 Find the torque about the point 0 (2, – 1, 3) of a force $\vec{F}$(3, 2, – 4) passing through the point P(1, – 1, 2).

Sol. The position vector of P relative to O is

$$\vec{r}=(\hat{i}-\hat{j}+2\hat{k})-(2\hat{i}-\hat{j}+3\hat{k})$$
$$=-\hat{i}-\hat{k}$$

Again force $\vec{F} = 3\hat{i} + 2\hat{j} - 4\hat{k}$

The required moment or torque is $\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & -1 \\ 3 & 2 & -4 \end{vmatrix} = 2\hat{i} - 7\hat{j} - 2\hat{k}$

Example 1.9 If $\vec{a}, \vec{b}, \vec{c}$ satisfy the relation $\vec{a} + \vec{b} + \vec{c} = 0$ show that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.

Sol. We have $\vec{a} + \vec{b} = -\vec{c}$

$\therefore \quad b \times (\vec{a} + \vec{b}) = -\vec{b} \times \vec{c}$

or, $\vec{b} \times \vec{a} + \vec{b} \times \vec{b} = -\vec{b} \times \vec{c}$

or, $\vec{b} \times \vec{a} = -\vec{b} \times \vec{c} = \vec{c} \times \vec{b}$

or, $\vec{a} \times \vec{b} = \vec{b} \times \vec{c}$

Similarly, $\vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

$\therefore \quad \vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

Example 1.10 Show that $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) = 0$

Sol. We have $\vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a} + \vec{c} \times \vec{b}$

$$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{c} - \vec{a} \times \vec{b} - \vec{a} \times \vec{c} - \vec{b} \times \vec{c}$$

$$= 0$$

Example 1.11 A particle is acted upon by constant forces $(5\hat{i} + 2\hat{j} + \hat{k})$ and $(2\hat{i} - \hat{j} - 3\hat{k})$ and is displaced from the origin to the point $(4\hat{i} + \hat{j} - 3\hat{k})$. Show that the total work done by the forces is 35 units.

Sol. The displacement produced in moving from origin to the point $(4\hat{i} + \hat{j} - 3\hat{k})$ is $(4\hat{i} + \hat{j} - 3\hat{k})$.

So, the total work done

$$= (5\hat{i} + 2\hat{j} + \hat{k}) \cdot (4\hat{i} + \hat{j} - 3\hat{k}) + (2\hat{i} - \hat{j} - 3\hat{k}) \cdot (4\hat{i} + \hat{j} - 3\hat{k})$$

$$= 20 + 2 - 3 + 8 - 1 + 9$$

$$= 35 \text{ units}$$

Example 1.12 A charge of 2.0 μC moves with a speed of 2.0×10^6 m/s along the positive x axis. A magnetic field B of strength $(0.20\,\hat{j} + 0.40\,\hat{k})$ tesla exists in space. Find the magnetic force acting on the charge.

Sol. The force on the charge $= q(\vec{v} \times \vec{B})$

$$= 2 \times 10^{-6} \times [2 \times 10^6\,\hat{i} \times (0.2\hat{j} + 0.4\hat{k})]$$

$$= (0.8\hat{k} - 1.6\hat{j}) \quad \text{Newton}$$

Example 1.13 If $\vec{A}(t)$ has a constant magnitude then show that $\frac{d\vec{A}}{dt}$ is perpendicular to $\vec{A}$.

Sol. We know that $\vec{A} \cdot \vec{A} = |A|^2 = \text{constant}$

$$\therefore \quad \vec{A} \cdot \frac{d\vec{A}}{dt} + \frac{d\vec{A}}{dt} \cdot \vec{A} = 0 \text{ or, } 2\vec{A} \cdot \frac{d\vec{A}}{dt} = 0$$

So, $\frac{d\vec{A}}{dt}$ is perpendicular to $\vec{A}$.

Example 1.14 If $\varphi(x, y, z) = 3x^2y - y^3z^2$, find $\vec{\nabla}\varphi$ at the point $(1, -2, -1)$. [WBUT 2004, 2006]

Sol. Here $\varphi(x, y, z) = 3x^2y - y^3z^2$

Again $\vec{\nabla}\varphi = \hat{i}\frac{\partial\varphi}{\partial x} + \hat{j}\frac{\partial\varphi}{\partial y} + \hat{k}\frac{\partial\varphi}{\partial z}$

$$\frac{\partial\varphi}{\partial x} = \frac{\partial}{\partial x}(3x^2y - y^3z^2) = 6xy$$

$$\frac{\partial\varphi}{\partial y} = \frac{\partial}{\partial y}(3x^2y - y^3z^2) = 3x^2 - 3y^2z^2$$

$$\frac{\partial\varphi}{\partial z} = \frac{\partial}{\partial z}(3x^2y - y^3z^2) = -2y^3z$$

$\therefore$ $\vec{\nabla}\varphi = \hat{i}6xy + \hat{j}(3x^2 - 3y^2z^2) - \hat{k}2y^3z$

Now at $(1, -2, -1)$, $\vec{\nabla}\varphi|_{1,-2,-1} = \hat{i}[(6\times(1)(-2)] + \hat{j}[3 - 3\times(-2)^2\times(-1)^2] - \hat{k}[2\times(-2)^3(-1)]$

$$= -12\hat{i} - 9\hat{j} - 16\hat{k}$$

Example 1.15 Find the directional derivative of $\psi(x, y, z) = xy^2z + 4x^2z$ at $(-1, 1, 2)$ in the direction $(2\hat{i} + \hat{j} - 2\hat{k})$.

Sol. Here $\vec{\nabla}\psi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy^2z + 4x^2z)$

$$= (y^2z + 8xz)\,\hat{i} + 2xyz\hat{j} + (xy^2 + 4x^2)\hat{k}$$

at $(-1, 1, 2)$ $\vec{\nabla}\psi = -14\hat{i} - 4\hat{j} + 3\hat{k}$

The unit vector in the direction of $2\hat{i} + \hat{j} - 2\hat{k}$ is

$$\hat{n} = \frac{2\hat{i} + \hat{j} - 2\hat{k}}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$$

Hence the directional derivative is $\vec{\nabla}\psi\cdot\hat{n} = (-14\hat{i} - 4\hat{j} + 3\hat{k})\cdot\left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}\right) = -38/3$

Example 1.16 Show that $\vec{\nabla}\psi$ is a vector perpendicular to the surface $\psi(x, z, y) = c$, where c is a constant.

Sol. Let the position vector $\vec{r} = \hat{i}\,x + \hat{j}y + \hat{k}z$ belong to the surface $\psi(x, z, y) =$ constant. Then $d\vec{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$, represents a vector which is tangent to the surface ψ.

Again $d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy + \frac{\partial\psi}{\partial z}dz = 0$

$$= \left(\hat{i}\,\frac{\partial\psi}{\partial x} + \hat{j}\,\frac{\partial\psi}{\partial y} + \hat{k}\,\frac{\partial\psi}{\partial z}\right)\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \vec{\nabla}\psi\cdot d\vec{r} = 0$$

So, $\vec{\nabla}\psi$ is perpendicular to $d\vec{r}$ and therefore perpendicular to the surface.

Example 1.17 Find a unit normal to the surface $z^2 = x^2 - y^2$ at the point $(1, 0, -1)$.

Sol. Let $\psi(x, y, z) = x^2 - y^2 - z^2$

$\therefore$ $\vec{\nabla}\psi = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)(x^2 - y^2 - z^2)$

$$= 2x\hat{i} - 2y\hat{j} - 2z\hat{k}$$

$$\vec{\nabla}\psi|_{1,0,-1} = 2\hat{i} + 2\hat{k} = 2(\hat{i} + \hat{k})$$

Now unit normal $\hat{n} = \left.\dfrac{\vec{\nabla}\psi}{|\vec{\nabla}\psi|}\right|_{1,0,-1} = \pm 2\dfrac{(\hat{i}+\hat{k})}{\sqrt{2^2+2^2}} = \pm\left[\dfrac{2(\hat{i}+\hat{k})}{2\sqrt{2}}\right] = \pm\left[\dfrac{\hat{i}}{\sqrt{2}} + \dfrac{\hat{j}}{\sqrt{2}}\right] = \pm\dfrac{1}{\sqrt{2}}(\hat{i}+\hat{j})$

Positive and negative signs imply that the unit normal is either outward or inward.

Example 1.18 In what direction from the point (1, 3, 2) is the directional derivative of $\psi = 2xz - y^2$ a maximum? What is the magnitude of this maximum?

Sol. Here
$$\vec{\nabla}\psi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2xz - y^2)$$
$$= 2z\hat{i} - 2y\hat{j} + 2x\hat{k}$$
$$\vec{\nabla}\psi|_{1,3,2} = 4\hat{i} - 6\hat{j} + 2\hat{k}$$

The directional derivative is maximum in the direction $\vec{\nabla}\psi = 4\hat{i} - 6\hat{j} + 2\hat{k}$

The magnitude of this maximum is $|\vec{\nabla}\psi| = \sqrt{(4)^2 + (-6)^2 + (2)^2} = \sqrt{56} = 2\sqrt{14}$

Example 1.19 Find the values of the constants a, b, c so that the directional derivative of $\psi = axy^2 + byz + cz^2x^3$ at (1, 2, – 1) has a maximum of magnitude 64 in a direction parallel to the z axis.

Sol. Here
$$\vec{\nabla}\psi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(axy^2 + byz + cz^2x^3)$$
$$= (ay^2 + 3x^2cz^2)\hat{i} + (2axy + bz)\hat{j} + (by + 2czx^3)\hat{k}$$

Now $\vec{\nabla}\psi|_{1,2,-1} = (4a + 3c)\hat{i} + (4a - b)\hat{j} + (2b - 2c)\hat{k}$

Since ψ has a maximum magnitude along the z axis, so

$\vec{\nabla}\psi|_{1,2,-1}\cdot\hat{k} = 64$ or, $(2b - 2c) = 64$

and $4a + 3c = 0$ Now solving these equations,

$4a - b = 0$ we have $a = 6$

$b = 24$

$c = -8$

Example 1.20 Show that $\vec{\nabla}r^n = nr^{n-2}\vec{r}$

Sol.
$$\vec{\nabla}r^n = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{n/2}$$
$$= \hat{i}\left\{\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}\cdot 2x\right\} + \hat{j}\left\{\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}\cdot 2y\right\} + \hat{k}\left\{\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}\cdot 2z\right\}$$
$$= n(x^2+y^2+z^2)^{n/2-1}(\hat{i}x + \hat{j}y + \hat{k}z)$$
$$= n(r^2)^{n/2-1}\vec{r} = nr^{n-2}\vec{r}$$

Example 1.21 Find the angle between the surfaces $x^2 + y^2 = 9 - z^2$ and $x^2 + y^2 = (z + 3)$ at the point (2, – 1, 2).

Sol. Let $\psi_1 = x^2 + y^2 + z^2 - 9$ and $\psi_2 = x^2 + y^2 - z - 3$

These two surfaces are constant.

Now $\vec{\nabla}\psi_1 = \vec{\nabla}(x^2 + y^2 + z^2 - 9) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

and $\vec{\nabla}\psi_1\big|_{2,-1,2} = 4\hat{i} - 2\hat{j} + 4\hat{k}$

Again $\vec{\nabla}\psi_2 = \vec{\nabla}(x^2 + y^2 - z - 3) = 2x\hat{i} + 2y\hat{j} - \hat{k}$

$\vec{\nabla}\psi_2\big|_{2,-1,2} = 4\hat{i} - 2\hat{j} - \hat{k}$

Since $\vec{\nabla}\psi_1$ and $\vec{\nabla}\psi_2$ are normal to the surfaces ψ_1 and ψ_2, the angle between the surfaces is the angle between the normal to the surfaces. Let θ be the angle between the surfaces.

So, $\vec{\nabla}\psi_1 \cdot \vec{\nabla}\psi_2 = |\vec{\nabla}\psi_1|\,|\vec{\nabla}\psi_2| \cos\theta$

or, $$\cos\theta = \frac{\vec{\nabla}\psi_1 \cdot \vec{\nabla}\psi_2}{|\vec{\nabla}\psi_1|\,|\vec{\nabla}\psi_2|} = \frac{(4\hat{i} - 2\hat{j} + 4\hat{k})\cdot(4\hat{i} - 2\hat{j} - \hat{k})}{\sqrt{4^2 + (-2)^2 + 4^2}\,\sqrt{4^2 + (-2)^2 + (-1)^2}} = \frac{16}{6\sqrt{21}}$$

or, $$\theta = \cos^{-1}\left(\frac{16}{6\sqrt{21}}\right) = 54.41°$$

Example 1.22 Find the unit vector perpendicular to $x^2 + y^2 - z^2 = 100$ at the point (1, 2, 3). [WBUT 2007]

Sol. Here $\psi = x^2 + y^2 - z^2 - 100$

or, $\vec{\nabla}\psi = \vec{\nabla}(x^2 + y^2 - z^2 - 100) = 2x\hat{i} + 2y\hat{j} - 2z\hat{k}$

or $\vec{\nabla}\psi|_{1,2,3} = 2\hat{i} + 4\hat{j} - 6\hat{k}$

Now unit normal $$\hat{n} = \frac{\vec{\nabla}\psi}{|\vec{\nabla}\psi|} = \frac{2\hat{i} + 4\hat{j} - 6\hat{k}}{\sqrt{2^2 + 4^2 + (-6)^2}} = \pm\frac{\hat{i} + 2\hat{j} - 3\hat{k}}{\sqrt{14}}$$

Example 1.23 Show that $\vec{\nabla}\cdot\vec{r} = 3$

Sol. Here $$\vec{\nabla}\cdot\vec{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(\hat{i}x + \hat{j}y + \hat{k}z)$$
$$= 1 + 1 + 1 = 3$$

Example 1.24 Show that $\vec{\nabla}(\vec{a}\cdot\vec{r}) = \vec{a}$, where $\vec{a}$ is a constant vector.

Sol. Here $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z,\ \vec{a}\cdot\vec{r} = (\hat{i}a_1 + \hat{j}a_2 + \hat{k}a_3)\cdot(\hat{i}x + \hat{j}y + \hat{k}z) = a_1x + a_2y + a_3z$

Now $$\vec{\nabla}(\vec{a}\cdot\vec{r}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$$
$$= \hat{i}a_1 + \hat{j}a_2 + \hat{k}a_3 = \vec{a}$$

Example 1.25 Find $\vec{\nabla}\cdot\vec{F}$ if $\vec{F} = 2x^2z\hat{i} - xy^2z\hat{j} + 3yz^2\hat{k}$

Sol. We have $$\vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(2x^2z\hat{i} - xy^2z\hat{j} + 3yz^2\hat{k})$$
$$= 4xz - 2xyz + 6yz$$

Example 1.26 Determine the constant a so that the vector field

$\vec{F} = (2x + 3y)\hat{i} + (3y - 2z)\hat{j} + (y + az)\hat{k}$ is solenoidal.

Sol. If vector $\vec{F}$ is solenoidal then $\vec{\nabla}\cdot\vec{F} = 0$

Here $\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(2x+3y) + \frac{\partial}{\partial y}(3y-2z) + \frac{\partial}{\partial z}(y+az) = 0$

$= 2+3+a = 0$

or, $a = -5$

Example 1.27 Show that the vector field $\vec{F} = \frac{\vec{i}x+\vec{j}y}{\sqrt{x^2+y^2}}$ is a "source" field.

Sol. For a source field, $\vec{\nabla}\cdot\vec{F}$ should be positive and for a sink field, $\vec{\nabla}\cdot\vec{F}$ should be negative.

Here $\vec{F} = \frac{\hat{i}x+\hat{j}y}{\sqrt{x^2+y^2}}$

$$\therefore \quad \vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\hat{i}x+\hat{j}y}{\sqrt{x^2+y^2}}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right)$$

$$= -\frac{x^2}{(x^2+y^2)^{3/2}} + \frac{1}{(x^2+y^2)^{1/2}} - \frac{y^2}{(x^2+y^2)^{3/2}} + \frac{1}{(x^2+y^2)^{1/2}}$$

$$= -\frac{(x^2+y^2)}{(x^2+y^2)^{3/2}} + \frac{2}{(x^2+y^2)^{1/2}}$$

$$= -\frac{1}{(x^2+y^2)^{1/2}} + \frac{2}{(x^2+y^2)^{1/2}} = \frac{1}{\sqrt{x^2+y^2}}$$

$\therefore$ $\vec{\nabla}\cdot\vec{F}$ is +ve, so $\vec{F}$ is a source field.

Example 1.28 Evaluate $\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right)$

Sol. We know that $\vec{\nabla}\cdot(\psi\vec{A}) = \vec{\nabla}\psi\cdot\vec{A} + \psi\vec{\nabla}\cdot\vec{A}$

Here we consider $\psi = \frac{1}{r^3}$ and $\vec{A} = \vec{r}$

So, $\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right) = \vec{\nabla}\frac{1}{r^3}\cdot\vec{r} + \frac{1}{r^3}\vec{\nabla}\cdot\vec{r}$

$$= -3r^{-5}\vec{r}\cdot\vec{r} + \frac{3}{r^3} \quad [\because \ \vec{\nabla}r^n = nr^{n-2}\vec{r} \text{ and } \vec{\nabla}\cdot r = 3]$$

$$= -3r^{-3} + \frac{3}{r^3} = -\frac{3}{r^3} + \frac{3}{r^3} = 0$$

Example 1.29 Find $\vec{\nabla}\cdot\vec{F}$ and $\vec{\nabla}\times\vec{F}$ where $\vec{F} = \vec{\nabla}(x^3+y^3+z^3-3xyz)$. [WBUT 2001]

Sol. Here, $\vec{F} = \vec{\nabla}(x^3+y^3+z^3-3xyz)$

$$= \hat{i}\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz) + \hat{j}\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz) + \hat{k}\frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)$$

$$= (3x^2-3yz)\hat{i} + (3y^2-3xz)\hat{j} + (3z^2-3xy)\hat{k}$$

Now $\vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot[(3x^2-3yz)\,\hat{i} + (3y^2-3xz)\,\hat{j} + (3z^2-3xy)\,\hat{k}]$

$$= 6x + 6y + 6z = 6(x + y + z)$$

Again $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix} = \hat{i}\left[\dfrac{\partial}{\partial y}(3z^2 - 3xy) - \dfrac{\partial}{\partial z}(3y^2 - 3xz)\right]$

$$+ \hat{j}\left[\frac{\partial}{\partial z}(3x^2 - 3yz) - \frac{\partial}{\partial x}(3z^2 - 3xy)\right] + \hat{k}\left[\frac{\partial}{\partial x}(3y^2 - 3xz) - \frac{\partial}{\partial y}(3x^2 - 3yz)\right]$$

$$= (-3x + 3x)\,\hat{i} + (-3y + 3y)\,\hat{j} + (-3z + 3z)\,\hat{k} = 0$$

Example 1.30 Prove that $\nabla^2 \ln r = \dfrac{1}{r^2}$ [WBUT 2007]

Sol. Here $\ln r = \ln (x^2 + y^2 + z^2)^{1/2} = \dfrac{1}{2}\ln (x^2 + y^2 + z^2)$

$$\nabla^2 \ln r = \vec{\nabla}\cdot\vec{\nabla}\ln r = \vec{\nabla}\cdot\vec{\nabla}\frac{1}{2}\ln (x^2 + y^2 + z^2)$$

$$= \vec{\nabla}\cdot\left[\frac{1}{2}\left\{\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right\}\ln (x^2 + y^2 + z^2)\right]$$

$$= \frac{1}{2}\vec{\nabla}\cdot 2\left(\frac{\hat{i}x + \hat{j}y + \hat{k}z}{x^2 + y^2 + z^2}\right) = \vec{\nabla}\cdot\frac{\vec{r}}{r^2}$$

So, $\nabla^2 \ln r = \vec{\nabla}\cdot\dfrac{\vec{r}}{r^2}$

We know that $\vec{\nabla}\cdot(\psi\vec{A}) = \vec{\nabla}\psi\cdot\vec{A} + \psi\vec{\nabla}\cdot\vec{A}$

So, $\nabla^2 \ln r = \vec{\nabla}\dfrac{1}{r^2}\cdot\vec{r} + \dfrac{1}{r^2}\vec{\nabla}\cdot\vec{r}$

$$= -2r^{-4}(\vec{r}\cdot\vec{r}) + \frac{3}{r^2} = -\frac{2}{r^2} + \frac{3}{r^2} = \frac{1}{r^2}$$

Example 1.31 Show that $\vec{\nabla} \times \vec{r} = 0$

Sol.
$$\vec{\nabla} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{i}\left(\frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y\right) + \hat{j}\left(\frac{\partial}{\partial z}x - \frac{\partial}{\partial x}z\right) + \hat{k}\left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\right) = 0$$

Example 1.32 Show that the vector $\vec{F} = (4xy - z^3)\,\hat{i} + 2x^2\,\hat{j} - 3xz^2\,\hat{k}$ is irrotational.

Sol. If $\vec{F}$ is an irrotational vector then $\vec{\nabla} \times \vec{F} = 0$

Now
$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 4xy-z^3 & 2x^2 & -3xz^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-3xz^2) - \frac{\partial}{\partial z}(2x^2)\right] + \hat{j}\left[\frac{\partial}{\partial z}(4xy - z^3) - \frac{\partial}{\partial x}(-3xz^2)\right] + \hat{k}\left[\frac{\partial}{\partial x}(2x^2) - \frac{\partial}{\partial y}(4xy - z^3)\right]$$

$$= 0 + \hat{j}(-3z^2 + 3z^2) + \hat{k}(4x - 4x) = 0$$

Example 1.33 Show that $\vec{\nabla}\cdot\vec{\nabla}\times\vec{F} = 0$

Sol. Let $\vec{F} = \hat{i}F_1 + \hat{j}F_2 + \hat{k}F_3$

Now $$\vec{\nabla}\times\vec{F} = \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \hat{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Again $$\vec{\nabla}\cdot\vec{\nabla}\times\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[\hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \hat{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \left(\frac{\partial^2 F_3}{\partial x\,\partial y} - \frac{\partial^2 F_2}{\partial x\,\partial z}\right) + \left(\frac{\partial^2 F_1}{\partial y\,\partial z} - \frac{\partial^2 F_3}{\partial y\,\partial x}\right) + \left(\frac{\partial^2 F_2}{\partial z\,\partial x} - \frac{\partial^2 F_1}{\partial z\,\partial y}\right) = 0$$

Example 1.34 Show that $\vec{F} = (2xy + z^3)\,\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force field. [WBUT 2004, 2006]

Sol. Here $$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix} = \hat{i}\left(\frac{\partial}{\partial y}3xz^2 - \frac{\partial}{\partial z}x^2\right) + \hat{j}\left[\frac{\partial}{\partial z}(2xy + z^3) - \frac{\partial}{\partial x}3xz^2\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}(2xy + z^3)\right]$$

$$= 0 + \hat{j}(3z^2 - 3z^2) + \hat{k}(2x - 2x) = 0$$

For a conservative force field, $\vec{\nabla}\times\vec{F} = 0$

Example 1.35 If the vectors $\vec{A}$ and $\vec{B}$ are irrotational then show that the vector $\vec{A}\times\vec{B}$ is solenoidal. [WBUT 2006]

Sol. If $\vec{A}$ and $\vec{B}$ are irrotational then

$$\vec{\nabla}\times\vec{A} = 0 \text{ and } \vec{\nabla}\times\vec{B} = 0$$

Now $$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = (\vec{\nabla}\times\vec{A})\cdot\vec{B} - (\vec{\nabla}\times\vec{B})\cdot\vec{A} = 0 - 0 = 0$$

Hence $(\vec{A}\times\vec{B})$ is solenoidal.

Example 1.36 Show that $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = -\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A})$

Sol. Let $\vec{\nabla} = \vec{P}$

So that $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{P}\times(\vec{P}\times\vec{A})$

Again $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}\,(\vec{a}\cdot\vec{c}) - (\vec{a}\cdot\vec{b})\,\vec{c}$

So, $\vec{P}\times(\vec{P}\times\vec{A}) = \vec{P}\,(\vec{P}\cdot\vec{A}) - (\vec{P}\cdot\vec{P})\,\vec{A}$

or, $\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \vec{\nabla}\,(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

Example 1.37 If $\vec{\omega}$ is a constant vector, $\vec{r}$ is the position vector and $\vec{v} = \vec{\omega}\times\vec{r}$. Prove that $\vec{\nabla}\cdot\vec{v} = 0$

Sol. $$\vec{\nabla}.\vec{v} = \vec{\nabla}\cdot(\vec{\omega}\times\vec{r})$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot [\hat{i}\,(\omega_2 z - \omega_3 y) + \hat{j}(\omega_3 x - \omega_1 z) + \hat{k}(\omega_1 y - \omega_2 x)$$

$$= 0 + 0 + 0 = 0$$

Example 1.38 Show that $\vec{\nabla} \times \vec{\nabla}\psi = 0$

Sol. $$\vec{\nabla} \times \vec{\nabla}\psi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\hat{i}\frac{\partial\psi}{\partial x} + \hat{j}\frac{\partial\psi}{\partial y} + \hat{k}\frac{\partial\psi}{\partial z}\right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y} & \frac{\partial\psi}{\partial z} \end{vmatrix} = \hat{i}\left[\frac{\partial^2\psi}{\partial y\,\partial z} - \frac{\partial^2\psi}{\partial z\,\partial y}\right] + \hat{j}\left[\frac{\partial^2\psi}{\partial z\,\partial x} - \frac{\partial^2\psi}{\partial x\,\partial z}\right] + \hat{k}\left[\frac{\partial^2\psi}{\partial x\,\partial y} - \frac{\partial^2\psi}{\partial y\,\partial x}\right] = 0$$

Example 1.39 If $\vec{\nabla} \cdot \vec{E} = 0$, $\vec{\nabla} \cdot \vec{B} = 0$, $\vec{\nabla} \times \vec{E} = -\frac{\partial\vec{B}}{\partial t}$, $\vec{\nabla} \times \vec{B} = \frac{\partial\vec{E}}{\partial t}$, then show that $\vec{E}$ and $\vec{B}$ satisfy $\nabla^2\psi = \frac{\partial^2\psi}{\partial t^2}$.

Sol. We have $\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \nabla \times \left(-\frac{\partial\vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = -\frac{\partial}{\partial t}\left(\frac{\partial\vec{E}}{\partial t}\right) = -\frac{\partial^2\vec{E}}{\partial t^2}$

But $\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2\vec{E} = -\nabla^2\vec{E}$

So, $-\nabla^2\vec{E} = -\frac{\partial^2\vec{E}}{\partial t^2}$ $\quad\therefore\ \nabla^2\vec{E} = \frac{\partial^2\vec{E}}{\partial t^2}$

Similarly, $\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \vec{\nabla} \times \left(\frac{\partial\vec{E}}{\partial t}\right) = \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E}) = \frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right) = -\frac{\partial^2\vec{B}}{\partial t^2}$

But $\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2\vec{B} = -\nabla^2\vec{B}$

So, $\nabla^2\vec{B} = \frac{\partial^2\vec{B}}{\partial t^2}$

i.e., $\vec{E}$ and $\vec{B}$ satisfy the equation $\nabla^2\psi = \frac{\partial^2\psi}{\partial t^2}$

Example 1.40 Show that $\nabla^2\left(\frac{1}{r}\right) = 0$ **[WBUT 2011]**

Sol. $$\nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

Now $$\frac{\partial^2}{\partial x^2}(x^2 + y^2 + z^2)^{-1/2} = \frac{\partial}{\partial x}\left[-\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2x)\right]$$

$$= -\frac{\partial}{\partial x}[x\,(x^2 + y^2 + z^2)^{-3/2}]$$

$$= -(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2}$$

$$= \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$$

Similarly, $\frac{\partial^2}{\partial y^2}(x^2+y^2+z^2)^{-1/2} = \frac{2y^2-z^2-x^2}{(x^2+y^2+z^2)^{5/2}}$

and $\frac{\partial^2}{\partial z^2}(x^2+y^2+z^2)^{-1/2} = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$

So, $$\nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(x^2+y^2+z^2)^{-1/2}$$

$$= \frac{2x^2-y^2-z^2+2y^2-z^2-x^2+2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}} = 0$$

Example 1.41 Show that $\vec{A} = (6xy+z^3)\,\hat{i} + (3x^2-z)\,\hat{j} + (3xz^2-y)\,\hat{k}$ is irrotational. Find ψ such that $\vec{A} = \vec{\nabla}\psi$. **[WBUT 2012]**

Sol. A vector $\vec{A}$ is called irrotational if $\vec{\nabla}\times\vec{A} = 0$

Here $$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix} = \hat{i}\left[\frac{\partial}{\partial y}(3xz^2-y) - \frac{\partial}{\partial z}(3x^2-z)\right]$$

$$+ \hat{j}\left[\frac{\partial}{\partial z}(6xy+z^3) - \frac{\partial}{\partial x}(3xz^2-y)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(3x^2-z) - \frac{\partial}{\partial y}(6xy+z^3)\right]$$

$$= \hat{i}\,(-1+1) + \hat{j}\,(3z^2-3z^2) + \hat{k}\,(6x-6x)$$

$$= 0$$

So, $\vec{A}$ is irrotational.

Here $$\vec{A} = \vec{\nabla}\psi = \hat{i}\frac{\partial\psi}{\partial x} + \hat{j}\frac{\partial\psi}{\partial y} + \hat{k}\frac{\partial\psi}{\partial z} = (6xy+z^3)\,\hat{i} + (3x^2-z)\,\hat{j} + (3xz^2-y)\,\hat{k}$$

Then $$\frac{\partial\psi}{\partial x} = 6xy+z^3,\quad \frac{\partial\psi}{\partial y} = 3x^2-z,\quad \frac{\partial\psi}{\partial z} = 3xz^2-y$$

Integrating first with respect to x, keeping y and z constant,

$$\psi = 6\frac{x^2}{2}y + xz^3 + C_1(y,z) = 3x^2y + xz^3 + C_1(y,z)$$

Integrating second with respect to y, keeping x, and z constant,

$$\psi = 3x^2y - yz + C_2(x, z)$$

Integrating third with respect to z, keeping x and y constant,

$$\psi = xz^3 - yz + C_3(x, y)$$

Comparison of all equations in ψ, shows that there will be a common value of ψ if we choose

$$C_1(x, z) = -yz, \quad C_2(x, z) = xz^3, \quad C_3(x, y) = 3x^2y$$

So that $\quad \psi = 3x^2y + xz^3 - yz + C \quad$ [where C is pure constant]

Example 1.42 Show that $\vec{E} = \frac{\vec{r}}{r^2}$ is irrotational. Find ψ such that $\vec{E} = -\vec{\nabla}\psi$ and such that $\psi(a) = 0$ where $a > 0$.

Sol.

$$\vec{\nabla} \times \vec{E} = \vec{\nabla} \times \frac{\vec{r}}{r^2} = \frac{1}{r^2}\vec{\nabla} \times \vec{r} + \vec{\nabla}\frac{1}{r^2} \times \vec{r}$$

$$= 0 + \left(-\frac{2}{r^4}\vec{r} \times \vec{r}\right) = 0$$

So E is irrotational.

Since E is irrotational, so $\quad \vec{E} = -\vec{\nabla}\psi = \frac{\vec{r}}{r^2} = \frac{\hat{r}}{r}$, or, $-\hat{r}\frac{\partial \psi}{\partial r} = \frac{\hat{r}}{r}$

or, $\quad \frac{\partial \psi}{\partial r} = -\frac{1}{r}$

Now integrating on both sides, $\psi = -\ln r + C$ where C is constant

Applying boundary condition, $\psi(a) = 0$, so $C = \ln a$

$$\therefore \quad \psi = -\ln r + \ln a = \ln\left(\frac{a}{r}\right)$$

Example 1.43 Show that $\nabla^2 f(r) = \frac{d^2 f(r)}{dr^2} + \frac{2}{r}\frac{df(r)}{dr}$

Sol. We know that

$$\nabla^2 f(r) = \vec{\nabla} \cdot \vec{\nabla} f(r)$$

$$= \vec{\nabla} \cdot \hat{r}\frac{df(r)}{dr} = \vec{\nabla} \cdot \frac{\vec{r}}{r}\frac{df(r)}{dr}$$

$$= \frac{1}{r}\frac{df(r)}{dr}(\vec{\nabla} \cdot \vec{r}) + \vec{\nabla}\left(\frac{1}{r}\frac{df(r)}{dr}\right) \cdot \vec{r}$$

[Here we have used $\quad \vec{\nabla} \cdot (\psi\vec{A}) = \psi\, \vec{\nabla} \cdot \vec{A} + \vec{\nabla}\psi \cdot \vec{A}$]

$$= \frac{3}{r}\frac{df(r)}{dr} + \hat{r}\frac{d}{dr}\left(\frac{1}{r}\frac{df(r)}{dr}\right) \cdot \vec{r}$$

$$= \frac{3}{r}\frac{df(r)}{dr} + \hat{r}\left[\frac{1}{r}\frac{d^2 f(r)}{dr^2} - \frac{1}{r^2}\frac{df(r)}{dr}\right] \cdot \vec{r}$$

$$= \frac{3}{r}\frac{df(r)}{dr} + \frac{d^2 f(r)}{dr^2} - \frac{1}{r}\frac{df(r)}{dr} \quad [\because \hat{r} \cdot \vec{r} = r]$$

$$= \frac{d^2 f(r)}{dr^2} + \frac{2}{r}\frac{df(r)}{dr}$$

Example 1.44 Find $f(r)$ such that $\nabla^2 f(r) = 0$

Sol. In the previous problem, we have seen that

$$\nabla^2 f(r) = \frac{d^2 f(r)}{dr^2} + \frac{2}{r}\frac{df(r)}{dr}$$

But, here $\nabla^2 f(r) = 0$ so $\frac{d^2 f(r)}{dr^2} + \frac{2}{r}\frac{df(r)}{dr} = 0$

or, $$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{df(r)}{dr}\right) = 0$$

Now integrating $$r^2 \frac{df(r)}{dr} = A \text{ (constant)}$$

or, $$\frac{df(r)}{dr} = \frac{A}{r^2}$$

Again integrating $$f(r) = -\frac{A}{r} + B \quad \text{(where } B \text{ is constant)}$$

So, $$f(r) = B - \frac{A}{r} = B + \frac{A'}{r} \quad (\text{where } A = -A')$$

Example 1.45 Calculate the amount of work done in moving a particle in a force field $\vec{F} = xy\,\hat{i} + yz\,\hat{j} + xz\,\hat{k}$ along the curve $\vec{r} = \hat{i}\,t + \hat{j}\,t^2 + \hat{k}\,t^3$, where t is varying from -1 to 1.

Sol. We know that position vector $r = \hat{i}x + \hat{j}y + \hat{k}z$

$$= \hat{i}\,t + \hat{j}\,t^2 + \hat{k}\,t^3$$

So $x = t$, $y = t^2$ and $z = t^3$, the parametric equation of the curve. Here, $\vec{F} = xy\,\hat{i} + yz\,\hat{j} + xz\,\hat{k} = t\,(t^2)\,\hat{i} + t^2\,(t^3)\,\hat{j} + t\,(t^3)\,\hat{k} = t^3\,\hat{i} + t^5\,\hat{j} + t^4\,\hat{k}$

and $$\frac{dr}{dt} = \hat{i} + 2t\hat{j} + 3t^2\,\hat{k}$$

Now work done $$W = \int_c \vec{F}.d\vec{r} = \int_{-1}^{+1} (t^3\,\hat{i} + t^5\,\hat{j} + t^4\,\hat{k}).(\hat{i} + 2t\,\hat{j} + 3t^2\hat{k})\,dt$$

$$= \int_{-1}^{+1} (t^3 + 2t^6 + 3t^6)\,dt = \int_{-1}^{+1} (t^3 + 5t^6)\,dt$$

$$= \left|\frac{t^4}{4} + 5\,\frac{t^7}{7}\right|_{-1}^{+1} = \frac{10}{7}$$

Example 1.46 Find the work done in moving a particle in the force field $\vec{F} = 3x^2\,\hat{i} + (2xz - y)\,\hat{j} + z\hat{k}$ along the straight line from (0, 0, 0) to (2, 1, 3).

Sol. Work done $$W = \int_c \vec{F}.\vec{dr} = \int_c [3x^2\,\hat{i} + (2xz - y)\,\hat{j} + z\hat{k}].(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int_c [3x^2\,dx + (2xz - y)\,dy + zdz]$$

We know that the equation of a straight line

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = t \text{ (say)}$$

or, $$\frac{x-0}{2-0}=\frac{y-0}{1-0}=\frac{z-0}{3-0}=t$$

So, $x = 2t, \quad y = t \quad$ and $\quad z = 3t$

The points (0, 0, 0) and (2, 1, 3) correspond to $t = 0$ and $t = 1$

So, work done $$W=\int_C [3x^2\,dx+(2xz-y)\,dy+z\,dz]$$

$$=\int_0^1 [12t^2\,(2dt)+(12t^2-t)\,dt+9\,t\,dt]$$

$$=\int_0^1 (36\,t^2+8t)\,dt=16.$$

Example 1.47 Show that the force $\vec{F}=(2xy+z^3)\,\hat{i}+x^2\,\hat{j}+3xz^2\,\hat{k}$ is conservative. If this force acting on a particle displaces it from the point (0, 1, 2) to (4, 2, 3), calculate the work done.

Sol. For the first part, see Example 1.34.

For the second part, $W=\int_A^B \vec{F}.\vec{dr}=\int_A^B F_x\,dx+F_y\,dy+F_z\,dz)$

$$=\int (2xy+z^3)\,dx+x^2dy+3xz^2\,dz$$

$$=\int_A^B (2xy\,dx+x^2\,dy)+(z^3\,dx+3xz^2\,dz)$$

$$=\int_{(0,1,2)}^{(4,2,3)} d\,(x^2y)+d(xz^3)=\int_{(0,1,2)}^{(4,2,3)} d\,(x^2y+xz^3)$$

$$=x^2y+xz^3\Big|_{0,1,2}^{4,2,3}=16\times 2+4\times 27-0$$

$$=140$$

Example 1.48 Show that for a conservative force field, work done in moving a particle from one point P_1 (x_1, y_1, z_1) in this field to another point P_2 (x_2, y_2, z_2) is independent of the path joining the two points.

Sol. For a conservative force field, $\vec{F}=\vec{\nabla}\psi$

Now work done $\quad dW=\vec{F}.\,d\vec{r}$

Total work done $$W=\int dW=\int_{P_1}^{P_2}\vec{F}.\,d\vec{r}=\int_{P_1}^{P_2}\vec{\nabla}\psi\,.\,d\vec{r}$$

$$=\int_{P_1}^{P_2}\left(\hat{i}\frac{\partial\psi}{\partial x}+\hat{j}\frac{\partial\psi}{\partial y}+\hat{k}\frac{\partial\psi}{\partial z}\right).\,(\hat{i}\,dx+\hat{j}\,dy+\hat{k}\,dz)$$

$$=\int_{P_1}^{P_2}\left(\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy+\frac{\partial\psi}{\partial z}dz\right)$$

$$=\int_{P_1}^{P_2}d\psi=\psi(P_2)-\psi(P_1)$$

Thus we see that work done depends only on the initial and final points but is independent of the path joining them.

Example 1.49 Show that the work done on a particle in moving it from A to B equals its change in kinetic energies at these points whether the force field is conservative or not.

Sol. Let a force acting on a particle displace it from the point A to another point B. Then work done

$$W = \int_A^B \vec{F} . \vec{dr}$$

We know that $\vec{F} = m \dfrac{d\vec{v}}{dt}$ where $\vec{v}$ is the velocity of the body.

So,
$$W = \int_A^B m \frac{d\vec{v}}{dt} . d\vec{r} = \int_A^B m \frac{dv}{dt} . \vec{v}\, dt$$

$$= m \int_A^B d\vec{v} . \vec{v} = m \int_A^B v\, dv = \frac{1}{2} m (v_B^2 - v_A^2)$$

So, work done on a particle in moving it from A to B equals its change in kinetic energies at these points and does not depend on the nature of the force.

Example 1.50 Evaluate $\oint_c \vec{F} . \vec{dr}$ where $\vec{F} = (2x - y + z)\hat{i} + (x + y - z^2)\hat{j} + (3x - 2y + 4z)\hat{k}$ and C is the closed curve in the xy plane, $x = 3 \cos t$, $y = 3 \sin t$ from $t = 0$ to 2π.

Sol. In the plane xy, $z = 0$, $\vec{F} = (2x - y)\hat{i} + (x + y)\hat{j} + (3x - 2y)\hat{k}$ and $\vec{dr} = \hat{i}\, dx + \hat{j}\, dy$

So, $\vec{F} . d\vec{r} = (2x - y)\, dx + (x + y)\, dy$

Again $x = 3 \cos t$ and $y = 3 \sin t$

or, $dx = -3 \sin t\, dt$ and $dy = 3 \cos t\, dt$

So, the total work done

$$W = \oint_c \vec{F} . d\vec{r} = \int_0^{2\pi} (6 \cos t - 3 \sin t)(-3 \sin t\, dt) + (3 \cos t + 3 \sin t)(3 \cos t\, dt)$$

$$= \int_0^{2\pi} (9 - 9 \sin t \cos t)\, dt = 9 \int_0^{2\pi} \left(1 - \frac{1}{2} \sin 2t\right) dt$$

$$= 9 \left[t + \frac{1}{4} \cos 2t\right]_0^{2\pi} = 9 \left[2\pi + \frac{1}{4} - \frac{1}{4}\right] = 18\pi$$

Example 1.51 Evaluate $\iint_S \vec{r} . \hat{n}\, dS$ over the surface S of the unit cube bounded by the coordinate planes and the planes $x = 1$, $y = 1$, $z = 1$

Sol. For any surface S, the projection of that surface on the plane xy (See Appendix A) will be $\iint_S \vec{A} . \hat{n}\, dS = \iint_R \vec{A} . \hat{n} \dfrac{dxdy}{|\hat{n} . \hat{k}|}$ where R is the region on the plane xy. Similarly, on the other planes, yz and zx

will be $\iint_R \vec{A}\,.\,\hat{n}\,\frac{dydz}{|\hat{n}.\,\hat{i}|}$ and $\iint_R \vec{A}\,.\hat{n}\,\frac{dzdx}{|\hat{n}.\hat{j}|}$ From Fig. 1.1W, over the surface $ABEF$, unit normal $\hat{n}=\hat{i}$ and $x=1$

So, $\vec{r}\,.\,\hat{n}=(\hat{i}x+\hat{j}y+\hat{k}z)\,.\,\hat{i}=x=1$

$$\therefore \quad \iint_{ABEF} \vec{r}.\,\hat{n}\,dS=\int_0^1\int_0^1\frac{dydz}{|\hat{n}.\,\hat{i}|}=\int_0^1\int_0^1 dydz=1$$

Over the surface $OCDG$, unit normal

$$\hat{n}=-\hat{i} \quad \text{and} \quad x=0$$

So, $\vec{r}.\,\hat{n}=-x=0$

$$\therefore \quad \iint_{OCDG} \vec{r}.\hat{n}\,dS=\int_0^1\int_0^2 0\,dydz=0$$

Over the surface $BCDE$, unit normal

$$\hat{n}=\hat{j} \quad \text{and} \quad y=1$$

So, $\vec{r}.\,\hat{n}=(\hat{i}x+\hat{j}y+\hat{k}z)\,.\,\hat{j}=y=1$

$$\therefore \quad \iint_{BCDE} \vec{r}.\,\hat{n}\,dS=\int_0^1\int_0^1\frac{dxdz}{|\hat{n}.\hat{j}|}=\int_0^1\int_0^1 dxdz=1$$

Over the surface $AOGF$, unit normal

$$\hat{n}=-\hat{j} \quad \text{and} \quad y=0$$

So, $\vec{r}.\,\hat{n}=-y=0$

$$\therefore \quad \iint_{AOGF} \vec{r}.\,\hat{n}\,dS=\int_0^1\int_0^1 0\,dxdz=0$$

Over the surface $EDGF$, unit normal

$$\hat{n}=\hat{k} \quad \text{and} \quad z=1$$

So, $\vec{r}.\,\hat{n}=(\hat{i}x+\hat{j}y+\hat{k}z)\,.\,\hat{k}=z=1$

$$\therefore \quad \iint_{EDGF} \vec{r}.\,\hat{n}\,dS=\int_0^1\int_0^1 1\,\frac{dxdy}{|\hat{n}.\hat{k}|}=\int_0^1\int_0^1 dxdy=1$$

Over the surface $OABC$, unit normal

$$\hat{n}=-\hat{k} \quad \text{and} \quad z=0$$

So, $\vec{r}.\,\hat{n}=(\hat{i}x+\hat{j}y+\hat{k}z)\,.\,(-\hat{k})=-z=0$

$$\iint_{OABC} \vec{r}.\,\hat{n}\,dS=\int_0^1\int_0^1 0\,dxdy=0$$

So, $\iint_S \vec{r}.\,\hat{n}\,dS=1+0+1+0+1+0=3$

[Note: By applying divergence theorem it can be easily shown that

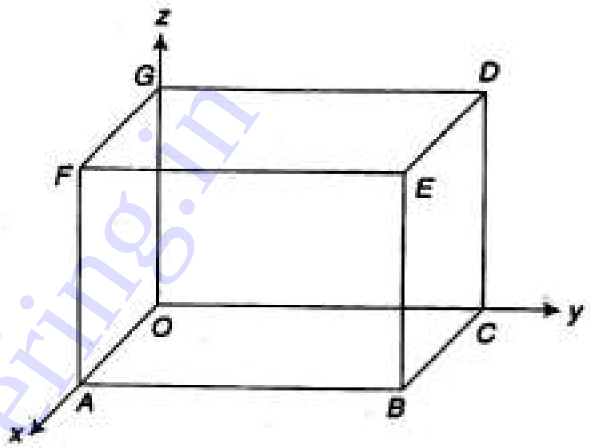


Fig. 1.1W Calculation of surface area of a unit cube bounded by coordinate planes and planes $x=1$, $y=1$ and $z=1$.

$$\iint_S \vec{r}\cdot\hat{n}\, dS = \int_V (\vec{\nabla}\cdot\vec{r})\, dV = \int_0^1\int_0^1\int_0^1 3\, dx dy dz = \left[3xyz\right]_{0,0,0}^{1,1,1} = 3$$

Example 1.52 Evaluate $\iiint_V (2x+y)\, dV$, where V is the enclosed volume bounded by the cylinder $z = 4 - x^2$ and the planes $x = 0$, $y = 0$, $y = 2$ and $z = 0$

Sol. Here $\iiint_V (2x+y)\, dV = \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{4-x^2} (2x+y)\, dx\, dy\, dz$

$$= \int_{x=0}^{2}\int_{y=0}^{2}\left[\int_{z=0}^{4-x^2} (2x+y)\, dz\right] dxdy = \int_{x=0}^{2}\int_{y=0}^{2} [2x(4-x^2) + y(4-x^2)]\, dxdy$$

$$= \int_{x=0}^{2}\left[2x(4-x^2)\,y + \frac{y^2}{2}(4-x^2)\right]_{y=0}^{2} dx = \int_{x=0}^{2} (16x - 4x^3 + 8 - 2x^2)\, dx$$

$$= 8x^2 - x^4 + 8x - \frac{2x^3}{3}\Big|_0^2 = \frac{80}{3}$$

Example 1.53 If $\vec{H} = \vec{\nabla}\times A$, prove that $\iint_S \vec{H}\cdot\hat{n}\, dS = 0$ for any closed surface S.

Sol. Here, $\vec{H} = \vec{\nabla}\times\vec{A}$

Now from divergence theorem, $\iint_S \vec{H}\cdot\hat{n}\, dS = \iiint_V (\vec{\nabla}\cdot\vec{H})dV$

But we have $\vec{\nabla}\cdot\vec{H} = \vec{\nabla}\cdot\vec{\nabla}\times\vec{A} = 0$ [since divergence of any curl is zero]

$\therefore \quad \iint_S \vec{H}\cdot\hat{n}\, dS = 0$

Example 1.54 If $\hat{n}$ is the unit outward drawn normal to any closed surface of area S, show that $\iiint$ div $\hat{n}\, dV = S$.

Sol. From divergence theorem $\iiint_V \text{div}\,\hat{n}\, dV = \iint_S \hat{n}\cdot\hat{n}\, dS = \iint_S dS = S$

Example 1.55 If S is any closed surface enclosing a volume V and $\vec{A} = ax\,\hat{i} + by\,\hat{j} + cz\,\hat{k}$, prove that $\iint_S \vec{A}\cdot\hat{n}\, dS = (a+b+c)\,V$.

Sol. Applying divergence theorem

$$\iint_S \vec{A}\cdot\hat{n}\, dS = \iiint_V (\vec{\nabla}\cdot\vec{A})\, dV$$

Here $\quad \vec{\nabla}\cdot\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(ax\,\hat{i} + by\,\hat{j} + cz\,\hat{k})$

$$= a + b + c$$

So $\quad \iiint_V (\vec{\nabla}\cdot\vec{A})\, dV = \iiint_V (a+b+c)\, dv = (a+b+c)\, V$

Example 1.56 Evaluate $\iint_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, dS$, where $\vec{A} = (y - z + 2)\,\hat{i} + (yz + 4)\,\hat{j} - xz\,\hat{k}$ and S is the surface of the cube $x = 0, y = 0, z = 0, x = 2, y = 2, z = 2$ above the xy plane.

Sol. Here $\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y - z + 2 & yz+4 & -xz \end{vmatrix} = \hat{i}\,(0 - y) + \hat{j}\,(-1 + z) + \hat{k}\,(0 - 1) = -y\,\hat{i} + (z - 1)\,\hat{j} - \hat{k}$

Now $\iint_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, dS = \iint_R (\vec{\nabla} \times \vec{A}) \cdot \hat{n} \dfrac{dxdy}{|\hat{n} \cdot \hat{k}|}$

above the xy plane $\hat{n} = \hat{k}$, so $\iint_R ((\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, dS = \iint_R (\vec{\nabla} \times \vec{A}) \cdot \hat{k} \dfrac{dxdy}{|\hat{k} \cdot \hat{k}|}$

Here $(\vec{\nabla} \times \vec{A}) \cdot \hat{k} = -1$, so $\iint_R (\vec{\nabla} \times \vec{A}) \cdot \hat{k}\, dx\, dy = -\int_0^2 \int_0^2 dxdy = -4$

Example 1.57 Evaluate $\int \vec{A} \cdot \hat{n}\, dS$ where $\vec{A} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$ and S is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$

Sol. By applying divergence theorem

$$\iint_S \vec{A}.\,\hat{n}\, dS = \iiint_V (\vec{\nabla} \cdot \vec{A})\, dV = \iiint_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right] dV$$

$$= \iiint_V (4z - y)\, dx\, dy\, dz = \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} (4z - y)\, dx\, dy\, dz$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} 2z^2 - yz|_0^1\, dxdy = \int_{x=0}^{1} \int_{y=0}^{1} (2 - y)\, dxdy = \int_{x=0}^{1} \left(2y - \frac{y^2}{2}\right)\Big|_0^1 dx.$$

$$= \int_0^1 \left(2 - \frac{1}{2}\right) dx = \int_0^1 \frac{3}{2}\, dx = \frac{3}{2}$$

Example 1.58 Evaluate $\iint_S r \cdot \hat{n}\, dS$ over the surface of a sphere of radius a with centre at (0, 0, 0).

Sol. By applying divergence theorem

$$\iint_S \vec{r} \cdot \hat{n}\, dS = \iiint_V (\vec{\nabla} \cdot \vec{r})\, dV = \iiint_V 3\, dV = 3V$$

The volume of the surface of a sphere of radius a is

$$V = \frac{4}{3}\pi a^3 \quad \text{So} \iint_S \vec{r} \cdot \hat{n}\, dS = 3 \times \frac{4}{3}\pi a^3 = 4\pi a^3$$

Example 1.59 Verify Stoke's theorem for $\vec{A} = (x + 3yz)\,\hat{j} + xy\,\hat{k}$ for the square surface as shown in Fig. 1.2W.

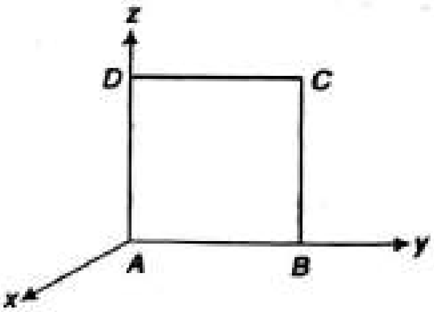


Fig. 1.2W Verification of Stoke's theorem.

Sol. Here $$\vec{\nabla}\times\vec{A}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 0 & x+3yz & xy\end{vmatrix}$$

$$=\hat{i}\,(x-3y)+\hat{j}\,(-y)+\hat{k}\,(1)$$

Since $x=0$, for this surface,

$$\int_S(\vec{\nabla}\times\vec{A})\cdot\hat{n}\,dS=\int_0^1\int_0^1[\hat{i}\,(x-3y)-\hat{j}\,y+\hat{k}]\cdot\hat{i}\,\frac{dydz}{|\hat{i}\cdot\hat{i}|}$$

$$=\int_0^1\int_0^1-3y\,dy\,dz\quad[\text{Here, } x=0]$$

$$=-\frac{3}{2}.$$

Now $$\int\vec{A}\cdot d\vec{r}=\int_{AB}\vec{A}\cdot d\vec{r}+\int_{BC}\vec{A}\cdot d\vec{r}+\int_{CD}\vec{A}\cdot d\vec{r}+\int_{DA}\vec{A}\cdot d\vec{r}$$

For the first part $\int_{AB}\vec{A}\cdot d\vec{r}=\int\vec{A}\cdot\hat{j}\,dy=\int 0\,dy=0$ [Here, $z=0$ $x=0$]

For the second part $\int_{BC}\vec{A}\cdot d\vec{r}=\int\vec{A}\cdot\hat{k}\,dz=\int 0\,dz=0$ [Here, $x=0$]

For the third part $\int_{CD}\vec{A}\cdot d\vec{r}=\int\vec{A}\cdot\hat{j}\,dy=\int_1^0 3y\,dy$ [Here, $z=1$]

$$=-\frac{3}{2}$$

For the fourth part $\int_{DA}\vec{A}\cdot d\vec{r}=\int\vec{A}\cdot\hat{k}dz=\int 0\,dz=0$

So, total $$\int\vec{A}\cdot\vec{dr}=0+0-\frac{3}{2}+0=-\frac{3}{2}$$

So, Stoke's theorem is verified.

Review Exercises

Part 1: Multiple Choice Questions

1. If $\hat{n}$ is the unit vector in the direction $\vec{A}$ then [WBUT 2004]
   (a) $\hat{n}=\frac{\vec{A}}{|\vec{A}|}$ (b) $\hat{n}=\vec{A}|\vec{A}|$ (c) $\hat{n}=\frac{|\vec{A}|}{\vec{A}}$ (d) None of these
2. Two vectors $\vec{A}$ and $\vec{B}$ are parallel when [WBUT 2004]
   (a) $\vec{A}\times\vec{B}=0$ (b) $\vec{A}\cdot\vec{B}=0$ (c) $\vec{A}\cdot\vec{B}=1$ (d) None of these
3. The angle between $\hat{i}$ and $(2\hat{i}+\hat{j})$ is [WBUT 2005]
   (a) $\cos^{-1}\frac{2}{5}$ (b) $\cos^{-1}\frac{2}{\sqrt{5}}$ (c) $\cos^{-1}\frac{2}{3}$ (d) None of these

4. Condition of coplanarity of three vectors $(\vec{\alpha}, \vec{\beta}, \vec{\gamma})$ is [WBUT 2006]
   (a) $\vec{\alpha}\cdot(\vec{\beta}+\vec{\gamma})=0$ (b) $\vec{\alpha}\cdot(\vec{\beta}\times\vec{\gamma})$ (c) $\vec{\alpha}\times(\vec{\beta}\times\vec{\gamma})$ (d) $\vec{\alpha}\cdot(\vec{\beta}-\vec{\gamma})=0$
5. The unit vector along the direction $2\hat{i}+3\hat{j}+4\hat{k}$ is
   (a) $\frac{1}{\sqrt{19}}(2\hat{i}+3\hat{j}+4\hat{k})$ (b) $\frac{1}{\sqrt{29}}(2\hat{i}+3\hat{j}+4\hat{k})$
   (c) $\frac{1}{\sqrt{29}}(2\hat{i}-3\hat{j}+2\hat{k})$ (d) $\frac{1}{\sqrt{19}}(2\hat{i}-3\hat{j}+4\hat{k})$
6. When the magnitude of $\vec{A}$ is constant which one of the following is true? [WBUT 2008]
   (a) $\frac{d\vec{A}}{dt}=0$ (b) $\vec{A}\cdot\frac{d\vec{A}}{dt}=0$ (c) $\vec{A}\times\frac{d\vec{A}}{dt}=0$ (d) $\left|\frac{d\vec{A}}{dt}\right|=0$
7. The angle between $\vec{\nabla}\varphi$ and the surface $\varphi=$ constant is [WBUT 2006]
   (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) π (d) zero
8. For arbitrary scalar and vector fields φ and $\vec{A}$, which of the following is always correct? [WBUT 2008]
   (a) $\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=0$ and $\vec{\nabla}\times(\vec{\nabla}\varphi)=0$ (b) $\vec{\nabla}(\vec{\nabla}\cdot\vec{A})=0$ and $\vec{\nabla}\cdot(\vec{\nabla}\varphi)=0$
   (c) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=0$ and $\vec{\nabla}\times(\vec{\nabla}\varphi)=0$ (d) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=0$ and $\vec{\nabla}\cdot(\vec{\nabla}\varphi)=0$
9. If $\vec{A}$ and $\vec{B}$ are irrotational then $\vec{A}\times\vec{B}$ is
   (a) solenoidal (b) irrotational (c) rotational (d) None of these
10. $\vec{\nabla}\cdot\vec{r}$ is equal to [WBUT 2004]
   (a) 2 (b) zero (c) 1 (d) 3
11. A fluid with velocity $\vec{v}$ is said to be incompressible when
   (a) $\vec{\nabla}\cdot\vec{v}=0$ (b) $\vec{\nabla}\times\vec{v}=0$ (c) $\vec{\nabla}\times(\vec{\nabla}\times\vec{v})=0$ (d) None of these
12. The value of $\oint_S \vec{r}\cdot\hat{n}\,dV$ is
   (a) V (b) $2V$ (c) zero (d) $3V$
13. A solenoidal field is one for which
   (a) grad $\vec{A}=0$ (b) div $\vec{A}=0$ (c) curl $\vec{A}=0$ (d) None of these
14. The value of a for which $\vec{A}=\hat{i}\,2ax+\hat{j}\,2y+\hat{k}\,4z$ is solenoidal is equal to
   (a) 2 (b) 3 (c) -3 (d) 1
15. For conservative field, a vector field $\vec{A}$ can be written as
   (a) $\vec{A}=\nabla^2\psi$ (b) $\vec{A}=\vec{\nabla}\cdot\vec{\nabla}\,\psi$ (c) $\vec{A}=-\vec{\nabla}\,\psi$ (d) None of these
16. For irrotational vector field $\vec{A}$
   (a) $\vec{\nabla}\times\vec{A}=0$ (b) $\vec{\nabla}\cdot\vec{A}=0$ (c) grad $\vec{A}=0$ (d) None of these
17. The values of $\vec{\nabla}\cdot\vec{\nabla}\times\vec{A}$ is
   (a) 1 (b) 2 (c) zero (d) -1

18. Gradient of ψ represents the rate of change which is
    (a) minimum (b) maximum
    (c) neither maximum nor minimum (d) None of these

[**Ans.** 1 (a), 2 (a), 3(b), 4 (b), 5 (b), 6(b), 7 (a), 8 (c), 9 (a), 10 (d), 11 (a), 12 (d), 13 (b), 14 (c), 15 (c), 16 (a), 17 (c), 18 (b)]

Short Questions with Answers

1. Define scalar field and vector field.

Ans. See section 1.7

2. What is solenoidal vector?

Ans. A vector field is said to be solenoidal if the divergence of that vector field is zero. So for solenoidal vector field ($\vec{A}$)

$$\vec{\nabla}\cdot\vec{A} = 0$$

3. What is irrotational or lamellar vector field?

Ans. A vector field is said to be irrotational if the curl of that vector field is zero. So for irrotational or lamellar vector field ($\vec{A}$)

$$\vec{\nabla}\times\vec{A} = 0$$

4. Define conservative field.

Ans. If the line integral of the vector field depends on the initial and final points but is independent of the path of the integral then the vector field is called a conservative field.

For conservative field, $\oint \vec{A}\cdot\vec{dl} = 0$ then $\vec{A}$ will be the gradient of a scalar function, i.e., $\vec{A} = \vec{\nabla}\psi$.

5. Prove that $\int_V \vec{\nabla}\psi\, dV = \int_S \psi\hat{n}\, dS$

Ans. Let $\vec{P} = \psi\vec{c}$ where $\vec{c}$ is a constant vector.

Then from divergence theorem $\int_V \vec{\nabla}\cdot(\psi\vec{c})\, dV = \int_S \psi\vec{c}\cdot\hat{n}\, dS$

Again we know that $\vec{\nabla}\cdot(\psi\vec{c}) = \vec{\nabla}\psi\cdot\vec{c} = \vec{c}\cdot\vec{\nabla}\psi$ and $\psi\vec{c}\cdot\hat{n} = \vec{c}\cdot(\psi\hat{n})$

So, $\int_V \vec{c}\cdot\vec{\nabla}\psi\, dV = \int_S \vec{c}\cdot(\psi\hat{n})\, dS$

Now taking $\vec{c}$ outside of the integration

$$\vec{c}\cdot\int_V \vec{\nabla}\psi\, dV = \vec{c}\cdot\int_S \psi\hat{n}\, dS$$

So, $\int_V \vec{\nabla}\psi\, dV = \int_S \psi\hat{n}\, dS$

6. Define line integral.

Ans. The integral of a point function along a curve is called line integral.

Let $\vec{r} = \vec{r}(x, y, z)$ be the equation of a curve. If ψ and $\vec{A}$ are scalar and vector fields respectively, and $\vec{dr}$ is the displacement then the integral may be $\int_C \psi\, dr$, $\int \vec{A}\cdot\vec{dr}$ and $\int \vec{A}\times\vec{dr}$

Each integral is called the line integral along the curve C.

7. What is gradient?

Ans. Gradient is the maximum variation of a scalar considering all directions. It is the maximum rate of growth of scalar ψ. Gradient is a vector that represents both the magnitude and the direction of the maximum space rate of increase of a scalar.

8. Define source and sink.

Ans. If the divergence of a vector field is positive, it indicates a net outward flow from the point. The given point is a 'source point'. But if divergence of a vector field is negative, it indicates a net flow towards the point. The given point is a 'sink point'.

9. State Stoke's theorem.

Ans. See Section 1.13.2

10. What is the physical meaning of divergence of a vector?

Ans. The physical meaning of the divergence of a vector is the limiting value of the net outward flow to some physical quantity like a fluid or electric flux through the surface area of unit volume as the volume tends to approach zero.

Part 2: Descriptive Questions

1. What are scalar and vector fields? Give examples.
2. (a) Define vector field. Give an example.
   (b) What do you mean by a scalar field? If $\psi(x, y, z) = 3x^2y - y^3z^2$, find $\vec{\nabla}\psi$ at the point $(1, -2, 1)$. **[WBUT 2004]**
3. Define divergence of a vector point function. What is its physical significance?
4. Explain the terms surface integral and volume integral.
5. Show that (i) curl grad $v = 0$ (ii) div curl $\vec{A} = 0$.
6. Define curl of a vector point function. What is its physical significance?
7. State Gauss' divergence theorem. Explain the physical significance of this theorem.
8. Show that $\vec{\nabla}\psi$ is perpendicular to the surface over which ψ is constant.
9. State Stoke's theorem. Define line integral of a vector.
10. If $\vec{A} = \vec{\nabla} \times \vec{B}$, show that $\iint_S \vec{A} \cdot \hat{n}\, dS = 0$ for any closed surface 'S'.
11. Show that $\int_C \varphi \vec{\nabla}\psi \cdot \vec{dr} + \int_C \psi \vec{\nabla}\varphi \cdot \vec{dr} = 0$; φ and ψ are scalar fields.
12. If a rigid body rotates about an axis passing through the origin with angular velocity $\vec{\omega}$ and with linear velocity $\vec{v} = \vec{\omega} \times \vec{r}$, then prove that $\vec{\omega} = \frac{1}{2}(\vec{\nabla} \times \vec{v})$.
13. Prove that $\nabla^2 f(r) = \dfrac{d^2 f(r)}{dr^2} + \dfrac{2}{r}\dfrac{df(r)}{dr}$.
14. Prove that $\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$. **[WBUT 2002]**
15. If the vector functions $\vec{F}$ and $\vec{G}$ are irrotational, show that $\vec{F} \times \vec{G}$ is solenoidal. **[WBUT 2006]**
16. The gradient of a scalar quantity is a vector quantity. Explain. **[WBUT 2002]**
17. Show that $\nabla^2(\log_e r) = \dfrac{1}{r^2}$. **[WBUT 2007]**

18. Given $\vec{F} = f(r)\,\vec{r}$, show that $\vec{\nabla} \times \vec{F} = 0$ and hence show that $\oint_C \vec{F}\cdot d\vec{r} = 0$ where C is a simple closed curve. [WBUT 2008]

19. Show that $\oint_S \vec{B}\cdot d\vec{S} = 0$ where $\vec{B}$ is the magnetic field and S is a closed surface. State the theorem that you have used. [WBUT 2006]

Part 3: Numerical Problems

1. Find the directional derivative of $\psi(x, y, z) = xy^2 z + 4x^2 z$ at $(-1, 1, 2)$ in the direction $(2\hat{i} + \hat{j} - \hat{k})$. $\left[\text{Ans. } -\dfrac{38}{3}\right]$
2. Find the torque about the point O $(3, -1, 3)$ of a force $\vec{F}$ $(4, 2, 1)$ passing through the point A $(5, 2, 4)$. [Ans. $\hat{i} + 2\hat{j} - 8\hat{k}$]
3. A fluid motion is given by $\vec{V} = (y+z)\,\hat{i} + (z+x)\hat{j} + (x+y)\,\hat{k}$. Show that the motion is irrotational.
4. Find the constant a so that $\vec{V}$ is a conservative vector field where
$$\vec{V} = (axy - z^2)\,\hat{i} + (a-z)\,x^2\,\hat{j} + (1-a)\,az^2\,\hat{k}.$$ [Ans. $a = 4$]
5. Find the work done in moving an object in the field $\vec{F} = (2xy + z^3)\,\hat{i} + x^2\,\hat{j} + 3xz^2\,\hat{k}$ from the point $(1, -2, 1)$ to $(3, 1, 4)$ independent of the path. [WBUT 2007]
6. Find the work done to move an object along a vector $\vec{r} = 3\hat{i} + 2\hat{j} - 5\hat{k}$ if the applied force is $\vec{F} = 2\hat{i} - \hat{j} - \hat{k}$. [WBUT 2005]
7. Find the unit vector perpendicular to $x^2 + y^2 - z^2 = 100$ at the point $(1, 2, 3)$. $\left[\text{Ans. } \dfrac{\pm\,\hat{i} + 2\hat{j} - 3\hat{k}}{\sqrt{14}}\right]$ [WBUT 2007]
8. Evaluate $\iint_S \vec{F}\cdot\hat{n}\,dS$, where $\vec{F} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$ and S is the surface of the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$. [Ans. -2] [WBUT 2002]
9. Evaluate $\iint_R \sqrt{x^2 + y^2}\,dx\,dy$ over the region R in the xy plane bounded by $x^2 + y^2 = 36$. [Ans. $144\,\pi$]
10. Show that the force field $\vec{F}$ defined by $\vec{F} = (y^2z^3 - 6\,xz^2)\,\hat{i} + 2xyz^3\,\hat{j} + (3xy^2z^2 - 6x^2z)\,\hat{k}$ is conservative.
11. For what value of a is the vector $\vec{A} = (x + 3y)\,\hat{i} + (y - 2z)\,\hat{j} + (x + az)\,\hat{k}$ solenoidal? [Ans. $a = -2$]
12. Evaluate $\iint_S \vec{r}\cdot\hat{n}\,dS$. [Ans. $3V$]
13. Find the unit vector perpendicular to $x^2 + y^2 - z^2 = 100$ at the point $(1, 2, 3)$. $\left[\text{Ans. } \dfrac{1}{\sqrt{14}}\hat{i} + \dfrac{2}{\sqrt{14}}\hat{j} - \dfrac{3}{\sqrt{14}}\hat{k}\right]$ [WBUT 2007]
14. If $\psi(x, y, z) = 3xy^2 - 5x^2z + 2z^2$ find $\nabla^2\psi$. [Ans. $6x - 10z + 4$]
15. Evaluate $\iint_S \vec{A}\cdot\hat{n}\,dS$, where $\vec{A} = (2x + 3z)\,\hat{i} - (xz + y)\,\hat{j} + (y^2 + 2z)\,\hat{k}$ and S is the surface of the sphere having centre at $(3, -1, 2)$ and radius 3. [Ans. $108\,\pi$]

CHAPTER

2

Electrostatics

2.1 INTRODUCTION

Electrostatics is an important branch of physics which deals with electric charge at rest. The electric force between two electrons is the same as the electric force between two protons placed at the same distance. The amount of charge on an electron is the same as that on a proton. The charge on a proton is positive and that on an electron is negative. The net charge on the electron–proton system is zero. Stationary charges produce an electric field that is constant with time, hence the term electrostatics. Both electrostatics and magnetostatics can be explained by using vector calculus.

2.2 QUANTIZATION OF CHARGE

In 1911, Millikan successfully showed that charges in tiny oil drops are exact multiples of elementary charges. The magnitude of charge on a proton or an electron ($e = 1.6 \times 10^{-19} c$) is called elementary charge. Quantization of charge means that all observable charges are integral multiple of elementary charge $e(= 1.6 \times 10^{-19} c)$.

2.3 CONSERVATION OF CHARGE

The law of conservation of charge states that for an isolated system, the net charge always remains constant. In β-decay a neutron converts itself into a proton and creates an electron. The net charge remains zero before and after the decay. In nuclear fission, the total charge is always conserved.

$$_0n^1 + {}_{92}U^{235} \rightarrow {}_{56}Ba^{141} + {}_{36}K^{92} + 3\,{}_0n^1$$

Before collision total charge = + 92 e and total charge after collision = (56 + 36) e = 92 e. So total charge is conserved.

2.4 COULOMB'S LAW

Statement

The force between two small charged bodies separated by a distance in air is

(a) directly proportional to the magnitude of each charge

(b) inversely proportional to the square of the distance between them
(c) directed along the line joining the charges

The distance between charges must be large compared to their linear dimension.

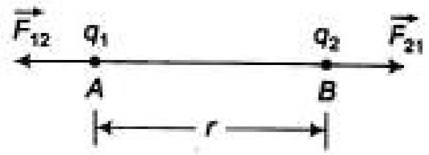


Fig. 2.1 Force between two charges.

In mathematical form, if q_1 and q_2 be two like charges and r is the distance between them [Fig. 2.1] then the force exerted on q_1 due to the charge q_2 is

$$\overrightarrow{F_{12}} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{r}_{21} \qquad \text{...(2.1)}$$

Here, $\hat{r}_{21}$ is unit vector pointing from q_2 to q_1 and ε_0 is the permittivity of free space. Experimentally, measured value of $\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9$ Nm^2/C^2

Similarly, the force exerted on q_2 due to the charge q_1 is

$$\overrightarrow{F_{21}} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{r}_{12} \qquad \text{...(2.2)}$$

Here, $\hat{r}_{12}$ is a unit vector pointing from q_1 to q_2. So $\vec{F}_{12} = -\vec{F}_{21}$. For two unlike charges, $\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{r}_{12}$ and $\overrightarrow{F_{21}} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{r}_{21}$ are attractive

***Relative permittivity* (ε_r)** The relative permittivity or dielectric constant of a medium is defined as the ratio of the force between two charges placed at a distance in vacuum (or air) to the force between the same charges placed at the same separation in that medium.

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0}\left[F_{\text{vacuum}} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2} \text{ and } F_{\text{medium}} = \frac{1}{4\pi\varepsilon}\frac{q_1 q_2}{r^2}\right] \qquad \text{...(2.3)}$$

2.5 PRINCIPLE OF SUPERPOSITION

The principle states that when a number of charges are interacting, the total force on a given charge is the vector sum of the individual forces exerted by all other charges on the given charge.

If $q_1, q_2, q_3, q_4, \ldots$ are the charges situated at $A, B, C, D, \ldots$, as shown in Fig. 2.2.

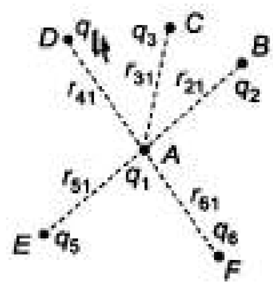


Fig. 2.2 Principle of superposition.

The total force on q_1 due to all other charges is

$$\vec{F}_1 = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{r_{21}^2}\hat{r}_{21} + \frac{q_1 q_3}{r_{31}^2}\hat{r}_{31} + \ldots + \ldots\right]$$
$$= \vec{F}_{12} + \vec{F}_{13} + \ldots$$

The total force on q_2 due to all other charges is

$$\vec{F}_2 = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_2 q_1}{r_{12}^2}\hat{r}_{12} + \frac{q_2 q_3}{r_{32}^2}\hat{r}_{32} + \ldots\right]$$

If there is a test charge q_0, then total force on the test charge q_0 due to all other charges is

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \ldots = \sum_{i=1}^{N} \vec{F}_i$$

$$= \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_0 q_i}{r_i^2} \hat{r}_i \qquad ...(2.4)$$

2.6 ELECTRIC FIELD

The electric field due to a charge is the space around the charge in which any other charge is acted upon by an electrostatic force.

If we have many charges $q_1, q_2, ... q_n$ at distances $r_1\ r_2 ... r_n$ respectively from a test charge q_0 then from the principle of superposition, the total force on q_0 is

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_0 q_i}{r_i^2} \hat{r}_i$$

$$= \frac{q_0}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{r_i^2} \hat{r}_i$$

The electric field intensity at the point is

$$\vec{E} = \frac{\vec{F}}{q_0} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{r_i^2} \hat{r}_i \qquad ...(2.5)$$

Thus, the electric field intensity at a point in the electric field is the force on a unit test charge placed at the point concerned.

2.7 CONTINUOUS CHARGE DISTRIBUTIONS

On a uniform charge body, there are three types of distribution of charge:

(i) Line charge distribution If q is the total charge over a conducting wire of length l and infinitesimally small thickness, then charge per unit length λ (line charge density) is

$$\lambda = \frac{q}{l} \text{ coulomb/m}$$

For non-uniform distribution of charge

$$\lambda = \underset{\Delta l \to 0}{\text{Lt}} \frac{\Delta q}{\Delta l} = \frac{dq}{dl}$$

where Δl is a small element of length which carries a charge Δq.

So, total charge over the whole length

$$q = \int_l \lambda\, dl \qquad ...(2.6)$$

(ii) Surface charge distribution If q is the charge uniformly distributed over the conducting surface S, then surface density of charge (charge per unit area) σ is

$$\sigma = \frac{q}{S} \text{ coulomb/m}^2$$

If Δq be the charge contained by a small element ΔS then surface charge density,

$$\sigma = \underset{\Delta S \to 0}{\text{Lt}} \frac{\Delta q}{\Delta S} = \frac{dq}{dS}$$

So, total charge on the surface $q = \int_S \sigma\, dS \qquad ...(2.7)$

(III) Volume charge distribution If q is the charge uniformly distributed over the volume V then the volume density of charge (charge per unit volume)

$$\rho = \frac{q}{V} \text{ coulomb/m}^3$$

If charge distribution is not uniform then

$$\rho = \underset{\Delta V \to 0}{\mathrm{Lt}} \frac{\Delta q}{\Delta V} = \frac{dq}{dV}$$

where ΔV is a small volume element which carries a charge Δq.

So, total charge over the whole volume

$$q = \int_V \rho\, dV \qquad \text{...(2.8)}$$

2.8 ELECTRIC POTENTIAL

The concept of potential is based on energy consideration. The electric potential at a point in an electric field near a charged conductor is defined as the amount of work done in bringing a unit positive charge from infinity to that point against the electrostatic force. A positively charged body always tends to move from higher potential to lower potential.

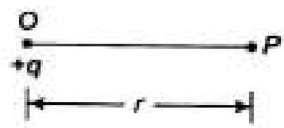


Fig. 2.3 Potential at point P due to charge $+q$ at O.

Potential at a point due to a point charge

The potential at P [Fig. 2.3] is given by

$$V = -\int_\infty^r \vec{E} \cdot \vec{dr}$$

$$= -\int_\infty^r \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} dr = \frac{q}{4\pi\varepsilon_0 r} \qquad \text{...(2.9)}$$

Electric field intensity as a gradient of potential

Let the electric field at a point $\vec{r}$ due to a charge distribution be $\vec{E}$ and electric potential at the same point be V. Suppose a test charge q_0 is displaced slightly from $\vec{r}$ to $\vec{r} + d\vec{r}$. Then the force on the test charge q_0 is

$\vec{F} = q_0\vec{E}$ and work done for small displacement $\vec{dr}$ is $dw = \vec{F} \cdot d\vec{r} = q_0 \vec{E} \cdot d\vec{r}$

The change in potential energy $= -d\omega = -q_0\vec{E} \cdot d\vec{r}$

So the change in potential $dV = -\vec{E} \cdot d\vec{r}$

Again $\qquad dV = \vec{\nabla} V \cdot d\vec{r} = -\vec{E} \cdot d\vec{r}$

or, $\qquad \vec{E} = -\vec{\nabla} V \qquad \text{...(2.10)}$

Hence, the electric field at a point is equal to the negative gradient of the electrostatic potential at the point.

2.9 ELECTRIC POTENTIAL ENERGY

We define electric potential energy of a system of point charges as the work required to assemble this system of charges by bringing them from infinite distances.

If two point charges q_1 and q_2 are separated by a distance r_{12} then potential energy of the system q_1 and q_2 is

$$U = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}}$$

If another charge q_3 is at a distance r_{13} from q_1 and distance r_{23} from q_2 then potential energy of the system $(q_1 + q_2 + q_3)$ is

$$U = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$

Generally, the potential energy for a system of n point charges is

$$U = \frac{1}{2} \times \frac{1}{4\pi\varepsilon_0} \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{q_i q_j}{r_{ij}} \quad ...(2.11)$$

In summation, each term is counted twice, so half factor is included here to avoid double counting each term for calculation of potential energy.

2.10 ELECTRIC FLUX

We know that any area element dS is a vector $\vec{dS}$. If $\hat{n}$ is the unit normal to the area element then

$$\vec{dS} = \hat{n}\, dS$$

The total number of lines of force passing through a surface placed in an electric field is known as electric flux (φ_E).

Consider a surface of area S inside an electric field $\vec{E}$ [Fig. 2.4]. The surface S is divided into a number of elementary areas $\vec{dS}$ (known as area vector). The component of the electric field along the area vector $\vec{dS}$ is given by

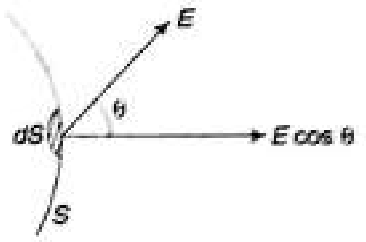


Fig. 2.4 Component of electric field $\vec{E}$ along $\vec{dS}$.

$$E_n = E\cos\theta$$

So, electric flux

$$d\varphi_E = E_n\, dS = E\cos\theta\, dS$$
$$= \vec{E}\cdot\vec{dS} = \vec{E}\cdot\hat{n}\, dS \quad ...(2.12)$$

The total electric flux through S is

$$\varphi_E = \int_S \vec{E}\cdot\hat{n}\, dS \quad ...(2.13)$$

2.11 SOLID ANGLE

In Fig. 2.5, the solid angle subtended by any surface dS at a point O, distance r away, is given by

$$d\omega = \frac{dS'}{r^2} = \frac{dS\cos\theta}{r^2}$$

where $dS' = dS\cos\theta$ is the projection of the surface dS.

For sphere $dS' = 4\pi r^2$ and $\theta = 0$, so solid angle subtended by the sphere at its center is

$$\omega = \int_S \frac{dS\cos\theta}{r^2} = 4\pi$$

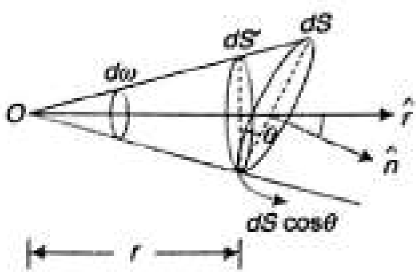


Fig. 2.5 Solid angle at point O.

2.12 GAUSS' LAW

Gauss' law states that the total electric flux through a closed surface in an electric field is equal to $\frac{1}{\varepsilon_0}$ times the total charge enclosed by the surface, where ε_0 is the permittivity of free space.

Mathematically, $\oint_S \vec{E} \cdot d\vec{S} = \frac{q}{\varepsilon_0}$ when S encloses q

$= 0$ when S does not enclose q ...(2.14)

Proof of Gauss' law

We consider a spherically symmetric closed surface. Suppose a charge q is placed at the center of a sphere a radius r and $\vec{E}$ is the electric field intensity at a point R on the surface [Fig. 2.6].

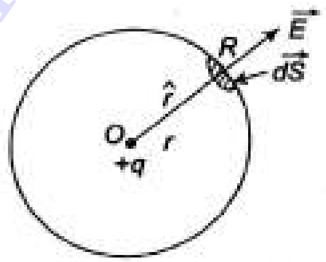


Fig. 2.6 Proof of Gauss' law.

From Coulomb's law,

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

The flux of electric field through $\vec{dS}$ is

$$d\varphi_E = \vec{E}\cdot\vec{dS} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}\cdot\vec{dS} = \frac{q}{4\pi\varepsilon_0 r^2}dS$$

The total flux of the electric field due to the internal charge q through the closed surface

$$\varphi_E = \int d\varphi_E = \frac{q}{4\pi\varepsilon_0}\int_s \frac{dS}{r^2} = \frac{q}{4\pi\varepsilon_0}\frac{4\pi r^2}{r^2} = \frac{q}{\varepsilon_0}$$

which is Gauss' law in electrostatics.

2.12.1 Differential Form of Gauss' Law

The integral form of Gauss' law is

$$\oint_S \vec{E}\cdot\vec{dS} = \frac{q}{\varepsilon_0} \quad \text{...(2.15)}$$

If ρ be the volume charge density over a small volume element dV within the closed surface S, then

$$q = \int_V \rho\, dV \quad \text{...(2.16)}$$

Again from Gauss' divergence theorem

$$\oint_S \vec{E}\cdot\vec{dS} = \int_V (\vec{\nabla}\cdot\vec{E})\, dV \quad \text{...(2.17)}$$

Now from Eqs (2.17) and (2.16) into Eq. (2.15)

$$\int_V (\vec{\nabla}\cdot\vec{E})\, dV = \frac{1}{\varepsilon_0}\int_V \rho\, dV \quad \text{...(2.18)}$$

or

$$\int_V \left(\vec{\nabla}\cdot\vec{E} - \frac{\rho}{\varepsilon_0}\right) dV = 0$$

This is true for any arbitrary volume V.

$$\therefore \quad \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{...(2.19)}$$

This is the differential form of Gauss' law.

In vacuum, electric displacement vector $\vec{D} = \varepsilon_0 \vec{E}$

So
$$\vec{\nabla} \cdot \vec{D} = \rho \qquad ...(2.20)$$

This is also the differential form of Gauss' law in terms of electric displacement vector.

2.12.2 Coulomb's Law from Gauss' Law

Here we would like to deduce Coulomb's law from Gauss' law. Here, S is the spherical gaussian surface. In Fig. 2.7, $\vec{E}$ and $\vec{dS}$ on the gaussian surface are directed radially outward. So $\vec{E} \cdot \vec{dS} = E\,dS$.

Now from Gauss' law

$$\oint_S \vec{E} \cdot \vec{dS} = \frac{q}{\varepsilon_0}$$

or,
$$E \oint_S dS = E \times 4\pi r^2 = \frac{q}{\varepsilon_0} \qquad [E \text{ is constant}]$$

or,
$$E = \frac{q}{4\pi\varepsilon_0 r^2}$$

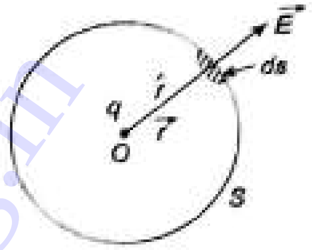


Fig. 2.7 Derivation of Coulomb's law from Gauss' law.

If another point charge q_0 be placed at the point at which $\vec{E}$ is calculated then the force on q_0 due to q is

$$\vec{F} = q_0 \vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q_0 q}{r^2} \hat{r} \qquad ...(2.21)$$

where $\hat{r}$ is the unit vector.

Equation (2.21) is Coulomb's law.

2.12.3 Application of Gauss' Law

For calculating electric field due to a charge distribution, Gauss' law provides the easiest way. Here, we consider some important applications of Gauss' law.

(i) Electric field due to uniformly charged sphere

Case I Field at a point outside the charged sphere

Let P be a point situated outside the charged sphere having charge q uniformly distributed throughout the volume of the sphere of radius R [Fig. 2.8]. In order to find the electric field intensity at P, a concentric sphere of radius r is drawn as gaussian surface, over which the electric field intensity is directed normal to every point of this surface. If $\vec{E}$ is the electric field intensity at the point P then the total flux through the gaussian surface is

$$\varphi_E = \oint_S \vec{E} \cdot \vec{dS} = E\,(4\pi r^2) = \frac{q}{\varepsilon_0}$$

or,
$$E = \frac{q}{4\pi\varepsilon_0 r^2}$$

For continuous charge distribution of density ρ within the sphere

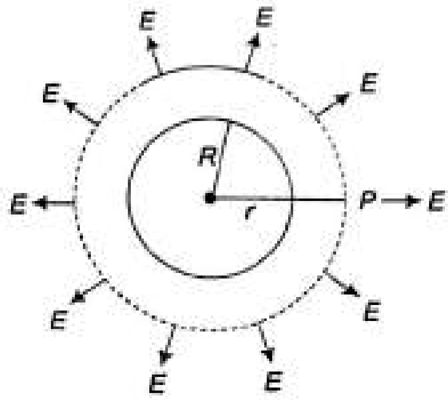


Fig. 2.8 Electric field outside of a charged sphere.

$$q = \frac{4}{3}\pi R^3 \rho$$

So,
$$E = \frac{1}{4\pi\varepsilon_0}\frac{\frac{4}{3}\pi R^3\rho}{r^2} = \frac{R^3\rho}{3\varepsilon_0 r^2} \quad ...(2.22)$$

Case II Field at a point on the surface of the sphere

For any point on the surface of the sphere $r = R$, from Eq. (2.22), we have

$$E = \frac{R^3\rho}{3\varepsilon_0 R^2} = \frac{R\rho}{3\varepsilon_0} \quad ...(2.23)$$

Case III Field at a point inside the charged sphere

We want to find the electric field at a point P inside the sphere at a distance r from the center [Fig. 2.9]. The total charge inside the gaussian surface of radius r

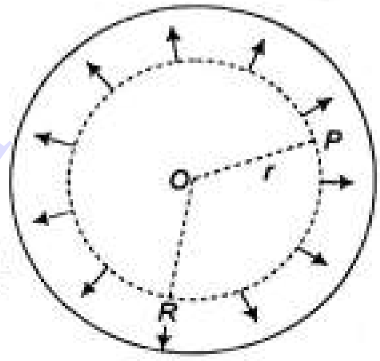


Fig. 2.9 Electric field inside a charged sphere.

$$q = \frac{4}{3}\pi r^3 \rho$$

The total electric flux over the gaussian surface is given by

$$\varphi_E = \oint_s \vec{E}\cdot d\vec{S} = E \times 4\pi r^2 = \frac{q}{\varepsilon_0} = \frac{4\pi r^3 \rho}{3\varepsilon_0}$$

or,
$$E = \frac{r\rho}{3\varepsilon_0} \quad ...(2.24)$$

The variation of electric field intensity in different cases as discussed is shown in Fig. 2.10.

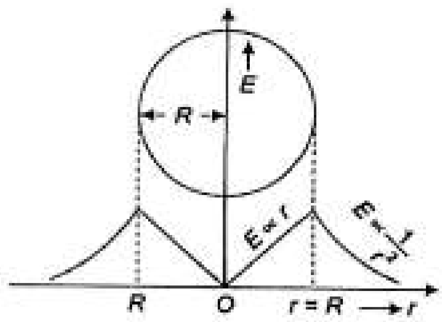


Fig. 2.10 Variation $\vec{E}$ with distance from the center of a charged sphere.

(ii) Electric field due to a charged spherical shell

Case I At a point outside the charged shell

Let P be a point outside the shell at a distance r [Fig. 2.11]. Here $r > R$

The total flux over the gaussian surface

$$\varphi_E = \oint_s \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

or,
$$E(4\pi r^2) = \frac{q}{\varepsilon_0}$$

or,
$$E = \frac{q}{4\pi\varepsilon_0 r^2} \quad ...(2.25)$$

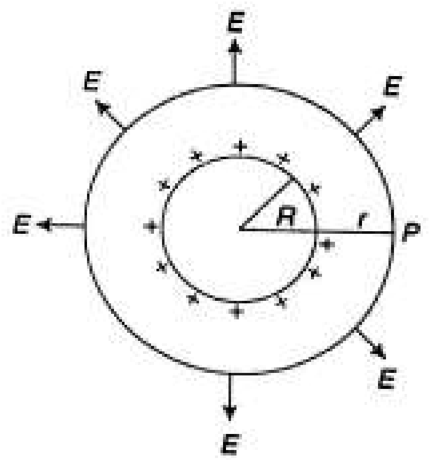


Fig. 2.11 Electric field due to a charged spherical shell.

Case II At a point on the surface of the charged shell

Here, $r = R$ and the sphere itself behaves as gaussian surface. The electric field intensity on the surface is then

$$E = \frac{q}{4\pi\varepsilon_0 R^2}$$

Again for spherical shell the surface density of charge

$$\sigma = \frac{q}{4\pi R^2}$$

$$\therefore \qquad E = \frac{\sigma}{\varepsilon_0} \qquad ...(2.26)$$

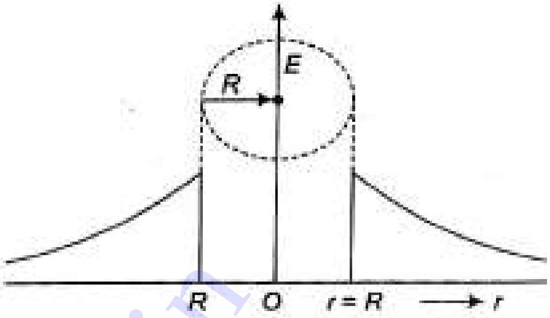


Fig. 2.12 Electric field at a point inside a spherical charged shell.

Case III At a point inside the shell

Here $r < R$, the total charge is situated on the surface of the shell of radius R and no charge is therefore enclosed by the gaussian surface, therefore,

$$\oint_s \vec{E} \cdot d\vec{S} = 0$$

$$\therefore \qquad E = 0 \qquad ...(2.27)$$

The variation of the electric field intensity with distance from the centre is shown in Fig. 2.12 with the help of Eqs (2.25), (2.26) and (2.27).

(III) Electric field intensity due to long uniformly charged cylinder

Case I At a point outside of the cylinder

Let P_1 be the point at a distance r from the axis of the cylinder of radius R [Fig. 2.13]. Imagine a cylindrical gaussian surface of length l through P_1. The field will have a cylindrical symmetry and the total flux over the gaussian surface

$$\varphi_E = \oint_s \vec{E} \cdot \vec{dS} = \frac{q}{\varepsilon_0} = \frac{\lambda l}{\varepsilon_0},$$

where λ is the line charge density.

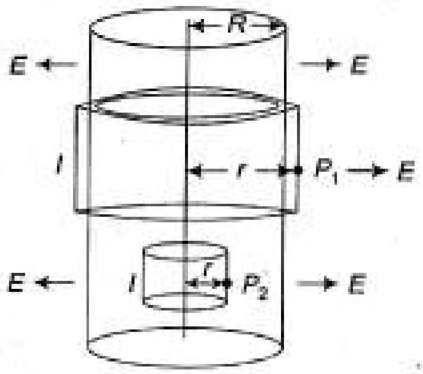


Fig. 2.13 Electric field at a point from the axis of a uniformly charged cylinder.

or, $$E \times 2\pi r l = \frac{\lambda l}{\varepsilon_0}$$

$$\therefore \qquad E = \frac{\lambda}{2\pi r \varepsilon_0} \qquad ...(2.28)$$

Case II At a point on the surface of the cylinder

Here $r = R$, the cylinder itself is the gaussian surface. The electric filed at the surface is obtained by replacing r by R in Eq. (2.28)

$$E = \frac{\lambda}{2\pi R \varepsilon_0} \qquad ...(2.29)$$

Case III At a point inside the cylinder

Let P_2 be the point at a distance r ($r < R$) from the axis of the cylinder [Fig. 2.13]. A coaxial cylinder of radius r and length l is constructed as gaussian surface such that the point P_2 lies on the curved surface of the cylinder.

The total charge enclosed by the gaussian surface

$$q' = \pi r^2 l \rho$$

[where, ρ is the volume charge density]

Again $(\pi R^2 l)\rho = \lambda l$

or, $$\rho = \frac{\lambda}{\pi R^2}.$$

From Gauss' law, the electric flux

$$\varphi_E = \oint_s \vec{E}\cdot\vec{dS} = \frac{q'}{\varepsilon_0} = \frac{\pi r^2 l \rho}{\varepsilon_0}$$

or, $$E \times (2\pi r l) = \frac{\pi r^2 l \rho}{\varepsilon_0}$$

or, $$E = \frac{r\rho}{2\varepsilon_0} = \frac{r}{2\varepsilon_0}\left(\frac{\lambda}{\pi R^2}\right) = \frac{\lambda r}{2\pi R^2 \varepsilon_0} \qquad \text{...(2.30)}$$

The variation of electric field intensity with distance r from the axis of a charged cylinder is shown in Fig. 2.14.

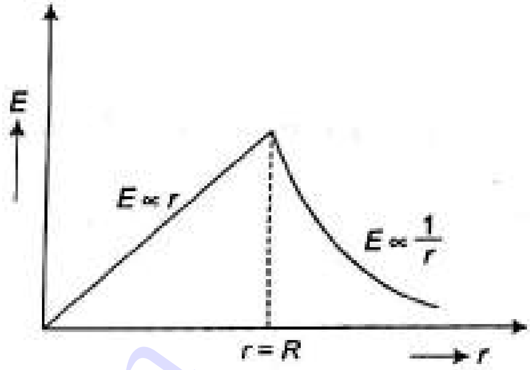


Fig. 2.14 Variation of $\vec{E}$ with distance r from the axis of a charged cylinder.

[Note: For a hollow charged cylinder, the charge inside the cylinder is zero and so the electric flux inside the cylinder, $\varphi_E = 0$ which gives $E = 0$.]

(iv) Electric field due to an infinite plane charge sheet

Here we consider an infinite thin plane charge sheet of positive charge having surface density of charge σ [Fig. 2.15]. To find the electric field at a point P_1, let us consider another point P_2 on the other side of the sheet so that two points P_1 and P_2 are equidistant from the sheet. We construct a cylindrical gaussian surface normally through the plane which extends equally on two sides of the plane.

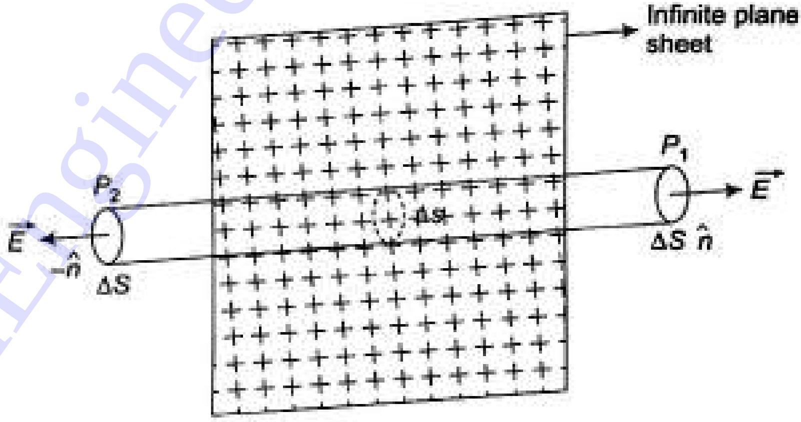


Fig. 2.15 Electric field due to an infinite charged sheet.

The flux of the electric field crossing through the gaussian surface

$$\varphi_E = E\,\Delta s + E\,\Delta S$$

$= 2E\ \Delta S$ [where ΔS is the area of cross section of each end face]

$$= \frac{\sigma \Delta S}{\varepsilon_0}$$

or, $$E = \frac{\sigma}{2\varepsilon_0} \qquad \text{...(2.31)}$$

We see that electric field is uniform and does not depend on the distance from the charge sheet.

(v) Electric field near a charged conducting surface

Here we consider a plane conducting sheet. All the charges of the conductor lie on the surface, so the electric field inside the conductor is zero. We have to find the electric field at a point P which is near but outside the conductor. To find the electric field, we construct a gaussian surface as follows [Fig. 2.16]. The total flux through the gaussian surface is

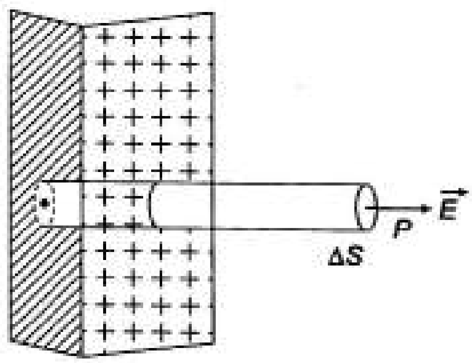


Fig. 2.16 Electric field at a point near a charged conducting surface.

$$\varphi_E = E\,\Delta S$$

and charge enclosed inside the closed surface is $\sigma\,\Delta S$. So from Gauss' law

$$E\,\Delta S = \frac{\sigma\,\Delta S}{\varepsilon_0}$$

or,
$$E = \frac{\sigma}{\varepsilon_0} \qquad \text{...(2.32)}$$

The electric field near a plane charged conductor is twice the electric field due to a non-conducting plane charge sheet. This is also known as *Coulomb's theorem*. The theorem states that the electric field at any point very close to the surface of a charged conductor is equal to charge density of the surface divided by free space permittivity.

2.13 POISSON'S AND LAPLACE'S EQUATIONS

The differential form of Gauss' law is

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{[From Eq. (2.19)]}$$

Again, the electric field ($\vec{E}$) at any point is equal to the negative gradient of the potential V

i.e.,
$$E = -\vec{\nabla}V \quad \text{[From Eq. (2.10)]}$$

Now combining these two equations, we have

$$\vec{\nabla}\cdot(-\vec{\nabla}V) = -\nabla^2 V = \frac{\rho}{\varepsilon_0}$$

or,
$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} \qquad \text{...(2.33)}$$

This is known as Poisson's equation.

Now in a charge-free region ($\rho = 0$), the Poisson's equation becomes

$$\nabla^2 V = 0 \qquad \text{...(2.34)}$$

This equation is known as Laplace's equation and is valid only in the charge-free region.

The laplacian operator ∇^2 is a scalar operator and its form in three coordinate systems are:

(a) Cartesian system (x, y, z): $\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

(b) Cylindrical coordinate system (ρ, ϕ, z):

$$\nabla^2 \equiv \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}$$

(c) Spherical polar coordinate system (r, θ, φ)

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

2.13.1 Application of Laplace's Equation (Effective 1D Problem)

(i) Potential between the plates of a parallel-plate capacitor

Let us consider a parallel-plate condenser (capacitor) having two plates, one at $z = 0$ and other at $z = d$ [Fig. 2.17]. The potential at the upper plate is V_A and potential at the lower plate is zero. So, the potential exists only along the z direction.

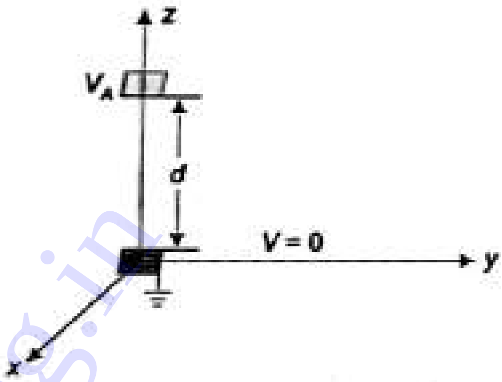


Fig. 2.17 Potential difference between the plates of a parallel-plate capacitor.

The Laplace's equation in the cartesian system is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Here $$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial y^2} = 0$$

Since the potential exists only along the z direction

So, $$\frac{\partial^2 V}{\partial z^2} = 0 \quad \text{or} \quad \frac{\partial V}{\partial z} = C \text{ (constant)}$$

or, $$V = Cz + D \quad (D \text{ is another constant}) \qquad \ldots(2.35)$$

Now applying boundary conditions, i.e.,

$$z = 0,\ V = 0$$

and $$z = d,\ V = V_A$$

The first condition gives $D = 0$

From the second condition at $z = d$, $V = V_A$

$$V_A = Cd$$

or $$C = \frac{V_A}{d}$$

So, from Eq. (2.35), $$V = \frac{V_A z}{d} \qquad \ldots(2.36)$$

Equation (2.36) is the solution of Laplace's equation, which gives the potential between the plates.

(ii) Potential of a coaxial cylindrical capacitor

Here, we consider a cylindrical capacitor of inner and outer radii a and b ($b > a$) as shown in Fig. 2.18. The potential of the inner cylinder is V_A and the potential of the outer cylinder is zero.

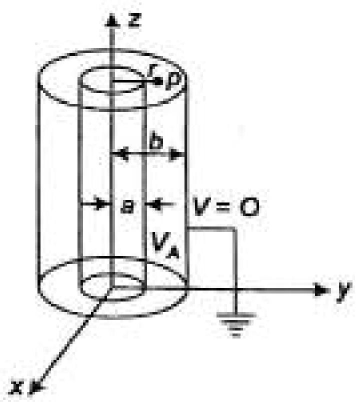


Fig. 2.18 Potential of a cylindrical capacitor.

Since the variation of potential exists only along the radial direction.

then $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0 \left[\because \frac{\partial^2 V}{\partial \phi^2} = \frac{\partial^2 V}{\partial z^2} = 0\right] \qquad \ldots(2.37)$$

Integrating Eq. (2.37) twice with respect to r, the potential at an arbitrary distance r is

$$V = C \ln r + D \qquad \ldots(2.38)$$

[where C and D are constants of integration]

Now applying boundary conditions, i.e.,

$$r = b, V = 0$$

and $$r = a, V = V_A$$

we have, from Eq. (2.38), $D = -C \ln b$ and $V_A = C \ln \frac{a}{b}$

so, $$C = \frac{V_A}{\ln \frac{a}{b}} \quad \text{and} \quad D = -V_A \frac{\ln b}{\ln \frac{a}{b}}$$

Now from Eq. (2.38), we have

$$V = \frac{V_A}{\ln \frac{a}{b}} \ln r - V_A \frac{\ln b}{\ln \frac{a}{b}}$$

$$= V_A \frac{\ln \frac{r}{b}}{\ln \frac{a}{b}} \qquad \ldots(2.39)$$

Equation (2.39) is the solution of Laplace's equation in cylindrical coordinate, which gives the potential inside a cylindrical capacitor.

(iii) Potential of a concentric spherical capacitor

Let us consider a spherical capacitor of inner and outer radii a and b ($b > a$) as shown in Fig. 2.19. The potential of the inner sphere is V_A and the outer sphere is zero.

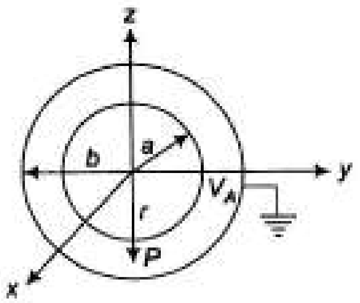


Fig. 2.19 Potential of concentric spherical capacitor.

Since the variation of potential exists only along the radial direction then from Laplace's equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0 \quad \left[\text{Here, } \frac{\partial V}{\partial \theta} = \frac{\partial^2 V}{\partial \varphi} = 0\right] \qquad \ldots(2.40)$$

Integrating Eq. (2.40) twice with respect to r, the potential at P will be

$$V = -\frac{C}{r} + D \qquad \ldots(2.41)$$

[where C and D are integrating constants]

Now applying boundary conditions, i.e.,

$$r = b, V = 0$$

and $$r = a, V = V_A$$

we have from Eq. (2.41) $D = \frac{C}{b}$

So, $$V = -\frac{C}{r} + \frac{C}{b} = C\left(\frac{1}{b} - \frac{1}{r}\right) \qquad \ldots(2.42)$$

and from second boundary condition ($r = a$, $V = V_A$), Eq. (2.41) gives $V_A = C\left(\frac{1}{b} - \frac{1}{a}\right)$

or, $$C = \frac{V_A}{\left(\frac{1}{b} - \frac{1}{a}\right)}$$

Now putting the values of c in Eq. (2.42)
we have

$$V = \frac{V_A}{\left(\frac{1}{b} - \frac{1}{a}\right)} \left(\frac{1}{b} - \frac{1}{r}\right) \qquad \ldots(2.43)$$

Equation (2.43) is the solution of Laplace's equation in spherical polar coordinate, which gives the potential inside a spherical capacitor.

Worked Out Problems

Example 2.1 Compare the electrostatics force and gravitational force between a proton and electron in a hydrogen atom. Given $e = 1.6 \times 10^{-19}$ C, $m_e = 9.1 \times 10^{-31}$ kg, $m_p = 1.7 \times 10^{-27}$ kg and $G = 6.67 \times 10^{-11}$ Nm^2 kg^{-2}

Sol. The electrostatic force between a proton and electron is

$$F_e = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} = 9 \times 10^9 \times \frac{(1.6 \times 10^{-19})^2}{r^2} \quad \left[\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9 \text{ Nm}^2/\text{C}^2\right]$$

The gravitational force between a proton and electron

$$F_g = G\frac{m_p m_e}{r^2} = 6.67 \times 10^{-11} \frac{1.7 \times 10^{-27} \times 9.1 \times 10^{-31}}{r^2}$$

or

$$\frac{F_e}{F_g} = \frac{9 \times 10^9}{6.67 \times 10^{-11}} \times \frac{(1.6 \times 10^{-19})^2}{9.1 \times 1.7 \times 10^{-58}} = 2.2 \times 10^{34}$$

So, electrostatics force between electron and proton is much greater than gravitational force.

Example 2.2 Two particles P and Q having charges 8.0×10^{-6} C and -2.0×10^{-6} C respectively are held fixed with a separation of 20 cm. Where should a third charged particle be placed so that it does not experience a net electric force?

Sol. Since the particles are charged with opposite sign, the point R where net electric force is zero, can't be between P and R.

Fig. 2.1W

Suppose $QR = x$ metre and charge on R is q.

The force at R due to P is $= \dfrac{8.0 \times 10^{-6} \times q}{4\pi\varepsilon_0 (x + 0.2)^2}$

The force at R due to Q is $= \dfrac{2.0 \times 10^{-6} \times q}{4\pi\varepsilon_0 x^2}$

They are oppositely directed and the resultant is zero.

So, $$\frac{8 \times 10^{-6}}{(0.2 + x)^2} = \frac{2 \times 10^{-6}}{x^2}$$

or, $$0.2 + x = 2x$$

or, $$x = 0.2 \text{ m}$$

Example 2.3 Four equal charges $+q$ are placed at the corner of a square. Find the point charge at the center of the square so that the system will remain in equilibrium.

Sol. The charge Q at the center (O) must be negative [Fig. 2.2W]. If the net force on a charge at D is zero, then by symmetry it follows that the net force experienced by charges at other points will also be zero.

The resultant force (F_R) at D due to all other charges at different corner will be

$$F_R = F_B + F_C \cos 45^\circ + F_A \cos 45^\circ$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q^2}{(DO)^2} + \frac{q^2}{a^2}\frac{1}{\sqrt{2}} + \frac{q^2}{a^2\sqrt{2}}\right]$$

$$= \frac{q^2}{4\pi\varepsilon_0}\left[\frac{1}{(\sqrt{2}a)^2} + \frac{2}{\sqrt{2}a^2}\right]$$

That force must be equal to $\dfrac{Qq}{4\pi\varepsilon_0\left(\frac{1}{\sqrt{2}}a\right)^2}$

So $$\frac{Qq \times 2}{4\pi\varepsilon_0 a^2} = \frac{q^2}{4\pi\varepsilon_0 a^2}\left(\frac{1}{2} + \frac{2}{\sqrt{2}}\right)$$

or, $$Q = q\frac{(1+2\sqrt{2})}{2}$$

So, the charge at the center will be $-\dfrac{q(1+2\sqrt{2})}{2}$.

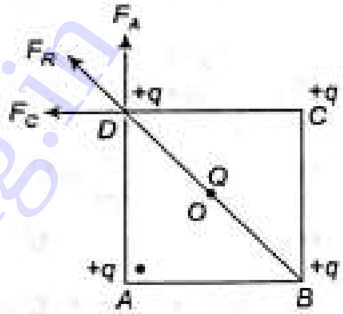


Fig. 2.2W Equilibrium of four equal charges at corners of a square when a charge is placed at the center.

Example 2.4 Two similar balls of mass m are hung from silk threads of length l and carry same charges. Prove that for a small angle φ, the separation of the charges, will be

$$x = \left(\frac{q^2 l}{2\pi\varepsilon_0 mg}\right)^{\frac{1}{3}}$$

Sol. In Fig. 2.3W, each ball of charge $+q$ are suspended from O by silk threads. Here $\theta = \dfrac{\varphi}{2}$

The restoring force $= mg \sin\theta$ and electrostatic repulsive force between the balls is $= \dfrac{qq}{4\pi\varepsilon_0 x^2}$

In equilibrium, $mg\sin\theta = \dfrac{q^2}{4\pi\varepsilon_0 x^2}$ where x is the separation of the balls.

From the figure for small θ, $\sin\theta = \dfrac{x/2}{l} = \dfrac{x}{2l}$

$\therefore$ $$mg\frac{x}{2l} = \frac{q^2}{4\pi\varepsilon_0 x^2}$$

or, $$x^3 = \frac{q^2 l}{2\pi\varepsilon_0 mg} \quad \text{or,} \quad x = \left(\frac{q^2 l}{2\pi\varepsilon_0 mg}\right)^{\frac{1}{3}}$$

Hence proved.

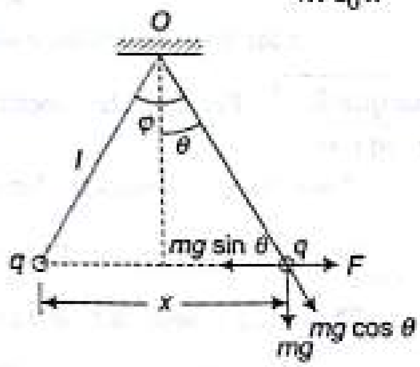


Fig. 2.3W Separation of two like and equal charges suspended from a point.

Example 2.5 An amount of charge Q is divided into two particles. Find the charge on each particle so that the effective force between them will be maximum.

Sol. Suppose the charge on one particle be q, then charge on the other is $(Q-q)$. If they are separated by a distance r then force

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q(Q-q)}{r^2}$$

For maximum F, $\frac{dF}{dq} = 0$

So, $\frac{d}{dq}\left[\frac{1}{4\pi\varepsilon_0}\frac{q(Q-q)}{r^2}\right] = 0$

or, $\frac{d}{dq}(qQ - q^2) = 0$

or, $Q - 2q = 0$ or, $q = \frac{Q}{2}$

So, for maximum F, Q is to be equally divided in the particles.

Example 2.6 Find out electric field intensity at any point on the axis of the uniformly charged rod.

Sol. In Fig. 2.4W, let L is the length of the rod AB uniformly charged (q). If λ be the linear charge density then $\lambda = \frac{q}{L}$ and charge on an elementary length dx is $dq = \lambda\, dx$. The electric field at P due to dx is

$$d\vec{E} = \frac{dq}{4\pi\varepsilon_0 x^2}\hat{x} \text{ along } \overrightarrow{BP}$$

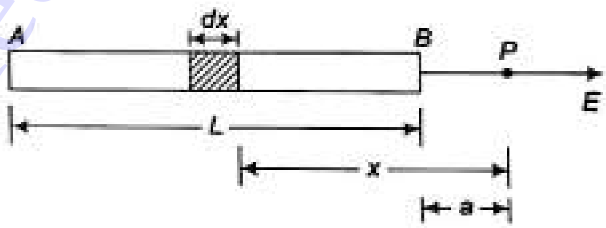


Fig. 2.4W Electric field at a point P on the axis of a uniformly charged rod AB.

For the entire charged rod, electric field acts in the same direction and total field at P is

$$E = \int_a^{a+L}\frac{\lambda}{4\pi\varepsilon_0}\frac{dx}{x^2} = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{a+L}\right] = \frac{\lambda L}{4\pi\varepsilon_0 a(a+L)}$$

If $a >> L$, $E = \frac{1}{4\pi\varepsilon_0}\frac{q}{a^2}$.

So under this condition, a charged rod behaves as a point charge.

Example 2.7 Find out the electric field intensity at any point on the perpendicular bisector of a uniformly charged rod.

Sol. Consider an elementary length dx at a distance x from the center of the rod of length L [Fig. 2.5W].

The charge on this element dx is $dq = \frac{q}{L}dx = \lambda\, dx$, where $\lambda = \frac{q}{L}$ the linear charge density.

The electric field at P due to this element is

$$dE = \frac{dq}{4\pi\varepsilon_0 (AP)^2} = \frac{dq}{4\pi\varepsilon_0 (a^2 + x^2)}$$

Again, the component of dE along OP is $dE\cos\theta$.

The resultant field at P due to the whole charged rod is

$$E=\int dE\cos\theta=\frac{1}{4\pi\varepsilon_0}\int_{-L/2}^{L/2}\frac{q\,a\,dx}{L(a^2+x^2)^{3/2}}$$

$$\left[\because \cos\theta=\frac{a}{\sqrt{x^2+a^2}}\right]$$

$$=\frac{q\,a}{4\pi\varepsilon_0 L}\int_{-L/2}^{L/2}\frac{dx}{(a^2+x^2)^{3/2}}$$

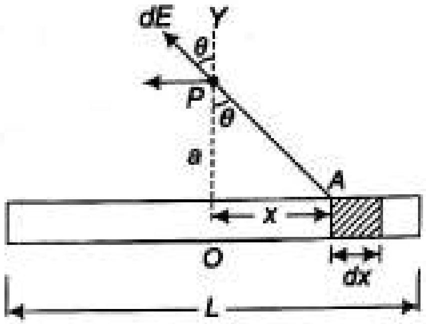


Fig. 2.5W Electric field at a point on the perpendicular bisector of a uniformly charged rod.

We have $x=a\tan\theta$ or $dx=a\sec^2\theta\,d\theta$

or, $$E=\frac{q\,a}{4\pi\varepsilon_0 L}\int\frac{a\sec^2\theta\,d\theta}{a^3\sec^3\theta}=\frac{qa^2}{4\pi\varepsilon_0 a^3 L}\int\cos\theta\,d\theta$$

$$=\frac{q}{4\pi\varepsilon_0 La}\sin\theta=\frac{q}{4\pi\varepsilon_0 La}\left[\frac{x}{(a^2+x^2)^{1/2}}\right]_{-L/2}^{L/2}$$

$$=\frac{1}{4\pi\varepsilon_0}\cdot\frac{2q}{a\sqrt{L^2+4a^2}}=\frac{q}{2\pi\varepsilon_0 a\sqrt{L^2+4a^2}}$$

Example 2.8 Find out the electric field intensity at a point on the axis of a uniformly charged ring.

Sol. Consider an elementary length dl of the ring (Fig. 2.6W). The charge of the ring $dq=\lambda\,dl$. The field at P due to dq of dl is

$$dE=\frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\text{ along }AP$$

The component of dE along the x axis is $dE_x=dE\cos\theta$

$$\therefore\qquad dE_x=\frac{1}{4\pi\varepsilon_0}\frac{\lambda\,dl}{r^2}\cos\theta$$

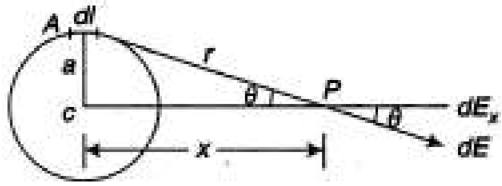


Fig. 2.6W Electric field intensity at a point on the axis of a charged ring.

Total field intensity at P due to the whole ring

$$E=\frac{\lambda}{4\pi\varepsilon_0}\int\frac{dl}{r^2}\cos\theta=\frac{\lambda}{4\pi\varepsilon_0 r^2}\int\frac{x}{r}\,dl$$

$$=\frac{\lambda}{4\pi\varepsilon_0}\frac{x}{r^3}\times 2\pi a$$

$$=\frac{\lambda}{4\pi\varepsilon_0}\frac{2\pi a x}{(a^2+x^2)^{3/2}}=\frac{qx}{4\pi\varepsilon_0(a^2+x^2)^{3/2}}\text{ along the }x\text{ axis}$$

When $x>>a$, $$E=\frac{q}{4\pi\varepsilon_0 x^2}$$

So, at large distance the ring behaves like a point charge.

Example 2.9 Find out the electric field intensity at any point on the axis of a uniformly charged disc.

Sol. In Fig. 2.7W, Let R be the radius of the disc and x be the distance of the field point P from the center O. We consider a concentric ring within radii r and $r+dr$. If σ is the surface charge density then total charge on a surface element dS is $\sigma\,dS$. The field at P due to ds is given by $d\vec{E}=\frac{\sigma\,dS}{4\pi\varepsilon_0 a^2}\hat{a}$, here $a^2=x^2+r^2$.

The component along the axis of the disc

$$dE_1 = \frac{1}{4\pi\varepsilon_o}\frac{\sigma\, dS}{(r^2+x^2)}\cos\theta$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\sigma\, dS}{(r^2+x^2)}\frac{x}{(x^2+r^2)^{1/2}}$$

$$= \frac{\sigma\, dS\, x}{4\pi\varepsilon_0 (r^2+x^2)^{3/2}}$$

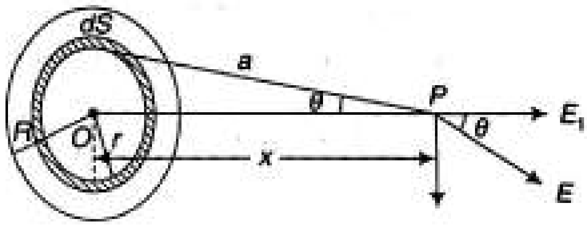


Fig. 2.7W Electric field at a point P on the axis of a charged disc.

Again $ds = 2\pi r\, dr$

or, $$dE_1 = \frac{\sigma\times 2\pi r\, dr\, x}{4\pi\varepsilon_0 (r^2+x^2)^{3/2}}$$

Total intensity at P is $E = \int dE_1 = \frac{2\pi x\sigma}{4\pi\varepsilon_0}\int_0^R \frac{r\, dr}{(r^2+x^2)^{3/2}}$

$$E = \frac{x\sigma}{2\varepsilon_0}\left[\frac{1}{x} - \frac{1}{(R^2+x^2)^{1/2}}\right]$$

$$= \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{x}{(R^2+x^2)^{1/2}}\right]$$

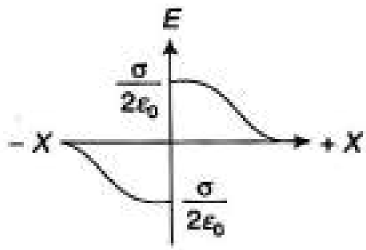


Fig. 2.8W Variation of field intensity with distance from the center of a charged disc.

For infinite charge sheet $R \to \alpha$

$\therefore$ $$E = \frac{\sigma x}{2E_0 |x|}$$

For positive σ, E is positive when x is positive and it is negative when x is negative. The magnitude of the field is independent of the distance x and is given by $\frac{\sigma}{2\varepsilon_0}$. The variation of field intensity with distance from the center of a uniformly charged disc is shown in Fig. 2.8W.

Example 2.10 Five equal charges of 40 nC each are placed at five vertices of a regular hexagon of 6 cm side. The sixth vertex is free. Determine the electric field at the center of the hexagon due to the distribution.

Sol. The field at the center due to the charges located at two opposite vertices is zero. Since there is no charge at F (free vertex) [Fig. 2.9W] so the resultant field will be due to charge q located at C. The field is directed from the center to the vacant corner and its magnitude is $\frac{q}{4\pi\varepsilon_0 a^2}$, where a is the distance of the center from each of the vertices. So, the field is $\frac{9\times 10^9 \times 40\times 10^{-9}}{(6\times 10^{-2})^2}$ N/C or, 10^5 N/C

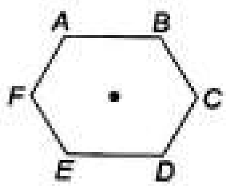


Fig. 2.9W Electric field at center due to five equal charges placed at five corners of a regular hexagon.

Example 2.11 Infinite number of positive charges, each of magnitude q is placed on the x axis at the point $x = 1, 2, 4, 8, \ldots$ What will be intensity of electric field at $x = 0$? Also calculate the electric field if alternative charges are of opposite signs.

Sol. The resultant intensity at $x = 0$ [Fig. 2.10W] is

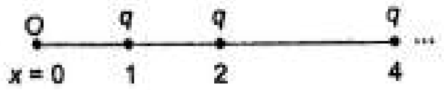


Fig. 2.10W Electric field at $x = 0$ due to equal charge at $x = 1, 2, 4, 8, \ldots$

$$E = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{1^2} + \frac{q}{2^2} + \frac{q}{4^2} + \frac{q}{8^2} + \ldots\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{\left(1 - \frac{1}{4}\right)}$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{4}{3}\text{ units}$$

If alternate charges are of opposite signs then electric intensity at $x = 0$ is

$$E_1 = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{1^2} - \frac{q}{2^2} + \frac{q}{4^2} - \frac{q}{8^2} + \ldots\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{\left(1 + \frac{1}{4}\right)} = \frac{q}{4\pi\varepsilon_0}\frac{4}{5}\text{ units}$$

Example 2.12 Three charges q_1, q_2 and q_3 are at the vertice of an equilateral triangle of 1 m side. The charges are $q_1 = -2\,\mu C$, $q_2 = 6\,\mu C$ and $q_3 = 4.5\,\mu C$. Find the total potential energy of this charge distribution.

Sol. The total potential energy, $U = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_2 q_3}{r_{23}} + \frac{q_1 q_3}{r_{13}}\right]$

$$= 9 \times 10^9 \times \frac{1}{a}(q_1 q_2 + q_2 q_3 + q_1 q_3) \quad [r_{12} = r_{23} = r_{31} = a \text{ (Say)}]$$

$$= 9 \times 10^9 \times [-2 \times 6 + 6 \times 4.5 - 2 \times 4.5] \times 10^{-12}$$

$$= 0.054 \text{ Joule}$$

Example 2.13 Three charges q, $2q$, and $4q$ are placed along a straight line of 6 cm length. Where should the charges be placed so that potential energy of the system is minimum. Find out the distance of the charges.

Sol. Let $2q$ be placed between the other two charges [Fig. 2.11W]. Suppose, the distance between q and $2q$ is x m. so, the distance between $2q$ and $4q$ is $(0.06 - x)$ m.

The potential energy of the whole system,

$$U = \frac{1}{4\pi\varepsilon_0}\left[\frac{q \times 2q}{x} + \frac{2q \times 4q}{0.06 - x} + \frac{q \times 4q}{0.06}\right]$$

For minimum value of U $\quad \frac{dU}{dx} = 0$

or, $\quad -\frac{1}{x^2} + \frac{4}{(0.06 - x)^2} = 0$

or, $\quad x = 0.02$ m

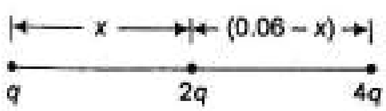


Fig. 2.11W Minimum PE due to charges q, $2q$, $4q$ placed on a line of 6 cm length.

So, the distance between q and $2q$ is 0.02 m and the distance between $2q$ and $4q$ is 0.04 m.

Example 2.14 If the electric field is given by $\vec{E} = 6\hat{i} + 3\hat{j} + 4\hat{k}$, calculate the electric flux through a surface of area of 20 units laying in the yz plane.

Sol. Since the surface area lies in the yz plane, the area vector $\vec{S}$ is directed along the x direction. So $\vec{S} = 20\hat{i}$.

Here $\vec{E} = 6\hat{i} + 3\hat{j} + 4\hat{k}$

$\therefore$ electric flux through the surface is $\varphi_E = \vec{E} \cdot \vec{S}$

$$= (6\hat{i} + 3\hat{j} + 4\hat{k}) \cdot 20\hat{i}$$

$$= 120 \text{ units}$$

Example 2.15 Find out electric field intensity at regions I, II, III due to two infinite plane parallel sheets of charge [Fig. 2.12W].

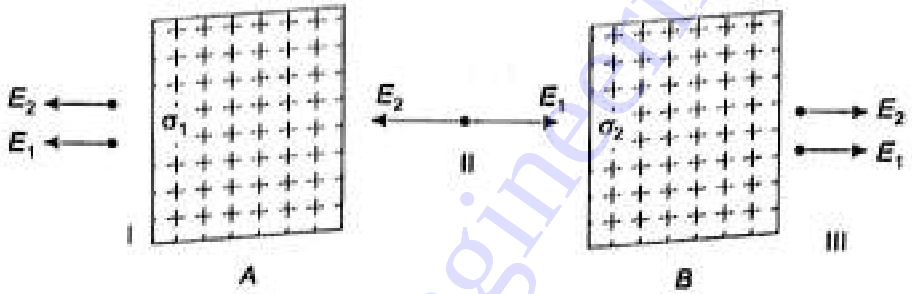


Fig. 2.12W Electric field at three points I, II and III due to two infinite plate charge sheets.

Sol. Let A and B be two infinite plane parallel charge sheets and σ_1, σ_2 be uniform surface densities of charge on A and B respectively. Here $\sigma_1 > \sigma_2$ (say).

In region I, the net electric field

$$E_I = E_1 + E_2 = \left(\frac{-\sigma_1}{2\varepsilon_0}\right) + \left(\frac{-\sigma_2}{2\varepsilon_0}\right) = -\frac{1}{2\varepsilon_0}(\sigma_1 + \sigma_2)$$

In region II, $E_{II} = E_1 - E_2 = \dfrac{\sigma_1}{2\varepsilon_0} - \dfrac{\sigma_2}{2\varepsilon_0} = \dfrac{1}{2\varepsilon_0}(\sigma_1 - \sigma_2)$

In region III, $E_{III} = E_1 + E_2 = \dfrac{\sigma_1}{2\varepsilon_0} + \dfrac{\sigma_2}{2\varepsilon_0} = \dfrac{1}{2\varepsilon_0}(\sigma_1 + \sigma_2)$

Now if $\sigma_1 = \sigma$ and $\sigma_2 = -\sigma$ then $E_I = 0$ $E_{III} = 0$, and $E_{II} = \dfrac{\sigma}{\varepsilon_0}$

Example 2.16 The electric field components in Fig. 2.13W are $E_x = \alpha x^{1/2}$, $E_y = E_z = 0$ in which $\alpha = 800 \text{ NC}^{-1} \text{ m}^{-1/2}$. Calculate electric flux through the cube and the charge within the cube. Assume that $a = 0.1$ m.

Sol. The electric flux is zero for each face of the cube except the two faces $ABCD$ and $EFGH$. The magnitude of electric field at the face $ABCD$, $E_1 = \alpha x^{1/2} = \alpha a^{1/2}$ and at the face $EFGH$, $E_2 = \alpha x^{1/2} = \alpha (2a)^{1/2}$.

So, flux $\varphi_1 = \vec{E}.\vec{dS} = E_1 S \cos 180°$

$= \alpha a^{1/2} \times a^2 (-1) = -a^{5/2} \alpha$

and flux $\varphi_2 = \vec{E}.\vec{dS} = E_2 S \cos 0°$

$= \alpha (2a)^{1/2} a^2 = 2^{1/2} a^{5/2} \alpha$

So, net flux $\varphi = (E_2 - E_1) = 2^{1/2} a^{5/2} \alpha - a^{5/2} \alpha$

$= a^{5/2} \alpha (2^{1/2} - 1)$

Now, putting the value $a = 0.1$ m and

$\alpha = 800$ NC^{-1} m$^{-1/2}$,

we have net flux = 1.05 Nm^2 C^{-1} and

charge $q = \varepsilon_0 \varphi$

So, $q = 8.85\ 10^{-12} \times 1.05$

$= 9.3 \times 10^{-12}$ C

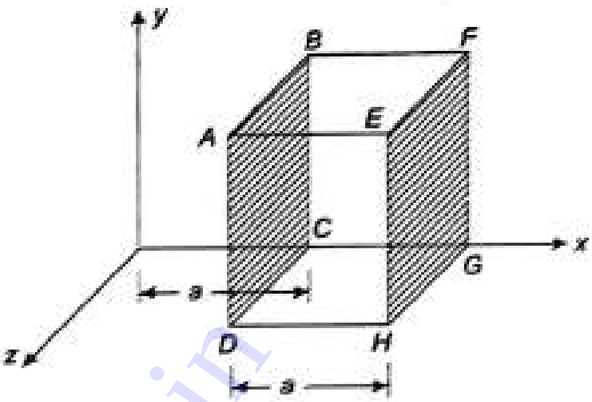


Fig. 2.13W Electric flux through a cube and charge inside it.

Example 2.17 Using Gauss' law in integral form, obtain the electric field due to the following charge distribution in spherical coordinates. [WBUT 2012]

$$\rho(r, \theta, \varphi) = \rho_0 \left(1 - \frac{r^2}{a^2}\right) \quad 0 < r < a$$

$$= 0 \qquad a < r < \infty$$

Sol. Consider region $0 < r < a$, in spherical coordinate volume element $dv = r^2 \sin\theta\, dr\, d\theta\, d\varphi$

Now, from Gauss' law $\int_S \vec{E} \cdot \vec{dS} = \frac{q}{\varepsilon_0}$

or, $$E \times 4\pi r^2 = \frac{1}{\varepsilon_0} \int_{r=0}^{r} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \rho_0 \left(1 - \frac{r^2}{a^2}\right) r^2 \sin\theta\, d\theta\, dr\, d\varphi$$

$$E \times 4\pi r^2 = \frac{\rho_0}{\varepsilon_0} \left(\frac{r^3}{3} - \frac{r^5}{5a^2}\right) 4\pi$$

or, $$E = \frac{\rho_0 (5a^2 r^3 - 3r^5)}{15 a^2 r^2 \varepsilon_0} \quad \text{for } 0 < r < a$$

For region $a < r < \infty$, apply Gauss' law

$$\oint_s \vec{E} \cdot \vec{dS} = \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \rho_0 \left(1 - \frac{r^2}{a^2}\right) r^2 \sin\theta\, dr\, d\theta\, d\varphi \qquad \text{[Charge enclosed up to } r = a\text{]}$$

or, $$E \times 4\pi r^2 = \rho_0 \left(\frac{a^3}{3} - \frac{a^5}{5a^2}\right) 4\pi$$

or, $$E = \frac{2\rho_0 a^3}{15 \varepsilon_0 r^2} \quad a < r < \infty$$

and at $r = a$ $$E = \frac{2}{15} \frac{\rho_0 a}{\varepsilon_0}$$

Example 2.18 If the potential in the region of space near the point (– 2 m, 4 m, 6 m) is $V = 80x^2 + 60y^2$ volt, what are the three components of the electric field at that point?

Sol.
$$\vec{E} = -\vec{\nabla} V = -\left(\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right)$$

or,
$$E_x = -\frac{\partial V}{\partial x},\ E_y = -\frac{\partial V}{\partial y},\ E_z = -\frac{\partial V}{\partial z}$$

So,
$$E_x = -\frac{\partial}{\partial x}(80x^2 + 60y^2) = -160x$$

or, E_x at (– 2 m) = – 160 × (– 2) = 320 Vm^{-1}

$$E_y = -\frac{\partial V}{\partial y} = -\frac{\partial}{\partial y}(80x^2 + 60y^2) = -120\,y$$

E_y at (4 m) = – 120 × 4 = – 480 Vm^{-1}

$$E_z = -\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}(80x^2 + 60y^2) = 0$$

Example 2.19 If $V\left(= \frac{1}{4\pi\varepsilon_0}\frac{q}{r}\right)$ is the potential, at a distance r, due to a point charge q, then determine the electric field due to point charge q, at a distance r.

Sol.
$$\vec{E} = -\vec{\nabla} V = -\hat{r}\frac{\partial}{\partial r}\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r}\right)$$
$$= -\hat{r}\frac{q}{4\pi\varepsilon_0}\left(-\frac{1}{r^2}\right) = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}.$$

Example 2.20 Three point charges q, $2q$ and $8q$ are placed on a 9 cm long straight line Fig. 2.14 W. Determine the position where the charges should be placed such that the potential energy of this system is minimum.

Sol. Let q charge be placed at a distance x from the charge $2q$.

Now potential energy,

$$U = \frac{2q^2}{x} + \frac{8q^2}{9-x}$$

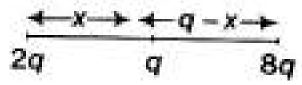


Fig. 2.14 W Placing of three charges q, $2q$, $8q$ on a line of 9 cm length to make PE minimum.

For minimum U,

$$\frac{dU}{dx} = 0 = -\frac{2q^2}{x^2} + \frac{8q^2}{(9-x)^2}$$

or, $(9-x)^2 = 4x^2$

or, $9 - x = \pm 2x$

or, $x = 3, -9$ cm

But $x = -9$ is not possible so $x = 3$ cm.

Example 2.21 Show that the potential function $V = V_0(x^2 - 2y^2 + z^2)$ satisfies Laplace's equation, where V_0 is a constant. [WBUT 2004]

Sol. Here $V = V_0(x^2 - 2y^2 + z^2)$

or, $$\vec{\nabla} V = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)[V_0(x^2 - 2y^2 + z^2)]$$

$$= V_0(2x\,\hat{i} - 4y\hat{j} + 2z\,\hat{k})$$

Again $$\nabla^2 V = \vec{\nabla}\cdot\vec{\nabla} V$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot[V_0(2x\hat{i} - 4y\hat{j} + 2z\,\hat{k})]$$

$$= V_0(2 - 4 + 2) = 0$$

or, $$\nabla^2 V = 0$$

So, the potential function V satisfies Laplace's equation

Example 2.22 A very long cylindrical object carries charge distribution proportional to the distance from the axis (r). If the cylinder is of radius a, then find the electric field both at $r > a$ and $r < a$ by the application of Gauss' law in electrostatics. [WBUT 2007]

Sol. Let A_1 and A_2 be the points [Fig. 2.15W] at a distance r such that (i) $r < a$ (ii) $r > a$

(i) ***Inside*** ($r < a$)

Here we consider a coaxial cylinder of radius $r < a$ and length l.

Total flux through the cylindrical surface

$$\oint_S \vec{E}\cdot\vec{dS} = \frac{q}{\varepsilon_0}$$ (q is the total charge enclosed within the cylinder).

Here, if $\rho(r)$ be the charge density then $\rho(r) = \lambda r$ where λ is constant.

Then total charge $$q = \int_0^r 2\pi r l\, dr\, \rho$$

$$= \int 2\pi r l\, dr\, \lambda r = 2\pi l \lambda \int_0^r r^2\, dr$$

$$= \frac{2}{3}\pi l \lambda r^3$$

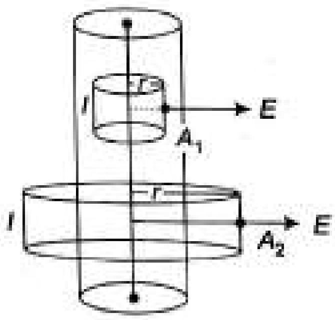


Fig. 2.15W Electric field at points where $r < a$ and $r > a$ due to long cylindrical charged body.

Now, from Gauss' law $$\oint_S \vec{E}\cdot\vec{dS} = \frac{2\pi l \lambda r^3}{3\varepsilon_0}$$

or, $$E \times 2\pi r l = \frac{2\pi l \lambda r^3}{3\varepsilon_0}$$

or, $$E = \frac{\lambda r^2}{3\varepsilon_0}$$

(ii) ***Outside*** ($r > a$)

$$\oint_S \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

But $$q = \int_0^a 2\pi r l\, dr\, \rho = 2\pi l \int_0^a r\, dr\, \lambda r = 2\pi\lambda l \int_0^a r^2\, dr$$

Now $$\oint_S \vec{E}\cdot\vec{dS} = \frac{2\pi\lambda l}{\varepsilon_0}\int_a^0 r^2\, dr$$

or,
$$E \times 2\pi r l = \frac{2}{3}\frac{\pi a^3}{\varepsilon_0}\lambda l$$

or,
$$E = \frac{1}{3}\frac{a^3 \lambda}{\varepsilon_0 r}$$

Example 2.23 The potential field at any point in free space is given by $V = 5x^2y + 3yz^2 + 6xz$ volt, where x, y, z are in meters. Calculate the volume charge density at point (2, 5, 3) m.

Sol. From Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

Here
$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

So,
$$\frac{\partial^2 V}{\partial^2 x^2} = \frac{\partial^2}{\partial x^2}(5x^2y + 3yz^2 + 6xz) = \frac{\partial}{\partial x}(10xy + 6z) = 10y$$

$$\frac{\partial^2 V}{\partial y^2} = \frac{\partial^2}{\partial y^2}(5x^2y + 3yz^2 + 6xz) = \frac{\partial}{\partial y}(5x^2 + 3z^2) = 0$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial^2}{\partial z^2}(5x^2y + 3yz^2 + 6xz) = \frac{\partial}{\partial z}(6yz + 6x) = 6y$$

or,
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 10y + 6y = 16y$$

Now at point (2, 5, 3),
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 80$$

or,
$$\nabla^2 V = 80 = -\frac{\rho}{\varepsilon_0}$$

or,
$$\rho = -80\,\varepsilon_0$$
$$= -80 \times 8.854 \times 10^{-12}\ \text{C/m}^3$$

Example 2.24 In a field $\vec{E} = -50y\,\hat{i} - 50x\,\hat{j} + 30\,\hat{k}$ V/m, calculate the differential amount of work done in moving $2\mu C$ charge a distance $5\,\mu$m from P_1 (1, 2, 3,) to P_2 (2, 4, 1).

Sol. We know that work done $dw = -q\vec{E}.\vec{dl}$

Here
$$\vec{dl} = 5\hat{e}_{P_1P_2}$$

$$\hat{e}_{P_1P_2} = \frac{(2-1)\,\hat{i} + (4-2)\,\hat{j} + (1-3)\,\hat{k}}{\sqrt{1^2 + 2^2 + (-2)^2}} = \frac{1}{3}(\hat{i} + 2\hat{j} - 2\hat{k})$$

or,
$$\vec{dl} = \frac{5}{3}(\hat{i} + 2\hat{j} - 2\hat{k})\ \mu\text{m}.$$

or,
$$dw = -q\vec{E}\cdot\vec{dl} = -2 \times 10^{-6}(-50y\,\hat{i} - 50x\hat{j} + 30\,\hat{k}).\frac{5}{3}(\hat{i} + 2\hat{j} - 2\hat{k}) \times 10^{-6}$$

At initial point (1, 2, 3)

$$dw = -2 \times 10^{-6}(-100\,\hat{i} - 50\,\hat{j} + 30\,\hat{k}).\frac{5}{3} \times 10^{-6}\,(\hat{i} + 2\hat{j} - 2\hat{k})$$

$$= -\frac{10}{3} \times 10^{-12}\,(-100 - 100 - 60)$$

$$= -\frac{10}{3} \times 260 \times 10^{-12}\ \text{Joule} = 8.66 \times 10^{-10}\ \text{Joule}$$

Example 2.25 Show that $V = \frac{1}{r}$ satisfies Laplace's equation.

Sol. The position vector $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$

Magnitude of $\vec{r}$, $|\vec{r}| = \sqrt{x^2 + y^2 + z^2} = (x^2 + y^2 + z^2)^{1/2}$

$\therefore$ $$\nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(x^2 + y^2 + z^2)^{-1/2}$$

Now, $$\frac{\partial^2}{\partial x^2}(x^2 + y^2 + z^2)^{-1/2} = \frac{\partial}{\partial x}\{-x(x^2 + y^2 + z^2)^{-3/2}\}$$

$$= \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

Similarly, $$\frac{\partial^2}{\partial y^2}(x^2 + y^2 + z^2)^{-1/2} = \frac{2y^2 - x^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

and $$\frac{\partial^2}{\partial z^2}(x^2 + y^2 + z^2)^{-1/2} = \frac{2z^2 - y^2 - x^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$\therefore$ $$\nabla^2\left(\frac{1}{r}\right) = \frac{2x^2 - y^2 - z^2 + 2y^2 - x^2 - z^2 - 2z^2 - y^2 - x^2}{(x^2 + y^2 + z^2)^{5/2}} = 0$$

So, $V = \frac{1}{r}$ satisfies Laplace's equation.

Example 2.26 The potential in a medium is given by $\varphi(r) = \dfrac{qe^{-r/\lambda}}{4\pi\varepsilon_0 r}$

(i) Obtain the corresponding electric field.

(ii) Find the charge density that may produce the potential mentioned above. [WBUT 2008]

Sol. Here $\varphi(r)$ is purely a function of r. So, electric field

(i) $$\vec{E} = -\vec{\nabla}\varphi = -\hat{r}\frac{\partial\varphi}{\partial r} = -\hat{r}\frac{q}{4\pi\varepsilon_0}\frac{\partial}{\partial r}\left(\frac{e^{-r/\lambda}}{r}\right)$$

$$= -\hat{r}\frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{\lambda r}e^{-r/\lambda} - \frac{1}{r^2}e^{-r/\lambda}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\hat{r}\left(\frac{1}{\lambda r} + \frac{1}{r^2}\right)e^{-r/\lambda}$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{\vec{r}}{\lambda r^2} + \frac{\vec{r}}{r^3}\right)e^{-r/\lambda} \qquad \left[\text{where } \hat{r} = \frac{\vec{r}}{r}\right]$$

(ii) We know that $\vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\varepsilon_0}$

or, $$\rho = \varepsilon_0 \vec{\nabla}\cdot\vec{E} = \varepsilon_0 \vec{\nabla}\cdot\left[\frac{q}{4\pi\varepsilon_0}\left(\frac{\vec{r}}{\lambda r^2}+\frac{\vec{r}}{r^3}\right)e^{-r/\lambda}\right]$$

$$= \frac{q}{4\pi}\left[\vec{\nabla}\cdot\left(\frac{\vec{r}}{\lambda r^2}\right)e^{-r/\lambda} + \vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right)e^{-r/\lambda}\right]$$

By applying $\vec{\nabla}\cdot(\varphi\vec{A}) = \vec{\nabla}\varphi\cdot\vec{A} + \varphi\vec{\nabla}\cdot\vec{A}$

We have $$\rho = \frac{q}{4\pi}\left[\frac{1}{\lambda}\vec{\nabla}\left(e^{-\frac{r}{\lambda}}\right)\cdot\frac{\vec{r}}{r^2} + \frac{1}{\lambda}e^{-r/\lambda}\left(\vec{\nabla}\cdot\frac{\vec{r}}{r^2}\right) + \vec{\nabla}e^{-r/\lambda}\cdot\frac{\vec{r}}{r^3} + e^{-r/\lambda}\left(\vec{\nabla}\cdot\frac{\vec{r}}{r^3}\right)\right]$$

Again $$\vec{\nabla}(e^{-r/\lambda}) = -\frac{1}{\lambda}e^{-\frac{r}{\lambda}}\hat{r}$$

$$\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right) = 0 \quad \text{and} \quad \vec{\nabla}\cdot\left(\frac{\vec{r}}{r^2}\right) = \frac{1}{r^2}$$

So, $$\rho = \frac{q}{4\pi}\left[-\frac{1}{\lambda^2}e^{-\frac{r}{\lambda}}\frac{1}{r} + \frac{1}{\lambda}e^{-\frac{r}{\lambda}}\frac{1}{r^2} - \frac{1}{\lambda}e^{-\frac{r}{\lambda}}\frac{1}{r^2}\right]$$

$$= -\frac{q}{4\pi\lambda^2 r}e^{-r/\lambda}.$$

Example 2.27 Show that $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative field. Find also the scalar potential. [WBUT 2003]

Sol. We know that for conservative force field $\vec{\nabla}\times\vec{F} = 0$

Here $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$

$$\therefore \quad \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (2xy + z^3) & x^2 & 3xz^2 \end{vmatrix} = (0-0)\hat{i} + (3z^2 - 3z^2)\hat{j} + (2x - 2x)\hat{k} = 0$$

So, $\vec{F}$ is a conservative field.

Again, we know that for a conservative field $\vec{F} = \vec{\nabla}V$, where V is the scalar potential.

$$\therefore \quad \vec{F}\cdot\vec{dr} = \vec{\nabla}V.\vec{dr} = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = dV$$

$\therefore$ here $\vec{F}\cdot\vec{dr} = (2xy + z^3)dx + x^2dy + 3xz^2\,dz$

So, $$dV = \vec{F}.\vec{dr} = (2xy + z^2)\,dx + x^2dy + 3xz^2\,dz$$
$$= 2xy\,dx + x^2dy + z^3\,dx + 3xz^2\,dz$$
$$= d\,(x^2y) + d\,(xz^3)$$
$$= d\,(x^2y + xz^3)$$

$$\therefore \quad V = \int dV = \int d\,(xy^2 + xz^3) = x^2y + xz^3 + \text{constant}.$$

Example 2.28 Find the electric field due to the following electric potential.

$$V = \frac{\sin\theta\cos\varphi}{r^2}$$

Sol. We know $\vec{E} = -\vec{\nabla} V$. Here, V is the function of r, θ, φ. So in spherical polar coordinates

$$\vec{E} = -\vec{\nabla} V = -\left[\frac{\partial V}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\hat{e}_\varphi\right]$$

$$\frac{\partial V}{\partial r} = -\frac{2\sin\theta\cos\varphi}{r^3},\ \frac{\partial V}{\partial \theta} = \frac{\cos\theta\cos\varphi}{r^2},\ \frac{\partial V}{\partial \varphi} = -\frac{\sin\theta\sin\varphi}{r^2}$$

So,
$$\vec{E} = \frac{2\sin\theta\cos\varphi}{r^3}\hat{e}_r - \frac{1}{r^3}\cos\theta\cos\varphi\,\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\sin\theta\sin\varphi}{r^2}\hat{e}_\varphi$$

$$= \frac{1}{r^3}[2\sin\theta\cos\varphi\,\hat{e}_r - \cos\theta\cos\varphi\,\hat{e}_\theta + \sin\theta\,\hat{e}_\varphi] \text{ units}$$

Example 2.29 Is it possible for the electric potential in a charge-free space to be given by (a) $V = x^2 + y^2 - y^2$ (b) $x^2 + y^2 - 2z^2$. If not, find the charge density.

Sol. (a) $V = x^2 + y^2 - z^2$

or,
$$\vec{\nabla} V = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 - z^2)$$

$$= 2x\,\hat{i} + 2y\,\hat{j} + -2z\,\hat{k}$$

and
$$\nabla^2 V = \vec{\nabla}\cdot\vec{\nabla} V = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(2x\,\hat{i} + 2y\,\hat{j} - 2z\,\hat{k})$$

$$= 2 + 2 - 2 = 2.$$

Now by using Poisson's equation, $\nabla^2 V = \dfrac{\rho}{\varepsilon_0}$

$\therefore$ $\rho = -2\,\varepsilon_0$

The space is not charge-free.

(b) $V = x^2 + y^2 - 2z^2$

or, $\vec{\nabla} V = 2x\,\hat{i} + 2y\,\hat{j} - 4z\,\hat{k}$

$\nabla^2 V = 2 + 2 - 4 = 0$

By Poisson's equation, $\nabla^2 V = -\dfrac{\rho}{\varepsilon_0} = 0$ or, $\rho = 0$

The space is a charge-free region.

Example 2.30 Region between the two coaxial cones is shown in Fig. 2.16W. A potential V_a exists at θ_1 and $V = 0$ at θ_2. The cone vertices are insulated at $r = 0$. Solve Laplace's equation to get potential at a cone at any angle θ.

Sol. The potential is constant with respect to r and φ. So, in spherical polar coordinates, Laplace's equation takes from

$$\frac{1}{r^2}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0$$

Now after integration with respect to θ

$$\sin\theta\frac{\partial V}{\partial\theta}=\text{constant}=C_1\ \text{(say)}$$

Again integrating $\qquad V=C_1\ln\left(\tan\frac{\theta}{2}\right)+C_2\ \text{(constant)}$

Here, boundary constants are $\theta=\theta_1,\ V=V_a$

$\theta=\theta_2,\ V=0$

Now using boundary conditions

$$V_a=C_1\ln\left(\tan\frac{\theta_1}{2}\right)+C_2$$

$$0=C_1\ln\left(\tan\frac{\theta_2}{2}\right)+C_2$$

After simplification $\qquad V=V_a\dfrac{\ln\left(\tan\frac{\theta}{2}\right)-\ln\left(\tan\frac{\theta_2}{2}\right)}{\ln\left(\tan\frac{\theta_1}{2}\right)-\ln\left(\tan\frac{\theta_2}{2}\right)}$

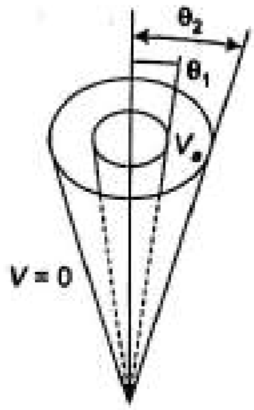


Fig. 2.16W Potential at a cone of angle θ due to two other equipotential cones at angles θ_1 and θ_2.

Review Exercises

Part 1: Multiple Choice Questions

1. Which of the following statements is not correct regarding electrostatic field vector E? [WBUT 2008]

   (a) $\oint_C \vec{E}\cdot\vec{dr}=0$, where c is a simple closed curve.

   (b) $\int_a^b \vec{E}\cdot\vec{dr}$ is independent of the path for given end points a and b.

   (c) $\vec{E}=\vec{\nabla}\times\vec{A}$ for some vector potential $\vec{A}$.

   (d) $\vec{E}=\vec{\nabla}\varphi$, for some scalar field φ.

2. Flux of the electric field for a point charge (q) at origin through a spherical surface centered at the origin is [WBUT 2006]

   (a) $\dfrac{2q}{\varepsilon_0}$ (b) $\dfrac{q}{\varepsilon_0}$ (c) $\dfrac{q}{4\pi\varepsilon_0}$ (d) zero

3. Two concentric spheres of radii r_1 and r_2 carry charges q_1 and q_2 respectively. If the surface charge density (σ) is the same for both spheres, the electric potential at the common center will be

   (a) $\dfrac{\sigma}{\varepsilon_0}\dfrac{r_1}{r_2}$ (b) $\dfrac{\sigma}{\varepsilon_0}\dfrac{r_2}{r_1}$ (c) $\dfrac{\sigma}{\varepsilon_0}(r_1-r_2)$ (d) $\dfrac{\sigma}{\varepsilon_0}(r_1+r_2)$

4. Six charges, each equal to $+q$, are placed at the corners of a regular hexagon of side a. The electric potential at the point where the diagonals of the hexagon intersect will be given by

(a) zero (b) $\frac{1}{4\pi\varepsilon_0}\frac{q}{a}$ (c) $\frac{1}{4\pi\varepsilon_0}\frac{6q}{a}$ (d) $\frac{1}{4\pi\varepsilon_0}\frac{\sqrt{3}q}{2a}$

5. In free space Poisson's equation is [WBUT 2005]

(a) $\nabla^2 V=0$ (b) $\nabla^2 V=\frac{\varepsilon_0}{\rho}$ (c) $\nabla^2 V=\alpha$ (d) None of these

6. The electric flux through each of the faces of a cube of 1 m side if a charge q coulomb is placed at its centre is [WBUT 2007]

(a) $\frac{q}{4\varepsilon_0}$ (b) $4\varepsilon_0 q$ (c) $\frac{q}{6\varepsilon_0}$ (d) $\frac{\varepsilon_0}{6q}$

7. Let (r, θ, φ) represent the spherical polar coordinates of a point in a region where the electrostatics potential V is given by $V = K\varphi^2$. Then the charge density in that region [WBUT 2007]

(a) is also a function of φ only (b) is constant in that region
(c) is a function of all the coordinates (r, θ, φ) (d) is a function of (r, θ) only

8. The electrostatic potential energy of a system of two charges q_1 and q_2 separated by a distance r is

(a) $\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$ (b) $\frac{\varepsilon_0}{4\pi}\frac{q_1 q_2}{r}$ (c) $\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r}$ (d) $\frac{1}{4\pi\varepsilon_0}\frac{q_1^2 q_2^2}{r^2}$

9. The magnitude of electric field $\vec{E}$ in the annular region of a charged cylindrical capacitor

(a) same anywhere (b) varies as $\frac{1}{r}$ (c) varies as $\frac{1}{r^2}$ (d) None of these

10. Electric field and potential inside a hollow charged conducting sphere are respectively

(a) $0, 4\pi\varepsilon_0\frac{q}{r}$ (b) $\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}, 0$ (c) $0, \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$ (d) $\frac{q}{4\pi\varepsilon_0 r^2}, \frac{q}{4\pi\varepsilon_0 r}$

11. For a closed surface which does not include any charge, the Gauss's law will be

(a) $\oint_S \vec{E}.\vec{ds}=0$ (b) $\oint_S \vec{E}.\vec{ds}=\frac{q}{\varepsilon_0}$ (c) $\oint_S \vec{E}.\vec{ds}=\frac{q}{4\pi\varepsilon_0}$ (d) None of these

12. Electrostatic field is

(a) conservative (b) non-conservative (c) rotational (d) None of these

13. Laplace's equation for an electrostatic field is

(a) $\nabla^2 V=\frac{\rho}{\varepsilon_0}$ (b) $\nabla^2 V=0$ (c) $\nabla^2 V=-\frac{\rho}{\varepsilon_0}$ (d) $\vec{\nabla} V=\frac{\rho}{\varepsilon_0}\hat{r}$

14. Electric field intensity at any point distant r from a plane charged conducting sheet varies as

(a) r^{-1} (b) r^0 (c) r^{-2} (d) r

15. In a region of space, if the electrostatic potential is constant, then the electric field at that region is

(a) zero (b) infinite (c) constant (d) None of these

16. If the flux of the electric field through *a* closed surface is zero,
    (a) the charge inside the surface must be zero
    (b) the electric field must be zero everywhere on the surface
    (c) the charge in the vicinity of the surface must be zero
    (d) None of these

[**Ans.** 1 (c), 2 (b), 3 (d), 4 (c), 5 (a), 6 (c), 7 (d), 8 (c), 9 (b), 10 (c), 11 (a), 12 (a), 13 (b), 14 (b), 15 (a), 16 (a)]

Short Questions with Answers

1. **What is an electric line of force? What is its importance?**

Ans. An electric line of force is an imaginary straight or curved path along which *a* positive test charge is supposed to move. The lines of force originate from a single positive charge and converge at an isolated negative charge.

The relative closeness of electric line of force in *a* certain region provides an estimate of the electric field strength in that region.

2. **Sketch the electric lines of force due to point charges (i) $q > 0$ (ii) $q < 0$.**

Ans.

Fig. 2.17W

3. **Electrostatic force are much stronger than gravitational force. Give an example.**

Ans. A charged glass rod can lift a piece of paper. This shows that the electrostatic force of attraction between the glass rod and paper is much stronger than the gravitational force of attraction between them.

4. **State the principle of superposition of electric forces.**

Ans. See Section 2.5.

5. **State the importance of Gauss' law.**

Ans. By applying Gauss' law one can calculate in a simple manner the field intensity due to many different symmetrical configurations of charge. Gauss' law is also important to gain information about the properties of conductors.

6. **Obtain Coulomb's theorem from lines of force concept.**

Ans. See Section 2.12.3(v).

7. **Why is electrostatic field called conservative field?**

Ans. A field is conservative when the work done is independent of the path followed and depends only on the initial and final position. For *a* close path work done is zero. In electric field, work done to bring a charge from one point to another point depends on initial and final points. So, the electric field is conservative.

8. Show that electric field is always perpendicular to the equipotential surface.

Ans. In Fig. 2.18W, S is an equipotential surface. A and B are two very close points on the surface. Let electric field $\vec{E}$ make an angle θ with the equipotential surface. The work done for moving a charge q from A to B along the surface is

$$W = qE\cos\theta \times AB$$

Again work done $\quad W = q(V_A - V_B)$

So, $\quad qE\cos\theta \times AB = q(V_A - V_B)$

But $\quad V_A = V_B$ (equipotential surface)

$\therefore \quad qE\cos\theta\,(AB) = 0$

$\therefore \quad \cos\theta = 0 \quad$ or, $\quad \theta = 90°$

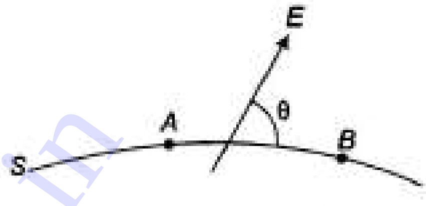


Fig. 2.18W Electric field due to equipotential surface.

9. If Coulomb's law involved $\frac{1}{r^3}$ dependence instead of $\frac{1}{r^2}$, would Gauss' law be still true?

Ans. No, Gauss' law would not hold good.

10. Can electric potential at any point in space be zero while intensity of electric field at that point is not zero?

Ans. Yes, at a point midway between two equal and opposite charges, electric potential is zero but electric field is not zero.

11. No two equipotential surfaces intersect each other. Why?

Ans. We know that two electric lines of force can't intersect, therefore two equipotential surfaces also can't intersect. This is because the electric lines of force and the equipotential surface are mutually perpendicular.

12. Define positive and negative electric flux.

Ans. The electric flux linked with a surface is said to be positive if the electric field vector appears to be leaving the surface.

The electric flux linked with a surface is said to be negative if the electric field vector appears to be entering the surface.

13. Show that Coulomb's law can be derived from Gauss' law.

Ans. See Section 2.12.2.

14. The electric potential is constant in *a* region. What can you say about the electric field there?

Ans. We know that $\quad E = -\dfrac{dV}{dr}$

If V is constant then $\dfrac{dV}{dr} = 0 \quad \therefore E = 0$

So, electric field is zero.

15. Is it possible for a metal sphere of 1 cm radius to hold a charge of one coulomb?

Ans. $V = 9 \times 10^9 \times \dfrac{1}{10^{-2}} = 9 \times 10^{11}$ volt

This is so high a potential that there will be an electrical breakdown of air. On account of ionization of air, the charge on the sphere will leak away.

Part 2: Descriptive Questions

1. What do you mean by conservation of charge? Explain.
   Find out the relation between electric field intensity and potential. What is equipotential surface?
2. State and explain Gauss' law in electrostatics. Obtain its differential form. [WBUT 2002]
3. Derive Coulomb's law from Gauss' law in electrostatics. [WBUT 2007]
4. (a) State Gauss' law in electrostatics and hence obtain Poisson's equation.
   (b) Derive Coulomb's law from Gauss' law. [WBUT 2008]
5. State Gauss' law. Use Gauss' law to find electric field intensity outside, inside and on the surface of a solid sphere.
6. Write down Laplace's equation in cylindrical coordinates and find the solution.
7. If in the region of space electric field is always in the x direction then prove that the potential is independent of y and z coordinates. If the field is constant there is no free charge in that region. [WBUT 2007]
8. (a) State and Prove Gauss' law in electrostatics.
   (b) Using Gauss' law, obtain an expression for the electric field around a charged hollow cylinder. [WBUT 2004]
9. Show that the potential $V = V_0 (x^2 - 2y^2 + z^2)$ satisfies Laplace's function where V_0 is a constant [WBUT 2004]
10. Write down Laplace's equation in spherical coordinate system and hence find the solution.
11. (a) State Gauss' law of electrostatics.
    (b) Use this law to calculate the electric field between two infinite extent parallel-plate capacitors carrying charge density σ and mutual separation d. Draw the necessary diagram. [WBUT 2006]
12. State Gauss' theorem in electrostatics. Using this theorem, derive an expression for the electric field intensity due to an infinite plane sheet of charge density σ coulomb/m^2.
13. Using Gauss' law, determine the electric field intensity due to a long thin wire of uniform linear charge density.
14. Derive Poisson's and Laplace's equations from fundamentals.

Part 3: Numerical Problems

1. Two point charges Q and q are placed at distance x and $\frac{x}{2}$ respectively from a third charge $4q$. All the three charges are on the same straight line. Calculate Q in terms of q such that the net force on q is zero. [**Ans.** $Q = 4q$]
2. Charge is distributed along the x axis from $x = 0$ to $x = L = 50.0$ cm in such a way that its linear charge density is given by $\lambda = ax^2$ where $a = 18.0\ \mu$ cm^{-3}. Calculate the total charge in the region $0 \le x \le 1$.
   [**Ans.** 0.75 μc] $\left[\textbf{Hints: } q = \int_0^L \lambda\, dx\right]$
3. Consider a uniform electric field $\vec{E} = 3 \times 10^3\, \hat{i}$ NC^{-1}. What is the flux of this field through a square of 10 cm on a side whose plane is parallel to the yz plane? [**Ans.** $\varphi_E = 30$ Nm^{2}C^{-1}]

4. A point charge of 2.0 μ C is at the centre of a cubic gaussian surface, 9.0 cm on edge. What is the net electric flux through the surface? [Ans. 2.26×10^5 Nm^2 C^{-1}]
5. The electric potential $V(x)$ in a region along the x axis varies with distance x (in meter) according to the relation $V(x) = 4x^2$. Calculate the force experienced by 1 μ C charge placed at point $x = 1$ m. [Ans. $F = 8 \times 10^{-6}$ N]
6. In a region of space, the electric field is given by $\vec{E} = 8\hat{i} + 4\hat{j} + 3\hat{k}$. Calculate the flux through a surface of 1000 units area in the xy plane. [Ans. 300 units]
7. Show that the potential function $V = x^2 + z - y^2$ satisfies Laplace's equation [WBUT question bank]
8. A circular wire of radius R has a linear charge density $\lambda = \lambda_0 \cos^2\theta$, where θ is the angle with respect to a fixed radius. Calculate total charge. [Ans. $\pi R\lambda_0$]
9. n charged spherical water drops, each having a radius r and charge q, coalesce into a single big drop. What is the potential of the big spherical drop? $\left[\text{Ans. } \frac{1}{4\pi\varepsilon_0}\frac{n^{2/3}q}{r}\right]$
10. An infinite line charge produces a field of 9×10^4 N C^{-1} at a distance of 2 cm. Calculate the linear charge density. [Ans. 10^{-7} cm^{-1}]
11. Determine the charge distribution at $r \neq 0$ which gives a spherically symmetrical potential $V(r) = \frac{e^{-\lambda r}}{r}$ where λ is a constant.
12. The volume charge density of a spherical body of radius a centered at the origin is given by

$$\rho(r, \theta, \varphi) = \frac{\rho_0}{r} \quad \text{where } \rho_0 \text{ is constant.}$$

Calculate the total charge in the sphere. [Ans. $\varphi = 2\pi\rho_0 a^2$]
13. Is is possible for the electric potential in a charge-free region to be given by (i) $V = x^2 + y^2 - z^2$? (ii) $V = x^2 + y^2 + z^2$? If not find the charge density. [Ans. (i) $-4\varepsilon_0$ (ii) $-6\varepsilon_0$] [WBUT Question Bank]
14. Two concentric spheres of radii a and b are kept in potential V_a and V_b. If the intervening space is vacuum then write the appropriate differential equation that the electrostatics potential satisfies. Solve this equation to find out the potential in any point between the spheres and also for a point outside the sphere. Calculate the total charge on the outer sphere. [WBUT 2007]
15. A uniformly charged conducting sphere of 2.4 m diameter has a surface charge density of 80.0 μC/m^2. (i) Find the charge on the sphere. (ii) What is the total electric flux leaving the surface of the sphere? [Ans. (i) 1.45 mC (ii) 1.6×10^8 Nm^2 C^{-1}]

CHAPTER 3

Dielectrics

3.1 INTRODUCTION

A dielectric is an insulating material in which all the electrons are tightly bound to the nuclei of the atom and there are no free electrons available for the conduction of current. The difference in the name between dielectric and insulator lies in the application for which these materials are used. When these materials are used to prevent the flow of electricity through them or the application of potential difference, then they are called insulators or passive dielectrics. On the other hand, if they are used for charge storage, they are called dielectrics or active dielectrics. Materials such as glass, rubber, mica, porcelain and polymers are examples of dielectrics.

3.2 POLARIZATION

Polarization is defined as the process of creating or inducing dipoles in a dielectric material by an external electric field.

In an atom, there is a positively charged nucleus at the center surrounded by orbiting electrons which are negatively charged. In the absence of an electric field an isolated atom does not have any dipole moment, since the centroids of positive and negative charge coincide [Fig. 3.1(a)]. Suppose now the atom is placed

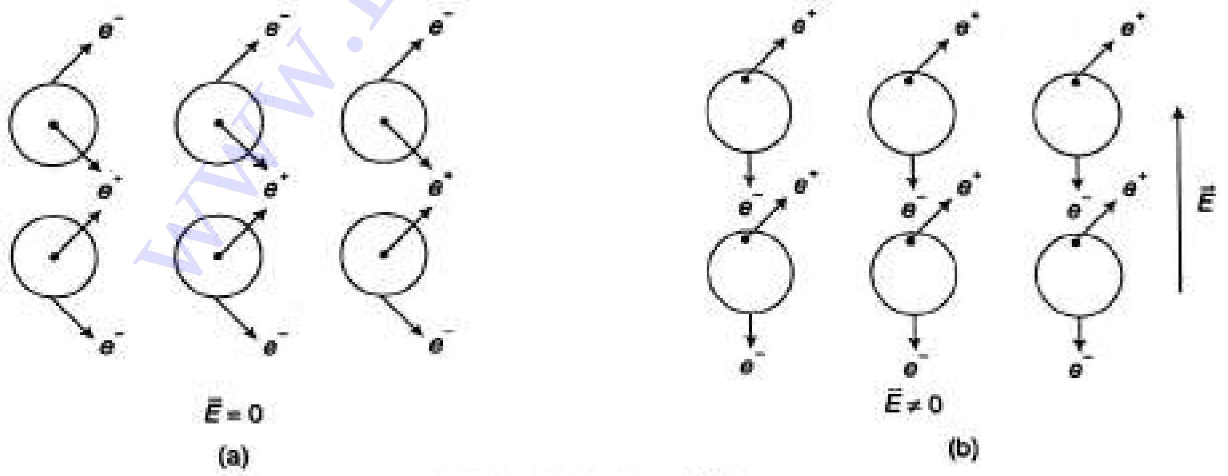


Fig. 3.1 (a) No polarization. (b) Polarization.

in an external electric field. The field will push the positively charged nucleus slightly in the direction of the field and the negatively charged electrons in the opposite direction [Fig. 3.1(b)]. The centroids of the positive and negative charges now no longer coincide and as a result an electric dipole is induced in the atom. The amount of dipole moment induced is proportional to the field because a large field displaces charges more than a smaller field. We say that the atoms are polarized under the influence of the external field.

3.3 TYPES OF DIELECTRICS

On the basis of the polarization concept, dielectrics are the materials that have either permanent dipoles or induced dipoles in the presence of an applied electric field. They are classified into two categories, namely, polar and non-polar dielectrics.

3.3.1 Non-polar Dielectrics

A dielectric material in which, there is no permanent dipole existence in the absence of an external field is called 'non-polar' dielectrics.

For non-polar dielectrics, the center of gravity of the positive and negative charges of the molecules coincide. So such molecules do not have any permanent dipole moment [Fig. 3.2(a)].

Examples O_2, H_2, N_2, CO_2, H_2O_2.

(a) (b)

Fig. 3.2 (a) Non-polar dielectrics. (b) Polar dielectrics.

3.3.2 Polar Dielectrics

A dielectric material in which there is an existence of permanent dipole even in the absence of an external field is called polar dielectrics.

For ~~non~~-polar dielectrics, the center of gravity of the positive charges is separated by finite distance from that of the negative charges of the molecules. So such molecules possess permanent electric dipole [Fig. 3.2(b)].

Examples H_2O, NaCl, HCl, CO.

3.3.3 Dielectric Constant

A capacitor consisting of two parallel conducting plates of area *A*, separated by a distance *d* by a vacuum space [Right side of Fig. 3.3], has a capacitance of

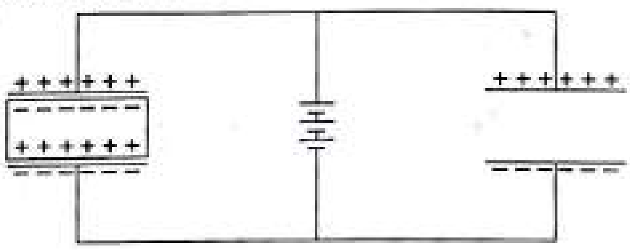


Fig. 3.3 Two identical capacitors: one evacuated and other filled with dielectric.

$$C_0 = \frac{\varepsilon_0 A}{d} \qquad ...(3.1)$$

where ε_0 is the permittivity of free space. If C is the capacitance when the space is filled with dielectric material [Left side of Fig. 3.3], then

$$C = \frac{\varepsilon A}{d} \qquad ...(3.2)$$

where ε is the permittivity of the dielectric. Now the dielectric constant of the material

$$K = \frac{C}{C_0} = \frac{\varepsilon}{\varepsilon_0} \qquad ...(3.3)$$

The dielectric constant of a material is the ratio of the capacitance of a given capacitor completely filled with that material to the capacitance of the same capacitor in vacuum. In other words, the ratio of permittivity of medium to that of the vacuum is also known as dielectric constant or relative permittivity (ε_r).

$$\varepsilon_r = K = \frac{\varepsilon}{\varepsilon_0} \qquad ...(3.4)$$

ε_r is a dimensionless quantity and varies widely from material to material. ε_r has a value unity for vacuum and for all other dielectrics ε_r is always greater than 1. For most materials the value of ε_r varies between 1 to 10.

3.3.4 Polarization Vector or Polarization Density

There are two kinds of dipoles in materials—those that are induced and those that are permanent and both cause polarization or charge separation. A dipole moment μ is defined as $\mu = qd$, where q is the magnitude of the charge and d is the distance separating the pair of opposite charges. Dipole moment is a vector, pointing from negative towards positive charges. Polarization vector measures the extent of polarization in a unit volume of dielectric matter. It is defined as the induced dipole moment per unit volume of the dielectric. If N is the number of molecules per unit volume, then the polarization vector or polarization density

$$P = N\mu \qquad ...(3.5)$$

The direction of P is along the direction of the applied field.

If a dielectric slab of thickness d and volume V is kept between the two plates of a capacitor [left-hand side of Fig. 3.3], then the dipole moment is $\mu = qd$, where $+q$ and $-q$ are induced charges on the respective faces of the slab.

The polarization is given

$$P = \frac{qd}{V} = \frac{qd}{Ad} \qquad [\because V = Ad, \text{ where } A \text{ is the area of the slab}]$$

$$\therefore \quad P = \frac{q}{A} = \sigma_P \text{ (surface charge density of the slab)} \qquad ...(3.6)$$

So, **polarization is also defined as the induced surface charge per unit area.** The unit of polarization is coulomb/m^2. Thus polarization density is equal to surface charge density on the dielectric slab. In general, if the polarization vector makes an angle θ with $\hat{n}$, the outward vector of the surface, the surface charge density

$$\sigma_P = \vec{P} \cdot \hat{n} = P \cos \theta \qquad ...(3.6a)$$

3.3.5 Susceptibility

The strength of polarization (P) is directly proportional to the applied electric field (E) for dielectrics and is given by

$$P = \varepsilon_0 \chi_e E \qquad ...(3.7)$$

The constant of proportionality is usually written as $\varepsilon_0 \chi_e$, where χ_e is known as electric susceptibility of the medium. χ_e is a dimensionless parameter.

Now $$\chi_e = \frac{P}{\varepsilon_0 E} \qquad ...(3.8)$$

Thus, **susceptibility is the ratio of polarization to the net electric field $\varepsilon_0 E$ as modified by the induced charges on the surface of the dielectric.**

3.4 RELATION BETWEEN DIELECTRIC CONSTANT AND ELECTRICAL SUSCEPTIBILITY

We consider a parallel-plate capacitor which has vacuum between its plates. When it is charged with a battery, the electric field of strength E_0 is set up between the plates of the capacitor [Fig. 3.4(a)]. If σ and $-\sigma$ are the surface charge densities of the two plates of the capacitor, then the electric field developed between the plates is given by

$$E_0 = \frac{\sigma}{\varepsilon_0} \qquad ...(3.9)$$

Fig. 3.4 (a) Capacitor with vacuum space. (b) Capacitor filled with dielectric.

If a dielectric slab is placed between the plates of the capacitor; then due to polarization charges, appear on the two faces of the slab and establish another field E_p within the dielectric [Fig. 3.4(b)]. This field will be in a direction opposite to the E_0. Under this situation, the net electric field in the dielectric is given by

$$E = E_0 - E_p \qquad ...(3.10)$$

If σ_s is the surface charge density on the slab, then by following Eq. (3.9), we can write

$$E_p = \frac{\sigma_p}{\varepsilon_0} \qquad ...(3.11)$$

Now, from Eqs. (3.9), (3.10) and (3.11),

$$E = \frac{\sigma}{\varepsilon_0} - \frac{\sigma_p}{\varepsilon_0} = \frac{1}{\varepsilon_0}(\sigma - \sigma_p)$$

or, $$\varepsilon_0 E = \sigma - \sigma_P = \sigma - P \qquad [\because P = \sigma_P]$$

or, $$\sigma = \varepsilon_0 E + P \tag{3.12}$$

Again, by Gauss' law, electric flux density or electric displacement vector D is given by

$$D = \sigma \tag{3.13}$$

Now, from Eq. (3.12)

$$D = \varepsilon_0 E + P \tag{3.14}$$

Again, from Eq. (3.7)

$$P = \varepsilon_0 \chi_e E \tag{3.15}$$

and from electrostatics we know

$$D = \varepsilon E = \varepsilon_0 \varepsilon_r E \tag{3.16}$$

Therefore, from Eqs. (3.15), (3.16) and (3.14)

$$\varepsilon_0 \varepsilon_r E = \varepsilon_0 E + \varepsilon_0 \chi_e E$$

or, $$\varepsilon_r = 1 + \chi_e$$

or $$\chi_e = \varepsilon_r - 1 \tag{3.17}$$

3.5 POLARIZABILITY

Let us consider an individual atom in a dielectric material and the material be subjected to an electric field E. The strength of the dipole induced in an atom is proportional to the actual field acting on the particle, and is given by

$$\mu = \alpha E \tag{3.18}$$

where α is the proportionality constant called polarizability. Its unit is Fm^2.

Note: In the case of gases, the molecules, for most of the time are far apart, so that local electric field (E_{loc}) is the same as the macroscopic field E.

If N be the number of atoms in a unit volume then polarization vector is

$$P = N\mu$$

$$= N \alpha E = \varepsilon_0 \chi_e E \qquad \text{[From Eq. (3.8)]}$$

Therefore, $$\alpha = \frac{\varepsilon_0 \chi_e}{N} \tag{3.19}$$

Polarizability measures the resistance of the particle to the displacement of its electron cloud.

3.6 TYPES OF POLARIZATION

Three basic types of polarization that contribute to the total magnitude of polarization in a material have been identified.

(i) Electronic polarization
(ii) Ionic polarization
(iii) Orientation polarization

Taking into account the three contributions, the total electrical dipole moment

$$\mu = (\alpha_e + \alpha_i + \alpha_0)\, E$$

or, polarization

$$P = N\mu = N(\alpha_e + \alpha_i + \alpha_0)\, E \qquad ...(3.20)$$

$$= N\,\alpha\, E$$

where α is total polarizability

α_e is electronic polarizability

α_i is ionic polarizability

α_0 is orientation polarizability

(i) Electronic polarization

Electronic polarization occurs due to the displacement of the positively charged nucleus and negatively-charged electron cloud in opposite directions within a dielectric material upon applying an external electric field E [Fig. 3.5a, b]. The dipole moment (μ_e) induced is proportional to the applied field and the proportionality constant is called electronic polarizability (α_e).

$$\mu_e = \alpha_e E \qquad ...(3.21)$$

If a material has N such atoms per unit volume, subjected to homogeneous field E, then the electronic polarization is

$$P = N\alpha_e E \qquad ...(3.22)$$

The electronic polarizability for a rare gas atom is given by

$$\alpha_e = \frac{\varepsilon_0(\varepsilon_r - 1)}{N} \qquad ...(3.23)$$

The electronic polarization can persist to extremely high field frequencies because electronic standing waves within atoms have very high natural frequencies.

Fig. 3.5 (a) No field applied. (b) Applied electric field.

(ii) Ionic polarization

This type of polarization occurs in ionic materials. In an ionic bond when two different atoms join together, there is transfer of electrons from an atom to another atom, like HCl shows in Fig. 3.6. Even in the absence of an applied field, an HCl molecule has a permanent dipole moment $e \times d$, where d is the distance of separation of ions. In the presence of an applied electric field, the resultant torque lines up the dipoles parallel to the field at absolute zero temperature. The distance between the ions increases from d to $d + x$. The field has induced an additional dipole moment $e \times x$ in the molecule. The induced dipole moment is proportional to the applied electric field and is given by

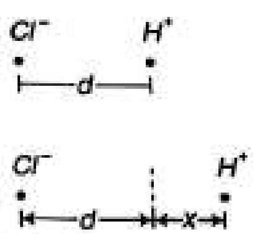


Fig. 3.6 Ionic Polarization.

$$\mu_i = \alpha_i E \quad ...(3.24)$$

where α_i is the ionic polarizability. Now if N is number of dipole per unit volume, then ionic polarization

$$P_i = N\alpha_i E \quad ...(3.25)$$

(iii) Orientation polarization

Orientation polarization occurs in dielectric materials which possess molecules with permanent dipole moment, for example, H_2O molecule (polar molecule). In the absence of an external electric field, the permanent dipoles are oriented randomly such that they cancel the effects of each other [Fig. 3.7(a)]. But under the influence of an external applied electric field, each of the dipoles undergo rotation so as to reorient along the direction of the field as shown in Fig. 3.7(b). Thus, the material itself develops electric polarization. This is known as orientation polarization, which depends upon temperature.

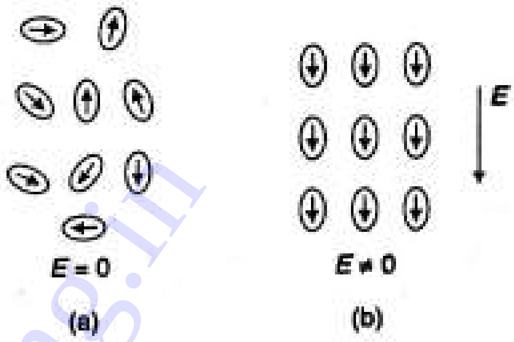


Fig. 3.7 Orientation polarization.

The orientation polarization P_0 is given by Langevin function (1905)

$$P_0 = N\mu L(x) \quad ...(3.26)$$

where $L(x)$ is known as Langevin function.

Here $x = \dfrac{\mu E}{k_B T}$, k_B is the Boltzmann constant and T is the temperature in Kelvin.

The value of $L(x)$ is $\coth x - \dfrac{1}{x} = \left[\dfrac{e^x + e^{-x}}{e^x - e^{-x}} - \dfrac{1}{x}\right]$

For $\quad x >> 1, P_0 \to 1$

complete alignment, but this does not occur in gases.

For most practical cases (x is small, i.e., at high temperature)

$$x << 1, L(x) = \frac{x}{3}$$

Or, $$P_0 = N\mu\frac{x}{3} = \frac{N\mu^2 E}{3k_B T} \quad ...(3.27)$$

The orientation plorizability α_0 is given by

$$\alpha_0 = \frac{\mu^2}{3k_B T} \quad ...(3.28)$$

3.7 POLARIZATION IN MONOATOMIC GASES

Let us consider one of constituent atom of a dielectric material (rare gases, such as helium and argon) in the absence of an electric field. Let the radius of the atom be a and its atoms number be Z as shown in Fig. 3.8(a). Here positive nucleus $+Ze$ is surrounded by an electronic cloud of charge $-Ze$. Also nucleus is point charge and electron cloud of charge $-Ze$ distributed homogeneously throughout a sphere of radius a. Therefore, the charge density for electron cloud is given by

$$\rho = \frac{-Ze}{\frac{4}{3}\pi a^3} = -\frac{3}{4}\left(\frac{Ze}{\pi a^3}\right) \quad ...(3.29)$$

Fig. 3.8 (a) Atom placed in free space. (b) Atom placed into external field.

When the atom is placed into an external electric field E, the nucleus and electron cloud move in the opposite direction [Fig. 3.8(b)], and experience a Lorentz force of magnitude ZeE. Equilibrium is established with nucleus shifted slightly relative to the center of the electron cloud by a distance d.

The nucleus experiences a force (F_N) in the direction of the electric field,

$$F_N = ZeE \quad \text{...(3.30)}$$

and opposing force F_G due to the electric field of the charge located within the sphere of radius d and concentrated at the center of the electron cloud. By Gauss' law, the electric field (E_G) at edge location of the nucleus due to electrons within the sphere of radius d is

$$E_G \times 4\pi d^2 = \text{Total charge } (q) \text{ enclosed in a sphere of radius } d.$$

or,

$$E_G \times 4\pi d^2 = \frac{4}{3}\pi d^3 \rho/\varepsilon_0$$

$$= \frac{4}{3}\pi d^3 \times \left(-\frac{3Ze}{4\pi a^3}\right)\Big/\varepsilon_0 \quad \text{[From Eq. (3.29)]}$$

$$= -\frac{Ze\left(\dfrac{d^3}{a^3}\right)}{\varepsilon_0}$$

Now $$|F_G| = ZeE_G = \frac{Z^2e^2d}{4\pi\varepsilon_0 a^3}$$

But $$|F_G| = |F_N|$$

So, $$d = \frac{4\pi\varepsilon_0 a^3}{Ze}E \quad \text{...(3.31)}$$

Thus, the displacement distance d is proportional to the external electric field E. Due to this displacement, the atom acts as a dipole.

For the single atom, the electronic polarizability of a monoatomic gas can be obtained from induced dipole moment

$$\mu = \alpha_e E = (Ze)\, d = (Ze)\,\frac{4\pi\varepsilon_0 a^3}{Ze}E$$

or $$\alpha_e = 4\pi\varepsilon_0 a^3 \quad \text{...(3.32)}$$

Thus, the electronic polarizability is proportional to the volume of the atom and is independent of temperature.

The polarization vector $P = N\mu$

$$\therefore \quad P = N\alpha_e E \qquad ...(3.33)$$

But we know that $P = \varepsilon_0 E(\varepsilon_r - 1)$ ...(3.34)

Now from Eqs. (3.33) and (3.34)

$$N\alpha_e E = \varepsilon_0 E(\varepsilon_r - 1)$$

or,

$$\alpha_e = \frac{\varepsilon_0(\varepsilon_r - 1)}{N} \qquad ...(3.35)$$

For He, the value of α_e is 0.18×10^{-40} Fm^2, and for Ne, the value of α_e is 0.35×10^{-40} Fm^2. So, bigger the size of the atom, the value of α_e is larger.

3.8 POLARIZATION IN POLYATOMIC GASES

Let us consider a gas containing N molecules per m^3. Assume that each molecule has a permanent electric dipole moment μ.

The polarization is due to the electronic polarization P_e (nucleus shifted slightly relative to the center of the electron cloud), the ionic polarization P_i (ionic nature of bond between atoms) and the orientation polarization P_0 (due to rotation and alignment of the polar molecules in the external electric field). The total polarization of a polyatomic gas is given by

$$P = P_e + P_i + P_0$$

$$= N\alpha_e E + N\alpha_i E + N\frac{\mu^2}{3KT}E$$

$$= N\left(\alpha_e + \alpha_i + \frac{\mu^2}{3KT}\right)E \qquad ...(3.36)$$

Again $P = \varepsilon_0\chi_e E = (\varepsilon_r - 1)\,\varepsilon_0 E$ ...(3.37)

So,

$$(\varepsilon_r - 1)\,\varepsilon_0 = N\left(\alpha_e + \alpha_i + \frac{\mu^2}{3KT}\right) \qquad ...(3.38)$$

P_e and P_i are essentially independent of the temperature but P_0 is temperature dependent.

At a temperature T and zero external electric field, the molecules will be randomly oriented, so, zero polarization.

When there is an external electric field, the molecules will try to align with the field. Each polar molecule can be considered to be a simple dipole. The force on the dipole provides the torque to rotate the molecule so that they will be in the lowest state where they are parallel to the field. If there was no thermal motion, all dipoles would line up along the external field direction.

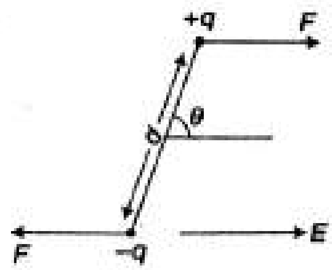


Fig. 3.9 Electric dipole placed in external electric field.

The electric force on the dipole produces a couple [Fig. 3.9] and the torque acting to rotate the dipole is

$$\tau = qEd\sin\theta = \mu E\sin\theta$$

or,
$$\vec{\tau} = \vec{\mu} \times \vec{E} \quad \text{...(3.39)}$$

The potential energy of the dipole for an arbitrary angle θ is given by

$$U(\theta) = -\mu E \cos\theta = -\vec{\mu} \cdot \vec{E} \quad \text{...(3.40)}$$

The dipole has the lowest potential energy when the dipole is parallel to the electric field and the highest potential energy when antiparallel to the field. For no thermal motion, all dipoles would line along the direction of the external electric field. But at a greater temperature, the thermal motion will be greater and there will be small alignment of the dipoles with the field.

Worked Out Problems

Example 3.1 Two parallel plates have equal and opposite charges. When the space between them is evacuated, the electric field intensity is 3×10^5 V/m and when the space is filled with dielectric, the electric intensity is 1.0×10^5 V/m. What is the induced charge density on the surface of the dielectric?

Sol. Given $E_0 = 3 \times 10^5$ V/m and $E = 1.0 \times 10^5$ V/m

We know that $E = E_0 - \dfrac{P}{\varepsilon_0}$

or,
$$P = \varepsilon_0(E_0 - E)$$
$$= 8.85 \times 10^{-12}\,(3 - 1) \times 10^5$$
$$= 1.77 \times 10^{-6}\ \text{C/m}^2$$

Again $P = \sigma_P$

So, $\sigma_P = 1.77 \times 10^{-6}\ \text{C/m}^2$

Example 3.2 Calculate the polarizability and relative permittivity in hydrogen gas with a density of 9.8×10^{26} atoms/m^3. [Given the radius of the hydrogen atom to be 0.50 Å].

Sol. Given $N = 9.8 \times 10^{26}$ atoms/m^3

$$a = 0.50 \times 10^{-10}\ \text{m}$$

We know that $\alpha_e = 4\pi\varepsilon_0 a^3$
$$= 4 \times 3.14 \times 8.85 \times 10^{-12} \times (0.50 \times 10^{-3})^3$$
$$= 1.38 \times 10^{-41}\ \text{Fm}^2$$

So, the polarizability $\alpha_e = 1.38 \times 10^{-41}\ \text{Fm}^2$

Again
$$\alpha_e = \frac{\varepsilon_0(\varepsilon_r - 1)}{N}$$

or,
$$\varepsilon_r = \frac{N\alpha_e}{\varepsilon_0} + 1$$

$$= \frac{9.8 \times 10^{26} \times 1.38 \times 10^{-41}}{8.85 \times 10^{-12}} + 1$$

$$= 1.001$$

So relative permittivity $\varepsilon_r = 1.001$

Example 3.3 In a dielectric material, $E_x = 5$ V/m and $\vec{P} = \frac{1}{10\pi}(3\hat{i} - \hat{j} + 4\hat{k})$ nC/m². Calculate (i) χ_e (ii) $\vec{E}$ (iii) $\vec{D}$

Sol. The polarization is given by

$$\vec{P} = \chi_e \varepsilon_0 \vec{E}$$

or,

$$\chi_e = \frac{\vec{P}}{\varepsilon_0 \vec{E}}$$

Here we consider only the x component

So, (i) $$\chi_e = \frac{P}{\varepsilon_0 E_x} = \frac{3 \times 10^{-9}}{10\pi} \times \frac{36\pi}{10^{-9} \times 5} = 2.16$$

(ii) $$\vec{E} = \frac{\vec{P}}{\varepsilon_0 \chi_e} = \frac{1}{10\pi}(3\hat{i} - \hat{j} + 4\hat{k}) \times \frac{36\pi}{10^{-9} \times 2.16}$$

$$= 5\hat{i} - \frac{5}{3}\hat{j} + \frac{20}{3}\hat{k} \text{ V/m}$$

(iii) $$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \frac{10^{-9}}{36\pi}\left(5\hat{i} - \frac{5}{3}\hat{j} + \frac{20}{3}\hat{k}\right) + \frac{10^{-9}}{10\pi}(3\hat{i} - \hat{j} + 4\hat{k})$$

$$= (139.7\,\hat{i} - 46.6\,\hat{j} + 186.3\,\hat{k}) \text{ pC/m}^2$$

Example 3.4 Calculate the dipole moment μ of a molecule of carbon tetrachloride (CCl_4) in a field 10^7 Vm^{-1}. [Given: Density = 1.60 gm/cm³, Molecular weight = 156, Relative permittivity $\varepsilon_r = 2.24$].

Sol. Molecular density $N = \frac{\text{Avogadro's number}}{\text{Molecular weight}} \times \text{Density}$

$$= \frac{6.02 \times 10^{23}}{156} \times 1.60$$

$$= 6.17 \times 10^{21} \text{ molecules/cm}^3$$

The dipole moment of a single molecule μ is

$$\mu = \frac{\varepsilon_0(\varepsilon_r - 1)}{N} E$$

$$= \frac{8.85 \times 10^{-12} \times 1.24 \times 10^7}{6.17 \times 10^{27}}$$

$$= 1.77 \times 10^{-32} \text{ C/m}$$

There are 74 electrons in each CCl_4 molecule

So, $\mu = 74\,ed$

or, $d = \dfrac{\mu}{74e} = \dfrac{1.77 \times 10^{-32}}{74 \times 1.6 \times 10^{-19}}$

$= 1.5 \times 10^{-15}$ m

Example 3.5 Dielectric constant of a gas at N.T.P is 1.00074. Calculate the dipole moment of each atom of the gas when it is held in an external field of 3×10^4 V/m.

Sol. Given $E = 3 \times 10^4$ V/m $= 3 \times 10^4$ N/C

and $K = \varepsilon_r = 1.00074$

We know $\varepsilon_r = 1 + \chi_e$

or, $\chi_e = \varepsilon_r - 1 = 1.00074 - 1 = 0.00074$

Polarization density $P = \varepsilon_0 \chi_e E$

$= 8.85 \times 10^{-12} \times 0.74 \times 10^{-3} \times 3 \times 10^4$

$= 1.96 \times 10^{-10}$ C/m

No. of atoms of gas per cubic meter

$$N = \frac{6.06 \times 10^{23}}{22.4 \times 10^{-3}} = 2.7 \times 10^{25}$$

So induced dipole moment of each atom

$$\mu = \frac{P}{N} = \frac{1.96 \times 10^{-10}}{2.7 \times 10^{25}}$$

$= 7.27 \times 10^{-36}$ C/m

Example 3.6 A dielectric cube of side L and center at the origin has a polarization vector given as $\vec{P} = \hat{i}x + \hat{j}y + \hat{k}z$. Find the volume and surface bound charge densities and show that the total bound charge vanishes in this case.

Sol. The bound surface charge density is $\sigma_b = \vec{P} \cdot \hat{n}$. For each of the six sides of the cube, there exists a surface charge density. For the side located at $x = L/2$, the surface charge density

$$\sigma_b^1 = \vec{P} \cdot \hat{i}|_{L/2} = (\hat{i}x + \hat{j}y + \hat{k}z) \cdot \hat{i}|_{L/2} = x|_{L/2} = L/2$$

$\therefore$ the total bound surface charge

$$q_{bs} = \int_s \sigma_b\, ds = 6\int_{-L/2}^{L/2}\int_{-L/2}^{L/2} \sigma_b\, dy\, dz = 3L^3$$

The bound volume charge density is

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -(1 + 1 + 1) = -3$$

$\therefore$ the total bound volume charge is

$$q_{bV} = \int \rho_b\, dV = -3\int dV = -3L^3$$

Hence, the total bound charge within the cube,

$$q = q_{bs} + q_{bv} = 3L^3 - 3L^3 = 0$$

So, total bound charge vanishes.

Example 3.7 The two plates of a parallel-plate capacitor are identical and carry equal amount of opposite charges. The separation between the plates is 5 mm and the space between the plates is filled with a dielectric of dielectric constant 3. The electric field within the dielectric is 10^6 V/m. Calculate (i) polarization vector $\vec{P}$, and (ii) displacement vector $\vec{D}$.

Sol. (i) The magnitude of the polarization vector is

$$\begin{aligned} P &= \varepsilon_0 (k-1) E \\ &= 8.85 \times 10^{-12} (3-1) \times 10^6 \\ &= 17.7 \times 10^{-6} \text{ C/m}^2 \\ &= 17.7 \ \mu \text{C/m}^2 \end{aligned}$$

(ii) The magnitude of the displacement vector is

$$\begin{aligned} D &= k\varepsilon_0 E \\ &= 3 \times 8.85 \times 10^{-12} \times 10^6 \\ &= 2.65 \times 10^{-7} \text{ C/m}^2 \\ &= 26.5 \ \mu \text{C/m}^2 \end{aligned}$$

Example 3.8 A dielectric cube of side L, centered at the origin, carries a "frozen-in" polarization $\vec{P} = k\vec{r}$, where k is a constant. Find all the bound charges and check that they add up to zero.

Sol. The bound volume charge density ρ_v is equal to

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 kr) = -3k$$

Since the bound volume charge density is constant, the total bound volume charge in the cube is equal to product of the charge density and the volume

$$q_{bv} = -3k L^3$$

The surface charge density σ_s is equal to

$$\sigma_s = \vec{P} \cdot \hat{n} = k\vec{r} \cdot \hat{n}$$

The scalar product between $\vec{r}$ and $\hat{n}$ can be evaluated easily (see Fig. 3.1W) and is equal to

$$\vec{r} \cdot \hat{n} = r \cos \theta = \frac{1}{2} L$$

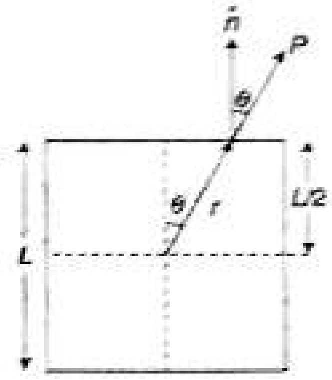


Fig. 3.1W Dielectric cube of side L.

Therefore the surface charge density is equal to

$$\sigma_s = k\vec{r} \cdot \hat{n} = \frac{1}{2} k \cdot L$$

The total surface charge density is equal to the product of the surface charge density and the total surface area of the cube

$$q_{bs} = \frac{1}{2} kL \times 6L^2 = 3\ kL^3$$

$\therefore$ the total bound charge on the cube is equal to

$$q = q_{bv} + q_{bs} = -3\ kL^3 + 3\ kL^3$$
$$= 0$$

Example 3.9 The space between the plates of a parallel-plate capacitor [Fig. 3.2W] is filled with two slabs of linear dielectric material. Each slab has thickness S, so that the total distance between the plates is $2S$. Slab 1 has a dielectric constant of 2, and slab 2 has a dielectric constant of 1.5. The free charge density on the top plate is σ and on the bottom plate is $-\sigma$.

(i) Find the electric displacement $\vec{D}$ in each slab.

(ii) Find the electric field $\vec{E}$ in each slab.

(iii) Find the polarization $\vec{P}$ in each slab.

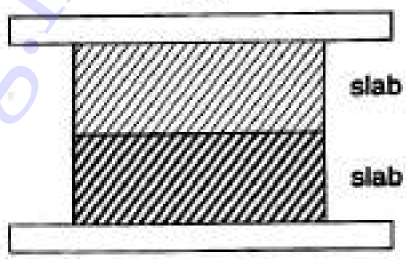


Fig. 3.2W Parallel-plate capacitor filled with two slabs of linear dielectric material.

Sol. (i) The electric displacement $\vec{D}_1$ in slab 1 can be calculated using Gauss' law. Consider a cylinder with cross-sectional area A and axis parallel to the z axis, being used as a gaussian surface. The top of the cylinder is located inside the top metal plate (where electric displacement is zero) and the bottom of the cylinder is located inside the dielectric of slab 1. The electric displacement is directed parallel to the z axis and pointed downwards. So, the displacement flux through this surface is equal to

$$\phi_D = D_1 A$$

The free charge enclosed by this surface is equal to

$$q_{\text{free}} = \sigma A$$

Combining these two we obtain

$$D_1 = \frac{\phi_D}{A} = \frac{q_{\text{free}}}{A} = \sigma$$

In vector notation $\vec{D}_1 = -\sigma\hat{k}$

Similarly for slab 2 $D_2 = -\sigma\hat{k}$

(ii) The electric field $\vec{E}_1$ in slab 1 is equal to

$$\vec{E}_1 = \frac{1}{k_1\varepsilon_0}\vec{D} = -\frac{\sigma}{k_1\varepsilon_0}\hat{k} = -\frac{\sigma}{2\varepsilon_0}\hat{k}$$

The electric field $\vec{E}_2$ in slab 2 is equal to

$$\vec{E}_2 = \frac{1}{k_2\varepsilon_0}\vec{D}_2 = -\frac{\sigma}{k_2\varepsilon_0}\hat{k} = -\frac{2\sigma}{3\varepsilon_0}\hat{k}$$

(iii) The polarization $\vec{P}_1$ of slab 1 is equal to

$$\vec{P}_1 = \vec{D}_1 - \varepsilon_0 \vec{E}_1$$
$$= -\sigma\hat{k} + \frac{\sigma}{2}\hat{k}$$
$$= -\frac{\sigma}{2}\hat{k}$$

The polarization $\vec{P}_2$ of slab 2 is equal to

$$\vec{P}_2 = \vec{D}_2 - \varepsilon_0 \vec{E}_2$$
$$= -\sigma\hat{k} + \frac{2\sigma}{3}\hat{k} = -\frac{\sigma}{3}\hat{k}$$

Example 3.10 The polarizability of a gas is 0.35×10^{-40} Fm^2. If the gas contains 2.7×10^{25} atoms/m^3 at 0°C and one atmospheric pressure, calculate its relative permittivity.

[Given $\alpha = 0.35 \times 10^{-40}$ Fm^2, $N = 2.7 \times 10^{25}$].

Sol. We know

$$\varepsilon_r = 1 + \frac{N\alpha}{\varepsilon_0}$$
$$= \frac{1 + 2.7 \times 10^{25} \times 0.35 \times 10^{-40}}{8.854 \times 10^{-12}}$$
$$= 1 + 0.1067 \times 10^{-3}$$
$$= 1.000107$$

So, the relative permittivity is 1.000107.

Example 3.11 A capacitor uses a dielectric material of relative permittivity $\varepsilon_r = 8$. It has an effective surface area of 0.036 m^2 with a capacitance of 6 μF. Calculate the field strength and dipole moment per unit volume if a potential difference of 15 V exists across the capacitor.

Sol. Field strength $E = \frac{V}{d}$ where $d = \frac{\varepsilon_0 \varepsilon_r A}{C}$

$$d = \frac{8.85 \times 10^{-12} \times 8 \times 0.036}{6 \times 10^{-6}}$$
$$= 0.42 \times 10^{-6} \text{ m}$$

or, field strength $$E = \frac{V}{d} = \frac{15}{0.42 \times 10^{-6}} = 35.3 \times 10^6 \text{ V/m.}$$

Dipole moment/unit volume $$= \varepsilon_0 (\varepsilon_r - 1) E$$
$$= 8.85 \times 10^{-12} (8 - 1) \times 3.5 \times 10^6$$
$$= 2.1 \times 10^{-5} \text{ C/m}^2$$

Review Exercises

Part 1: Multiple Choice Questions

1. In vacuum, electric susceptibility is
   (a) greater than 1 (b) zero (c) less than 1 (d) None of these
2. The relation between three electric vectors E, D and P is
   (a) $\vec{D} = \varepsilon_0 (\vec{E} + \vec{P})$ (b) $\vec{D} = \vec{E} + \varepsilon\vec{P}$ (c) $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ (d) $\vec{D} = \frac{1}{\varepsilon_0} (\vec{E} + \vec{P})$
3. The relation between electrical susceptibility and dielectric constant is
   (a) $\chi_e = \varepsilon_0 k$ (b) $\chi_e = k - 1$ (c) $\chi_e = k + 1$ (d) $\chi_e = \frac{k}{\varepsilon_0} - 1$
4. The dimension of polarizability in SI unit is
   (a) Fm^2 (b) Fm (c) Fm^{-1} (d) Fm^{-2}
5. Dielectrics are the substances which are
   (a) semiconductor (b) conductors (c) insulators (d) None of these
6. The electronic polarizability for a rare gas atom is
   (a) $\alpha_e = \frac{(\varepsilon_r - 1)}{\varepsilon_0 N}$ (b) $\alpha_e = N(\varepsilon_r - 1)$ (c) $\alpha_e = \frac{\varepsilon_0(\varepsilon_r - 1)}{N}$ (d) $\alpha_e = \frac{N}{\varepsilon_r - 1}$
7. A medium behaves like dielectric when the
   (a) displacement current is much greater than the conduction current
   (b) displacement current is zero
   (c) conduction current is almost zero
   (d) displacement current is equal to the conduction current
8. The total polarization of a polyatomic gas is
   (a) $P = N(\alpha_e + \alpha_i)$ (b) $P = N\left(\alpha_e + \alpha_i + \frac{\mu}{KT}\right) E$
   (c) $P = N\left(\alpha_e + \alpha_i + \frac{\mu^2}{3KT}\right) E$ (d) $P = \frac{N\mu E}{KT}$
9. The potential energy of the dipole for an arbitrary angle θ
   (a) $U(\theta) = -\vec{\mu} \times \vec{E}$ (b) $U(\theta) = -\vec{\mu} \cdot \vec{E}$ (c) $U(\theta) = \vec{E} \times \vec{\mu}$ (d) None of these
10. The relation between electronic polarizability and atomic radius for monatomic gases is
    (a) $\alpha_e = a^3$ (b) $\alpha_e = 4\pi\varepsilon_0 a^3$ (c) $\alpha_e = 4\pi\varepsilon_0 a^2$ (d) $\alpha_e = 4\pi\varepsilon_0 a$
11. For polar dielectrics, the orientation polarizability α_0 is given by
    (a) $\alpha_0 = \frac{3KT}{\mu^2}$ (b) $\alpha_0 = \frac{\mu^2}{3KT}$ (c) $\alpha_0 = \mu KT$ (d) None of these
12. Electrical susceptibility χ_e is
    (a) $\chi_e = \frac{P}{\varepsilon_0 E}$ (b) $\chi_e = \frac{P}{3\varepsilon_0 E}$ (c) $\chi_e = \varepsilon_0 EP$ (d) $\chi_e = \frac{3\varepsilon_0 E}{P}$

13. Generally, the dielectrics are
    (a) metallic materials of low specific resistance and have negative temperature coefficient of resistance
    (b) metallic materials of high specific resistance and have negative temperature coefficient of resistance.
    (c) metallic materials of high specific resistance and have positive temperature coefficient of resistance.
    (d) None of these
14. The ionic polarizability is
    (a) independent of temperature (b) depends on temperature
    (c) depends on square of the temperature (d) None of these

[**Ans.** 1 (b), 2 (c), 3 (b), 4 (a), 5 (c), 6 (c), 7 (a), 8 (c), 9 (b), 10 (b), 11 (b), 12 (a), 13 (b), 14 (a)]

Short Questions with Answers

1. Define polarization.

Ans. Polarization is defined as the process of creating or inducing dipoles in a dielectric material by an external electric field.

2. Define electrical susceptibility.

Ans. The electrical susceptibility is the ratio of polarization (P) to the net electric field ($\varepsilon_0 E$) as modified by the induced charges on the surface of the dielectric.

3. What are non-polar and polar dielectrics?

Ans. A dielectric, in the atoms and molecules of which, the center of gravity of positive and negative charges coincides, is called non-polar dielectric.

A dielectric, in the atoms and molecules of which, the center of gravity of positive and negative charges does not coincide, is called polar dielectric.

4. Define dielectric strength.

Ans. The dielectric strength of a dielectric is defined as the maximum value of the electric field that can be applied to the dielectric without its electric breakdown.

5. What do we mean by 'dielectric constant of glass is 8.5'?

Ans. Dielectric constant of a glass is 8.5 means that the ratio of the capacitance of a capacitor with glass as dielectric to the capacitance of the capacitor with air as dielectric.

6. Define polarizability.

Ans. Polarizability is the ability of an atom or a molecule to become polarized in the presence of an electric field.

7. What is electronic polarization?

Ans. Under the action of an external field, the electron clouds of atoms are displaced with respect to heavy fixed nuclei to a distance less than the dimensions of the atom. This is known as electronic polarization.

Part 2: Descriptive Questions

1. What are polar and non-polar dielectrics? What is meant by polarization of dielectric?
2. Show that $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$.
3. Derive a relation between electric susceptibility and atomic polarizability on the basis of microscopic description of matters at atomic level.
4. Define the following terms: (i) dipole moment, (ii) electrical susceptibility, (iii) relative dielectric constant, and (iv) polarization.
5. Explain the phenomenon of polarization of dielectric medium and show that $K = 1 + \chi_e$, where the symbols have their usual meanings.
6. Show that electronic polarizability α_e is

$$\alpha_e = \frac{\varepsilon_0(\varepsilon_r - 1)}{N},$$ where the symbols have their usual meanings.

7. Derive an expression for electronic polarization of a dielectric medium.
8. What are non-polar and polar dielectrics? Find out the relation between dielectric constant and electrical susceptibility.
9. Define polarization. Show that electronic polarizability is proportional to the volume of the atom and is independent of temperature.
10. Explain polarization in polyatomic gases.

Part 3: Numerical Problems

1. Copper is a FCC crystal with a lattice constant 3.6 Å and atomic number 29. If the average displacement of the electrons relative to the nucleus is 1×10^{-18} m. Applying an electric field, calculate the electronic polarization. **[Ans.** $P = 3.94 \times 10^{-7}$ C/m^2**]**
2. A sphere of radius R carries a polarization $\vec{P}(\vec{r}) = k\vec{r}$ where k is constant and $\vec{r}$ is the vector from the center.
   (a) Calculate the bound charges σ_b and ρ_s.
   (b) Find the field inside and outside the sphere. **[Ans.** $\sigma_b = KR$, $\rho_b = -3k, 0$**]**
3. Two parallel plates of a capacitor having equal and opposite charges are separated by 6.0 mm thick dielectric materials of dielectric constant 2.8. If the electric field strength inside be 10^5 V/m, determine polarization vector and displacement vector. **[Ans.** $P = 1.6 \times 10^{-6}$ C/m^2, $D = 2.5 \times 10^{-6}$ C/m^2**]**
4. Determine the electric susceptibility at 0°C for a gas whose dielectric constant at 0°C is 1.000041. **[Ans.** $\chi_e = 4.1 \times 10^{-5}$**]**
5. The electronic polarizability of argon atom is 1.75×10^{-40} Fm^2. What is the static dielectric constant of solid argon, if its density is 1.8×10^3 kg/m^3 (Given atomic weight of $A_r = 39.95$ and $N = 6.025 \times 10^{26}$/K mole). **[Ans.** $\varepsilon_r = 1.5367$**]**
6. The dielectric constant of helium at 0°C is 1.0000684. If the gas contains 2.7×10^{25} atoms/m^3, find the radius of the electron cloud. **[Ans.** 0.6×10^{-10} m**]**
7. A gas containing 2.7×10^{25} atoms per m^3 has polarizability of 0.2×10^{-40} Fm^2. Calculate the relative permittivity of the gas. **[Ans. 1.000061]**

CHAPTER 4

Magnetostatics

4.1 INTRODUCTION

Moving charges, or current, are the sources of magnetic fields in the same way as static charges are the sources of electric fields. By using Biot–Savart law and Ampere's law we can calculate magnetic fields due to different current distribution. The magnetic field, like the electric field, is a vector field.

4.2 ELECTRIC CURRENT

Electric charge in motion produces electric current, and the current-carrying medium may be called a conductor. Electric current is simply a flow of charge. If a charge ΔQ crosses an area in time Δt, then average electric current

$$I_{av} = \frac{\Delta Q}{\Delta t}$$

The current at time t is

$$I = \underset{\Delta t \to 0}{\text{Lt}} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt} \text{ coulomb/s} \quad ...(4.1)$$

If one coulomb of charge crosses an area in one second, the current is one ampere. The SI unit of current is ampere (A).

4.3 CURRENT DENSITY

A charged particle placed is an electric field $\vec{E}$ experiences a force $\vec{F}$. If the electric field $\vec{E}$ is constant, then the particle will have an average velocity and the average velocity of a charged particle is called drift velocity, $\vec{v}_d$.

Now we define a vector quantity known as *electric current density at a point*. To define current density, we consider a medium of uniform area of cross section S and volume charge density ρ. Then current I at a given point becomes

$$I = v_d \rho S \quad \text{...(4.2)}$$

For uniformly distributed current, the magnitude of current density

$$J = \frac{I}{S} = v_d \rho \quad \text{...(4.3)}$$

But if current density is not uniform, then we define it as

$$J = \underset{\Delta S \to 0}{\text{Lt}} \frac{\Delta I}{\Delta S}$$

Surface ΔS is normal to current direction. The total current through the entire surface S is

$$I = \int_S \vec{J} \cdot \vec{dS} \quad \text{...(4.4)}$$

For many materials, it is found that the current density in the steady state is linearly proportional to the applied electric field intensity. Therefore

$$\vec{J} \propto \vec{E} \quad \text{or,} \quad \vec{J} = \sigma \vec{E}\,* \quad \text{...(4.5)}$$

The constant of proportionality is known as the conductivity of the medium at a given temperature.

The drift velocity is directed along the direction of electric field and is related to by a constant called the mobility μ,

$$\vec{v}_d = \mu \vec{E} \quad \text{...(4.6)}$$

Mobility (μ) is defined as the drift velocity per unit electric field.

4.4 EQUATION OF CONTINUITY FOR CURRENT

Let us consider a volume V of the conductor enclosed by a surface S. If ρ is the volume charge density then the total charge (Q) within the volume is given by

$$Q = \int_V \rho \, dV$$

From conservation of charge (charge can neither be created nor destroyed), the amount of incoming flow of charge $\left(I = \oint_S \vec{J} \cdot \vec{dS}\right)$ must be equal to the rate of decrease of the total charge $\left(-\frac{dQ}{dt}\right)$ inside the volume.

i.e.
$$\oint_S \vec{J} \cdot \vec{dS} = -\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho \, dV = -\int_V \frac{\partial \rho}{\partial t} dV$$

By applying Gauss' divergence theorem

$$\int_V (\vec{\nabla} \cdot \vec{J})\, dV = -\int_V \frac{\partial \rho}{\partial t} dV$$

Therefore,

$$\int_V \left(\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t}\right) dV = 0$$

[*Note: From Eqs (4.2) and (4.6), we have $I = v_d \rho S = \mu \rho S E$

So, current density $J = \frac{I}{S} = \mu \rho E = \sigma E$ where $\sigma = \rho \mu$ is called electrical conductivity.

If $\rho = ne$ then $\sigma = ne\mu$ where n is the number of electrons per unit volume.]

For any arbitrary volume V, the integral must be zero

So, $$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \quad \ldots(4.7)$$

This is known as the ***equation of continuity*** and represents the mathematical statement of local charge conservation.

If the region does not contain a source or sink of charge then $\frac{\partial \rho}{\partial t} = 0$ [for steady current] and Eq. (4.7) reduces to

$$\vec{\nabla} \cdot \vec{J} = 0 \quad \ldots(4.8)$$

Equation (4.7) represents the condition of steady current flow.

4.5 FORCE ON A MOVING CHARGE IN A STATIC MAGNETIC FIELD

If a charged particle moves across a magnetic field, it is accelerated at right angles to its direction of motion.

The particle experiences a force at right angles to its velocity, with a magnitude proportional to the component of velocity, charge and magnetic field.

So, we can write the infinitesimal magnetic force $\vec{dF}$ on an infinitesimal charge dq moving with a velocity $\vec{v}$ in a steady magnetic field as

$$d\vec{F} = dq\,(\vec{v} \times \vec{B}) \quad \ldots(4.9)$$

Since the electric force on an infinitesimal charge dq in an electric field is $dq\,\vec{E}$, so the total electromagnetic force on an infinitesimal charge is

$$d\vec{F} = dq\,(\vec{E} + \vec{v} \times \vec{B}) \quad \ldots(4.10)$$

This is known as ***Lorentz force***.

Now for a single particle of charge e, the Lorentz force will be

$$\vec{F} = e\,(\vec{E} + \vec{v} \times \vec{B}) \quad \ldots(4.11)$$

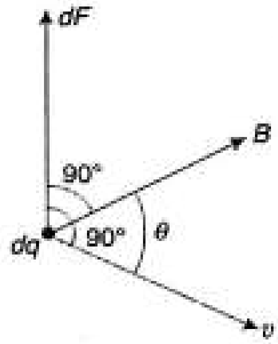


Fig. 4.1 Force on a moving charge in a constant magnetic field.

In the absence of an electric field ($\vec{E}$), Lorentz force (magnetic force) for a single particle of charge e is

$$\vec{F} = e\,(\vec{v} \times \vec{B}) \quad \ldots(4.12)$$

The magnitude of the Lorentz force is

$$F = evB \sin\theta \quad \ldots(4.13)$$

where θ is the angle between $\vec{v}$ and $\vec{B}$ [Fig. 4.1].

No workforce If infinitesimal charge dq moves through a small amount dl then $\vec{dl} = \vec{v}\,dt$, the work done is [from Eq. (4.9)]

$$dW = d\vec{F} \cdot \vec{dl} = dq\,(\vec{v} \times \vec{B}) \cdot \vec{dl} = dq\,(\vec{v} \times \vec{B}) \cdot \vec{v}\,dt$$
$$= 0$$

Since, $\vec{v} \times \vec{B}$ is perpendicular to $\vec{v}$, so $(\vec{v} \times \vec{B}) \cdot \vec{v} = 0$

So, magnetic force does no work on a charged particle to move with a velocity v in a static magnetic field $\vec{B}$.

4.6 FORCE ON CURRENT ELEMENT PLACED IN A STATIC MAGNETIC FIELD

We know from Lorentz force [Eq. (4.9)] that

$$d\vec{F} = dq\,(\vec{v} \times \vec{B})$$

where $\overrightarrow{dF}$ is the infinitesimal force on an infinitesimal charge dq moving with a velocity v in a steady magnetic field B [Fig. 4.2]. Again, if dq is the amount of charge flow through the cross section of a conductor in time dt, then electric current

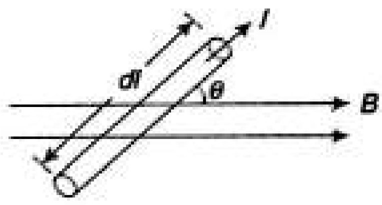


Fig. 4.2 Force on a current element which is in static magnetic field.

$$I = \frac{dq}{dt}$$

So,

$$d\vec{F} = I\,dt\,(\vec{v} \times \vec{B})$$
$$= I\,(dt\,\vec{v} \times \vec{B})$$

$v\,dt$, a segment of length (dl) gives the indication of the distance travelled by a particle in time dt, then

$$d\vec{F} = I\,(\vec{dl} \times \vec{B}) \quad \text{...(4.14)}$$

For a finite length of the conductor, the magnetic force

$$\vec{F} = I \int (\vec{dl} \times \vec{B}) \quad \text{...(4.15)}$$

Considering the current as the vector along the length dl, the magnetic force per unit length

$$\vec{F} = \vec{I} \times \vec{B} \quad \text{...(4.16)}$$

Thus magnetic force depends only on the total current and applied magnetic field and is independent of the amount of charge carried by each particle. The direction of current is perpendicular to the plane containing $\vec{B}$ and $\vec{I}$.

4.7 BIOT–SAVART LAW

Steady currents produce magnetic fields which are constant in time. The right-hand thumb rule, Fig. 4.3(a, b) gives the direction of the magnetic field. According *to* **the thumb rule, if the current flows in the thumb's direction, right-handed fingers curl around in the direction of the magnetic field.** The symbol ⊙ gives the direction of the magnetic field perpendicular to the plane of the paper and ⊗ gives the direction of the magnetic field into the plane of the paper. If the fingers are curled along the current, the stretched thumb will point towards the magnetic field [Fig. 4.3(c)].

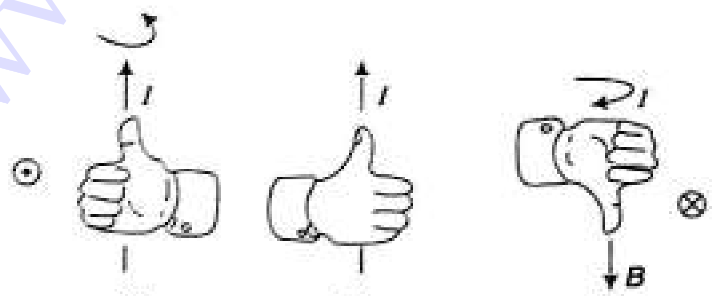


Fig. 4.3 Direction magnetic field by using the right-hand thumb rule.

The Biot–Savart law states that the magnetic field $\overrightarrow{dB}$ due to a current element $I\,\overrightarrow{dl}$ [Fig. 4.4(a, b)] is given by

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\vec{dl} \times \hat{r}}{r^2} \qquad \text{...(4.17)}$$

where $\hat{r}$ is the unit vector from the point of interest $I\,\vec{dl}$ towards the point of interest and r is the distance between the current element $I\,\vec{dl}$ and the point of observation.

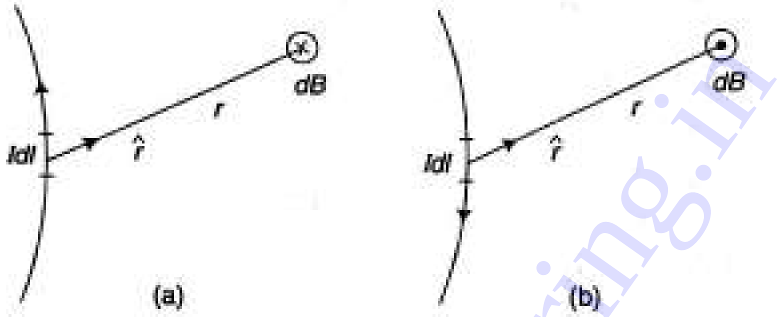


Fig. 4.4 Biot–Savart law for current element *Idl*.

The total field B due to the whole conductor can be obtained after taking the integration

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{I\,\vec{dl} \times \hat{r}}{r^2} \qquad \text{...(4.18)}$$

or,

$$\vec{B} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J} \times \hat{r}}{r^2}\,dV \qquad \text{...(4.19)}$$

where $\quad I\,\vec{dl} = \vec{J}dV$

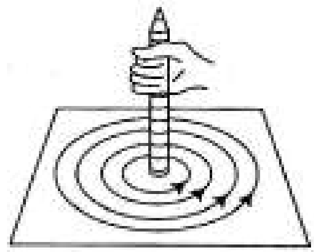


Fig. 4.5 Maxwell's cork-screw rule.

The constant μ_0 is called the permeability of free space and its value

$$\mu_0 = 4\pi \times 10^{-7}\ \text{N/A}^2.$$

Equation (4.18) is the integral form of the ***Biot–Savart law***.

The direction of magnetic field can also be obtained by **Maxwell's cork-screw** rule. Maxwell's cork-screw rule points that **if the direction of the current through a conductor is represented by the linear motion of the cork-screw motion then the direction of the magnetic field can be represented by the direction of rotation of the cork** [Fig. 4.5].

4.8 APPLICATIONS OF BIOT–SAVART LAW

(i) Magnetic field due to a long straight wire

In the diagram [Fig. 4.6] $(\vec{dl} \times \hat{r})$ points into (X) the paper. From Biot–Savart law, dB at P is

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\vec{dl} \times \hat{r}}{r^2}$$

where Idl is the small current element at a distance l from O. The magnitude of the magnetic field dB at the point P at a distance x from the wire due to the current element Idl is

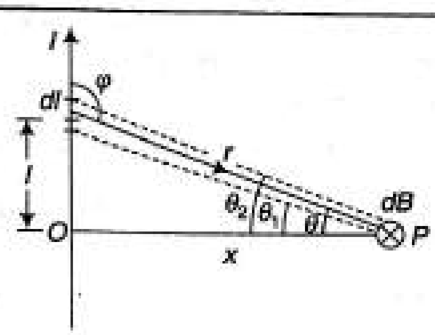


Fig. 4.6 Application of Biot–Savart law in case of a straight wire.

$$dB = \frac{\mu_0}{4\pi}\left|\frac{I\,\vec{dl}\times\hat{r}}{r^2}\right|$$

$$= \frac{\mu_0}{4\pi}\frac{I\,dl\sin\varphi}{r^2} \quad \text{...(4.20)}$$

Also, $l = x\tan\theta$

$$dl = x\sec^2\theta\, d\theta \text{ and } \frac{x}{r} = \cos\theta, \text{ so } \frac{1}{r^2} = \frac{\cos^2\theta}{x^2}.$$

and $\sin\varphi = \sin(90° - \theta) = \cos\theta$

Thus, from Eq. (4.20), we have

$$dB = \frac{\mu_0}{4\pi}\frac{I(x\sec^2\theta\, d\theta)\cos\theta}{(x\sec\theta)^2} = \frac{\mu_0 I}{4\pi x}\cos\theta\, d\theta$$

Now, total magnetic field B at P in terms of the initial and final angles θ_1 and θ_2 is

$$\therefore \quad B = \int dB = \frac{\mu_0 I}{4\pi x}\int_{\theta_1}^{\theta_2}\cos\theta\, d\theta = \frac{\mu_0 I}{4\pi x}(\sin\theta_2 - \sin\theta_1) \quad \text{...(4.21)}$$

Now for infinite wire $\theta_1 = -\frac{\pi}{2}$ and $\theta_2 = \frac{\pi}{2}$

Magnetic field $B = \frac{\mu_0 I}{4\pi x}\left(\sin\frac{\pi}{2} + \sin\frac{\pi}{2}\right)$

$$= \frac{\mu_0}{4\pi}\left(\frac{2I}{x}\right) \quad \text{...(4.22)}$$

Equation (4.21) shows that magnetic field due to a straight infinite wire is inversely proportional to the distance from the wire.

(ii) Magnetic field at a point on the axis of a circular loop

Here, we consider the center of the loop to be at the origin and its axis is along the x direction [Fig. 4.7].

Now according to Biot–Savart law, the magnetic field at P due to the current element Idl of the loop is given by

$$dB = \frac{\mu_0}{4\pi}\frac{I\,dl}{r^2}$$

The total magnetic field at P is due to the current element $I\,dl$. On both sides of the loop is $dB' = 2\,dB\sin\theta$. Perpendicular component $dB\cos\theta$ due to current elements of both sides of the loop cancels out which is shown in Fig. 4.7.

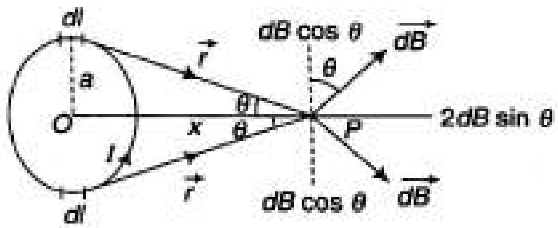


Fig. 4.7 Magnetic field on at a point on the axis of a circular current loop.

So, the total magnetic field at P due to current element Idl on both sides is

$$dB' = 2\,dB\sin\theta = 2\times\frac{\mu_0}{4\pi}\frac{I\,dl}{r^2}\sin\theta$$

Again $r^2 = x^2 + a^2$ and $\sin\theta = \frac{a}{r} = \frac{a}{(x^2+a^2)^{1/2}}$

So, $$dB' = 2\frac{\mu_0}{4\pi}\frac{I\,dl\,a}{(x^2+a^2)^{3/2}}$$

Hence, total magnetic field at P is

$$B' = \int dB' = \frac{2\mu_0}{4\pi}\frac{Ia}{(x^2+a^2)^{3/2}}\int_0^{\pi a} dl$$

$$= \frac{\mu_0}{4\pi}\frac{2\pi I a^2}{(x^2+a^2)^{3/2}} \qquad \text{...(4.23)}$$

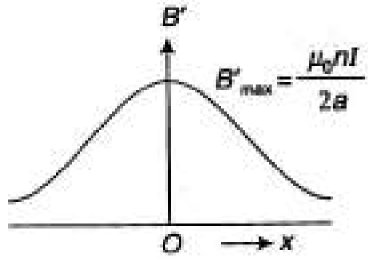


Fig. 4.8 Variation of magnetic field on the axis of a circular were.

Now for n number of turns, the total magnetic field will be

$$B' = \frac{\mu_0}{4\pi}\frac{2\pi nIa^2}{(x^2+a^2)^{3/2}} \qquad \text{...(4.24)}$$

At the center of the loop $x = 0$ so,

$$B'_{max} = \frac{\mu_0}{4\pi}\left(\frac{2\pi n I}{a}\right) \qquad \text{...(4.25)}$$

The variation of magnetic field (B') with the axis of the coil is shown in Fig. 4.8. The graph shows that the magnetic field is maximum at the center of the coil.

(iii) *Magnetic field along the axis of a solenoid*

A solenoid is a wire wound closely in the form of a helix around a right circular cylinder. Generally, the length of the solenoid is large as compared to the transverse dimension. To find out the magnetic field $\vec{B}$ at an axial point P at a distance l from O of the solenoid of radius a and carrying a current I, we consider an elementary length dx at a distance x from O [Fig. 4.9]. The current in the section dx of the coil is $nIdx$, where n is the number of turns (N) per unit length, i.e., $\frac{N}{L}$.

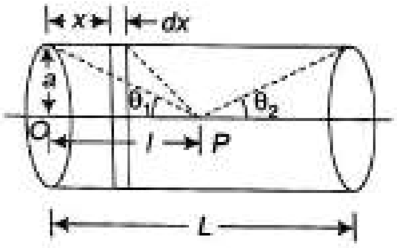


Fig. 4.9

The field at P due to the element dx is

$$dB = \frac{\mu_0(ndx)\,Ia^2}{2[(l-x)^2+a^2]^{3/2}} \qquad \text{...[4.26]}$$

The total magnetic field B at P due to the entire solenoid is

$$B = \int dB = \int_0^L \frac{\mu_0 nIa^2}{2}\frac{dx}{[(l-x)^2+a^2]^{3/2}}$$

$$= \frac{\mu_0 nI}{2}\left[\frac{x-l}{\sqrt{(l-x)^2+a^2}}\right]_0^L$$

$$= \frac{\mu_0 nI}{2}\left[\frac{l}{\sqrt{l^2+a^2}} + \frac{L-l}{\sqrt{(L-l)^2+a^2}}\right] \qquad \text{...(4.27)}$$

Again, from Fig. 4.9, $\cos\theta_1 = \frac{l}{\sqrt{l^2+a^2}}$ and $\cos\theta_2 = \frac{L-l}{\sqrt{(L-l)^2+a^2}}$

So from Eq. (4.27), total magnetic field

$$B = \frac{\mu_0 nI}{2}(\cos\theta_1 + \cos\theta_2) \qquad \text{...(4.28)}$$

Case I For an infinite solenoid $\theta_1 = \theta_2 = 0$; then $B = \mu_0 nI = \mu_0 \dfrac{NI}{L}$ ...(4.29)

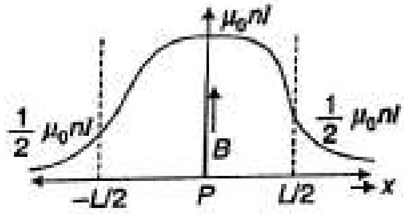


Fig. 4.10 Variation *B* along the axis of a solenoid.

Case II If P is at the right end ($\theta_1 = 0$, $\theta_2 = 90°$) or at the left end ($\theta_1 = 90°$, $\theta_2 = 0$) of the solenoid then

$$B = \frac{\mu_0 nI}{2} = \frac{\mu_0 N}{2L} I \qquad ...(4.30)$$

The variation of the magnetic field along the axis of the solenoid is shown in Fig. 4.10. Figure 4.10 shows that for a long solenoid, the magnetic field is maximum at center (P) and just half at the ends of the solenoid.

4.9 FORCE BETWEEN TWO STRAIGHT PARALLEL WIRES

Let C_1 and C_2 be two long parallel wires carrying currents I_1 and I_2 respectively in the same direction [Fig. 4.11(a)]. The separation between the wires is d. The magnetic field at dl, a small element of the wire C_2 due to the current I_1 in C_1 is

$$B = \frac{\mu_0}{4\pi}\left(\frac{2I_1}{d}\right) \qquad ...(4.31)$$

The direction of B is perpendicular to C_2. The magnetic force at the element dl due to B is

$$d\vec{F} = I_2 \, \vec{dl} \times \vec{B}$$

or,

$$|d\vec{F}| = I_2 \, dl \, \frac{\mu_0}{4\pi}\left(\frac{2I_1}{d}\right)$$

$$= \frac{\mu_0 I_1 I_2}{2\pi d} \, dl \qquad ...(4.32)$$

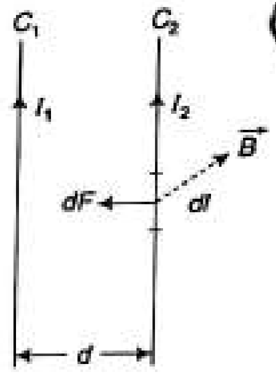


Fig. 4.11 (a) Magnetic force between the parallel wires carrying current.

The vector product $(\vec{dl} \times \vec{B})$ has a direction towards the wire C_1. So, the direction of the force $d\vec{F}$ is towards the wire C_1. The force per unit length of the wire C_2 due to the wire C_1 is

$$F = \frac{\mu_0 I_1 I_2}{2\pi d} \qquad ...(4.33)$$

If the parallel wires carrying currents are in opposite directions, the force will be repulsive in nature.

4.10 MAGNETIC FORCE BETWEEN TWO FINITE ELEMENTS OF CURRENT

From Fig. 4.11(b), the magnetic field due to current I_1 of the conductor A on dl_2 at a distance r is

$$dB = \frac{\mu_0}{4\pi} \frac{I_1 \vec{dl}_1 \times \hat{r}}{r^2}$$

where $I_1 \vec{dl}_1$ is the current element of the conductor A.

Now, the force on current I_2, due to current I_1 is

$$d\vec{F} = I_2 \vec{dl}_2 \times d\vec{B} = \frac{\mu_0}{4\pi} I_1 I_2 \frac{\vec{dl}_2 \times (\vec{dl}_1 \times \hat{r})}{r^2} \qquad ...(4.34)$$

where $I_2 \vec{dl}_2$ is the current elements of the conductor B and $\hat{r}$ is the unit vector in the direction of r.

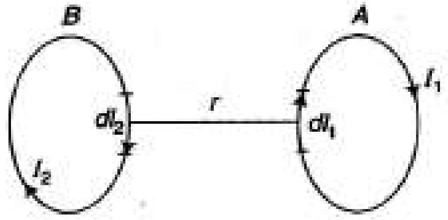


Fig. 4.11 (b) Magnetic field between two finite current-carrying elements.

4.11 DIVERGENCE OF MAGNETIC FIELD

We know from Biot–Savart law, the magnetic field at P [Fig. 4.12] is

$$\vec{B} = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J} \times \hat{r}}{r^2} dV \qquad ...(4.35)$$

where dV $(dx'dy'dz')$ is the volume element or source element of current. The position vectors of the source and field points are suppose $\vec{r}'$ $(\hat{i}x' + \hat{j}y' + \hat{k}z')$ and $\vec{r}$ $(\hat{i}x + \hat{j}y + \hat{k}z)$. Again current density $\vec{J}$ is the function of (x', y', z') and magnetic field $\vec{B}$ is the function of (x, y, z).

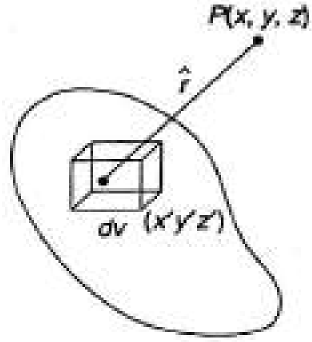


Fig. 4.12 Divergence of magnetic field.

Now taking divergence of equation

$$\vec{\nabla} \cdot \vec{B}(r) = \frac{\mu_0}{4\pi} \int_V \vec{\nabla} \cdot \left(\vec{J}(r') \times \frac{\hat{r}}{r^2} \right) dV$$

Now using vector identity

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot \vec{\nabla} \times \vec{A} - \vec{A} \cdot \vec{\nabla} \times \vec{B}$$

we have
$$\vec{\nabla} \cdot \vec{B}(r) = \frac{\mu_0}{4\pi} \int_V \left[\frac{\hat{r}}{r^2} \cdot (\vec{\nabla} \times J(r') - \vec{J}(r') \cdot \vec{\nabla} \times \frac{\hat{r}}{r^2} \right] dV$$

But $\vec{\nabla} \times \vec{J}(r') = 0$ because $\vec{\nabla}$ operator derivatives with respect to the field point $(\vec{r})$ while $\vec{J}$ is the function of the source point$(\vec{r}')$ only.

Again $\vec{\nabla} \times \frac{\hat{r}}{r^2} = 0$ from vector calculus.

Hence, $$\vec{\nabla} \cdot \vec{B} = 0 \qquad ...(4.36)$$

Thus, the magnetic field is solenoidal.

Physical significance Since magnetic lines of force are continuous, the magnetic flux entering any region of volume is equal to the magnetic flux leaving the volume. Hence the net flux over the volume is equal to zero. Divergence of magnetic field B is defined as the flux of B through the surface enclosing per unit volume. Since, net flux per unit volume is zero, so mathematically

$$\vec{\nabla} \cdot \vec{B} = 0$$

which is known as the differential form of Gauss' law in magnetostatics. Comparing it with Gauss' law in electrostatics $\left(\vec{\nabla} \cdot E = \frac{\rho}{\varepsilon_0} \right)$, we may conclude that monopole does not exist.

The magnetic flux $\varphi_B = \oint_S \vec{B} \cdot \vec{dS}$

By applying the divergence theorem,

$$\oint_S \vec{B} \cdot \vec{dS} = \int_V (\vec{\nabla} \cdot \vec{B})\, dV$$
$$= 0 \qquad \ldots(4.36a)$$

Equation (4.36a) states that there are no magnetic flux sources, and magnetic flux lines always close upon themselves. So, there is no source or sink of magnetic flux (law of conservation of magnetic flux), i.e., magnetic monopole does not exist.

Equations (4.36) and (4.36a) are differential and integral forms of Gauss' law in magnetostatics.

4.12 AMPERE'S CIRCUITAL LAW

Ampere's circuital law states that the line integral of the magnetic field $\vec{B}$ around any closed path is equal to μ_0 times the net current enclosed by the path.

Mathematically, $$\int_c \vec{B} \cdot \vec{dl} = \mu_0 I \qquad \ldots(4.37)$$

Amperian loop To explain net current we consider a loop known as Amperian loop that encloses four wires of currents i_1, i_2, i_3 and i_4 but i_5 and i_6 is outside the loop [Fig. 4.13]. Since the direction of the loop is clockwise, the positive side of the plane (lower plane) is away from the viewer, i.e., into the plane of the paper. So i_1 and i_3 are positive and i_2 and i_4 are negative. Hence, total current is $i_1 + i_3 - (i_2 + i_4)$. Any current outside the loop is not included in writing the right-hand side of Eq. (4.37).

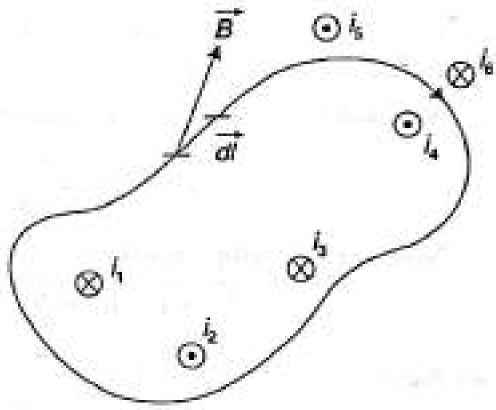


Fig. 4.13 Amperian loop.

Ampere's law is valid for a closed path of any shape. If the path does not include the current then

$$\int_c \vec{B} \cdot \vec{dl} = 0$$

To find magnetic field, there must be two conditions: (i) At each point on the closed path, $\vec{B}$ is either tangential or normal to the path. (ii) If $\vec{B}$ is tangential then at all points of the path, $\vec{B}$ must have the same value.

Ampere's law plays the same role in magnetostatics as Gauss' law plays in electrostatics and is very helpful in determining the magnetic field around a conductor for symmetrical distribution.

4.12.1 Differential Form of Ampere's Law

The total current (I) enclosed by a path enclosing a surfaces S is

$$I = \iint_S \vec{J} \cdot \vec{dS}$$

where $\vec{J}$ is the current density in an element $\vec{dS}$ of the surface bounded by the closed path.

Now, Ampere's law in terms of current density $\vec{J}$ is

$$\int_c \vec{B} \cdot \vec{dl} = \mu_0 \iint_S \vec{J} \cdot \vec{dS} \qquad \ldots(4.38)$$

which is the integral form of the Ampere's circuital law.

Now applying Stoke's law

$$\int_c \vec{B}\cdot\vec{dl} = \iint_S (\vec{\nabla}\times\vec{B})\cdot\vec{dS} \qquad ...(4.39)$$

Now from Eqs. (4.38) and (4.39), we get

$$\iint_S (\vec{\nabla}\times\vec{B})\cdot\vec{dS} = \mu_0 \iint_S \vec{J}\cdot\vec{dS}$$

or, $$\iint_S [\vec{\nabla}\times\vec{B} - \mu_0\vec{J}]\cdot\vec{dS} = 0$$

Since the surface element $\vec{dS}$ is arbitrary, so

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \qquad ...(4.40)$$

which is the differential form of Ampere's law.

In electrostatics $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$, $\vec{\nabla}\times\vec{E} = 0$

In magnetostatics $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

4.13 APPLICATIONS OF AMPERE'S CIRCUITAL LAW

Ampere's circuital law is applicable for line currents, sheet currents or volume currents.

4.13.1 Long Straight Cylindrical Wire

Let us consider an infinitely long conducting wire of radius R, carrying current I as shown in Fig. 4.14. Suppose the current distribution is uniform throughout the cross section of the wire. Now applying Eq. (4.37) to an amperian loop at A_1 [Fig. 4.14] of radius r is

$$\oint \vec{B}.\vec{dl} = \mu_0 I_1 \qquad ...(4.41)$$

where $$I_1 = \frac{I}{\pi R^2}\times\pi r^2 = I\frac{r^2}{R^2}$$

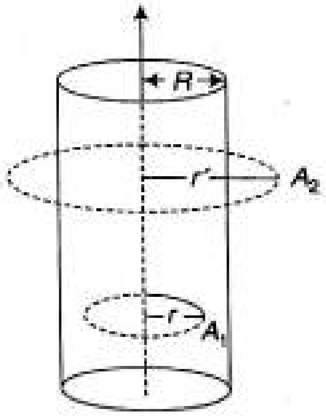


Fig. 4.14 Magnetic field due to a long straight wire of radius *R*.

Now, from Eq. (4.40)

$$\oint \vec{B}\cdot\vec{dl} = I\frac{r^2}{R^2}\mu_0$$

or, $$B\times 2\pi r = \mu_0\frac{I\,r^2}{R^2}$$

so that $$B = \mu_0\frac{Ir}{2\pi R^2} \qquad ...(4.42)$$

within the wire. Now, outside the wire, applying Eq. (4.37) to an amperian loop at A_2 [Fig. 4.14] of radius $r' > R$ is

$$\int_c \vec{B}\cdot\vec{dl} = \mu_0 I$$

or, $$B\times 2\pi r' = \mu_0 I$$

or, $$B = \frac{\mu_0 I}{2\pi r'} \qquad ...(4.43)$$

At the surface of the wire, $r' = R$, $B = \dfrac{\mu_0 I}{2\pi R}$ ...(4.44)

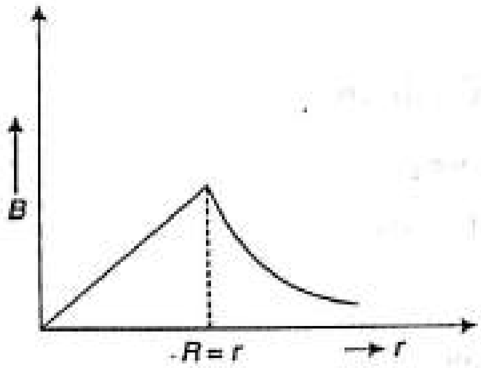


Fig. 4.15 Variation of magnetic field with distance of a current carrying cylindrical wire.

The variation of the magnetic field with distance from the axis of the cylinder is shown in Fig. 4.15.

Note:

(i) For long straight conducting wire of infinite length carrying current I, the magnetic field at any point at a perpendicular distance r from the wire will be $B = \frac{\mu_0 I}{2\pi r}$.

(ii) For a hollow cylinder, since the current I exists only on the surface of the cylinder and inside the cylinder current is zero, so magnetic field inside the cylinder will be zero.

4.13.2 Magnetic Field Inside a Long Solenoid

When a current (I) carrying wire is wound tightly on the surface of a cylindrical tube, we get a solenoid. Generally, the length (L) of the solenoid is large as compared to the transverse dimension. If N is the total number of turns over a length L, we get $\frac{N}{L} = n$ as the number of turns per unit length of the solenoid. Keeping the product nI fixed, if we make n very large and corresponding I very small, then we obtain a surface current of value nI over the curved surface of the cylinder. It turns out that the magnetic field inside a closely wound solenoid is almost uniform over its cross section and can be taken to be negligible outside the volume of the solenoid. Ampere's law thus can easily applied to find out the value of $\vec{B}$ inside the solenoid.

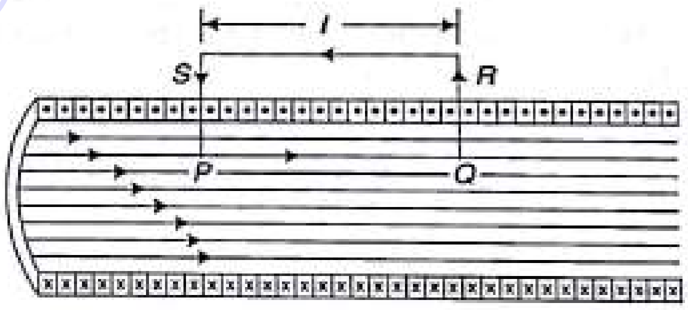


Fig. 4.16 Magnetic field inside a solenoid.

In Fig. 4.16, we draw a rectangle $PQRS$ of length l. The line PQ is parallel to the solenoid axis and hence parallel to the field $\vec{B}$ inside the solenoid. Thus,

$$\int_P^Q \vec{B} \cdot \vec{dl} = Bl$$

Along QR, RS and SP, $\vec{B}.\vec{dl}$ is zero everywhere as $\vec{B}$ is either zero (outside the solenoid) or perpendicular to $\vec{dl}$ (inside the solenoid).

Thus, from $PQRSP$,

$$\oint \vec{B} \cdot \vec{dl} = Bl \qquad \text{...(4.45)}$$

If n be the number of turns per unit length, then total of nl turns cross the rectangle $PQRS$. Each turn carries a current I. Hence net current passes through the area $PQRS$ is nlI.

Now, from Eq. (4.45) we have from Ampere's law

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 \, nlI$$

or, $$Bl = \mu_0 \, nlI$$

or, $$B = \mu_0 \, nI \qquad \text{...(4.46)}$$

The above equations gives the magnetic field inside a long closely wound solenoid. The relation does not depend on the diameter or the length of the solenoid and magnetic field B is constant over the solenoid cross section.

4.13.3 Magnetic Field Due to Toroid

An endless solenoid in the form of circular shape is called toroid. The magnetic field in such a toroid can be obtained by using Ampere's law.

Let P be a point on the concentric circular path at which magnetic field $\vec{B}$ is to be calculated. By symmetry, the field will have equal magnitude at all points of this circle [Fig. 4.17]. Let the distance of P from the center be r. The field B is tangential at every point of the circle. Hence,

$$\oint \vec{B} \cdot \vec{dl} = B \int dl = B \times 2\pi r$$
$$= \mu_0 NI \qquad \text{...(4.47)}$$

where N is the total number of turns and the current crossing the area bounded by the circle is NI.

So, $$B = \frac{\mu_0 NI}{2\pi r} \qquad \text{...(4.48)}$$

Thus B is inversely proportional to r. If the cross section of the toroid is very small, the variation in r can be neglected and $\frac{N}{2\pi r}$ can be written as n, the number of turns per unit length. So

$$B = \mu_0 nI \qquad \text{...(4.49)}$$

The field at an external point (P') of the toroid, from Ampere's law

$$\oint B \cdot dl = 0 \text{ or, } B = 0 \qquad \text{...(4.50)}$$

Thus the field outside the toroid is zero.

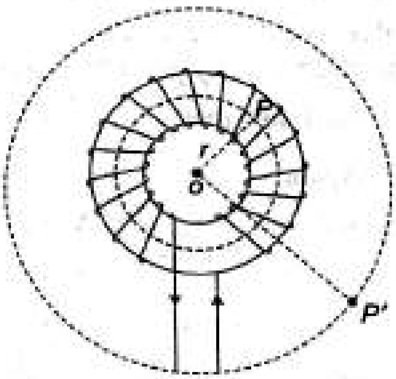


Fig. 4.17 Magnetic field due to a toroid.

4.14 MAGNETIC POTENTIALS

4.14.1 Magnetic Scalar Potential

In electrostatics, scalar potential V plays an important role to find electric field intensity $\vec{E}$. Since $\vec{\nabla} \times \vec{E} = 0$, $\vec{E}$ can be expressed as the gradient of a scalar quantity V. The two powerful relations are:

$$\vec{E} = -\vec{\nabla} V \text{ and} \qquad \text{...(4.51)}$$

$$\nabla^2 V = 0 \qquad \text{...(4.52)}$$

If in some region of space, current density $J = 0$, then from Ampere's circuital law in magnetostatics, $\vec{\nabla} \times \vec{B} = 0$. We may therefore express $\vec{B}$ as a gradient of scalar quantity V_m

i.e. $$\vec{B} = -\vec{\nabla} V_m \qquad \text{...(4.53)}$$

where V_m is called the magnetic scalar potential.

Since $\vec{\nabla} \cdot \vec{B} = 0$ so $\vec{\nabla} \cdot (-\vec{\nabla} V_m) = 0$ or, $\nabla^2 V_m = 0$ ...(4.54)

We see that V_m satisfies *Laplace's equation* in homogeneous magnetic materials, it is not defined in any region where the current density exists.

The magnetic scalar potential may be defined as a scalar whose negative gradient at any point gives the magnetic induction at that point due to a close loop of carrying current.

The magnetic scalar potential is useful in describing the magnetic field around a current source

4.14.2 Magnetic Vector Potential

Gauss' law in magnetostatics state that always $\vec{\nabla} \cdot \vec{B} = 0$. Again we know that divergence of any curl = 0, i.e., $\vec{\nabla} . \vec{\nabla} \times \vec{A} = 0$. Since the curl of $\vec{B}$ is not necessarily zero (only if $\vec{J} = 0$, $\vec{\nabla} \times \vec{B} = 0$), so $\vec{B}$ can't be the gradient of a scalar potential in general but as the curl of a vector field, in the form

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad \text{... (4.55)}$$

The vector function $\vec{A}$ which satisfies Eq. (4.54) is known as *vector potential*. The vector potential $\vec{A}$ is as important in magnetostatics as the scalar potential function V in electrostatics. The vector potential $\vec{A}$ does not have any physical significance. It can help to determine $\vec{B}$ at a given point, since $\vec{B}$ is the space derivative of $\vec{A}$. The magnetic vector potential may be defined as a vector, the curl of which gives the magnetic induction produced at any point by a closed-loop carrying current.

We know that $$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \qquad \text{...(4.56)}$$

Again $$\vec{B} = \vec{\nabla} \times \vec{A}$$

So, $$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \vec{\nabla} \times \vec{A} = \mu_0 \vec{J} \qquad \text{...(4.57)}$$

but, $$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} . \vec{A}) - \nabla^2 \vec{A}$$

For steady current, we take $\vec{\nabla} . \vec{A} = 0$

So, $$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = -\nabla^2 \vec{A}$$

Now from Eq. (4.57) for dc current only, $\nabla^2 \vec{A} = -\mu_0 \vec{J}$ ...(4.58)

Equation (4.58) is the same as Poisson's equation in electrostatics,

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} \qquad \text{...(4.59)}$$

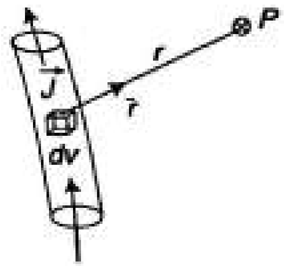


Fig. 4.18 Magnetic vector potential at a distance r from a current element.

where V is the electrostatics potential and satisfies

$$V = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho\, dV}{r} \qquad \text{...(4.60)}$$

Similarly for Eq. (4.58) we have the general solution

$$\vec{A} = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}}{r} dV \qquad \text{...(4.61)}$$

The magnetic vector potential is useful for studying radiation in transmission lines, wave guides, antennas.

Here r is the distance from the current element to the point at which the magnetic vector potential is being calculated [Fig. 4.18]. Thus the field $\vec{B}$ produced by a current can be calculated by first determining $\vec{A}$ using Eq. (4.61) and substituting this in Eq. (4.55).

Worked Out Problems

Example 4.1 How many electrons pass through a wire in 1 minute if the current passing through the wire is 200 mA?

Sol. We know $I = \frac{q}{t} = \frac{ne}{t}$

or, $n = \frac{It}{e} = \frac{200 \times 10^{-3} \times 60}{1.6 \times 10^{-19}} = 7.5 \times 10^{19}.$

Example 4.2 What is the drift velocity of electrons in a Cu conductor having a cross-sectional area of 5×10^{-6} m^2 if the current is 10 A? Assume that there are 8×10^{28} electrons/m^3.

Sol. Here, area of cross section $A = 5 \times 10^{-6}$ m^2

Current $I = 10$ A

Number density of free electrons, $n = 8 \times 10^{28}$ electrons/m^3.

We know $v_d = \frac{I}{neA} = \frac{10}{8 \times 10^{28} \times 1.6 \times 10^{-19} \times 5 \times 10^{-6}} = 1.56 \times 10^{-4}$ ms^{-1}.

Example 4.3 Calculate the magnetic field at the center of a regular hexagon [Fig. 4.1W] of side a meter and carrying a current I A.

Sol. The magnetic field at O due to part AB of the hexagon is

$$B' = \frac{\mu_0}{4\pi}\frac{I}{r}(\sin\theta_1 + \sin\theta_2)$$

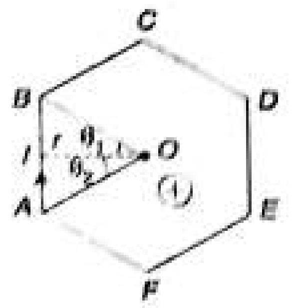


Fig. 4.1W Magnetic field at the center of a hexagon.

Here $r = \frac{\sqrt{3}}{2}a$ and $\theta_1 = \theta_2 = 30^\circ$, $\frac{\mu_0}{4\pi} = 10^{-7}$

So
$$B' = 10^{-7}\frac{I}{\frac{\sqrt{3}}{2}a}(\sin 30^\circ + \sin 30^\circ)$$

$$= 10^{-7} \times \frac{2I}{\sqrt{3}a}\left(\frac{1}{2} + \frac{1}{2}\right)$$

$$= 10^{-7} \times \frac{2I}{\sqrt{3}a}$$

So, total magnetic field at O of the hexagon is

$$B = 6B' = 6 \times 10^{-7} \times \frac{2I}{\sqrt{3}a} = \frac{4\sqrt{3}}{a} \times 10^{-7} \text{ Tesla}$$

Example 4.4 A circular segment QR of a wire $PQRS$ [Fig. 4.2W] of 0.1 m radius subtends an angle of 60° at its center. A current of 6 amperes is flowing through it. Find the magnitude and direction of the magnetic field at the center of the segment.

Sol. Magnetic field due to current in a circular segment making an angle θ at the center is

$$dB = \frac{\mu_0}{4\pi}\frac{I\,dl}{r^2}$$

Here, $dl = r\,d\theta$ So, $dB = \frac{\mu_0}{4\pi}\frac{Ird\theta}{r} = \frac{\mu_0}{4\pi}\frac{Id\theta}{r}$

So $B = \frac{\mu_0}{4\pi}\frac{I\theta}{r}$

$$= 10^{-7} \times 6 \times \frac{\pi}{3} \times \frac{1}{0.1} \text{ Tesla}$$

$$= 6.28 \times 10^{-6} \text{ Tesla}$$

$$= 6.28\ \mu \text{ Tesla}$$

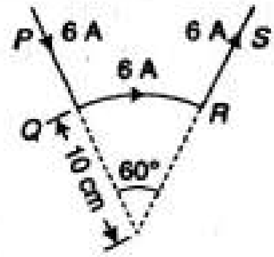


Fig. 4.2W Magnetic field at the center of a circular element of a current-carrying wire.

Example 4.5 A magnetic field $4 \times 10^{-3}\,\hat{k}$ tesla exerts a force of $(4\hat{i} + 3\hat{j}) \times 10^{-10}$ N or a particle having change of 1×10^{-9} C and moving in the xy plane. Calculate the velocity of the particle.

Sol. Lorentz force $\vec{F} = q\,(\vec{v} \times \vec{B})$

Here $(4\hat{i} + 3\hat{j}) \times 10^{-10} = 1 \times 10^{-9}\,[(v_x\hat{i} + v_y\hat{j})] \times 4 \times 10^{-3}\,\hat{k}$

or, $(4\hat{i} + 3\hat{j}) \times 10^{-10} = 4 \times 10^{-12}\,[(v_x(-\hat{j}) + v_y(\hat{i})]$

or, $$v_x = -\frac{3 \times 10^{-10}}{4 \times 10^{-12}} = -\frac{3}{4} \times 10^2 = -75$$

$$v_y = \frac{4 \times 10^{-10}}{4 \times 10^{-12}} = 100$$

So, $v = -75\,\hat{i} + 100\,\hat{j}$ ms^{-1}

Example 4.6 A test charge having a charge of 0.4 C is moving with a velocity of $4\hat{i} - \hat{j} + 2\hat{k}$ m/s through an electric field of intensity $10\,\hat{i} + 10\,\hat{k}$ and a magnetic field $2\hat{i} - 6\hat{j} - 6\hat{k}$. Determine the magnitude and direction of the Lorentz force acting on the test charge. [WBUT 2007]

Sol. Total Lorentz force = $q\,\vec{E} + q\,(\vec{v} \times \vec{B})$

Here electric force = $q\,\vec{E} = 0.4\,(10\,\hat{i} + 10\,\hat{k})$

and magnetic force = $q\,(\vec{v} \times \vec{B})$ $= 0.4 \begin{vmatrix} i & j & k \\ 4 & -1 & 2 \\ 2 & -6 & -6 \end{vmatrix}$

$$= 0.4\,[(6 + 12)\,\hat{i} + (4 + 24)\,\hat{j} + (-24 + 2)\,\hat{k}]$$

$$= 0.4\,(18\,\hat{i} + 28\,\hat{j} - 22\,\hat{k})$$

Now total Lorentz force, $\vec{F} = q\,\vec{E} + q\,(\vec{v} \times \vec{B})$

$$= 0.4\,(10\,\hat{i} + 10\,\hat{k}) + 0.4\,(18\,\hat{i} + 28\,\hat{j} - 22\,\hat{k})$$

$$= 0.4\,(28\,\hat{i} + 28\,\hat{j} - 12\,\hat{k})$$

The magnitude of the force $= 0.4\sqrt{(28)^2 + (28)^2 + (-12)^2} = 16.6$ N

Suppose, the total force makes an θ with the x axis, then

$$\vec{F}.\hat{i} = 0.4\,(28\,\hat{i} + 28\,\hat{j} - 12\,\hat{k}).\hat{i} = 0.4 \times 28$$

or, $(0.4)\sqrt{(28)^2+(28)^2+(-12)^2}\cos\theta = 0.4\times 28$

or, $$\cos\theta = \frac{28}{\sqrt{(28)^2+(28)^2+(-12)^2}}$$

or, $$\cos\theta = \frac{7}{\sqrt{107}}$$

$\therefore$ $$\theta = \cos^{-1}\left(\frac{7}{\sqrt{107}}\right) = 47.41°$$

Example 4.7 A straight wire carrying a current of 10 A is bent into a semicircular arc of π cm radius as shown in Fig. 4.3W. What is the magnetic field and direction of the magnetic field at center O of the arc?

Sol. The magnetic field at center O due to each straight portion of the wire is O. The magnetic field at center O is only due to half the circular loop. The magnitude of magnetic field

$$B = \frac{\mu_0}{4\pi}\left(\frac{\pi I}{a}\right) = \frac{\mu_0 I}{4a} = \frac{4\pi\times 10^{-7}\times 10}{4\times\pi\times 10^{-2}} = 10^{-4}\text{ T}$$

The current in the loop is anticlockwise and the direction, of the field is perpendicular to the paper.

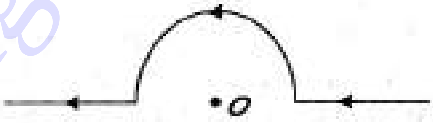


Fig. 4.3W Magnetic field at the center of a semicircular wire carrying current.

Example 4.8 The wire loop $ABCDA$ formed by joining two semicircular wires of radii R_1 and R_2 carries a current I as shown in Fig. 4.4W. Find out the magnetic field at center O.

Sol. The magnetic field due to a semicircular loop of radius R_1 is

$$B_1 = \frac{\mu_o}{4\pi}\left(\frac{\pi I}{R_1}\right)$$

[Here, direction of current is anticlockwise]

and direction of the field is normal to the plane of the loop, directed upward. For a bigger loop, direction of the current is clockwise. The value of the magnetic field due to semi-circular loop of radius R_2 is

$$B_2 = \frac{\mu_0}{4\pi}\left(\frac{\pi I}{R_2}\right)$$

Here direction of the magnetic field is into the plane of the paper. So, net magnetic field

$$B = B_1 - B_2 = \frac{\mu_0}{4\pi}\pi I\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$
$$= \frac{\mu_0 I}{4}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

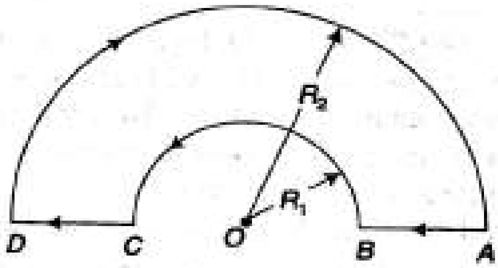


Fig. 4.4W Magnetic field at the center of two concentric semicircular wires carrying current.

and direction is perpendicular to the plane of the paper, hence directed upward.

Example 4.9 A current-carrying straight wire cannot move but a current-carying square loop adjacent to it can move under the influence of a magnetic force. Show that the square loop in Fig. 4.5W will move towards the wire.

Sol. In Fig. 4.5W, we see that the force acting on arms AB and DC are equal and opposite. But the force on arm AD is given by

$$F_1 = \frac{\mu_0}{4\pi}\left(\frac{2Ii}{a}\right)$$

which is directed towards the wire. The force on arm BC is given by

$$F_2 = \frac{\mu_0}{4\pi}\left(\frac{2Ii}{b}\right)$$

which is directed away from the wire. Here, $b > a$ hence $F_1 > F_2$. So the loop will move towards the wire.

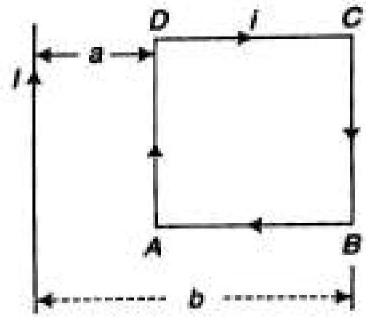


Fig. 4.5W Current-carrying square loop moves under the influence of magnetic force due to the current-carrying wire.

Example 4.10 A long solenoid of 40 cm length has 300 turns. If the solenoid carries a current of 3.5 A, calculate (i) magnetic field at the center of the solenoid, and (ii) magnetic field of the axis at one end of the solenoid.

Sol. (i) The magnetic field at the center of the solenoid is

$$B = \mu_0\, n\, I = \mu_0 \frac{N}{l} I = 4\pi \times 10^{-7} \times \frac{300}{0.4} \times 3.5 = 3.3 \times 10^{-3} \text{ Tesla}$$

(ii) The magnetic field at one end of the solenoid

$$B = \frac{1}{2}\mu_0\, n\, I = \frac{1}{2}\mu_0 \frac{N}{l} I = \frac{1}{2} \times \frac{300}{0.4} \times 3.5$$

$$= 1.65 \times 10^{-3} \text{ Tesla.}$$

Example 4.11 In Fig. 4.6W, in between two rails, a metal wire of mass m slides without friction. The distance of separation between the two rails is l. The break lies in a vertical uniform magnetic field B (perpendicular to the paper). Find the velocity of the wire as a function of time t, if a constant current I flows along one rail and then through the wire and back down the other rail.

Sol. The force exerted on the wire of length l is

$$F = B\,I\,l \sin 90° = B\,I\,l$$

The direction of the force will be to the left according to Fleming's left-hand rule.

The acceleration of the wire

$$a = \frac{F}{m} = \frac{B\,I\,l}{m}$$

Let initial velocity of wire $u = 0$, then velocity at any time t is $v = at = \frac{B\,I\,l}{m}\,t.$

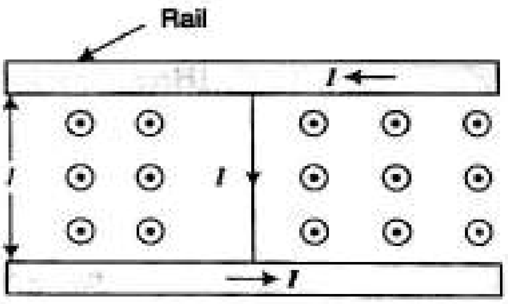


Fig. 4.6W A metal wire slides without friction between two rails carrying current in an opposite direction.

Example 4.12 Two straight wires, each 2 m long, are parallel to one another and are separated by a distance of 2 cm. If each carries a current of 8 A, calculate the force experienced by either of the wires.

Sol. The force per unit length experienced by each wire carrying currents I_1, I_2 separated by a distance d is given by

$$F = \frac{\mu_0 I_1 I_2}{2\pi a}$$

Here $I_1 = I_2 = 8$ A $d = 2$ cm $= 0.02$ m

so, $F = \dfrac{4\pi \times 10^{-7} \times 8 \times 8}{2\pi \times 0.02} = 64 \times 10^{-5}$ N/m

The total force on either of the wires is

$$F = 2 \times 64 \times 10^{-5} \text{ N} = 128 \times 10^{-5} \text{ N}$$

Example 4.13 In the Bohr model of a hydrogen atom, an electron is revolving in a circular path of 0.4 Å radius with a speed of 10^6 m/s. What is the value of magnetic field at the center of the orbit?

Sol. Here $r = 0.4$ Å $= 0.4 \times 10^{-10}$ m, $v = 10^6$ m/s.

We know that time period $T = \dfrac{2\pi r}{v} = \dfrac{2\pi \times 0.4 \times 10^{-10}}{10^6} = 8\pi \times 10^{-17}$ s

Again, current $I = \dfrac{e}{T} = \dfrac{1.6 \times 10^{-19}}{8\pi \times 10^{-17}} = \dfrac{2}{\pi} \times 10^{-3}$ A

The magnetic field at the center of the orbit

$$B = \frac{\mu_o}{4\pi}\left(\frac{2\pi I}{r}\right) = 10^{-7} \times \frac{2 \times \pi}{0.4 \times 10^{-10}} \times \frac{2}{\pi} = 10 \text{ Tesla.}$$

Example 4.14 The volume current density distribution in cylindrical coordinates is

$$J(r, \varphi, z) = 0 \qquad 0 < r < a$$

$$= J_0\left(\frac{r}{a}\right)\hat{e}_z \qquad a < r < b$$

$$= 0 \qquad b < r < \infty$$

Find the magnetic field in various regions [Fig. 4.7W].

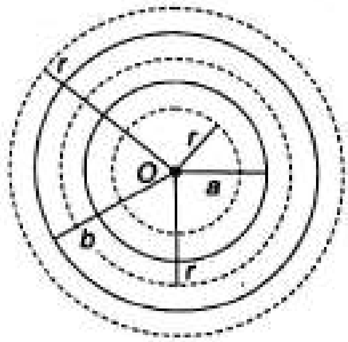


Fig. 4.7W

Sol. From Ampere's circuital law

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 \int_S J \cdot d\vec{S}$$

For region $0 < r < a$, $\quad J = 0$

So, $\oint \vec{B} \cdot \vec{dl} = 0$ or, $B = 0$

For region $a < r < b$ $\quad J(r, \varphi, z) = J_0\left(\frac{r}{a}\right)\hat{e}_z$

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 \int_{\varphi=0}^{2\pi} \int_a^r J_0\left(\frac{r}{a}\right) \cdot r\, dr\, d\varphi$$

or, $$B \times 2\pi r = \mu_0 \frac{J_0}{a}\left[\frac{r^3}{3}\right]_a^r [\varphi]_0^{2\pi} = \mu_0 \frac{2\pi}{3a} J_0 (r^3 - a^3)$$

or, $$B = \frac{\mu_0 J_0}{3ar}(r^3 - a^3)$$

Now at $r = b$, $\quad B = \dfrac{\mu_0 J_0}{3ab}(b^3 - a^3)$

For region $b < r < \alpha$, $J = 0$

we have $\oint \vec{B} \cdot \vec{dl} = \mu_0 \int_{\varphi=0}^{2\pi} \int_{r=a}^{b} \frac{J_0 r}{a} r \, dr \, d\varphi$

or, $$B = \frac{\mu_0 J_0}{3ar}(b^3 - a^3)$$

Example 4.15 Show that $\oint_S \vec{B} \cdot \vec{dS} = 0$ where $\vec{B}$ is the magnetic field and S is a closed surface. [WBUT 2006]

Sol. We have by applying divergence theorem

$$\oint_S \vec{B} \cdot \vec{dS} = \int_V (\vec{\nabla} \cdot \vec{B}) \, dV$$

But $\vec{\nabla} \cdot \vec{B} = 0$ [$\because \vec{B}$ is solenoidal field]

$$\therefore \quad \oint_S \vec{B} \cdot \vec{dS} = \int_V (\vec{\nabla} \cdot \vec{B}) \, dV = 0$$

$$\therefore \quad \oint_S \vec{B} \cdot \vec{dS} = 0$$

which shows that the lines of induction are continuous, meaning, it has no sources or sinks.

Example 4.16 If the vector potential $\vec{A} = (x^2 + y^2 - z^2)\,\hat{j}$ at position (x, y, z), find the magnetic field at (1, 1, 1). [WBUT 2007]

Sol. Here, vector potential $\vec{A} = (x^2 + y^2 - z^2)\,\hat{j}$

We know that $$\vec{B} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & (x^2 + y^2 - z^2) & 0 \end{vmatrix}$$

$$= \hat{i}\left[-\frac{\partial}{\partial z}(x^2 + y^2 - z^2)\right] + \hat{k}\left[\frac{\partial}{\partial x}(x^2 + y^2 - z^2)\right] = 2z\,\hat{i} + 2x\,\hat{k}$$

At (1, 1, 1) $\quad \vec{B} = 2\hat{i} + 2\hat{k}$

Example 4.17 If the vector potential $\vec{A} = \frac{1}{2}(\vec{a} \times \vec{r})$, where $\vec{a}$ is a constant vector, find the associated magnetic field.

Sol. Let $\quad \vec{a} = \hat{i}\,a_1 + \hat{j}\,a_2 + \hat{k}\,a_3$

So $\quad \vec{a} \times \vec{r} = \hat{i}\,(z\,a_2 - y\,a_3) + \hat{j}\,(x\,a_3 - z\,a_1) + \hat{k}\,(y\,a_1 - x\,a_2)$

We know that $$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{2}\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z\,a_2 - y\,a_3 & x\,a_3 - z\,a_1 & y\,a_1 - x\,a_2 \end{vmatrix}$$

$$= \frac{\hat{i}}{2}\left[\frac{\partial}{\partial y}(y a_1 - x a_2) - \frac{\partial}{\partial z}(x a_3 - z a_1)\right] + \frac{\hat{j}}{2}\left[\frac{\partial}{\partial z}(z a_2 - y a_3) - \frac{\partial}{\partial x}(y a_1 - x a_2)\right]$$

$$+ \frac{\hat{k}}{2}\left[\frac{\partial}{\partial x}(x a_3 - z a_1) - \frac{\partial}{\partial y}(z a_2 - y a_3)\right]$$

$$= a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k} = \vec{a}$$

Example 4.18 Two circular coils having identical turns and radii in the ratio 1:3 are joined in series. Find the ratio of the magnetic fields at the center of the coils.

Sol. Here, $B = \dfrac{\mu_0 N I}{2R}$. Since the coils are connected in series, therefore I is constant. N is also given to be constant.

So, $B \propto \dfrac{1}{R}$ $\therefore \dfrac{B_1}{B_2} = \dfrac{3}{1}$ or, $B_1 : B_2 = 3:1$

Review Exercises

Part 1: Multiple Choice Questions

1. A wire of length L carrying a current I is bent into a circle. The magnitude of the magnetic field at the center of the circle is

   (a) $\dfrac{\pi \mu_0 I}{L}$ (b) $\dfrac{\mu_0 I}{2\pi L}$ (c) $\dfrac{\mu_0 I}{2L}$ (d) $\dfrac{2\pi \mu_0 I}{L}$

2. In the region around a moving charge, there is
   (a) electric field only
   (b) magnetic field only
   (c) neither electric field nor magnetic field
   (d) electric as well as magnetic field

3. The magnetic field at the origin due to a current element $i\,\vec{dl}$ placed at a position $\vec{r}$ is

   (a) $\dfrac{\mu_0}{4\pi}\dfrac{i\,\vec{dl}\times\vec{r}}{r^2}$ (b) $\dfrac{\mu_0}{4\pi}\dfrac{i\,\vec{dl}\times\vec{r}}{r^3}$ (c) $\dfrac{\mu_0}{4\pi}\dfrac{i\,\vec{dl}\times\hat{r}}{r^3}$ (d) zero

4. A current-carrying straight wire is kept along axis of a circular loop carrying current. The straight wire
   (a) will exert an inward force on the circular loop
   (b) will exert an outward force on the circular loop
   (c) will not exert any force on the circular loop
   (d) None of these

5. A moving charge produces
   (a) electric field only (b) magnetic field only
   (c) Both of them (d) None of these

6. Which of the following statements is not characteristic of a static magnetic field? [WBUT 2006]
   (a) It is solenoid.
   (b) It is conservative.
   (c) Magnetic flux lines are always closed.
   (d) It has no sink or source.

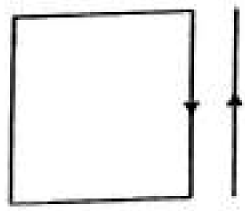


Fig. 4.8W Moving of current-carrying square loop.

7. A current-carrying straight wire cannot move, but a current-carrying square loop adjacent to it can move under the influence of a magnetic force [Fig. 4.8W]. [WBUT 2008]
   The square loop will
   (a) remain stationary
   (b) move towards the wire
   (c) move away from the wire
   (d) None of these

8. The direction of magnetic induction due to a straight infinitely long current carrying wire is [WBUT 2008]
   (a) perpendicular to the wire
   (b) parallel to the wire
   (c) at an inclination of 30° to the wire
   (d) None of these

9. The equation of continuity in a steady charge distribution is
   (a) $\vec{\nabla} \cdot \vec{J} = 0$
   (b) $\vec{\nabla} \times \vec{J} = 0$
   (c) $\vec{\nabla} \cdot \vec{J} = \rho$
   (d) $\vec{\nabla} \cdot \vec{J} = \dfrac{\rho}{\varepsilon_0}$

10. The work done by the Lorentz force $\vec{F}$ on a charged particle is
   (a) $\vec{F} \cdot d\vec{r}$
   (b) zero
   (c) $\dfrac{q}{\varepsilon_0}$
   (d) $q\,F$

11. $\vec{\nabla} \cdot \vec{J} + \dfrac{\partial \rho}{\partial t} = 0$
   (a) integral form of law of conservation of charge
   (b) differential form of law of conservation of charge
   (c) Poisson's equation
   (d) None of these

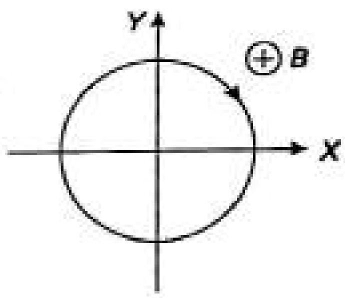


Fig. 4.9W A current-carrying loop in a magnetic field.

12. A conduction loop carrying a current I is placed in a uniform magnetic field pointing into the plane of the paper as shown in Fig. 4.9W. The loop will have a tendency to [WBUT 2007]
   (a) contract
   (b) expand
   (c) move towards positive the x axis
   (d) move towards negative the x axis

13. A copper wire is bent in the form of a sine wave of wavelength λ and peak-to-peak value as shown in Fig. 4.10W. A magnetic field of flux density B tesla acts perpendicular to the plane of the figure in the entire region. If the wire carries a steady current I ampere, the magnetic force on the wire is [WBUT 2007]
   (a) $I\sqrt{(a^2 + \lambda^2)}\,B$
   (b) $I a B$
   (c) $I(a + \lambda)\,B$
   (d) $I \lambda B$

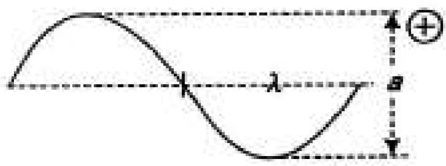


Fig. 4.10W Current-carrying sinusoidal wire in a magnetic field.

14. If the current density $\vec{J} = k\,\hat{r}$ where $\hat{r}$ is a unit vector along $x\,\hat{i} + y\,\hat{j}$, the current through the surface $x^2 + y^2 = a^2$, bounded by $z = 0$ and $z = h$ is [WBUT 2007]
    (a) $\pi a^2 hk$ (b) zero (c) $2\pi ahk$ (d) $\frac{a^3 k}{\pi h}$
15. If $\vec{B} = \vec{\nabla} \times \vec{A}$, $\vec{B}$ and $\vec{A}$ are any vectors then [WBUT 2005]
    (a) $\vec{\nabla} \cdot \vec{B} = 0$ (b) $\vec{\nabla} \cdot \vec{B} = +1$ (c) $\vec{\nabla} \cdot \vec{B} = -1$ (d) None of these
16. Magnetic field due to an infinitely long straight conductor carrying current I is
    (a) $\frac{\mu_0}{4\pi}\left(\frac{2\pi I}{a}\right)$ (b) $\frac{\mu_0}{4\pi}\left(\frac{2I}{a}\right)$ (c) $\frac{1}{4\pi\mu_0}\frac{I}{a}$ (d) zero
17. Two thin, long parallel wires, separated by a distance 'd' carry a current of I A, in the same direction. They will
    (a) attract each other with a force of $\frac{\mu_0 I^2}{2\pi d^2}$ (b) repel each other with a force of $\frac{\mu_0 I^2}{2\pi d^2}$
    (c) attract each other with a force of $\frac{\mu_0 I^2}{2\pi d}$ (d) None of these
18. A long wire carries a steady current. It is bent into a circle of one turn and the magnetic field at the center of the coil is B. It is then bent into a circular loop of n turns. The magnetic field at the center of the coil will be
    (a) $n^2 B$ (b) $n^3 B$ (c) $n B$ (d) $2\,n B$
19. A 1.5 m long solenoid 0.4 cm in diameter possesses 10 turns per cm length. A current of 5 A flows through it. The magnetic field at the axis inside the solenoid is
    (a) $4\pi \times 10^{-4}$ Tesla (b) $2\pi \times 10^{-3}$ Tesla
    (c) $2\pi \times 10^{-6}$ Tesla (d) None of these
20. A long straight wire along the z axis carries a current I in the negative z direction. The magnetic vector field $\vec{B}$ at a point having coordinates (x, y) in the $z = 0$ plane is
    (a) $\frac{\mu_0 I}{4\pi}\left(\frac{x\,\hat{i} - y\,\hat{j}}{x^2 + y^2}\right)$ (b) $\frac{\mu_0 I}{2\pi}\left(\frac{y\,\hat{i} - x\,\hat{j}}{x^2 + y^2}\right)$
    (c) $\frac{\mu_0 I}{2\pi}\left(\frac{y\,\hat{i} + x\,\hat{j}}{2x^2 + y^2}\right)$ (d) None of these

[Ans. 1 (a), 2 (d), 3 (b), 4 (c), 5 (c), 6 (b), 7 (c), 8 (a), 9 (a), 10 (b), 11 (b), 12 (b), 13 (d), 14 (b), 15 (a), 16 (b), 17 (c), 18 (a), 19 (b), 20 (b)]

Short Questions with Answers

1. The net charge on a current-carrying conductor is zero. Then, why does it experience a force in a magnetic field?

Ans. In a conductor, positive ions are stationary. So, they do not experience any force. But the free electrons drift towards the positive end of the conductor with some drift velocity and experience a magnetic field.

2. Why does a solenoid contract when current is passed through it?

Ans. When current is passed through a solenoid, the currents in the different turns of the solenoid flow in the same direction. Again we know that when currents in two parallel conductors flow in the same direction, the conductors attract each other. So, the solenoid contracts.

3. Define current density.

Ans. Current density is defined as the current through an infinitesimal area at any point inside a conductor, the area held perpendicular to the direction of flowing positive charge.

4. What is Lorentz force? Show that Lorentz force does not work on a charged particle.

Ans. See Section 4.5.

5. State Ampere's law both in integral and differential form.

Ans. See Section 4.12.

6. Compare between Lorentz electric force and Lorentz magnetic force.

Ans. Lorentz electric force $F_e = qE$, direction along the field does not depend on velocity and work is done. Lorentz magnetic force $F_m = Bq\,v\sin\theta$, direction perpendicular to plane containing B and v and depends on velocity of the charge, no workforce.

7. Define magnetic scalar potential and magnetic vector potential.

Ans. See Section 4.14.

8. A proton moving through a magnetic field region experiences maximum force. When does this occur?

Ans. When the proton moves perpendicular to the magnetic field, $\theta = 90°$, $\vec{v} \times \vec{B}$ will be maximum. So $\vec{F}$ is maximum.

9. Write the one condition under which an electric charge does not experience a force in a magnetic field.

Ans. Either the electric charge is at rest or it is moving parallel to the direction of the magnetic field.

10. Define an ampere in terms of the force between current-carrying conductors.

Ans. One ampere is that current which if passed in each of the two parallel conductors of infinite length and 1 m apart in vacuum, causes each conductor to experience a force of 2×10^{-7} $\mathrm{Nm^{-1}}$ length of conductor.

11. Apply Ampere's law qualitatively to the three parts as shown in Fig. 4.11W.

Ans. For paths I and III, $\oint B.\,\vec{dl} = \mu_0 I$

For path II, the net current is zero, i.e., for path II, $\oint \vec{B} \cdot \vec{dl} = 0$

12. A charge 4C is moving with a velocity $\vec{v} = (2\hat{j} + 3\hat{k})$ in a magnetic field $B = (2\hat{j} + 3\hat{k})$ $\mathrm{Wbm^{-2}}$. Find the force acting on the charge.

Ans. Here, $\vec{v}$ and $\vec{B}$ are parallel vectors.

So $\vec{v} \times \vec{B} = 0 \quad \therefore \vec{F} = q(\vec{v} \times \vec{B}) = 0$

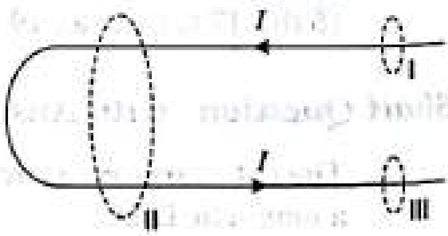


Fig. 4.11W

Part 2: Descriptive Questions

1. State Biot–Savart law. Using Biot–Savart law, calculate the field at the center of a circular current-carrying coil. [WBUT 2002]
2. (a) State Biot–Savart law.
   (b) Using Biot–Savart law, obtain an expression for the magnetic flux intensity at the center of a long current-carrying solenoid
   (c) Show that the field at the end of such a solenoid is half of that at the center. [WBUT 2003]
3. (a) State Biot–Savart law in magnetostatics. Find the magnetic field of an infinitely long straight wire at a transverse distance of d from the expression of $\vec{B}$ found in Biot–Savart law.
   (b) Express Biot–Savart law in terms of current density and hence show that the magnetic field is solenoidal.
   (c) Express Ampere's circuital law in terms of vector potential. (You may use $\vec{\nabla}\cdot\vec{A} = 0$, where $\vec{A}$ is the vector potential.) [WBUT 2008]
4. Find the magnetic induction $\vec{B}$ at a point on the axis of an infinitely long solenoid carrying a current I, number of turns per unit length being n. [WBUT 2007]
5. Find the magnetic field of a circular loop carrying field due to a long solenoid at a point. [WBUT 2007]
6. (a) State Ampere's circuital law.
   (b) By applying Ampere's circuital law, find out magnetic field due to a long solenoid at a point (i) inside the solenoid, and (ii) outside the solenoid.
7. What is Lorentz force? Show that Lorentz force does not work on a charged particle.
8. What do you mean by magnetic vector potential? Why is it called so? [WBUT 2002]
9. Show that $\oint \vec{B}\cdot\vec{dS} = 0$, when $\vec{B}$ is the magnetic field and S is a closed surface. State the theorem that you have used. [WBUT 2006]
10. Show that the equation of continuity is given by $\vec{\nabla}\cdot\vec{J} + \dfrac{\partial\rho}{\partial t} = 0$, where $\vec{J}$ and ρ have their usual meaning. [WBUT 2005]
11. (a) Give an example of an electrical circuit carrying a non-steady current when Ampere's circuital law is not applicable.
    (b) Write the expression of the magnetic field due to a current-carrying conductor. Draw a diagram necessary to explain the symbols. Show that this field is solenoidal. [WBUT 2006]
12. Starting from the definition of current density, derive the equation of continuity in current electricity. [WBUT 2007]
13. State Ampere's law in magnetostatics in integral form and from that deduce its differential form. [WBUT 2007]
14. Write down the condition of steady-state current. Show that Ampere's law implies that the current is in the steady state. [WBUT 2007]
15. Prove that the magnetic field inside a toroid having n numbers of turns per unit length and carrying a current I is $\mu_0 nI$.
16. Find the force per unit length of a current-carrying conductor placed in a uniform magnetic field. Hence find the force between the straight conductors carrying currents.

Part 3: Numerical Problems

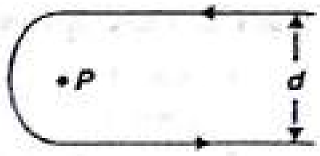


Fig. 4.12W

1. Calculate the drift speed of the electrons when 1 A of current exists in a copper wire of 2 mm^2 cross section. The number of free electrons in 1 cm^3 of copper is 8.5×10^{22}. [Ans. 36×10^{-3}]
2. Find the magnetic field at the point P in Fig. 4.12W. The curve portion is a semicircle and the straight wires are long. $\left[\text{Ans. } \dfrac{\mu_o}{2d}\left(\dfrac{2}{\pi}+\dfrac{1}{d}\right)\right]$
3. Consider a coaxial cable which consists of an inner wire of radius 'a' surrounded by an outer shell of inner and outer radii 'b' and 'c' respectively. The inner wire carries an equal current in opposite direction. Find the magnetic field at a distance 'r' from the axis, where
   (a) $r < a$ (b) $a < r < b$ (c) $b < r < c$ (d) $r > c$.
   Assume that the current density is uniform in the inner wire and also uniform in the outer shell.
   [WBUT (Question Bank)] $\left[\text{Ans. (a) } B = \dfrac{\mu_0 Ir}{2\pi a^2} \text{ (b) } \dfrac{\mu_0 I}{2\pi r} \text{ (c) } \dfrac{\mu_0 Ir}{2\pi(c^2-b^2)} \text{ (d) } B = 0\right]$
4. Find the magnetic field B due to a semicircular wire of 10.00 cm radius carrying a current of 5.0 A at its center of curvature. [Ans. 1.6×10^{-4} Tesla]
5. Two parallel wires carry equal currents of 10 A along the same direction and are separated by a distance of 2.0 cm. Find the magnetic field at a point which is 2.0 cm away from each of these wires. [Ans. 1.7×10^{-9} Tesla]
6. An infinite wire carrying a current I is bent in the form of a parabola. Find the magnetic field at the focus of the parabola. $\left[\text{Ans. } B = \dfrac{\mu_0 I}{4a}\right]$
7. If the magnetic scalar potential $\varphi_m = x^2 + y^2 - z^2$ at any point (x, y, z) in current free space, then find the magnetic field at the point (1, 2, 2). [Ans. $-2\,\hat{i} - 4\hat{j} + 4\,\hat{k}$] [Hints: Let $\vec{B} = -\vec{\nabla}\,\varphi_m$]
8. If the vector potential $\vec{A} = (2z+5)\,\hat{i} + (3x-2)\,\hat{j} + (4x-1)\,\hat{k}$, find the magnetic field. [Ans. $B = -2\hat{i} + 3\,\hat{k}$]
9. A solenoid has 4 layers of 1200 turns each. Its length and mean radius are 3 m and 0.25 m respectively. Find the magnetic field at the center if a current of 2.5 A flows through it. [Ans. $B = 5.02 \times 10^{-3}$ Tesla]
10. Figure 4.13W shows a current-carrying system of straight wire and loop. Determine the magnetic field at the center O of the loop. Given R is the radius of the loop and I is the current flowing in the system. $\left[\text{Ans. } B = \dfrac{\mu_0 I}{2R}\left(1-\dfrac{1}{\pi}\right)\right]$

Fig. 4.13W

11. A proton enters a magnetic field of 4 Tesla with a velocity of 2.5×10^6 ms^{-1} at an angle of 30° with the direction of the field. Find the magnitude of the force acting on the proton. [Ans. $F = 8 \times 10^{-13}$ N]
12. A particle of charge q moves with a velocity $v = a\,\hat{i}$ in a magnetic field $B = b\hat{j} + c\,\hat{k}$ where a, b and c are constants. Find the magnitude of the force experienced by the particle. [Ans. $qa\,(b^2 + c^2)^{1/2}$]

CHAPTER

5

Electromagnetic Field Theory

5.1 INTRODUCTION

In previous chapters we have discussed the fundamentals of electrostatics and magnetostatics. The static electric and magnetic fields are produced by charges at rest and steady current, respectively, and they are independent of each other.

In the present chapter, we discuss the time-varying fields. A time-varying electric field produces a magnetic field and a time-varying magnetic field produces an electric field. Michael Faraday gave the fundamental postulate for electromagnetic induction that relates the time-varying magnetic field with an electric field.

In this chapter, we deal with the interaction between electric and magnetic fields and obtain the four Maxwell's equations. Maxwell provided a mathematical theory that showed a close relationship between all electrical and magnetic phenomena, and form the foundation of electromagnetic theory. The combined Maxwell's equations yield wave equations and predict the existence of electromagnetic waves.

5.2 MAGNETIC FLUX

The magnetic flux linked with a surface held in a magnetic field is defined as the number of magnetic field lines crossing the surface normally.

The magnetic flux linked with a surface ds [Fig. 5.1] held inside a magnetic field is given by

$$d\varphi = B_n\, ds \qquad \text{...(5.1)}$$

where B_n is the normal component of the magnetic field B along the direction of a surface element ds.

Again from Fig. 5.1, $B_n = B\cos\theta$

so,
$$d\varphi = B\cos\theta\, ds$$
$$= \vec{B}\cdot\vec{ds} \qquad \text{...(5.2)}$$

If the magnetic field B is uniform over the surface area S, then total flux

$$\varphi = \vec{B}\cdot\vec{S} \qquad \text{...(5.3)}$$

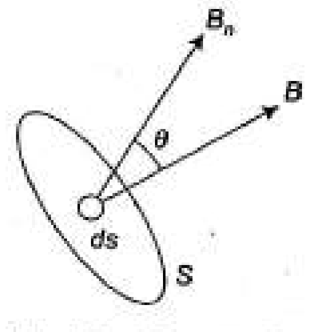


Fig. 5.1 Magnetic flux associated with a surface S.

If the normal is drawn in the direction of the magnetic field then the flux is taken as positive and if the normal is drawn opposite to the direction of the field then magnetic flux is taken as negative. The SI unit of magnetic flux is Weber (Wb) or Tesla m^2. Magnetic flux density of magnetic field induction is the magnetic flux per unit area, i.e., $B = \frac{\varphi}{A}$.

The SI unit of B is Tesla (T).

5.3 FARADAY'S LAW OF ELECTROMAGNETIC INDUCTION

Faraday observed that if a bar magnet moved either away or toward the axis of a conducting loop with no battery, then a current is produced in the loop. The current exists as long as the magnet is moving. The current flowed in the circuit when the flux through the circuit is altered. Faraday called this phenomenon *electromagnetic induction. Electromagnetic induction* is the process in which an emf is induced in a circuit placed in a magnetic field when the magnetic flux linked with the circuit changes. **Faraday's law tells us whenever the flux (φ) of magnetic field through the area bounded by a close conducting loop changes, an emf (ε) is produced in the loop.** Mathematically,

$$|\varepsilon| \propto \frac{d\varphi}{dt} \qquad ...(5.4)$$

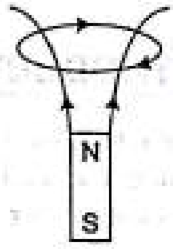


Fig. 5.2 The direction of induced emf according to Lenz's law.

The direction of the induced emf is provided by *Lenz's law*. This law states that *the direction of the induced emf is such that the magnetic flux associated with the current generated by it opposes the original change of flux causing emf.* To explain Lenz's law, we consider a magnet in the direction as shown in Fig. 5.2, i.e., towards the loop. As the magnet gets closer to the loop, the magnetic field increases and hence, the flux of the magnet field through the area of the loop increases. Thus increasing the magnetic flux through the coil, the induced current will flow in the direction shown, so that its own flux opposes the increase in the flux of the magnet. The induced current produces an induced emf. The induced emf is often called the back emf.

So, $$\varepsilon = -\frac{d\varphi}{dt} \qquad ...(5.5)$$

The direction of the current that produces a field towards the magnet can easily be obtained by using the right-hand thumb rule.

Lenz's Law and Conservation of Energy

According to Lenz's law, induced emf opposes the change that produces it. For change in magnetic flux, we have to perform mechanical work. So mechanical energy is converted into electrical energy. Thus, Lenz's law is in accordance with the law of conservation of energy.

5.4 ELECTROMOTIVE FORCE

In Fig. 5.3, we see a rod of length L sliding on a pair of parallel conducting tracks AB and DC. The arrangement is kept in a uniform magnetic field B which is normally out of the plane paper. Suppose, the rod moves parallel to the track with a velocity v making an angle θ with the magnetic flux (φ) link with the loop will change with time and we get an induced emf. Now applying Faraday's law, in unit time the area of the loop increases by the area of the parallelogram with sides L and v, the rate of change of flux is

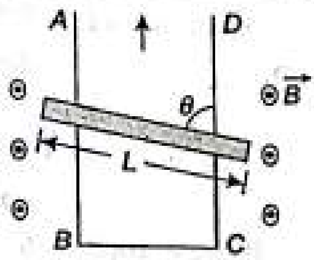


Fig. 5.3 Motional emf.

$$\frac{\partial \varphi}{\partial t} = BLv \sin\theta \qquad \text{...(5.6)}$$

If the resistance of the loop is R, the current will be

$$I = \frac{Blv \sin\theta}{R} \text{ in the clockwise direction.}$$

The value of the induced emf will be $(\varepsilon) = BLv \sin\theta$ ...(5.7)

Note: *Fleming's right-hand rule* The direction of motional emf is given by either Lenz's law or Fleming's right-hand rule. **Fleming's right-hand rule states that if the thumb and the first two fingers of your right-hand are spread out in mutually perpendicular directions then the first finger points in the direction of the magnetic field and the thumb in the direction of motion of the conductor, and the central fingers points in the direction of the induced emf and thus the induced current.**

5.5 INTEGRAL FORM OF FARADAY'S LAW

If at any time t, the flux linked with the closed coil is φ then according to Faraday's law, the induced emf in the coil

$$\varepsilon = -\frac{d\varphi}{dt} \qquad \text{...(5.8)}$$

If $\vec{E}$ be the field induced in space then the induced emf ε around the closed path C is given by integration of $\vec{E}$ and can be written as

$$\varepsilon = \oint_C \vec{E} \cdot \vec{dl} \qquad \text{...(5.9)}$$

$$\therefore \quad \oint_C \vec{E} \cdot \vec{dl} = -\frac{d\varphi}{dt} \qquad \text{...(5.10)}$$

The total flux through the circuit is equal to the integral of normal component of flux density $\vec{B}$ over the surface bounded by the circuit.

$$\varphi = \int_S \vec{B} \cdot \vec{dS} \qquad \text{...(5.11)}$$

$$\therefore \quad \oint_C \vec{E} \cdot \vec{dl} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{dS} \qquad \text{...(5.12)}$$

This is the integral form of Faraday's law of electromagnetic induction. Here, we use $\frac{\partial \vec{B}}{\partial t}$ instead of $\frac{d\vec{B}}{dt}$, because $\vec{B}$ is a function of both position and time.

5.5.1 Differential Form of Faraday's Law

Using Stoke's theorem,

$$\oint_C \vec{E} \cdot \vec{dl} = \int_S (\vec{\nabla} \times \vec{E}) \cdot \vec{dS} \qquad \text{...(5.13)}$$

Now, from Eq. (5.12), we have

$$\int_S (\vec{\nabla} \times \vec{E}) \cdot \vec{dS} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{dS} \qquad \text{...(5.14)}$$

or, $$\int_S \left(\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t}\right) \cdot \vec{dS} = 0 \quad \text{...(5.15)}$$

Eq. (5.15) must hold for any arbitrary surface S

so, $$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{...(5.16)}$$

This is the differential form of Faraday's law of electromagnetic induction.

The sources of the electromagnetic field are of two kinds. First one is the electrostatics field in which energy is conserved during a cyclic process and such a field has no curl. The second one is the magnetic field in which energy is transferred in a cyclic process and such a field is specified by the curl sources and has no divergence.

Taking divergence of Eq. (5.16), we have

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{B}) \quad \text{... (5.17)}$$

Since divergence of any curl is always zero, this is possible if

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{... (5.18)}$$

So, $\vec{B}$ is solenoidal.

Thus, Faraday's law gives two important results:

(a) The electric field is no longer a conservative field when the magnetic field varies with time.

(b) Magnetic free pole does not exist. All magnetic poles occur in pairs.

The time-varying electric and magnetic fields are thus interrelated and these two fields combine to form a single field known as *electromagnetic field*.

According to **Helmholtz theorem, any vector field is uniquely determined if its divergence and curl sources are given.** Electromagnetic fields have both types of sources.

5.6 DISPLACEMENT CURRENT

The concept of displacement current was introduced by Maxwell to account for production of magnetic field in empty space. The current for production of magnetic field is called *displacement current*. The current carried by conductors due to flow of charges is called *conduction current*. In empty space, conduction current is zero.

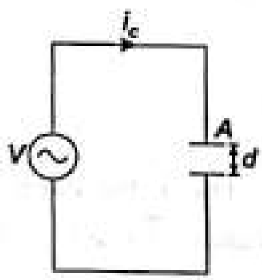


Fig. 5.4 Charging of a capacitor.

The displacement current is different from the conduction current in the sense that the former exists only when the electric field varies with time. For steady electric field in a conducting wire, the displacement current is zero. The current arising due to time-varying electric field between the plates of a capacitor is known as the displacement current.

Figure 5.4 shows a circuit connecting a time-varying voltage source V to a pure capacitor (C). The current through a capacitor is called displacement current. Actually the displacement current does not flow through the capacitor, the displacement is only an apparent current representing the rate of transport of charge from one plate to another. When a voltage is applied to a capacitor the current through it is

$$I = \frac{dQ}{dt} = C\frac{dV}{dt} \quad \text{... (5.19)}$$

where Q is the instantaneous charge, equal to CV. Again we know that for a parallel-plate capacitor

$$C = \frac{\varepsilon_0 A}{d} \quad \text{... (5.20)}$$

where A is the area of the parallel plate, d is the separation between the plate and ε_0 is the free space permittivity.

So,
$$I = \frac{\varepsilon_0 A}{d}\frac{dV}{dt} \quad \text{... (5.21)}$$

The relation between the electric field (ε_0) in the capacitor with potential

$$E = \frac{V}{d} \quad \text{... (5.22)}$$

Now from Eq. (5.21)

$$I = \varepsilon_0 A \frac{dE}{dt} \quad \text{... (5.23)}$$

or,
$$\frac{I}{A} = \varepsilon_0 \frac{\partial E}{\partial t} = \frac{\partial D}{\partial t} \quad \text{... (5.24)}$$

where $D = \varepsilon_0 E$ is known as electric displacement.

Now $\frac{I}{A}$ gives the current density (J_d)

So,
$$J_d = \frac{\partial D}{\partial t} \quad \text{... (5.25)}$$

J_d is called the displacement current density. The displacement current $\frac{\partial D}{\partial t}$ is zero outside the plates but has a definite value between the plates. This definite value is exactly equal to the value of conduction current outside the plates.

5.7 MODIFIED AMPERE'S LAW

The differential form of Ampere's circuital law is

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad \text{... (5.26)}$$

which is applied to a steady magnetic field only. Since the divergence of any curl is zero, we have from Eq. (5.26)

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{B} = \mu_0 \vec{\nabla} \cdot \vec{J} = 0$$

or,
$$\vec{\nabla} \cdot \vec{J} = 0 \quad \text{... (5.27)}$$

However, the equation of continuity

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \quad \text{... (5.28)}$$

shows that Eq. (5.28) is true only if

$$\frac{\partial \rho}{\partial t} = 0,\ \rho = \text{constant of time.}$$

So, Ampere's circuital law is valid only for static charge density. Ampere's circuital law in case of time-varying field does not hold good. Maxwell added another term to Ampere's law and ensured that it is valid for a time-varying field.

Let us add an unknown M to Eq. (5.26)

$$\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J} + \vec{M}) \quad \text{...(5.29)}$$

Taking divergence on both sides of Eq. (5.29), we have

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{B} = \mu_0 \vec{\nabla} \cdot (\vec{J} + \vec{M}) = 0$$

or, $$\vec{\nabla} \cdot \vec{J} + \vec{\nabla} \cdot \vec{M} = 0$$

or, $$\vec{\nabla} \cdot \vec{M} = -\vec{\nabla} \cdot \vec{J} = \frac{\partial \rho}{\partial t} \quad \text{from Eq. (5.28)} \quad \text{... (5.30)}$$

Again, from Gauss' law in electrostatics

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

or, $$\rho = \varepsilon_0 \vec{\nabla} \cdot \vec{E} \quad \text{... (5.31)}$$

Now from Eqs. (5.30) and (5.31),

$$\vec{\nabla} \cdot \vec{M} = \varepsilon_0 \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{E})$$

we get $$\vec{M} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{\partial \vec{D}}{\partial t} \quad \text{...(5.32)}$$

After modification, Ampere's law becomes

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \quad \text{...(5.33)}$$

$$= \mu_0 (\vec{J} + \vec{J}_d) \quad \text{...(5.34)}$$

The term $\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$ is known as the displacement current density.

Thus, displacement current density is entirely different from conduction current density. Displacement current is taken as a current only because it produces a magnetic field. Even in perfect vacuum, displacement current exists although there is no charge of any type. The presence of the term $\vec{J}_d$ enabled Maxwell to predict that an electromagnetic field should propagate through space in form of waves.

5.8 CONTINUITY PROPERTY OF CURRENT

From the conservation of charge, it is necessary that the total current density $(\vec{J} + \vec{J}_d)$ should obey the continuity equation. The individual term may or may not be continuous. As an example, in the case of charging of a parallel-plate capacitor, there is a conduction current in the region outside the plates of the capacitor. In empty space between the plates of the capacitor, conduction current is zero, but displacement current has a definite value between the plates. Thus, the individual terms are discontinuous but the sum of conduction current and displacement has the same value both inside and outside the plates.

MAXWELL'S EQUATIONS

5.9.1 Maxwell's Equations in Differential Form

Maxwell's equations represent the four basic laws of electricity and magnetism. These four laws are (i) Gauss' law in electrostatics, (ii) Gauss' law in magnetostatics, (iii) Faraday's law of electromagnetic induction, and (iv) Ampere's law with Maxwell's correction. All the four Maxwell's equations along with their salient features are being discussed here.

(i) Maxwell's first equation

The first equation represents Gauss' law in electrostatics, which may be written as

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad ...(5.35)$$

This is a time-independent steady-state equation which relates the spatial variation or divergence of an electric field with charge density. This relation is true both for stationary and moving charges.

(ii) Maxwell's second equation

The second Maxwell's equation represents Gauss' law in magnetostatics. Mathematically,

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad ...(5.36)$$

Equation (5.36) states that an isolated magnetic pole does not exist. This is also a time-independent or steady-state equation which gives the spatial variation or divergence of magnetic induction.

(iii) Maxwell's third equation

The third Maxwell's equation represents Faraday's law of electromagnetic induction. Mathematically,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad ...(5.37)$$

This is a time-dependent equation. Equation (5.37) shows that a time-varying magnetic field acts as a source of electric field. It relates the spatial variation of electric field with time variation of a magnetic field.

(iv) Maxwell's fourth equation

The fourth Maxwell's equation represents modified Ampere's (Ampere's law with Maxwell's correction). Mathematically,

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \qquad ...(5.38)$$

$$= \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

This is a time-independent equation. Equation (5.38) shows that a time-varying electric field acts as a source of magnetic field. The equation relates the spatial variation of a magnetic field with conduction current density and displacement current density.

Maxwell's equations are the basic equations for electromagnetism.

5.9.2 Maxwell's Equation in Integral Form

The four Maxwell's equations (5.35, 5.36, 5.37, 5.38) can be converted into an integral form.

(i) Maxwell's first equation

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

By taking the volume integral of both sides of $\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0}$ over a volume V with a closed surface S and using divergence theorem,

$$\left.\begin{aligned}\int_V (\vec{\nabla}\cdot\vec{E})\,dV &= \frac{1}{\varepsilon_0}\int_V \rho\,dV\\ \text{or,}\quad \oint_S \vec{E}\cdot\vec{dS} &= \frac{q}{\varepsilon_0}\ \text{where } q=\int_V \rho\,dV\end{aligned}\right\}\qquad ...(5.39)$$

We can infer from this equation that the electric lines of force do not constitute continuous close path.

(ii) Maxwell's second equation

$$\vec{\nabla}\cdot\vec{B}=0$$

By taking the volume integral of both sides of $\vec{\nabla}\cdot\vec{B}=0$ over a volume V with a closed surface S and using divergence theorem,

$$\left.\begin{aligned}\int_V (\vec{\nabla}\cdot\vec{B})\,dV &= 0\\ \text{or,}\quad \oint_S \vec{B}\cdot\vec{dS} &= 0\end{aligned}\right\}\qquad ...(5.40)$$

We can infer from this equation that there is no magnetic flux sources, and magnetic flux lines always close upon themselves. It is the law of conservation of magnetic flux, i.e., magnetic monopole does not exist.

(iii) Maxwell's third equation

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

By taking the surface integral of both sides of $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$ over an open surface S with a contour C and applying Stoke's theorem

$$\left.\begin{aligned}\int_S (\vec{\nabla}\times\vec{E})\cdot\vec{dS} &= -\int_S \frac{\partial\vec{B}}{\partial t}\cdot\vec{dS}\\ \text{or,}\quad \oint_C \vec{E}\cdot\vec{dl} &= -\frac{\partial}{\partial t}\int_S \vec{B}\cdot\vec{dS}\end{aligned}\right\}\qquad ...(5.41)$$

Equation (5.41) is the expression of Faraday's law, which states that a changing magnetic field $\vec{B}$ produces an electric field $\vec{E}$ such that line integral of $\vec{E}$ around a closed curve equals the negative rate of change of magnetic flux of a surface bounded by C.

(iv) Maxwell's fourth equation

$$\vec{\nabla}\times\vec{B}=\mu_0\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right)$$

By taking the surface integral of both sides of $\vec{\nabla}\times\vec{B}=\mu_0\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right)$ over an open surface S with a contour C and applying Stoke's theorem

$$\left.\begin{aligned}\int_S (\vec{\nabla}\times\vec{B})\cdot\vec{dS} &= \mu_0\int_S\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right)\cdot\vec{dS}\\ \text{or,}\quad \oint_C \vec{B}\cdot\vec{dl} &= \mu_0\left[\int_S \vec{J}\cdot\vec{dS}+\frac{\partial}{\partial t}\int_S \vec{D}\cdot\vec{dS}\right]\end{aligned}\right\}\qquad ...(5.42)$$

Equation (5.42) is the expression of modified Ampere's circuital law which states that the circulation of the magnetic field intensity around any closed path is equal to μ_o times the sum of conduction current and displacement current.

5.9.3 Physical Significance of Maxwell's Equations

(i) The first equation

The first Maxwell's equation known as Gauss' law in electrostatics, states that "The total electric flux through any closed surface is equal to the total charge enclosed by the surface divided by free space permittivity." Mathematically,

$$\oint_S \vec{E}\cdot\vec{dS} = \frac{q}{\varepsilon_0} \quad \text{[integral form]}$$

or,

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{[differential form]}$$

If a closed surface does not include any charge, then obviously the total flux over the surface is zero. The equation represents that the electric lines of force are not closed lines. The electric field lines start on positive charges (sources) and end on negative charges (sink). Gauss' law is valid not only for static charges but also for charges in motion.

(ii) The second equation

The second Maxwell's equation is known as Gauss' law in magnetostatics and states that there are no magnetic flux sources and magnetic flux lines always close upon themselves. Mathematically,

$$\oint_S \vec{B}\cdot\vec{dS} = 0 \quad \text{[integral form]}$$

or,

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \text{[differential form]}$$

Magnetic field is solenoidal means, it has no sources or sinks. The total magnetic flux through a closed surface is equal to zero, i.e., the magnetic flux entering into the volume is equal to the magnetic flux leaving the volume. The magnetic monopole does not exist, magnetic poles exist in pairs.

(iii) The third equation

The third Maxwell's equation is known as Faraday's law of electromagnetic induction and states that the line integral of $\vec{E}$ around a closed circuit is equal to the negative rate of change of magnetic flux linking the circuit. Mathematically,

$$\oint_C \vec{E}\cdot\vec{dl} = -\frac{\partial}{\partial t}\int_S \vec{B}\cdot\vec{dS} \quad \text{[integral form]}$$

or,

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \text{[differential form]}$$

It shows that the time variation of a magnetic field generates the electric field. So Faraday's law of electromagnetic induction shows how the electric and magnetic fields are interrelated. The time-varying magnetic fields acts as a source of electric field.

(iv) The fourth equation

The fourth Maxwell's equation is known as Ampere's law with Maxwell's correction. It relates the spatial variation of magnetic field with conduction current density and displacement current density. Mathematically,

$$\oint_C \vec{B}\cdot\vec{dl} = \mu_0\left[\int_S \vec{J}\cdot\vec{dS} + \frac{\partial}{\partial t}\int_S \vec{D}\cdot\vec{dS}\right] \quad \text{[integral form]}$$

or,
$$\vec{\nabla} \times \vec{B} = \mu_0 \left[\vec{J} + \frac{\partial \vec{D}}{\partial t}\right] \qquad \text{[differential form]}$$

We see that both the conduction current density and displacement current density are two possible sources of magnetic field. The term $\frac{\partial \vec{D}}{\partial t}$ which arises from the variation of electric displacement with time is known as displacement current density and its introduction in $\vec{\nabla} \times \vec{B}$ equation was one of the major contributions of Maxwell.

Thus we see that the interrelation between electric and magnetic field generates a single field known as electromagnetic field which gives rise to propagation of electromagnetic waves.

5.9.4 Maxwell's Equations in Free Space

In free space, the following physical conditions are satisfied

$$\rho = 0, J = \sigma E = 0 \text{ as conductivity } \sigma = 0$$

Under these conditions, Maxwell's equations take the following form:

(i) $\vec{\nabla} \cdot \vec{E} = 0$ ...(5.43)

(ii) $\vec{\nabla} \cdot \vec{B} = 0$ ...(5.44)

(iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ ...(5.45)

(iv) $\vec{\nabla} \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t}$ ...(5.46)

5.10 WAVE EQUATIONS IN FREE SPACE

The time-varying electric and magnetic fields give rise to the phenomenon of electromagnetic wave propagation. Here we deduce the relevant wave equation.

In free space, Maxwell's equations are:

(i) $\vec{\nabla} \cdot \vec{E} = 0$

(ii) $\vec{\nabla} \cdot \vec{B} = 0$

(iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

(iv) $\vec{\nabla} \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t}$

$$= \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad (\because \vec{D} = \varepsilon_0 \vec{E})$$

- *For Electric Field*

  Take curl on both sides of Eq. (iii)

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t}$$

or,
$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) \qquad ...(5.47)$$

But
$$\vec{\nabla} \cdot \vec{E} = 0 \text{ and } \vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

So, from Eq. (5.47)

$$-\nabla^2 \vec{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

or, $$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \quad ...(5.48)$$

This is the three-dimensional wave equation for the vector field $\vec{E}$ in free space.

- *For Magnetic Field*

Taking curl on both sides of Eq. (iv)

$$\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \vec{\nabla} \times \left(\frac{\partial \vec{E}}{\partial t}\right)$$

or, $$\vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E}) \quad ...(5.49)$$

But $$\vec{\nabla} \cdot \vec{B} = 0 \text{ and } \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

So, from Eq. (5.49)

$$-\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right)$$

or, $$\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \quad ...(5.50)$$

This is the three-dimensional wave equation for the vector field $\vec{B}$ in free space.

Thus the fields satisfy the same formal partial differential equations for waves

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} \quad ...(5.51)$$

The three-dimensional wave function ψ depends on x, y, z, t and c is the velocity of the wave.

Thus, we conclude that the field vectors $\vec{E}$ and $\vec{B}$ are propagated in free space as waves whose speed is given by

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \quad [\text{here, } \mu_0 = 4\pi \times 10^{-7} \text{ weber/amp, } \varepsilon_0 = 8.854 \times 10^{-12} \text{ C}^2/\text{Nm}^2]$$

$$= 2.9978 \times 10^8 \text{m/s}$$

which is the velocity of light in free space. So, we may conclude that light waves are electromagnetic waves.

From Eqs (5.49) and (5.50), we see that the electric field and the magnetic field satisfy the same wave equation, so they oscillate exactly in the same phase.

5.11 TRANSVERSE NATURE OF ELECTROMAGNETIC WAVE

The electromagnetic wave equations in free space are:

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \text{ and } \nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}$$

We assume that the plane wave fields are of the form,

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \quad \text{...(5.52)}$$

and
$$\vec{B}(\vec{r}, t) = \vec{B}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \quad \text{...(5.53)}$$

where $\vec{E}_0$ and $\vec{B}_0$ are vector constant in time and $\vec{k}$ is the propagation vector.

From Maxwell's first equation in free space

$$\vec{\nabla}\cdot\vec{E} = 0$$

or,
$$\vec{\nabla}\cdot\vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} = 0$$

which gives the relation

$$\vec{k}\cdot\vec{E} = 0 \quad \text{...(5.54)}$$

where $\vec{\nabla}$ is the operator, after operation on $\vec{E}$ its value (ik). Similarly, from Maxwell's second equation in free space

$$\vec{\nabla}.\vec{B} = 0$$

or,
$$\vec{\nabla}\cdot\vec{B}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

which gives
$$\vec{k}\cdot\vec{B} = 0 \quad \text{...(5.55)}$$

Equations (5.54) and (5.55) show that both $\vec{E}$ and $\vec{B}$ are perpendicular to the propagation vector $\vec{k}$.

Again from Maxwell's third equation

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

or,
$$\vec{\nabla}\times\vec{E}_0 e^{i(\vec{k}.\vec{r} - \omega t)} = -\frac{\partial}{\partial t}\vec{B}_0 e^{i(\vec{k}.\vec{r} - \omega t)}$$

or,
$$i\,\vec{k}\times\vec{E} = -(-i\omega)\,\vec{B}$$

Therefore,
$$\vec{k}\times\vec{E} = \omega\vec{B} \quad \text{...(5.56)}$$

So, $\vec{B}$ is perpendicular to both $\vec{k}$ and $\vec{E}$. Therefore, from Eqs (5.54), (5.55) and (5.56), $\vec{E}$, $\vec{B}$ and $\vec{k}$ are mutually perpendicular to each other.

Now, considering only the magnitude of E, B and k, we have

$$\frac{E_0}{B_0} = \frac{\omega}{k} = c \quad \text{...(5.57)}$$

where c is the velocity of light.

Thus, we may conclude that (i) electromagnetic waves travel with the speed of light, and (ii) electromagnetic waves are transverse waves. The ratio of electric to magnetic field in electromagnetic waves equal to the speed of light.

From Fig. 5.5 (a, b) we see that both electric field and magnetic field are perpendicular to the direction of motion of the wave. Thus an electromagnetic wave is a transverse wave.

Figure 5.5 (b) is a graphical representation of the sinusoidal wave showing $\vec{E}$ and $\vec{B}$ vectors. At any fixed point, the electrical and magnetic field vectors vary sinusoidally with time. The energy flow in the +ve x direction (i.e., $\vec{E}\times\vec{B}$). Radio waves, light waves, x-rays, γ-rays are examples of electromagnetic waves.

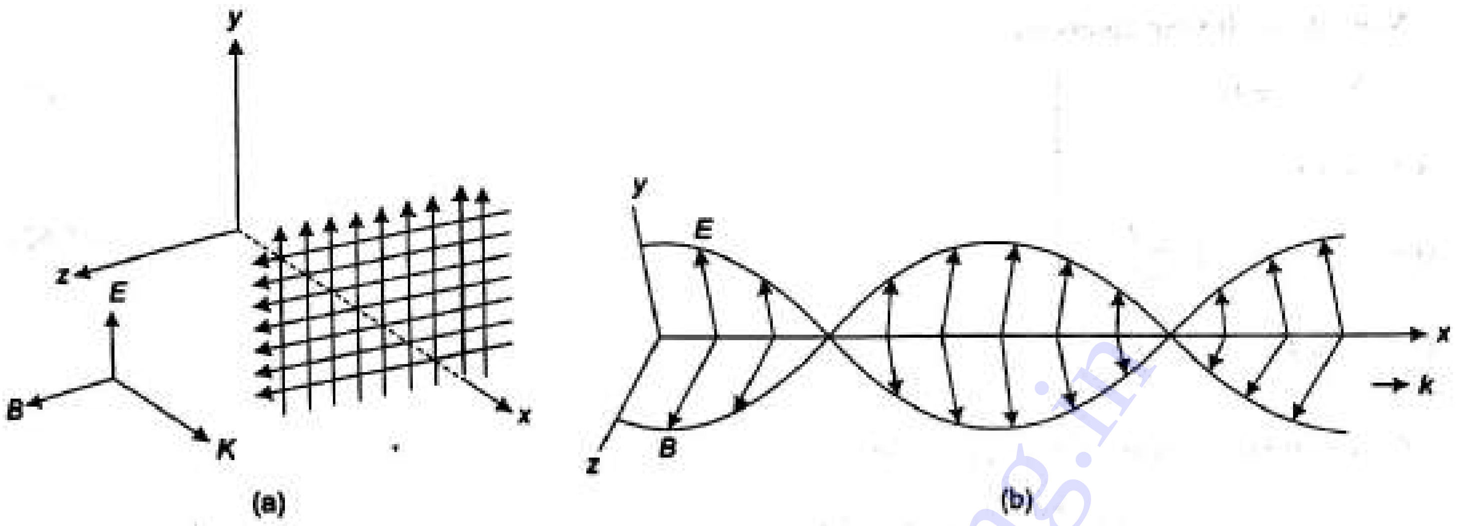


Fig. 5.5 (a) Transverse nature of electromagnetic waves. (b) Representation of the sinusoidal wave showing $\vec{E}$ and $\vec{B}$ vectors.

5.12 POTENTIALS OF ELECTROMAGNETIC FIELD

We know that magnetic field is solenoidal, i.e., $\vec{\nabla} \cdot \vec{B} = 0$, so $\vec{B}$, in terms of vector potential $\vec{B} = \vec{\nabla} \times \vec{A}$.

Again from Faraday's laws of electromagnetic induction

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) \quad [\text{Taking } \vec{B} = \vec{\nabla} \times \vec{A}]$$

Therefore, $$\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0 \qquad ...(5.58)$$

Since the curl of the gradient of a scalar function is zero, so,

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\varphi \qquad ...(5.59)$$

or, $$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t} \qquad ...(5.60)$$

Here, $\vec{A}$ and φ are magnetic vector potential and scalar potential. The electric field $\vec{E}$ and magnetic field $\vec{B}$ can be found if we determine $\vec{A}$ and φ.

5.13 ELECTROMAGNETIC WAVES IN A CHARGE-FREE CONDUCTING MEDIA AND SKIN DEPTH

Inside the conductor $\rho = 0$. Because there is no permanent charge inside the conductor, it can only be redistributed on the surface of the conductor. The propagation of EM waves through conducting, homogeneous, isotropic medium of permittivity ε, permeability μ and conductivity σ hold the relations

$$\left.\begin{aligned} \vec{D} &= \varepsilon\vec{E} \\ \vec{B} &= \mu\vec{H} \\ \vec{J} &= \sigma\vec{E} \end{aligned}\right\} \qquad ...(5.61)$$

Now Maxwell's equations are:

$$\left.\begin{aligned}
&\text{(i)}\ \vec{\nabla}\cdot\vec{E}=0\\
&\text{(ii)}\ \vec{\nabla}\cdot\vec{H}=0\\
&\text{(iii)}\ \vec{\nabla}\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t}\\
&\text{(iv)}\ \vec{\nabla}\times\vec{H}=\sigma\vec{E}+\varepsilon\frac{\partial\vec{E}}{\partial t}
\end{aligned}\right\}\qquad\ldots(5.62)$$

Taking curl on both sides of Eq. 5.62 (iii)

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E}=-\mu\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H})$$

or,

$$-\nabla^2\vec{E}=-\mu\sigma\frac{\partial\vec{E}}{\partial t}-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}\qquad\text{[Using Eqs. (i) and (iv)]}$$

After arranging, we get

$$\nabla^2\vec{E}-\mu\sigma\frac{\partial\vec{E}}{\partial t}-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}=0\qquad\ldots(5.63)$$

Similarly taking curl of Eq. (5.62) (iv), we get

$$\nabla^2\vec{H}-\mu\sigma\frac{\partial\vec{H}}{\partial t}-\mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}=0\qquad\ldots(5.64)$$

Both Eqs. (5.63) and (5.64) are known as Helmholtz equations for electric field and magnetic field.

Let us now find the plane wave solutions of Maxwell's equations for a conducting medium. We assume that the field vector $\vec{E}(\vec{r},t)$ and $\vec{H}(\vec{r},t)$ vary harmonically with time,

i.e.,

$$\left.\begin{aligned}
E(\vec{r},t)&=E_0\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\\
H(\vec{r},t)&=H_0\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}
\end{aligned}\right\}\qquad\ldots(5.65)$$

and

Substituting Eq. (5.65) in Eq. (5.63)

$$-k^2E(r,t)+\varepsilon\mu\omega^2\vec{E}(r,t)+i\sigma\mu\omega\vec{E}(r,t)=0$$

i.e.,

$$k^2=\varepsilon\mu\omega^2\left(1+\frac{i\sigma}{\varepsilon\omega}\right)\qquad\ldots(5.66)$$

The propagation vector is complex, and may be expressed as

$$k=\alpha+i\beta$$

$$=\left[\varepsilon\mu\omega^2\left(1+\frac{i\sigma}{\varepsilon\omega}\right)\right]^{1/2}\qquad\ldots(5.67)$$

From Eq. (5.67)

$$\left.\begin{aligned} \alpha^2-\beta^2 &= \varepsilon\mu\omega^2 \\ \text{and}\qquad 2\alpha\beta &= \sigma\mu\omega \end{aligned}\right\} \qquad \ldots(5.68)$$

Solving these equations,

$$\left.\begin{aligned} \alpha &= \omega\sqrt{\frac{\varepsilon\mu}{2}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right]} \\ \text{and}\qquad \beta &= \omega\sqrt{\frac{\varepsilon\mu}{2}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]} \end{aligned}\right\} \qquad \ldots(5.69)$$

For good conductor, if the frequency is not too high, $\frac{\sigma}{\varepsilon\omega} >> 1$.

$\therefore$ propagation vector, $k = \sqrt{\mu\sigma\omega i} = \sqrt{\mu\sigma\omega}\,(\cos 45° + i\sin 45°)$

$$= \sqrt{\mu\sigma\omega}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$$= \alpha + i\beta$$

So

$$\alpha = \beta = \sqrt{\frac{\mu\sigma\omega}{2}} \qquad \ldots(5.70)$$

$$= \frac{1}{\delta} \quad \text{where} \quad \delta = \sqrt{\frac{2}{\mu\sigma\omega}}$$

Since $k = \alpha + i\beta$, Eq. (5.65) can be written as

$$\left.\begin{aligned} \vec{E} &= \vec{E}_0\, e^{-\beta r}\, e^{i(\alpha r-\omega t)} \\ \vec{H} &= \vec{H}_0\, e^{-\beta r}\, e^{i(\alpha r-\omega t)} \end{aligned}\right\} \qquad \ldots(5.71)$$

These equation indicate that a plane wave cannot propagate in a conducting medium without attenuation.

5.14 SKIN DEPTH OR DEPTH OF PENETRATION (δ)

From Eq. (5.71)

$$\vec{E} = \vec{E}_0\, e^{-r/\delta}\, e^{i\left(\frac{r}{\delta}-\omega t\right)}$$

At $r = \delta$ the amplitude decreases in magnitude to $\frac{1}{e}$ times its value at the surface which is called **skin depth** or **penetration depth** [Fig. 5.6].

The phenomenon that the alternating fields and hence currents are confined within a small region of a conducting medium inside the surface is known as the ***skin effect*** and the small distance from the surface of the conductor is known as ***skin depth***.

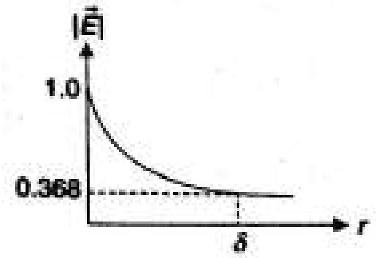


Fig. 5.6 Variation of intensity with distance in a change free conducting medium.

Significance of skin depth The attenuation of electromagnetic waves in a conduction medium is due to the conversion of the electromagnetic energy of the wave into Joule's heat because the electric field of the wave induces currents in a conducting medium which produce the heat. The energy in the form of electromagnetic waves carried by a current propagates in the space surrounding the conductors that partially penetrates the conductor surface to maintain the motion of the electrons. So, the current is maintained in the parts of the conductor which receive electromagnetic energy from the surrounding space. This energy can penetrate the conductor only by such small distance, called the skin depth (δ) and current may exist near the surface of the conductor only within the limits of this depth. The skin depth in copper for 1 mm microwaves is 10^{-4} m and for visible light 10^{-6} m. A poor conductor can be made a good conductor with a thin coating of copper or silver. A silver coating on a piece of glass may be an excellent conductor at microwave frequency. The performance of gold and gold-coated brass waveguides will be same if the coating thickness is equal to skin depth. This method is useful to reduce the cost of the material.

5.14.1 Electromagnetic Shielding

We may enclose a volume with a thin layer of good conductor to act as an electromagnetic shield. Depending on the application, the electromagnetic shield may be necessary to prevent waves from radiating out of the shielded volume or to prevent waves from penetrating into the shielded volume.

5.14.2 Phase Velocity

The phase velocity in the conducting medium is given by

$$v = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}} \qquad \text{...(5.72)}$$

In the conductor, α and β are large. The wave attenuates greatly as it progresses and the phase shift per unit length is also large. The phase velocity of the wave is small. Phase velocity depends on frequency, so dispersion takes place in the conducting medium.

5.15 ELECTROMAGNETIC ENERGY FLOW AND POYNTING VECTOR

The electromagnetic waves carry energy when they propagate and there is an energy density associated with both the electric and magnetic fields. As electromagnetic waves propagate through the space from the source to the receiver, there exists a simple and direct relationship between the rate of energy transfer and the amplitude of electric and magnetic field strengths. The relation may be obtained from Maxwell's equations

$$\left.\begin{aligned}
&\text{(i)}\quad \vec{\nabla}\cdot\vec{D} = 0\\
&\text{(ii)}\quad \vec{\nabla}\cdot\vec{B} = 0\\
&\text{(iii)}\quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \text{and}\\
&\text{(iv)}\quad \vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}
\end{aligned}\right\} \qquad \text{...(5.73)}$$

Taking dot product of Eqs. (5.73) (iii) and (iv) with $\vec{H}$ and $\vec{E}$ respectively, we have

$$\vec{H}\cdot\vec{\nabla}\times\vec{E} = -\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} \quad ...(5.74)$$

and

$$\vec{E}\cdot\vec{\nabla}\times\vec{H} = \vec{E}\cdot\vec{J} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} \quad ...(5.75)$$

Form Eqs. (5.74) and (5.75)

$$\vec{H}\cdot\vec{\nabla}\times\vec{E} - \vec{E}\cdot\vec{\nabla}\times\vec{H} = -\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{E}\cdot\vec{J} - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} \quad ...(5.76)$$

$$= -\left(\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right) - \vec{E}\cdot\vec{J}$$

or,

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\left(\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right) - \vec{E}\cdot\vec{J} \quad ...(5.77)$$

$$\left[\because \vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot\vec{\nabla}\times\vec{A} - \vec{A}\cdot\vec{\nabla}\times\vec{B}\right]$$

From relations $\vec{B} = \mu\vec{H}$ and $\vec{D} = \varepsilon\vec{E}$, we have

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\left[\vec{H}\cdot\frac{\partial}{\partial t}(\mu\vec{H}) + \vec{E}\cdot\frac{\partial}{\partial t}(\varepsilon\vec{E})\right] - \vec{E}\cdot\vec{J}$$

$$= -\left[\frac{1}{2}\mu\frac{\partial}{\partial t}(H^2) + \frac{1}{2}\varepsilon\frac{\partial}{\partial t}(E^2)\right] - \vec{E}\cdot\vec{J}$$

$$= -\left[\frac{\partial}{\partial t}\left(\frac{1}{2}\vec{H}\cdot\vec{B}\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{E}\cdot\vec{D}\right)\right] - \vec{E}\cdot\vec{J}$$

or,

$$\vec{E}\cdot\vec{J} = \frac{\partial}{\partial t}\left[\frac{1}{2}(\vec{H}\cdot\vec{B} + \vec{E}\cdot\vec{D})\right] - \vec{\nabla}\cdot(\vec{E}\times\vec{H}) \quad ...(5.78)$$

Integrating over the volume V

$$\int_V (\vec{E}\cdot\vec{J})\,dV = -\frac{\partial}{\partial t}\int_V \left\{\frac{1}{2}(\vec{H}\cdot\vec{B} + \vec{E}\cdot\vec{D})\right\} dV - \int_V \vec{\nabla}\cdot(\vec{E}\times\vec{H})\,dV$$

or,

$$\int_V (\vec{E}\cdot\vec{J})\,dV = -\frac{\partial}{\partial t}\int_V \left\{\frac{1}{2}(\vec{H}\cdot\vec{B} + \vec{E}\cdot\vec{D})\right\} dV - \oint_S (\vec{E}\times\vec{H})\cdot d\vec{S} \quad ...(5.79)$$

$$\left[\because \int_V \vec{\nabla}\cdot(\vec{E}\times\vec{H})\,dV = \oint_S (\vec{E}\times\vec{H})\cdot d\vec{S}\right]$$

or,

$$\oint (\vec{E}\times\vec{H})\cdot d\vec{S} = -\frac{\partial}{\partial t}\int_V \left\{\frac{1}{2}(\vec{H}\cdot\vec{B} + \vec{E}\cdot\vec{D})\right\} dV - \int_V (\vec{E}\cdot\vec{J})\,dV$$

$$= -\frac{\partial}{\partial t}\int_V \left\{\frac{1}{2}(\mu H^2 + \varepsilon\vec{E}^2)\right\} dV - \int_V (\vec{E}\cdot\vec{J})\,dV \quad ...(5.80)$$

Equation (5.80) is known as **Poynting theorem**. This is also known as the **work-energy theorem** of electrodynamics.

Interpretation of each term

(a) $\frac{\partial}{\partial t}\int_V \left\{\frac{1}{2}(\mu H^2 + \varepsilon E^2)\, dV\right.$: The terms $\frac{1}{2}\mu H^2$ and $\frac{1}{2}\varepsilon E^2$ represent the energy stored in electric and magnetic fields respectively and their sum will be equal to the total energy stored in electromagnetic field. This expression represents the rate of decrease of energy stored within volume V due to electric and magnetic fields.

(b) $\int_V (\vec{E}\cdot\vec{J})\, dV$ or $\int \sigma E^2 dV$: This term represents the total ohmic power dissipated within the volume. This is a generalisation of Joule's law.

(c) $\oint_S (\vec{E}\times\vec{H})\cdot\vec{dS}$: This term represents the rate at which electromagnetic energy is leaving the volume V through the closed surface S.

The vector $(\vec{E}\times\vec{H})$ is known as the Poynting vector $\vec{P}$ or $\vec{P} = (\vec{E}\times\vec{H})$.

Poynting Vector: The amount of energy flowing through unit area, perpendicular to the direction of energy propagation per unit time, i.e., the rate of energy transport per unit area, is called the Poynting vector.

Poynting Theorem: It states that the vector product $\vec{P} = (\vec{E}\times\vec{H})$ at any point is a measure of the rate of energy flow per unit area at that point. The direction of energy flow is in the direction of the vector represented by the product $(\vec{E}\times\vec{H})$ and is perpendicular to both $\vec{E}$ and $\vec{H}$.

5.16 AVERAGE POWER CALCULATION USING POYNTING VECTOR

The Poynting vector $\vec{P} = \vec{E}\times\vec{H}$ gives the instantaneous rate of energy flow. Since the vector $\vec{E}$ and $\vec{H}$ vary harmonically with time, the average flow can be found by taking the average of $\vec{P} = \vec{E}\times\vec{H}$ over a complete period, i.e.,

$$\langle P\rangle = \langle R_e\, E \times R_e\, H\rangle \quad \text{...(5.81)}$$

where R_e stands for the real part.

Here E and H are complex quantities, so

$$\left.\begin{aligned} E &= (E_1 + iE_2)\, e^{-i\omega t} \\ \text{and} \quad H &= (H_1 + iH_2)\, e^{-i\omega t} \end{aligned}\right\} \quad \text{...(5.82)}$$

where E_1, E_2, H_1 and H_2 are real

Now $$R_e E = E_1 \cos\omega t + E_2 \sin\omega t$$

and $$R_e H = H_1 \cos\omega t + H_2 \sin\omega t$$

So $$R_e E \times R_e H = (E_1\times H_1)\cos^2\omega t + (E_1\times H_2)\cos\omega t\sin\omega t + (E_2\times H_1)\sin\omega t\cos\omega t + (E_2\times H_2)\sin^2\omega t$$

Now, over a complete period of oscillation

$$\langle\cos^2\omega t\rangle = \langle\sin^2\omega t\rangle = \frac{1}{2}$$

and $$\langle \sin \omega t \cos \omega t \rangle = 0$$

Therefore,
$$\langle R_e E \times R_e H \rangle = \frac{1}{2}[(E_1 \times H_1) + (E_2 \times H_2)] \quad ...(5.83)$$

Let as now compute $R_e(E \times H^*)$

$$E = (E_1 + iE_2)\, e^{-i\omega t} = (E_1 + iE_2)(\cos \omega t - i \sin \omega t)$$

$$H = (H_1 + iH_2)\, e^{-i\omega t} = (H_1 + iH_2)(\cos \omega t - i \sin \omega t)$$

or,
$$H^* = (H_1 - iH_2)(\cos \omega t + i \sin \omega t)$$

Therefore,
$$\begin{aligned} R_e(E \times H^*) = {} & (E_1 \times H_1)\cos^2 \omega t + (E_1 \times H_2)\cos \omega t \sin \omega t \\ & + (E_2 \times H_2)\cos^2 \omega t - (E_2 \times H_1)\cos \omega t \sin \omega t \\ & - (E_1 \times H_2)\cos \omega t \sin \omega t + (E_1 \times H_1)\sin^2 \omega t \\ & + (E_2 \times H_1)\cos \omega t \sin \omega t + (E_2 \times H_2)\sin^2 \omega t \end{aligned}$$

$$R_e(E \times H^*) = (E_1 \times H_1) + (E_2 \times H_2) \quad ...(5.84)$$

Hence,
$$\langle R_e E \times R_e H \rangle = \frac{1}{2} R_e(E \times H^*) \quad ...(5.85)$$

So, average Poynting vector

$$\langle P \rangle = \frac{1}{2} R_e(E \times H^*)$$

The average power
$$\langle P \rangle = \frac{1}{2} R_e(E \times H^*) \quad ...(5.86)$$

Worked Out Problems

Example 5.1 A rectangular loop of sides 8 cm and 2 cm having a resistance of 1.6 Ω is placed in a magnetic field of 0.3 Tesla directed normal to the loop. The magnetic field is gradually reduced at the end of 0.02 Ts^{-1}. Find out the induced current.

Sol. Induced emf $e = \frac{d\varphi}{dt} = \frac{d}{dt}(BA) = A\frac{dB}{dt}$

$$= (8 \times 2 \times 10^{-4}) \times 0.02$$

$$= 3.2 \times 10^{-5}\ \text{V}$$

Now induced current $I = \frac{e}{R} = \frac{3.2 \times 10^{-5}}{1.6} = 2.0 \times 10^{-5}\ \text{A}$

Example 5.2 A metal bar slides without friction on two parallel conducting rails at distance l apart. A resistor R is connected across the rails and a uniform magnetic field B, pointing into this plane fills the entire region. If the bar moves to the right at a constant speed v then what is the current in the resistor?

[WBUT 2008]

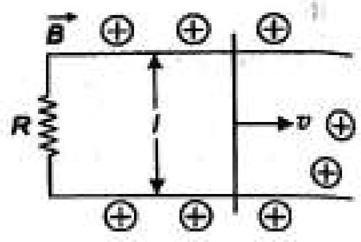


Fig. 5.1W

Sol. Induced emf $|e| = \frac{d\varphi}{dt} = \frac{d}{dt}(BA) = B\frac{dA}{dt}$

If the wire of length l moves a distance dx in time dt then $A = l\,dx$

or, $$|e| = B\frac{d}{dt}(l\,dx) = Bl\frac{dx}{dt} = Blv$$

and induced current $$i = \frac{e}{R} = \frac{Blv}{R}$$

Example 5.3 Flux φ (in Weber) in a closed circuit of resistance 10 Ω varies with time t (in seconds) according to the equation

$$\varphi = 6t^2 - 5t + 1$$

Find induced current at $t = 0.25$ second

Sol. The induced emf $e = -\frac{d\phi}{dt} = -\frac{d}{dt}(6t^2 - 5t + 1) = -12t + 5$

At $t = 0.25$ s, $e = -12 \times 0.25 + 5 = -3 + 5 = 2$V

Now induced current $I = \frac{e}{R} = \frac{2}{10} = 0.2$ A

Example 5.4 A metallic wheel with 6 metallic spokes, each 0.5 m long is rotating at a speed of 120 revolutions per minute in a plane perpendicular to a magnetic field of strength 0.2×10^{-4} Tesla. Find the magnitude of the induced emf between the axle and rim of the wheel.

Sol. If a conductor of length l is rotating perpendicularly to a magnetic field (B) about the fixed point with a constant angular velocity ω, then we can easily calculate the induced emf in the conductor.

Let dl be a small element of the conductor and its velocity be v. Then induced emf in the element is

$$de = Bv\,dl$$

Now, total emf induced in the conductor of length l is

$$e = \int de = \int_0^l Bv\,dl = \int_0^l Bl\omega\,dl = B\omega\frac{l^2}{2} \quad [v = \omega l]$$

$$= \frac{1}{2}B\omega l^2$$

In our problem $\omega = 2\pi \times \frac{120}{60} = 4\pi$ rad/s

and induced emf $e = \frac{1}{2}Bl^2\omega$

$$= \frac{1}{2} \times 0.2 \times 10^{-4} \times (0.5)^2 \times 4\pi = 3.14 \times 10^5 \text{ V.}$$

Example 5.5 An ac voltage source $V = V_0 \sin \omega t$ is connected across a parallel-plate capacitor C. Verify that the displacement current in the capacitor is the same as the conduction current in the wire.

Sol. For a parallel-plate capacitor, if A is the area and d is the separation between the plates then

$$C = \frac{\varepsilon_0 A}{d}$$

Again electric field $E = \frac{V}{d}$

So, $D = \varepsilon_0 E = \frac{\varepsilon_0 V}{d} = \frac{\varepsilon_0 V_0 \sin \omega t}{d}$

The displacement current

$$I_d = \int_s \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \cdot \vec{ds} = \varepsilon_0 \frac{A}{d} V_0 \omega \cos \omega t$$

$$= CV_0 \omega \cos \omega t$$

Again conduction current $I = \frac{dQ}{dt} = \frac{d}{dt}(CV) = C\frac{dV}{dt}$

$$= C\frac{d}{dt}(V_0 \sin \omega t)$$

$$= CV_0 \omega \cos \omega t$$

So, both currents are same.

Example 5.6 A parallel-plate capacitor with circular plates of 10 cm radius separated by 5 mm is being charged by an external source. The charging current is 0.2 A. Find (i) the rate of change of potential difference between the plates, and (ii) obtain the displacement current.

Sol. Here capacitance $C = \frac{\varepsilon_0 A}{d} = \frac{8.85 \times 10^{-12} \times \pi (0.1)^2}{5 \times 10^{-3}}$

$$= 5.56 \times 10^{-11}\ F$$

Given $I = \frac{dQ}{dt} = C\frac{dV}{dt} = 0.2\ A$

So, $\frac{dV}{dt} = \frac{0.2}{C} = \frac{0.2}{5.56 \times 10^{-11}} = 3.6 \times 10^{11}\ V/s.$

Again displacement current $I_d = \varepsilon_0 \frac{d\varphi}{dt} = \varepsilon_o A\frac{dE}{dt} = \frac{dQ}{dt} = I$

So displacement current is equal to 0.2 A.

Example 5.7 Show that $\frac{\sigma}{\varepsilon}\rho + \frac{\partial \rho}{\partial t} = 0$, where σ is the electric conductivity and ε is the electric permittivity of the medium.

Sol. From equation of continuity, $\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$ and from Ohm's law, $\vec{J} = \sigma \vec{E}$.

Now $$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

or, $$\vec{\nabla}\cdot(\sigma\vec{E})+\frac{\partial\rho}{\partial t}=0$$

or, $$\sigma\vec{\nabla}\cdot\vec{E}+\frac{\partial\rho}{\partial t}=0$$

Again we know $$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon}\quad \text{so,}\ \frac{\sigma\rho}{\varepsilon}+\frac{\partial\rho}{\partial t}=0\quad \text{(Proved)}$$

Example 5.8 Given $\vec{E}=\hat{i}E_0\cos\omega\left(\frac{z}{c}-t\right)+\hat{j}E_0\sin\omega\left(\frac{z}{c}-t\right)$, determine the magnetic field $\vec{B}$.

Sol. We know $$\vec{B}=\frac{\vec{k}\times\vec{E}}{\omega}=\frac{k}{\omega}(\hat{k}\times\vec{E})\quad [\because \vec{k}=k\,\hat{k}]$$

So, $$\vec{B}=\frac{k}{\omega}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 0 & 1\\ E_0\cos\omega\left(\frac{z}{c}-t\right) & E_0\sin\omega\left(\frac{z}{c}-t\right) & 0\end{vmatrix}$$

$$=\frac{k}{\omega}\left[-\hat{i}\,E_0\sin\omega\left(\frac{z}{c}-t\right)+\hat{j}\,E_0\cos\omega\left(\frac{z}{c}-t\right)\right]$$

$$=-\hat{i}\,\frac{E_0}{c}\sin\omega\left(\frac{z}{c}-t\right)+\hat{j}\,\frac{E_0}{c}\cos\omega\left(\frac{z}{c}-t\right)\quad\left[\because \frac{k}{\omega}=c\right]$$

Hence, magnetic field $$\vec{B}=-\hat{i}\,\frac{E_0}{c}\sin\omega\left(\frac{z}{c}-t\right)+\hat{j}\,\frac{E_0}{c}\cos\omega\left(\frac{z}{c}-t\right)$$

Example 5.9 A wave has a wavelength of 4 mm and the electric field associated with it has an amplitude of 40 V/m. Determine the amplitude and frequency of oscillations of the magnetic field.

Sol. The relation between electric and magnetic field

$$B_0=\frac{E_0}{c}=\frac{40}{3\times10^8}=13.3\times10^{-8}\ \text{Tesla}$$

Frequency of oscillation $$f=\frac{c}{\lambda}=\frac{3\times10^8}{4\times10^{-3}}=0.75\times10^{11}\ \text{Hz}$$

Example 5.10 Calculate the skin depth for radio waves of 3 m wavelength (in free space) in copper, the electrical conductivity of which is 6×10^7 S/m. [Given permeability of free space $\mu_0=4\pi\times10^{-7}$ H/m]

Sol. Given $$\lambda=3\ \text{m},\ \sigma=6\times10^7\ \text{S/m},\ \mu_0=4\pi\times10^{-7}\ \text{H/m}$$

Skin depth $$\delta=\sqrt{\frac{2}{\omega\sigma\mu}}$$

$$\omega=2\pi f=2\pi\frac{c}{\lambda}=\frac{2\pi\times3\times10^8}{3}=2\pi\times10^8\ \text{rad/s}$$

Now, skin depth $$\delta=\sqrt{\frac{2}{\omega\mu\sigma}}=\sqrt{\frac{2}{2\pi\times10^8\times6\times10^7\times4\pi\times10^{-7}}}$$

$$=6.5\times10^{-6}\ \text{m}.$$

Example 5.11 The earth is considered to be a good conductor when $\frac{\omega\varepsilon}{\sigma} << 1$. Calculate the highest frequencies for which the earth can be considered a good conductor if << 1 means less than 0.1.

[Assume $\sigma = 5 \times 10^{-3}$ mho/m, $\varepsilon = 10\ \varepsilon_0$]

Sol. Here $\frac{\omega\varepsilon}{\sigma} < 0.1$

or,
$$\omega < \frac{0.1\sigma}{\varepsilon} < \frac{0.1 \times 5 \times 10^{-3}}{8.854 \times 10^{-12} \times 10}$$

$\therefore\ \omega < 5.65 \times 10^6$

Highest frequency for which the earth can be considered as a good conductor is

$$f = \frac{\omega}{2\pi} = \frac{5.65 \times 10^6}{2\pi} = 0.9 \text{ MHz.}$$

Example 5.12 Find the skin depth δ at a frequency of 1.6 MHz in *Al*, where $\sigma = 38.2$ Ms/m and $\mu_r = 1$. Also find the propagation constant and wave velocity.

Sol.
$$\sigma = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi \times 1.6 \times 10^6 \times 4\pi \times 10^{-7} \times 38.2 \times 10^6}}$$

$$= 6.43 \times 10^{-2} \text{ mm}$$

The propagation constant $k = \alpha + i\beta$

But
$$\alpha = \beta = \frac{1}{\delta} = 15.53 \times 10^3 \text{ m}^{-1}$$

$\therefore$
$$k = 15.53 \times 10^3 + i\ 15.53 \times 10^3$$

$$= 21.96 \times 10^3 < 45° \text{ m}^{-1}$$

Wave velocity
$$v = \frac{\omega}{\beta} = \omega\delta = 2\pi \times 1.6 \times 10^6 \times 6.43 \times 10^{-5} \text{ m/s}$$

$$= 647.2 \text{ m/s}$$

Example 5.13 Calculate the value of Poynting vector at the surface of the sun if the power radiated by the sun is 3.8×10^{26} W and its radius is 7×10^8 m.

Sol. Here, Power = 3.8×10^{26} W and $r = 7 \times 10^8$ m

If P is the average Poynting vector at the surface of the sun then

$$P = \frac{\text{Power}}{4\pi r^2} = \frac{3.8 \times 10^{26}}{4 \times 3.14 \times (7 \times 10^8)^2}$$

$$= 6.17 \times 10^7 \text{ W/m}^2$$

Example 5.14 Calculate the strength of the electric and magnetic field of radiation if the earth's surface receives sunlight of energy per unit time per unit area is 3 cal/min cm^2.

Sol. Here, solar energy which the earth receives is 3 cal/min cm^2

i.e., $I = 3$ cal/(min cm^2)

or, $$I = \frac{3 \times 4.2 \times 10^4}{60} = 2100 \text{ J/m}^2\text{s}$$

$\therefore$ the poynting vector, $\vec{P} = \vec{E} \times \vec{H}$

$$= EH \sin 90°$$

$$= 2100 \text{ J/m}^2\text{s}$$

Again $$\frac{E}{H} = \sqrt{\frac{\mu_0}{\epsilon_0}} = \sqrt{\frac{4\pi \times 10^{-7}}{8.85 \times 10^{-12}}} = 377$$

$\because$ $$EH \times \frac{E}{H} = 2100 \times 377$$

$$E^2 = 7917 \times 10^2$$

or, $$E = 890 \text{ V/m}$$

and $$H^2 = \frac{2100}{377} = 5.57$$

or, $$H = 2.36 \text{ A/m}$$

Example 5.15 Find the magnetic field B and Poynting vector P of electromagnetic waves in free space if the components of the electric fields are $E_x = E_y = 0$ and $E_z = E_0 \cos kx \sin \omega t$.

Sol. From Faraday's law $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

But $$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & E_z \end{vmatrix} = \hat{i}\frac{\partial E_z}{\partial y} - \hat{j}\frac{\partial E_z}{\partial x}.$$

Now $E_z = E_0 \cos kx \sin \omega t$

So $$\frac{\partial E_z}{\partial y} = 0 \text{ and } \frac{\partial E_z}{\partial x} = \frac{\partial}{\partial x}(E_0 \cos kx \sin \omega t) = -E_0 k \sin kx \sin \omega t$$

$\therefore$ $$\vec{\nabla} \times \vec{E} = +E_0 k \sin kx \sin \omega t\, \hat{j} = -\frac{\partial \vec{B}}{\partial t}$$

$\therefore$ $$\vec{B} = +\frac{E_0 k}{\omega} \sin kx \cos \omega t\, \hat{j}$$

Now Poynting vector $\vec{P} = \vec{E} \times \vec{H}$

$$= E_0 \cos kx \sin \omega t\, \hat{k} \times \frac{E_0 k}{\mu_0 \omega} \sin kx \cos \omega t\, \hat{j}$$

$$= \frac{E_0^2 k}{\mu_0 \omega} \times \frac{1}{4} \sin 2kx \sin 2\omega t\, (\hat{k} \times \hat{j})$$

Example 5.16 Consider a monochromatic plane wave, where the electric field is given by

$$\vec{E} = E_0 e^{i(kz-\omega t)} \hat{i}$$

where E_0 is an arbitrary constant vector.

(i) Show that the electric field vector lies in a direction perpendicular to the propagation.

(ii) Determine the corresponding magnetic field.

Sol. From Maxwell's equation

$$\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$

or

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = -\mu_0 \left[\hat{i} \frac{\partial H_x}{\partial t} + \hat{j} \frac{\partial H_y}{\partial t} + \hat{k} \frac{\partial H_z}{\partial t} \right]$$

Comparing both sides

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0} \frac{\partial E_x}{\partial z} \quad \text{and} \quad H_x = 0, H_z = 0$$

$$\therefore \quad \frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0} \frac{\partial}{\partial z} \left[E_0 e^{i(kz-\omega t)} \right]$$

$$= -\frac{ikE_0}{\mu_0} e^{i(kz-\omega t)}$$

$$\therefore \quad H_y = -i \frac{k}{\mu_0} E_0 \int e^{i(kz-\omega t)} dt = \frac{E_0 k}{\mu_0 \omega} e^{i(kz-\omega t)}$$

Here, the electric field propagates in the x direction and the magnetic field propagates in the y direction, whereas the wave propagates in the z direction. So, we can say that the electric field vector lies in a direction perpendicular to the propagation.

(ii) The corresponding magnetic field

$$\vec{H} = \frac{E_0 k}{\mu_0 \omega} e^{i(kz-\omega t)} \hat{j}$$

Example 5.17 Show that for frequency $\leq 10^9$ Hz, a sample of silicon will act like a good conductor. For silicon, one may assume $\frac{\varepsilon}{\varepsilon_0} = 12$ and $\sigma = 2$ mho/cm. Also calculate the penetration depth for this sample at frequency 10^6 Hz.

Sol. A material will be good conductor if $\frac{\sigma}{\omega\varepsilon} >> 1$

Here $\sigma = 2$ mhos/cm $= 200$ mhos/m

$\omega = 2\pi f = 2\pi \times 10^9$

$\varepsilon = 12\, \varepsilon_0$

Now $$\frac{\sigma}{\omega\varepsilon} = \frac{200 \times 2}{2 \times \pi \times 10^9 \times 12\,\varepsilon_0} = \frac{400 \times 9 \times 10^9}{12 \times 10^9} = 300$$

So $\frac{\sigma}{\omega\varepsilon} >> 1$; a sample of silicon will act like a conductor at frequency $\leq 10^9$ Hz.

The penetration depth for good conductor

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

$$= \sqrt{\frac{2}{2\pi \times 10^6 \times 4\pi \times 10^{-7} \times 200}} = 3.6 \times 10^{-2}\ \text{m}$$

$$= 3.6\ \text{cm}$$

Example 5.18 Calculate the skin depth for a frequency 10^{10} Hz for silver.

Given $\sigma = 2 \times 10^7\ \text{Sm}^{-1}$ and $\mu = 4\pi \times 10^{-7}\ \text{Hm}^{-1}$

Sol. Here $\omega = 2\pi f = 2\pi \times 10^{10}$, $\sigma = 2 \times 10^7\ \text{Sm}^{-1}$ and $\mu = 4\pi \times 10^{-7}\ \text{Hm}^{-1}$

Now skin depth $$\delta = \left(\frac{2}{\omega\mu\sigma}\right)^{1/2}$$

$$= \left(\frac{2}{2\pi \times 10^{10} \times 4\pi \times 10^{-7} \times 2 \times 10^7}\right)^{1/2}$$

$$= 1.12 \times 10^{-6}\ \text{m}$$

Example 5.19 Discuss the behavior of copper to electromagnetic waves of frequency 0.5×10^{16} Hz and 7.0×10^{20} Hz. Given the conductivity of copper $\sigma = 5.8 \times 10^7$ mho m^{-1} and permittivity $\varepsilon = 9 \times 10^{-12}\ \text{C}^2/\text{Nm}^2$

Sol. Here $$\omega = 2\pi \times 0.5 \times 10^{16}$$

$$\frac{\sigma}{\omega\varepsilon} = \frac{5.8 \times 10^7}{2\pi \times 0.5 \times 10^{16} \times 9 \times 10^{-12}} = 205$$

Since $\frac{\sigma}{\omega\varepsilon} > 100$, the conduction current dominates. Hence for frequency 0.5×10^{16} Hz copper is a conductor.

Now for frequency $f = 7 \times 10^{20}$ Hz, $\omega = 2\pi \times 7 \times 10^{20}$

$$\therefore \quad \frac{\sigma}{\omega\varepsilon} = \frac{5.8 \times 10^7}{2\pi \times 7 \times 10^{20} \times 9 \times 10^{-12}} = 1.47 \times 10^{-3}$$

Since $\frac{\sigma}{\omega\varepsilon} < 100$, the displacement current dominates. Hence for frequency 7×10^{20} Hz, copper is a dielectric.

Example 5.20 Calculate the value of Poynting vector for a 60 W lamp at a distance of 0.5 m from it.

Sol. Total average power emitted by the lamp = 60 W

The light emitted by the lamp will spread out in the form of a sphere, the radius of which is equal to the distance from it.

∴ radius of the sphere $R = 0.5$ m

Let P be the average Poynting vector over the surface of the sphere, then

$$P = \frac{\text{Power}}{4\pi R} = \frac{60}{4\pi (0.5)^2} = 19.1 \text{ W/m}^2.$$

Review Exercises

Part 1: Multiple Choice Questions

1. The magnetic flux linked with a coil at any instant 't' is given by $\varphi_t = 5t^3 - 100\,t + 200$, the emf induced in the coil at $t = 2$ seconds is
   (a) 200 V (b) 40 V (c) 20 V (d) –20 V
2. A cylindrical conducting rod is kept with its axis along the positive z axis, where a uniform magnetic field exists parallel to the z axis. The current induced in the cylinder is
   (a) clockwise as seen from the + z axis (b) zero
   (c) anticlockwise as seen from the – z axis (d) None of these
3. In an electromagnetic wave in a free space, the electric and magnetic fields are [WBUT 2008]
   (a) parallel to each other (b) perpendicular to each other
   (c) inclined at an acute angle (d) inclined at an obtuse angle
4. If $\vec{E}$ and $\vec{B}$ are the electric field and the magnetic field of electromagnetic waves traveling in vacuum with propagation vector then [WBUT 2006]
   (a) $\vec{k} \cdot \vec{E} = 0$ (b) $\vec{k} \times \vec{E} = 0$ (c) $\vec{B} \times \vec{E} = 0$ (d) $\vec{k} \times \vec{E} = -\vec{B}$
5. The velocity of a plane electromagnetic wave is given by
   (a) $c = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}}$ (b) $c = \dfrac{1}{\mu_0 \varepsilon_0}$ (c) $c = \mu_0 \varepsilon_0$ (d) $\dfrac{\varepsilon_0}{\mu_0}$
6. $\vec{\nabla} \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0}$ represent
   (a) Ampere's law (b) Laplace's equation
   (c) Gauss' law in electrostatics (d) Faraday's law of electromagnetic induction
7. The differential form of Faraday's law of electromagnetic induction is
   (a) $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ (b) $\vec{\nabla} \times \vec{E} = \dfrac{\partial \vec{B}}{\partial t}$
   (c) $\vec{\nabla} \times \vec{B} = \dfrac{\partial \vec{E}}{\partial t}$ (d) $\vec{\nabla} \times \vec{B} = -\dfrac{\partial \vec{E}}{\partial t}$

8. Maxwell's electromagnetic wave equations in terms of an electric field vector $\vec{E}$ in free space is [WBUT 2005]
   (a) $\nabla^2 \vec{E} = -\mu_0 \varepsilon_0 \dfrac{\partial^2 \vec{E}}{\partial t^2}$ (b) $\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \dfrac{\partial^2 \vec{E}}{\partial t^2}$
   (c) $\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t}$ (d) $\vec{\nabla} E = \mu_0 \varepsilon_0 \dfrac{\partial^2 \vec{E}}{\partial t^2}$
9. Displacement current arises due to [WBUT 2004, 2007]
   (a) positive charge only (b) negative charge only
   (c) time-varying electric field (d) None of these
10. In electromagnetic induction
   (a) mechanical energy is converted into magnetic energy
   (b) mechanical energy is converted into electrical energy
   (c) magnetic energy is converted into mechanical energy
   (d) magnetic energy is converted into electrical energy
11. Waves originating from a point source and traveling in an isotropic medium are described as [WBUT 2007]
   (a) $\varphi = \varphi_0 \exp i(kr - \omega t)$ (b) $\varphi = \varphi_0 \exp i(kr - \omega t)/r$
   (c) $\varphi = \varphi_0 \exp i(kr - \omega t)/r^2$ (d) $\varphi = \varphi_0 \exp i(kr + \omega t)/r$
12. Electromagnetic wave is propagated through a region of vacuum, which does not contain any charge or current. If the electric vector is given by $\vec{E} = \vec{E}_0 \exp i(kx - \omega t)\,\hat{j}$ then the magnetic vector is [WBUT 2007]
   (a) in the x direction (b) in the y direction
   (c) in the z direction (d) rotating uniformly in the xy plane
13. Steady current produces
   (a) magnetostatic field (b) electrostatic field
   (c) time varying electric field (d) time-varying magnetic field
14. A conducting rod is moved with a constant velocity v in a magnetic field. A potential difference appears across the two ends
   (a) if $\vec{v} \parallel \vec{l}$ (b) if $\vec{v} \parallel \vec{B}$ (c) if $\vec{l} \parallel \vec{B}$ (d) None of these
15. A bar magnet is released from rest along the axis of a very long vertical copper tube. After some time, the magnet
   (a) will stop in the tube (b) will move with almost constant speed
   (c) will move with an acceleration g (d) will oscillate
16. The dimension of $\mu_0 \varepsilon_0$ is
   (a) $L^{-2}T^{-2}$ (b) $L^{-2}T^{2}$ (c) LT^{-1} (d) $L^{-1}T^{-1}$
17. The modified Ampere's circuital law is
   (a) $\oint \vec{B} \cdot \vec{dl} = \mu_0 (I + I_d)$ (b) $\oint \vec{B} \cdot \vec{dl} = \mu_0 I$
   (c) $\oint \vec{B} \cdot \vec{dl} = \varepsilon_0 I$ (d) $\oint \vec{B} \cdot \vec{dl} = \varepsilon_0 (I + I_d)$

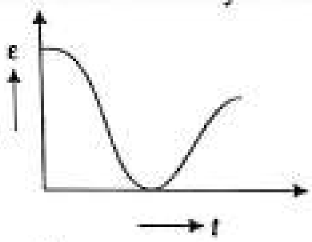
18. The electromagnetic wave is called transverse wave because [WBUT 2008]
    (a) the electric field and magnetic field are perpendicular to each other
    (b) the electric field is perpendicular to the direction of propagation
    (c) the magnetic field is perpendicular to the direction of propagation
    (d) both the electric field and magnetic field are perpendicular to the direction of propagation

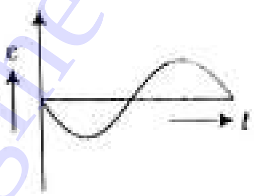
19. The solution of a plane electromagnetic wave $\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \dfrac{\partial^2 \vec{B}}{\partial t^2}$ is

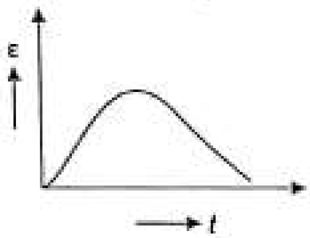
    (a) $\vec{B} = \vec{B}_0 e^{i(\omega t - \vec{k}\cdot\vec{r})}$ (b) $\vec{B} = \vec{B}_0 e^{i(\omega t + \vec{k}\cdot\vec{r})}$
    (c) $\vec{B} = \vec{B}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ (d) $B = \vec{B}_0 e^{-i(\omega t + \vec{k}\cdot\vec{r})}$

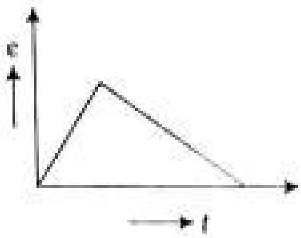
20. When a magnet is being moved towards a coil, the induced emf does not depend upon
    (a) the number of turns of the coil (b) the motion of the magnet
    (c) the magnetic moment of the magnet (d) the resistance of the coil

21. The variation of induced emf (ε) with time (t) in a coil if a short bar magnet is moved along its axis with a constant velocity is best represented as

(a)


(b)


(c)


(d)


22. The SI unit of Poynting vector is
    (a) Wm (b) Wm^{-1} (c) Wm^2 (d) Wm^{-2}

23. The Poynting vector is given by the expression
    (a) $\vec{E} \times \vec{H}$ (b) $\vec{H} \times \vec{E}$ (c) $\vec{E} \cdot \vec{H}$ (d) None of these

24. Skin depth for a conductor in reference to electromagnetic wave varies
    (a) inversely as frequency (b) directly as frequency
    (c) inversely as square root of frequency (d) directly as square of frequency

25. The ratio of the phase velocity and velocity of light is
    (a) one (b) less than one (c) greater than one (d) zero

26. The value of skin depth (δ) in a conducting medium is

    (a) $\delta = \sqrt{\dfrac{2\sigma}{\mu\omega}}$ (b) $\delta = \sqrt{\dfrac{2}{\mu\omega\sigma}}$ (c) $\delta = \sqrt{\dfrac{1}{\mu\omega\sigma}}$ (d) $\delta = \sqrt{\dfrac{2\mu}{\omega\sigma}}$

27. Skin depth is proportional to

(a) ω (b) μ (c) $\frac{1}{\sqrt{\sigma}}$ (d) $\sqrt{\sigma}$

[**Ans.** 1 (b), 2 (b), 3 (b), 4 (a), 5 (a), 6 (c), 7 (a), 8 (b), 9 (c), 10 (d), 11 (a), 12 (a), 13 (a), 14 (d), 15 (b), 16 (b), 17(a), 18 (d), 19 (c), 20 (d), 21 (b), 22 (d), 23 (a), 24 (c), 25 (b), 26 (b), 27 (c)]

Short Questions with Answers

1 State Faraday's laws of electromagnetic induction.

Ans. (i) Whenever there is a change in the magnetic flux linked with a coil an emf is set up in it and stays as long as the magnetic flux linked with it is changing.

(ii) The magnitude of the induced emf is proportional to the rate of change of magnetic flux linked with the coil, i.e.,

$$\varepsilon \propto \frac{d\varphi}{dt}$$

2 What is the difference between conduction current and the displacement current?

Ans.

Conduction Current	*Displacement Current*
(i) It does obey Ohm's law.	(i) It does not obey Ohm's law.
(ii) Conduction current is due to the actual flow of charge in a conductor.	(ii) Displacement current is due to the time-varying electric field in a dielectric.
(iii) Conduction current density $\vec{J} = \sigma \vec{E}$.	(iii) Displacement current density $\vec{J}_D = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

3. An electron moves along the line AB which lies in the same plane as a circular loop of conduction wire, as shown in Fig. 5.2W. Find out the direction of the induced current in the loop.

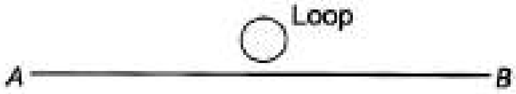


Fig. 5.2W

Ans. The electron moves from A to B so the direction of the current will be from B to A. The magnetic field generated in the loop due to the motion of the current will be directed into the plane of the paper. To oppose this, the current in the coil must be anticlockwise, in accordance with Lenz's law.

4. Why is electromagnetic wave called transverse wave?

Ans. An electromagnetic wave is called a transverse wave because by the direction of the propagation, the electric field and magnetic field are mutually perpendicular to each other.

5. What is displacement current?

Ans. See Section 5.6.

6. Why do birds fly off a high-tension wire when current is switched on?

Ans. When current begins to increase from zero to maximum value, a current is induced in the body of the bird. This produces a repulsive force and the bird flies off.

7. Two similar circular coaxial loops carry equal currents in the same direction. If the loops be brought nearer, what will happen to the currents in them?

Ans. When the loops are brought closer, there is an increase of magnetic flux. An induced emf is produced. According to Lenz's law, the induced emf has to oppose the change of magnetic flux. So, the current in each loop will decrease.

8. Two coils are being moved out of a magnetic field. One coil is moved rapidly and the other slowly. In which case is more work done and why?

Ans. More work will be done in the case of a rapidly moving coil. This is because the induced emf will be more in this coil as compared to slow moving coil.

9. Define skin depth.

Ans. Skin depth is defined as the distance in the conductor over which the electric field vector of the wave propagating in the medium decays to $1/e$ times its value at the surface.

10. What is Poynting vector?

Ans. The cross product of the electric vector $\vec{E}$ and the magnetic field vector $\vec{H}$ is known as a Poynting vector. Mathematically Poynting vector $\vec{P} = \vec{E} \times \vec{H}$

11. What is the effect of frequency on skin depth?

Ans. We know that skin depth

$$\delta = \left(\frac{2}{\omega\mu\sigma}\right)^{1/2}$$

Thus skin depth is inversely proportional to the square root of the frequency. So, skin depth decreases with increase in frequency.

12. How is skin depth useful in practical situation?

Ans. A poor conductor can be made a good conductor with a thin coating of copper or silver. A silver coating on a piece of glass may be an excellent conductor at microwave frequency. The performance of gold and gold-coated brass wave guides will be the same if the coating thickness is equal to skin depth. So in a practical situation, skin depth method is useful to reduce the cost of the material.

Part 2: Descriptive Questions

1. (a) Write down Faraday's law of electromagnetic induction. [WBUT 2004]
   (b) Express it in differential form. [WBUT 2006]
2. (a) Write down Maxwell's equations in differential form and explain the physical significance of each equation. [WBUT 2002, 2004]
   (b) Show that the wave equation in free space for electric field $\vec{E}$ is given by $\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$ [WBUT 2004]
3. (a) State Maxwell's equations. From these equations, derive the wave equations for an electromagnetic wave. What is the velocity of this wave?
   (b) Assuming a plane wave solution, establish the relation between the propagation vector ($\vec{k}$), electric field ($\vec{E}$) and magnetic field ($\vec{B}$). [WBUT 2008]
4. (a) Distinguish between the conduction current and displacement current.
   (b) Write down Faraday's law of electromagnetic induction.

5. (a) Write down Maxwell's field equations, explaining the term used. Show that in vacuum, both electric and magnetic vectors obey wave equation. Assuming a plane wave solution show that magnetic field is always orthogonal to the electric field.
   (b) Find the displacement current within a parallel-plate capacitor in series with a resistor which carries current I. Area of the capacitor plates are A and the dielectric is vacuum. [WBUT 2006]
6. (a) Write and explain differential and integral forms of Maxwell's equations.
   (b) Explain the significance of displacement current.
7. (a) Write down Maxwell's field equations.
   (b) From those equations identify Gauss' law, Ampere's law and Faraday's law.
   (c) How does velocity of light depend on the properties of vacuum? [WBUT 2005]
8. Use Faraday's law of electromagnetic induction and the fact that magnetic induction $\vec{B}$ can be derived from a vector potential $\vec{A}$. Show that the electric field can be expressed as

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t} \quad \text{where } \varphi \text{ is the scalar potential}$$ [WBUT 2007]

9. (a) What is displacement current? Distinguish between conduction and displacement current.
   (b) Show that $\vec{E} = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ and $\vec{B} = \vec{B}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ are possible solutions of Maxwell's electromagnetic wave equations in terms of electric and magnetic field.
10. Starting from Maxwell's equations in free space, show that the magnetic field $\vec{B}$ and the electric field $\vec{E}$ in an electromagnetic wave travel with the same speed.
11. Write down Maxwell's equations in integral form and explain the physical significance of each equation.
12. Show that $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ and $\vec{\nabla} \times \vec{B} = \mu_0\left[\vec{J} + \varepsilon_o \frac{\partial \vec{E}}{\partial t}\right]$
13. Show that in free space, the electric field $\vec{E}$, magnetic field $\vec{B}$ and propagation vector $\vec{k}$ are perpendicular to each other.
14. The electric field associated with an electromagnetic wave is $\vec{E} = \hat{i}\, E_0 \cos(kz - \omega t) + \hat{j}\, E_0 \sin(kz - \omega t)$, where E_0 is a constant. Find the corresponding magnetic field $\vec{B}$.
15. (a) State Lenz's law. Explain it from the principle of conservation of energy.
    (b) A wire is rotated about one of its ends at right angles to a magnetic field. Deduce the expression for induced emf.
16. Define skin depth. Show that in case of a semi-infinite solid conductor, the skin depth δ is given by

$$\delta = \frac{1}{\sqrt{\omega\mu\sigma}}$$

    where symbols have their usual meanings.
17. What is Poynting vector? Show that Poynting vector measures the flow of energy per unit area per second in an electromagnetic wave.
18. State and prove Poynting theorem.
19. Show that average power $\langle P \rangle = \frac{1}{2} R_e(E \times H^*)$

20. What is Poynting vector? Find the expression of Poynting vector. What is the physical interpretation of this vector?
21. A plane electromagnetic wave is incident normally on a metal of electrical conductivity σ. Show that the electromagnetic wave is damped inside the conductor and find the skin depth.

Part 3: Numerical Problems

1. A parallel-plate capacitor with plate area A and separation d between the plates is charged by a constant current I. Consider a plane surface of area $\frac{A}{4}$ parallel to the plates and drawn symmetrically between the plates. Calculate the displacement current through this area. $\left[\textbf{Ans. } I_D = \frac{I}{4}\right]$
2. A parallel-plate capacitor with circular plates of radius $a = 5.5$ cm is being charged at a uniform rate so the electric field between the plates changes at a constant rate $\frac{dE}{dt} = 1.5 \times 10^{12}$ V/ms. Find the displacement current for the capacitor. [**Ans.** $I_D = 0.13$A]
3. The magnetic field in a plane electromagnetic wave is given by $B_y = 2 \times 10^{-7} \sin(0.5 \times 10^3 x + 1.5 \times 10^{11} t)$ Tesla. (a) What is the wavelength of the wave? (b) Write an expression for the electric field. [**Ans.** $\lambda = 1.26$ cm, $E_0 = 60$ Vm^{-1}]
4. Capacitance of a parallel-plate capacitor is $2\mu F$. Calculate the rate at which the potential difference between the two plates must charge to get a displacement current of 0.4 A. [**Ans.** $\frac{dV}{dt} = 2 \times 10^5$ V/s]
5. A current of 5 A is passed through a solenoid of 50 cm length, 3.0 cm radius and having 200 turns. When the switch is open, the current becomes zero within 10^{-3} s. Calculate the emf induced across the switch. [**Ans.** $e = 1.42$ volt] $\left[\textbf{Hints: } e = N\frac{d\varphi}{dt},\ \varphi = BA = \mu_0 ni A,\ n = \frac{N}{l}\right]$
6. A rectangular loop of 8 cm side and 2 cm having a resistance of 1.6 Ω is placed in a magnetic field and gradually reduced at the rate of 0.02 Tesla/s. Find out the induced current. [**Ans.** 2×10^{-5} A]
7. A 50 cm long bar PQ is moved with a speed of 4 ms^{-1} in a magnetic field $B = 0.01$ Tesla as shown in Fig. 5.3W. Find out the induced emf. [**Ans.** 0.02 V]

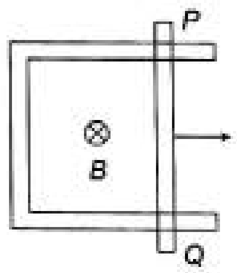


Fig. 5.3W

8. Calculate the skin depth for a frequency 10^{10} Hz for silver. Given $\sigma = 2 \times 10^7$ Sm^{-1} and $\mu = 4\pi \times 10^{-7}$ Hm^{-1}. [**Ans.** $\delta = 1.12 \times 10^{-6}$ m]
9. Find the skin depth δ at a frequency of 1.6 MHz in aluminium, where $\sigma = 38.2$ mS/m and $\mu_r = 1$. Also find wave velocity. [**Ans.** $\delta = 6.4 \times 10^{-5}$ m, $v = 6.47 \times 10^2$ m/s]
10. A laser beam has a diameter of 2 mm, what is the amplitude of the electric and magnetic field in the beam in vacuum if the power of the laser is 1.5 mW? [Given $\mu_0 = 4\pi \times 10^{-7}$ H/m, $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m]. [**Ans.** $E_0 = 600$ V/m $H_0 = 1.59$ amp/m]
11. Find the depth of penetration of a megacycle wave into copper which has conductivity of $\sigma = 5.8 \times 10^7$ mho/m and a permeability equal to that of free space. [**Ans.** $\delta = 6.6 \times 10^{-5}$ m]

CHAPTER

6

Classical Mechanics

6.1 INTRODUCTION

In its literal meaning, the very phrase 'classical mechanics' contains the topic of newtonian mechanics. But for some practical reasons, Newtonian mechanics is discussed separately. The newtonian mechanics for its discussion relies on the three laws of Newton. On the other hand, classical mechanics goes a step further. Some luminaries (like Euler, D'Alembert, Lagrange, Hamilton and others) made remarkable contributions to the development of the subject of classical mechanics by introducing some superior methods for solving real physical problems. These methods are basically discussed in the topic of classical mechanics. The entire subject of classical mechanics, which deals with all kinds of motion of material bodies at the macroscopic level throughout the universe, is basically based on Newton's laws of motion. But the aforesaid superior methods introduced by the said luminaries avoid explicit mention of Newton's laws of motion. So, under the topic of classical mechanics we will discuss the D'Alembert's principle along with lagrangian and hamiltonian formulations. The generalized version of classical mechanics which is known as quantum mechanics was formulated in the early twentieth century to account for these phenomena at the microscopic level. In this chapter we will try to explore the superiority of lagrangian and hamiltonian formulations over the newtonian formulation in solving mechanical problems.

6.1.1 Limitations of Newtonian Mechanics

Though the subject of classical mechanics is entirely based on Newton's laws of motion, it often becomes very difficult to solve problems of mechanics by considering the forces acting on a system and the accelerations of the system under these forces. For analysis of the nature of motion of any system one has to proceed through two essential steps: at first one has to write the differential equation of motion of the system by considering the forces which are acting on the system and then one has to solve the said differential equation. But in newtonian mechanics, while using newtonian formulations, one comes across many limitations which are noted below:

(a) Newtonian mechanics is valid only in inertial frames of reference which are by definition cartesian-like and move with constant velocities. newtonian mechanics cannot be applied in case of non-inertial frames of reference.

(b) For writing the differential equation of motion, a particular coordinate system has to be selected by considering the symmetry of the system.

(c) The newtonian formulation is not invariant under all coordinate systems. While coordinates are transformed from one system to another, the equations of motion change their forms.
As an example, when a particle of mass m moves under the influence of an external force F_x, one can write its differential equation in cartesian coordinates as

$$m\frac{d^2x}{dt^2} = F_x \qquad \text{...(6.1)}$$

But in the polar coordinates (r, θ), the equation of motion of the particle is given by

$$m\frac{d^2r}{dt^2} = mr\left(\frac{d\theta}{dt}\right)^2 \qquad \text{...(6.2)}$$

[i.e., $m\ddot{r} = mr\,\dot{\theta}^2$]

which is not in the form $m\frac{d^2r}{dt} = F_r$. Here, the force depends on both the parameters r and θ. Such a coupled dependency of coordinates is very much inconvenient. This inconvenience can be avoided if one selects a generalized coordinate system without making any reference to a specific coordinate system.

(d) The newtonian formulation (while using Newton's laws) require complete specification of all the forces which act on the body whose motion is under consideration at all instants of time. But it has been observed that sometimes the independent form of some forces cannot be known though the effects of them on the concerned system are well known. There is no scope to substitute these effects of the forces in newtonian formulation. Consequently, it becomes difficult to find the solution of the equation of motion in order to know the nature of motion of the particles of the system.

(e) Because of the aforesaid reasons, one cannot deal with the constrained motion by making use of the newtonian formulation.

(f) In newtonian formulation, the equations of motion involve vector quantities like force, momentum, etc., which introduce complexity in solving the problems.

(g) The greatest limitation faced in this formulation is that the mechanical problems are always tried to resolve geometrically rather than analytically. And in case of constraint motion, determination of all the reaction forces is difficult in newtonian formulation. One can easily avoid all these aforementioned difficulties if one uses a new formulation called lagrangian formulation.

6.1.2 Constraints of Motion

The constraints are the obstacles or geometrical restrictions on the motion of a particle or a system of particles. The forces which are responsible for the constraints are called the forces of constraints. When there is a constraint, the particle motion is restricted to occur only along some specified path, or on a surface (plane or curved) arbitrarily oriented in space. Let us mention some examples of constraint motion as given below:

(a) The motion of a bead along a horizontally stretched wire. It is restricted to move on a straight path.

(b) The motion of the bob of a simple pendulum. It is restricted to oscillate in the vertical plane.

(c) The motion of a train on the rails. It is confined along the rails only.

(d) The motion of the gas molecules which are confined in a container. The motion is restricted by the walls of the container.

(e) The motion of a rigid body is such that the distance between only two particles of the body remains always constant.

To describe the motion of a body (e.g., a bead) in space we require three coordinates but to describe its motion on a stretched string we require only one coordinate. Thus, if we impose constraints on a mechanical system we can reduce the number of coordinates required to describe its motion. And in this way we can simplify the mathematical description of its motion. The conditions imposed on the system by the constraints can be written down mathematically as a relation satisfied by the coordinates of the particles of the system at any time. This is the way in which the constraints imposed on a system reduce the number of coordinates needed to simplify the configuration of a system. Let us consider, as an example, the motion of a simple pendulum which is confined to move in a vertical plane as shown in Fig. 6.1.

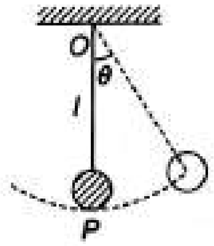


Fig. 6.1 Motion of a simple pendulum on a vertical plane.

We need only two coordinates (in cartesian coordinate system x and y; and in polar coordinates system, r and θ), to locate the position of the bob at any time with respect to the point of suspension. The constraint of motion of the pendulum is that the distance (l) of the bob P from the point of suspension O remains always constant.

The mathematical expression of the constraint when expressed in cartesian coordinates is given by

$$x^2 + y^2 = l^2 \quad \text{...(6.3)}$$

If it is expressed in plane polar coordinates, it is given by

$$r = l \quad \text{...(6.4)}$$

It is seen that the mathematical expression describing the constraint is simpler in case of plane polar coordinates. One coordinate θ, in plane polar coordinates, would suffice to describe the constraint motion, whereas in cartesian coordinates, we require two coordinates (namely x and y). So, choice of a suitable coordinate system is also very much important in describing constrained motion. Proper choice of coordinates helps us reduce the number of parameters and make expressions (describing the motion of the restricted system) simpler.

If a particle is restricted to move on the surface of a sphere, then we can write the mathematical expression of the restriction in cartesian coordinates as follows:

$$x^2 + y^2 + z^2 = a^2 \quad \text{...(6.5)}$$

And in spherical polar coordinates, it can be expressed as

$$r = a \quad \text{...(6.6)}$$

where a is the radius of the sphere. In this example we can see that the spherical polar coordinates can be used with a greater advantage.

In the above two examples, we have so far considered a single particle (or body) system. Let us now consider a two-body system. A double pendulum is shown in Fig. 6.2.

In this case, we would require four coordinates (two for each pendulum) if we had chosen cartesian coordinates to describe the system completely. But if we choose plane polar coordinates, we require only two coordinates, namely, θ_1 and θ_2.

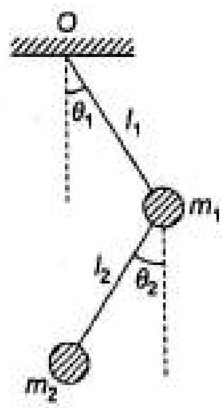


Fig. 6.2 A double pendulum having two bobs of masses m_1 and m_2 suspended from a rigid support O.

6.1.3 Classification of Constraints

The constraints of motion can be classified on the basis of any parameter of interest. We will classify the constraints in this case on the basis of only two parameters, namely, time and velocity. Depending on time, constraints are classified as *scleronomic constraints* and *rheonomic constraints*.

(i) Scleronomic constraints These are the constraints which do not explicitly depend on time.

(ii) Rheonomic constraints These are the constraints which explicitly depend on time.

We can represent scleronomic constraints in an equational form as follows:

$$f(x_1, y_1, z_1; x_2, y_2, z_2 : x_n, y_n, z_n) = 0$$

Similarly, a rheonomic constraint can be expressed as follows:

$$f(x_1, y_1, z_1; x_2, y_2, z_2 : x_n, y_n, z_n; t) = 0$$

Depending on velocity the constraints can be classified as *holonomic* and *non-holonomic* constraints.

(i) Holonomic constraints If the relation of the constraints can be expressed as an equation and if they are independent of velocity then such constraints are called holonomic constraints. When expressed in equational form we get.

$$f(x_1, y_1, z_1; x_2, y_2, z_2 : x_n, y_n, z_n; t) = 0$$

(ii) Non-holonomic constraints If the constraints cannot be expressed in equational form and they are dependent on velocity then such constraints are called non-holonomic constraints.

Examples of various constraints:

(*a*) *Scleronomic constraints* The motion of a point mass in a simple pendulum with a rigid support. The equation of constraint $x^2 + y^2 = l^2$ is independent of time.

(*b*) *Rheonomic constraints* A bead sliding on a moving wire is an example of a rheonomic constraint. If the effective length 'l' of a simple pendulum varies with time because of change of temperature, then it becomes an example of rheonomic constraint as $l = f(t)$.

(*c*) *Holonomic constraints* The constraints involved in the motion of a rigid body is an example of holonomic constraint because in this case

$$(\bar{r}_i - \bar{r}_j)^2 - c_{ij}^2 = 0$$

where $\bar{r}_i$ and $\bar{r}_j$ are respectively the position vector of the i^{th} and j^{th} particles and c_{ij} is the distance between them.

(*d*) *Non-holonomic constraints* If a gas is kept in a spherical container with radius a and r is the position vector of a gas molecule, then the condition of constraint for motion of a molecule is given by

$$|\bar{r}| \le a \Rightarrow r - a \le 0$$

It is thus an example of a non-holonomic constraint.

6.1.4 Degrees of Freedom

The number of independent ways in which a mechanical system can execute its motion without violating any constraint imposed on it is called degrees of freedom of the said system. In other words, the degrees of freedom is the least possible number of coordinates to describe the motion of the system completely.

A bead constrained to move along a stretched string has one degree of freedom as it requires only one coordinate for describing its motion.

Let us consider again the simple pendulum constrained to move on a vertical plane. If we choose the cartesian coordinate system, we require two coordinates (x and y) to describe its motion completely but if we choose the plane polar coordinate system instead, we require only one coordinate (θ) and as it is the least number of coordinates for describing its motion completely its degree of freedom is one.

We can look at the degrees of freedom from another point of view as stated below:

The kinetic energy of a particle when it moves in one-dimensional, two-dimensional and three-dimensional spaces (i.e., along the x axis, in the xy plane and in the xyz space) is given by $\frac{1}{2}mx^2$, $\frac{1}{2}m(x^2 + y^2)$ and $\frac{1}{2}m(x^2 + y^2 + z^2)$ respectively. **So, the degrees of freedom may also be defined as the number of squared terms appearing in the expression of the kinetic energy of the system.**

If we impose more constraints on the motion of a body, we get the number of coordinates required to describe its motion reduced. So, imposing constraints is a way of simplifying the problems mathematically in the sense that the number of equations of motion are reduced to the same number as the degrees of freedom.

In a system of n particles subjected to k constraints which are expressible in k equations of the form

$$g_1(x_1, y_1, z_1; x_2, y_2, z_2 : x_n, y_n, z_n, t) = a_1$$
$$g_2(x_1, y_1, z_1; x_2, y_2, z_2 : x_n, y_n, z_n, t) = a_2$$
$$\ldots \qquad \ldots \qquad \ldots$$
$$g_k(x_1, y_1, z_1 ; x_2, y_2, z_2 : x_n, y_n, z_n, t) = a_k$$

The degrees of freedom n_f is given by $3n - k$; i.e., $n_f = 3n - k$.

6.1.5 Generalized Coordinates

A set of independent coordinates which is sufficient in number to specify unambiguously the configuration of the system is known as the set of generalized coordinates. In other words, the generalized coordinates are such coordinates which are capable of replacing coordinates of any coordinate system like cartesian, spherical polar and cylindrical polar coordinate systems.

We sometimes wish to introduce not all coordinates with respect to a fixed coordinate system but some of them may be selected with respect to a new origin or a moving coordinate system. As for example, while dealing with rigid body motion, we specify three cartesian coordinates to locate the center of mass of the rigid body with respect to an external origin and three angular coordinates relative to the origin at the center of mass of the rigid body. Thus, the generalized coordinates should be all chosen as the conventional orthogonal position coordinates or all may be angular coordinates.

6.1.6 Choice of Generalized Coordinates

While one chooses generalized coordinates for a system, one must be guided by the following three principles:

(a) Their values should be capable of determining the configuration of the concerned system.

(b) They should be in a position to vary arbitrarily and independently of each other, without violating the constraints of the system.

(c) There is no uniqueness in the choice of generalized coordinates. The choice should fall on a set of coordinates that will be able to give us a reasonable mathematical simplification of the concerned problem.

6.1.7 Notation for Generalized Coordinates

The generalized coordinates are denoted by the letter q of Roman alphabet with a numerical subscript, i.e., a set of n generalized coordinates can be expressed as $q_1, q_2, q_3, \ldots, q_n$ respectively. Alternatively, the set of n generalized coordinates can be expressed by q with a letter subscript to it and specifying within a bracket the numerical values that the letter subscript is allowed to take.

e.g., $$q_j \; (j = 1, 2, 3, \ldots, n).$$

Thus, when a particle moves in a plane, it may be described by cartesian coordiantes (x, y) or the polar coordinates (r, θ), and so on. And we can have the following mapping between the set (q_1, q_2) of generalized coordinates and any one of the sets (x, y) or (r, θ), as follows:

$$\begin{aligned} q_1 &= x \\ q_2 &= y \end{aligned} \quad \text{or} \quad \begin{aligned} q_1 &= r \\ q_2 &= \theta \end{aligned}$$

where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

When the problem involves some spherical symmetry, it is convenient to use spherical polar coordinates while transforming generalized coordinates to any specific coordinates.

So, we can write,

$$\left.\begin{aligned} q_1 &= r = (x^2 + y^2 + z^2)^{\frac{1}{2}} \\ q_2 &= \theta = \cot^{-1} \frac{z}{\sqrt{(x^2 + y^2)}} \\ \text{and} \quad q_3 &= \phi = \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}\right|$$

or,

$$\left.\begin{aligned} q_1 &= \theta = \cot^{-1} \frac{z}{\sqrt{(x^2 + y^2)}} \\ q_2 &= \phi = \tan^{-1}\left(\frac{y}{x}\right) \\ \text{and} \quad q_3 &= r = (x^2 + y^2 + z^2)^{\frac{1}{2}} \end{aligned}\right|$$

If one prefers to accept a coordinate system which is moving uniformly with a velocity v in the x direction, the generalized coordinates are given by (q_1, q_2, q_3) as follows:

$$\left.\begin{aligned} q_1 &= x - \dot{x}t \\ q_2 &= y \\ \text{and} \quad q_3 &= z \end{aligned}\right| \qquad \text{where } \dot{x} = v = \text{constant.}$$

So, we can see that one of our generalized coordinates (q_1, q_2, q_3) is velocity dependent, i.e., $q_1 = q_1(v)$. It is also dependent on time, i.e., $q_1 = q_1(t)$

$\therefore$ we can write $$q_1 = q_1(x, v, t) = q_1(x, \dot{x}, t)$$

With a bit of analysis we can show that each coordinate of a generalized coordinate system can be expressed as a function of all the (or some of them) coordinates of another system including time and/or velocities.

Let us now consider a single particle system in a plane. If A be a point in the XY plane as shown in Fig. 6.3.

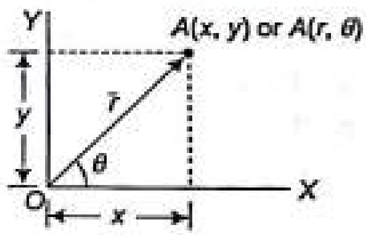


Fig. 6.3 A single particle system in XY plane.

then the point A can be represented by $A(x, y)$ in cartesian coordinates and by $A(r, \theta)$ in polar coordinates. And from the diagram we can write,

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

So, $$r = \sqrt{x^2+y^2} \quad \text{and} \quad \theta = \tan^{-1}(y/x)$$

Hence, $$x = x(r, \theta) \quad \text{and} \quad y = y(r, \theta)$$

and $$r = r(x, y) \quad \text{and} \quad \theta = \theta(x, y)$$

So, we can show x and y as functions of r and θ and vice versa.

Now, we can, in general, represent the point A in the XY plane by two generalized coordinates q_1 and q_2 as $A(q_1, q_2)$. And (q_1, q_2) may represent either (x, y) or (r, θ) or any other coordinate pair belonging to any other coordinate system.

Let us now extend our idea by considering a system of two particles A and B in the XY plane as shown in Fig. 6.4. If the distance AB between the points A and B be constant (i.e., the points A and B are constrained) then we can write,

$$AB = \{(x_2-x_1)^2 + (y_2-y_1)^2\}^{1/2}$$
$$= \text{constant} = r \text{ (say)}$$

where (x_1, y_1) and (x_2, y_2) are the cartesian coordinates of the points A and B respectively.

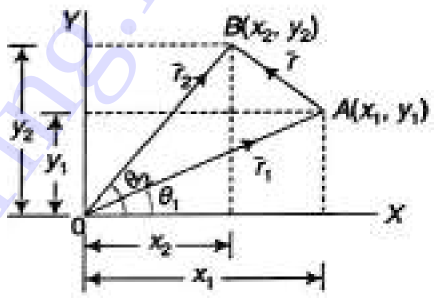


Fig. 6.4 A system of two particles A and B in the XY plane.

Similarly, the system of points A and B can be denoted by (r_1, θ_1) and (r_2, θ_2) in the polar coordinate system. So, in general, the two points can be represented by the ordered pairs (q_1, q_2) and (q_3, q_4) in the generalized coordinate system.

Now, we can write the following expressions from Fig. 6.4:

$$\bar{r}_2 = \bar{r}_1 + \bar{r} \Rightarrow \bar{r} = \bar{r}_2 - \bar{r}_1$$

where $$\bar{r}_1 = \bar{f}_1(x_1, y_1, \theta_1),$$

$$\bar{r}_2 = \bar{f}_2(x_2, y_2, \theta_2)$$

and $$\bar{r} = \bar{f}_3(\bar{r}_1, \bar{r}_2) = \bar{f}_3(x_1, y_1, x_2, y_2, r_1, r_2, \theta_1, \theta_2)$$

So, the above four equations indicate that any coordinate of one coordinate system can be expressed as a function of the coordinates of any other coordinate system. Hence, by choosing a generalized coordinate system, we can write

$$q_1 = q_1(x_1, y_1, x_2, y_2; t)$$

and $$q_2 = q_2(x_1, y_1, x_2, y_2; t)$$

Also, $$q_1 = q_1(r_1, \theta_1, r_2, \theta_2; t)$$

and $$q_2 = q_2(r_1, \theta_1; r_2, \theta_2; t)$$

Now, if we consider a system of n particles, then we can write, in general

$$\left.\begin{aligned} q_1 &= q_1(x_1, y_1, z_1; x_2, y_2, z_2; \ldots, x_n, y_n, z_n; t) \\ q_2 &= q_2(x_1, y_1, z_1; x_2, y_2, z_2; \ldots, x_n, y_n, z_n; t) \\ &\ldots \\ q_{3n} &= q_{3n}(x_1, y_1, z_1; x_2, y_2, z_2; \ldots, x_n, y_n, z_n; t) \end{aligned}\right\} \quad \ldots(6.7)$$

As the system contains n particles, it requires $3n$ space coordinates as well as one time coordinate. The set of above $3n$ equations forms a set of $3n$ transformation equations which transform the coordinates of the cartesian coordinate system to generalized coordinates q_j (where $j = 1, 2, \ldots, 3n$) of the generalized coordinate system. Similarly, the generalized coordinates can be transformed into cartesian coordinates as given below:

$$\left.\begin{aligned} x_1 &= x_1(q_1, q_2, q_3, \ldots, q_{3n}, t) \\ y_2 &= y_2(q_1, q_2, q_3, \ldots, q_{3n}, t) \\ &\ldots \quad \ldots \quad \ldots \\ z_n &= z_n(q_1, q_2, q_3, \ldots, q_{3n}, t) \end{aligned}\right\} \quad \ldots(6.8)$$

The necessary and sufficient condition for the transformation from a set of coordinates q_j ($j = 1, 2, 3, \ldots, 3n$) to another set $(x_1, y_1, z_1; x_2, y_2, z_2, \ldots, x_n, y_n, z_n)$ to be effective is that the value of the Jacobian determinant J will not be equal to zero.

i.e.,
$$J\left(\frac{q_1, q_2, \ldots, q_{3n}}{x_1, y_1, \ldots, z_n}\right) \equiv \left(\frac{\partial(q_1, q_2, \ldots, q_{3n})}{\partial(x_1, y_1, \ldots, z_n)}\right)$$

$$= \begin{vmatrix} \frac{\partial q_1}{\partial x_2} & \frac{\partial q_2}{\partial x_1} & \ldots & \frac{\partial q_{3n}}{\partial x_1} \\ \frac{\partial q_1}{\partial x_2} & \frac{\partial q_2}{\partial x_2} & \ldots & \frac{\partial q_{3n}}{\partial x_2} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial q_1}{\partial z_n} & \frac{\partial q_2}{\partial z_n} & \ldots & \frac{\partial q_{3n}}{\partial z_n} \end{vmatrix} \neq 0$$

If this condition fails to hold then Eq. (6.7) does not define a consistent set of generalized coordinates.

6.1.8 Notations for Generalized Variables

Let us now develop notations for generalized variables like generalized displacement, generalized velocity, generalized acceleration, generalized momentum, generalized force, etc., in terms of generalized coordinates q_j ($j = 1, 2, 3\ldots\ldots, 3n$).

Generalized displacement

Let $\delta\bar{r}_i$ represent a small displacement of the i^{th} particle of the system of n particles. $\bar{r}_i$ represents the cartesian coordinates of the i^{th} particle.

$\therefore$
$$\bar{r}_i = \bar{r}_i(q_1, q_2, \ldots, q_{3n}, t)$$

For scleronomic system, we get

$$\bar{r}_i = \bar{r}_i(q_1, q_2, \ldots, q_{3n})$$

Hence,
$$\delta\bar{r}_i = \sum_{j=1}^{3n} \frac{\partial \bar{r}_i}{\partial q_j}\delta q_j \quad \ldots(6.9)$$

where δq_i are called the generalized displacements or virtual arbitrary displacements and $\delta t = 0$.

Generalized velocity

The generalized velocity associated with the generalized position coordinate q_j is denoted by $\dot{q}_j$. For an unconstrained system of n particles, we get

$$\bar{r}_i = \bar{r}_i(q_1, q_2, \ldots, q_{3n}, t)$$

$\therefore$
$$\dot{\bar{r}}_i = \sum_{j=1}^{3n} \frac{\partial \bar{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \bar{r}_i}{\partial t} \quad \ldots(6.10)$$

If the n-particle system contains k constraints, the number of generalized coordinates is $3n-k=f$ and in that case

$$\dot{\bar{r}}_i = \sum_{j=1}^{f} \frac{\partial \bar{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \bar{r}_i}{\partial t} \qquad \text{...(6.11)}$$

If the dimensions of a generalized coordinate is that of momentum then the generalized velocity will have the dimension of force, and so on. The term $\frac{\partial \bar{r}_i}{\partial t}$ appears in the expression only when there are coordinates depending on time or in some cases where it is convenient to introduce moving coordinate axes.

Generalized accelerations

Let us again consider Eq. (6.10)

$$\dot{\bar{r}}_i = \sum_{j=1}^{3n} \frac{\partial \bar{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \bar{r}_i}{\partial t}$$

Now, differentiating it with respect to time, we get

$$\ddot{\bar{r}}_i = \frac{d}{dt}\left(\sum_{j=1}^{3n} \frac{\partial \bar{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \bar{r}_i}{\partial t}\right)$$

or,

$$\ddot{\bar{r}}_i = \sum_{j=1}^{3n} \frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right)\dot{q}_j + \sum_{j=1}^{3n} \frac{\partial \bar{r}_i}{\partial q_j}\ddot{q}_j + \frac{d}{dt}\left(\frac{\partial r_i}{\partial t}\right)$$

or,

$$\ddot{\bar{r}}_i = \sum_{j=1}^{3n} \frac{\partial \dot{\bar{r}}_i}{\partial q_j}\dot{q}_j + \sum_{j=1}^{3n} \frac{\partial \bar{r}_i}{\partial q_j}\ddot{q}_j + \frac{\partial \dot{\bar{r}}_i}{\partial t}$$

Now, putting the value of $\dot{\bar{r}}_i$ from Eq. (6.10) and changing the index j to k, we get

$$\ddot{\bar{r}}_i = \sum_{j=1}^{3n} \frac{\partial}{\partial q_j}\left(\sum_{k=1}^{3n} \frac{\partial \bar{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \bar{r}_i}{\partial t}\right)\dot{q}_j + \sum_{j=1}^{3n} \frac{\partial \bar{r}_i}{\partial q_j}\ddot{q}_j + \frac{\partial}{\partial t}\left(\sum_{k=1}^{3n} \frac{\partial \bar{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \bar{r}_i}{\partial t}\right)$$

or

$$\ddot{\bar{r}}_i = \sum_{j=1}^{3n}\sum_{k=1}^{3n} \frac{\partial^2 \bar{r}_i}{\partial q_j\,\delta q_k}\dot{q}_k\dot{q}_j + \sum_{j=1}^{3n} \frac{\partial^2 \bar{r}_i}{\partial q_j\,\partial t}\dot{q}_j + \sum_{j=1}^{3n} \frac{\partial \bar{r}_i}{\partial q_j}\ddot{q}_j + \sum_{k=1}^{3n} \frac{\partial^2 \bar{r}_i}{\partial q_k\,\partial t}\dot{q}_k + \frac{\partial^2 \bar{r}_i}{\partial t_i^2}$$

or,

$$\ddot{\bar{r}}_i = \sum_{j=1}^{3n} \frac{\partial \bar{r}_i}{\partial q_j}\ddot{q}_j + \sum_{j=1}^{3n}\sum_{k=1}^{3n} \frac{\partial^2 \bar{r}_i}{\partial q_j\,\partial q_k}\dot{q}_j\dot{q}_k + 2\sum_{j=1}^{3n} \frac{\partial^2 \bar{r}_i}{\partial q_j\,\partial t}\dot{q}_j + \frac{\partial^2 \bar{r}_i}{\partial t^2} \qquad \text{...(6.12)}$$

Now, if we look closely into Eq. (6.12) from the dimensional point of view, we get the generalized accelerations as $\ddot{q}_j$. So, $\ddot{q}_j$ are the generalized accelerations.

Generalized momentum

The kinetic energy T of a system of n free particles in terms of cartesian coordinates is given by

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i \dot{\bar{r}}_i^2 = \sum_{i=1}^{n} \frac{1}{2} m_i (\dot{\bar{r}}_i \cdot \dot{\bar{r}}_i) \qquad \text{...(6.13)}$$

Now substituting for $\dot{\bar{r}}_i$ from Eq. (6.10), we get

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i \left[\sum_{j=1}^{3n} \frac{\partial \bar{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \bar{r}_i}{\partial t}\right] \cdot \left[\sum_{k=1}^{3n} \frac{\partial \bar{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \bar{r}_i}{\partial t}\right]$$

or,

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{3n}\sum_{k=1}^{3n} m_i \frac{\partial \bar{r}_i}{\partial q_j}\cdot\frac{\partial \bar{r}_i}{\partial q_k}\dot{q}_j\dot{q}_k + \frac{1}{2}\sum_{i=1}^{n} m_i \left[\sum_{j=1}^{3n} \frac{\partial \bar{r}_i}{\partial q_j}\dot{q}_j + \sum_{k=1}^{3n} \frac{\partial \bar{r}_i}{\partial q_k}\dot{q}_k\right] \cdot \frac{\partial \bar{r}_i}{\partial t} + \frac{1}{2}\sum_{i=1}^{n} m_i \left(\frac{\partial \bar{r}_i}{\partial t}\right)^2$$

The second term on the right-hand side consists of two sums which are identical so that

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{3n}\sum_{k=1}^{3n} m_i \frac{\partial \bar{r}_i}{\partial q_j}\cdot\frac{\partial \bar{r}_i}{\partial q_k}\dot{q}_j\dot{q}_k + \sum_{i=1}^{n}\sum_{k=1}^{3n} m_i \left(\frac{\partial \bar{r}_i}{\partial q_k}\dot{q}_k\right)\cdot\left(\frac{\partial \bar{r}_i}{\partial t}\right) + \frac{1}{2}\sum_{i=1}^{n} m_i \left(\frac{\partial \bar{r}_i}{\partial t}\right)^2 \quad ...(6.14)$$

Thus, the general kinetic energy in terms of generalized velocities comprises three distinct terms—the first term contains quadratic terms of velocities, the second term contains linear terms of velocities and the third term is independent of generalized velocities. If we denote these terms respectively by $T^{(2)}$, $T^{(1)}$ and $T^{(0)}$, we can represent Eq. (6.14) as

$$T = T^{(2)} + T^{(1)} + T^{(0)} \quad ...(6.15)$$

$T^{(1)}$ and $T^{(0)}$ will vanish when $\frac{\partial \bar{r}_i}{\partial t} = 0$, i.e., when our defining Eq. (6.8) of generalized coordinates do not have explicit dependence on time (i.e., when no moving coordinates are involved). And in such a case the term $T^{(2)}$ survives and contains cross-terms also (i.e., terms $q_j q_k$ where $j \neq k$) unlike the kinetic energy Eq. (6.13) in cartesian coordinates which is purely a quadratic function of linear velocities containing only the squared terms. Such a form is called homogeneous quadratic form or canonical form. If the term $T^{(2)}$ is free from cross-terms which will happen when $\frac{\partial \bar{r}_i}{\partial q_j}\cdot\frac{\partial \bar{r}_i}{\partial q_k} = 0$ for $j \neq k$ then the generalized coordinate system in q_j's is referred as **an orthogonal system.**

Now, if we consider $T = \sum_{i=1}^{n}\frac{1}{2}m_i \dot{\bar{r}}_i^2$ and rewrite T in the following form:

$$T = \sum_{i=1}^{n}\frac{1}{2}m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$$

then we can show that the linear momentum $(m_i\dot{x}_i)$ associated with the linear velocity $\dot{x}_i$ is given by

$$p_{xi} = \frac{\partial T}{\partial \dot{x}_i} = m\dot{x}_i$$

Then, we can similarly define the generalized momentum associated with generalized coordinate q_k as

$$p_k = \frac{\partial T}{\partial \dot{q}_k} \quad ...(6.16)$$

The generalized momentum p_k need not always have the dimension (MLT^{-1}) of linear momentum, for if q_k is an angular coordinate, p_k is the corresponding angular momentum with dimensions (ML^2T^{-1}).

Now, differentiating Eq. (6.14) with respect to $\dot{q}_k$, we obtain

$$p_k = \sum_{i=1}^{n}\sum_{j=1}^{3n}\frac{1}{2}m_i\frac{\partial \bar{r}_i}{\partial q_j}\cdot\frac{\partial \bar{r}_i}{\partial q_k}\dot{q}_j + \sum_{i=1}^{n} m_i\frac{\partial \bar{r}_i}{\partial q_k}\cdot\frac{\partial \bar{r}_i}{\partial t} \quad ...(6.17)$$

If the generalized system is stationary then the last term will again be absent. It is again a linear function of generalized velocities. Let us now compute generalized momenta associated with the plane polar coordinates (r, θ) as follows: For a single particle system in two dimensions, Eq. (6.13) reduces to

$$T = \frac{1}{2}m(\dot{\bar{r}}.\dot{\bar{r}})$$

or, $$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

and Eq. (6.14) reduces to

$$T=\frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2} m\frac{\partial \bar{r}}{\partial q_j}\cdot\frac{\partial \bar{r}}{\partial q_k}\dot{q}_j\dot{q}_k \quad \text{[as the generalized coordinate system is stationary]}$$

or,
$$T=\frac{1}{2}m\sum_{j=1}^{2}\left(\frac{\partial \bar{r}}{\partial q_j}\cdot\frac{\partial \bar{r}}{\partial q_1}\dot{q}_j\dot{q}_1+\frac{\partial \bar{r}}{\partial q_j}\cdot\frac{\partial \bar{r}}{\partial q_2}\dot{q}_j\dot{q}_2\right)$$

or,
$$T=\frac{1}{2}m\left(\frac{\partial \bar{r}}{\partial q_1}\frac{\partial \bar{r}}{\partial q_1}\dot{q}_1^2+\frac{\partial \bar{r}}{\partial q_1}\cdot\frac{\partial \bar{r}}{\partial q_2}\dot{q}_1\dot{q}_2+\frac{\partial \bar{r}}{\partial q_1}\frac{\partial \bar{r}}{\partial q_2}\dot{q}_1\dot{q}_2+\frac{\partial \bar{r}}{\partial q_2}\cdot\frac{\partial \bar{r}}{\partial q_2}\dot{q}_2^2\right)$$

We are considering a single particle system in two dimensions in the polar coordinate system. So we have two polar coordinates r and θ. We can assume either $q_1 = r$ and $q_2 = \theta$, or $q_1 = \theta$ and $q_2 = r$. In either of the cases, we get

$$T=\frac{1}{2}m\left(\frac{\partial \bar{r}}{\partial r}\cdot\frac{\partial \bar{r}}{\partial r}\dot{r}^2+2\frac{\partial \bar{r}}{\partial r}\frac{\partial \bar{r}}{\partial \theta}\dot{r}\dot{\theta}+\frac{\partial \bar{r}}{\partial \theta}\cdot\frac{\partial \bar{r}}{\partial \theta}\dot{\theta}^2\right)$$

or,
$$T=\frac{1}{2}m\left[\left(\hat{i}\frac{\partial x}{\partial r}+\hat{j}\frac{\partial y}{\partial r}\right)\cdot\left(\hat{i}\frac{\partial x}{\partial r}+\hat{j}\frac{\partial y}{\partial r}\right)\dot{r}^2+2\left(\hat{i}\frac{\partial x}{\partial r}+\hat{j}\frac{\partial y}{\partial r}\right)\cdot\left(\hat{i}\frac{\partial x}{\partial \theta}+\hat{j}\frac{\partial y}{\partial \theta}\right)\dot{r}\dot{\theta}\right.$$
$$\left.+\left(\hat{i}\frac{\partial x}{\partial \theta}+\hat{j}\frac{\partial y}{\partial \theta}\right)\cdot\left(\hat{i}\frac{\partial x}{\partial \theta}+\hat{j}\frac{\partial y}{\partial \theta}\right)\dot{\theta}^2\right]$$

From the relation $x = r\cos\theta$ and $y = r\sin\theta$, we get

$$\frac{\partial x}{\partial r}=\cos\theta,\quad \frac{\partial x}{\partial \theta}=-r\sin\theta,\quad \frac{\partial y}{\partial r}=\sin\theta \text{ and } \frac{\partial y}{\partial \theta}=r\cos\theta$$

Putting the values of these derivatives in the above equation we get the kinetic energy in polar coordinates (r, θ) as follows:

$$T=\frac{1}{2}m(\cos^2\theta+\sin^2\theta)\dot{r}^2+2\times\frac{1}{2}m(-r\cos\theta\sin\theta+r\cos\theta\sin\theta)\dot{r}\dot{\theta}$$
$$+\frac{1}{2}m(r^2\sin^2\theta+r^2\cos^2\theta)\dot{\theta}^2$$

or
$$T=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)$$

which is a familiar expression.

$\therefore$ the associated generalized momenta are given by Eq. (6.16) as follows:

$$p_r=\frac{\partial T}{\partial \dot{r}}=m\dot{r}\quad \text{(Linear momentum)}$$

and
$$p_\theta=\frac{\partial T}{\partial \dot{\theta}}=mr^2\dot{\theta}\quad \text{(Angular momentum)}$$

However, the product of any generalized momentum and the associated (also known as conjugate) coordinate must have a dimension of angular momentum.

Generalized force

The definition of a generalized force associated with generalized displacement is given as follows:

Let us consider the amount of work done on the system by all forces $\bar{F}_i$ while causing all arbitrary displacements $\delta\bar{r}_i$ of the system. We can write

$$\delta w = \sum_{i=1}^{n} \overline{F}_i \cdot \delta \overline{r}_i$$

or,
$$\delta w = \sum_{i=1}^{n} \overline{F}_i \cdot \sum_{j=1}^{3n} \frac{\partial \overline{r}_i}{\partial q_j} \delta q_j$$

or,
$$\delta w = \sum_{i=1}^{n} \sum_{j=1}^{3n} \left(\overline{F}_i \cdot \frac{\partial \overline{r}_i}{\partial q_j} \right) \delta q_j$$

or,
$$\delta w = \sum_{j=1}^{3n} Q_j \, \delta q_j \qquad \text{...(6.18)}$$

where
$$Q_j = \sum_{i=1}^{n} \overline{F}_i \cdot \frac{\partial \overline{r}_i}{\partial q_j} \qquad \text{...(6.19)}$$

Let us note that Q_j is dependent on the force acting on the particles and the coordinate q_j and possibly on the time t.

As $Q_j \delta q_j$ gives the dimension of work, it is natural to call Q_j as the generalized force associated with a generalized coordinate q_j. So, the product of Q_j with the arbitrary displacement δq_j is equal to the work done corresponding to that displacement. And the generalized force need not always have the dimension of force as δq_j will not always have the dimension of distance. If we consider again a particle having coordinates (r, θ), then the components of the force acting on the particle is given by

$$\overline{F} = F_r \hat{r} + F_\theta \hat{\theta}$$

where $\hat{r}$ and $\hat{\theta}$ are the unit vectors acting along the directions of increasing r and θ respectively.

We can express the vector $\overline{r}$ as $\overline{r} = r\,\hat{r}$. The generalized forces associated with r– and θ–coordinates are respectively given by

$$Q_r = \overline{F} \cdot \frac{\partial \overline{r}}{\partial r} = (F_r \hat{r} + F_\theta \hat{\theta}) \cdot \frac{\partial r}{\partial r} \hat{r}$$

∴
$$Q_r = F_r \qquad \text{...(6.20)}$$

Again,
$$Q_\theta = \overline{F} \cdot \frac{\partial \overline{r}}{\partial \theta} = (F_r \hat{r} + F_\theta \hat{\theta}) \cdot \left(r \frac{\partial \hat{r}}{\partial \theta} \right)$$

or,
$$Q_\theta = r F_\theta \qquad \text{...(6.21)}$$

$$\left[\because \frac{\partial \hat{r}}{\partial \theta} = \hat{\theta} \right]$$

We have from Fig. 6.5,

$$\hat{r} = \hat{i} \cos\theta + \hat{j} \sin\theta$$

and
$$\hat{\theta} = -\hat{i} \sin\theta + \hat{j} \cos\theta$$

∴
$$\frac{\partial \hat{r}}{\partial \theta} = -\hat{i} \sin\theta + \hat{j} \cos\theta = \hat{\theta}$$

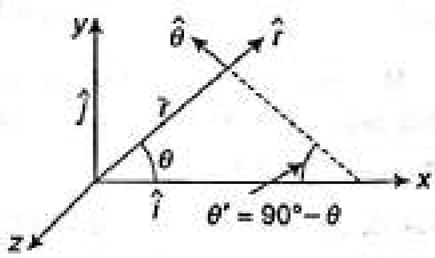


Fig. 6.5 Unit vector $\hat{r}$ and $\hat{\theta}$ in polar coordinate system.

From Eqs (6.20) and (6.21), we find that Q_r is the component of the force in the r direction and Q_θ is the torque acting on the particle to the direction of increasing θ. Q_r has the dimension of (MLT^{-2}) and Q_θ has the dimension of (ML^2T^{-2}) and rQ_r and θQ_θ have the dimension of work.

Generalized potential

If the forces acting on the system under consideration be derivable from a scalar potential V depending on the position coordinates only then V is the potential energy associated with the system and in such a case, we can write

$$\delta w = -\delta V$$

or,

$$\delta w = -\sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}\delta x_i + \frac{\partial V}{\partial y_i}\delta y_i + \frac{\partial V}{\partial z_i}\delta z_i\right) \quad ...(6.22)$$

And in terms of generalized coordinates q_j we can write

$$\delta w = -\delta V = -\sum_{k=1}^{3n}\frac{\partial V}{\partial q_k}\delta q_k$$

or,

$$\delta w = -\sum_{k=1}^{3n} Q_k\,\delta q_k \quad ...(6.23)$$

Thus, we have

$$Q_k = -\frac{\partial V}{\partial q_k} \quad ...(6.24)$$

In this sense, the definition of Q_k as the generalized force is justified. When the system is not conservative and the potential depends on the generalized velocities q_j, the generalized force in such a case is defined as

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial q_j}\right) \quad ...(6.25)$$

where U may be called velocity dependent potential or simply generalized potential as the generalized force Q_j can be derived from it.

6.2 LAGRANGIAN FORMULATION

In Section 6.1.1 we have discussed in detail the limitations of newtonian mechanics. We observed there that Newton's laws of motion are dependent on the type of reference frames. In non-inertial frames of reference, the Newton's laws of motion are not valid. The equations involved in newtonian mechanics are usually in vector form, so they are inconvenient to handle. To overcome all these difficulties, a set of equations called Lagrange's equations of motion are used. This set of equations can be written in any system of coordinates.

6.2.1 System Point and Configuration Space

The description of the motion of a single particle in three-dimensional space is a simple, mechanical problem. Any problem involving two particles (where each particle is described by a set of three coordinates) can be reduced to a single particle problem simply by regarding that the said single particle moves in a six-dimensional space. Though it is not possible to physically visualize a space having more than three dimensions, one can logically do so and this way one can convert a many-body problem to a single-body problem. Thus, in general, a problem involving n particles can be treated as a problem of a single particle moving along a trajectory of $3n$ dimensional space. So, we have constructed a $3n$ dimensional space to give meaning to the concept of the system point. **When an n particle problem is thus converted to a single particle problem, the $3n$ dimensional space is regarded as configuration space and the associated single particle is regarded as the system point.** The motion of the system point in the 'configuration space' is called the motion of the system between any two given instants of time. As stated above, the configuration space has no necessary connection with a real three-dimensional space.

6.2.2 Principle of Virtual Work

When the sum of all the forces acting on a mechanical system is zero, the system is regarded to be in an equilibrium state. The system can have several such equilibrium states at a given point of time without violating the constraints acting on it. **The virtual displacement is such a displacement which takes the system from one equilibrium state to another infinitely close equilibrium state at the same instant of time, i.e., it is an infinitesimal displacement when the change of time ($dt = 0$).** The virtual displacement $\delta\bar{r}_i$ differs from the real displacement $d\bar{r}_i$ in the sense that the virtual displacement does not involve any change in time while the real displacement involves a change in time dt. And during this time interval dt the forces and the constraints may change.

Let us consider a simple pendulum. The tension in the string is the force of constraint. Let l be the length of the pendulum. When this length (l) remains constant, the bob moves in the arc of a vertical circle having a constant radius l. If a very small displacement of the bob takes place, the force of constraint remains perpendicular to it. And the work done by the force of constraint is zero. On the other hand, for a pendulum whose length changes with time, the bob does not move in the arc of a circle and hence, the displacement is no more perpendicular to the force of constraint. And the work done by the force of constraint in any real displacement is not zero. The virtual displacement at an instant t is a displacement of the bob along the arc of a circle whose radius is equal to the length of the pendulum at the same instant (t). And when the bob makes a virtual displacement, the length l of the pendulum remains constant. This is because the virtual displacement in any instant of time must be consistent with the constraint at that instant. No passage of time takes place during any virtual displacement. It is an infinitesimal imaginary displacement which is normal to the force of constraint. Hence, the virtual work done by the force of constraint is zero. This is the principle of virtual work. This principle can be stated as follows:

A system of particles, will be in a state of equilibrium if and only if the total virtual work done by the forces of constraint in any arbitrary infinitesimal virtual displacement is zero.

6.2.3 D'Alembert's Principle

This principle is based on the principle of virtual work. The system is subjected to an infinitesimal displacement consistent with the forces of constraints imposed on the system of particles at a given instant of time t. This change in the configuration of the system is not associated with a change in time, i.e., there is no actual displacement during which the forces of constraint may change.

Now, let us suppose that the system is in an equilibrium state, i.e., the total force $\bar{F}_i$ acting on each particle is zero; then the work done by this force in a small virtual displacement $\delta\bar{r}_i$ will be zero. That is, for the whole system of n particles, we can have

$$\sum_{i=1}^{n} \bar{F}_i \cdot \delta\bar{r}_i = 0$$

Let this total force $\bar{F}_i$ acting on the i^{th} particle be expressed as the sum of the applied force $\bar{F}_i^a$ and the force of constraint $\bar{F}_i^c$. Then the above equation takes the following form:

$$\sum_{i=1}^{n} (\bar{F}_i^a \cdot \delta\bar{r}_i) + \sum_{i=1}^{n} (\bar{F}_i^c \cdot \delta\bar{r}_i) = 0$$

Now, let us consider a system of n particles for which the virtual work done by the forces of constraints becomes zero. An example of such a system can be that of a group of n particles which are constraint to move on a smooth horizontal surface so that the forces of constraints on them are normal to the surface while the virtual displacement will be tangential to the surface. In such a situation, the virtual work done by the forces of constraints will be equal to zero.

Thus, the above equation becomes,

$$\sum_{i=1}^{n} \overline{F}_i^a \cdot \delta \overline{r}_i = 0 \qquad ...(6.26)$$

This equation gives us the mathematical expression for the principle of virtual work.

In order to interpret the equilibrium of the system, D' Alembert adopted the idea of a 'reversed force'. He assumed that a system will be in equilibrium under the action of a force which is equal to the sum of the actual force $\overline{F}_i$ and the reversed effective force $\dot{\overline{p}}_i$.

Thus, we can have

$$\overline{F}_i + (-\dot{\overline{p}}) = 0 \quad [\because \overline{F} = m\overline{a} = m\dot{\overline{v}} = \dot{\overline{p}}]$$

or,

$$\overline{F}_i - \dot{\overline{p}}_i = 0$$

where $-\dot{\overline{p}}_i$ appears as an effective force called the **reversed effective force of inertia** on the i^{th} particle. And for the whole system of n particles we can write

$$\sum_{i=1}^{n} (\overline{F}_i - \dot{\overline{p}}) \cdot \delta \overline{r}_i = 0 \qquad ...(6.27)$$

Again writing $\overline{F}_i$ as the sum of applied force $\overline{F}_i^a$ and force of constraint $\overline{F}_i^c$, we get

$$\overline{F}_i = \overline{F}_i^a + \overline{F}_i^c$$

Putting this expression of $\overline{F}_i$ in Eq. (6.27), we get

$$\sum_{i=1}^{n} (\overline{F}_i^a - \dot{\overline{p}}_i) \cdot \delta \overline{r}_i + \sum_{i=1}^{n} \overline{F}^c \cdot \delta \overline{r}_i = 0 \qquad ...(6.28)$$

Since we are considering a system for which the virtual work done by the forces of constraint in zero, Eq. (6.28) reduces to

$$\sum_{i=1}^{n} (\overline{F}_i^a - \dot{\overline{p}}_i) \cdot \delta \overline{r}_i = 0 \qquad ...(6.29)$$

Equation (6.29) is the principle of virtual work in this case and it is known as **D'Alembert's principle**. The key point of this equation (i.e., Eq. (6.29)) is that we have got rid of the forces of constraints.

In order to satisfy Eq. (6.29), we cannot equate separately the coefficient of δr_i to zero since $\delta \overline{r}_i$'s are not independent of each other. And hence it becomes necessary to transform $\delta \overline{r}_i$'s into corresponding set of displacements of generalized coordinates, δq_j which are independent of each other. The coefficient of each δq_j will be equated to zero. The D'Alembert's principle does not involve with it the forces of constraints. Anyway it is now sufficient to specify all the applied forces only. Further, it is valid for all scleronomic and rheonomic systems that are either holonomic or homogenous non-holonomic. The inertial force $-\dot{\overline{p}}$ is introduced in order to reduce the dynamical problem to an equivalent statical problem. This inertial force can be considered as an inertial force arising in an accelerated frame of reference, e.g., in a freely falling accelerated frame $\overline{F}_i - \dot{\overline{p}}_i = 0$ and hence gravity is nullified by the action of the inertial force even though the whole system is in motion and not in either static or dynamic equilibrium.

6.2.4 Lagrange's Equations from D'Alembert's Principle

We have discussed about coordinate transformation in Section 6.1.7. The coordinate transformation equations in vector form can be written as

$$\overline{r}_i = \overline{r}_i (q_1, q_2, q_3, \ldots, q_f, t)$$

f being degree of freedom and $f = 3n - k$, k being the number of constraints and all q's are independent of each other. Taking the derivative of $\overline{r}_i$ with respect to time, we get

$$\frac{d\bar{r}_i}{dt} = \frac{\partial \bar{r}_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial \bar{r}_i}{\partial q_2}\frac{\partial q_2}{dt} + \cdots + \frac{\partial \bar{r}_i}{\partial q_f}\frac{dq_f}{dt} + \frac{\partial \bar{r}_i}{\partial t}\frac{dt}{dt}$$

or, $$\bar{v}_i = \sum_{j=1}^{f} \frac{\partial \bar{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \bar{r}_i}{\partial t} \qquad ...(6.30)$$

Again, we can connect the infinitesimal displacement $\delta \bar{r}_i$ with δq_j as follows:

$$\delta \bar{r}_i = \sum_{j=1}^{f} \frac{\partial \bar{r}_i}{\partial q_j}\delta q_i + \frac{\partial \bar{r}_i}{\partial t}\delta t$$

But in this equation the last term on the right-hand side is zero since in virtual displacement, only the coordinate displacement is considered but not that of time.

So the above equation of $\delta \bar{r}_i$ will reduce to

$$\delta \bar{r}_i = \sum_{j=1}^{f} \frac{\partial \bar{r}_i}{\partial q_j}\delta q_j \qquad ...(6.31)$$

We can now write the D'Alembert's principle (i.e., Eq. (6.29)) as follows and modify it with the help of Eq. (6.31):

$$\sum_{i=1}^{n} (\bar{F}_i^a - \dot{\bar{p}}_i).\, \delta \bar{r}_i = 0 \quad ...(6.29)$$

or, $$\sum_{i=1}^{n} (\bar{F}_i^a - \dot{\bar{p}}_i) \cdot \sum_{j=1}^{f} \frac{\partial \bar{r}_i}{\partial q_j}\partial q_j = 0$$

or, $$\sum_{i=1}^{n}\sum_{j=1}^{f} \bar{F}_i^a \cdot \frac{\partial \bar{r}_i}{\partial q_j}\delta q_i - \sum_{i=1}^{n}\sum_{j=1}^{f} \dot{\bar{p}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}\delta q_j = 0 \qquad ...(6.32)$$

Let us now write $$\sum_{i=1}^{n} \bar{F}_i^a \cdot \frac{\partial \bar{r}_i}{\partial q_j} = Q_j$$

where Q_j's are the components of the generalized force as we have defined it in Section 6.1.8. Q_j need not have the dimensions of force but the product $(Q_j\, \delta q_j)$ must have the dimensions of force.

Thus, we can now write Eq. (6.32) as given below:

$$\sum_{j=1}^{f} Q_j\, \delta q_j - \sum_{i=1}^{n}\sum_{j=1}^{f} \dot{\bar{p}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}\delta q_i = 0 \qquad ...(6.33)$$

The second term of Eq. (6.33) can be evaluated as follows:

$$\sum_{i=1}^{n}\sum_{j=1}^{f} \dot{\bar{p}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}\delta q_j = \sum_{i=1}^{n}\sum_{j=1}^{f} \left(m_i \ddot{\bar{r}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}\right)\delta q_j$$

or, $$\sum_{i=1}^{n}\sum_{j=1}^{f} \dot{\bar{p}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}\delta q_j = \sum_{i=1}^{n}\sum_{j=1}^{f} \left\{\frac{d}{dt}\left(m_i \dot{\bar{r}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}\right) - m_i \dot{\bar{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right)\right\}\delta q_j$$

or, $$\sum_{i=1}^{n}\sum_{j=1}^{f} \dot{\bar{p}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}\delta q_j = \sum_{i=1}^{n}\sum_{j=1}^{f} \left\{\frac{d}{dt}\left(m\bar{v}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}\right) - m_i \bar{v}_i \cdot \frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right)\right\}\delta q_j \qquad ...(6.34)$$

Again $$\frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right) = \frac{\partial}{\partial q_j}\left(\frac{d\bar{r}_i}{dt}\right)$$

or, $$\frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right) = \frac{\partial}{\partial q_j}(\bar{v}_i) \quad ...(6.35)$$

Let us now differentiate Eq. (6.30) with respect to $\dot{q}_j$ to obtain,

$$\frac{\partial}{\partial \dot{q}_j}(\bar{v}_i) = \frac{\partial}{\partial \dot{q}_j}\left(\sum_{j=1}^{f} \frac{\partial \bar{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \bar{r}_i}{\partial t}\right)$$

or, $$\frac{\partial}{\partial \dot{q}_i}(\bar{v}_i) = \frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial \bar{r}_i}{\partial q_1}\dot{q}_1 + \frac{\partial \bar{r}_i}{\partial q_2}\dot{q}_2 + ... + \frac{\partial \bar{r}_i}{\partial q_j}\dot{q}_j + ... + \frac{\partial \bar{r}_i}{\partial q_f}\dot{q}_f + \frac{\partial \bar{r}_i}{\partial t}\right)$$

or, $$\frac{\partial \bar{v}_i}{\partial \dot{q}_i} = \frac{\partial \bar{r}_i}{\partial q_i} \quad ...(6.36)$$

Let us now modify Eq. (6.33) with the help of Eqs. (6.34), (6.35) and (6.36)

$$\sum_{j=1}^{f} Q_j\,\delta q_j - \sum_{i=1}^{n}\sum_{j=1}^{f} \dot{\bar{p}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}\delta q_j = 0 \quad ...(6.33)$$

or, $$\sum_{i=1}^{n}\sum_{j=1}^{f} \dot{\bar{p}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}\delta q_j = \sum_{j=1}^{f} Q_j\,\delta q_j$$

or, $$\sum_{i=1}^{n}\sum_{j=1}^{f}\left\{\frac{d}{dt}\left(m_i\bar{v}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}\right) - m_i\bar{v}_i \cdot \frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right)\right\}\delta q_j = \sum_{j=1}^{f} Q_j\,\delta q_j$$

[by using Eq. (6.34)]

or, $$\sum_{i=1}^{n}\sum_{j=1}^{f}\left\{\frac{d}{dt}\left(m_i\bar{v}_i \cdot \frac{\partial \bar{v}_i}{\partial \dot{q}_j}\right) - m_i\bar{v}_i \cdot \frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right)\right\}\delta q_i = \sum_{j=1}^{f} Q_j\,\delta q_j$$

[by using Eq. (6.36)]

or, $$\sum_{i=1}^{n}\sum_{j=1}^{f}\left\{\frac{d}{dt}\left(m_i\bar{v}_i \cdot \frac{\partial \bar{v}_i}{\partial \dot{q}_i}\right) - m_i\bar{v}_i \cdot \frac{\partial \bar{v}_i}{\partial q_i}\right\}\delta q_j = \sum_{j=1}^{f} Q_j\,\delta q_j$$

[by using Eq. (6.35)]

or, $$\sum_{j=1}^{f}\left[\sum_{i=1}^{n}\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}m_i\bar{v}_i^2\right)\right\} - \sum_{i=1}^{n}\frac{\partial}{\partial q_j}\left(\frac{1}{2}m_i\bar{v}_i^2\right)\right]\delta q_j = \sum_{j=1}^{f} Q_j\,\delta q_j$$

or, $$\sum_{j=1}^{f}\left[\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_j}\sum_{i=1}^{n}\frac{1}{2}m_i\bar{v}_i^2\right\} - \frac{\partial}{\partial q_j}\sum_{i=1}^{n}\frac{1}{2}m_i\bar{v}_i^2\right]\delta q_j = \sum_{j=1}^{f} Q_j\,\delta q_j$$

or, $$\sum_{j=1}^{f}\left[\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\left(\sum_{i=1}^{n} T_i\right) - \frac{\partial}{\partial q_j}\left(\sum_{i=1}^{n} T_i\right)\right]\delta q_j = \sum_{j=1}^{f} Q_j\,\delta q_j$$

where $T_i = \frac{1}{2}m_i\bar{v}_i^2$

or, $$\sum_{j=1}^{f}\left[\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}(T) - \frac{\partial}{\partial q_j}(T)\right]\delta q_j = \sum_j Q_j\,\delta q_j$$

where $T = \sum_{i=1}^{n} T_i$

or,
$$\sum_{j=1}^{f}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial \dot{q}_j}-Q_j\right]\delta q_j=0 \quad \text{...(6.37)}$$

Since the constraints are holonomic, q_j are independent of one another and hence to satisfy Eq. (6.37), the coefficient of each δq_j should separately be zero, i.e.,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}-\frac{\partial T}{\partial q_j}-Q_j=0$$

or,
$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}-\frac{\partial T}{\partial q_j}=Q_j \quad \text{...(6.38)}$$

As j ranges from 1 to f (where f represents the degrees of freedom), there will be f such second-order differential equations of motion of the system of n particles. Let us now find the values of Q_j in case of conservative and non-conservative force fields.

(i) Conservative system

Now, if the system is a conservative one, $\bar{F}_i$ or Q_j will be derivative of the potential function V, which is again a function of the coordinates only. That is, in this case $V = V(\bar{r})$. In such a case, we can write

$$\bar{F}_i=-\bar{\nabla}_i V=-\frac{\partial V}{\partial r_i}\hat{n}$$

where $\hat{n}$ is a unit vector normal to the equipotential surface.

The generalized force can be expressed as follows:

$$Q_j=\sum_{i=1}^{n}\bar{F}_i\cdot\frac{\partial \bar{r}_i}{\partial q_j}=-\sum_{i=1}^{n}\bar{\nabla}_i V\cdot\frac{\partial \bar{r}_i}{\partial q_j}$$

or,
$$Q_j=-\sum_{i=1}^{n}\left(\frac{\partial V}{\partial r_i}\right)\left(\frac{\partial \bar{r}_i}{\partial q_j}\right).\hat{n}$$

or,
$$Q_j=-\frac{\partial V}{\partial q_j} \quad \text{...(6.39)}$$

Now, from Eqs (6.38) and (6.39), we can have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}=-\frac{\partial V}{\partial q_j}$$

or,
$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}-\frac{\partial}{\partial q_j}(T-V)=0$$

or,
$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}(T-V)\right)-\frac{\partial}{\partial q_j}(T-V)=0 \quad [\text{since } V \text{ is independent of } \dot{q}_j]$$

Now writing $T - V = L$, where L is called the lagrangian for the conservative system, the set of above equations becomes

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}-\frac{\partial L}{\partial q_j}=0,\quad j=1,2,3\ldots f \quad \text{...(6.40)}$$

Each equation of the set of Eq. (6.40) is known as Lagrange's equation of motion for a conservative system.

(ii) Non-conservative system

If the potentials are velocity dependent (called generalized potential) then though the system is not a conservative one, yet the above mentioned form of Lagrange's equations can be retained provided Q_j, the generalized forces, are derived from a function $U(q_j, \dot{q}_j)$, called the generalized or velocity-dependent potential, such that we have,

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \qquad ...(6.41)$$

Now, substituting Q_j by this value of it in Eq. (6.38), we can write

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

or,

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}(T-U)\right) - \frac{\partial}{\partial q_j}(T-U) = 0$$

Now, if we put lagrangian $L = T - U$ then the above equation becomes

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0; \quad j = 1, 2, 3, ..., f$$

which is exactly of the same form of Eq. (6.40).

Some characteristics of Lagrange's equations

Some characteristics of Lagrange's equations are noted below:

(a) Lagrange's equations of motion for a system of particles are derived by considering the energy of the system instead of considering the forces.

(b) While deriving Lagrange's equations the effects of the forces of constraint are eliminated.

(c) As the lagrangian L (= $T - V$) is a scalar quantity, Lagrange's equations are in scalar form. But Newton's equations can appear in vectorial form which are difficult to handle.

(d) Lagrange's equations are independent of coordinate systems.

(e) Since all Lagrange's equations are second-order differential equations, their solutions contain two arbitrary constants and all equations are of the same form.

6.2.5 Cyclic or Ignorable Coordinates

The lagrangian as described in the previous section is a function of the generalized coordinates q_j; generalized velocities $\dot{q}_j$ and time t. Now, if the lagrangian of a system of particles does not contain a particular coordinate q_k, then obviously in case of such a system of particles; $\frac{\partial L}{\partial q_k} = 0$. Such a coordinate ($q_k$) which does not appear in the expression of the lagrangian is called an **ignorable** (or a **cyclic**) **coordinate**.

6.2.6 Conservation of the Linear Momentum in Case of a Cyclic Coordinate

For a conservative system of particles, the equations of Lagrange are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \quad j = 1, 2, ..., f$$

where the lagrangian $L = T - V$.

Let us now assume that the q_k coordinate is cyclic, i.e., it is missing in the expression of L. Then for this coordinate, the Lagrange's equation reduces to $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0$

or, $$\frac{\partial L}{\partial \dot{q}_j} = \text{constant} \quad \text{...(6.42)}$$

Now, we can write

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}(T - V) = \frac{\partial T}{\partial \dot{q}_j} \quad [\because V \neq V(\dot{q}_j)]$$

or, $$\frac{\partial L}{\partial \dot{q}_j} = p_j \quad \text{...(6.43)}$$

Now, considering Eqs (6.42) and (6.43), we can write

$$p_j = \text{constant}$$

i.e., the linear momentum corresponding to a cyclic or ignorable coordinate q_k is conserved.

6.3 HAMILTONIAN FORMULATION

In the previous section, we have derived Lagrange's equations of motion for a system of particles. These equations are a set of second-order, first-degree differential equations. In this section, we shall discuss an alternative set of $2f$ first-order, first-degree differential equations known as Hamilton's equations of motion. Lagrange's equations are derived on the basis of the lagrangian L. Similarly, the Hamilton's equations of motion will be derived on the basis of another function H which is known as the hamiltonian. In case of lagrangian formulations, the independent variables are the generalized coordinates as well as time. Though the generalized velocities also appear explicitly in this formulation, but they are simply time derivatives of the generalized coordinates. So they are treated ultimately as some dependent variables. This fact is reflected in the definition of configuration space where a single particle supplies only three initial values namely the three position coordinates. In the hamiltonian approach, the aforesaid dependency of velocities are eliminated by introducing a new independent variable—known as the generalized momentum p_j. In this approach, we shall use the hamiltonian H instead of the lagrangian. This hamiltonian is a function of generalized coordinates, generalized momenta as well as time, i.e., $H(q_j, p_j, t)$ while the lagrangian is $L(q_j, \dot{q}_j, t)$. So there is a change of basis from the set $(q_j, \dot{q}_j, t)$ to the set (q_j, p_j, t).

6.3.1 Phase Space

While defining configuration space, we considered three space coordinates of real three-dimensional space for each particle along with time t. So in a system of n particles without any constraint, we had $3n$ space coordinates in the corresponding configuration space. But having provided equal status to 'coordinates' and 'momenta', the configuration space no more remains sufficient to represent the history of the system of particles. So, the path adopted by the system of particles during its motion must now be represented by a space of $6n$ coordinates in which three coordinates will represent the three position coordinates of real three-dimensional space and three momenta of the same space for a single particle. So, the new space will have $6n$ dimensional space called phase space. While there will be k constraints the configuration space will have f dimensions and phase space will have $2f$ dimensions where f is the degree of freedom and is given by $f = 3n - k$.

6.3.2 Hamilton's Equations of Motion

The hamiltonian H is a function of generalized coordinates, generalized momenta and time, i.e., $H = H(q_j, p_j, t)$. And the generalized mometa p_j are given by

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

which show that for each generalized coordinate q_j there is one corresponding generalized momentum p_j.

The mechanical state of the system under consideration can thus be described completely, provided the variables q_j and p_j are given as functions of time. The hamiltonian H is defined by the following equation:

$$H = \sum_{j=1}^{2f} p_j \dot{q}_j - L \qquad ...(6.44)$$

H is a constant of motion with the condition that the lagrangian L does not depend on time explicitly. We can express the hamiltonian H as

$$H = \sum_j p_j \dot{q}_j - L(q_j, \dot{q}_j)$$

$$\therefore \qquad H = H(q_j, p_j)$$

As in hamiltonian formulation we provide generalized momenta an independent status, having them placed on equal footing along with the generalized coordinates, the hamiltonian is regarded, in general, as a function of the position coordinates q_j, the momenta p_j and time t, i.e.,

$$H = H(q_j, p_j, t)$$

Hence, in differential form we can write

$$dH = \sum_{j=1}^{2f} \frac{\partial H}{\partial q_j} dq_j + \sum_{j=1}^{2f} \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial t} dt \qquad ...(6.45)$$

Again, H is defined as

$$H = \sum_{j=1}^{2f} p_j \dot{q}_j - L$$

Hence,

$$dH = \sum_{j=1}^{2f} \dot{q}_j \, dp_j + \sum_{i=1}^{2f} p_j \, d\dot{q}_j - dL \qquad ...(6.46)$$

But the lagrangian is given by

$$L = L(q_j, \dot{q}_j, t)$$

Hence,

$$dL = \sum_{j=1}^{2f} \frac{\partial L}{\partial q_j} dq_j + \sum_{j=1}^{2f} \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j + \frac{\partial L}{\partial t} dt \qquad ...(6.47)$$

Now putting the value of dL from the Eq. (6.47) in Eq. (6.46), we have

$$dH = \sum_{j=1}^{2f} \dot{q}_j \, dp_j + \sum_{j=1}^{2f} p_j \, d\dot{q}_j - \sum_{j=1}^{2f} \frac{\partial L}{\partial q_j} dq_j - \sum_{j=1}^{2f} \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j - \frac{\partial L}{\partial t} dt \qquad ...(6.48)$$

Now recognizing, $\frac{\partial L}{\partial \dot{q}_j} = p_j$ and $\frac{\partial L}{\partial q_j} = \dot{p}_j$ and putting these in Eq. (6.48)

we have

$$dH = \sum_{j=1}^{2f} \dot{q}_j \, dp_j + \sum_{j=1}^{2f} p_j \, d\dot{q}_j - \sum_{j=1}^{2f} \dot{p}_j \, dq_j - \sum_{j=1}^{2f} p_j \, d\dot{q}_j - \frac{\partial L}{\partial t} dt$$

or,
$$dH = \sum_{j=1}^{2f} \dot{q}_j \, dp_j - \sum_{j=1}^{2f} \dot{p}_j \, dq_j - \frac{\partial L}{\partial t} dt \qquad ...(6.49)$$

Now, comparing the coefficients in Eqs (6.49) and (6.45), we arrive at the following set of equations:

$$\left.\begin{aligned} \dot{q}_j &= \frac{\partial H}{\partial p_j} \\ \dot{p}_j &= -\frac{\partial H}{\partial q_j} \end{aligned}\right\} \qquad ...(6.50)$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \qquad ...(6.51)$$

The set of Eqs. (6.50) are known as the Hamilton's canonical equations of motion or simply Hamilton's equations of motion. They make a set of $2f$ first-order, first-degree differential equations of motion.

On integration of these $2f$ differential equations, we get $2f$ constants of integration which can be evaluated in terms of the initial conditions of the system of n particles.

6.3.3 A Coordinate Cyclic in Hamiltonian

A coordinate which is cyclic in the lagrangian is also cyclic in the hamiltonian.

Let us assume that q_j does not appear in the lagrangian L. Then we can write

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} = 0 \quad \left[\because L \neq L(q_i) \text{ and } p_j = \frac{\partial L}{\partial q_j}\right]$$

So, we have $\qquad p_j = \text{constant}.$

But if p_j is constant, then from Eq. (6.50), we can have

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} = 0$$

or,
$$\frac{\partial H}{\partial q_j} = 0 \Rightarrow H \neq H(q_j)$$

i.e., q_j does not appear in H.

So, q_j is cyclic in the hamiltonian. If a position coordinate is cyclic in H, it would reduce the number of variables by two in the hamiltonian formulation.

6.3.4 Physical Significance of the Hamiltonian

(a) The lagrangian L possesses the dimensions of energy. Similarly, the hamiltonian H also possesses the dimensions of energy.

(b) In all circumstances, it is not equal to the total energy E of the system of particles. Only in a special case the hamiltonian is equal to the total energy, i.e., when the system is conservative and the coordinate transformation equations are independent of time; then $H = E$.

(c) In all other cases except the case (b), H is a constant of motion but not equal to the total energy of the system of particles.

(d) When L is not an explicit function of t, H is also not an explicit function of t.

Let us first write,

$$H = H(q_j, p_j, t)$$

Hence, by differentiating H with respect to t, we get

$$\frac{dH}{dt} = \sum_{j=1}^{2f} \frac{\partial H}{\partial q_j}\dot{q}_j + \sum_{j=1}^{2f} \frac{\partial H}{\partial p_j}\dot{p}_j + \frac{\partial H}{\partial t}$$

From Eq. (6.50), we have

$$\frac{\partial H}{\partial q_j} = -\dot{p}_j \quad \text{and} \quad \frac{\partial H}{\partial p_j} = \dot{q}_j$$

Therefore, we can write

$$\frac{dH}{dt} = -\sum_{j=1}^{2f} \dot{p}_j\dot{q}_j + \sum_{j=1}^{2f} \dot{p}_j\dot{q}_j + \frac{\partial H}{\partial t}$$

or, $$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

or, $$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \quad \text{[by Eq. (6.51)]}$$

Now, if L is not a function of t, then

$$\frac{\partial L}{\partial t} = 0, \quad \text{so we get } \frac{dH}{dt} = 0$$

This means that t will also not appear in the expression of the hamiltonian H, i.e, t must be cyclic in the hamiltonian for it to be a constant of motion. Our interest is more in those systems for which the hamiltonian H is both a constant of motion and equal to the total energy E of the system of particles.

(e) The lagrangian formulation can be suitably used in the macroscopic world only (i.e., in classical mechanics) but the hamiltonian formulation can be used in both the microscopic as well as the macroscopic world, i.e., in quantum as well as classical mechanics).

6.3.5 Use of Lagrangian and Hamiltonian Methods

In order to solve any mechanical problem by either lagrangian or hamiltonian formulation, one has to construct the lagrangian function L first. It is expressed in terms of the generalized coordinates and generalized velocities. In order to construct the lagrangian function one needs to express the kinetic energy and the potential energy as functions of these generalized quantities. Then the hamiltonian function can be derived from the expression of the lagrangian function and then the hamiltonian is constructed in terms of generalized coordinates and generalized momenta. In the following section named Worked Out Problems we shall show how to solve mechanical problems by using lagrangian and hamiltonian formulations.

Worked Out Problems

Example 6.1 Find the degrees of freedom of the following systems:

(a) The bob of a simple pendulum oscillating in a vertical plane.

(b) Two particles, connected by a rigid rod moving freely in a plane.

(c) Three particles connected by a rigid rod moving freely in a plane.

(d) A particle moving on the circumference of a circle. [WBUT 2006]

Sol. (a) A simple pendulum is shown in Fig. 6.1W. The motion of bob of this simple pendulum can be described by the angular coordinate θ. The radial coordinate $r = l$ (a constant). This is the equation of constraint and l is the effective length of the pendulum. Hence, the simple pendulum (i.e., the bob of it) has one degree of freedom.

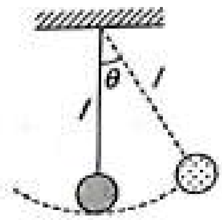


Fig. 6.1W Simple pendulum.

(b) In this case, the motion is confined in a plane. So the two particles require $(x_1\, y_1)$ and (x_2, y_2), i.e., 4 coordinates. There is only one constraint in the present case which is $(x_2 - x_1)^2 + (y_2 - y_1)^2 = l^2$

where l is the distance between the particles. So the number of degrees of freedom is given by

$$f = 2 \times 2 - 1 = 3$$

(c) The given three particles may be connected in two different ways:

(i) *Linear arrangement of particles*

In this case, the distance between particles 1 and 2, and 2 and 3 remain constant. Hence the number of constraints is given by $k = 2$.

$\therefore$ the degrees of freedom f is given by

$$f = 3 \times 2 - 2 = 4$$

(ii) *Triangular arrangement of particles*

In this case, the number of constraints $k = 3$ as there are three equations representing three sides of the triangle. So, the required number of degrees of freedom is given by

$$f = 3 \times 2 - 3 = 3$$

(d) The locus of the particle is a circle. So, the equation of the constraint is given by

$$x^2 + y^2 = a^2 \quad \text{or} \quad r = a$$

where a is the radius of the circle. In cartesian coordinates, one of the two variables i.e., either x or y is sufficient to find the position of the particle. And in polar coordinates, one variable θ is sufficient to locate the position of the particle.

$\therefore$ the degrees of freedom is given by

$$f = 2 - 1 = 1$$

Example 6.2 A mass m is tied with a horizontal spring as shown in Fig. 6.2W. It is able to move to and fro on a frictionless plane in one line. If the spring is pulled and released, the mass executes simple harmonic motion. Find its time period by using lagrangian method.

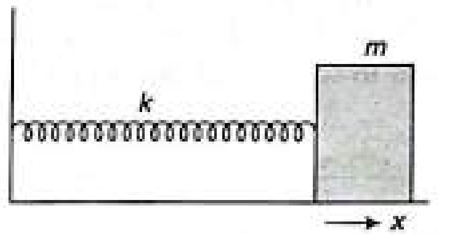


Fig. 6.2W Spring and mass system.

Sol. In this case, force $F = -kx$,

where k is the spring constant. So, the kinetic energy (T) and potential energy (V) of the mass m is given by

$$T = \frac{1}{2} m\dot{x}^2 \text{ and } V = -\int_0^x F\,dx = \frac{1}{2} kx^2$$

$\therefore$ the lagrangian $\quad L = T - V$

or, $$L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 \qquad \ldots(1)$$

Now, the Lagrange's equation of motion for a simple harmonic oscillator is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \qquad \ldots(2)$$

Now, differentiating Eq. (1) with respect to $\dot{x}$ and x, we get

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \text{and} \quad \frac{\partial L}{\partial x} = -kx$$

Putting these, in Eq. (2) we get

$$\frac{d}{dt}(m\dot{x}) - (-kx) = 0$$

or, $$m\ddot{x} + kx = 0$$

or, $$\ddot{x} + \frac{k}{m}x = 0$$

or, $$\ddot{x} + \omega^2 x = 0 \quad \text{where } \omega = \sqrt{\frac{k}{m}}$$

$\therefore$ the time period of oscillation of the spring-mass system is given by

$$T_p = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

Example 6.3 Obtain the equation of motion of a simple pendulum form its lagrangian representation. Hence deduce the expression for its time period for small amplitude motion. [WBUT 2006]

Sol. The simple pendulum in oscillation has been shown in Fig. 6.3W. Let at any instant of time during its oscillation, the string OB make an angle θ with its equilibrium position OA.

The kinetic energy of the bob is given by

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(l\dot{\theta})^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

where l is the length of the string and v, the velocity of the bob is given by $v = l\dot{\theta}$.

While passing from B to A the bob falls freely through a vertical distance CA. The potential energy of the bob of the simple pendulum is thus, given by

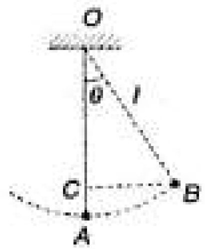


Fig. 6.3W A simple pendulum.

$$V = mg(OA - OC) = mg(l - l\cos\theta)$$

or, $$V = mgl(1 - \cos\theta)$$

We have considered the horizontal level passing through A as reference level.

$\therefore$ the lagrangian L is given by

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta) \qquad \ldots(1)$$

The Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad [\because \theta \text{ is the only generalized coordinate}] \qquad ...(2)$$

From Eq. (1), we get

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \text{ and } \frac{\partial L}{\partial \theta} = -mgl \sin\theta$$

Putting these values of $\frac{\partial L}{\partial \dot{\theta}}$ and $\frac{\partial L}{\partial \theta}$ in Eq. (2), we get

$$\frac{d}{dt}(ml^2\dot{\theta}) + mgl\sin\theta = 0$$

or,
$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \qquad ...(3)$$

This is the required differential equation of motion of the simple pendulum.

Now, for small oscillation of the bob, we can write,

$$\sin\theta \approx \theta$$

∴ Eq. (3) reduces to

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

or,
$$\ddot{\theta} + \omega^2\theta = 0 \qquad ...(4)$$

where $\omega = \sqrt{\frac{g}{l}}$

So, Eq. (4) represents simple harmonic motion whose time period T_p is given by

$$T_p = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

Example 6.4 Derive the equations of motion of a bead which is sliding on a uniformly rotating wire in a horizontal space.

Sol. Let P be the position of a bead [Fig. 6.4W] which is sliding along the wire OQ. The wire OQ is rotating with a uniform angular velocity ω. Let the angular displacement $\theta = 0$ at $t = 0$, so, we can write

$$\theta = \omega t$$

This equation helps us to determine the angular position θ at any time t. To locate the bead in the XY plane we require the variable r (radius vector). The transformation equations relating the cartesian and plane polar coordinates are given by

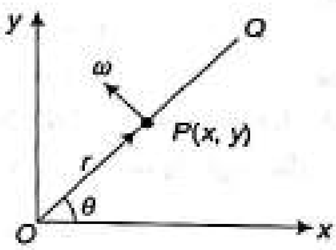


Fig. 6.4W A bead is sliding on a wire which is rotating on a horizontal plane.

$$x = r\cos\theta = r\cos\omega t$$

and
$$y = r\sin\theta = r\sin\omega t$$

Differentiating these equations with respect to t, we get

$$\dot{x} = \dot{r}\cos(\omega t) - r\omega\sin(\omega t)$$

and $\dot{y} = \dot{r}\sin(\omega t) + r\omega\cos(\omega t)$

Hence, the kinetic energy of the bead is given by

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

or,

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\omega^2)$$

As the plane of rotation is horizontal, the potential energy is constant anywhere in the plane. Let it be V (= constant).

Then, the lagrangian L is given by

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\omega^2) - V$$

or,

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V \quad [\because \omega = \dot{\theta}] \quad ...(1)$$

The Lagrange's equation for r and θ are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \quad ...(2)$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad ...(3)$$

Now, differentiating Eq. (1), we get

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}, \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta},$$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 \text{ and } \frac{\partial L}{\partial \theta} = 0$$

Putting these value in Eqs. (2) and (3), we get

$$m\ddot{r} - mr\dot{\theta}^2 = 0 \quad ...(4)$$

$$mr^2\ddot{\theta} = 0 \quad ...(5)$$

Equation (4) can be rewritten as

$$\ddot{r} = r\dot{\theta}^2 = r\omega^2 \quad ...(6)$$

Equation (6) is an expression which indicates that the bead moves under the action of the centrifugal force $m\omega^2 r$ directed outwards along the wire.

Example 6.5 Derive the equation of motion of a system of two masses connected by an inextensible string passing over a small smooth pulley (called Atwood's machine).

Sol. Figure 6.5W shows the principle of Atwood's machine consisting of two masses m_1 and m_2 suspended over a frictionless pulley P of radius 'a' and connected by an inextensible string of length l.

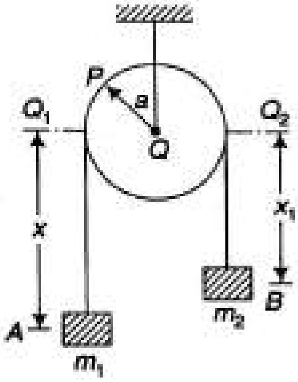


Fig. 6.5W Atwood's machine

Let $Q_1A = x$, and $Q_2B = x_1$

From the diagram, we can write

$$l = x + x_1 + \pi a \quad [\because a \text{ is radius}]$$

$\therefore \quad x_1 = l - \pi a - x$

$\therefore \quad \dot{x}_1 = -\dot{x}$ and $\ddot{x}_1 = -\ddot{x}$

Now, the kinetic energy of the mass m_1 is

$$T_1 = \frac{1}{2} m_1 \dot{x}^2$$

and the kinetic energy of m_2 is

$$T_2 = \frac{1}{2} m_2 \dot{x}_1^2 = \frac{1}{2} m_2 \dot{x}^2$$

$\therefore$ the total kinetic energy of the system is given by

$$T = T_1 + T_2 = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2$$

or, $$T = \frac{1}{2} (m_1 + m_2)\, \dot{x}^2$$

With reference to the horizontal line passing through Q, the potential energy of m_1 is given by

$$V_1 = -m_1 g x$$

and that of m_2 is $\quad V_2 = -m_2 g\,(l - \pi a - x)$

$\therefore$ total potential energy is given by

$$V = V_1 + V_2 = -m_1 g x - m_2 g\,(l - \pi a - x)$$

The lagrangian of the system is given by

$$L = T - V = \frac{1}{2}(m_1 + m_2)\,\dot{x}^2 + m_1 g x + m_2 g\,(l - \pi a - x) \qquad \text{...(1)}$$

$$\therefore \quad \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\,\dot{x} \qquad \text{...(2)}$$

and $$\frac{\partial L}{\partial x} = (m_1 - m_2)\, g \qquad \text{...(3)}$$

Then Lagrange's equation of motion is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

or, $\quad (m_1 + m_2)\,\ddot{x} - (m_1 - m_2)\, g = 0$

or, $$\ddot{x} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g \qquad \text{...(4)}$$

Equation (4) is the required equation of motion of the system.

Example 6.6 Find the equation of motion of a free particle.

Sol. If a particle is not acted upon by any external force, the particle is called a free particle. So the potential energy of this particle $V = 0$.

Hence, the lagrangian is given by

$$L = T - V = \frac{1}{2} m\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

where x, y and z are cartesian coordinates.

The coordinates (x, y, z) are cyclic,

$$\therefore \qquad \frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial z} = 0$$

So, the Lagrange's equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow m\ddot{x} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \Rightarrow m\ddot{y} = 0$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0 \Rightarrow m\ddot{z} = 0$$

Example 6.7 Derive the lagrangian and Lagrange's equation of a freely falling body.

Sol. Let us assume that a body of mass m is falling freely under gravitational force. The point P is at a height h. A body of mass m is falling from the point P. Let it reach the point Q at time t [Fig. 6.6W].

$\therefore$ the potential energy of the body is given by

$$V = mg\,(h - x)$$

and kinetic energy is given by

$$T = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 = \frac{1}{2} m\dot{x}^2$$

$\therefore$ its lagrangian is given by

$$L = T - V = \frac{1}{2} m\dot{x}^2 - mg\,(h - x)$$

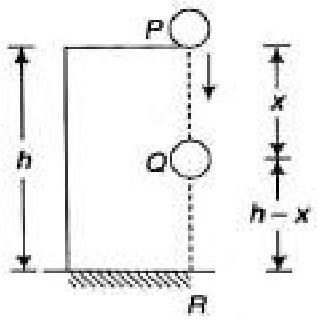


Fig. 6.6W A freely falling body.

As there is only one variable x, the Lagrange's equation is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \qquad \ldots(1)$$

Now, differentiating the lagrangian, we get

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \text{ and } \frac{\partial L}{\partial x} = mg$$

Putting these values in Eq. (1), we get

$$\frac{d}{dt}(m\dot{x}) - mg = 0$$

or, $$\ddot{x} - g = 0 \qquad \ldots(2)$$

$\therefore$ Eq. (2) is the required equation.

Example 6.8 Find equation of motion of a bead which is sliding on a wire which is rotating on a vertical plane with uniform angular velocity.

Sol. Let us assume that XY is a vertical plane where the y axis is along the vertical direction.

The gravitational force acting on the bead is acting along the negative y axis [Fig. 6.7W].

The potential energy of the bead is given by

$$V = mgy = mgr \sin \theta$$

Let the angular displacement $\theta = 0$ at $t = 0$. So, we can write,

$$\theta = \omega t \quad \text{...(1)}$$

This equation helps us to find θ at any instant of time. To locate the bead in the XY vertical plane, we require the variable r (the radius vector).

The transformation equations are:

$$x = r \cos\theta = r \cos \omega t$$

and $$y = r \sin \theta = r \sin \omega t$$

Now, differentiating these two equations with respect to t, we get

$$\dot{x} = \dot{r} \cos \omega t - r\omega \sin \omega t$$

and $$\dot{y} = \dot{r} \sin \omega t + r\omega \cos \omega t$$

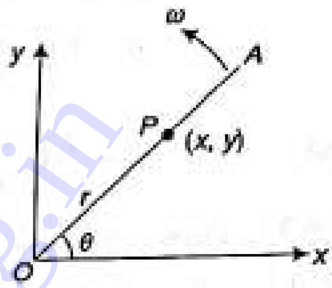


Fig. 6.7W A bead sliding over a wire which is rotating uniformly on a vertical plane.

Hence, the kinetic energy of the bead is given by

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

or, $$T = \frac{1}{2} m (\dot{r}^2 + r^2 \omega^2)$$

So, the lagrangian is given by

$$L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2 \omega^2) - mg\, r \sin\theta$$

or, $$L = \frac{1}{2} m (\dot{r}^2 + r^2\omega^2) - mgr \sin \omega t$$

$\therefore$ the Lagrange's equation for r is given by

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$$

or, $$\frac{d}{dt} (m\dot{r}) - mr\, \omega^2 + mg \sin \omega t = 0$$

or, $$\ddot{r} - r\omega^2 + g \sin \omega t = 0 \quad \text{...(2)}$$

The Eqs (1) and (2) are the required equations of motion of the bead in a vertical plane.

Example 6.9 Find the equation of motion of a simple pendulum whose point of suspension is moving along the x axis with an acceleration f.

Sol. The system is shown in Fig. 6.8W.

The point of suspension O' moves along OX line with an acceleration of f. Hence $X'\ Y'$ (with origin at O') is a non-inertial frame and XY is an inertial frame of reference.

The equation of constraints are given by

$$x' = l \sin \theta$$
$$y' = l \cos \theta$$

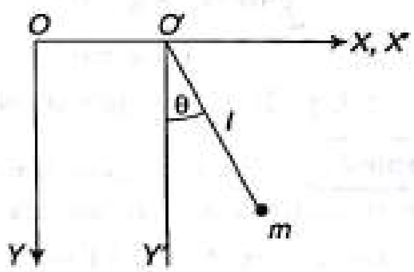


Fig. 6.8W A simple pendulum with sliding point of suspension.

The motion is, here, described by the generalized coordinate θ.

These equations of constraints can be transferred from the non-inertial frame to inertial frame of reference as follows:

$$x = x' + \frac{1}{2}ft^2 \text{ and } y = y'$$

or,
$$x = l\sin\theta + \frac{1}{2}ft^2 \text{ and } y = l\cos\theta$$

Again,
$$\dot{x} = l\cos\theta\,\dot{\theta} + ft \text{ and } \dot{y} = -l\sin\theta\,\dot{\theta}$$

The kinetic energy of the bob is given by

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m[(l\cos\theta\,\dot{\theta} + ft)^2 + (-l\sin\theta\,\dot{\theta})^2]$$

or,
$$T = \frac{1}{2}m[l^2\cos^2\theta\,\dot{\theta}^2 + f^2t^2 + 2lft\cos\theta\,\dot{\theta} + l^2\sin^2\theta\,\dot{\theta}^2]$$

or,
$$T = \frac{1}{2}m[l^2\dot{\theta}^2 + f^2t^2 + 2lft\cos\theta\,\dot{\theta}]$$

The potential energy of the bob is given by

$$V = -mgy = -mgl\cos\theta \quad [\text{Assuming } V = 0 \text{ at } y = 0]$$

$\therefore$ the lagrangian of the system is given by

$$L = T - V = \frac{1}{2}m[l^2\dot{\theta}^2 + f^2t^2 + 2lft\cos\theta\,\dot{\theta}] + mgl\cos\theta$$

Now, Lagrange's equation of motion is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

or,
$$\frac{d}{dt}(ml^2\dot{\theta} + mlft\cos\theta) - (-mlft\sin\theta\,\dot{\theta}) + mgl\sin\theta = 0$$

or,
$$ml^2\ddot{\theta} + mlf\cos\theta - mlft\sin\theta\,\dot{\theta} + mlft\sin\theta\,\dot{\theta} + mgl\sin\theta = 0$$

or,
$$ml^2\ddot{\theta} + lfm\cos\theta + mgl\sin\theta = 0$$

or,
$$l\ddot{\theta} + f\cos\theta + g\sin\theta = 0$$

or,
$$\ddot{\theta} + \frac{f}{l}\cos\theta + \frac{g}{l}\sin\theta = 0 \quad \text{...(1)}$$

Equation (1) represents the equation of motion of a simple pendulum whose point of suspension is moving along the x axis with an acceleration f.

Example 6.10 Obtain the Lagrange's equation of motion for a particle in central force field in the space.

Sol. Let $P(r, \theta)$ be the position of the particle of mass m moving under a central force field in space [Fig. 6.9W]. Then the lagrangian of the system is given by

$$L = T - V(r) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \quad \text{...(1)}$$

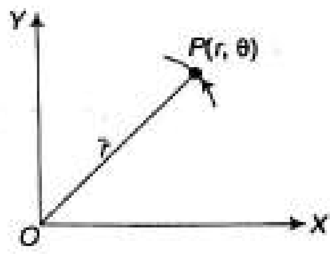


Fig. 6.9W Motion of a particle under central force.

where the velocity of the particle in plane polar coordinate is given by

$$v^2 = \dot{r}^2 + r^2\,\dot{\theta}^2$$

$\therefore$ kinetic energy of the particle is given by

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\,(\dot{r}^2 + r^2\,\dot{\theta}^2)$$

And the potential energy is given by

$$V = -\int_{-\infty}^{r} F\,dr = \int_{-\infty}^{r} -\frac{k}{r^2}\,dr = -\frac{k}{r}$$

Since the force F is attractive in nature and varies inversely as the square of the distance from the origin,

i.e., $F = -\dfrac{k}{r^2}$ where k is a constant of proportionality.

$\therefore$ the lagrangian is given by

$$L = T - V$$

or, $$L = \frac{1}{2}m\,(\dot{r}^2 + r^2\,\dot{\theta}^2) + \frac{k}{r}$$

Differentiating L, we get

$$\left.\begin{aligned} \frac{\partial L}{\partial \dot{r}} &= m\dot{r}, & \frac{\partial L}{\partial \dot{\theta}} &= mr^2\dot{\theta} \\ \frac{\partial L}{\partial r} &= mr\,\dot{\theta}^2 - \frac{k}{r^2}, & \frac{\partial L}{\partial \theta} &= 0 \end{aligned}\right\} \quad \ldots(1)$$

The equations of motion for this case are:

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \end{aligned}\right\} \quad \ldots(2)$$

From Eqs (1) and (2), we get

$$\frac{d}{dt}(m\dot{r}) - mr\,\dot{\theta}^2 + \frac{k}{r^2} = 0$$

and $$\frac{d}{dt}(mr^2\,\dot{\theta}) = 0$$

Rearranging again, we get

$$\left.\begin{aligned} m\ddot{r} - mr\,\dot{\theta}^2 + \frac{k}{r^2} = 0 \\ 2\,mr\,\dot{r}\dot{\theta} + mr^2\,\ddot{\theta} = 0 \end{aligned}\right\} \quad \ldots(3)$$

The two equations of the equation set (3) are the required equations of motion for the particle that moves under a central attractive force.

Example 6.11 Find Lagrange's equations of motion of a spherical pendulum.

Sol. In case of a spherical pendulum, the bob is constrained to move on the surface of the sphere. The position of the bob can be located by spherical polar coordinates r, θ and ϕ as shown in Fig. 6.10W. The distance r of the bob from the origin is the radius of the sphere on which the particle (bob) moves. The force in this case is not central, but is constant in the vertical direction. Among the three coordinates r, θ and ϕ, here r is a constant. So, the bob can be located by θ and ϕ only and they are the generalized coordinates. The transformation equations which relate the spherical polar coordinates to cartesian coordinates are given by

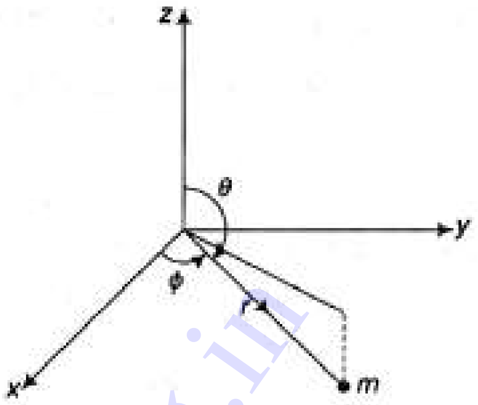


Fig. 6.10W A spherical pendulum of a bob of mass m. The bob traces on the lower hemispherical surface. The greatest circle lies on the horizontal XY plane.

$$x = r \sin\theta \cos\phi,\ y = \sin\theta \sin\phi \text{ and } z = r\cos\theta$$

Using these transformation equations and keeping in mind the fact that $\dot{r} = 0$, the kinetic and potential energies are expressed as follows:

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2} mr^2 (\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) \text{ and}$$

$$V = mgz = mgr\cos\theta$$

So, the lagrangian of the system is given by

$$L = T - V = \frac{1}{2} mr^2 (\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) - mgr\cos\theta \quad ...(1)$$

Now, differentiating L partially with respect to θ, $\dot{\theta}$, ϕ and $\dot{\phi}$, we get

$$\left.\begin{aligned} \frac{\partial L}{\partial \theta} &= mr^2 \sin\theta\cos\theta\, \dot{\phi}^2 + mgr\sin\theta, \\ \frac{\partial L}{\partial \dot{\theta}} &= mr^2\dot{\theta}, \\ \frac{\partial L}{\partial \phi} &= 0 \\ \text{and} \quad \frac{\partial L}{\partial \dot{\phi}} &= mr^2\sin^2\theta\, \dot{\phi} \end{aligned}\right\} \quad ...(2)$$

The Lagrange's equations for such a system can be written as

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= 0 \\ \text{and} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} &= 0 \end{aligned}\right\} \quad ...(3)$$

By using Eq. (2), Eq. (3) can be written as follows:

$$\frac{d}{dt}(mr^2\,\dot{\theta}) - mr^2 \sin\theta\cos\theta\,\dot{\varphi}^2 - mgr\sin\theta = 0$$

and

$$\frac{d}{dt}(mr^2 \sin^2\theta\,\dot{\varphi}) = 0$$

On solving these two equations further, we get

$$\left.\begin{aligned} mr^2\,\ddot{\theta} - mr^2 \sin\theta\cos\theta\,\dot{\varphi}^2 - mgr\sin\theta &= 0 \\ \text{and} \qquad mr^2\sin^2\theta\,\ddot{\varphi} &= 0 \end{aligned}\right\} \quad \ldots(4)$$

Equations of set (4) are the required equations of motion for a spherical pendulum.

Example 6.12 Use Hamilton's canonical equations of motion to obtain the equation of motion of simple pendulum. [WBUT 2006]

Sol. A simple pendulum is shown in Fig. 6.11W. Referring to the figure and assuming m to be the mass of the bob, l, the effective length and θ, angular displacement for small oscillations, we can express the kinetic energy as

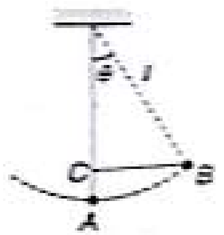


Fig. 6.11W A simple pendulum.

$$T = \frac{1}{2} ml^2\,\dot{\theta}^2$$

and the potential energy as

$$V = mgl\,(1 - \cos\theta)$$

So, the lagrangian is given by

$$L = T - V = \frac{1}{2} ml^2\,\dot{\theta}^2 - mgl\,(1-\cos\theta) \quad \ldots(1)$$

Now, the generalized momentum p_θ can be written as

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\,\dot{\theta} \quad \ldots(2)$$

$\therefore$ the hamiltonian is given by

$$H = \sum_i p_i\,\dot{q}_i - L = p_\theta\,\dot{\theta} - L$$

or,

$$H = ml^2\,\dot{\theta}^2 - \left[\frac{1}{2} ml^2\,\dot{\theta}^2 - mgl\,(1-\cos\theta)\right]$$

or,

$$H = \frac{1}{2} ml^2\,\dot{\theta}^2 + mgl\,(1-\cos\theta) \quad \ldots(3)$$

Hence $H = T + V = E$

where E is the total energy

From Eq. (2), we can write

$$\dot{\theta} = \frac{p_\theta}{ml^2}$$

$$\therefore \quad H = \frac{1}{2} ml^2 \left(\frac{p_\theta}{ml^2}\right)^2 + mgl\,(1-\cos\theta) \quad \text{[from Eq. (3)]}$$

or, $$H = \frac{p_\theta^2}{2ml^2} + mgl\,(1 - \cos\theta) \qquad ...(4)$$

$\therefore$ $$\frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2}, \frac{\partial H}{\partial \theta} = mgl \sin\theta \qquad ...(5)$$

Now, Hamilton's equations for $\dot{\theta}$ and $\dot{p}_\theta$ are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} \quad \text{and} \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta}$$

In this case, we have

$$\dot{\theta} = \frac{p_\theta}{ml^2} \qquad ...(6)$$

and $$\dot{p}_\theta = -mgl \sin\theta \qquad ...(7)$$

Equations (6) and (7) are Hamilton's equations for simple pendulum.

From Eq. (2), we have

$$p_\theta = ml^2\,\dot{\theta}$$

$\therefore$ $$\dot{p}_\theta = ml^2\,\ddot{\theta}$$

Putting this value of $\dot{p}_\theta$ in Eq. (7), we get

$$ml^2\,\ddot{\theta} = -mgl \sin\theta$$

or, $$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

or, $$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \qquad ...(8)$$

Equation (8) is the equation of a simple pendulum which represents SHM for small θ.

For small θ, it is $\ddot{\theta} + \frac{g}{l}\theta = 0$ ...(9)

Example 6.13 Derive the Hamilton's equations of motion for a spring-mass system which oscillates on a frictionless horizontal plane. (The spring is unstretched at $x = 0$.)

Sol. The spring-mass system is shown in Fig. 6.12W. The kinetic energy of the mass m is given by

$$T = \frac{1}{2}m\dot{x}^2$$

where x is its displacement from its mean position at any time t.

The potential energy is given by

$$V = \frac{1}{2}kx^2$$

where k is the spring constant.

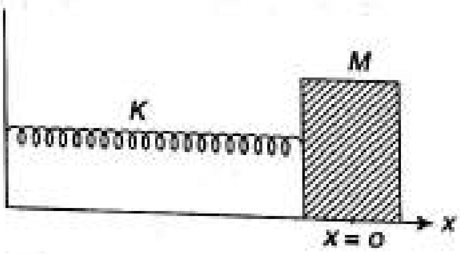


Fig. 6.12W A spring-mass system. If the mass m is pulled or pushed and then released, it executes SHM along the x axis.

$\therefore$ the lagrangian L of the system is given by

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$\therefore \quad p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\text{or,} \quad \dot{x} = \frac{p_x}{m} \qquad \text{...(1)}$$

Now, the hamiltonian is given by

$$H = \sum_i p_i \dot{q}_i - L = p_x \dot{x} - \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2$$

$$\text{or,} \quad H = p_x \cdot \frac{p_x}{m} - \frac{1}{2} m \cdot \left(\frac{p_x}{m}\right)^2 + \frac{1}{2} kx^2 \quad \text{[by Eq. (1)]}$$

$$\text{or,} \quad H = \frac{p_x^2}{2m} + \frac{1}{2} kx^2$$

$$\text{or,} \quad \frac{\partial H}{\partial p_x} = \frac{p_x}{m}, \frac{\partial H}{\partial x} = kx$$

Now, the Hamilton's equations are given by

$$\dot{x} = \frac{\partial H}{\partial p_x} \quad \text{or,} \quad \dot{x} = \frac{p_x}{m}$$

$$\text{and} \quad \dot{p}_x = -\frac{\partial H}{\partial x} \quad \text{or,} \quad \dot{p}_x = -kx$$

These are the desired Hamilton's equations of motion.

Example 6.14 Derive the Hamilton's equations of motion for a particle which is acted upon by a central force.

Sol. The particle which is being acted upon by a central force is shown in Fig. 6.13W.

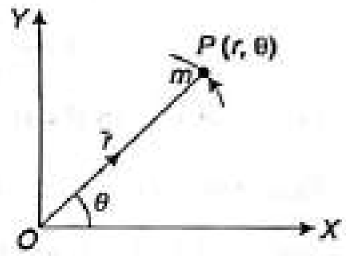


Fig. 6.13W The particle P with mass m is acted upon by a central force.

The lagrangian of the system is given by

$$L = T - V(r) = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \qquad \text{...(1)}$$

For an attractive system the potential $V(r)$ is given by

$$V(r) = -\frac{k}{r} \qquad \text{...(2)}$$

$$\therefore \quad L = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r} \qquad \text{...(3)}$$

Now, from Eq. (3), we get

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \text{...(4)}$$

$$\text{and} \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2 \dot{\theta} \qquad \text{...(5)}$$

$\therefore$ the hamiltonian H is given by

$$H = \sum_i p_i \dot{q}_i - L = p_r \dot{r} + p_\theta \dot{\theta} - L$$

Now, using Eqs (4) and (5), we get

$$H = (m\dot{r})\,\dot{r} + (mr^2\dot{\theta})\,\dot{\theta} - L$$

or, $$H = m\dot{r}^2 + mr^2\dot{\theta}^2 - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{k}{r}$$

or, $$H = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{k}{r}$$

or, $$H = \frac{1}{2m}(m\dot{r})^2 + \frac{1}{2mr^2}(mr^2\dot{\theta})^2 - \frac{k}{r}$$

or, $$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{k}{r} \qquad ...(6)$$

The hamiltonian equations for the generalized coordinates q_j and p_j are:

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \quad \text{and} \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

In the present case, the generalized coordinates are r, θ, p_r and p_θ.

$\therefore$ the hamiltonian equations for r, θ, p_r and p_θ are given by

$$\dot{r} = \frac{\partial H}{\partial p_r}, \quad \dot{\theta} = \frac{\partial H}{\partial p_\theta}, \quad \dot{p}_r = -\frac{\partial H}{\partial r} \quad \text{and} \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta}$$

Now, using Eq. (6), we get

$$\dot{r} = \frac{p_r}{m},$$

$$\dot{\theta} = \frac{p_\theta}{mr^2},$$

$$\dot{p}_r = \frac{p_\theta}{mr^3} - \frac{k}{r^2},$$

and $$\dot{p}_\theta = 0$$

These are the required hamiltonian equations.

Example 6.15 Find the lagrangian as well as hamiltonian equations of motion for a compound pendulum.

Sol. The compound pendulum has been shown in Fig. 6.14W. It is pivoted at the point O and capable of oscillating about this point. When the compound pendulum oscillates, its center of mass gets displaced. Let at time t its angular displacement be θ.

$\therefore$ its kinetic energy, $T = \frac{1}{2}I\dot{\theta}^2$

and its potential energy, $V = -mgl\cos\theta$

[Here, O is the reference point]

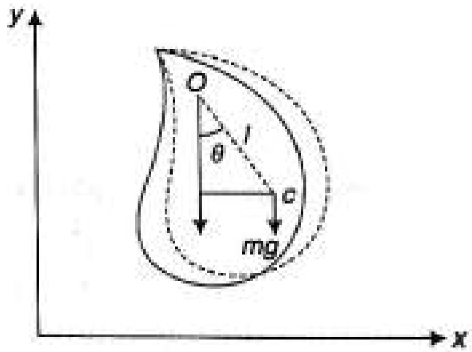


Fig. 6.14W Motion of a compound pendulum.

where $OC = l$ and I is the moment of inertia of the body about the axis of rotation.

∴ the lagrangian L is given by

$$L = T - V = \frac{1}{2} I \dot{\theta}^2 + mgl \cos\theta \qquad ...(1)$$

The Lagrange's equation of motion is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

or,
$$\frac{d}{dt}(I\dot{\theta}) + mgl \sin\theta = 0$$

or,
$$\ddot{\theta} + \frac{mgl}{I}\sin\theta = 0$$

If θ is very small then we can write $\sin\theta \approx \theta$. So the equation reduces to

$$\ddot{\theta} + \frac{mgl}{I}\theta = 0 \qquad ...(2)$$

This equation (i.e., Eq. (2)) is the Lagrange's equation of motion. It is a simple harmonic motion and its time period is given by

$$T = 2\pi\sqrt{\frac{I}{mgl}}$$

Differentiating Eq. (1) with respect to $\dot{\theta}$, we get

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}$$

or,
$$\dot{\theta} = \frac{p_\theta}{I}$$

∴ the hamiltonian H is given by

$$H = \sum_j p_j \dot{q}_j - L = p_\theta \dot{\theta} - L$$

or,
$$H = p_\theta \dot{\theta} - \frac{1}{2} I\dot{\theta}^2 - mgl\cos\theta \quad \text{[by Eq. (1)]}$$

or,
$$H = p_\theta \frac{p_\theta}{I} - \frac{1}{2} I . \frac{p_\theta^2}{I^2} - mgl\cos\theta$$

$$H = \frac{p_\theta^2}{2I} - mgl\cos\theta$$

∴ the hamitonian equations of motion are given by

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{I} \quad \text{or} \quad p_\theta = I\dot{\theta} \qquad ...(3)$$

and
$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgl\sin\theta \qquad ...(4)$$

Equations (3) and (4) are the desired hamiltonian equations of motion.

Differentiating Eq. (3) with respect to time and combining with Eq. (4), we get

$$I\ddot{\theta} = -mgl\sin\theta$$

or, $$\ddot{\theta}+\frac{mgl}{I}\sin\theta=0$$

If θ is very small then $\sin\theta \approx \theta$

$\therefore$ $$\ddot{\theta}+\frac{mgl}{I}\theta=0 \quad \Rightarrow T=2\pi\sqrt{\frac{I}{mgl}}$$

Example 6.16 Derive the hamiltonian and hamiltonian equation of motion for a particle falling freely under the influence of gravity.

Sol. Let us consider a body of mass m which is falling freely under the influence of gravitational force. The body initially remains at point P. Then it falls freely and at time t it reaches the point Q [Fig. 6.15W]. Let

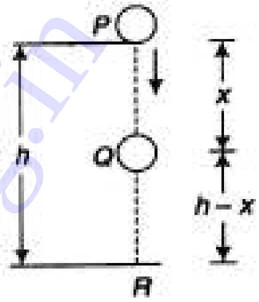


Fig. 6.15W A freely falling body under gravity.

$$PQ=x, \quad \therefore QR=h-x$$

The potential energy at Q is given by

$$V=mg\,(h-x)$$

and kinetic energy at Q is given by

$$T=\frac{1}{2}m\left(\frac{dx}{dt}\right)^2=\frac{1}{2}m\dot{x}^2$$

$\therefore$ the lagrangian of the system is given by

$$L=T-V=\frac{1}{2}m\dot{x}^2-mg\,(h-x)$$

$\therefore$ $$p_x=\frac{\partial L}{\partial \dot{x}}=m\dot{x} \quad \text{or} \quad \dot{x}=\frac{p_x}{m}$$

So, the hamiltonian H is given by

$$H=p_x\dot{x}-L=p_x\frac{p_x}{m}-\left[\frac{1}{2}m\frac{p_x^2}{m^2}-mg\,(h-x)\right]$$

or, $$H=\frac{p_x^2}{m}-\frac{p_x^2}{2m}+mg\,(h-x)$$

or, $$H=\frac{p_x^2}{2m}+mg\,(h-x)$$

$\therefore$ the hamiltonian equations of motion are given by

$$\dot{p}_x=-\frac{\partial H}{\partial x}=mg \quad \text{and} \quad \dot{x}=\frac{\partial H}{\partial p_x}=\frac{p_x}{m}$$

Now, $$\dot{x}=\frac{p_x}{m} \quad \Rightarrow p_x=m\dot{x}$$

$\therefore$ $$mg=m\ddot{x} \quad [\because \dot{p}_x=mg]$$

$\therefore$ $$\ddot{x}-g=0$$

This is the expected equation of motion for a freely falling body.

Review Exercises

Part 1: Multiple Choice Questions

1. If the constraint relations are independent of velocity, then the constraints are called
   (a) scleronomic (b) conservative (c) bilateral (d) holonomic
2. A system is called scleronomous, if the constraint equations do not contain
   (a) velocity (b) time (c) velocity and time (d) displacement
3. When a particle placed on the surface of a sphere moves on the surface, the constraint is
   (a) holonomic (b) non-holonomic (c) scleronomic (d) rheonomic
4. The constraint in case of rigid body motion is
   (a) holonomic (b) non-holonomic (c) rheonomic (d) None of these
5. The constraint associated with the motion of a pendulum is
   (a) dissipative (b) holonomic (c) rheonomic (d) All of these
6. The constraint involving rolling without sliding is
   (a) holonomic (b) non-holonomic (c) rheonomic (d) conservative
7. The characteristics of generalized coordinates are that
   (a) they are independent of each other
   (b) they are necessarily spherical coordinates
   (c) they may be cartesian coordinates
   (d) Both (a) and (c)
8. The dimensions of generalized momentum
   (a) are those of linear momentum
   (b) are those of angular momentum
   (c) may be either those of linear or angular momentum
   (d) None of these
9. If the lagrangian is free of a particular generalized coordinate q_j, then q_j is called
   (a) superfluous (b) cyclic (c) cartesian (d) spherical
10. The number of generalized coordinates necessary to describe the motion of a particle in a circular wire of radius a is/are
    (a) two (b) three (c) one (d) None
11. The virtual displacement $\delta \bar{r}_i$
    (a) does not involve time
    (b) may involve time
    (c) may or may not involve time
    (d) None of these
12. The double pendulum requires n generalized coordinates to be described where n is
    (a) one (b) three (c) two (d) none of these
13. A generalized coordinate q_j is absent in the lagrangian of a system. The corresponding conserved quantity is
    (a) energy (b) velocity (c) momentum (d) force
14. When a system is conservative, the hamiltonian is given by
    (a) $H = T - V$ (b) $H = T + V$ (c) $H = V - T$ (d) $H = TV$

15. If a generalized coordinate q_j is cyclic in the lagrangian then in the hamiltonian it is
    (a) not cyclic (b) may or may not be cyclic
    (c) cyclic (d) unpredictable
16. The hamiltonian of a system is a function of
    (a) q_j, p_j, t (b) $p_j, \dot{q}_j, t$ (c) $q_j, \dot{q}_j, t$ (d) $p_j, \dot{q}_j, t$
17. The lagrangian of a system is a function of
    (a) q_j, p_j, t (b) $q_j, \dot{q}_j, t$ (c) $\dot{p}_j, q_j, t$ (d) $p_j, \dot{q}_j, t$
18. A system consists of three point masses and forms a triangle. If the mutual distances between the masses do not vary with time, then the degrees of freedom of the system are
    (a) zero (b) three (c) six (d) None of these
19. A rigid body has constraints classified in the following group:
    (a) Rheonomic and holonomic (b) Rheonomic and non-holonomic
    (c) Scleronomic and holonomic (d) Scleronomic and non-holonomic

    [WBUT 2007]
20. The generalized coordinate can be of any dimension. However,
    (a) the product of the generalized coordinate and the generalized momentum is always ml^2t
    (b) the dimension of the work done depends on the dimension of the generalized coordinate
    (c) the generalized momentum is always of the dimension ml/t^2
    (d) the generalized force is always of the dimension ml/t^2 [WBUT 2007]
21. If the constraints are time dependent, then the relation between the hamiltonian H and the total energy E is given by
    (a) $H = E$ (b) $H \neq E$ (c) $H = E - V$ (d) $H = E - T$

[Ans. 1 (d), 2 (b), 3 (b), 4 (a), 5 (d), 6 (b), 7 (d), 8 (c), 9 (b), 10 (c), 11 (a), 12 (c), 13 (c), 14 (b), 15 (c), 16 (a), 17 (b), 18 (c), 19 (c), 20 (a), 21 (b)]

Short Questions with Answers

1. Discuss the nature of constraints and specify the force of constraint in case of a pendulum with variable length and a particle moving on an ellipsoidal under gravity.

Ans. In case of a pendulum with variable length, the length can be expressed as $l = l(t)$. And its equation of constraint can be expressed as $|\bar{r}_1 - \bar{r}_2|^2 = l^2(t)$ which is dependent on time. So, the constraint is rheonomic and holonomic. The force of constraint is the tension in the string.

A particle which is moving on the surface of an ellipsoid under gravity will leave the ellipsoid after reaching a certain point on the ellipsoid. Hence, the constraint is non-holonomic. And the force of reaction of the ellipsoidal surface on the particle is the force of constraint.

2. What difficulties do the constraints introduce while solving mechanical problems?

Ans. The difficulties introduced by the constraints while solving mechanical problems are of two kinds. They are as follows:

(a) The equation representing the constraints for a system of 'n' particles with position vectors $\bar{r}_1$, $\bar{r}_2, \bar{r}_3, \ldots, \bar{r}_n$ is

$$f(\bar{r}_1, \bar{r}_2, \ldots \bar{r}_n, t) = 0$$

Usually, these n coordinates are not all independent but they are connected by the aforementioned equation. So, the n equations which are connected by this constraint equation are not all independent. Hence, in order to have a solution of the problem, these equations are to be written again by considering the effect of the constraints. This is the first difficulty.

(b) In many situations, the nature of the forces of constraint remain unknown and they belong to the unknown parameters of the problems. These unknown forces of constraints introduce some complications in the process of solution of the problem. This is the second difficulty.

3. What is meant by degrees of freedom of a dynamical system?

Ans. The term *degrees of freedom* of a dynamical system means the number of independent ways in which that dynamical system can execute motion without violating the constraint imposed on the said system. It is defined as the minimum number of independent coordinates which can specify the position of the said dynamical system completely. It can also be defined as the number of squared terms in the expression of the kinetic energy of the system.

If a system has n particles and k constraints, then the minimum number of coordinates to specify its position is $(3n - k)$. Hence, the degrees of freedom of the system of n particles is given by

$$f = 3n - k$$

4. What do you mean by generalized coordinates? State their advantages.

Ans. If a rigid body consists of n particles and there are k constraints associated with the body then its degrees of freedom is

$$f = 3n - k$$

This suggests that the system can be described by a minimum number of f independent coordinates where the effects of the constraints are included. These f independent coordinates q_j (j = 1, 2, 3, ..., f) are called the generalized coordinates. Use of generalized coordinates gives us the following advantages:

(i) The generalized coordinates may not essentially be cartesian, spherical, polar, etc. Thus, the dependence on the description of the configuration of the dynamical system is eliminated.

(ii) The individual variation (Δq_j) may be considered as the generalized coordinates q_j are all independent.

(iii) A dynamical system described in terms of generalized coordinates may be treated as a free system.

5. What are the limitations of newtonian mechanics?

Ans. See Section 6.1.1.

6. What are system-point and configurations space?

Ans. See Section 6.2.1.

7. State characteristics of lagrangian equations.

Ans. See Section 6.2.4.1.

8. What is phase space? Describe.

Ans. See Section 6.3.1.

9. What is meant by cyclic coordinate?

Ans. The lagrangian (L) of a system is, in general, a function of all the generalized coordinates (q_j) as well as time t, i.e.,

$$L = L(q_1, q_2, q_3, \ldots, q_f; t)$$

If a generalized coordinate remains absent in the expression of the lagrangian (i.e., it does not appear explicitly) then that coordinate is called a cyclic coordinate. Hence, for a cyclic coordinate, we get

$$\frac{\partial L}{\partial q_j} = 0$$

10. What are the advantages of the hamiltonian formulation over the lagrangian formulation?

Ans. (i) In lagrangian formulation, the two variables q_j and $\dot{q}_j$ are not given equal status since $\dot{q}_j$ is simply the time derivative of q_j. But in case of hamiltonian formulation, both the coordinate (q_j) and the momentum (p_j where $p_j = m_j\dot{q}_j$) are given equal status. This provides a frequent freedom of choosing coordinates and momenta.

(ii) The fact that, for a conservative system, the numerical value of the hamiltonian (H) is equal to the total energy is very important in case of energy changes in atoms and molecules.

(iii) Assigning equal status to q_j and p_j in hamiltonian formulation provides a convenient basis for the development of quantum mechanical theories as well as statistical mechanics.

(iv) While quantizing a dynamical system, the knowledge of the hamiltonian is extremely important. While we set up Schrödinger equation in wave mechanics, we simply replace generalized momenta by corresponding differential operators.

11. State the conservation theorem of generalized momentum.

Ans. The conservation theorem of generalized momentum states that the generalized momentum corresponding to a cyclic coordinate is constant.

If q_j be a cyclic coordinate then one can find that $\frac{\partial L}{\partial q_j} = 0$

And the lagrangian equations of motion for a conservative system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

reduces to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0$$

This imples $\qquad \frac{\partial L}{\partial \dot{q}_j} = \text{constant}$

The pair (q_j, p_j) is called the canonical or conjugate variables.

Part 2: Descriptive Questions

1. State the principle of virtual work. Derive the mathematical expression for the principle of virtual work.
2. State the principle of D'Alembert. Derive a mathematical expression to represent D'Alembert's principle.
3. What is the advantages of using 'generalized coordinates'? Obtain the expression of 'generalized force'.
4. What are the advantages of Lagrange's equation over Newton's equation of motion?

5. Obtain Lagrange's equation of motion from D'Alembert's principle. Discuss the effect of dissipative forces.
6. Obtain Hamilton's canonical equations by using the relations between the hamiltonian and the lagrangian of a system.
7. What do you mean by cyclic coordinates and superfluous coordinates?
8. From the lagrangian obtain hamiltonian. Is the hamiltonian equal to total energy? Is this equality of hamiltonian and total energy valid in general?
9. What do you mean by configuration space and degrees of freedom of a dynamical system? What are generalized coordinates and what are their advantages? Write down the equation of transformation from generalized coordinates to cartesian coordinates.
10. Write down the hamiltonian and derive the hamiltonian equation of motion of a simple pendulum.
11. Write down the hamiltonian and derive the hamiltonian equations of motion for a compound pendulum.
12. Derive the expression of kinetic energy of a system of particles in terms of generalized velocities.
13. Show that the kinetic energy for a scleronomous system is a homogeneous quadratic function of the generalized velocities.

Part 3: Numerical Problems

1. Use Lagrange's equations to find the equations of motion of a compound pendulum which oscillates in a vertical plane about a fixed horizontal axis. Hence find the period of oscillation of the compound pendulum.
2. A particle of mass m moves in a two-dimensional potential $V(x, y) = -k_1x^2 + k_2x^2$. Write down the Lagrange's equation of motion.
3. The bob of a simple pendulum moves in a horizontal circle. Find the angular frequency of the circular motion in terms of angle θ and the length l of the rod.
4. Find out the degrees of freedom of the CH_4 molecule.
5. Use Lagrange's equation of motion to determine the motion of a mass m which slides without friction down an inclined plane.
6. The kinetic energy of an electron of charge e moving in the field of the nucleus of charge + 2e in spherical polar coordinates is given by $T = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin\theta^2 \dot{\phi}^2)$ and potential energy V by $V = \frac{Ze^2}{r}$. Obtain the Lagrange's equation of motion. What is the hamiltonian of the system?
7. The lagrangian of a system is given by $L = \frac{1}{2} \alpha \dot{q}^2 - \frac{1}{2} \beta q^2$, where α and β are constants. Obtain lagrangian equation of motion and also find the hamiltonian of the system.

CHAPTER 7

Quantum Mechanics

7.1 INTRODUCTION

In the book *Basic Engineering Physics*, the old quantum theory has been discussed. This theory was a collection of the results of researches in this field between the years 1900 and 1925, which predated the modern quantum mechanics. The theory was never complete or self-consistent, but was a collection of heuristic prescriptions, which are now understood to be the first quantum corrections to classical mechanics.

The motions of particles which are subjected to external forces are discussed in newtonian (or classical) mechanics. And in newtonian mechanics it is also taken for granted that one can measure correctly the properties of particles like mass, position, velocity, acceleration, etc. Of course, this idea is valid in our common sense which we acquire through the experiences of our daily life. Classical mechanics is able to explain correctly the dynamical behavior of the material bodies in terms of the values it predicts for observable quantities and the values of the same quantities which are measured experimentally.

On the other hand, quantum mechanics also deals with the values of observable quantities related to dynamical systems but in this case, the term 'observable quantity' bears different significance in the light of the uncertainty principle in the atomic realm. The uncertainty principle plays an important role in quantum mechanics. According to quantum mechanics, simultaneous measurement of the position and momentum of a moving particle with precision is impossible while according to classical mechanics, both the position and momentum of a moving particle can be ascertained accurately at every instant of time. And for this reason, classical mechanics is considered to be deterministic, whereas quantum mechanics is considered to be probabilistic. For example, according to classical mechanics, the radius of the electronic orbit in the ground state of hydrogen atom is exactly 5.3×10^{-11} m while according to quantum mechanics, this is most probable value of the said radius.

Classical mechanics is a special case of quantum mechanics; in other words, it is an approximate version of quantum mechanics. The certainties which one comes across in classical mechanics are only apparent. The agreement which is observed between the predicted value and the experimental value of an object is due to the fact that the macroscopic material bodies consist of a large number of individual atoms and the departure of the collective behavior of the atoms from their individual behavior is unnoticeable. Thus, unlike classical mechanics, quantum mechanics includes two sets of physical principles; one for the macroscopic world and the other one for the microscopic world.

Quantum mechanics was not developed uniquely through a single formulation like classical mechanics. Rather initially it was developed through two different formulations by two renowned physicists, namely, Werner Heisenberg and Erwin Schrödinger. In 1925, on one hand, Heisenberg formulated the quantum theory in terms of observable quantities alone (like intensity and frequencies of spectral lines) to replace position, momentum and other dynamical variables of classical mechanics by matrices, while keeping the form of the equations of motion superficially the same as in classical mechanics. This formulation of quantum mechanics is known as *matrix mechanics*. On the other hand, in 1926, Schrödinger formulated the quantum theory on the basis of de Broglie hypothesis of matter waves even before it was experimentally verified by Davison and Germer in 1927. Schrödinger's quantum theory is known as wave mechanics. He proposed that the wave function Ψ describing the matter waves satisfies a partial differential equation, and gave the prescription to write down the equation for any particular system of particles. Despite the vastly different appearance of the two aforesaid theories, it was very soon recognised that they are completely equivalent. Schrödinger's equation is applicable only to non-relativistic particles. So, a new wave equation, which meets the requirements of the theory of relativity, was formulated by Paul Adrien Maurice Dirac in 1928 which is known as relativistic wave mechanics. Out of the two alternative forms of quantum mechanics — wave mechanics and matrix mechanics — it is the former one which lends itself more easily to the solution of wide variety of practical problems. For this reason, we will be dealing with Schrödinger's version of quantum mechanics, i.e., wave mechanics in this chapter. It was again Dirac who ultimately set up a general formalism of quantum mechanics in the year 1930 to unify the two formulations of Heisenberg and Schrödinger.

7.2 WAVE FUNCTION, PROBABILITY AND PROBABILITY DENSITY

We now know well that nature is strikingly symmetrical in many ways. Our observable universe is entirely composed of matter and energy. Light is one of the various forms of energy. We have learnt through Compton and photoelectric effects that light has dual, wave as well as particle nature. So, one can expect from the concept of symmetry of nature that matter also may have dual, wave as well as particle nature (i.e., character). The particle nature of matter is well known to every one. Louis de Broglie proposed the wave nature of matter in 1924 in his hypothesis which was later named after him and Davison and Germer confirmed this experimentally in case of electrons in the year 1927. When light propagates through free space, its motion in space and time can be represented by the electromagnetic wave equation as given below:

$$\nabla^2 \bar{\xi} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\bar{\xi} \qquad \text{...(7.1)}$$

where c is the velocity of light in the vacuum and $\bar{\xi}$ is either the electric vector $\overline{E}(\bar{r}, t)$ or the magnetic vector $\overline{B}(\bar{r}, t)$ of the light wave. So, again from the concept of symmetry of nature one can expect a similar equation to represent the motion of a wave-particle in space and time. This is what was done by Erwin Schrödinger in 1926. The idea that the stationary states of an electron in an atom corresponding to the standing matter waves was taken up and used by him as the foundation of wave mechanics. He developed an equation, now known as Schrödinger's equation, to represent the motion of the matter-wave (in space and time) related to a free particle which is given below:

$$-\frac{\hbar^2}{2m}\nabla^2\,\Psi(\bar{r}, t) = i\,\hbar\frac{\partial \Psi}{\partial t}(\bar{r}, t) \qquad \text{...(7.2)}$$

where h is Planck's constant and $\hbar = h/(2\pi)$ and $\Psi(\bar{r}, t)$ is a function of space and time which satisfies the Eq. (7.2). The function $\Psi(\bar{r}, t)$ is known as Schrödinger's wave function (i.e., the function which represents Schrödinger's matter wave in space and time).

7.2.1 Probability and Probability Density

When the double-slit experiment is carried out with light, such as the Young's double-slit experiment, one considers the superposition of the secondary waves arising from the two slits on the screen of recording of the interference pattern. The light intensity on this screen is determined by the square of the amplitude of the wave, which is formed as a result of superposition of the two secondary waves coming from the two slits. The intensity of light on the screen is also proportional to the number of photons reaching the screen per unit area. The probability (or chance) of finding a photon at a given point on the screen depends on the intensity of light wave at the point. This intensity (as stated above) is proportional to the square of the resultant amplitude of the waves (i.e., the electric field) at the point. Hence, the probability of finding a photon on a given point is proportional to the square of the resultant electric field at the point. Now comparing the duality (i.e., dual behavior) of photon with that of a material particle (e.g., an electron) one can arrange a similar double-slit experiment with monoenergetic material particles (i.e., electrons) as shown in Fig. 7.1.

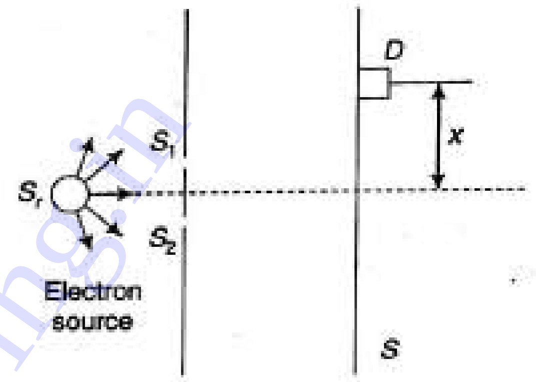


Fig. 7.1 Electron double-slit set-up for interference of electron waves. S_r: source of electrons, S_1, S_2: double slits, S: the screen and D: the detector.

By putting forward similar arguments in case of electron double-slit experiment as we have done in case of the double-slit experiment of photon (i.e., Young's double-slit experiment), we can conclude that the probability of finding an electron per unit area of the screen S of Figure 7.1 is proportional to the square of the wave function (or state function) Ψ for the electron.

If one compares Eqs (7.1) and (7.2), the logic which has been put forward above will be more clear. The wave function ξ (and hence the electric field vector $\bar{E}$) represents wave behavior of the photon while Schrödinger's wave function Ψ represents the wave behavior of the electron. So, if $|\bar{E}|^2$ gives the probability of finding the photon on the screen of Young's double-slit experiment, then $|\Psi|^2$ will give the probability of finding the electron on the screen of the electron double-slit experiment [Fig. 7.1].

Therefore in general, we can state that if $\Psi(\bar{r}, t)$ represents the wave function of a material particle, then $|\Psi(\bar{r}, t)|^2$ will represent the probability of finding that particle at point (x, y, z) at time t (since $\bar{r} = \hat{i}x + \hat{j}y + \hat{k}z$). The wave function $\Psi(\bar{r}, t)$ plays the role of probability amplitude. This function is generally a complex quantity. Since probabilities are real positive numbers (varying from +0 to +1), the probability of finding a particle, at a given position (x, y, z) at a given time (t), is expected to be proportional to $|\Psi|^2$. And it will be equal to $|\Psi|^2$ when the wave function Ψ is a normalized one.

Probability density of a particle

It is the probability of finding the concerned particle in unit volume of a given space at a given instant of time.

It is usually expressed as the square of the absolute value of the wave function when it is normalized. If the normalized wave function Ψ_n is complex, then

$$|\Psi_n|^2 = (\Psi_n)(\Psi_n^*)$$

where Ψ_n^* is the complex conjugate of Ψ_n.

So, the probability density P can be expressed as follows:

$$P = \Psi_n^*(\bar{r}, t)\, \Psi_n(\bar{r}, t) = |\Psi_n(\bar{r}, t)|^2 \qquad ...(7.3)$$

Therefore, the probability of finding a particle having normalized wave function Ψ_n in a volume element dV at the point (x, y, z) at time t is given by

$$P_V = \Psi_n^* \Psi_n \, dV$$

Since at a given point of time, the particle must be somewhere in the space, the total probability P_t to find the particle in space must be equal to unity.

$$P_t = \int_{-\infty}^{+\infty} \Psi_n^* \Psi_n \, dV = 1 \qquad \text{...(7.4)}$$

Now, if the particle always lies in a point of a straight line, then the probability of finding it within a distance dx is given by

$$P \, dx = \Psi_n^* \Psi_n \, dx$$

Since the particle must be somewhere on the straight line (in this case the x axis), the total probability of finding it on the line (i.e., total probability for one-dimensional motion) is given by

$$\int_{-\infty}^{+\infty} \Psi_n^* \Psi_n \, dx = 1 \qquad \text{...(7.5)}$$

7.3 NORMALIZATION OF WAVE FUNCTION AND ORTHOGONALITY OF WAVE FUNCTIONS

One should note that the wave function $\Psi(x, t)$ which describes the complete space-time behavior of a particle in one-dimensional motion has an appreciable amplitude in a region where the particle is likely to be found with a greater probability. One can assume that the quantity $|\Psi(x, t)|^2 \, dx = \Psi^*(x, t) \, \Psi(x, t) \, dx$ is proportional to the probability of finding the particle in the interval between x and $x + dx$ at the time t where $\Psi^*(x, t)$ is the complex conjugate of $\Psi(x, t)$. The total probability of finding the particle anywhere in the one-dimensional space is given by

$$P_t \propto \int_{-\infty}^{+\infty} |\Psi(x, t)|^2 \, dx$$

Let us define the position probability density (or simply probability density) as

$$P(x, t) = |\Psi(x, t)|^2 \Big/ \left[\int_{-\infty}^{+\infty} |\Psi(x, t)|^2 \, dx \right] \qquad \text{...(7.6)}$$

Hence, the total probability will be

$$\begin{aligned} P_t &= \int_{-\infty}^{+\infty} P(x, t) \, dx \\ &= \int_{-\infty}^{+\infty} |\Psi(x, t)^2| \, dx \Big/ \left[\int_{-\infty}^{+\infty} |\Psi(x, t)|^2 \, dx \right] \\ &= 1 \end{aligned} \qquad \text{...(7.7)}$$

This is what should be, since the total probability P_t must be always equal to unity.

Since $|\Psi(x, y)|^2$ is necessarily positive, Eq. (7.6) indicates that the probability density $P(x, t)$ is always positive. This is consistent with the expected behavior and definition of probability.

If $\Psi(x, t)$ is multiplied by one complex constant quantity N so that

$$\Psi_n(x, t) = N \, \Psi(x, t) \quad \text{where } \Psi_n(x, t) \text{ satisfies the following relation}$$

$$\int_{-\infty}^{+\infty} |\Psi n(x, t)|^2 \, dx = |N|^2 \int_{-\infty}^{+\infty} |\Psi(x, t)|^2 \, dx = 1 \qquad ...(7.8)$$

Then $\Psi_n(x, t)$ is said to be the normalized wave function and N is called the norm of the unnormalized wave function $\Psi(x, t)$ required to normalize it.

From Eq. (7.8), one can have

$$|N|^2 = 1 \Big/ \int_{-\infty}^{+\infty} |\Psi(x, t)|^2 \, dx \qquad ...(7.9)$$

N is also called the normalization constant. And obviously for a wave function to be normalizable $\int |\Psi(x, t)|^2 \, dx$ or in general $\int |\Psi(\bar{r}, t)|^2 \, dV$ over all space must be finite. This condition is known as the square-integrability of the wave function.

Orthogonality Let us consider two wave functions $\Psi_m(x, t)$ and $\Psi_n(x, t)$ which satisfy Schrödinger's Eq. (7.2) with $y = z = 0$.

In case of this pair of wave functions one can show that

$$\int_{-\infty}^{+\infty} \Psi_m(x, t) \, \Psi_n(x, t) \, dx = 0 \qquad ..(7.10)$$

where $m \neq n$.

This property of two wave functions is called **orthogonality** of these functions. To be physically meaningful, both the wave functions must satisfy the condition of nomalization.

i.e., $$\int_{-\infty}^{+\infty} |\Psi_m(x, t)|^2 \, dx = \int_{-\infty}^{+\infty} \Psi_m^*(x, t) \, \Psi_m(x, t) \, dx = 1$$

and $$\int_{-\infty}^{+\infty} |\Psi_n(x, t)|^2 \, dx = \int_{-\infty}^{+\infty} \Psi_n^*(x, t) \, \Psi_n(x, t) \, dx = 1$$

7.4 PHYSICAL SIGNIFICANCE AND INTERPRETATION OF WAVE FUNCTION

As the wave function of a material particle represents the physical behavior of the particle while it is in motion, it must have some physical significance, otherwise it will not be able to represent the dynamical behavior of the particle. Some important physical significances of the wave function are listed below:

(a) Wave function gives information regarding the space-time behavior of the concerned particle.

(b) The square of the wave function gives the measure of the probability of finding the particle about a position.

(c) The magnitude of the wave function is large in regions where the probability of finding the particle is high and small in the region where the probability of finding the particle is low.

(d) The wave function representing a moving particle must be continuous and single valued at each point of space. Also its value at each point of space must be finite.

Interpretation of wave function One can give a physical interpretation of the wave function as follows: During the propagation of any wave some physical quantities vary with respect to space and time. As for example, when a light wave propagates, the electric and the magnetic fields of it vary in space and time and when a sound wave propagates the pressure in the medium of propagation varies in space and time. Similarly, when a particle moves the matter wave related to it (i.e., the de Broglie wave) also propagates. And during the time of propagation of this wave, some physical quantities also keep on varying in space and time. All the information about these varying physical quantities are carried by the wave function Ψ of the matter wave. Ψ is generally a complex quantity and is a function of position coordinates (x, y, z) as well as time (t). In case of a moving body, the value of Ψ is related to the probability (i.e., chance) of finding the particle at a particular point in space (x, y, z) at a given time t. The wave function Ψ can also give information regarding the future behavior of the particle. The wave function Ψ actually contains all the information which the uncertainty principle allows one to know about the related particle but this wave function cannot be measured directly. As stated earlier, the probability of any event must be a real positive quantity whose value lies within the range between 0 and 1. Since wave function is generally a complex quantity which can have both positive and negative values, the probability related to the existence of the particle at a point in space and time can be expressed by $|\Psi|^2$. This is because for any allowed value of Ψ, $|\Psi|^2$ will always be positive. In order to understand this, one can express Ψ as follows:

$$\Psi = a + ib$$

where i represents the square root of (-1), i.e., $i = \sqrt{-1}$ and a and b are two real quantities which can be expressed as the function of position coordinates (x, y, z) and time t. Now, the complex conjugate of Ψ (denoted by Ψ^*) is given by

$$\Psi^* = a - ib$$

So, the square of the modulus of Ψ can be written as

$$|\Psi|^2 = (\Psi^*)(\Psi) = (a - ib)(a + ib)$$
$$= a^2 + b^2$$

The value of $a^2 + b^2$ is always real and positive. Thus, $|\Psi|^2$ is always a positive real quantity. If Ψ_n be normalized value of Ψ, then $|\Psi|^2$ represents the probability density of the particle. For a particle that is represented by the wave function Ψ (or the normalized wave function Ψ_n), the probability of experimentally observing it at the position (x, y, z) at time t is given by $|\Psi_n|^2$. At any point in space where $|\Psi_n| > 0$, there is a definite probability of detecting it. It was Max Born who first gave this statistical interpretation of the wave function in 1926.

Let us now consider an ensemble of a large number of identical particles where each of them is described by the same normalized wave function Ψ_n. Then the actual density of particles at position (x, y, z) at a given time t is equal to $|\Psi_n|^2$.

If one assumes that a particle is moving along the x axis, then its wave function is a function of x and t only. If this function is normalized and denoted by $\Psi_n(x, t)$, then the probability density $P(x, t)$ is taken to be $|\Psi_n(x, t)|^2$. And the probability of finding the particle at a particular time t in a region between x and $x + dx$ is equal to $|\Psi_n|^2\, dx$. This can be expressed as follows:

$$p(x, t)\,dx = \Psi_n^*(x, t)\; \Psi_n(x, t)\,dx$$
$$= |\Psi_n(x, t)|^2\,dx \qquad ...(7.11)$$

where the probability density $p(x, t)$ is defined as the probability per unit length (along the x axis for finding the particle around the point $(x, 0, 0)$ at the time t.

The probability for finding the particle in the region between the points $(x_1, 0, 0)$ and $(x_2, 0, 0)$ is given by

$$p(x_1, x_2, t) = \int_{x_1}^{x_2} p(x, t)\,dx$$

$$= \int_{x_1}^{x_2} \Psi_n^*(x, t)\; \Psi_n(x, t)\,dx$$

or,
$$p(x_1, x_2, t) = \int_{x_1}^{x_2} |\Psi_n(x, t)|^2 dx \qquad ...(7.12)$$

In a three-dimensional case, the probability of finding the particle in a small volume dV around the point (x, y, z) at time t is given by

$$p(x, y, z, t) = \int_V |\Psi_n(x, y, z, t)|^2 dV \qquad ...(7.13)$$

7.5 OPERATORS IN QUANTUM MECHANICS

The word 'operation' means action, so an operator is an actor which can change a physical quantity having taken some action on it. Basically **an operator is a mathematical rule that can change a given function into a new function.**

If o be an operator, then it is mathematically expressed as $\hat{o}$. As an example, if $\hat{o}$ be the differential operator $\left(\frac{d}{dx}\right)$, then when it operates on the function $f(x) = 2x^2 + 3x + 1$ one gets the following result:

$$\frac{d}{dx}f(x) = \frac{d}{dx}(2x^2 + 3x + 1) = 4x + 3 = f_1(x) \text{ (say)}$$

An operator can be either linear or complex in nature. If $\hat{\alpha}$ be a linear operator then it exhibits the following properties:

$$\hat{\alpha}(cf) = c\hat{\alpha}(f)$$

and
$$\hat{\alpha}(f + g) = \hat{\alpha}f + \hat{\alpha}g$$

where c is a constant and f and g are two functions.

An operator can generate a new function by operating on a given function. In the example given above, the operator $\hat{o} = \frac{d}{dx}$ has generated the new function $f_1(x)$ by operating on the function $f(x)$.

In quantum mechanics, each dynamical variable is represented by an operator and the former provides a link with the latter (i.e., the operator) through the correspondence principle. Usually if α be a dynamical variable then the corresponding operator is denoted by $\hat{\alpha}$ where the symbol (^) is known as caret. The operators used in quantum mechanics are linear operators. The linear operators obey distributive law. Thus, if $f(x)$ and $g(x)$ are two functions of x and $\hat{\alpha}$ is a linear operator, then one gets

$$\hat{\alpha}\{f(x) + g(x)\} = \hat{\alpha}f(x) + \hat{\alpha}g(x)$$

The linear operators also obey associative laws. Sums and products of two linear operators are also linear operators. And the sum of two linear operators is commutative, i.e., $(\hat{\alpha} + \hat{\beta})f(x) = (\hat{\beta} + \hat{\alpha})f(x)$. But the product of two linear operators may or may not be commutative. For example, the product of the two linear operators $\hat{\alpha} = \frac{d}{dx}$ and $\hat{\beta} = \frac{d^2}{dx^2}$ is commutative while the product of the two linear operators $\hat{\alpha}_1 = x$ and $\hat{\beta}_1 = \frac{d}{dx}$ is not commutative.

In order to verify the above statements let us consider the function $f(x)$ and perform the following operations:

(a) $$\hat{\alpha}\hat{\beta}f(x) = \left(\frac{d}{dx}\right)\left(\frac{d^2}{dx^2}\right)f(x) = \frac{d^3}{dx^3}f(x)$$

and $$\hat{\beta}\hat{\alpha}f(x) = \left(\frac{d^2}{dx^2}\right)\left(\frac{d}{dx}\right)f(x) = \frac{d^3}{dx^3}f(x)$$

Hence $(\hat{\alpha}\hat{\beta} - \hat{\beta}\hat{\alpha})f(x)$

$$= \left(\frac{d}{dx}\frac{d^2}{dx^2} - \frac{d^2}{dx^2}\frac{d}{dx}\right)f(x)$$

$$= \frac{d}{dx}\frac{d^2}{dx^2}f(x) - \frac{d^2}{dx^2}\frac{df(x)}{dx}$$

$$= \frac{d^3}{dx^3}f(x) - \frac{d^3}{dx^3}f(x) = 0$$

i.e., $$[\hat{\alpha}, \hat{\beta}] = \left[\frac{d}{dx}, \frac{d^2}{dx^2}\right] = \left(\frac{d}{dx}\frac{d^2}{dx^2} - \frac{d^2}{dx^2}\frac{d}{dx}\right) = 0$$

So, the operators $\frac{d}{dx}$ and $\frac{d^2}{dx^2}$ commute.

(b) Let $\hat{\alpha}_1 = x$ and $\hat{\beta}_1 = \frac{d}{dx}$ and let $f(x)$ be a function. Then one can write

$$\hat{\alpha}_1\hat{\beta}_1 f(x) = x\frac{d}{dx}f(x) = x\cdot\frac{df(x)}{dx}$$

and $$\hat{\beta}_1\hat{\alpha}_1 f(x) = \frac{d}{dx}(x)f(x) = \frac{d}{dx}\{x(f(x))\}$$

$$= x\frac{df(x)}{dx} + f(x)$$

Hence $$(\hat{\alpha}_1\hat{\beta}_1 - \hat{\beta}_1\hat{\alpha}_1)f(x) = \left(x\frac{d}{dx} - \frac{d}{dx}x\right)f(x)$$

$$= x\frac{d}{dx}f(x) - x\frac{d}{dx}f(x) - f(x)$$

$$= -1.f(x)$$

As $f(x)$ is an arbitrary function, one can remove it from both sides and get the following expression:

$$\left[x, \frac{d}{dx}\right] = x\frac{d}{dx} - \frac{d}{dx}x = -1$$

So, the operators x and $\frac{d}{dx}$ are not commutative. **In order to get a function operated upon by an operator, the function is placed at the right side of the operator.**

7.5.1 Commutator

If $\hat{\alpha}$ and $\hat{\beta}$ be two operators, the commutator of this pair of operators is denoted by $[\hat{\alpha}, \hat{\beta}]$ and it is defined as

$$[\hat{\alpha}, \hat{\beta}] = \hat{\alpha}\hat{\beta} - \hat{\beta}\hat{\alpha}.$$

If $[\hat{\alpha}, \hat{\beta}] = 0$, i.e., $\hat{\alpha}\hat{\beta} = \hat{\beta}\hat{\alpha}$, then the pair of operators is known as commutative.

If $[\hat{\alpha}, \hat{\beta}] \neq 0$, i.e., $\hat{\alpha}\hat{\beta} \neq \hat{\beta}\hat{\alpha}$, then the pair of operators is known as non-commutative. As for example, the pair of operators $\hat{x}$ and $\hat{p}_x$ is non-commutative, i.e., $[\hat{x}, \hat{p}_x] \neq 0$, actually $[\hat{x}, \hat{p}_x] = i\hbar$. This implies that the simultaneous measurement of these variables (i.e., x and p_x) with absolute precision is not possible. One should note that this incompatibility is valid for position and momentum in the same direction. When the operators are in different direction, they commute.

7.5.2 Some Quantum Mechanical Operators

Before discussing the quantum mechanical operators, let us consider two wave equations given below:

$$y = a \sin (kx - wt)$$

and
$$\Psi = Ae^{i(kx - wt)}$$

The first equation represents a wave of a simple harmonic oscillator while the second one represents the de Broglie wave of a particle and both the waves propagate along the x axis. The second equation contains a complex expression because a wave representing a moving particle can be complex.

Now, in order to understand the role of operators in wave mechanics, let us differentiate the wave function $\Psi(x, t)$ for a free particle given by

$$\Psi(x, t) = Ae^{i(kx - wt)} \text{ with respect to } x \text{ and } t.$$

So, we get
$$\frac{\partial}{\partial x}\Psi(x, t) = A\frac{\partial}{\partial x}\{Ae^{i(kx - wt)}\}$$

$$\frac{\partial}{\partial t}\Psi(x, t) = A\frac{\partial}{\partial t}\{Ae^{i(kx - wt)}\}$$

Now, multiplying the first equation by $-i\hbar$

$$-i\hbar\frac{\partial}{\partial x}\Psi(x, t) = A(-i\hbar)(ik)Ae^{i(kx - wt)}$$

or,
$$-i\hbar\frac{\partial}{\partial x}\Psi(x, t) = (\hbar k)\Psi(x, t)$$

or,
$$-i\hbar\frac{\partial}{\partial x}\Psi(x, t) = p_x\Psi(x, t) \quad ...(7.14)$$

$[\because p = p_x = \hbar k]$

Again, multiplying the second equation by $i\hbar$, we get

$$i\hbar\frac{\partial}{\partial t}\Psi(x, t) = A(i\hbar)(-iw)Ae^{i(kx - wt)}$$

or,
$$i\hbar\frac{\partial}{\partial t}\Psi(x, t) = \hbar w\Psi(x, t)$$

or,
$$i\hbar\frac{\partial}{\partial t}\Psi(x, t) = E\Psi(x, t) \quad ...(7.15)$$

$[\because E = \hbar w]$

As the wave function $\Psi(x, t)$ in Eqs (7.14) and (7.15) is arbitrary, by dropping $\Psi(x, t)$ from both sides of the said equations, we get

$$-i\hbar\frac{\partial}{\partial x} = p_x \quad \text{and} \quad i\hbar\frac{\partial}{\partial t} = E$$

So, the momentum and energy operators can be written as

$$\hat{p}_x \equiv -i\hbar\frac{\partial}{\partial x} \qquad \text{...(7.16)}$$

$$\hat{E} \equiv +i\hbar\frac{\partial}{\partial t} \qquad \text{...(7.17)}$$

These two operators are mathematical operators for representing the dynamical variables of momentum and the energy of a particle.

In addition, since the coordinate x is a multiplying operator, we can express it in operator form as follows:

$$\hat{x} = x \qquad \text{...(7.18)}$$

The operator representation of Eqs. (7.16) and (7.18) is called Schrödinger representation or coordinate representation.

The operators corresponding to the other dynamical variables that are functions of the coordinate x and momentum p_x are obtained by substituting them (i.e., x and p_x) in classical expressions by $\hat{x}(=x)$ and $\hat{p}_x$ $\left(=-i\hbar\frac{\partial}{\partial x}\right)$ respectively.

Thus, if $\alpha(\hat{x}, \hat{p}_x)$ is a dynamical variable, and function of x and p_x, then the operator representation of α can be found as follows:

$$\hat{\alpha}(\hat{x}, p_x) = \alpha(\hat{x}, \hat{p}_x) = \alpha\left(\hat{x}, -i\hbar\frac{\partial}{\partial x}\right) \qquad \text{...(7.19)}$$

As for example, by applying the above rule (i.e., Eq. 7.19) we can find operators for kinetic energy (E_k) and angular momentum about the x axis (L_x) as follows:

(i) Operator representation of kinetic energy

$$E_k = E_k(p_x) = \frac{p_x^2}{2m}$$

Now, $\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$, putting this for p_x

we get $$\hat{E}_k = \frac{\left(-i\hbar\frac{\partial}{\partial x}\right)^2}{2m}$$

or, $$\hat{E}_k = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \qquad \text{...(7.20)}$$

(ii) Operator representation of L_x

$$L_x = y\,p_z - z\,p_y \qquad [\because \bar{L} = \bar{r}\times\bar{p},\ \bar{r} = \hat{i}x + \hat{j}y + \hat{k}z \text{ and } \bar{p} = \hat{i}\,p_x + \hat{j}\,p_y + \hat{k}\,p_z]$$

$\therefore$ $$\hat{L}_x = \hat{y}\,\hat{p}_z - \hat{z}\,\hat{p}_y$$

or, $$\hat{L}_x = y\left(-i\hbar\frac{\partial}{\partial z}\right) - z\left(-i\hbar\frac{\partial}{\partial y}\right)$$

or, $$\hat{L}_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \quad ...(7.21)$$

Similarly we can find the operator representation of L_y and L_z as

$$\hat{L}_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \quad ...(7.22)$$

and $$\hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \quad ...(7.23)$$

7.5.3 Operator Representation of Three-Dimensional Variables

One-dimensional operators like $\hat{x} = x$ and $\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$ lead us to write the three-dimensional position and momentum operators as follows:

$$\hat{r} = \overline{r} \quad \text{and} \quad \hat{p} = -i\hbar\overline{\nabla} \quad ...(7.24)$$

where $\overline{r} = \hat{i}x + \hat{j}y + \hat{k}z$ and $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

As we have seen in a one-dimensional case, the operator representation of a three-dimensional dynamical variable $\alpha(\overline{r}, \overline{p})$ which is a function of position and momentum can be obtained by using the operator forms of $\overline{r}$ and $\overline{p}$ in the expression of $\alpha(\overline{r}, \overline{p})$ by using the operators $\hat{r}$ and $\hat{p}$ as given in Eq. (7.24):

$$\hat{\alpha}(\overline{r}, \overline{p}) = \alpha(\hat{r}, \hat{p}) = \alpha(\hat{r}, -i\hbar\overline{\nabla}) \quad ...(7.25)$$

As an example, the hamiltonian operator $\hat{H}$ for three-dimensional motion can be obtained by using the classical expression of hamiltonian function

$$H = \frac{p^2}{2m} + V(r)$$

Hence, $$\hat{H} = \frac{(\hat{p})^2}{2m} + V(r)$$

or, $$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(r) \quad ...(7.26)$$

7.5.4 Measurement of Observables

Observables are dynamical quantities like position, linear momentum, angular momentum, kinetic energy, total energy, etc., which can be measured by conducting experiments on a physical system. There is an operator corresponding to each observable in quantum mechanics which operates upon the wave function to produce a new function. And the wave function contains every information regarding the system it represents. By using the operators one can extract information regarding the observables from the wave function. And also one can theoretically predict the average value (or expectation value) of the experimental observations regarding a physical quantity by using these operators.

7.5.5 Eigen Functions and Eigen Values: Definition of Eigen Function and Eigen Value

If an operator operates (i.e., acts) on a wave function and as a result of operation produces the same wave function multiplied by a constant factor, then the wave function is called an *eigen function* and the constant multiplier is called the *eigen value* of the operator.

Let the operator $\hat{A}$ act on the wave function Ψ and returns the same function Ψ multiplied by α,

i.e., $$\hat{A}\,\Psi = \alpha\,\Psi \qquad \text{...(7.27)}$$

So, according to the definition given above Ψ is the eigen function of the operator $\hat{A}$ and α the corresponding eigen value. If the operator $\hat{A}$ represents an observable A, the measurement of this observable will yield the value α when the state of the particle is represented by the wave function Ψ which is an eigen function of the operator $\hat{A}$. And in this case, the wave function Ψ represents a state of the system which is called the eigen state of the observable A. One observable may have several eigen states. The system is found to be in any one of the eigen states of it with a characteristic eigen value whenever one performs an experiment to measure the observable.

Equation (7.27), i.e., $\hat{A}\,\Psi = \alpha\,\Psi$ is called the eigen value equation. As for an example, the function $f(x) = \sin(wx + \phi)$ is an eigen function of the operator $\hat{D} = \frac{d^2}{dx^2}$ with an eigen value $(-w^2)$, because

$$\hat{D}\,f(x) = \frac{d^2}{dx^2}\{\sin(wx + \phi)\}$$
$$= (-w^2)\sin(wx + \phi)$$

or, $$\hat{D}\,f(x) = (-w^2)\,f(x)$$

So, $(-w^2)$ is the eigen value of $\hat{D}$.

The time-independent Schrödinger's equation, i.e., the equation

$$\hat{H}\,u(x) = Eu(x) \qquad \text{...(7.28)}$$

with $\left[\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right]$ is also an eigen value equation where $u(x)$ is an eigen function of the operator $\hat{H}$ (known as hamiltonian operator) corresponding to an energy eigen value E. Since the hamiltonian function H represents the total energy of the system, the function $u(x)$ represents one energy eigen state of the system having the value of the energy equal to E. The hamiltonian operator $\hat{H}$ may have many eigen functions representing energy eigen states with characteristic energy eigen values when there are several eigen states with the same eigen value, the states, in that case, are called the **degenerate states**.

Let us consider the wave function $\Psi = u(x)\,e^{-i(Et/\hbar)}$ and let $\hat{H}$ operate on it.

So, $$\hat{H}\,\Psi = \hat{H}\,[u(x)\,e^{-i(Et/\hbar)}]$$

or, $$\hat{H}\,\Psi = e^{-i(Et/\hbar)}\,\hat{H}\,u(x)$$

or, $$\hat{H}\,\Psi = e^{-i(Et/\hbar)}\,Eu(x) \quad \text{[by Eq. (7.28)]}$$

or, $$\hat{H}\,\Psi = E[u(x)\,e^{-i(Et/\hbar)}]$$

or, $$\hat{H}\,\Psi = E\Psi \qquad \text{...(7.29)}$$

The above equation [i.e., Eq. (7.29)] shows that a stationary state wave function Ψ is an eigen function of the hamiltonian operator $\hat{H}$ corresponding to an eigen value E.

7.5.6 Expectation Value (or Expected Average)

As stated earlier, wave function related to an observable is the solution of Schrödinger's equation. An operator is used to extract information about the corresponding observable by operating on the wave function. In order to measure an observable one can conduct one single experiment on a large number of identical systems or one can measure the same (i.e., single) observable independently a large number of times. The arithemetic average of all the experimental results so obtained in either case can be theoretically predicted by making use of the wave function that is expected to contain all information about the identical systems in the former case or the system concerned in the earlier case. The average, thus, obtained is called the expected average or the expectation value or mathematical expectation of the concerned observable.

If one wants to determine the expected value of the momentum (p_x) of a particle (moving along the x axis) at a time t, one can perform an experiment with a large number of so moving identical particles. Let the number of so moving particles (which are described by the same wave function Ψ) be n. Let n_i be the number of particles lying on the position denoted by x_i at time t. So, the average momentum can be expressed as follows:

$$p_{x(av)} = \frac{\sum_i n_i p_{xi}}{\sum_i n_i} = \frac{\sum_i n_i p_{xi}}{n}$$

$$p_{x(av)} = \sum_i \left(\frac{n_i}{n}\right) p_{xi} \qquad \text{...(7.30)}$$

The average of Eq. (7.30) is the weighted average and the weight in this case is $\left(\frac{n_i}{n}\right)$. This weight (or ratio) can be regarded as the probability (P_i) of finding a particle with momentum p_{xi}.

Hence the expression for the average value can be written as follows:

$$p_{x(av)} = \sum_i p_i p_{xi} \qquad \text{...(7.31)}$$

The above expression of Eq. (7.31) is true for discrete distribution of particles. If the distribution of particles be continuous, then one can replace the summation symbol by the integration symbol in the expression of Eq. (7.31). And in such case, the average value $p_{x(av)}$ of momentum can be calculated if the probability density function $p(x, t)$ be such that the probability of finding a particle in the region between x and $x + dx$ is $p(x, t)\, dx$ at a time t. Then the average momentum of a particle can be expressed by the following equation:

$$p_{x(av)} = \int_{-\infty}^{+\infty} p_x\, p(x, t)\, dx \qquad \text{...(7.32)}$$

Now, if the wave function is known one can easily calculate the probability density. The probability density is given by the following expression:

$$p(x, t) = |\Psi_n(x, t)|^2 = \Psi_n^*(x, t)\, \Psi_n(x, t)$$

where $\Psi_n(x, t)$ is the normalized value of the $\Psi(x, t)$. On substitution of this value of $p(x, t)$ in Eq. (7.32), one can obtain

$$p_{x(av)} = \int_{-\infty}^{+\infty} p_x\, \Psi_n^*(x, t)\, \Psi_n(x, t)\, dx$$

This theoretically calculated average is called the *expected average* or *expectation value* of the observable which is denoted by the symbol $<p_x>$. Then the expression for the expected average can be written as

$$<p_x> = \int_{-\infty}^{+\infty} \Psi_n^*(x,t)\, p_x\, \Psi_n(x,t)\, dx$$

or,
$$<p_x> = \int_{-\infty}^{+\infty} \Psi_n^*(x,t)\, p_x\, \Psi_n(x,t)\, dx \quad ...(7.33)$$

Similarly one can calculate the expectation value of the position of a particle which is given by the following expression:

$$<x> = \int_{-\infty}^{+\infty} \Psi_n^*(x,t)\, \hat{x}\, \Psi_n(x,t)\, dx \quad ...(7.34)$$

In general, for any physical quantity $f(x, p_x, t)$ which is a function of x, p_x and t, the expected value is given by

$$<f(x, p_x, t)> = \int_{-\infty}^{+\infty} \Psi_n^*(x,t)\, \hat{f}(x, p_x, t)\, \Psi_n(x,t)\, dx$$

or,
$$<f(x, p_x, t)> = \int_{-\infty}^{+\infty} \Psi_n^*(x,t) f(\hat{x}, \hat{p}_x, t)\, \Psi_n(x,t)\, dx$$

or,
$$<f(x, p_x, t)> = \int_{-\infty}^{+\infty} \Psi_n^*(x,t) f\left(x, -i\hbar\frac{\partial}{\partial x}, t\right) \Psi_n(x,t)\, dx \quad ...(7.35)$$

where $\hat{f}(x, p_x, t) = f(\hat{x}, \hat{p}_x, t) = f\left(x, -i\hbar\dfrac{\partial}{\partial x}, t\right)$. ...(7.36)

For the matter of a particle in three-dimensional space, the wave function $\Psi_n(\bar{r}, t)$ of the particle is a function of the position vector $\bar{r}$ and time t and the physical quantities are generally functions of position vector $\bar{r}$, momentum $\bar{p}$ and time t.

The expected average of one such physical variable (quantity) $g(\bar{r}, \bar{p}, t)$ is given by

$$<g> = \int_{-\infty}^{+\infty} \Psi_n^*(\bar{r},t)\, \hat{g}(\bar{r}, \bar{p}, t)\, \Psi_n(\bar{r},t)\, dV \quad ...(7.37)$$

where dV is a differential volume element.

For calculation of the mathematical expectation or expectation value or expected average one can follow the rule given below:

In order to calculate the expected average, place the operator representing the dynamical variable to operate on the normalized wave function $\Psi_n(\bar{r})$ and multiply the result by $\Psi_n^*(\bar{r})$ and then integrate over the spatial coordinates.

7.5.7 Correspondence Principle or Ehrenfest Theorem-Operator Correspondence

The notion of the expectation values (or mathematical expectations) of dynamical variables [as has been given in Eqs. (7.33), (7.34) and (7.35)] can be used to check the validity of the correspondence principle which is one of the basic postulates of quantum mechanics and known as postulate of operator-variable correspondence or simply operator correspondence. In fact this postulate states about the correspondence

> ***Postulate 2 of Quantum Mechanics***
> ***The classical dynamical variables relating to the motion of a particle are represented by mathematical operators in quantum mechanics.***

of dynamical variables of classical mechanics to the operators related to the expected averages of quantum mechanics.

According to the principle of correspondence, one can expect that the relationships of classical mechanics between various dynamical variables (e.g., coordinate and momentum) of a particle will also hold good between the expected averages of these quantities for the quantum mechanical wave packet associated with the particle. **This is known as Ehrenfest theorem.**

The measure of the positional uncertainty (Δx) of a particle is given by the width of the wave packet. Since the momentum of the particle is given by $p = \hbar k$, the range of variation of the wave number (Δk) gives the measure of uncertainty in case of the momentum of the particle. For a classical particle (where large distance and momenta are involved) one can ignore the uncertainty principle in the measurement of position and momentum. It was shown by Ehrenfest that the motion of a wave packet agrees with the motion of a particle in classical mechanics if the position and momentum vectors are considered as the expected averages of these quantities.

One can show that the time rate of change of the expected average of the position coordinate x of a particle corresponds to the expected average of the velocity of the same. It can be expressed as follows:

$$\frac{d}{dt} < x > = \frac{1}{m} < p_x > \qquad ...(7.38)$$

One can also show that the time rate of change of the expected average of p_x corresponds to the x component of the expected average of the force acting on the particle. It can be expressed as follows:

$$\frac{d}{dt} < p_x > = -\left\langle \frac{\partial V}{\partial x} \right\rangle = < F_x > \qquad ...(7.39)$$

Equation (7.39) confirms the wave packet description of a moving particle for which the time rate of change of the momentum equals to the negative of the expected average of the gradient of potential, that is equal to the impressed force acting on the particle. It can be seen from this equation the wave packet description reproduces Newton's second law of motion. In fact, the Eqs. (7.38) and (7.39) together express the Ehrenfest theorem and it presents a correspondence between the wave packet formalism and the classical dynamics describing the motion of a particle.

7.6 FUNDAMENTAL POSTULATES OF QUANTUM MECHANICS

In the approach of developing the theory of wave mechanics which has been done so far, it has been tried to make plausible various new ideas on the basis of underlying physical principles involved. However, there is another alternative approach in which the formal mathematical structure of wave mechanics is founded on a number of fundamental postulates. The theory can be developed more logically on the basis of these postulates and derives justification from the success of it, in accounting for a wide variety of atomic and subatomic phenomena.

Let us enumerate and discuss briefly the fundamental postulates all of which have been introduced so far. Five fundamental postulates of wave mechanics, which are listed below, are applicable in general to any quantum mechanical system. But here we shall restrict ourselves to a single particle system for the sake of simplicity.

Postulate 1 There is a wave function associated with a particle [denoted by $\Psi(\vec{r}, t)$ or $\Psi(x, y, z, t)$] which can completely describe the space-time behavior of the particle in consistence with the uncertainty principle.

Postulate 2 In quantum mechanics, the dynamical variables related to the motion of a particle are represented by mathematical operators.

Postulate 3 For a dynamical variable α, the possible results of measurement are given only by the eigen values of the operator $\hat{\alpha}$ satisfying the following eigen value equation

$$\hat{\alpha}\,\Psi_e = a_e\,\Psi_e$$

where Ψ_e is the eigen function of the operator $\hat{\alpha}$ belonging to the eigen value a_e. The eigen functions Ψ_e are single-valued in space and square-integrable. And they also form a complete set of orthogonal wave functions.

Postulate 4 The probability $p\,dV$ of finding a particle in the volume element dV is given by the equation

$$p\,dV = \Psi_n^*\,\Psi_n\,dV = |\Psi_n|^2\,dV$$

where Ψ_n is the normalized wave function, $p = |\Psi_n|^2$ and it is called the probability density. In a finite volume V of space, the probability of finding the particle is given by

$$\int_V p\,dV = \int_V |\Psi_n|^2\,dV$$

And the probability of finding the particle in entire space is given by

$$P = \int_{\text{all space}} \Psi_n^*\,\Psi_n\,dV = 1$$

Postulate 5 The expected average of the results of a large number of measurements of a dynamical variable α of a particle is given by

$$<\alpha> = \int_V \Psi_n^*\,\hat{\alpha}\,\Psi_n\,dV$$

where $\hat{\alpha}$ is the operator representing the variable α and Ψ_n is normalized wave function.

The operators corresponding to some observables or dynamical variables are listed in the following table:

Table 7.1 Some observables and their corresponding operators.

Observable	*Operator*
(a) Position	$\hat{r} = \vec{r}$ (or $\hat{x} = x, \hat{y} = y, \hat{z} = z$)
(b) Momentum	$\hat{p} = -i\hbar\vec{\nabla}\left(\text{or, } \hat{p}_x = -i\hbar\frac{\partial}{\partial x}, \hat{p}_y = -i\hbar\frac{\partial}{\partial y}, \hat{p}_z = -i\hbar\frac{\partial}{\partial z}\right)$
(c) Kinetic Energy	$\hat{E}_k = \frac{-\hbar^2}{2m}\nabla^2 = \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$
(d) Potential Energy	$\hat{V}(\vec{r}) = V(\vec{r})$ or $\hat{V}(x, y, z) = V(x, y, z)$
(e) Total Energy	$\hat{E} = i\hbar\frac{\partial}{\partial t}$
(f) Hamiltonian	$\hat{H} = \frac{-\hbar^2}{2m}\nabla^2 + V(\vec{r})$ or, $\hat{H} = \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + V(x, y, z)$
(g) Angular Momentum ($\vec{L} = \vec{r} \times \vec{p}$)	$\hat{L} = -i\hbar\,(\vec{r} \times \vec{\nabla})$

7.7 BASIC CONSIDERATIONS FOR DEVELOPING SCHRÖDINGER'S EQUATION

The laws of classical mechanics are not able to explain the origin of quantized motion in atomic or subatomic domain. The old quantum theory is also not adequate in this respect as the rules of quantization in this theory are introduced on ad hoc basis. According to Heisenberg, the failure of the old quantum theory is due to the formulation of this theory on the basis of some classical concepts which have no meaning in the context of quantized motion of particles. Thus, in order to overcome this discrepancy, the electronic orbits of atoms and the velocity of the rotating electrons in these orbits are considered. If, however, one tries to determine the position (or, the velocity) of the electron in the atomic orbits, one has to perform some experiment (e.g., gamma-ray microscopic experiment) which will disturb the system (i.e., the electron) and will introduce some uncertainties in the quantity to be measured as predicted by the principle of uncertainty.

Thus, attempts should be made to develop atomic mechanics not on the basis of unobservable classical concepts, but on the basis of some new concepts which are in fact based on observable quantities. For example, when one performs experiments to investigate an atomic structure, one measures the energies of the atomic states through measurement of wavelengths of the spectral line, intensities, etc., and not through the positions or velocities of the electrons. So the new atomic mechanics should be built on the basis of quantities like the intensities of spectral lines and the energies of the atomic states.

It was on the basis of such thinking that Schrödinger developed wave mechanics (in 1926) on the basis of de Broglie hypothesis of wave-particle duality. The concepts introduced by de Broglie to relate dynamical variables connected with the particle motion (e.g., momentum) with the characteristics of waves (e.g., wavelength) were extended by him in a more general way to a quantum system to develop a wave equation to describe the motion of atomic particles, like electrons. Schrödinger's wave equation has become the foundation of non-relativistic wave mechanics. Later (in 1928) Dirac laid the foundation of relativistic wave mechanics. And he then generalized the quantum mechanics by unifying Heisenberg's matrix mechanics and Schrödinger's wave mechanics.

7.7.1 Time-Dependent Schrödinger's Equation

According to the second postulate of quantum mechanics, the dynamical variables of classical mechanics relating to the motion of a particle can be represented by mathematical operators in quantum mechanics. If one keeps this point in view, one can easily derive Schrödinger's time-dependent equation.

Let us consider a particle of mass m, moving along the x axis with a fixed momentum p_x and total energy E in a free region. As no force is acting on the particle, it is in a region of uniform potential energy which can be assumed to be zero. Hence, in this case, the total energy will be equal to the kinetic energy. And it is $E = p_x^2/(2m)$ for non-relativistic motion. Now, keeping in view, the aforesaid second postulate, one can write the corresponding quantum mechanical expression of the classical expression $E = p_x^2/(2m)$ as follows:

$$\hat{E} = \frac{\hat{p}_x^2}{2m}$$

Now, multiplying this equation by the wave function of a free particle $\Psi(x, t)$ from right side, i.e., allowing the operators to act upon $\Psi(x, t)$, one gets the following equation:

$$\hat{E}\,\Psi(x, t) = \frac{\hat{p}_x^2}{2m}\,\Psi(x, t)$$

or, $$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \frac{\left(-i\hbar\frac{\partial}{\partial x}\right)^2}{2m}\Psi(x,t)$$

or, $$\frac{-\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} = i\hbar\frac{\partial}{\partial t}\Psi(x,t) \qquad ...(7.40)$$

This equation is Schrödinger's one-dimensional time-dependent equation for a free particle having wave function $\Psi(x,t) = Ae^{i(kx-\omega t)}$. The wave function $\Psi(x,t)$ conveys the necessary information to locate the particle in time and space, so $\Psi(x,t)$ is a function of x and t.

Now, if the particle is not free one it will be acted upon by some external force which will be characterized by a non-zero potential energy of the particle. Let the potential energy be $V(x, t)$. So, one can write the operator equation as

$$\hat{E} = \frac{\hat{p}_x^2}{2m} + \hat{V}(x,t)$$

So, the corresponding Schrödinger's equation will be given by

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \frac{(-i\hbar)^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x,t)\,\Psi(x,t)$$

or, $$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x,t)\,\Psi(x,t)$$

or, $$\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x,t)\,\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t) \qquad ...(7.41)$$

This equation [Eq. (7.41)], is Schrödinger's one-dimensional equation for one particle system.

In a three-dimensional case, Schrödinger's equation for one particle system is given by

$$\frac{-\hbar^2}{2m}\nabla^2\,\Psi(\bar{r},t) + V(\bar{r},t)\,\Psi(\bar{r},t) = i\hbar\frac{\partial}{\partial t}\Psi(\bar{r},t) \qquad ...(7.42)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the laplacian operator named after the famous mathematician Laplace. By using the hamitonian operator $\hat{H} = \frac{-\hbar^2}{2m}\nabla^2 + V(\bar{r},t)$, one can write the time-dependent Schrödinger's equation of three dimensions as

$$\hat{H}\,\Psi(\bar{r},t) = i\hbar\frac{\partial}{\partial t}\Psi(\bar{r},t) \qquad ...(7.43)$$

7.7.2 Time-Independent Schrödinger's Wave Equation

In the last subsection we have seen that in case of time-dependent Schrödinger's equation, the potential energy (V) of a moving particle is a function of both time and space coordinates. But, in many physical situations the potential energy of a moving body does not depend explicitly on time. In such cases, the potential energy as well as the forces which act on the particle is a function of position ($\bar{r}$) only. So $V = V(\bar{r})$. In such cases, the time-dependent Schrödinger's equation of three dimensions given in Eq. (7.42) can be rewritten as follows:

$$\frac{-\hbar^2}{2m}\nabla^2\Psi(\bar{r},t)+V(\bar{r})\,\Psi(\bar{r},t)=i\hbar\frac{\partial\Psi(\bar{r},t)}{\partial t} \qquad ...(7.44)$$

[since $V = V(\bar{r})$ only]

Equation (7.44) is a partial differential equation of variables which are functions of $\bar{r}$ and t. While the potential energy V is a function of position vector $\bar{r}$ only, the total energy E is a constant quantity. So the equation is separable into the time-dependent and time-independent parts. And because of this situation, the wave function $\Psi(\bar{r}, t)$ can be expressed as the product of two separate functions $\phi(\bar{r})$ and $f(t)$ where the former is a function of $\bar{r}$ only and the later is a function of t only. Hence $\Psi(\bar{r}, t)$ can be written as

$$\Psi(\bar{r},t)=\phi(\bar{r})\,f(t)$$

Now, putting this value of $\Psi(\bar{r}, t)$ in Eq. (7.44), one can get

$$\frac{-\hbar^2}{2m}\nabla^2\phi(\bar{r})f(t)+V(\bar{r})\,\phi(\bar{r})f(t)=i\hbar\frac{\partial}{\partial t}\{\phi(\bar{r})f(t)\}$$

Now, dividing both sides by $\phi(\bar{r})f(t)$, one gets,

$$-\frac{\hbar^2}{2m}\frac{1}{\phi(\bar{r})}\nabla^2\phi(\bar{r})+V(\bar{r})=i\hbar\frac{1}{f(t)}\frac{\partial}{\partial t}f(t) \qquad ...(7.45)$$

The left-hand side of Eq. (7.45) is a function $\bar{r}$ only while right-hand side is a function of t only. This is possible only when both of the sides are separately equal to a constant quantity. This constant is the total energy E of the particle. Thus, one can write two equations, when one equates the left side to E, the following equation is obtained:

$$\frac{-\hbar^2}{2m}\nabla^2\phi(\bar{r})+V(\bar{r})\,\phi(\bar{r})=E\,\phi(\bar{r})$$

or,

$$\nabla^2\phi(\bar{r})+\frac{2m}{\hbar^2}[E-V(\bar{r})]\,\phi(\bar{r})=0 \qquad ...(7.46)$$

This is the three-dimensional time-independent Schrödinger's wave equation. From this equation, the one-dimensional time-independent Schrödinger's equation can be written as follows:

$$\frac{d^2}{dx^2}\phi(x)+\frac{2m}{\hbar^2}[E-V(x)]\,\phi(x)=0 \qquad ...(7.47)$$

Equation (7.46) can be again written as follows

$$\frac{-\hbar^2}{2m}\nabla^2\phi(\bar{r})+V(\bar{r})\phi(\bar{r})=E\phi(\bar{r})$$

or,

$$\hat{H}\,\phi(\bar{r})=E\phi(\bar{r}) \qquad ...(7.48)$$

where $\hat{H}=\frac{-\hbar^2}{2m}\nabla^2+V(\bar{r})$

Equation (7.48) is the eigen value equation where E and $\phi(r)$ are the eigen value and eigen function respectively.

Thus, when the hamiltonian operator of time-independent Schrödinger's wave equation operates on the wave function $\phi(\bar{r})$ it reproduces the same wave function $\phi(\bar{r})$ multiplied by the total energy. The eigen value E of the hamiltonian operator which is only the possible value of total energy of a quantum mechanical system.

Now, equating the right side of Eq. (7.45) to E, one can get

$$i\hbar \frac{d}{dt} f(t) = E f(t) \quad \left[\because \text{ there is only one independent variable } \frac{\partial}{\partial t} = \frac{d}{dt}\right]$$

or, $$\frac{d}{dt} f(t) = -\frac{iE}{\hbar} f(t)$$

or, $$\frac{df(t)}{f(t)} = \frac{iE}{\hbar} dt$$

Now, integrating both sides, one can get

$$\ln \{f(t)\} = -\frac{iE}{\hbar} t + \ln c \text{ where } c \text{ is a constant.}$$

or, $$\ln \left\{\frac{f(t)}{c}\right\} = -\frac{iE}{\hbar} t$$

or, $$f(t) = ce^{-(iE/\hbar)t} \qquad \text{...(7.49)}$$

Thus, from the equation $\Psi(\bar{r}, t) = \phi(\bar{r}) f(t)$

one can write

$$\Psi(\bar{r}, t) = \phi(\bar{r})\, ce^{-\frac{iEt}{\hbar}} \qquad \text{...(7.50)}$$

This gives the solution of Schrödinger's equation.

Here, $\phi(\bar{r})$ and $ce^{-\frac{iEt}{\hbar}}$ are the amplitude and phase of the wave function $\Psi(\bar{r}, t)$ respectively. The time-independent form of Schrödinger's wave Eq. (7.46) is sometimes known as the amplitude equation.

Every solution of the time-independent Schrödinger's wave equation gives a definite energy value. If one writes, $\Psi = \Psi_n(\bar{r})$ as the solution for the energy value $E = E_n$, then the particular solution is given by

$$\Psi_n(\bar{r}, t) = \Psi_n(\bar{r})\, ce^{iE_n t/\hbar} \qquad \text{...(7.51)}$$

$\Psi_n(\bar{r}, t)$ belongs to the definite energy value E_n.

One can find the probability of finding the particle with energy value E_n as follows:

$$p_n = |\Psi_n(\bar{r}, t)|^2 = \Psi_n^*(\bar{r}, t)\, \Psi_n(\bar{r}, t)$$

$$= \Psi_n^*(\bar{r})\, c^* e^{\frac{iE_n t}{\hbar}}\, \Psi_n(\bar{r})\, ce^{-\frac{iE_n t}{\hbar}}$$

or, $$p_n = |\Psi_n(\bar{r})|^2 = [\Psi_n(\bar{r})]^2 \qquad \text{...(7.52)}$$

$$[\because \Psi_n^*(\bar{r}) = \Psi_n(\bar{r}) \text{ and } c^*c = 1]$$

Thus, the probability of finding the particle with energy E_n is independent of time t.

7.8 APPLICATIONS OF SCHRÖDINGER'S EQUATION

Schrödinger's equation is able to describe any system of quantum particles in space and time. As it is a differential equation, it is to be solved for finding all the information about the quantum system of particles. The solution of Schrödinger's equation is known as Schrödinger's wave function as this equation is basically a wave equation which represents the de Broglie wave of the system. In this section of the present chapter, attempts will be made to determine the wave function of a particle which remains trapped in a certain region of space by applying the theory of Schrödinger. The possible energy values of the particles also will be calculated.

7.8.1 Particle in a Box with Infinitely Deep Potential Well

The simplest quantum-mechanical problem is that of a particle trapped in a box with infinitely hard walls. One may specify the motion of the particle along a straight line, then the problem will be one-dimensional. And when no such restriction in motion will be imposed then it will be a three-dimensional problem. The one-dimensional case will be discussed first and then it will be followed by the three-dimensional case.

(i) One-dimensional case (1D box)

Let us consider a particle of mass m capable of moving along the x axis only between two infinitely hard walls and face an infinitely deep potential as shown in Fig. 7.2.

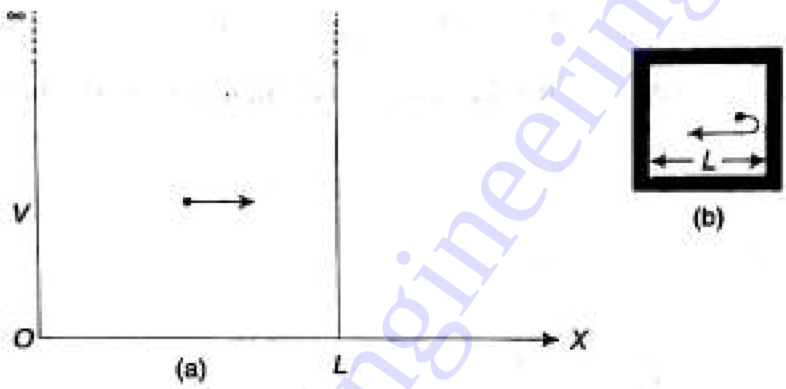


Fig. 7.2 One-dimensional infinite potential well.

The variable x can assume any value between 0 and L (i.e., $x_{min} = 0$ and $x_{max} = L$). The two hard walls act as infinitely high potential barriers. The particle is said to be in a one-dimensional box. The potential energy is constant within this region of length L. But it is infinite on both sides of the box. Inside the box, the potential energy can be assumed to be zero. So, one can write

$$V = 0 \quad \text{for} \quad 0 < x < L \qquad \text{...(7.53)}$$

$$V = \infty \quad \text{for} \quad x \geq L \quad \text{and} \quad x \leq 0 \qquad \text{...(7.54)}$$

The particle will need infinite amount of energy to overcome these two barriers. Since a particle cannot have infinite energy, it remains confined within the box with $0 \leq x \leq L$. It is not possible to find the particle outside the box and therefore, its wave function $\Psi(x, t)$ is zero outside. So, the wave function $\Psi(x, t)$ must satisfy the following condition:

$$\Psi(x, t) = 0 \quad \text{for} \quad x \geq L \quad \text{and} \quad x \leq 0 \qquad \text{...(7.55)}$$

The wave function $\Psi(x, t)$ represents a stationary wave as V is independent of time. So, it can be expressed as

$$\Psi(x, t) = \phi(x) f(t)$$

The time-dependent part of Ψ is given by the following equation:

$$f(t) = e^{-iEt/\hbar} \qquad \text{...(7.56)}$$

Equation (7.55) implies that

$$\phi(x) = 0 \quad \text{for} \quad x \geq L \quad \text{and} \quad x \leq 0 \qquad \text{...(7.57)}$$

The function $\phi(x)$ can be determined having solved Schrödinger's time-independent equation. This equation is

$$\frac{-\hbar^2}{2m}\frac{d^2\phi}{dx^2} + V\phi = E\phi \qquad \text{...(7.58)}$$

As in this case, $V = 0$, one can write,

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} = E\phi$$

or,
$$\frac{d^2\phi}{dx^2} + \frac{2mE}{\hbar^2}\phi = 0 \qquad \text{...(7.59)}$$

The general solution of Eq. (7.59) is given by

$$\phi = A\sin\left(\sqrt{\frac{2mE}{\hbar^2}}\,x\right) + B\cos\left(\sqrt{\frac{2mE}{\hbar^2}}\,x\right) \qquad \text{...(7.60)}$$

where A and B are two constants whose value can be determined from the boundary conditions given below:

$$\phi = 0 \quad \text{at} \quad x = 0$$

and
$$\phi = 0 \quad \text{at} \quad x = L$$

Putting $x = 0$ and $\phi = 0$ in Eq. (7.60), one gets

$$B = 0$$

$\therefore$
$$\phi = A\sin\left(\sqrt{\frac{2mE}{\hbar^2}}\,x\right) \qquad \text{...(7.61)}$$

Now, putting $x = L$ and $\phi = 0$ and $B = 0$ in the Eq. (7.60), one can get

$$0 = A\sin\sqrt{\frac{2mE}{\hbar^2}}\,L + 0$$

or,
$$A\sin\sqrt{\frac{2mE}{\hbar^2}}\,L = 0 \qquad \text{...(7.62)}$$

In Eq. (7.62) $A \neq 0$, because in such case $\phi(x) = 0$ for all values of x. Then one can write

$$\sin\sqrt{\frac{2mE}{\hbar^2}}\,L = 0$$

And hence, one can have

$$\sqrt{\frac{2mE}{\hbar^2}}\cdot L = n\pi \qquad \text{...(7.63)}$$

where $n = 1, 2, 3, \ldots$

Now, using Eqs. (7.63) and (7.61), one can write

$$\phi(x) = A\sin\left(\frac{n\pi}{L}x\right)$$

As $\phi(x)$ is dependent on n, one can write the above equation as

$$\phi_n(x) = A\sin\left(\frac{n\pi}{L}x\right) \qquad \text{...(7.64)}$$

Squaring both sides of Eq. (7.63), one can write

$$E = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$$

As E is dependent on n, one can write

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2} \qquad ...(7.65)$$

Here, one can regard n as a quantum number which determines the possible energy values E_n corresponding to different states represented by the function $\phi_n(x)$.

Now, multiplying $\phi_n(x)$ by the time-dependent part of the wave function, one can write

$$\Psi_n(x, t) = \phi_n(x)\, e^{-iE_n t/\hbar}$$

So, in general form, the wave function can be written as

$$\Psi_n(x, t) = A \sin\left(\frac{n\pi}{L} x\right) e^{-iE_n t/\hbar} \qquad ...(7.66)$$

To find out the value of A, one has to apply the condition of normalization of the wave function as follows:

$$\int_{-\infty}^{+\infty} \Psi_n^* \Psi_n \, dx = 1$$

or,
$$\int_0^L \Psi_n^* \Psi_n \, dx = 1$$

Now using Eq. (7.66), one gets

$$\int_0^L A^2 \sin^2\left(\frac{n\pi}{L} x\right) dx = 1$$

or,
$$\frac{A^2 L}{2} = 1$$

or,
$$A = \sqrt{\frac{2}{L}} \qquad ...(7.67)$$

Putting this value of A in Eq. (7.66)

$$\Psi_n(x, t) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right) e^{-iE_n t/\hbar} \qquad ...(7.68)$$

This wave function represents a stationary state where the total energy of the particle is E_n, given by Eq. (7.65).

Since $E_n = \dfrac{n^2 \pi^2 \hbar^2}{2mL^2}$ and $n = 1, 2, 3, \ldots$, the energy of the particle cannot be continuously distributed; the possible energy values (E_n) are discrete. The lowest energy corresponds to $n = 1$.

$\therefore$ the lowest energy is given by

$$E_1 = \frac{\pi^2 \hbar^2}{2mL^2} \qquad ...(7.69)$$

One can express E_n in terms of E_1 as

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2} = n^2 E_1$$

i.e.,
$$E_n = n^2 E_1 \quad ...(7.70)$$

and
$$E_{n+1} = (n+1)^2 E_1 \quad ...(7.71)$$

Now, subtracting Eq. (7.70) from Eq. (7.71), one obtains

$$E_{n+1} - E_n = (2n+1) E_1 \quad ...(7.72)$$

or,
$$E_{n+1} - E_n = (2n+1) \cdot \frac{\pi^2 \hbar^2}{2mL^2} \quad ...(7.73)$$

Now, in case of a macroscopic body, mL^2 is very large and the difference between two successive energy levels becomes extremely small. Hence, the possible levels (in such case) seem to have a continuous distribution. So, in a macroscopic world, the energy distribution seems to be continuous.

The probability density p_n of the particle is given by the following equation:

$$p_n = |\Psi_n(x, t)|^2 = \Psi_n^*(x, t)\, \Psi_n(x, t)$$

or,
$$p_n = \left(\frac{2}{L}\right) \sin^2\left(\frac{n\pi}{L} x\right) \quad ...(7.74)$$

[by Eq. (7.68)]

When $\sin^2\left(\frac{n\pi x}{L}\right) = 1$, i.e., when $\frac{n\pi x}{L} = (2q+1)\frac{\pi}{2}$ and $q = 1, 2, 3, ...$, the probability density p_n becomes maximum.

In other words, when $x = (2q+1)\frac{L}{2n}$ for $q = 1, 2, 3, 4, ...$, the probability density p_n becomes maximum.

Figure (7.3) shows the variation of Ψ_n and $|\Psi_n|^2$ with respect to x for $n = 1$, $n = 2$ and $n = 3$.

As the value of n increases, the probable positions increase, which means that as the energy of a particle increases the peak number in the probability density curve also increases. According to classical theory, the particle should be found everywhere in the box with the same probability. But according to the quantum theory, the probability of existence of the particle is maximum at the antinodes and it is minimum at the nodes.

Expected average of the position

The expected average of position of the particle is given by

$$\langle x \rangle = \int_{-\infty}^{+\infty} \Psi_n^* \hat{x}\, \Psi_n\, dx = \int_0^L |\Psi_n|^2\, x\, dx$$

or,
$$\langle x \rangle = \frac{2}{L} \int_0^L x \sin^2\left(\frac{n\pi x}{L}\right) dx$$

or,
$$\langle x \rangle = \frac{2}{L}\left[\frac{x^2}{4} - \frac{x \sin(2x\pi x/L)}{4n\pi/L} - \frac{\cos(2x\pi x/L)}{8(n\pi/L)^2}\right]_0^L$$

or,
$$\langle x \rangle = \frac{2}{L}\left(\frac{L^2}{4}\right)$$

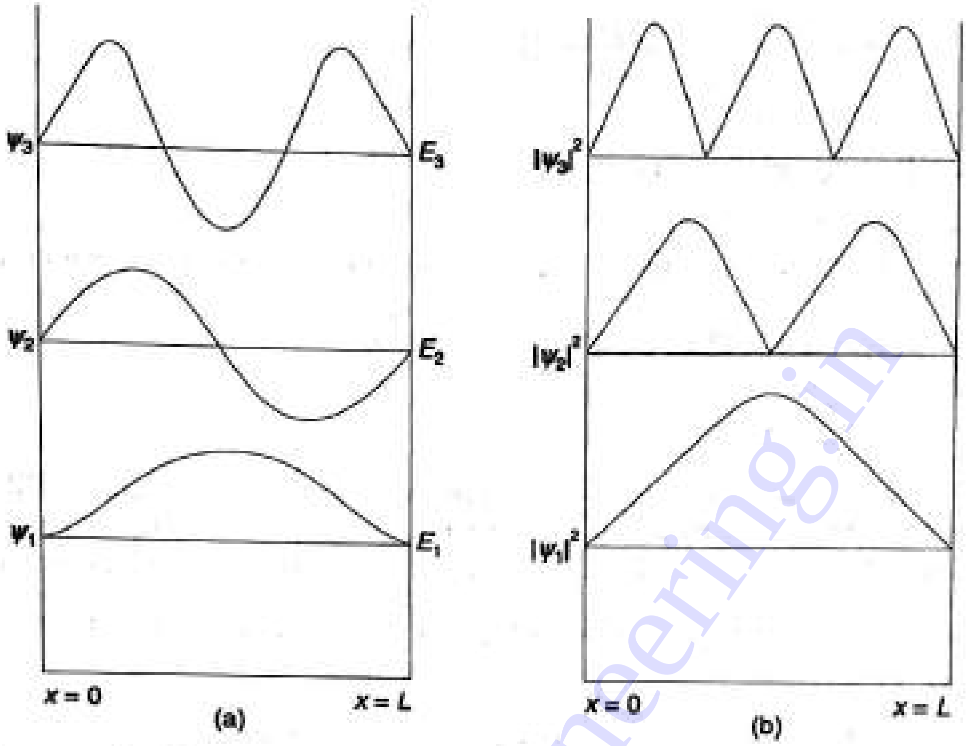


Fig. 7.3 (a) The energy band corresponding to normalized wave functions in case of ground, first excited and second excited states in one-dimensional box. (b) Probability densities of $|\Psi_1|^2$, $|\Psi_2|^2$ and $|\Psi_3|^2$.

or,
$$< x > = \frac{L}{2}$$

So, for any value of n, the expected average of the position (x) of the particle is always $L/2$ for a box of linear dimension or length L.

It is clear from this fact that the average position of the particle is the middle point of the box in all quantum states of the particle.

Expected average of the momentum

The expected average or the mathematical expectation of the momentum of the particle is given by

$$< p_x > = \int_{-\infty}^{+\infty} \Psi_n^* p_x \Psi_n \, dx = \int_0^L \Psi_n^* p_x \Psi_n \, dx$$

or,
$$< p_x > = \int_0^L \Psi_n^* \left(-i\hbar \frac{\partial}{\partial x} \right) \Psi_n \, dx \qquad \text{...(7.75)}$$

where $\Psi_n(x, t) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{iEnt}{\hbar}}$

Substituting the values of Ψ_n and Ψ_n^* in Eq. (7.75), one obtains

$$< p_x > = -\frac{2 i\hbar n\pi}{L^2} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx$$

or, $$< p_x > = \left[-\frac{i\hbar}{L} \sin^2 \left(\frac{n\pi x}{L} \right) \right]_0^L$$

or, $$< p_x > = 0 \qquad \text{...(7.76)}$$

Thus, the expected average of the momentum is zero.

Momentum eigen values

The momentum (p_{xn}) of a particle in a state of energy eigen value E_n is given by the following equation:

$$p_{xn}^2 = 2mE_n = 2m \frac{n^2 \pi^2 \hbar^2}{2mL^2} = \left(\frac{n\pi\hbar}{L} \right)^2$$

or, $$p_{xn} = \pm \left(\frac{n\pi\hbar}{L} \right) \qquad \text{...(7.77)}$$

From Eq. (7.77) one can conclude that the momentum eigen values are also discrete like the corresponding energy eigen values. The ($\pm$) sign indicates that for any value of the quantum number n, the particle has momenta in two mutually opposite directions. And at any energy level E_n, the magnitude of the momentum is $\left(\frac{n\pi\hbar}{L} \right)$ but its direction can be either positive or negative, this means that the particle moves to and fro in the box.

The average momentum for any value of the quantum number n is given by the following equation:

$$p_{xn(av)} = \frac{1}{2} \left[\left(+ \frac{n\pi\hbar}{L} \right) + \left(- \frac{n\pi\hbar}{L} \right) \right]$$

or, $$p_{xn(av)} = 0 \qquad \text{...(7.78)}$$

This is consistent with the expected average of p_x, i.e., $< p_x >$ as can be seen in Eq. (7.76).

Eigen function of the momentum

The eigen function Ψ'_n for an operator $\hat{\alpha}$ is given by

$$\hat{\alpha}\, \Psi'_n = \alpha\, \Psi'_n$$

where α is the eigen value of $\hat{\alpha}$. So for the momentum operator $\hat{p}_x$ the corresponding eigen value p_{xn} with respect to the eigen function Ψ'_n is given by

$$\hat{p}_x\, \Psi'_n = p_{xn}\, \Psi'_n \qquad \text{...(7.79)}$$

where the momentum operator $\hat{p}_n$ is given by $\hat{p}_x = -i\hbar \frac{\partial}{\partial x}$.

The energy eigen function is given by the following expression:

$$\Psi_n = \phi_n(x)\, e^{-i E_n t/\hbar}$$

or, $$\Psi_n = \sqrt{\frac{2}{L}} \sin \left(\frac{n\pi x}{L} \right) e^{-i E_n t/\hbar}$$

This energy eigen function does not satisfy Eq. (7.79). So, the energy eigen function Ψ_n is different from the momentum eigen function Ψ'_n.

One can write $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$, so the wave function $\phi_n(x)$ can be written as

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin \left(\frac{n\pi x}{L} \right)$$

$$= \sqrt{\frac{2}{L}} \left(\frac{e^{\frac{in\pi x}{L}} - e^{\frac{-in\pi x}{L}}}{2i} \right)$$

or, $$\phi_n(x) = \frac{1}{2i}\sqrt{\frac{2}{L}}\, e^{in\pi x/L} - \frac{1}{2i}\sqrt{\frac{2}{L}}\, e^{-in\pi x/L}$$

or, $$\phi_n(x) = \phi_n^+(x) - \phi_n^-(x) \qquad ...(7.80)$$

where $\phi_n^+(x) = \frac{1}{2i}\sqrt{\frac{2}{L}}\, e^{in\pi x/L}$ ...(7.81)

and $\phi_n^-(x) = \frac{1}{2i}\sqrt{\frac{2}{L}}\, e^{-in\pi x/L}$ ...(7.82)

Now, each of $\phi_n^+(x)$ and $\phi_n^-(x)$ of Eqs. (7.81) and (7.82), satisfies the eigen value Eq. (7.79). Hence $\phi_n^+(x)$ and $\phi_n^-(x)$ are the momentum eigen functions, i.e., eigen functions of $\hat{p}_x$, with eigen values $\left(+\frac{n\pi\hbar}{L}\right)$ and $\left(-\frac{n\pi\hbar}{L}\right)$ respectively.

(ii) Three-dimensional case (3D box)

Let us suppose that a particle of mass m is confined in a rectangular box $ABCDHGFE$ with its edges parallel to the three axes (i.e., x axis, y axis and z axis) as has been shown in the diagram of Fig. 7.4.

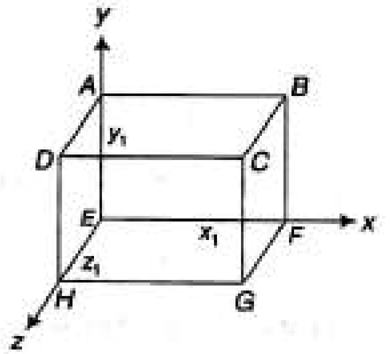


Fig. 7.4 Particle in a three-dimensional rectangular box $ABCDHGFE$ with length of edges x_1, y_1, and z_1 respectively.

Let the three sides of the said box be x_1, y_1, and z_1 respectively. The particle can move freely inside the box having ranges $0 < x < x_1$, $0 < y < y_1$ and $0 < z < z_1$ where the potential is zero (i.e., $V = 0$). The potential function $V(\vec{r}) = V(x, y, z)$ is having a constant value $V = 0$ in the regions given below:

$V(x, y, z) = 0$ for $0 < x < x_1$,

$V(x, y, z) = 0$ for $0 < y < y_1$ and

$V(x, y, z) = 0$ for $0 < z < z_1$.

And the potential outside the box (also on the walls of it) is always infinite,

i.e., $V(x, y, z) = \infty$ for $0 \geq x \geq x_1$,

$V(x, y, z) = \infty$ for $0 \geq y \geq y_1$ and

$V(x, y, z) = \infty$ for $0 \geq z \geq z_1$

The time-independent Schrödinger's wave equation for the particle inside the box is given by the following equation:

$$\nabla^2 \Psi + \frac{2mE}{\hbar^2}\Psi = 0 \quad [\because V = 0]$$

or, $$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} + \frac{2mE\Psi}{\hbar^2} = 0 \qquad ...(7.83)$$

Equation (7.83) can be solved by using the method of separation of variables. Let us assume that the function Ψ can be expressed as the product of three variables X, Y and Z where $X = X(x)$, $Y = Y(y)$ and $Z = Z(z)$. So, each of the three functions X, Y and Z depends only on one coordinate.

$$\therefore \qquad \Psi(x, y, z) = X(x)\, Y(y)\, Z(z) \qquad \text{...(7.84)}$$

Hence,
$$\frac{\partial^2 \Psi}{\partial x^2} = YZ \frac{d^2}{dx^2} X,$$
$$\frac{\partial^2 \Psi}{\partial y^2} = ZX \frac{d^2}{dy^2} Y \quad \text{and} \quad \frac{\partial^2}{\partial z^2} = XY \frac{d^2}{dz^2} Z$$

Now, substituting these in Eq. (7.82) and then dividing by (XYZ), one gets

$$\frac{1}{X}\frac{d^2}{dx^2}X + \frac{1}{Y}\frac{d^2}{dy^2}Y + \frac{1}{Z}\frac{d^2}{dz^2}Z + \frac{2mE}{\hbar^2} = 0 \qquad \text{...(7.85)}$$

Now, it is to be noted that each of the terms of this equation [i.e., Eq. (7.85)] depends on a different variable and the three variables are independent of each other. And the last term of the Eq. (7.85) is a constant. The only way for this equation to be valid for all the variables of x, y and z in the intervals $(0, x_1)$, $(0, y_1)$ and $(0, z_1)$ respectively is to be a constant.

That is each differential term in the equation will be separately constant.

Thus, one can express them as

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\alpha^2, \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = -\beta^2$$

$$\text{and} \quad \frac{1}{Z}\frac{d^2Z}{dz^2} = -\gamma^2$$

where, α, β and γ are some constants.

$$\therefore \qquad \alpha^2 + \beta^2 + \gamma^2 = \frac{2mE}{\hbar^2}$$

These three equations with dependent variables X, Y and Z can be rewritten as follows:

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0 \qquad \text{...(7.86)}$$

$$\frac{d^2Y}{dy^2} + \beta^2 Y = 0 \qquad \text{...(7.87)}$$

$$\frac{d^2Z}{dz^2} + \gamma^2 Z = 0 \qquad \text{...(7.88)}$$

The general solutions of Eqs (7.86), (7.87) and (7.88) are given below:

$$X = A_1 \sin \alpha x + B_1 \cos \alpha x,$$
$$Y = A_2 \sin \beta y + B_2 \cos \beta y \text{ and}$$
$$Z = A_3 \sin \gamma z + B_3 \cos \gamma z$$

The values of the constants A_1, A_2, A_3, B_1, B_2 and B_3 can be obtained by applying the boundary condition. The boundary conditions require that the wave function vanishes at the box walls where the potential is infinite (as stated earlier).

So, $\qquad \Psi(0, y, z) = \Psi(x, 0, z) = \Psi(x, y, 0) = 0 \qquad$ ...(7.89)

and $\qquad \Psi(x_1, y, z) = \Psi(x, y_1, z) = \Psi(x, y, z_1) = 0 \qquad$ ...(7.90)

Now, by applying the boundary conditions of relations (6.89) to the above equations of X, Y and Z, one gets,

$$B_1 = B_2 = B_3 = 0$$

Again by applying the boundary conditions of relations (7.90) to the above equations of X, Y and Z, one gets,

$$\sin(\alpha x_1) = 0 \Rightarrow \alpha x_1 = n_x \pi \quad \text{or} \quad \alpha = n_x \pi / x_1,$$

$$\sin(\beta y_1) = 0 \Rightarrow \beta y_1 = n_y \pi \quad \text{or} \quad \beta = n_y \pi / y_1$$

and

$$\sin(\gamma z_1) = 0 \Rightarrow \gamma z_1 = n_z \pi \quad \text{or} \quad \gamma = n_z \pi / z_1$$

where n_x, n_y and n_z are integers and none is equal to zero.

Hence,

$$X = A_1 \sin \frac{n_x \pi x}{x_1} \qquad \text{...(7.91)}$$

$$Y = A_2 \sin \frac{n_y \pi y}{y_1} \qquad \text{...(7.92)}$$

and

$$Z = A_3 \sin \frac{n_z \pi z}{z_1} \qquad \text{...(7.93)}$$

Now, substituting these values of X, Y and Z in Eq. (7.84), one gets

$$\Psi(x, y, z) = A_1 A_2 A_3 \sin \frac{n_x \pi x}{x_1} \sin \frac{n_y \pi y}{y_1} \sin \frac{n_z \pi z}{z_1}$$

or,

$$\Psi(x, y, z) = A \sin \frac{n_x \pi x}{x_1} \sin \frac{n_y \pi y}{y_1} \sin \frac{n_z \pi z}{z_l}$$

where $A = A_1 A_2 A_3$ and A is the normalization constant.

Now, A can be obtained by using the normalization condition as given below.

$$\int_0^{x_1}\int_0^{y_1}\int_0^{z_1} \Psi^* \Psi \, dx\, dy\, dz = 1$$

or,

$$A^2 \int_0^{x_1}\int_0^{y_1}\int_0^{z_1} \sin^2\left(\frac{n_x \pi x}{x_1}\right) \sin^2\left(\frac{n_y \pi y}{y_1}\right) \sin^2\left(\frac{n_z \pi z}{z_1}\right) dx\, dy\, dz = 1$$

or,

$$A^2 \cdot \left(\frac{x_1}{2}\right)\left(\frac{y_1}{2}\right)\left(\frac{z_1}{2}\right) = 1$$

or,

$$A = \frac{2\sqrt{2}}{\sqrt{(x_1 y_1 z_1)}}$$

So, the normalized wave function $\Psi_n(x, y, z)$ is given by the following expression:

$$\Psi_n(x, y, z) = \frac{2\sqrt{2}}{\sqrt{(x_1 y_1 z_1)}} \sin\left(\frac{n_x \pi x}{x_1}\right) \sin\left(\frac{n_y \pi y}{y_1}\right) \sin\left(\frac{n_z \pi z}{z_1}\right) \qquad \text{...(7.94)}$$

Now, one has

$$\alpha^2 + \beta^2 + \gamma^2 = \frac{2mE}{\hbar^2}$$

or,

$$\frac{n_x^2 \pi^2}{x_1^2} + \frac{n_y^2 \pi^2}{y_1^2} + \frac{n_z^2 \pi^2}{z_1^2} = \frac{2mE}{\hbar^2}$$

or,
$$E = \frac{\hbar^2 \pi^2}{2m}\left[\frac{n_x^2}{x_1^2} + \frac{n_y^2}{y_1^2} + \frac{n_z^2}{z_1^2}\right] \qquad ...(7.95)$$

Now, it is clear from the above equation [i.e., Eq. (7.95)] that E is dependent on n_x, n_y and n_z, so one can make use of the suffixes with E and rewrite Eq. (7.93) as follows:

$$E_{n_x n_y n_z} = \frac{h^2}{8m}\left[\left(\frac{n_x}{x_1}\right)^2 + \left(\frac{n_y}{y_1}\right)^2 + \left(\frac{n_z}{z_1}\right)^2\right] \qquad ...(7.96)$$

It can be stated from the above discussion that $\Psi_n\,(x, y, z)$ represents the state of the particle with total energy $E_{n_x n_y n_z}$ as given in Eq. (7.96). The three integers n_x, n_y and n_z are called the quantum numbers and are required to specify each energy state completely. For each set of values of the constants n_x, n_y and n_z, one gets a new wave function $\Psi_n\,(x, y, z)$ which can be represented by $\Psi_{n(n_x n_y n_z)}(x, y, z)$. And since $\Psi_n\,(x, y, z)$ cannot be zero within the box, no quantum number of the set (n_x, n_y, n_z) can be zero. So, the ground state energy is given by

$$E_g = E_{111} = \frac{h^2}{8m}\left(\frac{1}{x_1^2} + \frac{1}{y_1^2} + \frac{1}{z_1^2}\right) \qquad ...(7.97)$$

The corresponding wave function of energy E_{111} is given by

$$\Psi_{n(111)} = \left(\sqrt{\frac{8}{x_1\, y_1\, z_1}}\right) \sin\left(\frac{\pi x}{x_1}\right) \sin\left(\frac{\pi y}{y_1}\right) \sin\left(\frac{\pi z}{z_1}\right) \qquad ...(7.98)$$

The momentum of the particle in a state is given by

$$p_{n_x n_y n_z} = \sqrt{2m\, E_{n_x n_y n_z}}$$

or,
$$p_{n_x n_y n_z} = \sqrt{\frac{h^2}{4}\left(\frac{n_x^2}{x_1^2} + \frac{n_y^2}{y_1^2} + \frac{n_z^2}{z_1^2}\right)}$$

or,
$$p_{n_x n_y n_z} = \frac{h}{2}\sqrt{\left(\frac{n_x}{x_1}\right)^2 + \left(\frac{n_y}{y_1}\right)^2 + \left(\frac{n_z}{z_1}\right)^2}$$

Degeneracy of the energy states

If one considers a cubical box, then $x_1 = y_1 = z_1$. Let $x_1 = y_1 = z_1 = L$

Then the wave function of the time-independent Schrödinger's equation and its corresponding energy value are given by the following two equations respectively:

$$\Psi_{n(n_x n_y n_z)}\,(x, y, z) = \sqrt{\frac{8}{L^3}} \sin\left(\frac{n_x\, \pi x}{L}\right) \sin\left(\frac{n_y\, \pi y}{L}\right) \sin\left(\frac{n_z\, \pi z}{L}\right) \qquad ...(7.99)$$

and
$$E_{n_x n_y n_z} = \frac{h^2}{8mL^2}\,(n_x^2 + n_y^2 + n_z^2) \qquad ...(7.100)$$

As can be seen from Eq. (7.100), in a cubical box, the total energy $E_{n_x n_y n_z} \propto (n_x^2 + n_y^2 + n_z^2)$, i.e. the total energy is proportional to the sum of squares of the three quantum numbers n_x, n_y and n_z. And the particle in such a cube can have the same value of energy for more than one set of values of the quantum numbers (n_x, n_y, n_z). This means that more than one wave functions can have the same energy value. The state represented by such a set of wave functions, is known as denegenerate state.

Let us take help of an example to understand the concept of degeneration more clearly. Let us consider the following three sets of values of n_x, n_y and n_z.

Table 7.2 Some quantum numbers, energy levels and their wavefunctions.

n_x	n_y	n_z	$n_x^2+n_y^2+n_z^2$	E	Ψ_n
1	1	2	6	E_{112}	$\Psi_{n(112)}$
1	2	1	6	E_{121}	$\Psi_{n(121)}$
2	1	1	6	E_{211}	$\Psi_{n(211)}$

For the aforementioned three different states, the particle has the same energy $\frac{3h^2}{4mL^2}$. So, one can write

$$E_{112} = E_{121} = E_{211} = \frac{3h^2}{4mL^2} \qquad ...(7.101)$$

For these three sets of quantum numbers (i.e., values of n_x, n_y and n_z) the wave functions are quite different. And hence one can write the following inequation

$$\Psi_{n(112)} \neq \Psi_{n(121)} \neq \Psi_{n(211)}$$

Such energy states like $\Psi_{n(112)}$, $\Psi_{n(121)}$ and $\Psi_{n(211)}$ related to the same value of energy are called degenerate states. Since in these cases the wave functions are different, the probability densities in these three states are different from each other. The states having the energy equal to $\frac{3h^2}{4mL^2}$ are considered to have a degeneracy of three fold. **The number of the independent wave functions (and hence the number of independent states) corresponding to the same value of energy is called the degeneracy of the energy states.**

In case of a one-dimensional box, the degeneracy of energy states is not observed. For a particle in a cubical box, the degree of degeneracy for few energy states (i.e., energy levels) are presented in the Table 7.3 which follows:

Table 7.3 Some energy levels with their degenerate quantum number sets.

Energy Levels	*Energy Values*	*Quantum Numbers*	*Degree of Degeneracy*
E_{111}	$\frac{3h^2}{8mL^2}$	(1, 1, 1)	No degeneracy
E_{112}, E_{121}, E_{122}	$\frac{6h^2}{8mL^2}$	(1, 1, 2), (1, 2, 1), (2, 1, 1)	Three-fold degeneracy
E_{221}, E_{212}, E_{122}	$\frac{9h^2}{8mL^2}$	(2, 2, 1), (2, 1, 2), (1, 2, 2)	Three-fold degeneracy
E_{311}, E_{131}, E_{113}	$\frac{11h^2}{8mL^2}$	(3, 1, 1), (1, 3, 1), (1, 1, 3)	Three-fold degeneracy
E_{222}	$\frac{12h^2}{8mL^2}$	(2, 2, 2)	No degeneracy
E_{123}, E_{132}, E_{231}, E_{213}, E_{312}, E_{321}	$\frac{14h^2}{8mL^2}$	(1, 2, 3), (1, 3, 2), (2, 3, 1), (2, 1, 3), (3, 1, 2), (3, 2, 1)	Six-fold degeneracy

The stationary wave function (or states) $\Psi_{n(112)}$, $\Psi_{n(121)}$ and $\Psi_{n(211)}$ for the particle in a cubical box of dimension L^3 are degenerate. The linear combination of them $\Psi_n = C_1\ \Psi_{n(112)} + C_2\ \Psi_{n(121)} + C_3\ \Psi_{n(211)}$ is also an eigen function related to the same energy value $\left(\frac{3h^2}{4mL^2}\right)$.

Worked Out Problems

Wave Function and Probability

Example 7.1 A particle which is moving along the y axis has the wave function $\Psi(y)$ as given below:

$$\Psi(y) = cy \quad \text{for} \quad 0 \le y \le 1$$
$$\Psi(y) = 0 \quad \text{for} \quad 0 \ge y \ge 1$$

Find the probability that the particle lies between $y = 0.35$ and $y = 0.75$.

Sol. The probability of finding a particle represented by the wave function $\Psi(y)$ in the length dy between y and $y + dy$ is given by

$$\Psi^*(y)\ \Psi(y)\ dy = |\Psi(y)|^2\ dy$$

Hence, the probability within the range $0.35 \le y \le 0.75$ is given by

$$P = \int_{0.35}^{0.75} |\Psi(y)|^2\ dy = c^2 \int_{0.35}^{0.75} y^2\ dy$$

$$= \frac{c^2}{3}[y^3]_{0.35}^{0.75} = \frac{c^2}{3}[0.422 - 0.043]$$

$$\therefore \qquad P = 0.126\ c^2$$

Example 7.2 Consider the wave function $\Psi = Ae^{\alpha x + i\beta t}$ where A, α and β are real positive quantities. Could it represent the wave nature of a particle?

Sol. Ψ to be an acceptable wave function, it must satisfy the normalization condition, i.e., we must get

$$\int_{-\infty}^{+\infty} |\Psi(x, t)|^2\ dx = 1 \qquad ...(1)$$

In the present case, we get

$$\int_{-\infty}^{+\infty} |\Psi|^2\ dx = \int_{-\infty}^{+\infty} \Psi^*\ \Psi\ dx = \int_{-\infty}^{+\infty} A^2\ e^{2\alpha x}\ dx$$

$$= \frac{A^2}{2\alpha}[e^{2\alpha x}]_{-\infty}^{+\infty} = \infty$$

By comparing the result with Eq. (1), we realize that the function Ψ is not a normalizable wave function as $\int_{-\infty}^{+\infty} |\Psi|^2\ dx = \infty$. So, the given function cannot represent the wave nature of any particle.

Example 7.3 (i) Show that $\Psi = Ax + B$ where A and B are constants, is a solution to Schrödinger's equation for $E = 0$ energy level of a particle in a box. (ii) Show, however, that the probability of finding a particle with this wave function is zero.

Sol. (i) Schrödinger's equation for a particle in a box is given by

$$\frac{d^2\Psi}{dx^2} + \frac{2m}{\hbar^2} E\,\Psi = 0 \quad \text{...(1)}$$

Now, differentiating $\Psi = Ax + B$ twice with respect to x gives $\frac{d^2\Psi}{dx^2} = 0$ for the left-hand side of the Schrödinger's equation. Also by putting $E = 0$ in Eq. (1), we get the same equation. So $\Psi = Ax + B$ is a solution to this Schrödinger's equation for $E = 0$.

(ii) The wave function Ψ equals to zero outside the box ($x < 0$ and $x > L$). In order that $\Psi = Ax + B$ may be continuous at $x = 0$, it must be true that $\Psi = 0$ at $x = 0$. So $A \cdot 0 + B = 0$ or $B = 0$. Similarly, in order that Ψ may be continuous at $x = L$, it must be true that $\Psi = 0$ for $L = 0$. So, $A \cdot L + 0 = 0$ or $A = 0$. With both A and B equal to zero, $\Psi = Ax + B = 0$. Thus, the wave function is equal to zero inside the box as well as outside the box and probability of finding the particle anywhere with this wave function is zero.

Example 7.4 A particle is represented by the wave function $\Psi(x) = e^{-|x|} \sin \alpha x$. What is the probability that its position to the right of the point $x = 1$ (i.e., the point (1, 0, 0))?

Sol. Given $\quad \Psi(x) = e^{-|x|} \sin \alpha x$

Let, $\quad \Psi_1(x) = e^{x} \sin(\alpha x)$ for $x < 0$

and $\quad \Psi_2(x) = e^{-x} \sin(\alpha x)$ for $x > 0$

$$\therefore \quad \int_{-\infty}^{+\infty} |\Psi(x)|^2\, dx = \frac{1}{c^2}, \quad \text{where } c \text{ is the normalization constant.}$$

$$\text{or,} \quad \int_{-\infty}^{0} |\Psi_1(x)|^2\, dx + \int_{0}^{+\infty} |\Psi_2(x)|\, dx = \frac{1}{c^2}$$

$$\text{or,} \quad \int_{-\infty}^{0} e^{2x} \sin^2 \alpha x\, dx + \int_{0}^{+\infty} e^{-2x} \sin^2 \alpha x\, dx = \frac{1}{c^2}$$

$$\therefore \quad \text{L.H.S} = \int_{-\infty}^{0} e^{2x} \left[\frac{1-\cos 2\alpha x}{2}\right] dx + \int_{0}^{+\infty} e^{-2x} \left[\frac{1-\cos 2\alpha x}{2}\right] dx$$

$$= \frac{1}{2} - \frac{1}{2}\int_{0}^{-\infty} e^{2x} \cos 2\alpha x\, dx - \frac{1}{2}\int_{0}^{+\infty} e^{-2x} \cos 2\alpha x\, dx$$

$$= \frac{1}{2} - \frac{1}{4(1+\alpha^2)} - \frac{1}{4(1+\alpha^2)}$$

$$\therefore \quad \frac{1}{2} - \frac{2}{4(1+\alpha^2)} = \frac{1}{c^2}$$

$$\text{or,} \quad \frac{1+\alpha^2-1}{2(1+\alpha^2)} = \frac{1}{c^2}$$

$$\therefore \qquad c = \sqrt{\frac{2(1+\alpha^2)}{\alpha^2}}$$

So, the normalized wave function is given by

$$\Psi(x) = \sqrt{\frac{2(1+\alpha^2)}{\alpha^2}}\, e^{-|x|} \sin \alpha x$$

The required probability is given by

$$P = \int_1^{\infty} |\Psi|^2\, dx = \frac{1}{c^2} \int_1^{\infty} e^{-2x} \sin^2 \alpha x\, dx$$

$$= \frac{e^{-2}}{4(1+\alpha^2)} [1 + \alpha^2 - \cos 2\alpha + \alpha \sin 2\alpha]$$

Normalization and Orthogonality

Example 7.5 A particle is in a cubic box with infinitely hard walls whose edges are L units long. The wave function of the particle is given by

$$\Psi = A \sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{2}\right) \sin\left(\frac{n_z \pi z}{L}\right)$$

Find the value of the normalization constant A.

Sol. The condition for normalization is that the value of the normalization constant should satisfy the following condition:

$$\int_{-\infty}^{+\infty} \Psi(x, y, z)\, \Psi(x, y, z)\, dV = 1$$

i.e., the total probability in the entire space must be 1.

$$\int_{-\infty}^{+\infty} \Psi(x, y, t)\, \Psi(x, y, z)\, dx\, dy\, dz = 1 \quad \text{where } dV = dx\, dy\, dz$$

or, $$\int_{-\infty}^{+\infty} A^2 \sin^2\left(\frac{n_x \pi x}{L}\right) \sin^2\left(\frac{n_y \pi y}{L}\right) \sin^2\left(\frac{n_z \pi z}{L}\right) dx\, dy\, dz = 1$$

or, $$A^2 \int_{-\infty}^{+\infty} \sin^2\left(\frac{n_x \pi x}{L}\right) dx \int_{-\infty}^{+\infty} \sin^2\left(\frac{n_y \pi y}{L}\right) dy \int_{-\infty}^{+\infty} \sin^2\left(\frac{n_z \pi z}{L}\right) dz = 1$$

Since the wave function is zero outside the box, we can write

$$A^2 \int_0^L \sin^2\left(\frac{n_x \pi x}{L}\right) dx \int_0^L \sin^2\left(\frac{n_y \pi y}{L}\right) dy \int_0^L \sin^2\left(\frac{n_z \pi z}{L}\right) dz = 1$$

or, $$A^2 \left(\frac{L}{2}\right)\left(\frac{L}{2}\right)\left(\frac{L}{2}\right) = 1$$

or, $$A^2 = \frac{8}{L^3}$$

$$\therefore \qquad A = \sqrt{\frac{8}{L^3}}.$$

Example 7.6 Two wave functions $\Psi_m(x) = A\sin\left(\frac{m\pi x}{2}\right)$ and $\Psi_n(x) = B\sin\left(\frac{n\pi x}{2}\right)$ represent the motion of a particle confined in a one-dimensional box with perfectly hard walls. A and B are normalization constants. Show that if m and n are two integers and $m \neq n$, then the two given wave functions are orthogonal to each other.

Sol. Let
$$I = \int_{-\infty}^{+\infty} \Psi_m^*(x)\,\Psi_n(x)\,dx$$

or,
$$I = \int_{-\infty}^{+\infty} A\sin\left(\frac{m\pi x}{2}\right) B\sin\left(\frac{n\pi x}{L}\right) dx$$

or,
$$I = AB\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx$$

[∵ the wave functions must be zero outside the box]

or,
$$I = \frac{AB}{2}\int_0^L 2\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx$$

or,
$$I = \frac{AB}{2}\int_0^L \left[\cos\frac{\pi}{L}(m-n)x - \cos\frac{\pi}{L}(m+n)x\right] dx$$

or,
$$I = \frac{AB}{2}\left[\frac{L}{\pi(m-n)}\sin\frac{\pi}{L}(m-n)x - \frac{L}{\pi(m+n)}\sin\pi\frac{(m+n)}{L}x\right]_0^L$$

or,
$$I = \frac{AB}{2}\times 0$$

∴
$$I = 0$$

i.e.,
$$\int_{-\infty}^{+\infty} \Psi_m^*(x)\,\Psi_n(x)\,dx = 0$$

Hence, $\Psi_m(x)$ and $\Psi_n(x)$ are orthogonal to each other.

Quantum Mechanical Operators

Example 7.7 Determine the expression for the following two operators:

(a) $\left(\frac{d}{dx} + x\right)^2$ (b) $\left(\frac{d}{dx}(\cos x)\right)$

Sol. Let Ψ be the corresponding wave function.

(a)
$$\left(\frac{d}{dx}+x\right)^2\Psi = \left(\frac{d}{dx}+x\right)\left(\frac{d}{dx}+x\right)\Psi$$
$$= \left(\frac{d}{dx}+x\right)\left(\frac{d\Psi}{dx}+x\Psi\right)$$
$$= \frac{d^2\Psi}{dx^2} + x\frac{d\Psi}{dx} + \Psi + x\frac{d\Psi}{dx} + x^2\Psi$$

or, $$\left(\frac{d}{dx}+x\right)^2 \Psi = \left\{\frac{d^2}{dx^2}+2x\frac{d}{dx}+(x^2+1)\right\}\Psi$$

$\therefore$ $$\left(\frac{d}{dx}+x\right)^2 = \frac{d^2}{dx^2}+2x\frac{d}{dx}+(x^2+1)$$

(b) $$\left(\frac{d}{dx}\cos x\right)\Psi(x) = \frac{d}{dx}(\cos x\,\Psi(x))$$

$$= \cos x\frac{d}{dx}\Psi + \Psi\frac{d}{dx}(\cos x)$$

$$= \cos x\frac{d}{dx}\Psi - \Psi\sin x$$

or, $$\left(\frac{d}{dx}\cos x\right)\Psi = \left(\cos x\frac{d}{dx} - \sin x\right)\Psi$$

Hence $$\left(\frac{d}{dx}\cos x\right) = \cos x\frac{d}{dx} - \sin x$$

Example 7.8 Obtain the expression for the eigen function of the momentum operator $\hat{p}_x = -i\hbar\frac{d}{dx}$ corresponding to an eigen value p_x.

Sol. The operator equation for calculation of the eigen function is given by

$$\hat{p}_x\Psi = p_x\Psi.$$

where Ψ is an wave function satisfying the corresponding Schrödinger's equation.

$\therefore$ $$-i\hbar\frac{d}{dx}\Psi = p_x\Psi$$

or, $$-i\hbar\frac{d\Psi}{\Psi} = p_x\,dx$$

or, $$-i\hbar\int\frac{d\Psi}{dx} = p_x x + c$$ where c is the constant of integration

or, $$\ln\Psi = -\frac{p_x x}{i\hbar} - \frac{c}{i\hbar}$$

or, $$\ln\Psi = -\frac{p_x}{i\hbar}x + \ln\alpha \quad \left[\text{where } \ln\alpha = -\frac{c}{i\hbar}\right]$$

or, $$\Psi = \alpha e^{(ip_x x)/\hbar}$$

This is the eigen function of $\hat{p}_x$.

Example 7.9 If $\hat{A} = \cos x$ and $\hat{B}\frac{d}{dx}$, then show that $\hat{A}$ and $\hat{B}$ do not commute.

Sol. Let Ψ be a wave function.

So, $$[\hat{A},\hat{B}]\Psi = \left[\cos x, \frac{d}{dx}\right]\Psi$$

$$= \left[\cos x\frac{d}{dx} - \frac{d}{dx}\cos x\right]\Psi$$

$$= \cos x \frac{d\Psi}{dx} - \frac{d}{dx}(\Psi \cos x)$$

$$= \cos x \frac{d\Psi}{dx} - \Psi \frac{d\cos x}{dx} - \cos x \frac{d\Psi}{dx}$$

$$= +\Psi \sin x$$

$\therefore \quad [\hat{A}, \hat{B}] = \sin x$

$\therefore \quad [\hat{A}, \hat{B}] \neq 0$ (in general)

$\therefore$ $\hat{A}$ and $\hat{B}$ do not commute.

Example 7.10 Prove that $[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$ where $\hat{A}$ and $\hat{B}$ are two operators.

Sol. Let Ψ be a wave function corresponding to both of the operators.

$\therefore \quad [\hat{A}, \hat{B}]\,\Psi = (\hat{A}\hat{B} - \hat{B}\hat{A})\,\Psi = \hat{A}\hat{B}\,\Psi - \hat{B}\hat{A}\,\Psi$ ...(1)

Again $\quad [\hat{B}, \hat{A}]\,\Psi = (\hat{B}\hat{A} - \hat{A}\hat{B})\,\Psi = \hat{B}\hat{A}\,\Psi - \hat{A}\hat{B}\,\Psi$

$$= -(\hat{A}\hat{B} - \hat{B}\hat{A})\,\Psi$$

$$= -[\hat{A}, \hat{B}]\,\Psi \quad \text{[by Eq. (1)]}$$

Hence $\quad [\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$

Example 7.11 If $[\hat{z}, \hat{p}_z] = i\hbar$, then prove that (a) $[\hat{L}_x, \hat{z}] = -i\hbar\hat{y}$ and (b) $[\hat{L}_x, \hat{p}_z] = -i\hbar\,\hat{p}_y$.

Sol. We know that $\quad \hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$

Now,

(a)
$$[\hat{L}_x, \hat{z}] = [\hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \hat{z}]$$
$$= [\hat{y}\hat{p}_z, \hat{z}] - [\hat{z}\hat{p}_y, \hat{z}]$$
$$= \hat{y}\hat{p}_z\hat{z} - \hat{z}\hat{y}\hat{p}_z - \hat{z}\hat{p}_z\hat{z} + \hat{z}\hat{z}\hat{p}_y$$
$$= \hat{y}\hat{p}_z\hat{z} - \hat{y}\hat{z}\hat{p}_z + \hat{y}\hat{z}\hat{p}_z - \hat{z}\hat{y}\hat{p}_z - \hat{z}\hat{p}_y\hat{z} + \hat{z}\hat{z}\hat{p}_y$$
$$= \hat{y}(\hat{p}_z\hat{z} - \hat{z}\hat{p}_z) + (\hat{y}\hat{z} - \hat{z}\hat{y})\hat{p}_z - \hat{z}(\hat{p}_y\hat{z} - \hat{z}\hat{p}_y)$$
$$= \hat{y}[\hat{p}_z, \hat{z}] + [\hat{y}, \hat{z}]\hat{p}_z - \hat{z}[\hat{p}_y, \hat{z}]$$
$$= -\hat{y}[\hat{z}, \hat{p}_z] + 0 - 0$$
$$= -i\hbar\hat{y}$$

(b)
$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$$

$\therefore$
$$[\hat{L}_x, \hat{p}_z] = [\hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \hat{p}_z]$$
$$= [\hat{y}\hat{p}_z, \hat{p}_z] - [\hat{z}\hat{p}_y, \hat{p}_z]$$
$$= \hat{y}\hat{p}_z\hat{p}_z - \hat{p}_z\hat{y}\hat{p}_z - \hat{z}\hat{p}_y\hat{p}_z + \hat{p}_z\hat{z}\hat{p}_y$$
$$= (\hat{y}\hat{p}_z - \hat{p}_y)\hat{p}_z - \hat{z}\hat{p}_y\hat{p}_z + \hat{p}_z\hat{z}\hat{p}_y$$
$$= [\hat{y}, \hat{p}_z]\hat{p}_z - \hat{z}\hat{p}_y\hat{p}_z + \hat{z}\hat{p}_z\hat{p}_y - \hat{z}\hat{p}_z\hat{p}_y + \hat{p}_z\hat{z}\hat{p}_y$$

or,
$$[\hat{L}_x, \hat{p}_z] = [\hat{y}, \hat{p}_z]\hat{p}_z - \hat{z}(\hat{p}_y\hat{p}_z - \hat{p}_z\hat{p}_y) - (\hat{z}\hat{p}_z - \hat{p}_z\hat{z})\hat{p}_y$$
$$= [\hat{y}, \hat{p}_z]\hat{p}_z - \hat{z}[\hat{p}_y, \hat{p}_z] - [\hat{z}, \hat{p}_z]\hat{p}_y$$
$$= 0 + 0 - [\hat{z}, \hat{p}_z]\hat{p}_y$$
$$= -(i\hbar)\hat{p}_y$$

$\therefore$
$$[\hat{L}_x, \hat{p}_z] = -i\hbar\,\hat{p}_y$$

Example 7.12 Show that (a) $[\hat{L}_x, \hat{x}] = 0$; (b) $[\hat{L}_x, \hat{p}_x] = 0$; (c) $[\hat{L}_x, \hat{L}_y] = -i\hbar\,\hat{L}_z$.

Sol. (a) $$\hat{L}_x = \hat{y}\,\hat{p}_z - \hat{z}\,\hat{p}_y$$

Now $$[\hat{L}_x, \hat{x}] = \hat{L}_x\,\hat{x} - \hat{x}\,\hat{L}_x$$

or, $$[\hat{L}_x, \hat{x}] = (\hat{y}\,\hat{p}_z - \hat{z}\,\hat{p}_y)\hat{x} - \hat{x}(\hat{y}\,\hat{p}_z - \hat{z}\,\hat{p}_y)$$

or, $$[\hat{L}_x, \hat{x}] = -i\hbar\left[\left(y\frac{\partial}{\partial z} - \hat{z}\frac{\partial}{\partial y}\right)x - x\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right]$$

Now, $$[\hat{L}_x, \hat{x}]\,\Psi = -i\hbar\left[\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)x - x\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right]\Psi$$

$$= -i\hbar\left[\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)x\,\Psi - x\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\Psi\right]$$

$$= -i\hbar\left[xy\frac{\partial}{\partial z}\Psi - zx\frac{\partial \Psi}{\partial y} - xy\frac{\partial \Psi}{\partial z} + zx\frac{\partial \Psi}{\partial y}\right]$$

$$= 0$$

or, $$[\hat{L}_x, \hat{x}]\,\Psi = 0$$

Hence $$[\hat{L}_x, \hat{x}] = 0.$$

(b) $$\hat{L}_x = \hat{y}\,\hat{p}_z - \hat{z}\,\hat{p}_y$$

Now $$[\hat{L}_x, \hat{p}_x] = \hat{L}_x\,\hat{p}_x - \hat{p}_x\,\hat{L}_x$$

or, $$[\hat{L}_x, \hat{p}_x] = (\hat{y}\,\hat{p}_z - \hat{z}\,\hat{p}_y)\hat{p}_x - \hat{p}_x(\hat{y}\,\hat{p}_z - \hat{z}\,\hat{p}_y)$$

or, $$[\hat{L}_x, \hat{p}_x] = -\hbar^2\left[\left(y\frac{\partial^2}{\partial z\,\partial x} - z\frac{\partial^2}{\partial y\,\partial x}\right) - \frac{\partial}{\partial x}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right]$$

Now $$[\hat{L}_x, \hat{p}_n]\,\Psi = -\hbar^2\left[\left(y\frac{\partial^2}{\partial z\,\partial x} - z\frac{\partial^2}{\partial y\,\partial x}\right)\Psi - \frac{\partial}{\partial x}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\Psi\right]$$

$$= -\hbar^2\left[y\frac{d^2\Psi}{\partial z\,\partial x} - z\frac{\partial^2\Psi}{\partial y\,\partial x} - \frac{\partial}{\partial x}\left(y\frac{\partial \Psi}{\partial z}\right) + \frac{\partial}{\partial x}\left(z\frac{\partial \Psi}{\partial y}\right)\right]$$

$$= -\hbar^2\left[y\frac{\partial^2\Psi}{\partial z\,\partial x} - z\frac{\partial^2\Psi}{\partial y\,\partial x} - y\frac{\partial^2\Psi}{\partial x\,\partial z} + z\frac{\partial^2\Psi}{\partial x\,\partial y}\right]$$

$$= 0$$

or, $$[\hat{L}_x, \hat{p}_n]\,\Psi = 0$$

Hence $$[\hat{L}_x, \hat{p}_n] = 0.$$

(c) $$[\hat{L}_x, \hat{L}_y]\,\Psi = (\hat{L}_x\,\hat{L}_y - \hat{L}_y\,\hat{L}_x)\Psi$$

or, $$[\hat{L}_x, \hat{L}_y]\,\Psi = (\hat{y}\,\hat{p}_z - \hat{z}\,\hat{p}_y)(\hat{z}\,\hat{p}_x - \hat{z}\,\hat{p}_z)\Psi - (\hat{z}\,\hat{p}_x - \hat{x}\,\hat{p}_z)(\hat{y}\,\hat{p}_z - \hat{z}\,\hat{p}_y)\Psi$$

or, $$[\hat{L}_x, \hat{L}_y]\,\Psi = -\hbar^2\left[\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\Psi - \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\Psi\right]$$

or, $$[\hat{L}_x, \hat{L}_y]\Psi = -\hbar^2\, y\frac{\partial}{\partial z}\left(z\frac{\partial\Psi}{\partial x} - x\frac{\partial\Psi}{\partial z}\right) - z\frac{\partial}{\partial y}\left(z\frac{\partial\Psi}{\partial x} - x\frac{\partial\Psi}{\partial z}\right)$$
$$-z\frac{\partial}{\partial x}\left(y\frac{\partial\Psi}{\partial z} - z\frac{\partial\Psi}{\partial y}\right) + x\frac{\partial}{\partial z}\left(y\frac{\partial\Psi}{\partial z} - z\frac{\partial\Psi}{\partial y}\right)$$

or, $$[\hat{L}_x, \hat{L}_y]\Psi = -\hbar^2\, y\frac{\partial\Psi}{\partial x} + yz\frac{\partial^2\Psi}{\partial z\,\partial x} - 0 - xy\frac{\partial^2\Psi}{\partial z^2} - 0 - z^2\frac{\partial^2\Psi}{\partial y\,\partial x} + 0 + zx\frac{\partial^2\Psi}{\partial y\,\partial z} - 0$$
$$-yz\frac{\partial^2\Psi}{\partial x\,\partial z} + 0 + z^2\frac{\partial^2\Psi}{\partial x\,\partial y} + 0 + xy\frac{\partial^2\Psi}{\partial z^2} - x\frac{\partial\Psi}{\partial y} - zx\frac{\partial^2\Psi}{\partial y\,\partial z}$$

or, $$[\hat{L}_x, \hat{L}_y]\Psi = -\hbar^2\left[y\frac{\partial\Psi}{\partial x} - x\frac{\partial\Psi}{\partial y}\right]$$

or, $$[\hat{L}_x, \hat{L}_y]\Psi = (-i\hbar)\left[y(-i\hbar)\frac{\partial\Psi}{\partial x} - x(-i\hbar)\frac{\partial\Psi}{\partial y}\right]$$

or, $$[\hat{L}_x, \hat{L}_y]\Psi = (-i\hbar)[\hat{y}\hat{p}_x - \hat{x}\hat{p}_y]$$

or, $$[\hat{L}_x, \hat{L}_y]\Psi = -i\hbar\,\hat{L}_z \quad [\because \hat{L}_z = yp_x - xp_y]$$

Example 7.13 Evaluate the following operators which are combinations of $\hat{x}$ and $\frac{\partial}{\partial x}$, or $\frac{\partial}{\partial t}$ and t:

(a) $\left[\hat{x}, \frac{\partial}{\partial x}\right]$ (b) $\left[(\hat{x})^2, \frac{\partial^2}{\partial x}\right]$ (c) $\left[(\hat{x})^3, \frac{\partial^2}{\partial x^2}\right]$ (d) $\left[t, \frac{\partial}{\partial t}\right]$

Sol. (a) $$\left[\hat{x}, \frac{\partial}{\partial x}\right]\Psi = \left(\hat{x}\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\hat{x}\right)\Psi$$
$$= x\frac{\partial\Psi}{\partial x} - \frac{\partial}{\partial x}(x\Psi) = x\frac{\partial\Psi}{\partial x} - x\frac{\partial\Psi}{\partial x} - \Psi$$

or, $$\left[\hat{x}, \frac{\partial}{\partial x}\right]\Psi = -\Psi = (-1)\Psi$$

or, $$\left[\hat{x}, \frac{\partial}{\partial x}\right]\Psi = \hat{u}_n\Psi, \text{ where } u_n = -1$$

$\therefore$ $$\left[\hat{x}, \frac{\partial}{\partial x}\right] = \hat{u}_n \text{ (= unit negative operator)}$$

(b) $$\left[\hat{x}, \frac{\partial^2}{\partial x^2}\right]\Psi = \left(\hat{x}\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x^2}\hat{x}\right)\Psi$$
$$= x\frac{\partial^2}{\partial x^2}\Psi - \frac{\partial^2}{\partial x^2}(x\Psi)$$
$$= x\frac{\partial^2\Psi}{\partial x^2} - x\frac{\partial^2\Psi}{\partial x^2} - \Psi\frac{\partial^2}{\partial x^2}(x)$$
$$= 0$$

$\therefore$ $$\left[\hat{x}, \frac{\partial^2}{\partial x^2}\right] = 0$$

(c)
$$\left[(\hat{x})^3, \frac{\partial^2}{\partial x^2}\right]\Psi = \left\{(\hat{x})^3 \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x^2}(\hat{x})^3\right\}\Psi$$
$$= x^3 \frac{\partial^2}{\partial x^2}\Psi - \frac{\partial^2}{\partial x^2}(x^3 \Psi)$$
$$= x^3 \frac{\partial^2}{\partial x^2}\Psi - x^3 \frac{\partial^2 \Psi}{\partial x^2} - \Psi \frac{\partial^2}{\partial x^2}(x^3)$$
$$= -\ 6x\Psi$$

$\therefore$
$$\left[(\hat{x})^3, \frac{\partial^2}{\partial x^2}\right] = -6\hat{x}$$

(d)
$$\left[t, \frac{\partial}{\partial t}\right]\Psi = \left(t\frac{\partial}{\partial t} - \frac{\partial}{\partial t}t\right)\Psi$$
$$= t\frac{\partial \Psi}{\partial t} - t\frac{\partial \Psi}{\partial t} - \Psi$$

or,
$$\left[t, \frac{\partial}{\partial t}\right]\Psi = -\Psi = \hat{u}_n \Psi$$

where $u_n = -1$ and $\hat{u}_n$ is unit negative operator

$\therefore$
$$\left[t, \frac{\partial}{\partial t}\right] = \hat{u}$$

Eigen Functions and Eigen Values

By the principle of eigen values and eigen functions, we get $\hat{\alpha}\,\Psi = \alpha\,\Psi$ where $\hat{\alpha}$ is the operator and α is its numerical eigen value.

Example 7.14 Find the eigen function and eigen value of the momentum operator.

Sol. The momentum operator is given by

$$\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$$

Let p_x be the eigen value of $\hat{p}_x$.

$\therefore$ $$-i\hbar\frac{\partial}{\partial x}\Psi = p_x\,\Psi(x)$$

or, $$-i\hbar\frac{d\Psi(x)}{\Psi(x)} = p_x\,dx$$ [Since there is no mention of other independent variables]

or, $$-i\hbar\int\frac{d\Psi(x)}{\Psi(x)} = p_x\int dx$$

or, $$-i\hbar\ \ln\ \Psi(x) = p_x x + c_1,$$ where c_1 is the constant of integration

or, $$\ln\ \Psi(x) = \frac{i}{\hbar}p_x x + \frac{i}{\hbar}c_1$$

or, $$\ln\ \Psi(x) = \frac{i}{\hbar}p_x x + c_2$$ where $c_2 = \frac{i}{\hbar}c_1$

or, $\Psi(x) = ce\left(\frac{ip_x x}{\hbar}\right)$ where $c_2 = \ln c$

which is the required eigen function.

The eigen value p_x can be any number. On putting the boundary conditions such that $\Psi(x)$ be a periodic function in some distance L (periodic potential).

$$e^{(i p_x x/\hbar)} = e^{ip_x (x+L)/\hbar}$$

Putting $x = 0$, we get

$$1 = e^{ip_x L/\hbar}$$

or, $$1 = \cos\left(\frac{p_x L}{\hbar}\right) + i \sin\left(\frac{p_x L}{\hbar}\right)$$

or, $\cos\left(\frac{p_x L}{\hbar}\right) + i \sin\left(\frac{p_x L}{\hbar}\right) = 1 + i\,0$

Hence $\cos\left(\frac{p_x L}{\hbar}\right) = 1$

or, $\cos\left(\frac{p_x L}{\hbar}\right) = \cos(2n\,\pi)$

where $n = 1, 2, 3, ...$

or, $$p_x = \frac{2n\pi\hbar}{L}$$

The eigen function is now given by

$$\Psi(x) = ce^{(ip_x x/\hbar)} = ce^{i(2\pi n\, x/L)}$$

So, the eigen functions are discrete and real.

Example 7.15 Show that the function $\Psi(x) = cx\,e^{\left(-\frac{1}{2}x^2\right)}$ is an eigen function of the operator $\left(x^2 - \frac{d^2}{dx^2}\right)$. Find the eigen value.

Sol. By the definition of eigen function and eigen value, one gets

$$\hat{\alpha}\,\Psi(x) = \alpha\,\Psi(x)$$

where α is eigen value of $\hat{\alpha}$,

i.e., the operator returns the same function multiplied by the eigen value.

Now, $\left(x^2 - \frac{d^2}{dx^2}\right)\left\{cx\,e^{-\frac{1}{2}x^2}\right\}$

$$= cx^3\,e^{-\frac{1}{2}x^2} - c\frac{d^2}{dx^2}\left\{xe^{-\frac{1}{2}x^2}\right\}$$

$$= cx^3\,e^{-\frac{1}{2}x^2} - c\frac{d}{dx}\left\{e^{-\frac{1}{2}x^2} - x^2\,e^{-\frac{1}{2}x^2}\right\}$$

$$= cx^3\,e^{-\frac{1}{2}x^2} + cx\,e^{-\frac{1}{2}x^2} + 2\,cx\,e^{-\frac{1}{2}x^2} - cx^3\,e^{-\frac{1}{2}x}$$

$$= 3\,cx\,e^{-\frac{1}{2}x^2}$$

$$= (3)\;\Psi(x) \quad \text{where } \Psi(x) = cx\,e^{-\frac{1}{2}x^2}$$

or, $\left(x^2 - \dfrac{d^2}{dx^2}\right) \Psi(x) = (3)\ \Psi(x)$

$\therefore$ the eigen value of the eigen function $\Psi(x) = cx\, e^{-\frac{1}{2}x^2}$ is 3.

Hence, $cx\, e^{-\frac{1}{2}x^2}$ is an eigen function of the operator $\left(x^2 - \dfrac{d^2}{dx^2}\right)$.

Expected Averages

Example 7.16 The wave function of a particle in a one-dimensional box of length L is given by

$$\Psi_n = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$$

Find the expected average or expectation value of x and x^2.

Sol. The expected average of x is given by

$$< x > = \int_0^L \Psi_n^*(x)\, \hat{x}\, \Psi_n(x)\, dx$$

or, $$< x > = \frac{2}{L} \int_0^L x \sin^2 \frac{n\pi x}{L}\, dx$$

or, $$< x > = \frac{L}{2}$$

The expected average of x^2 is given by

$$< x > = \int_0^L \Psi_n^*(x)\, (\hat{x})^2\, \Psi(x)\, dx$$

or, $$< x^2 > = \frac{2}{L} \int_0^L x^2 \sin^2 \frac{n\pi x}{L}\, dx$$

$$= \frac{L^2}{3} - \frac{L^2}{2x^2\pi^2}$$

$\therefore$ $$< x^2 > = L^2 \left(\frac{1}{3} - \frac{1}{2x^2\pi^2}\right)$$

Example 7.17 $\Psi(x) = \left[\dfrac{2(1+\alpha^2)}{\alpha^2}\right]^{\frac{1}{2}} e^{-|x|} \sin \alpha x$ is the wave function for a particle. Calculate the expected average of position x and square of the momentum p_x.

Sol. We have,

$$\Psi(x) = \left[\frac{2\,(1+\alpha^2)}{\alpha^2}\right]^{\frac{1}{2}} e^{x} \sin \alpha x, \quad \text{for } x < 0$$

and $$\Psi(x) = \left[\frac{2\,(1+\alpha^2)}{\alpha^2}\right]^{\frac{1}{2}} e^{-x} \sin \alpha x, \quad \text{for } x > 0$$

Now $$< x > = \int_{-\infty}^{+\infty} \Psi^*(x)\, \hat{x}\, \Psi(x)\, dx$$

or, $$< x > = \frac{2(1+\alpha^2)}{\alpha^2}\left[\int_{-\infty}^{0} x\, e^{2x} \sin^2 \alpha x\, dx + \int_{0}^{+\infty} x e^{-2x} \sin^2 \alpha x\, dx\right] = 0$$

$\therefore$ $$< x > = 0$$

Also, $$< (p_x)^2 > = -\hbar^2 \int_{-\infty}^{+\infty} \Psi^*(x) \frac{d^2}{dx^2} \Psi(x)\, dx$$

or, $$< (p_x)^2 > = -\hbar^2 \left[\int_{-\infty}^{0} e^x \sin \alpha x \frac{d^2}{dx^2}(e^x \sin \alpha x)\, dx + \int_{0}^{\infty} e^{-x} \sin \alpha x \frac{d^2}{dx^2}(e^{-x} \sin \alpha x)\, dx\right]$$

$\therefore$ $$< (p_x)^2 > = \hbar^2(1+\alpha^2)$$

Example 7.18 For the one-dimensional motion of a particle with normalized wave function is $\Psi_n = A \sin \alpha x$. Calculate the expected average of the kinetic energy for this particle which is confined in a region between $x = 0$ and $x = 10$.

Sol. The kinetic energy operator is given by

$$\hat{E}_k = \frac{\hat{p}_x^2}{2m} = \frac{1}{2m}\left(-i\hbar \frac{d}{dx}\right)\left(-i\hbar \frac{d}{dx}\right)$$

or, $$\hat{E}_k = \frac{-\hbar^2}{2m} \frac{d^2}{dx^2}$$

$\therefore$ the expected average of the kinetic energy is

$$< E_k > = \int_{0}^{10} \Psi_n^* \hat{E}_k \Psi\, dx$$

or, $$< E_k > = \frac{-A^2 \hbar^2}{2m} \int_{0}^{10} \sin \alpha x \frac{d^2}{dx^2} (\sin \alpha x)\, dx$$

or, $$< E_k > = \frac{\alpha^2 A^2 \hbar^2}{2m} \int_{0}^{10} \sin^2 \alpha x\, dx$$

or, $$< E_k > = \frac{\alpha^2 A^2 \hbar^2}{2m} \int_{0}^{10} \frac{1-\cos 2\alpha x}{2}\, dx$$

or, $$< E_k > = \frac{\alpha^2 A^2 \hbar^2}{4m} \left[x - \frac{\sin 2\alpha x}{2\alpha}\right]_0^{10}$$

$\therefore$ $$< E_k > = \frac{\alpha^2 A^2 \hbar^2}{4m} \left[10 - \frac{\sin 20\alpha}{2\alpha}\right]$$

Miscellaneous Problems

Example 7.19 Compute the lowest energy of a neutron confined to the nucleus of an atom where the nucleus is considered a box with size of 10^{-14} m. ($h = 6.26 \times 10^{-34}$ Js, $m = 1.6 \times 10^{-24}$ g).

Sol. Consider the nucleus as a cubical box of size 10^{-14} m.

$\therefore$ $$x = y = z = 10^{-14}\text{ m} = 1\text{ (say)}$$

For the neutron to be in the lowest energy state $n_x = n_y = n_z = 1$

Now, $$E = \frac{\pi^2 \hbar^2}{2m}\left[\frac{n_x^2}{l_x^2} + \frac{n_y^2}{l_y^2} + \frac{n_z^2}{l_z^2}\right] \qquad [\because\ l_x = l_y = l_z = l]$$

or, $$E = \frac{\pi^2 \hbar^2}{2m}\left[\frac{3}{l^2}\right]$$

or, $$E = \frac{3h^2}{8ml^2} = \frac{3 \times (6.62 \times 10^{-34})^2}{8 \times 1.6 \times 10^{-27} \times 10^{-28}}$$

$\therefore$ $$E = 10.29 \times 10^{-23}\text{ J} = 6.43\text{ MeV}$$

Example 7.20 Show that $\Psi = \phi e^{-i\omega t}$ is a wave function of a stationary state.

Sol. For Ψ to be the wave function of a stationary state, the value of $|\Psi|^2$ at each point in space must be constant, i.e., independent of time.

Now, $$|\Psi|^2 = \Psi^* \Psi = (\phi e^{-i\omega t})^* (\phi e^{-i\omega t})$$

or, $$|\Psi|^2 = (\phi^*)(e^{i\omega t})(\phi e^{-i\omega t})$$

or, $$|\Psi|^2 = |\phi|^2 e^o = |\phi|^2$$

where $|\phi|^2$ is not a function of time, so, $|\psi|^2$ is also independent of time. Hence, $\Psi = \phi e^{-i\omega t}$ is a wave function of stationary state.

Example 7.21 Find the probability that a particle trapped in a box of size L (units) can be found between 0.45 L and 0.55 L for the ground and the first excited states.

Sol. The part between 0.45 L and 0.55 L of the box is one-tenth of the size of the box and is centered on the middle of the box. Classically, we could expect the particle to be in this region 10 per cent of the time. The quantum mechanics gives quite different predictions that depend on the quantum number of the particles state. The probability of finding the particle between x_1 and x_2 when it is in the n^{th} state is given by,

$$P = \int_{x_1}^{x_2} |\Psi_n|^2\, dx = \frac{2}{L}\int_{x_1}^{x_2} \sin^2 \frac{n\pi x}{L}\, dx$$

or, $$P = \left[\frac{x}{L} - \frac{1}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right]_{x_1}^{x_2}$$

For the ground state $n = 1$

$\therefore$ we have $$P = 0.198 = 19.8\%$$

For the 1st excited state $n = 2$

$\therefore$ $$P = 0.0065 = 0.65\%$$

The low figure is consistent with the probability density $|\Psi_n|^2 = 0$ at $x = L$.

Example 7.22 A system has two energy eigen states ε_0 and $3\varepsilon_0$. Ψ_1 and Ψ_2 are the corresponding normalized wave functions. At an instant the system is in a superposed state $\Psi = c_1\Psi_1 + c_2\Psi_2$ and $c_1 = \frac{1}{\sqrt{2}}$.

(i) Find the value of c_2 if Ψ is normalized.
(ii) What is the probability that an energy measurement would yield a value of $3\varepsilon_0$?
(iii) Find out the expectation value of the energy. [WBUT 2007]

Sol. (i) $\int_{-\infty}^{+\infty} \Psi^*(x)\, \Psi(x)\, dx = 1$

or, $\int_{-\infty}^{+\infty} (c_1\Psi_1 + c_2\Psi_2)^* (c_1\Psi_1 + c_2\Psi_2)\, dx = 1$

or, $\int_{-\infty}^{+\infty} c_1^* c_1\, \Psi_1^* \Psi_1\, dx + \int_{-\infty}^{+\infty} c_2^* c_2\, \Psi_2^* \Psi_2\, dx + \int_{-\infty}^{+\infty} (c_1^* c_2\, \Psi_1^* \Psi_2 + c_1 c_2^*\, \Psi_1^* \Psi_2)\, dx = 1$

or, $\int_{-\infty}^{+\infty} |c_1|^2\, \Psi_1^* \Psi_1\, dx + \int_{-\infty}^{+\infty} |c_2|^2\, \Psi_2^* \Psi_2\, dx + 0 = 1$

or, $\boxed{c_1^2 + c_2^2 = 1}$

or, $\frac{1}{2} + c_2^2 = 1$

or, $c_2^2 = \frac{1}{2}$. Hence $c_2 = \frac{1}{\sqrt{2}}$

(ii) The probability of finding the energy $3\varepsilon_0$ is given by

$$c_2^* c_2 = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$$

(iii) $\int_{-\infty}^{+\infty} (c_1\Psi_1 + c_2\Psi_2)^* \hat{H}(c_1\Psi_1 + c_2\Psi_2)\, dx$

$$= \int_{-\infty}^{+\infty} (c_1^* \Psi_1^* + c_2^* \Psi_2^*)(c_1 \hat{H}\Psi_1 + c_2\hat{H}\Psi_2)\, dx$$

$$= \int_{-\infty}^{+\infty} (c_1^* \Psi_1^* + c_2^* \Psi_2^*)(\varepsilon_0 c_1 \Psi_1 + 3\varepsilon_0 c_2 \Psi_2)\, dx$$

$$= \int_{-\infty}^{+\infty} (\varepsilon_0 c_1^* c_1\, \Psi_1^* \Psi_1 + \varepsilon_0 c_2^* c_1\, \Psi_2^* \Psi_1 + 3\varepsilon_0 c_1^* c_2\, \Psi_1^* \Psi_2 + 3\varepsilon_0 c_2^* c_2\, \Psi_2^* \Psi_2)\, dx$$

$$= \int_{-\infty}^{+\infty} (\varepsilon_0 c_1^2\, \Psi_1^* \Psi_1) + 0 + 0 + 3\varepsilon_0 c_2^2\, \Psi_2^* \Psi_2)\, dx$$

$$= \varepsilon_0 \int_{-\infty}^{+\infty} |c_1 \Psi_1|^2\, dx + 3\varepsilon_0 \int_{-\infty}^{+\infty} |c_2 \Psi_2|^2\, dx$$

$$= \varepsilon_0 c_1^2 + 3\varepsilon_0 c_2^2 \quad [\because \Psi_1 \text{ and } \Psi_2 \text{ are normalized}]$$

$$= \varepsilon_0 \cdot \frac{1}{2} + 3\varepsilon_0 \cdot \frac{1}{2}$$

$$= 2\varepsilon_0$$

Example 7.23 A particle of mass m is confined within the range between $x = 0$ and $x = L$.

(i) Write down Schrödinger's equation for the particle.

(ii) Solve the equation to find out the normalized eigen functions.

(iii) Show that the eigen functions corresponding to two different eigen values are orthogonal.

(iv) If p_x be the momentum, then find $< p_x >$ as well as $< p_x^2 >$ in the ground state. [WBUT 2008]

Sol. (i) As the particle is confined in the range between $x = 0$ and $x = L$, it may be in a deep potential well or in any potential dependent on x. In the first case V = constant and in the second case $V = V(x)$. So, Schrödinger's time-independent equation for the particle is given by

$$\frac{d^2}{dx^2}\Psi(x) + \frac{2m}{\hbar^2}[E - V(x)]\,\Psi(x) = 0$$

(ii) If the particle be in deep potential well, then V = constant and let $V = 0$.

So, the wave equation becomes

$$\frac{d^2}{dx^2}\Psi(x) + \frac{2mE}{\hbar^2}\Psi(x) = 0$$

or, $$\frac{d^2\Psi(x)}{dx^2} + k^2\Psi(x) = 0 \quad \text{where } k^2 = \frac{2mE}{\hbar^2}$$

The general solution of the equation is given by

$$\Psi(x) = A\sin(kx) + B\cos(kx)$$

Now, the boundary conditions are

at $x = 0$, $\Psi(x) = 0$ and

at $x = L$, $\Psi(x) = 0$

$\therefore$ when $x = 0$, we get $B = 0$

So, the equation becomes

$$\Psi(x) = A\sin(kx)$$

Again, at $x = L$, $\Psi(x) = 0$, so we get

$$\Psi(L) = A\sin(kL)$$

or, $A\sin(kL) = 0$ or, $\sin(kL) = \sin n\pi$

or, $kL = n\pi$

$\therefore$ $$k = \frac{n\pi}{L} \quad \text{where } n = 1, 2, 3, \ldots$$

Hence the permissible wave function is given by

$$\Psi(x) = A\sin\left(\frac{n\pi x}{L}\right)$$

Let $\Psi_n(x)$ be the normalized wave function

$\therefore$ $$\Psi_n(x) = \frac{1}{N}\Psi(x) \quad \text{where } N \text{ is the norm.}$$

Now $$\int_{-\infty}^{+\infty} \Psi_n^*(x)\, \Psi_n(x)\, dx = 1$$

or, $$\frac{1}{N^2}\int_{-\infty}^{+\infty} \Psi^*(x)\, \Psi(x)\, dx = 1$$

or, $$\int_0^L A^2 \sin^2\left(\frac{n\pi x}{L}\right) dx = N^2$$

or, $$\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{N^2}{A^2}$$

or, $$\frac{1}{2}\int_0^L \left(1 - \cos\left(\frac{2n\pi x}{L}\right)\right) dx = \frac{N^2}{A^2}$$

or, $$[x]_0^L - \int_0^L \cos\left(\frac{2n\pi x}{2}\right) dx = \frac{2N^2}{A^2}$$

or, $$L - \left[\frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right]_0^L = \frac{2N^2}{A^2}$$

or, $$L - \frac{L}{2n\pi}\sin(2n\pi) = \frac{2N^2}{A^2}$$

or, $$L - 0 = \frac{2N^2}{A^2}$$

$\therefore$ $$L = \frac{2N^2}{A^2}$$

$$2N^2 = A^2 L \quad \Rightarrow \quad N = \sqrt{\frac{L}{2}}\, A$$

$\therefore$ $$\Psi_n(x) = \sqrt{\frac{2}{L}}\cdot\frac{1}{A}\, A \sin\left(\frac{n\pi x}{L}\right)$$

$\Rightarrow$ $$\Psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

(iii) Let us consider two eigen functions $\Psi_n(x)$ and $\Psi_m(x)$ where $m \neq n$

Let $$I = \int_{-\infty}^{+\infty} \Psi_m^*(x)\, \Psi_n(x)\, dx$$

or, $$I = \int_{-\infty}^{+\infty} \frac{2}{L}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{L}\int_0^L \left[\cos\frac{\pi}{L}(m-n)x - \cos\frac{\pi}{L}(m+n)x\right] dx$$

$$= \frac{1}{L}\left[\frac{L}{(m-n)\pi}\sin\frac{\pi}{L}(m-n)x - \frac{L}{(m+n)\pi}\sin\frac{\pi}{L}(m+n)x\right]_0^L$$

$$= \frac{1}{L} \times \frac{L}{\pi} [0 + 0 + 0 + 0]$$

or, $$I = \int_{-\infty}^{+\infty} \Psi'^*_m(x)\, \Psi'_n(x)\, dx = 0$$

Hence $\Psi'_m(x)$ and $\Psi'_n(x)$ are orthogonal.

(iv) The ground-state wave function is $\Psi'_1(x)$.

$\therefore$ $$\Psi'_1(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right)$$

$$< p_x > = \int_{-\infty}^{+\infty} \Psi'^*_1(x)\, \hat{p}_x\, \Psi'_1(x)\, dx$$

or, $$< p_x > = \frac{2}{L} \int_0^L \sin\left(\frac{\pi x}{L}\right)\left(-i\hbar \frac{\partial}{\partial x}\right) \sin\left(\frac{\pi x}{L}\right) dx$$

$$= \left(\frac{2}{L}\right)(-i\hbar)\left(\frac{\pi}{2}\right) \int_0^L \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi x}{L}\right) dx$$

$$= -\frac{i\hbar\pi}{2L} \int_0^L \left(\sin\left(\frac{2\pi x}{L}\right)\right) dx$$

$$= -\frac{i\hbar\pi}{2L}\left[-\frac{L}{2\pi}\cos\left(\frac{2\pi x}{L}\right)\right]_0^L$$

$$= \frac{i\hbar}{4}(1 - 1) = 0$$

$$< p_x > = 0$$

$$< p_x^2 > = \int_0^L \Psi'_1(x)\, \hat{p}_x^2\, \Psi'_1(x)\, dx$$

$$= \frac{2}{L} \int_0^L \sin\left(\frac{\pi x}{L}\right)\left(-\hbar^2 \frac{\partial^2}{\partial x^2}\right) \sin\left(\frac{\pi x}{L}\right) dx$$

$$= \frac{2\hbar^2}{L} \int_0^L \left(\frac{\pi}{L}\right)^2 \sin^2\left(\frac{\pi x}{L}\right) dx$$

$$= \frac{2\hbar^2\pi^2}{L^3} \int_0^L \sin^2\left(\frac{\pi x}{L}\right) dx$$

$$= \frac{\hbar^2\pi^2}{L^3} \int_0^L \left[1 - \cos\left(\frac{2\pi x}{L}\right)\right] dx$$

$$= \frac{\hbar^2\pi^2}{L^3} \times L + \frac{\hbar^2\pi^2}{L^3} \int_0^L \cos\left(\frac{2\pi x}{L}\right) dx$$

$$= \frac{h^2 \pi^2}{L^2} + \frac{h^2\pi^2}{L^3} \times \frac{L}{2\pi}\left[\sin\left(\frac{2\pi x}{L}\right)\right]_0^L$$

$$< p_x^2 > = \frac{h^2\pi^2}{L^2}$$

$$\therefore \qquad < p_x^2 > = \frac{h^2}{4L^2}$$

Example 7.24 Show that $[x^n, \hat{p}_n] = + i\hbar\, nx^{n-1}$, n being a positive integer.

Sol. Let Ψ be the state function.

$$\therefore \qquad \hat{p}_x x^n \Psi(x) = (-i\hbar)\frac{\partial}{\partial x}(x^n \Psi(x))$$

$$= (-i\hbar)\, nx^{n-1}\, \Psi + (-i\hbar)\, x^n \frac{\partial \Psi}{\partial x}$$

$$= \left\{-i\hbar \frac{\partial}{\partial x}(x^n)\right\}\Psi + x^n\left(-i\hbar\frac{\partial}{\partial x}\Psi\right)$$

$$= -\hat{p}_x x^n \Psi + x^n \hat{p}_x \Psi + 2\,\hat{p}_n x^n \Psi$$

or, $\qquad \hat{p}_x x^n \Psi = [x_n, \hat{p}_n]\Psi + 2\,\hat{p}_n x^n \Psi$

or, $\qquad [x^n, \hat{p}_x]\Psi = -\hat{p}_x x^n \Psi$

or, $\qquad [x^n, \hat{p}_x]\Psi = -\left(-i\hbar\frac{\partial}{\partial x}x^n\right)\Psi$

or, $\qquad [x_n, \hat{p}_n] = (i\hbar\, nx^{n-1})\,\Psi$

$\therefore \qquad [x^n, \hat{p}_x] = i\hbar\, nx^{n-1}$

Example 7.25 Show that the eigen value of a hermitian operator is real.

Sol. An operator $\hat{A}$ is hermitian if it satisfies the condition,

$$\int \Psi_1^* (\hat{A}\Psi_2)\, dV = \int (\hat{A}\,\Psi_1)^*\, \Psi_2\, dV$$

where Ψ_1^* is the complex conjugate of Ψ_1. Ψ_1 and Ψ_2 are two eigen functions of the operator $\hat{A}$. A hermitian operator is linear and has real eigen value.

Now $\qquad \hat{A}\Psi = \lambda\Psi$

$$\hat{A}^*\Psi^* = \lambda^*\Psi^*$$

$$\therefore \quad \int \Psi^*\hat{A}\,\Psi\, dx = \int \Psi^*\lambda\,\Psi\, dx = \lambda\int \Psi^*\,\Psi\, dx$$

and $\quad \int \Psi\hat{A}^*\,\Psi^* dx = \int \Psi^*\lambda^*\,\Psi^*\, dx = \lambda^*\int \Psi\,\Psi^*\, dx$

$$\therefore \quad \lambda\int \Psi^*\,\Psi\, dx = \lambda^*\int \Psi\,\Psi^*\, dx \qquad [\because \hat{A} \text{ is hermitian}]$$

or, $\qquad \lambda = \lambda^*$

$\therefore$ λ is real, i.e., if the eigen value of a hermitian operator is real.

Review Exercises

Part 1: Multiple Choice Questions

1. The probability of finding a particle in a distance dx around the point x is given by
   (a) Ψ^* (b) $\Psi^* \Psi\, dx$ (c) $\Psi \Psi^*$ (d) Ψ
2. For a stationary state, the probability density is
   (a) function of time
   (b) dependent on wave function
   (c) independent of time
   (d) independent of space coordinates
3. Schrödinger's wave equation for a moving particle contains
   (a) second-order time derivative
   (b) first-order time derivative
   (c) third-order time derivative
   (d) None of above
4. A free particle has
   (a) definite momentum but indefinite energy
   (b) definite energy and indefinite momentum
   (c) definite energy and definite momentum
   (d) energy and momentum both indefinite
5. The energy, which a particle moving in a one-dimensional box can have, is
   (a) directly proportional to the quantum number
   (b) inversely proportional to the quantum number
   (c) directly proportional to the square of the quantum number
   (d) inversely proportional to the square of the quantum number
6. For a particle trapped in a box of length l, the value of the expected average is
   (a) $\frac{1}{l}$ (b) $\frac{2}{l}$ (c) $\frac{l}{2}$ (d) None of these
7. The expected average of the momentum of a particle trapped in a box of length l is
   (a) $\frac{h}{l}$ (b) $\frac{h}{2l}$ (c) 1 (d) 0
8. A particle is freely moving inside a box. Which of the following is incorrect for the energy of the particle?
   (a) The energy is directly proportional to the square of the quantum number n.
   (b) The energy is inversely proportional to the length of the box l.
   (c) The energy is inversely proportional to the mass of the particle.
   (d) The energy is directly proportional to the potential of the box.
9. Which of the following functions is an eigen function of the operator $\frac{d^2}{dx^2}$?
   (a) $\Psi = c \ln x$ (b) $\Psi = cx^2$ (c) $\Psi = \frac{c}{x}$ (d) $\Psi = ce^{-mx}$

10. The ground-state energy of a particle moving in a one-dimensional potential box is given in terms of length l of the box by
(a) $\frac{2h^2}{8ml^2}$ (b) $\frac{h^2}{8ml^2}$ (c) $\frac{h}{8ml^2}$ (d) zero

11. Which one of the following is not an acceptable wave function of a quantum particle?
(a) $\Psi = e^x$ (b) $\Psi = e^{-x}$ (c) $\Psi = x^n$ (d) $\Psi = \sin x$

12. The wave function of motion of a particle in a one-dimensional box of length l is given by $\Psi_n = \frac{A \sin n\pi x}{L}$ where A is the norm for a wave function. The value of A is
(a) $\sqrt{\frac{1}{L}}$ (b) $\sqrt{\frac{2}{L}}$ (c) $\frac{1}{L}$ (d) $\frac{2}{L}$

13. The spacing between the n^{th} energy state and the next energy state in a one-dimensional potential box increases by
(a) $(2n-1)$ (b) $(2n+1)$ (c) $(n-1)$ (d) $(n+1)$

14. If E_1 be the energy of the ground state of a one-dimensional potential box of length l and E_2 be the energy of the ground state when the length of the box is halved, then
(a) $E_2 = 2E_1$ (b) $E_2 = E_1$ (c) $E_2 = 4E_1$ (d) $E_2 = 3E_1$

15. The energy of a particle which is confined in a cubic box of side l is given by

$E = \frac{h^2}{8ml^2}(n_x^2 + n_y^2 + n_z^2)$. If n_x, n_y, n_z may have either of the three values –1, 2, and 3, then the degree of degeneracy of this energy level is given by
(a) 3 (b) 6 (c) 2 (d) 4

16. The normalized wave function for a particle in a rectangular box of dimensions a, b, c is given by
(a) $\Psi_n = \sqrt{\frac{abc}{2}} \sin\frac{n_x \pi x}{a} \sin\frac{n_y \pi x}{b} \sin\frac{n_z \pi z}{c}$ (b) $\Psi_n = \sqrt{\frac{2}{abc}} \sin\frac{n_x \pi x}{a} \sin\frac{n_y \pi x}{b} \sin\frac{n_z \pi x}{c}$
(c) $\Psi_n = \sqrt{\frac{1}{abc}} \sin\frac{n_x \pi x}{a} \sin\frac{n_y \pi x}{b} \sin\frac{n_z \pi x}{c}$ (d) $\Psi_n = \sqrt{\frac{8}{abc}} \sin\frac{n_x \pi x}{a} \sin\frac{n_y \pi x}{b} \sin\frac{n_z \pi x}{c}$

17. The waves representing a free particle in three dimensions are
(a) standing waves (b) progressive waves (c) transverse waves (d) polarized waves

18. If the quantum numbers n_x, n_y and n_z are each equal to zero for a particle trapped in a rectangular potential well, then the particle
(a) may be absent from the well (b) may be present in the well
(c) may or may not be present in the well (d) None of these

19. Schrödinger's time-independent wave equation is
(a) $\hat{H}\Psi = E\Psi^2$ (b) $\hat{H}\Psi^2 = E\Psi$ (c) $\hat{H}\Psi^3 = E\Psi^2$ (d) $\hat{H}\Psi = E\Psi$

20. For the function $e^{\beta x}$, the eigen value of the operator $\frac{d}{dx}$ is given by
(a) β (b) β^2 (c) $\frac{\beta^2}{2}$ (d) $\frac{\beta}{2}$

21. Which of the following is a linear operator?

(a) log ( ) (b) $\sqrt{\ }$ (c) $\frac{d}{dx}$ (d) exp ( )

22. The expected average of momentum of a particle in an infinite well is

(a) $\hbar$ (b) $\frac{\hbar}{2}$ (c) $k\hbar$ (d) 0

23. The wave functions $\Psi_m(x)$ and $\Psi_n(x)$ are orthogonal to each other. Which of the following relations must hold for them?

(a) $\int_{-\infty}^{+\infty} \Psi_m^* \Psi_n\, dx = 1$ (b) $\int_{-\infty}^{+\infty} \Psi_m \Psi_n\, dx = 0$ (c) $\int_{-\infty}^{+\infty} \Psi_m^* \Psi_n\, dx = 0$ (d) $\int_{-\infty}^{+\infty} \Psi_m \Psi_n\, dx = 1$

24. If $\Psi(x, t)$ be a normalized wave function we must have

(a) $\int_{-\infty}^{+\infty} \Psi^* \Psi\, dx = 1$ (b) $\int_{-\infty}^{+\infty} \Psi^* \Psi\, dx = 0$ (c) $\int_{-\infty}^{+\infty} \Psi^* \Psi\, dx = \frac{1}{2}$ (d) $\int_{-\infty}^{-\infty} \Psi\, \Psi\, dx = 1$

25. Which of the following functions is an eigen function of the momentum operator?

(a) $A \ln x$ (b) $A\sqrt{x/m}$ (c) Ax^2 (d) None of these

[Ans. 1 (b), 2 (c), 3 (b), 4 (c), 5 (c), 6 (c), 7 (d), 8 (d), 9 (d), 10 (c), 11 (c), 12 (b), 13 (b), 14 (d), 15 (b), 16 (d), 17 (b), 18 (a), 19 (d), 20 (a), 21 (c), 22 (d), 23 (c), 24 (a), 25 (d)]

Short Questions with Answers

1. **What is a wave function? Mention four points on its physical significance.**

Ans. Schrödinger's equation represents the wave behavior of a particle while it moves. The solution of this differential equation is a function of space and time coordinates. As this function [usually denoted by $\Psi(\bar{r}, t)$] is related to the wave behavior of a particle it is called a wave function.

The following are the four points regarding the physical significance of the wave function $\Psi(\bar{r}, t)$:

(a) It gives information regarding the space-time behavior of a particle.

(b) $|\Psi(\bar{r}, t)|^2 = \Psi^*(\bar{r}, t)\, \Psi(\bar{r}, t)$ measures the probability of finding a particle in space and time.

(c) The magnitude of the wave function is large where the probability of finding the particle is high.

(d) The wave function must be continuous and single-valued at each point in space.

2. **Define adjoint of an operator.**

Ans. Let us consider two operators $\hat{Q}$ and $\hat{Q}^\dagger$ which satisfy the following relation:

$$\int \phi^* \hat{Q}\, \Psi\, dV = \int (\hat{Q}^\dagger \phi)^* \Psi\, dV$$

In this case $\hat{Q}^\dagger$ is said to be the adjoint of $\hat{Q}$. In other words so far as the value of the integral is concerned, it does not make any difference whether $\hat{Q}$ acts on ψ or its adjoint $\hat{Q}^\dagger$ acts on the other function.

3. **Define self-adjoint or hermitian operator.**

Ans. The physical quantities are real. The eigen values of operators representing a physical quantity must be real. This condition limits the type of functions which can serve as operators for the physical quantities. An operator $\hat{Q}$ which corresponds to a physical quantity must be in a position to satisfy the following condition:

$$< q > = < q >^*$$

or, $$\int \phi^* \hat{Q}\, \Psi\, dV = \int (\hat{Q}\, \phi)^*\, \Psi\, dV$$

where ϕ and Ψ are two arbitrary wave functions. **The operators whose eigen values are real are known as real or hermitian or self-adjoint operators.**

An operator $\hat{Q}$ is considered to be self-adjoint or hermitian if $\hat{Q} = \hat{Q}^\dagger$.

4. Show that the expected average of hermitian operator is real.

Ans. Let $\hat{Q}$ be a hermitian operator and Ψ_1 and Ψ_2 be two eigen functions of $\hat{Q}$ corresponding to the eigen values q_1 and q_2 respectively. Then

$$\hat{Q}\,\Psi_1 = q_1 \Psi_1 \quad \text{and} \quad \hat{Q}\,\Psi_2 = q_2 \Psi_2$$

The condition of self-adjointness (or hermitianity) is given by

$$\int \Psi_1^*\, \hat{Q}\, \Psi_2\, dV = \int (\hat{Q}\, \Psi_1)^*\, \Psi_2\, dV$$

or, $$\int \Psi_1^*\, q_2\, \Psi_2\, dV = \int q_1^*\, \Psi_1^*\, \Psi_2\, dV$$

or, $$(q_2 - q_1^*) \int \Psi_1^*\, \Psi_2\, dV = 0$$

This is true for any two eigen function of $\hat{Q}$. So, if $q_2 = q_1$, then $q_1^* = q_2$

Hence q_1 is real.

5. Find the eigen function and eigen values of the operator $\hat{L}_z = -i\hbar \dfrac{d}{d\phi}$

Ans. The eigen value equation of $\hat{L}_z$ is given by

$$\hat{L}_z\, \Psi(\phi) = q\, \Psi(\phi) \quad \Rightarrow \quad -i\hbar \frac{d\Psi}{d\phi} = q\Psi$$

or, $$\frac{d\Psi}{\Psi} = -\frac{q}{i\hbar}\, d\phi \quad \Rightarrow \quad \int \frac{d\Psi}{\Psi} = +\frac{iq}{\hbar} \int d\phi + \ln c$$

or, $$\Psi = c e^{\left(\frac{iq}{\hbar}\phi\right)}$$

The function Ψ is a periodic function of variable ϕ with a period of 2π, i.e., $\Psi(\phi) = \Psi(\phi + 2\pi)$. This implies that the eigen value q is an integral multiple of $\hbar$ and Ψ as given by the above mentioned formula is the eigen function of $\hat{L}_z$.

6. (a) Write down Schrödinger's time-dependent equation. **[WBUT 2006]**

(b) State quantum mechanical postulates.

Ans. (a) Refer to Section 7.7.1.

(b) Refer to Section 7.6.

7. Show that a normalized wave function must have unit norm.

Ans. We know that Schrödinger's wave equation is a linear and homogeneous equation in Ψ and its derivatives $\left(\text{like } \dfrac{\partial \Psi}{\partial x} \text{ and } \dfrac{\partial \Psi}{\partial t}\right)$ are also linear. For this reason, if any solution of this equation is multiplied by a constant quantity then the new wave function so resulted will also be a solution. So, in order to avoid this arbitrariness, it becomes necessary to impose a normalization condition.

Let us assume that $\Psi_1(\bar{r}, t)$ is a solution of Schrödinger's equation and

$$\int_{-\infty}^{+\infty} |\Psi_1(\bar{r}, t)|^2\, dV = N^2 \qquad ...(1)$$

where $|\Psi_1(\bar{r}, t)|^2 = \Psi_1^*(\bar{r}, t)\, \Psi_1(\bar{r}, t)$ is a positive real number and the value of the integral is also so. N is, therefore, a real positive number and it is called the norm (or normalization constant) of $\Psi_1(\bar{r}, t)$.

Let us now consider another wave function $\Psi(\bar{r}, t)$ which is related to the present function $\Psi_1(\bar{r}, t)$ through the following equation:

$$\Psi(\bar{r}, t) = \frac{1}{N}\, \Psi_1(\bar{r}, t) \qquad ...(2)$$

where $\Psi(\bar{r}, t)$ is also a solution of Schrödinger's equation, as it merely differs from the present solution $\Psi_1(\bar{r}, t)$ by a factor of $(1/N)$ which is a constant.

Now, substituting this value of $\Psi_1(\bar{r}, t)$ in Eq. (1), one gets

$$\int_{-\infty}^{+\infty} N^2 |\Psi(\bar{r}, t)|^2\, dV = N^2$$

or,

$$\int_{-\infty}^{+\infty} |\Psi(\bar{r}, t)|^2\, dV = 1 \qquad ...(3)$$

The wave function $\Psi(\bar{r}, t)$ satisfies Eq. (3) and it is called the normalized wave function.

Hence, a normalized wave function must possess a unit norm, i.e., the normalization constant of a normalized wave function is unity.

8. What do you mean by an orthogonal wave function? Discuss briefly.

Ans. Let us consider a set of wave functions where each member of the set satisfies Schrödinger's wave equation, namely, $\Psi_1(\bar{r})$, $\Psi_2(\bar{r})$, $\Psi_3(\bar{r})$, etc. Such that any two members [e.g., $\Psi_m(\bar{r})$ and $\Psi_n(\bar{r})$] of the set satisfy the condition given below:

$$\int \Psi_m^*(\bar{r})\, \Psi_n(\bar{r})\, dV = 0 \qquad ...(1)$$

where $m \neq n$, then

the functions in the aforesaid set are said to be orthogonal in analogy with the condition of orthogonality of two vectors.

9. Derive Schrödinger's wave equation from the fundamental postulates of quantum mechanics.

Ans. Refer to Section 7.7.1.

10. What is the need for normalizing a wave function? Write down the normalization condition of two wave functions. What will be the condition for single wave function?

Ans. As we know Schrödinger's wave function predicts the position of the related particle in space and time. The wave function must have a finite value at any point in space and time.

The square of the wave function (i.e., $|\Psi|^2$) is proportional to the probability of finding the particle in space.

But according to the definition of probability in statistics, its value can range from 0 to +1 only. For this reason, the wave function needs to be normalized so that it can satisfy the demand of probability theory.

The normalization condition of any two wave functions which are the solutions of the same Schrödinger's equation is given by,

$$\Psi_n = \frac{1}{N}\Psi \quad \text{and} \quad \int_{-\infty}^{+\infty} \Psi_n^* \, \Psi_n \, dV = 1$$

where Ψ_n is the normalized wave function and Ψ is unnormalized one. The normalization condition for a single wave function is given by

$$\int_{-\infty}^{+\infty} \Psi_n^* \, \Psi_n \, dV = 1$$

Part 2: Descriptive Questions

1. (a) What is the physical interpretation of wave function?
   (b) How is wave function related to the probability of finding a particle at any point in space at a given time?
   (c) What is probability density?
   (d) Discuss the probability density of a particle represented by a wave which is generated by the superposition of two de Broglie waves.
2. (a) What do you mean by normalization of a wave function? What is the use of normalization?
   (b) How can you normalize a wave function by multiplying it with a suitable constant?
   (c) Can you normalize every wave function?
   (d) Consider two wave functions Ψ and $\Psi \exp(i\,\theta)$ where θ is a real quantity. What is the probability density of a particle for each of the states?
3. (a) What do you mean by stationary states? Why are they called so?
   (b) What is the condition for which the system can be represented by stationary states?
4. (a) Write down Schrödinger's time-dependent wave equation for three-dimensional and one-dimensional cases.
   (b) Obtain the time-independent form of Schrödinger's equation form its time-dependent form.
   (c) Under what condition can we obtain the time-independent form of Schrödingers's equation?
5. (a) What are the limitations of an any acceptable wave function?
   (b) Consider a particle moving along the x direction and represented by a normalized wave function Ψ_n. What is the probability of finding the particle in a region which is at infinity from the origin?
6. Write down Schrödinger's equation for one-dimensional motion of a free particle in a one-dimensional potential box. Find the eigen function and eigen energy. [WBUT 2002]
7. (a) Write down Schrödinger's equation for a particle of mass m in a rectangular box with infinitely hard walls whose edges are a, b, and c. What are the boundary conditions?
   What conditions are responsible for the quantization of energy of the particle?
   (b) The wave functions of a particle are given by

$$\Psi(x, y, z) = A \sin\left(\frac{n_x \pi x}{a}\right) \sin\left(\frac{n_y \pi y}{b}\right) \sin\left(\frac{n_z \pi z}{c}\right)$$

where n_x, n_y and n_z belong to the set $\{1, 2, 3, \ldots\}$

Find the value of the normalization constant A. [WBUT 2003]

8. Obtain the time-independent Schrödinger's equation. Show that the energy eigen values of a particle enclosed in a potential box is quantized. [WBUT 2004]

9. (a) Write down the time-independent Schrödinger's equation for a free particle of mass m and momentum $\overline{p}$.

   (b) Why in case of moving electron quantum mechanics is used while for moving cars we use newtonian mechanics? Explain. [WBUT 2005]

10. (a) Give the physical interpretation of the wave function $\Psi(x)$.

    (b) Obtain the expression for the stationary energy levels for a particle of mass m which is free to move in a region of zero potential between two rigid walls with $x = 0$ and $x = L$. Are the energy levels degenerate?

    (c) Find the corresponding wave function pertaining to part (b) of this question and normalize it. [WBUT 2005]

11. Write down Schrödinger's equation for a particle confined in a one-dimensional box. [WBUT 2006]

12. If the wave function $\Psi(x)$ of a quantum mechanical particle is given by

$$\Psi(x) = a \sin \frac{\pi x}{L}, \quad \text{for } 0 \le x \le L$$
$$= 0 \quad \text{for } 0 \ge x \ge L$$

Then determine the value of a. Also determine the value of x where the probability of finding the particle is maximum. [WBUT 2007]

13. A particle of mass m is confined within the space between $x = 0$ and $x = L$.
    (i) Write down Schrödinger's equation for such a system.
    (ii) Solve the equation to find out the normalized eigen functions.
    (iii) Show that the eigen functions corresponding to two different eigen values are orthogonal.
    (iv) If p_x denotes the momentum then find $< p_x >$ as well as $< p_x^2 >$ in the ground state. [WBUT 2008]

14. What are the values of $\left[\hat{x}, \frac{\partial}{\partial x}\right]$, $[\hat{L}_x, \hat{x}]$ and $[\hat{p}_x, \hat{p}_y]$? [WBUT 2004]

15. Show that (i) $[\hat{p}_x, \hat{x}_n] = -i\hbar\, n\, x^{n-1}$ and (ii) $[\hat{p}_x, f(x)] = -i\hbar \frac{df}{dx}$.

16. (a) State the basic postulates of quantum mechanics.

    (b) Define degeneracy and non-degeneracy. Prove that the lowest state of a free particle in a cubical box is not degenerate.

    (c) What do you mean by orthogonal wave function? Write down the condition of orthogonality.

Part 3: Numerical Problems

1. A particle exists inside a one-dimensional potential box of length L. Calculate the probability of finding the particle in a region between $\frac{L}{4}$ and $\frac{3L}{4}$ when the particle is in the lowest energy state.

   [**Ans.** 0.818]

2. Examine whether the operator $\hat{L}$ is linear in the following two cases:

   (i) $\hat{L} f(x) = f(x) + \sin x$

   (ii) $\hat{L} f(x) = f(-x)$ [**Ans.** (i) non-linear, (ii) linear]

3. Show that $\frac{\partial}{\partial x}$ and $\frac{\partial^2}{\partial x^2}$ are commutative.

4. A particle is moving along the x axis has the wave function

$$\begin{aligned} \Psi(x) &= ax \quad \text{for} \quad 0 \le x \le 1 \\ &= 0 \quad \text{for} \quad 0 \ge x \ge 1 \end{aligned}$$

   Find the expected average $< x >$ of the position of the particle. $\left[\textbf{Ans. } \frac{a^2}{4}\right]$

5. Prove that the wave function $\Psi(x, t) = A\cos(kx - \omega t)$ does not satisfy the time-dependent Schrödinger's equation for a free particle.

6. For an electron of mass 9.1×10^{-31} kg moving in the one-dimensional infinitely deep potential well of width 0.1 nm, find (i) the least possible energy, (ii) the first three eigen values in electron volt, (iii) the energy difference between the ground state and the first excited state, and (iv) the frequency of the emitted radiation due to the transition between the two states.

   [**Ans.** (i) 37.73 eV, (ii) $E_1 = 37.73$ eV, $E_2 = 150.95$ eV, $E_3 = 339.63$ eV, (iii) 113.22 eV, (iv) $\nu = 27.3 \times 10^{15}$ Hz]

7. Calculate the normalization constant for a wave function (at $t = 0$) given by

$$\Psi(x) = Ae^{-\sigma^2 x^2/2} \cdot e^{ikx}$$

   $\left[\textbf{Ans. } (\sigma/\sqrt{\pi})^{\frac{1}{2}}\right]$

8. Determine the probability density for the wave function

$$\Psi(x) = Ae^{\frac{-\sigma^2 x^2}{2}} e^{ikx}$$

   $\left[\textbf{Ans. } p = \frac{\sigma}{\sqrt{\pi}} \exp(-\sigma^2 x^2)\right]$

9. Consider two stationary state solutions $\Psi_1(\vec{r})$ and $\Psi_2(\vec{r})$ corresponding to energy states E_1 and E_2 respectively for time-independent Schrödinger's equation.

   Prove that $$\int_V \Psi_1^*(\vec{r})\, \Psi_2(r)\, dV = 0$$

10. Prove that for two linear operators $\hat{\alpha}$ and $\hat{\beta}$, $[\hat{\alpha}, \hat{\beta}] + [\hat{\beta}, \hat{\alpha}] = 0$.

11. If $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ are three linear operators, then prove that

   $[\hat{\alpha}, [\hat{\beta}, \hat{\gamma}]] + [\hat{\beta}, [\hat{\gamma}, \hat{\alpha}]] + [\hat{\gamma}, [\hat{\alpha}, \hat{\beta}]] = 0$

12. The minimum energy possible for a particle trapped in a one-dimensional infinite potential well is 10 eV. What are the next three energy levels? [**Ans.** 40 eV, 90 eV, 160 eV]

13. Find the commutator $[\hat{A}, \hat{B}]$ where $\hat{A} = x^3$ and $\hat{B} = x\dfrac{d}{dx}$. **[Ans. $-3x^3$]**
14. Calculate the expected average of p_x^2 for the wave function $\Psi = \sqrt{\dfrac{2}{L}} \sin\left(\dfrac{\pi x}{L}\right)$ in the region from $x = 0$ to $x = L$. **[Ans. $(\hbar\pi)^2/L$]**
15. Prove the following relation:

$$[\hat{A}, \hat{B} + \hat{C}] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]$$

CHAPTER 8

Statistical Mechanics

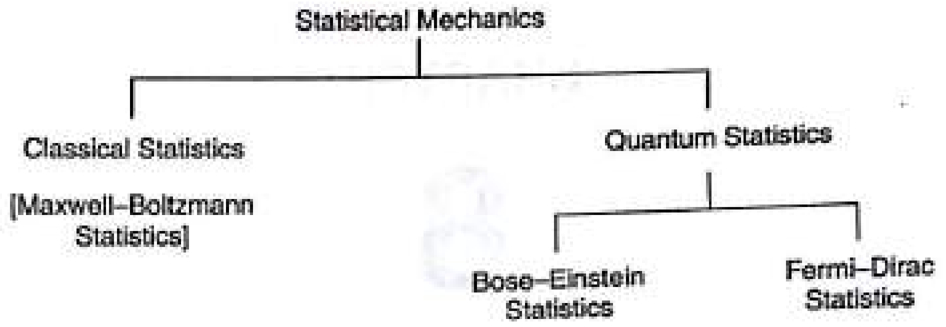
8.1 INTRODUCTION

Thermodynamics is the study of the relationship between macroscopic properties of systems, such as temperature, volume, pressure, magnetization, compressibility, etc. Statistical mechanics, also called statistical physics, is concerned with understanding how the various macroscopic properties arise as a consequence of the microscopic nature of the system. In essence, it makes macroscopic deduction from microscopic models. According to Lev Landau, statistical physics and thermodynamics together form a unit. All the concepts and quantities of thermodynamics follow most naturally, simply and rigorously from the concepts of statistical physics. Although the general statement of thermodynamics can be formulated non-statistically, their application to specific cases always requires the use of statistical physics.

Statistical approach can be applied only to such systems which contain a large number of individuals identical in some sense, but are distinguishable by means of some property. The purpose of statistical mechanics is to provide special statistical or probability laws for the distribution of identical microscopic particles (atoms and molecules) of a macroscopic system in thermal equilibrium, without considering the detailed calculation of position and velocity of individual particles and to derive all equilibrium properties of the system. Suppose, for example, 1 cm^3 of a gas under normal temperature and pressure contains 2.7×10^{19} molecules. Assuming that each molecule has three degrees of freedom, we will have to set up and solve $3 \times 2.7 \times 10^{19}$ equations of motion for 1 cm^3 of the gas. Theoretically, some equations can be solved and the solutions would be in terms of mass, position, velocity and energy of each molecule, etc., and not in terms of thermodynamic variables such as pressure, entropy, etc. Thus there is a gap between the results of classical mechanics and thermodynamics. Statistical mechanics provides the link between mechanical properties and thermodynamical properties of a system.

In statistical mechanics, the properties and laws of motion of individual atoms, molecules and elementary particles, are assumed to be known. If these laws are governed by classical mechanics, we call it classical statistics and if they are governed by quantum mechanics, we call it quantum statistics.

Classical statistics was developed by Maxwell in 1859 and then refined by Boltzmann and hence correctly called Maxwell–Boltzmann (or MB) statistics. Classical statistics is applicable to a system of identical, weakly interacting, structureless and neutral particles. Such hypothetical particles are called boltzons. Quantum particles are divided into two classes. One with integral spin (i.e., $n\hbar$) and are called Bose particles



(or simply bosons) and other with half integral spin $\left[\text{i.e.,} \left(n+\frac{1}{2}\right)\hbar\right]$ and are called Fermi particles (or simply fermions). Bose statistics was refined by Einstein and hence called Bose–Einstein (or BE) statistics and Fermi statistics was refined by Dirac and hence called Fermi–Dirac (or FD) statistics.

8.2 CONCEPT OF PHASE SPACE

In classical mechanics, the position of a point particle in three-dimensional space is determined by three cartesian coordinates x, y, z and its state of motion is given by the velocity components v_x, v_y and v_z. For many purposes it is more convenient to use the corresponding momenta $p_x\, p_y\, p_z$ (for $m\dot{x}$, $m\dot{y}$ and $m\dot{z}$ where m is the mass of the particle). To describe both the positions and the state of motion of the particle, it is required to set up a six-dimensional space called the phase space, in which the six coordinates x, y, z, p_x, p_y, p_z are marked out along six mutually perpendicular axes. A point in this space describes both the position and the motion of the particle at some particular instant. Therefore, if a position of a phase point is known at any instant of time, its trajectory is also known.

8.2.1 Phase Space of *N* Particles System

For N particles system, the mechanical state of the whole system can be determined completely in terms of $3N$ position coordinates $q_1 \ldots q_N$ and $3N$ momenta coordinates $p_1 \ldots p_N$. The 6-N-dimensional space is called the phase space or Γ space of the system. A point in Γ space represents a state of the entire system. For mono-atomic gas it is also called molecular phase space or μ space.

8.2.2 Smallest Cell in Phase Space

In the μ space of particle, a volume element

$$d\tau = dx\, dy\, dz\, dp_x\, dp_y\, dp_z$$

is the volume of a six-dimensional cell having sides dx, dy, dz, dp_x, dp_y, dp_z. Such a cell of minimum volume is called a unit cell in the μ space.

So, $$(d\tau)_{\min} = (\delta x\, \delta p_x)_{\min}\, (\delta y\, \delta p_y)_{\min}\, (\delta z\, \delta p_z)_{\min}$$

But uncertainty principle tells us that the minimum value of each of the products is approximately equal to Planck's constant h. Therefore,

$$(d\tau)_{\min} = (h)(h)(h) = h^3$$

In classical description, h can be chosen as an arbitrarily small constant having the dimensions of angular momentum. But quantum theory imposes a limitation on the accuracy with which a simultaneous specification of coordinate x and its corresponding momentum p_x is possible and from uncertainty principles, phase space is actually a cell of minimum volume h^3.

8.3 CONCEPT OF ENERGY LEVELS AND ENERGY STATES

In classical mechanics, the 'state' of a system is defined by the values of the coordinates and momenta of all its constituent particles. In quantum statistics, a quantum state is a mathematical object that fully describes a quantum system. One typically imagines some experimental apparatus and procedure which prepares this quantum state; the mathematical object then reflects the set-up of the apparatus. Quantum states can be statistically mixed, corresponding to an experiment involving a random change of the parameters. States obtained in this way are called mixed states, as opposed to pure states, which cannot be described as a mixture of others. When performing a certain measurement on a quantum state, the result is in general described by probability distribution and the form that this distribution takes is completely determined by the quantum state and the observable describing the measurement. However, unlike in classical mechanics, the result of a measurement on even a pure quantum state is only determined probabilistically.

A quantum system or particle that is bound, confined spacially, can only take on certain discrete values of energy, as opposed to classical particles, which can have any energy. These are called energy levels. The term is most commonly used for energy levels of electrons in atoms or molecules, which are bound by the electric field of the nucleus. The energy spectrum of a system with energy levels is said to be quantized.

Energy levels are said to degenerate, if the same energy level is obtained by more than one quantum mechanical state. They are then called degenerate energy levels. There may be several energy states corresponding to the same energy.

8.4 MACROSTATE AND MICROSTATE

Macrostate The macrostate of a system may be defined as the number of phase points in each cell of the phase space. For a macrostate, we specify only bulk quantities which can be determined by macroscopic measurements, such as pressure, temperature, etc. Thus an isolated system consisting of a fixed number (N) of non-interacting identical particles, having a fixed internal energy (E) and occupying a fixed space of volume (V) is said to be macroscopic state or macrostate of the system. The macrostate is also called the thermodynamic state.

Microstate A state of the system in which we specify the states of all the constituent particles is called a microstate. From a dynamical point of view, each state of a system can be defined as precisely as possible by specifying all of the dynamical variables of the system. Such a state is called a microscopic state. In general, the number of different meaningful ways in which the total energy (E) of the system can be distributed among its constituent particles called microstate. One of the fundamental hypothesis of statistical mechanics is that microstates are equally probable, which means one microstate would occur as often as another over a long period of time.

Example Suppose we have four distinguishable particles a, b, c, d and we wish to distribute them into two exactly similar compartments in an open box. Since both the compartments are exactly alike, the particles have the same *a priori* probability of going into either of them. The following table will give the various macrostates and microstates.

Macrostates	*Possible arrangement*		*No. of microstates*
	Compartment 1	*Compartment 2*	
0, 4	*O*	*abcd*	1
1, 3	*a*	*bcd*	4
	b	*cda*	
	c	*dab*	
	d	*abc*	
2, 2	*ab*	*cd*	6
	ac	*bd*	
	ad	*bc*	
	bc	*ad*	
	bd	*ac*	
	cd	*ab*	
3, 1	*bcd*	*a*	
	cda	*b*	4
	dab	*c*	
	abc	*d*	
4, 0	*abcd*	*O*	1

Each distinct arrangement is known as the microstate of the system.

8.5 THERMODYNAMIC PROBABILITY AND ENTROPY

Thermodynamical probability is defined as the number of possible microstate corresponding to any given macrostate. Thermodynamic probability is, in general, very large, unlike mathematical probability, which when normalized, as always is less than one. In our previous example, the thermodynamic probability corresponding to the macrostate (2, 2) is equal to the number of microstates in (2, 2) which is 6.

We know that when an isolated system undergoes an irreversible process, there is a net increase of the entropy of the system. The maximum value of the entropy is reached when the system arrives at a state of equilibrium. Boltzmann related the entropy S and the thermodynamic probability W by the equation

$$S = K_B \ln \frac{W}{W_o} \quad \text{....(8.1)}$$

where K_B is the Boltzmann constant and W_o the initial probability. From Eq. (8.1), we have

$$S = K_B \ln W - K_B \ln W_o$$
$$= K_B \ln W - S_o$$

The quantity $S_o = K_B \ln W_o$ can be taken as the zero from which the entropy is counted.

Hence, $$S = K_B \ln W \quad \text{...(8.2)}$$

8.6 EQUILIBRIUM MACROSTATE

A macroscopic state which does not tend to change in time except for random fluctuations is known as equilibrium macrostate. The macrostate of a system in equilibrium is time-independent, except for ever-present fluctuations. The macrostate of a system can be described by certain macroscopic parameters, i.e., parameters which characterize the properties of the system on a large scale. The equilibrium macrostate of a system is independent of its past history. For example, consider an isolated gas of N molecules in a box. These molecules may originally have been confined by a partition to one half of the box. But after the partition is removed and equilibrium has been attained, the macrostate of the gas is the same in both cases; it corresponds merely to the uniform distribution of all the molecules through the entire box.

8.7 MB, BE AND FD STATISTICS

In statistical mechanics, Maxwell–Boltzmann statistics describes the statistical distribution of material particles over various energy states in thermal equilibrium, when the temperature is high enough and density is low enough to render quantum effects negligible.

Fermi–Dirac (FD) and Bose–Einstein (BE) statistics apply when quantum effects have to be taken into account. Quantum effects appear if the concentration of particles $\left(\frac{N}{V}\right) \geq n_q$ (where n_q is the quantum concentration). The quantum concentration is when the interparticle distance is equal to the thermal de Broglie wavelength, i.e., when the wave functions of the particles are touching but not overlapping.

8.8 MAXWELL – BOLTZMANN (MB) STATISTICS

8.8.1 Basic Postulates

(i) Particles are identical and distinguishable.

(ii) They do not possess any kind of spin.

(iii) They do not obey the Pauli exclusion principle.

So any state can accommodate any number of particles. For example, the molecules of a gas. The particles that obey MB statistics are called boltzons. Example: Ideal gas molecules.

8.8.2 Maxwell–Boltzmann (MB) Distribution Function

Consider a system having energy levels $\varepsilon_1, \varepsilon_2, \varepsilon_3$ ..... and the number of particles in these levels be n_1, n_2, n_3 ...... Moreover, let each level be degenerated, i.e., let each level be associated with several sublevels. Let g_1, g_2, g_3, ..., be the number of sublevels corresponding to the energies $\varepsilon_1, \varepsilon_2, \varepsilon_3$, ..., respectively. So the available states of the system are the sublevels. Let us now consider the i^{th} level. The probability that a particular sublevel (or state) is occupied is $\left(\frac{1}{g_i}\right)$. Hence the average number of particles in that state is $\left(\frac{n_i}{g_i}\right)$, which is the occupation number for the i^{th} state.

In classical statistics, the number of ways in which the total number of particles, N can be grouped into levels having respectively n_1, n_2, n_3 ....., number of particles. The number of ways in which we can do so is equal to the number of permutations of N things, out of which n_1 are alike, n_2 others are alike, n_3 others are alike and so on. This is given by

$$M = \frac{N!}{n_1!\,n_2!\,n_3!\,\ldots} = \frac{N!}{\prod_i n_i!} \qquad \ldots(8.3)$$

where $\prod_i$ denotes the product of n_i for all values of i. Since classical particles are assumed to be distinguishable, then the total number of ways in which n_i particles occupy g_i sublevels in $g_i^{n_i}$. If $P_{\text{classical}}$ is the number of ways in which g_1 sublevels are occupied by n_1 particles, g_2 by n_2, g_3 by n_3 and so on, then

$$P_{\text{classical}} = g_1^{n_1} g_2^{n_2} g_3^{n_3} \ldots = \prod_i g_i^{n_i}$$

The total probability W of the distribution is the product of M and $P_{\text{classical}}$

$$W = \frac{N!}{\prod_i n_i!} \prod_i g_i^{n_i} = N! \prod_i \frac{g_i^{n_i}}{n_i!} \qquad \ldots(8.4)$$

Taking natural logarithm on both sides, we have

$$\ln W = \ln N! + \ln \prod_i \frac{g_i^{n_i}}{n_i!}$$

$$\ln W = \ln N! + \sum_i n_i \ln g_i - \sum_i \ln n_i!$$

Applying Stirling's theorem, [$\ln N! = N \ln N - N$ (where N is very large)], we have

$$\ln W = N \ln N - N + \sum_i n_i \ln g_i - \left[\sum_i (n_i \ln n_i - n_i)\right]$$

$$= N \ln N - N + \sum_i n_i \ln g_i - \sum_i n_i \ln n_i + \sum_i n_i$$

$$= N \ln N - \cancel{N} + \sum_i (n_i \ln g_i - n_i \ln n_i) + \cancel{N}$$

$$= N \ln N + \sum_i n_i \ln \left(\frac{g_i}{n_i}\right) \qquad \ldots(8.5)$$

The most probable distribution can be obtained by maximizing $\ln W$, i.e., by setting

$$\delta \ln W = \sum \frac{\partial \ln W}{\partial n_i} \delta n_i = 0 \qquad \ldots(8.6)$$

subject to the constraints that the total number of particles and total energy of the system are conserved:

$$N = n_1 + n_2 + n_3 + \ldots = \sum_i n_i = \text{const.}$$

$$E = n_1 \varepsilon_1 + n_2 \varepsilon_2 + \ldots = \sum_i n_i \varepsilon_i = \text{const.}$$

In differential form $\sum_i \delta n_i = 0$ ...(8.7)

and $\sum_i \varepsilon_i \, \delta n_i = 0$ ...(8.8)

Differentiating Eq. (8.5) w.r.t. n_i, we get from Eq. (8.5)

$$\sum_i \ln \left(\frac{g_i}{n_i}\right) \delta n_i = 0 \qquad \ldots(8.9)$$

From Eqs (8.7), (8.8) and (8.9), we have

$$\sum_i \left(\ln \frac{g_i}{n_i} - \alpha - \beta \varepsilon_i \right) \delta n_i = 0 \qquad ...(8.10)$$

where α, β are Lagrange's multipliers. Since coefficients δn_i are arbitrary, so from Eq. (8.10) for all values of i, the term in the bracket must be zero.

Thus, $\ln \frac{g_i}{n_i} - \alpha - \beta \varepsilon_i = 0$

or, $$\ln \frac{g_i}{n_i} = \alpha + \beta \varepsilon_i$$

or, $$\frac{n_i}{g_i} = e^{-(\alpha + \beta \varepsilon_i)} = \frac{1}{e^{\alpha + \beta \varepsilon_i}}$$

or, $$n_i = \frac{g_i}{e^{\alpha + \beta \varepsilon_i}} \qquad ...(8.11)$$

This result is known as the Maxwell–Boltzmann distribution of distinguishable particles among the energy levels.

Assuming Maxwell–Boltzmann distribution, the average speed, root mean square speed, most probable speed of the molecules of an ideal gas can be calculated. The total internal energy, specific heat at constant volume of an ideal gas can also be derived very easily.

8.8.3 Limitations of Maxwell–Boltzmann Statistics

If for a given density, the temperature of a perfect gas is sufficiently low, then Boltzmann statistics is no longer applicable and a new statistics must be set up. The MB distribution function suffers from the following major objections:

(i) The particles are assumed to be distinguishable, although in actual practice many elementary particles like electrons are indistinguishable. In BE statistics there is no restriction on the number of particles that can occupy the same energy state.

(ii) Any number of particles are assumed to occupy the same quantum state while many particles such as electrons obey the Pauli exclusion principle which does not allow a quantum state to accept more than one particle.

These objections are removed in the distribution function known as Fermi–Dirac (FD) distribution. The particles that obey the FD statistics are called Fermions. Bose–Einstein statistics (BE) give the statistical behavior of indistinguishable particles. The particles that obey BE statistics are known as boson, e.g., phonons, photons, etc.

8.9 BOSE–EINSTEIN (BE) STATISTICS

Bose–Einstein and Fermi–Dirac statistics apply when quantum effects have to be taken into account and the particles are considered indistinguishable. BE statistics was developed in 1924 by S N Bose for light quanta (photons) and generalized by A Einstein to find energy distribution among indistinguishable and identical particles, each having a spin angular momentum $m_s \hbar$ given by any of the values

$$m_s \hbar = 0, \hbar, 2\hbar, 3\hbar, ...$$

where m_s is the spin quantum number.

8.9.1 Basic Postulates

(i) The particles of the system are identical and indistinguishable.

(ii) The Pauli exclusion principle is not applicable, so any quantum state can accommodate any number of particles.

(iii) BE statistics is applicable to particles with integral spin angular momentum in units of $\hbar$.

(iv) These particles have symmetric wave function.

The particles that obey BE statistics are called bosons.

Examples Photons, phonons, α-particles, π-mesons, k-mesons, higgs-boson, ${}_1H^2$, ${}_2He^4$, ${}_6C^{12}$, ${}_8O^{16}$ are examples of bosons.

8.9.2 Bose–Einstein (BE) Distribution Function

For BE statistics, since particles are indistinguishable, the number of ways in which n_i bosons can occupy g_i sublevels (no restriction on the number of particles occupying a sublevel) be obtained by taking the permutations of $\{n_i + (g_i - 1)\}$ things, out of which a group of n_i things and another group of $(g_i - 1)$ things are alike. So the number is

$$W_i = \frac{(n_i + g_i - 1)!}{n_i!\,(g_i - 1)!} \qquad \text{...(8.12)}$$

The total number W of independent ways obtaining a distribution of particles among the quantum states in the various energy levels is the product of expressions given by Eq. (8.12) for $i = 1, 2, ..., n$.

So, $$W = \prod_i \frac{(n_i + g_i - 1)!}{n_i!\,(g_i - 1)!} \qquad \text{...(8.13)}$$

Since n_i and g_i are very large numbers compared to unity so the Eq. (8.13) can be written as

$$W = \prod_i \frac{(n_i + g_i)!}{n_i!\,g_i!} \qquad \text{...(8.14)}$$

Taking natural logarithm on both sides and applying Stirling's approximation, we get

$$\ln W = \sum_i [(n_i + g_i)\ln(n_i + g_i) - n_i \ln n_i - g_i \ln g_i] \qquad \text{...(8.15)}$$

The most probable distribution can be obtained by maximizing $\ln W$, i.e., by setting $\delta \ln W = \sum_i \frac{\partial \ln W}{\partial n_i}\delta n_i$ $= 0$ subject to the constraints that the total number of particles and total energy of the system are conserved:

$$\sum n_i = \text{const.}$$

$$\sum_i \varepsilon_i n_i = \text{const.}$$

In differential form $$\sum_i \delta n_i = 0 \qquad \text{...(8.16)}$$

and $$\sum \varepsilon_i\,\delta n_i = 0 \qquad \text{...(8.17)}$$

So by applying $\delta \ln W = 0$, Eq. (8.15) gives

$$\sum_i [\ln(n_i + g_i)\,\delta n_i - \ln n_i\,\delta n_i] = 0 \quad [\delta g_i = 0] \qquad \text{...(8.18)}$$

Using Eqs (8.16), (8.17) and (8.18), we have

$$\Sigma\left[\ln\frac{n_i+g_i}{n_i}-\alpha-\beta\,\varepsilon_i\right]\delta n_i=0 \quad ...(8.19)$$

where α, β are Lagrange's multipliers.

Since coefficients δn_i are arbitrary, so from Eq. (8.19) for all values of i, the term in the braket must be zero.

Thus, $$\ln\left(\frac{n_i+g_i}{n_i}\right)-\alpha-\beta\,\varepsilon_i=0$$

or, $$\frac{n_i}{g_i}=\frac{1}{e^{\alpha+\beta\varepsilon_i}-1} \quad ...(8.20)$$

This relation gives the most probable distribution of particles for a system obeying BE statistics and is known as Bose–Einstein distribution law.

If μ is the chemical potential and K_B is the Boltzmann constant, then in thermal equilibrium for bosons at temperature T, we can write

$$\beta=\frac{1}{K_BT} \quad \text{and} \quad \alpha=-\mu\beta$$

Substituting this value in Eq. (8.20), we obtain

$$n_i=\frac{g_i}{e^{(\varepsilon_i-\mu)/K_BT}-1} \quad ...(8.21)$$

The energy distribution function $f(\varepsilon_i)$ is the number of particles per quantum state in the energy level ε_i. Therefore, BE distribution function becomes (if we drop the index i)

$$f(\varepsilon)=\frac{n}{g}=\frac{1}{e^{(\varepsilon-\mu)/K_BT}-1} \quad ...(8.22)$$

If $\varepsilon >> K_BT$, the BE distribution reduces to the MB distribution.

8.10 FERMI–DIRAC (FD) STATISTICS

This statistics was developed by E Fermi for electrons and its relation to quantum mechanics was established by PAM Dirac in 1926. The objective of this statistics is to find the energy distribution among indistinguishable identical particles, each having a spin angular momentum $m_s\hbar$ given by any of the values

$$m_s\,\hbar=\frac{1}{2}\hbar,\ \frac{3}{2}\hbar,\ \frac{5}{2}\hbar\ ...$$

8.10.1 Basic Postulates

(i) The particles of the system are identical and indistinguishable.

(ii) The Pauli exclusion principle is applicable, so each quantum state can accommodate either no particle or only one particle.

(iii) FD statistics is applicable to particles with $\frac{1}{2}$ integral spin angular momentum in units of $\hbar$.

(iv) These particles have antisymmetric wave function.

The particles those obey FD statistics are called fermions.

Examples Protons, neutrons, electrons, μ-mesons, ${}_1H^3$, ${}_2He^3$, ${}_3Li^7$, ${}_6C^{13}$ are examples of fermions.

8.10.2 Fermi–Dirac (FD) Distribution Function

For FD statistics, since the particles are indistinguishable, we should know the number of ways in which n_i fermions can occupy g_i sublevels. Since fermions are governed by the Pauli exclusion principle (no two particles can occupy the same sublevel), g_i must be greater than n_i. Thus, the number of ways in which n_i fermions can occupy g_i sublevels is equal to the number of ways in which n_i things can be taken at a time from g_i different things. This is equal to ${}^{g_i}C_{n_i} = \frac{g_i!}{n_i!(g_i - n_i)!}$ considering them all, we can write the probability of the entire distribution of the particles is given by

$$W = \prod_i \frac{g_i}{n_i!\,(g_i - n_i)!} \qquad ...(8.23)$$

We shall now turn to the derivation of the distribution for fermions, the FD distribution. We shall proceed exactly on the same line as in the case of MB and BE statistics. Taking logarithm of Eq. (8.23) and applying Stirling's formula, we get

$$\ln W = \Sigma\,[g_i \ln g_i - (g_i - n_i) \ln (g_i - n_i) - n_i \ln n_i] \qquad ...(8.24)$$

Under the conditions stipulated by Eqs (8.7) and (8.8), we find the maximum of the quantity ln W, using Lagrange's undetermined multipliers. For this purpose, we equate to zero the derivatives of the quantity ln $W - \alpha \Sigma n_i - \beta \Sigma n_i \varepsilon_i$ [see Eq. (8.10)].

$$\Sigma\,[\ln n_i - \ln (g_i - n_i) + \alpha + \beta \varepsilon_i]\,\delta n_i = 0 \qquad ...(8.25)$$

Hence, we obtain

$$\frac{n_i}{g_i} = \frac{1}{e^{\alpha + \beta \varepsilon_i} + 1} \qquad ...(8.26)$$

This relation gives the most probable distribution of particles for a system obeying FD statistics and is known as Fermi–Dirac distribution law.

For fermions in statistical equilibrium at temperature T,

$$\alpha = -\frac{\varepsilon_f}{K_B T} \quad \text{and} \quad \beta = \frac{1}{K_B T}$$

where ε_f = Fermi energy of the system and K_B is the Boltzmann constant

Now substituting the values of β and α in Eq. (8.26), we get (if we drop the index i)

$$f(\varepsilon) = \frac{n}{g} = \frac{1}{e^{(\varepsilon - \varepsilon_f)/K_B T} + 1} \qquad ...(8.27)$$

This is known as Fermi–Dirac distribution function.

8.11 CLASSICAL STATISTICS AS A SPECIAL CASE OF QUANTUM STATISTICS

MB, BE and FD statistics can be represented by a single equation as

$$\frac{n_i}{g_i} = \frac{1}{e^{\alpha + \beta \varepsilon_i} + k} \quad \text{...(8.28)}$$

$k = 0$ gives classical or MB statistics
$k = -1$ gives BE statistics
$k = +1$ gives FD statistics

When $e^{\alpha + \beta \varepsilon_i} >> 1$ gives $\frac{n_i}{g_i} = \frac{1}{e^{\alpha + \beta \varepsilon_i}}$ which is the general form of classical distribution. The inequality in equation $e^{\alpha + \beta \varepsilon_i} >> 1$ shows that $\frac{n_i}{g_i} < 1$ which means that classical distribution is true only in the limiting case of small number of particles per quantum state. Classical distribution is thus valid only for rarefied gases.

For higher temperatures, since $\frac{n_i}{g_i} << 1$, we can neglect the value of k. So at higher temperatures, the three statistics gives the same result.

8.12 DENSITY OF STATES OR QUANTUM STATES IN ENERGY RANGE BETWEEN $\mathcal{E}$ AND $\mathcal{E} + d\mathcal{E}$

The probability $g(p)$ that a molecule has a momentum between p and $p + dp$ is equal to the number of cells in phase space within which such a molecule may exist. If each cell has the infinitesimal volume h^3, then

$$g(p)\, dp = \frac{\iiint \iiint dx\, dy\, dz\, dp_x\, dp_y\, dp_z}{h^3}$$

where the numerator is the phase space volume occupied by the particles with specified momenta.

Here $\iiint dx\, dy\, dz = V$

where V is the volume occupied by the gas in ordinary position space and

$$\iiint dp_x\, dp_y\, dp_z = 4\pi p^2\, dp$$

where $4\pi p^2\, dp$ is the volume of a spherical shell of radius p and thickness dp in momentum space.

Hence $$g(p)\, dp = \frac{4\pi V p^2\, dp}{h^3} \quad \text{...(8.29)}$$

Now the system contains identical particles of two types; one having a clockwise spin and another having an anticlockwise spin. If one cell contains one particle, then to accommodate them and taking spin into consideration, we need twice the number of cells. Therefore,

$$g(p)\, dp = 2\,\frac{4\pi p^2\, dp}{h^3}\, V = \frac{8\pi p^2\, dp\, V}{h^3} \quad \text{...(8.30)}$$

Again for free particle $p = \sqrt{2m\varepsilon}$. ε is the energy of the free particle

or, $$p^2 = 2mE \quad \text{...(8.31)}$$

Differentiating Eq. (8.31), we get

$$2p\, dp = 2m\, d\varepsilon$$

or, $$p\, dp = m\, d\varepsilon \quad \text{...(8.32)}$$

Substituting the values of p and $p\,dp$ from Eqs (8.31) and (8.32) in Eq. (8.30) we get the number of quantum states or cells lying within the energy range ε and $\varepsilon + d\varepsilon$ as $g(\varepsilon)\,d\varepsilon$ instead of $g(p)\,dp$ as

$$g(\varepsilon)\,d\varepsilon = \frac{8\pi V}{h^3}\sqrt{2m\varepsilon}\,m\,d\varepsilon$$

$$= \frac{8\sqrt{2}\,\pi V}{h^3} m^{3/2}\varepsilon^{1/2}\,d\varepsilon \qquad ...(8.33)$$

8.13 FERMI DISTRIBUTION AT ZERO AND NON-ZERO TEMPERATURE

Figure 8.1 shows the variation of $f(\varepsilon)$ with ε for different values of temperature.

The FD distribution function which gives the average occupation of the energy level is given by

$$f(\varepsilon) = \frac{1}{e^{(\varepsilon-\varepsilon_F)/K_BT}+1} \qquad ...(8.34)$$

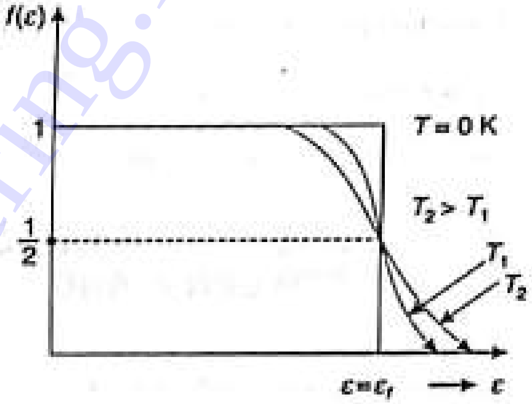


Fig. 8.1 Fermi distribution at zero and non-zero temperature.

Case I When $T = 0$ $f(\varepsilon)$ has two possible values

$$f(\varepsilon) = \frac{1}{e^{-\infty}+1} = 1 \quad \text{if } \varepsilon < \varepsilon_F$$

and

$$= \frac{1}{e^{\infty}+1} = 0 \quad \text{if } \varepsilon > \varepsilon_F$$

Thus, at absolute zero ($T = 0$ K) of temperature all possible quantum states of energy less then ε_F are occupied and all those of energy more than ε_F are empty.

Accordingly, the Fermi energy ε_F is defined as the energy of the highest occupied level at absolute zero.

Expanding Eq. (8.34) near $\varepsilon = \varepsilon_F$ we get at finite ($T > 0$ K) temperature as

$$f(\varepsilon) = \frac{1}{e^{(\varepsilon-\varepsilon_F)/K_BT}+1} = \frac{1}{2} - \frac{\varepsilon-\varepsilon_F}{4K_BT} + ... \qquad ...(8.35)$$

(i) If $\varepsilon \le \varepsilon_F - 2K_BT$ $\qquad f(\varepsilon) = 1$

(ii) If $\varepsilon \ge \varepsilon_F + 2K_BT$ $\qquad f(\varepsilon) = 0$

(iii) If $\varepsilon = \varepsilon_F$ $\qquad f(\varepsilon) = 1/2$

The spread region of Fermi distribution, i.e., the region of ε when $f(\varepsilon)$ changes from unity at zero (from $\varepsilon_F - 2K_BT$ to $\varepsilon_F + 2K_BT$) narrows as the temperature decreases and at absolute zero becomes a sharp discontinuity.

Thus, Fermi level is that energy level for which the probability of occupation at $T > 0$ is $\frac{1}{2}$, i.e., 50% of the quantum states are occupied and 50% are empty. The distribution is said to degenerate at low temperature and non-degenerate at high temperature when the step-like distribution is lost.

8.13.1 Fermi Temperature

It may be defined as the ratio of the Fermi energy (ε_F) at absolute zero to Boltzmann constant K_B.

$\therefore$ Fermi temperature $$\theta_F = \frac{\varepsilon_F}{K_B} \quad ...(8.36)$$

8.13.2 Free Electrons in a Metal

The conduction electrons in a metal may be considered as nearly free-moving in a constant potential field, like particles of an ideal gas. Electrons have half-integral spin and hence the formula of Fermi–Dirac statistics are applicable to an ideal gas electron.

Since there can be maximum of one particle per quantum state, the function $f(\varepsilon)$ is the ratio of the number of the quantum state of energy ε occupied by electrons to the total number of quantum states available in the energy level ε. From Eq. (8.27) $f(\varepsilon)$ is given by

$$f(\varepsilon) = \frac{1}{e^{(\varepsilon - \varepsilon_F)/(K_B T)} + 1} \quad ...(8.37)$$

The number of particles having energies in the range between ε and $\varepsilon + d\varepsilon$ is given by

$$n(\varepsilon)\, d\varepsilon = f(\varepsilon)\, g(\varepsilon)\, d\varepsilon \quad ...(8.38)$$

where $g(\varepsilon)\, d\varepsilon$ is the number of quantum states of energy between ε and $\varepsilon + d\varepsilon$.

Substituting the expression for $f(\varepsilon)$ in Eq. (8.38), we get

$$n(\varepsilon)\, d\varepsilon = \frac{g(\varepsilon)\, d\varepsilon}{e^{(\varepsilon - \varepsilon_F)/K_B T} + 1} \quad ...(8.39)$$

The energy of an electron (non-relativistic) having momentum p is

$$\varepsilon = \frac{p^2}{2m}$$

So from Eq. (8.33) we can write the number of quantum states or cells lying within the energy range ε and $\varepsilon + d\varepsilon$ is given by

$$g(\varepsilon)\, d\varepsilon = \frac{8\pi V}{h^3}\sqrt{2m\varepsilon}\; m\, d\varepsilon \quad ...(8.40)$$

where V is the volume of the Fermi-phase space of conductor. So from Eqs (8.39) and (8.40), we can write

$$n(\varepsilon)\, d\varepsilon = \frac{8\pi V}{h^3}\sqrt{2m\varepsilon} \times \frac{1}{e^{(\varepsilon - \varepsilon_F)/K_B T} + 1}$$

or, $$n(\varepsilon)\, d\varepsilon = \frac{8\sqrt{2}\,\pi V}{h^3}\, \frac{m^{3/2}\,\varepsilon^{1/2}\, d\varepsilon}{e^{(\varepsilon - \varepsilon_F)/K_B T} + 1} \quad ...(8.41)$$

This is the Fermi–Dirac law of energy distribution for free electrons in the metal.

8.13.3 Total Number of Particles and Total Energy at Absolute Zero of Temperature

At absolute zero temperature, the total number of electrons (N) is equal to the total number of energy states occupied by the electrons from 0 to ε_F since each energy state can have only one electron.

Therefore,
$$N = \int_0^{\varepsilon_F} n(\varepsilon)\, d\varepsilon = \int_0^{\varepsilon_F} g(\varepsilon)\, d\varepsilon \quad [\text{since } f(\varepsilon) = 1]$$

$$= 4\pi V\left(\frac{2m}{h^2}\right)^{3/2}\int_0^{\varepsilon_F} \varepsilon^{1/2}\, d\varepsilon \quad [\text{using Eq. (8.40)}]$$

$$N = \frac{8\pi V}{3}\left(\frac{2m\varepsilon_F}{h^2}\right)^{3/2}$$

This is the expression of total number of electrons in a metal at absolute zero.

The total energy E_t of an electron at absolute zero temperature is given by

$$E_t = \int_0^{\varepsilon_F} \varepsilon N(\varepsilon) d\varepsilon = \int_0^{\varepsilon_F} \varepsilon g(\varepsilon)\, d\varepsilon$$

or,
$$E_t = \int_0^{\varepsilon_F} \varepsilon\left(\frac{8\pi V}{h^3}\right)\sqrt{2m\varepsilon}\,(m d\varepsilon) \quad [\text{by Eq. 8.40}]$$

or,
$$E_t = \frac{8\sqrt{2}\pi V m^{3/2}}{h^3}\int_0^{\varepsilon_F} \varepsilon^{3/2} d\varepsilon$$

or,
$$E_t = \frac{8\sqrt{2}\pi V\, m^{3/2}}{h^3}\left(\frac{2}{5}\varepsilon_F^{5/2}\right)$$

or,
$$E_t = \frac{8\pi V}{3}\left(\frac{2m\varepsilon_F}{h^2}\right)^{3/2}\left(\frac{3}{5}\varepsilon_F\right)$$

or,
$$E_t = \frac{3}{5}N\varepsilon_F \quad \left[\because N = \frac{8\pi V}{3}\left(\frac{2m\varepsilon_F}{h^2}\right)^{3/2}\right]$$

Again, N is given by

$$N = \frac{8\pi V}{3}\left(\frac{2m\varepsilon_F}{h^2}\right)^{3/2}$$

or,
$$\varepsilon_F = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3} \qquad ...(8.42)$$

If $n = \dfrac{N}{V}$ = number of free electrons per unit volume, i.e., the free electron density, then

$$\varepsilon_F = \frac{h^2}{2m}\left(\frac{3n}{8\pi}\right)^{2/3} \qquad ...(8.43)$$

This is the expression of the Fermi energy at absolute zero temperature. Equation (8.43) shows that Fermi energy depends only on electrons concentration $\left(\dfrac{N}{V}\right)$ and it is totally independent on the size and volume of the conductor. The values of ε_F calculated from Eq. (8.43) for a number of metals are of the order of several electron volts. But according to classical statistics all electrons in a metal at absolute zero would have zero energy. Thus we see that the results of quantum statistics are appreciably different from those of classical statistics.

8.13.4 Fermi Energy at Non-Zero Temperature

The variation of ε_F with temperature T is given by Somerfield by the following equation:

$$\varepsilon_F(T) = \varepsilon_f(0)\left[1 - \frac{\pi^2 K_B^2 T^2}{12\varepsilon_F^2(0)}\right] \qquad ...(8.44)$$

where $\varepsilon_F(T)$ and $\varepsilon_F(0)$ are Fermi energy at finite temperature T and at zero temperature. The variation of ε_F with temperature T is shown in Fig 8.2. It is seen that as T is increased ε_F decreases. But the rate of decrease with temperature is very small over a large range of temperature. Hence for all practical purposes, $\varepsilon_F(T)$ may be considered constant and equal to $\varepsilon_F(0)$.

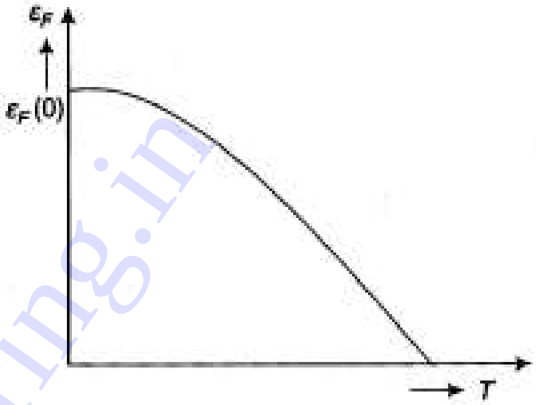


Fig. 8.2 Variation of ε_F with temperature for the free electrons in Cu.

8.13.5 Average Energy of Free Electrons in a Metal at Zero Kelvin

The average integral energy of free electron in a metal at absolute zero is given by

$$\bar{\varepsilon} = \frac{1}{N}\int_0^\infty \varepsilon n(\varepsilon)\, d\varepsilon \qquad ...(8.45)$$

$$= \frac{1}{N}\int_0^{\varepsilon_F} \varepsilon f(\varepsilon)\, g(\varepsilon)\, d\varepsilon + \frac{1}{N}\int_{\varepsilon_F}^\infty \varepsilon f(\varepsilon)\, g(\varepsilon)\, d\varepsilon$$

Now at $T = 0$ K the value of $f(\varepsilon) = 1$ if $\varepsilon \le \varepsilon_F$

and at $T = 0$ K the value of $f(\varepsilon) = 0$ if $\varepsilon_F \ge \varepsilon$

So, $$\bar{\varepsilon} = \frac{1}{N}\int_0^{\varepsilon_F} \varepsilon\, g(\varepsilon)\, d\varepsilon$$

Again $$g(\varepsilon)\, d\varepsilon = 4\pi V\left(\frac{2m}{h^2}\right)^{3/2} \varepsilon^{1/2}\, d\varepsilon \quad \text{[from Eq. (8.40)]}$$

So, $$\bar{\varepsilon} = \frac{1}{N}\times 4\pi V\left(\frac{2m}{h^2}\right)^{3/2}\int_0^{\varepsilon_F} \varepsilon^{3/2}\, d\varepsilon$$

$$= \frac{4\pi V}{N}\left(\frac{2m}{h^2}\right)^{3/2}\times\frac{\varepsilon_F^{5/2}}{5/2}$$

$$= \left[\frac{4\pi V}{N}\left(\frac{2m}{h^2}\right)^{3/2}\frac{2}{5}\,\varepsilon_F^{3/2}\right]\varepsilon_F \qquad ...(8.46)$$

We know, the Fermi energy at absolute zero ($T = 0K$)

$$\varepsilon_F = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3}$$

or, $$\varepsilon_F^{3/2} = \left(\frac{h^2}{2m}\right)^{3/2} \frac{3N}{8\pi V}$$

Substituting the value of $\varepsilon_F^{3/2}$ in Eq. (8.46), we have

$$\bar{\varepsilon} = \frac{3}{5}\varepsilon_F \quad \text{...(8.47)}$$

Therefore, the average electron energy is equal to $\frac{3}{5}$th of the Fermi energy at absolute zero.

8.13.6 Fermi Velocity and Average Velocity of a Free Electron at Zero Kelvin

We know that $$\frac{1}{2}mv_F^2 = \varepsilon_F = \frac{h^2}{2m}\left(\frac{3n}{8\pi}\right)^{2/3}$$

where v_F is the Fermi velocity and n is free electron density, i.e., $n = \frac{N}{V}$.

So $$v_F = \frac{h}{m}\left(\frac{3n}{8\pi}\right)^{1/3} \quad \text{...(8.48)}$$

Now if $\bar{v}$ be the average speed of an electron at 0 K, then

$$\bar{v} = \frac{1}{N}\int_0^{v_F} v n(v)\, dv \quad \text{...(8.49)}$$

where $n(v)\,dv$ is the number of particles within the velocity range v and $v + dv$.

Now $$\varepsilon = \frac{1}{2}mv^2 \quad \text{or,} \quad d\varepsilon = mv\,dv$$

Again at 0 K $$f(\varepsilon) = 1 \quad \text{if} \quad \varepsilon \le \varepsilon_F$$

So, $$n(\varepsilon)\,d\varepsilon = g(\varepsilon)\,d\varepsilon = \frac{8\pi V}{h^3}\sqrt{2m\varepsilon}\, m d\varepsilon$$

Therefore, $$n(v)\,dv = 4\pi V\left(\frac{2m}{h^2}\right)^{3/2}\left(\frac{1}{2}mv^2\right)^{1/2}(mv\,dv) = 8\pi V\left(\frac{m}{h}\right)^3 v^2\,dv.$$

So from Eq. (8.49), we have

$$\bar{v} = \frac{1}{N}\int_0^{v_F} 8\pi V\left(\frac{m}{h}\right)^3 v^3\,dv = \frac{8\pi V}{N}\left(\frac{m}{h}\right)^3\left(\frac{v_F^4}{4}\right) = \frac{3}{4}\left(\frac{8\pi}{3n}\right)\left(\frac{m}{h}\right)^3 v_F^4 \quad \left[\text{where } \frac{N}{V} = n \text{ free electron density}\right]$$

Again from Eq. (8.48)

$$v_F^3 = \left(\frac{h}{m}\right)^3 \left(\frac{3n}{8\pi}\right)$$

Hence

$$\bar{v} = \frac{3}{4}\left(\frac{8\pi}{3n}\right)\left(\frac{m}{h}\right)^3 \left(\frac{h}{m}\right)^3 \left(\frac{3n}{8\pi}\right) v_F$$

$$= \frac{3}{4} v_F \qquad \text{...(8.50)}$$

So the average speed of the electron in a metal is equal to $\frac{3}{4}$ times the Fermi velocity at absolute zero.

8.14 DERIVATION OF PLANCK'S LAW OF RADIATION FROM BE STATISTICS

Let T be the absolute temperature of a black-body chamber of volume V. The chamber is supposed to be filled with photons. Each photon has unit spin angular momentum. Hence photons are bosons and so we can use the BE statistics to derive Planck's law of radiation.

Photons are continuously emitted and absorbed by the walls of the chamber at constant temperature and constant volume. So, the number of photons is not constant. Therefore, $\sum_i n_i =$ const. or $\Sigma\, \delta n_i = 0$ is not valid for photon gas although the total energy of the photons remains constant. Hence the multiplier $\alpha = 0$. Thus, we have from BE distribution law [i.e., Eq. (8.20)]

$$n_i = \frac{g_i}{e^{\beta \varepsilon_i} - 1}$$

$$= \frac{g_i}{e^{\varepsilon/K_B T} - 1} \quad \text{where } \beta = \frac{1}{K_B T} \qquad \text{...(8.51)}$$

Again energy of each photon $\varepsilon = h\nu$. So from Eq. (8.51) [omitting index i]

$$n = \frac{g}{e^{h\nu/K_B T} - 1}$$

The number of photon in the frequency range ν and $\nu + d\nu$ is obtained by replacing g by $g(\nu)\, d\nu$ and n by $n(\nu)\, d\nu$. Hence we get from the above equation

$$n(\nu)\, d\nu = \frac{g(\nu)\, d\nu}{e^{h\nu/K_B T} - 1} \qquad \text{...(8.52)}$$

where $g(\nu)\, d\nu$ is the number of quantum states in the frequency range ν and $\nu + d\nu$.

The number of quantum states corresponding to the momenta in the range between p and $p + dp$ for particles is given by [See Eq. (8.30)]

$$g(p)\, dp = 2\, \frac{4\pi V p^2\, dp}{h^3} \qquad \text{...(8.53)}$$

For photons of frequency ν, we have

Energy $\varepsilon = h\nu$ and momentum $p = \dfrac{h\nu}{c}$

On substituting these values in Eq. (8.53), the number of quantum states with frequency range between v and $v + dv$ is given by

$$g(v)\,dv = \frac{8\pi V\left(\frac{hv}{c}\right)^2 \frac{h}{c}\,dv}{h^3}$$

$$= \frac{8\pi V v^2\,dv}{c^3} \qquad \text{...(8.54)}$$

Hence, substituting the value of $g(v)\,dv$ in Eq. (8.52), we have

$$n(v)\,dv = \frac{8\pi V v^2\,dv}{c^3}\,\frac{1}{e^{hv/K_BT} - 1} \qquad \text{...(8.55)}$$

The energy of each photon = hv. The energy per unit volume or energy density, within the frequency range v and $v + dv$ is given by

$$U_v\,dv = \frac{hv}{V}\cdot n(v)\,dv \qquad \text{...(8.56)}$$

or,

$$U_v\,dv = \frac{hv}{V}\cdot\frac{8\pi V v^2\,dv}{c^3}\,\frac{1}{e^{hv/K_BT} - 1}$$

$$= \frac{8\pi h v^3}{c^3}\,\frac{dv}{e^{hv/K_BT} - 1} \qquad \text{...(8.57)}$$

This is Planck's law of radiation in terms of the frequency v. In terms of λ, Eq. (8.57) can be written as

$$U_\lambda\,d\lambda = \frac{8\pi hc}{\lambda^5}\,\frac{d\lambda}{e^{hc/\lambda K_BT} - 1} \qquad \text{...(8.58)}$$

This is Planck's law of radiation in terms of the wavelength λ. On longer wavelength side $\left(\frac{hc}{\lambda} << K_BT\right)$, Eq. (8.58) becomes

$$U_\lambda\,d\lambda = \frac{8\pi K_BT}{\lambda^4}\,d\lambda \qquad \text{...(8.59)}$$

This is the Rayleigh–Jeans formula.

On shorter wavelength side $\frac{hc}{\lambda} >> K_BT$, Eq. (8.58) becomes

$$U_\lambda\,d\lambda = \frac{8\pi hc}{\lambda^5}\,e^{-hc/\lambda K_BT}\,d\lambda \qquad \text{...(8.60)}$$

This is the Wien radiation formula.

8.15 COMPARATIVE STUDY OF THREE STATISTICAL DISTRIBUTION FUNCTIONS

	MB	*BE*	*FD*
Nature of particles	Identical, distinguishable,	Identical, indistinguishable	Identical, indistinguishable
Category of particles	classical (called boltzons)	(called bosons)	(called fermions)
Properties of particles	Any spin, particles adequately far apart so wave functions do not overlap	0 or integral spin (1, 2, 3, 4, ... etc.), wave functions are symmetric under interchange of two bosons	Half integral spin (1/2, 3/2/ 5/2, ...), wavefunctions are antisymmetric under interchange of two fermions
Examples	Examples: molecules of a gas	Examples: Photon, meson, muon	Examples: Proton, electron, neutron
Distribution Function	$f_{MB} = \frac{1}{e^{\alpha + \beta \varepsilon_i}}$	$f_{BE} = \frac{1}{e^{\alpha + \beta \varepsilon_i} - 1}$	$f_{FD} = \frac{1}{e^{\alpha + \beta \varepsilon_i} + 1}$
No. of particles per energy state	No upper limit	No upper limit	Fermions obey the Pauli exclusion principle, i.e., maximum of one particle per quantum state

Worked Out Problems

Example 8.1 Distribute two particles in three different states according to (i) MB Statistics, (ii) BE statistics, and (iii) FD Statistics.

Sol. *MB Statistics*

Since the particles are distinguishable, so total number of microstates will be

$$W = \frac{N!\, g_i^{n_i}}{n_i!} = 2!\,\frac{3^2}{2!} = 9 \text{ microstates} \quad [g_i = 3,\, n_i = 2,\, N = 2]$$

In table form:

Here A, B are two distinguishable particles.

1	2	3
AB	0	0
0	*AB*	0
0	0	*AB*
A	*B*	0
B	*A*	0
0	*A*	*B*
0	*B*	*A*
A	0	*B*
B	0	*A*

So total number of microstates is 9.

BE Statistics

Since the particles are indistinguishable, so total number of microstates will be

$$W = \frac{(n_i + g_i - 1)!}{n_i!\,(g_i - 1)!} = \frac{(2+3-1)!}{2!\,(3-1)!} = \frac{4!}{2!\,2!} = \frac{4.3.2.1}{2.2} = 6 \text{ microstates}$$

In table form:

Here, the two particles are indistinguishable. Let us denote them both as A.

1	*2*	*3*
AA	0	0
0	*AA*	0
0	0	*AA*
A	0	*A*
0	*A*	*A*
A	*A*	0

So total number of microstates is 6.

FD Statistics

Since the particles are indistinguishable and according to the Pauli exclusion principle each state can accommodate only one particle so total number of microstates will be

$$W = \frac{g_i!}{n_i!\,(g_i - n_i)!} = \frac{3!}{2!\,(3-2)!} = 3 \text{ microstates} \quad [g_i = 3,\ n_i = 2]$$

In table form:

Here, the two particles are indistinguishable. Let us denote them both as *A*.

1	*2*	*3*
A	0	0
0	*A*	0
0	0	*A*

So total number of microstates is 3.

Example 8.2 Three distinguishable particles each of which can be in one of the ε, 2ε, 3ε, 4ε energy states have total energy 6ε. Find all possible number of distributions of all particles in the energy states. Find the number of microstates in each case. [WBUT 2007]

Sol. Let the particles be *A*, *B* and *C*. Particles are distinguishable. The possible microstates will be

Macrostate	ε	2ε	3ε	4ε	*Total energy*	*Microstates*
(2, 0, 0, 1)	*AB*	0	0	*C*	6ε	3
	AC	0	0	*B*	6ε	
	BC	0	0	*A*	6ε	
(0, 3, 0, 0)	0	*ABC*	0	0	6ε	1

(1, 1, 1, 0)	A	B	C	0	6 ε	
	B	C	A	0	6 ε	
	C	A	B	0	6 ε	6
	B	A	C	0	6 ε	
	A	C	B	0	6 ε	
	C	B	A	0	6 ε	

So, total number of microstates = 10 and total number of macrostates = 3.

Example 8.3 A system has non-degenerate single-particle states with 0, 1, 2, 3 energy units. Three particles are to be distributed in these states so that the total energy of the system is 3 units. Find the number of microstates if the particles obey (i) MB Statistics, (ii) BE Statistics, and (iii) FD Statistics.

[WBUT 2008, 2012]

Sol. (i) *MB Statistics*

Since the particles are distinguishable, let the particles be *A*, *B*, *C*. The possible microstates will be

Macrostate	*0 ε*	*1 ε*	*2 ε*	*3 ε*	*Total energy*	*Microstates*
(0, 3, 0, 0)	0	ABC	0	0	3ε	1
(2, 0, 0, 1)	AB	0	0	C	3ε	3
	AC	0	0	B	3ε	
	BC	0	0	A	3ε	
(1, 1, 1, 0)	A	B	C	0	3ε	6
	A	C	B	0	3ε	
	B	A	C	0	3ε	
	B	C	A	0	3ε	
	C	A	B	0	3ε	
	C	B	A	0	3ε	

So, total number of microstates = 10

(ii) *BE Statistics*

Since the particles are indistinguishable. Let us denote them as *A*.

Macrostate	*0 ε*	*1 ε*	*2 ε*	*3 ε*	*Total energy*	*Microstates*
(0, 3, 0, 0)	0	AAA	0	0	3ε	1
(2, 0, 0, 1)	AA	0	0	A	3ε	1
(1, 1, 1, 0)	A	A	A	0	3ε	1

So, total number of microstates = 3

(iii) *FD Statistics*

Since the particles are indistinguishable and according to the Pauli principle, each state can accommodate only one particle, so total number of microstates will be

Macrostate	*0*	ε	2ε	3ε	*Total energy*	*Microstates*
(1, 1, 1, 0)	*A*	*A*	*A*	0	3ε	1

So total number of microstates = 1

Example 8.4 Consider a two-particle system, each of which can exist in states ε_1, ε_2, and ε_3. What are the possible states if the particles are (i) bosons, and (ii) fermions ? **[WBUT 2006]**

Sol. *(i) BE Statistics*

Particles are indistinguishable and any number of particles can be accommodated in one quantum state. Let us denote them as *A*.

ε_1	ε_2	ε_3	*Microstates*
AA	0	0	1
0	*AA*	0	1
0	0	*AA*	1
A	*A*	0	1
0	*A*	*A*	1
A	0	*A*	1

So, total number of microstates = 6

(ii) FD Statistics

Particles are indistinguishable and one state can accommodate only one particle.

ε_1	ε_2	ε_3	*Microstates*
A	0	*A*	1
0	*A*	*A*	1
A	*A*	0	1

So, total number of microstates = 3

Example 8.5 Six distinguishable particles are distributed over three non-degenerate levels of energies 0, ε and 2ε. Calculate the total number of microstates of the system. Find the total energy of the distribution for which the probability is maximum.

Sol. There is only one state associated with non-degenerate energy levels. Let N_1, N_2 and N_3 be the number of the particles in three energy states. The total number of particles $N_1 + N_2 + N_3 = 6$ (given). As the particles are distinguishable, the number of microstates, i.e., the number of ways of choosing N_1, N_2 and N_3 particles from 6 particles is

$$W = \frac{6}{N_1!\, N_2!\, N_3!}$$

where W is the thermodynamical probability. The probability will be maximum when $N_1!\, N_2!\, N_3!$ is minimum. This is true when $N_1 = N_2 = N_3 = 2$. The corresponding total energy distribution is $O \times N_1 + \varepsilon \times N_2 + 2\varepsilon \times N_3 = 2\varepsilon + 4\varepsilon = 6\varepsilon$.

Example 8.6 Evaluate the temperature at which there is one per cent probability that a state, with an energy 0.5 eV above the Fermi energy will be occupied by an electron.

Sol. The FD distribution is $f(\varepsilon) = \dfrac{1}{e^{(\varepsilon - \varepsilon_F)/K_BT} + 1}$

ε_F is the Fermi energy

K_B = Boltzmann constant

Given $\varepsilon = (\varepsilon_F + 0.5)$ eV

Thus, $$f(\varepsilon) = \frac{1}{100} = \frac{1}{1 + e^{0.5/K_BT}}$$

or, $$0.01 = \frac{1}{1 + e^x} \quad \text{where } x = \frac{0.5}{K_BT}$$

or, $$0.01 + 0.01x = 1$$

or, $$e^x = \frac{0.99}{0.01} = 99$$

or, $$x = 2.303 \log_{10} 99 = \frac{0.5}{K_BT}$$

So, $$T = \frac{0.109 \times 1.6 \times 10^{-19}}{1.38 \times 10^{-23}} = 1264 \text{ K}$$

Example 8.7 Calculate the Fermi energy at 0 K of metallic silver containing one free electron per atom. The density and atomic weight of silver is 10.5 g/cm^3 and 108 respectively.

Sol. Fermi energy $$\varepsilon_F = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3}$$

Here $$\frac{N}{V} = \frac{6.02 \times 10^{23}}{108} \times 10.5$$
$$= 5.9 \times 10^{28} \text{ m}^{-3}$$

So, $$\varepsilon_F = \frac{(6.6 \times 10^{-34})^2}{2 \times 9.1 \times 10^{-31}} \times \left(\frac{3}{8 \times 3.14} \times 5.9 \times 10^{28}\right)^{2/3}$$
$$= 8.8 \times 10^{-19} \text{ Joules} = \frac{8.8 \times 10^{-19}}{1.6 \times 10^{-19}} \text{ eV}$$
$$= 5.5 \text{ eV}$$

Example 8.8 The Fermi energy of silver is 5.51 eV (i) What is the average energy of the free electrons in silver at 0 K? (ii) What is the speed of the electron corresponding to the above average energy?

Sol. The average electron energy at 0 K is

$$\bar{\varepsilon} = \frac{3}{5}\varepsilon_F$$
$$= \frac{3}{5} \times 5.51 = 3.306 \text{ eV.}$$

If v is the velocity of the electron, then its kinetic energy

$$\frac{1}{2}mv^2 = 3.306 \times 1.6 \times 10^{-19}$$

or,
$$v = \left(\frac{2 \times 3.306 \times 1.6 \times 10^{-19}}{9.1 \times 10^{-31}}\right)^{1/2} \text{ m/s}$$

$$= (1.16 \times 10^{12})^{1/2} = 1.08 \times 10^6 \text{ m/s}$$

Example 8.9 Consider a free electron gas at zero degree Kelvin and show that the de Broglie wavelength associated with an electron is given by

$$\lambda_F = 2\left(\frac{\pi}{3n_0}\right)^{1/3}$$

where n_0 is the number of electrons per c.c. of the gas.

Sol. The momentum (p) of the electrons is given by

$$p = \sqrt{2m\varepsilon_F}$$

But
$$\varepsilon_F = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3}$$

Again, de Broglie wavelength λ_F is given by

$$\lambda_F = \frac{h}{p}$$

$$= \frac{h}{\sqrt{2m\varepsilon_F}} = \frac{h}{\sqrt{2m\dfrac{h^2}{2m}\left(\dfrac{3N}{8\pi V}\right)^{2/3}}}$$

$$= \sqrt{\left(\frac{8\pi V}{3N}\right)^{2/3}} = 2\left(\frac{\pi V}{3N}\right)^{1/3}$$

$$= 2\left(\frac{\pi}{3n_0}\right)^{1/3} \quad \left[\text{where } n_0 = \frac{N}{V}\right]$$

Example 8.10 Estimate the temperature of the sun, if λ_m for the sun is 490 nm.

Sol. From Wien's displacement law of radiation, we know that

$$\lambda_m T = 0.2896$$

Here,
$$T = \frac{0.2896}{\lambda_m} = \frac{0.2896}{490 \times 10^{-9}} \simeq 5910 \text{ K}$$

So, the temperature of the sun is 5910 K.

Example 8.11 The Fermi energy for sodium at $T = 0$ K is 3.1 eV. Find its value for aluminum, given that the free electron density in aluminum is approximately 7 times that in sodium.

Sol.
$$\frac{\text{Electron density in Al}}{\text{Electron density in Na}} = \frac{n_2}{n_1} = 7$$

For Na, Fermi energy $\varepsilon_{F_1} = \frac{h^2}{2m}\left(\frac{3n_1}{8\pi}\right)^{2/3}$ ...(1)

For Al $\varepsilon_{F_2} = \frac{h^2}{2m}\left(\frac{3n_2}{8\pi}\right)^{2/3}$ ...(2)

Dividing Eq. (2) by Eq. (1)

$$\frac{\varepsilon_{F_2}}{\varepsilon_{F_1}} = \left(\frac{n_2}{n_1}\right)^{2/3} = (7)^{2/3}$$

so,
$$\varepsilon_{F_2} = \varepsilon_{F_1} \times 7^{2/3} = 3.1 \times 3.66$$
$$= 11.35 \text{ eV.}$$

Review Exercises

Part 1: Multiple Choice Questions

1. Statistical methods give greater accuracy when the number of observations is
   (a) very large (b) very small (c) average (d) None of these
2. The relation between entropy and thermodynamical probability is given by
   (a) $S = \ln W$ (b) $S = e^W$ (c) $S = K_B \ln W$ (d) None of these
3. The dimension of phase space volume are
   (a) (length × momentum)^3 (b) (time × momentum)^3
   (c) (energy × time)^3 (d) Both (a) and (c) are true
4. The number of macrostates for N particles in MB distribution are
   (a) N (b) $\frac{N}{2}$ (c) $N+1$ (d) None of these
5. The number of macrostates for N particles in two compartments obeying MB statistics are
   (a) $\frac{N}{2}$ (b) $N+1$ (c) 2^N (d) $N-1$
6. The number of possible arrangements of two fermions in 3 cells is [WBUT 2008]
   (a) 9 (b) 6 (c) 3 (d) 1
7. If n_i is the number of identical and indistinguishable particles in the i^{th} energy state with degeneracy g_i then classical statistics can be applied if [WBUT 2008]
   (a) $\frac{n_i}{g_i} = 1$ (b) $\frac{n_i}{g_i} << 1$ (c) $\frac{n_i}{g_i} >> 1$ (d) $g_i = 0$
8. A coin and a six-faced dice are thrown. The probability that the coin shows tail and the dice shows five is [WBUT 2005]
   (a) $\frac{7}{12}$ (b) $\frac{1}{8}$ (c) $\frac{1}{12}$ (d) $\frac{1}{6}$

9. Fermi–Dirac distribution approaches Maxwell–Boltzmann distribution at
   (a) low temperature and high pressure (b) low temperature and low particle mass
   (c) high temperature and high particle mass (d) high density and low energy range
10. MB statistics is applicable for
    (a) ideal gas (b) electron (c) proton (d) photon
11. Amongst the following statements, which one is wrong? [WBUT 2006]
    (a) Maxwell–Boltzmann statistics deal with distinguishable particles.
    (b) Fermi–Dirac Statistics deal with indistinguishable particles.
    (c) Bose–Einstein statistics deal with indistinguishable particles.
    (d) Bosons are subjected to the Pauli exclusion principle.
12. Which one of the following are bosons?
    (a) Photon (b) Phonon (c) Electron (d) Both (a) and (b)
13. The spin angular momentum of photon
    (a) $\hbar$ (b) $\frac{\hbar}{8}$ (c) 0 (d) $2\hbar$
14. The FD distribution function is expressed as
    (a) $f(\varepsilon) = \frac{1}{e^{(\varepsilon-\varepsilon_F)/K_BT}+1}$ (b) $f(\varepsilon) = \frac{1}{e^{(\varepsilon-\varepsilon_F)/K_BT}-1}$
    (c) $f(\varepsilon) = \frac{1}{e^{(\varepsilon-\varepsilon_F)/K_BT}}$ (d) None of these
15. Average energy $\bar{\varepsilon}$ of an electron in a metal at $T = 0$ K is [WBUT 2007]
    (a) ε_F (b) $\frac{1}{2}\varepsilon_F$ (c) $\frac{3}{5}\varepsilon_F$ (d) $\frac{5}{3}\varepsilon_F$
16. Which one of the following is a fermion?
    (a) Photon (b) Electron (c) Phonon (d) α particle
17. The number of ways in which 4 fermions can be arranged in 5 compartments is
    (a) 3 (b) 7 (c) 5 (d) 9
18. Planck's radiation law for black-body radiation can be derived from
    (a) MB statistics (b) FD statistics (c) BE statistics (d) None of these
19. The rest mass of the photon is
    (a) same as that of electron (b) zero
    (c) infinity (d) None of these
20. The maximum energy that can be occupied by an electron at $T = 0$ K is
    (a) Fermi energy (b) Bohr energy (c) chemical potential (d) None of these
21. The mathematical formula for Fermi energy at 0 K is
    (a) $\frac{\hbar^2}{8\pi V}\left(\frac{3N}{2m}\right)^{2/3}$ (b) $\frac{8\pi V}{\hbar^2}\left(\frac{3N}{2m}\right)^{2/3}$ (c) $\frac{\hbar^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3}$ (d) None of these

22. A perfect black body is one that
    (a) reflects all radiation (b) absorbs all radiation
    (c) transmits all radiation (d) None of these
23. For $T > 0$ K, the probability of occupancy of an electron at Fermi energy level
    (a) 1 (b) 0 (c) $\frac{1}{2}$ (d) 2

[**Ans.** 1 (a), 2 (c), 3 (d), 4 (c), 5 (c), 6 (c), 7 (b), 8 (c), 9 (c), 10 (a), 11 (d), 12 (d), 13 (a), 14 (a), 15 (c), 16 (b), 17 (c), 18 (c), 19 (b), 20 (a), 21 (c), 22 (b), 23 (c)]

Short Questions with Answers

1. Define macrostate and microstate.

Ans. Refer to Section 8.4

2. Define thermodynamical probability.

Ans. Refer to Section 8.5

3. Define phase space.

Ans. Refer to Section 8.2

4. What is an ensemble?

Ans. The collection of very large number of identical interacting macroscopic systems is called an ensemble. There are three types of ensemble (i) micro canonical ensemble – systems cannot exchange energy or matter with each other. N, V and E remain constant, (ii) Canonical ensemble – systems can exchange energy but not matter with each other. N, V and T remain constant. (iii) Grand canonical ensemble – systems can exchange both energy and matter with each other. μ, V and T remain constant, where μ is the chemical potential.

5. What is the most probable distribution?

Ans. The most probable distribution of the particles among the energy states in equilibrium is that for which the probability of occurrence is maximum. For mathematical convenience, we consider the condition for maximum value of ln W. The condition for most probable distribution is

$$d \ln W = 0.$$

6. What are the limitations of MB statistics?

Ans. MB statistics are applicable only to an isolated gas of identical molecules in equilibrium. The expression for MB distribution does not give a correct expression for the entropy of an ideal gas. MB statistics cannot be applied to a system of indistinguishable particles.

7. Why may the number of photons inside an enclosure not remain constant?

Ans: The total energy of photons inside an enclosure remains constant but the total number of photons at particular temperature T may not remain constant. Photons may be absorbed and emitted by the walls of the container without any restriction. Suppose a photon of energy $2h\nu$ may be absorbed by the well and two photons each of energy $h\nu$ may be emitted. So the photon may be destroyed and created, i.e., the number of photons inside an enclosure may not remain constant.

Part 2: Descriptive Questions

1. Define (i) Phase space, (ii) Macrostate, and (iii) Microstate.

2. Define thermodynamical probability. Obtain a relation between entropy and thermodynamical probability.
3. Three distinguishable particles, each of which can be in one of the ε, 2ε, 3ε, 4ε energy states having total energy 6ε. Find all possible distributions of particles in energy states. Find the number of microstates in each case. [WBUT 2007]
4. What are fermions and bosons? Give two examples of each. [WBUT 2004]
5. (a) Sketch the Fermi distribution for $T = 0$ K and $T > 0$ K and explain.
   (b) What is the occupation probability at $\varepsilon = \varepsilon_F$?
   (c) Calculate how the degeneracy function $g(\varepsilon)$ depends on ε for a fermionic gas.
   (d) Express the Fermi level in a metal in terms of free electron density. [WBUT 2008]
6. (a) Compare MB, BE and FD statistics mentioning at least three characteristics.
   (b) Draw the Fermi distribution curve for (i) $T = 0$ K and (ii) $T > 0$ K. Explain their significance.
   (c) Indicate which statistics will be applicable for [WBUT 2005]
   (i) ${}_1H^1$ (ii) e^- (iii) π^0 (iv) photon
   [Ans. (i) FD (ii) FD (iii) BE (iv) BE]
7. (i) What do you mean by Fermi energy? Show that it is independent of temperature.
   (ii) Prove that at 0 K the average energy is $\frac{3}{5}\varepsilon_F$.
8. Calculate the total number of particles in a fermionic gas in terms of the Fermi level at absolute zero temperature. [WBUT 2007]
9. Give an expression of BE statistics and hence obtain Planck's formula for black-body radiation. [WBUT 2002]

Part 3: Numerical Problems

1. Consider a two-particle system, each of which can exist in states ε_1, ε_2, ε_3. What are the possible states, if the particles are (i) bosons, and (ii) fermions? [WBUT 2006] [Ans. 6, 3]
2. Three identical particles can be in any of the five states. What are the number of possible ways of distributing them in various states according to MB, FD and BE statistics? [Ans. 125, 10, 35]
3. Assume in tungsten (at. wt = 183.8, density = 17.3 g/cc) there are two free electrons per atom. Calculate the Fermi energy and electron density. [Ans. 7.2 eV, 1.264×10^{23}/cc]
4. Find the Fermi level at absolute zero for Cu. Given that molar mass of Cu, $M = 63.55 \times 10^{-3}$ kg/mol. Its density $\rho = 8.93 \times 10^3$ kg/m^3. Avogadro's number $N = 6.023 \times 10^{23}$/mol. $h = 6.63 \times 10^{-34}$ Js. Mass of electron $m = 7.1 \times 10^{-31}$ kg. [Ans. 7.03 eV]
5. Find the Fermi temperature for the valance electrons in Cu. Given ε_F for copper = 7.03 eV. Boltzmann constant $K_B = 1.38 \times 10^{-23}$ J/K. [Ans. 81.5×10^3 K]
6. For silver, the electron concentration $\left(\frac{N}{V}\right)$ is 5.86×10^{28} m^{-3}. Find its Fermi energy. [Ans. 5.511 eV]
7. Determine the wavelength λ_m corresponding to the maximum emissivity of a black body at temperature T equal to 300 K. [Ans. 9660 nm]

APPENDIX

A

The Surface Integrals

Let a small outward surface of a body in a vector field $\vec{F}$ be divided into a finite number n of elementary area ΔS_i where $i = 1 \ldots n$ and take a point $Q(x_i, y_i, z_i)$ within Δs_i. If $\hat{n}_i$ is the positive unit normal to Δs_i and $\vec{F}_i(x_i, xy_i, z_i)$ is the vector field [Fig. A.1] then normal flux of $\vec{F}_i$ at Q is

$$\sum_{i=1}^{n} \vec{F}_i . \hat{n}_i \Delta S_i \qquad \ldots(1)$$

Taking the limit of this sum as $n \to \infty$, $\Delta S_i \to O$, then

$$\underset{\Delta s \to 0}{\text{Lt}} \sum_{i=1}^{n} \vec{F}_i . \hat{n}_i \Delta S_i = \iint_s \vec{F} . \hat{n} \, dS \qquad \ldots(2)$$

The projection of ΔS_i on the plane XY is

$$\Delta S_i |\hat{n}_i . \hat{k}| = |\hat{n}_i . \hat{k}| \Delta S_i$$

Here $|\hat{n}_i . \hat{k}| \Delta S_i = \Delta x_i \Delta y_i$ or, $\Delta S_i = \dfrac{\Delta x_i \Delta y_i}{|\hat{n}_i . \hat{k}|}$

Thus the sum (1) becomes

$$\sum_{i=1}^{n} \vec{F}_i . \hat{n}_i \frac{\Delta x_i \Delta y_i}{|\hat{n}_i . \hat{k}|}$$

In the limit $n \to \infty$, then Δx_i and Δy_i approach zero,

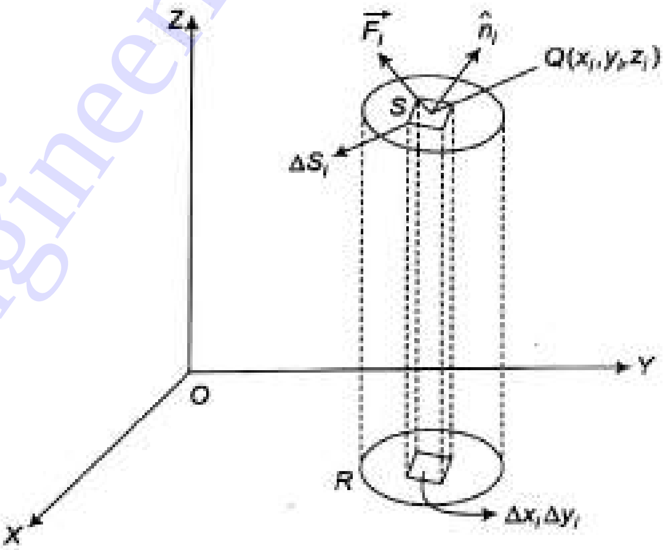


Fig. A.1 Projection of ΔS_i on the plane XY.

and we have

$$\underset{\Delta x \to 0}{\text{Lt}} \underset{\Delta y \to 0}{\text{Lt}} \sum_{i=1}^{n} \vec{F}_i . \hat{n}_i \frac{\Delta x_i \Delta y_i}{|\hat{n}_i . \hat{k}|} = \iint_R \vec{F} . \hat{n} \frac{dx\, dy}{|\hat{n}_i . \hat{k}|}$$

So, from expression (2)

$$\iint_s \vec{F} . \hat{n} \, ds = \iint_R \vec{F} . \hat{n} \frac{dxdy}{|\hat{n} . \hat{k}|}$$

APPENDIX

B

Angular Concept

What is an Angle?

The answer of this easy question is not that easy. Let us try to understand its meaning. The concept of an angle does not exist in a one-dimensional case. In a two-dimensional case, by angle we mean separation of two intersecting lines. To be more specific – an angle is one sense of separation between two intersecting lines which remains same if even the lines are extended up to infinity [Fig. B.1].

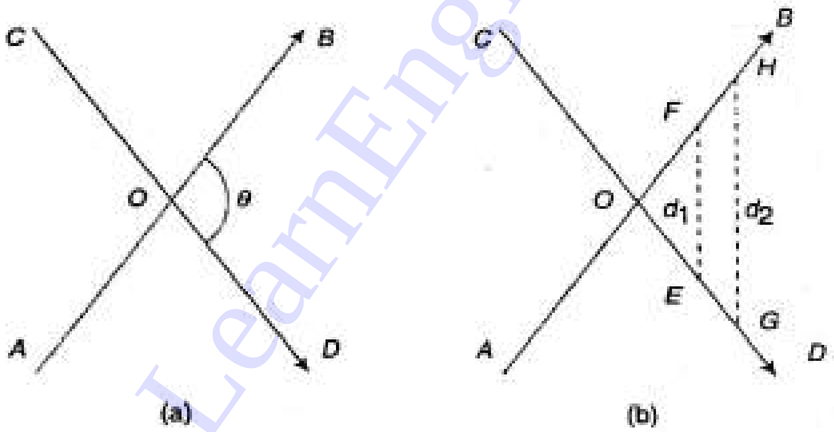


Fig. B.1 **Separation of two intersecting lines (a) in terms of angle (θ), (b) in terms of the distance between two equidistant points on two lines from the point of intersection.**

If one wants to express the separation in terms of distance one can follow the method given below:

Let us select two pairs of equidistant points which are equidistant from the point of intersection O. For one pair of points E and F the distance is d_1 and for the other pair of points G and H it is d_2, where $d_1 \neq d_2$ but the angular separation θ between the two lines remains the same throughout.

The unit of measurement of angle is radian (symbol is ^{c} or rad) in all systems of measurement like second which is the unit for time in all systems of measurement. If the angle between the two intersecting lines is 90° or 100^g (g stands for grade), then in radian one can express it as $\frac{\pi^c}{2}$ or $\frac{\pi}{2}$ rad. Radian is called a pseudounit as it does not have any representation in dimensional analysis.

Solid angle As has been stated above the counterpart of a two-dimensional angle (planar angle or simply angle) is the solid angle in three dimension. It is usually denoted by the Greek letter Ω. **It is the cone that an object subtends at a point in space.** It is a measure of how big an object appears to an observer looking from the said point in space. For example, a small object, which is nearer to the observer could subtend the same solid angle as a large object which is far away from the observer. The solid angle is proportional to the surface area S, of a projection of that object onto a sphere centered at the point where the observer lies, divided by the square of the radius, R, of the said sphere, or it is exactly equal to the surface area, S, of the aforesaid projection of an object on to a sphere of unit radius centered at the point where the observer lies. Symbolically, one can represent it by the following equation:

$$\Omega = K\frac{S}{R^2}$$

where K is a constant of proportionality. A solid angle is related to the surface of a sphere in the same way as an ordinary angle or planar angle is related to the circumference of a circle. If the proportionality constant is chosen to be unity, then the unit of solid angle in the SI system is steradian (denoted by sr). Thus, the solid angle of a sphere measured from a point in its interior is $4\pi\, sr$, and the solid angle subtended at the center of a cube by one of its six surfaces is one-sixth of that (i.e., $4\pi/6\, sr$) or $2\pi/3\, sr$.

In general, one can define a solid angle as follows [Fig. B.2]:

$$d\Omega = \frac{dA'}{r^2} = \frac{dA\cos\theta}{r^2}$$

or,
$$d\Omega = \frac{dA'}{r^2} = \frac{\vec{r}\cdot d\vec{A}}{r^3}$$

where $d\Omega$ is the solid angle subtended by any surface area dA at the point O in space. Here dA' is the projection of dA on the plane which is normal to OR and passes through the point O'

i.e.,
$$dA' = dA\cos\theta.$$

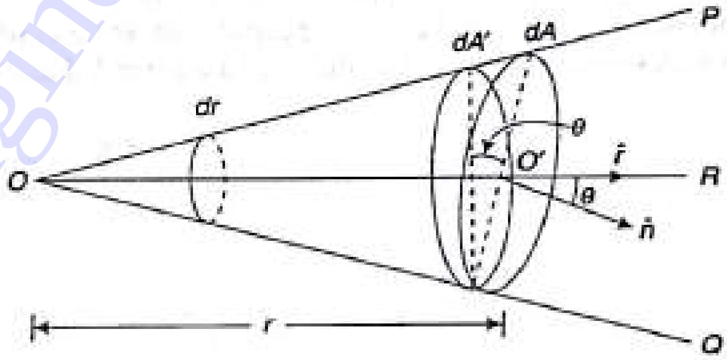


Fig. B.2 A solid angle $d\Omega$ at the point O subtended by the surface dA' which is normal to the bisector OR of angle $\angle POQ$.

For a spherical surface whose radius of curvature is $OO' = r$

$$dA' = 4\pi r^2 \quad \text{and} \quad \theta = 0,$$

so the solid angle subtended by the surface of the sphere having radius r is given by

$$\Omega = \int_s \frac{dA\cos\theta}{r^2} = 4\pi\, sr.$$

APPENDIX C

Useful Physical Constants

Rest mass of electron	m_e	9.1×10^{-31} kg
Charge of electron	e	-1.6×10^{-19} C
Specific charge of electron	e/m_e	1.758×10^{-11} C/kg
Rest mass of proton	m_p	1.673×10^{-27} kg
Charge of proton	e_p	$+1.6 \times 10^{-19}$ C
Speed of light in vacuum	c	3×10^{8} m/s
Planck's constant	h	6.63×10^{-34} Js
Boltzmann constant	k	1.38×10^{-23} J/K
Universal gas constant	R	8.314 J/(k mol)
Gravitational constant	G	6.6720×10^{-11} Nm^2/kg^2
Avogadro's number	N_A	6.023×10^{23} mol^{-1}
Refractive index of water	μ_w	1.33
Refractive index of glass	μ_g	1.50
Viscosity of water (20°C)	η_w	1.002×10^{-3} Ns/m^2
Standard atmospheric pressure	P_S	1.013×10^{5} Nm^{-2}
Bohr radius	r	5.29×10^{-11} m = 0.53 Å
Compton wavelength of electron	λ_c	2.42×10^{-12} m
Permittivity of free space	ε_0	8.85×10^{-12} F/m
Permeability of free space	μ_0	$4\pi \times 10^{-7}$ H/m
Stefan's radiation constant	σ	5.67×10^{-8} W/m^2 K^4
Rydberg constant	R_d	1.09737×10^{7} m
Young's modulus of iron	Y_S	19.5×10^{11} Nm^{-2}
Modulus of rigidity (iron)	η	7.5×10^{10} Nm^{-2}

APPENDIX D

Derivation of Gradient, Divergence, Curl and Laplacian

Cartesian Coordinate System

Let $\psi(x, y, z,)$ and $\vec{A}(x, y, z)$ be respectively a scalar and a vector function of x, y, and z.

(i) The gradient of ψ is given by

$$\overline{\nabla}\,\psi = \hat{i}\,\frac{\partial\psi}{\partial x} + \hat{j}\,\frac{\partial\psi}{\partial y} + \hat{k}\,\frac{\partial\psi}{\partial z} \tag{1}$$

(ii) The divergence of $\vec{A}$ is given by

$$\overline{\nabla}\cdot\overline{A} = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)\cdot(\hat{i}\,A_x + \hat{j}\,A_y + \hat{k}\,A_z)$$

or,

$$\overline{\nabla}\cdot\overline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{2}$$

(iii) Curl of $\overline{A}$ is given by

$$\overline{\nabla}\times\overline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

or,

$$\overline{\nabla}\times\overline{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{j}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \tag{3}$$

(iv) Laplacian of ψ is given by

$$\nabla^2\psi = \overline{\nabla}\cdot\overline{\nabla}\psi = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\,\frac{\partial\psi}{\partial x} + \hat{j}\,\frac{\partial\psi}{\partial y} + \hat{k}\,\frac{\partial\psi}{\partial z}\right)$$

or,

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} \tag{4}$$

Cylindrical Polar Coordinate System

Let $\psi(\rho, \phi, z)$ and $\vec{A}(\rho, \phi, z)$ be respectively a scalar and a vector function of ρ, ϕ and z.

Before derivation of the expressions of del ($\overline{\nabla}$), gradient, divergence, curl and Laplacian in cylindrical polar coordinate system, let us find the expression of the unit vectors $\hat{\rho}$, $\hat{\phi}$ and $\hat{k}$.

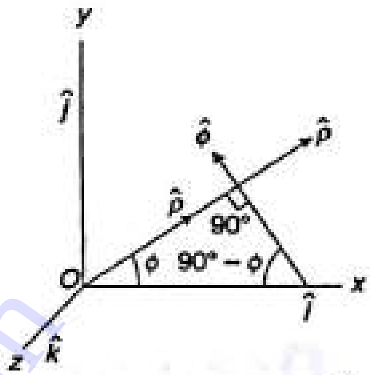


Fig. D.1 Unit vectors $\hat{\rho}$ and $\hat{\phi}$ in the XY plane.

From Fig. D.1, we can write

$$\hat{\rho} = \hat{i}\cos\phi + \hat{j}\sin\phi \tag{5}$$

and
$$\hat{\phi} = -\hat{i}\sin\phi + \hat{j}\cos\phi \tag{6}$$

Now, differentiating Eqs (5) and (6), we obtain,

$$\left.\begin{array}{l}\dfrac{\partial\hat{\rho}}{\partial\rho} = 0,\ \dfrac{\partial\hat{\rho}}{\partial\varphi} = \hat{\varphi},\ \dfrac{\partial\hat{\rho}}{\partial z} = 0\\[2ex] \dfrac{\partial\hat{\phi}}{\partial\rho} = 0,\ \dfrac{\partial\hat{\phi}}{\partial\varphi} = -\hat{\rho},\ \dfrac{\partial\hat{\phi}}{\partial z} = 0\\[2ex] \dfrac{\partial\hat{k}}{\partial\rho} = 0,\ \dfrac{\partial\hat{k}}{\partial\varphi} = 0,\ \dfrac{\partial\hat{k}}{\partial z} = 0\end{array}\right\} \tag{7}$$

(I) Gradient of ψ

By using the method of partial differentiation on the scalar function ψ, we get

$$d\psi = \frac{\partial\psi}{\partial\rho}d\rho + \frac{\partial\psi}{\partial\varphi}d\phi + \frac{\partial\psi}{\partial z}dz \tag{8}$$

Again,
$$d\psi = (d\,\overline{r})\cdot(\overline{\nabla}\,\psi) \tag{9}$$

where $d\overline{r} = \hat{\rho}d\rho + \hat{\varphi}(\rho d\phi) + \hat{k}dz$

Since $\overline{\nabla}\psi$ is a vector quantity, we can express it as follows:

$$\overline{\nabla}\psi = \hat{\rho}(\overline{\nabla}\psi)_\rho + \hat{\varphi}(\overline{\nabla}\psi)_\phi + \hat{k}(\overline{\nabla}\psi)_z \tag{10}$$

$\therefore$
$$d\psi = (d\overline{r})\cdot(\overline{\nabla}\psi)$$
$$= \left[\hat{\rho}d\rho + \hat{\phi}(\rho d\phi) + \hat{k}dz\right]\cdot\left[\hat{\rho}(\overline{\nabla}\psi)_\rho + \hat{\varphi}(\overline{\nabla}\psi)_\phi + \hat{k}(\overline{\nabla}\psi)_z\right]$$

or,
$$d\psi = (\overline{\nabla}\psi)_\rho\,d\rho + (\overline{\nabla}\psi)_\phi\,\rho d\phi + (\overline{\nabla}\psi)_z\,dz \tag{11}$$

Now, comparing Eqs (8) and (11), we can write

$$(\overline{\nabla}\psi)_\rho = \frac{\partial\psi}{\partial\rho}$$

$$(\overline{\nabla}\psi)_\phi\,\rho = \frac{\partial\psi}{\partial\varphi} \Rightarrow (\overline{\nabla}\psi)_\varphi = \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}$$

and $$(\overline{\nabla}\psi)_z = \frac{\partial\psi}{\partial z}$$

Putting these values of $(\overline{\nabla}\psi)_\rho$, $(\overline{\nabla}\psi)_\varphi$ and $(\overline{\nabla}\psi)_z$ in Eq. (10), we get

$$\overline{\nabla}\psi = \hat{\rho}\frac{\partial\psi}{\rho} + \hat{\varphi}\frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \hat{k}\frac{\partial\psi}{\partial z} \tag{12}$$

Equation (12) in the expression of gradient of ψ in cylindrical coordinate system.

$$\therefore \qquad \overline{\nabla} = \hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\varphi} + \hat{k}\frac{\partial}{\partial z} \tag{13}$$

where Eq. (13) is the expression of del $(\overline{\nabla})$ in cylindrical coordinate system.

(ii) Divergence of $\overline{A}$

$$\overline{\nabla}\cdot\overline{A} = \left(\hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(\hat{\rho}A_\rho + \hat{\varphi}A_\varphi + \hat{k}A_z)$$

Now, using the set of Eq. (7), we can expand the above equation to obtain,

$$\overline{\nabla}\cdot\overline{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$[\because \hat{\rho}\cdot\hat{\varphi} = \hat{\varphi}\cdot\hat{k} = \hat{k}\cdot\hat{\rho} = 0] \tag{14}$$

This is the expression of divergence of $\overline{A}$ in cylindrical polar coordinate system.

(iii) Curl of $\overline{A}$

The set of unit vectors $\{\hat{\rho}, \hat{\varphi}, \hat{z}\}$ is an orthogonal set of unit vectors.

$$\therefore \qquad \left.\begin{aligned} \hat{\rho}\times\hat{\varphi} &= \hat{k}, \\ \hat{\varphi}\times\hat{k} &= \hat{\rho} \\ \text{and} \qquad \hat{k}\times\hat{\rho} &= \hat{\varphi} \end{aligned}\right\} \tag{15}$$

curl of $\overline{A}$ is given by

$$\overline{\nabla}\times\overline{A} = \left(\hat{\rho}\frac{\partial}{\partial\rho} + \hat{\varphi}\frac{1}{\rho}\frac{\partial}{\partial\varphi} + \hat{k}\frac{\partial}{\partial z}\right)\times(\hat{\rho}A_\rho + \hat{\varphi}A_\varphi + \hat{k}A_z)$$

Now, by using sets of Eqs (7) and (15) we can expand the above equation, and ultimately we will obtain,

$$\overline{\nabla}\times\overline{A} = \hat{\rho}\left[\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\varphi}{\partial z}\right] - \hat{\phi}\left[\frac{\partial A_z}{\partial\rho} - \frac{\partial A_\rho}{\partial z}\right] + \frac{\hat{k}}{\rho}\left[\frac{\partial(A_\phi)}{\partial\rho} - \frac{\partial_A}{\partial\phi}\right]$$

$$= \frac{1}{\rho}\left[\left\{\frac{\partial A_z}{\partial\varphi} - \frac{\partial(\rho A_\varphi)}{\partial z}\right\} - \rho\hat{\varphi}\left\{\frac{\partial A_z}{\partial\rho} - \frac{\partial A_\rho}{\partial z}\right\} + \hat{k}\left\{\frac{\partial(\rho A_\varphi)}{\partial\rho} - \frac{\partial A_\rho}{\partial\varphi}\right\}\right] \tag{16}$$

or,
$$\overline{\nabla}\times\overline{A}=\frac{1}{\rho}\begin{vmatrix}\hat{\rho} & \rho\hat{\phi} & \hat{k}\\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z}\\ A_\rho & \rho A_\phi & A_z\end{vmatrix} \tag{17}$$

(iv) Laplacian

To obtain the expression of the laplacian in cylindrical polar coordinate system, let us take divergence of the gradient of $\psi(\rho, \phi, z)$.

$$\overline{\nabla}\cdot(\overline{\nabla}\psi)=\overline{\nabla}\cdot\overline{\nabla}\psi=\nabla^2\psi$$

or,
$$\nabla^2\psi=\overline{\nabla}\cdot(\overline{\nabla}\psi)$$

or,
$$\nabla^2\psi=\left(\hat{\rho}\frac{\partial}{\partial\rho}+\hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{\rho}\frac{\partial\psi}{\partial\rho}+\hat{\phi}\frac{1}{\rho}\frac{\partial\psi}{\partial\phi}+\hat{k}\frac{\partial\psi}{\partial z}\right)$$

or,
$$\nabla^2\psi=\frac{\partial^2\psi}{\partial\rho^2}+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}+\frac{\partial^2\psi}{\partial z^2} \tag{18}$$

[By using set of Eq. (7) and the fact that $\hat{\rho}\cdot\hat{\varphi}=\hat{\phi}\cdot\hat{k}=\hat{k}\cdot\hat{\rho}=0$]

Hence,
$$\nabla^2=\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial z^2} \tag{19}$$

which is the expression the laplacian in cylindrical polar coordinate system

Spherical Polar Coordinate System

Let $\psi(r, \theta, \phi)$ and $\overline{A}(r, \theta, \phi)$ be respectively a scalar and a vector function of r, θ and ϕ. Before derivation of the expressions of del (∇), gradient divergence, curl and laplacian in spherical polar coordinate system, let us first find the expression of the unit vectors $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ in terms of the cartesian unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$.

The positional vector $\overline{r}$ is given by

$$\overline{r}=\hat{i}r\sin\theta\cos\phi+\hat{j}r\sin\theta\sin\phi+\hat{k}r\cos\theta$$

So, the unit vector $\hat{r}$ is given by

$$\hat{r}=\overline{r}/r=\hat{i}\sin\theta\cos\phi+\hat{j}\sin\theta\sin\phi+\hat{k}\cos\theta$$

Again,
$$\frac{\partial\overline{r}}{\partial r}=\hat{i}\sin\theta\cos\phi+\hat{j}\sin\theta\sin\phi+\hat{k}\cos\theta$$

So, in general, we can express the unit vectors as follows:

$$\hat{r}=\left[\frac{\partial\overline{r}}{\partial r}\right]\Big/\left[\left|\frac{\partial\overline{r}}{\partial r}\right|\right]=\hat{i}\sin\theta\cos\phi+\hat{j}\sin\theta\sin\phi+\hat{k}\cos\theta \tag{20}$$

$$\hat{\theta}=\left[\frac{\partial\overline{r}}{\partial\theta}\right]\Big/\left[\left|\frac{\partial\overline{r}}{\partial\theta}\right|\right]=\hat{i}\cos\theta\cos\phi+\hat{j}\cos\theta\sin\phi-\hat{k}\sin\theta \tag{21}$$

and $$\hat{\phi} = \left[\frac{\partial \bar{r}}{\partial \phi}\right] \Big/ \left|\frac{\partial \bar{r}}{\partial \phi}\right| = -\hat{i} \sin\theta \sin\phi + \hat{j} \sin\theta \cos\phi \qquad (22)$$

As $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ form a set of orthogonal set of unit vectors, we can write

$$\left.\begin{aligned} \hat{r} \times \hat{\theta} &= \hat{\phi}, \\ \hat{\theta} \times \hat{\phi} &= \hat{r} \\ \text{and} \quad \hat{\phi} \times \hat{r} &= \hat{\theta} \end{aligned}\right\} \qquad (23)$$

Also, we have

$$\hat{r} \cdot \hat{\theta} = \hat{\theta} \cdot \hat{\phi} = \hat{\phi} \cdot \hat{r} = 0 \qquad (24)$$

(i) The gradient of ψ

Differentiating $\psi(r, \theta, \phi)$ partially with respect to r, θ and ϕ, we get

$$d\psi = \frac{\partial \psi}{\partial r} dr + \frac{\partial \psi}{\partial \theta} d\theta + \frac{\partial \psi}{\partial \varphi} d\phi \qquad (25)$$

We have already expressed $d\psi$ in cartesian coordinate system as

$$d\psi = (d\bar{r} \cdot \bar{\nabla}\psi)$$

The direction of $d\bar{r}$ may be any direction including that of $\bar{r}$. Let us denote $d\bar{r}$ by $d\bar{l}$ to emphasize on this nature of $d\bar{r}$.

$$\therefore \quad d\psi = (d\bar{r} \cdot \bar{\nabla}\psi) = (d\bar{l} \cdot \bar{\nabla}\psi)$$

Again, $$d\bar{l} = \hat{r}\, dr + \hat{\theta} r\, d\theta + \hat{\varphi} r \sin\theta d\phi \qquad (26)$$

Since $\bar{\nabla}\psi$ is a vector quantity, thus we can write it as

$$\bar{\nabla}\psi = \hat{r}\,(\bar{\nabla}\psi)_r + \hat{\theta}\,(\bar{\nabla}\psi)_\theta + \hat{\phi}\,(\bar{\nabla}\psi)_\phi \qquad (27)$$

where $(\bar{\nabla}\psi)_r$, $(\bar{\nabla}\psi)_\theta$ and $(\bar{\nabla}\psi)_\varphi$ are the components of $\bar{\nabla}\psi$ along $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ respectively.

$$d\psi = (d\bar{l}) \cdot (\bar{\nabla}\psi)$$

$$= (\hat{r}dr + \hat{\theta} r\, d\theta + \hat{\varphi} r \sin\theta d\phi) \cdot \left[\hat{r}\,(\bar{\nabla}\psi)_r + \hat{\theta}\,(\bar{\nabla}\psi)_\theta + \hat{\varphi}\,(\bar{\nabla}\psi)_\phi\right]$$

or, $$d\psi = (\bar{\nabla}\psi)_r\, dr + (\bar{\nabla}\psi)_\theta\, r\, d\theta + (\bar{\nabla}\psi)_\varphi\, r \sin\theta d\phi \qquad (28)$$

Comparing Eqs (25) and (28), we can write

$$(\bar{\nabla}\psi)_r = \frac{\partial \psi}{\partial r}$$

$$r(\vec{\nabla}\psi)_\theta = \frac{\partial \psi}{\partial \theta} \quad \Rightarrow \quad (\vec{\nabla}\psi)_\theta = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

and
$$r\sin\theta(\vec{\nabla}\psi)_\varphi = \frac{\partial \psi}{\partial \varphi} \quad \Rightarrow \quad (\vec{\nabla}\psi)_\varphi = \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \varphi}$$

Therefore, Eq. (27) can be written as

$$\vec{\nabla}\psi = \hat{r}\frac{\partial \psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial \psi}{\partial \theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \varphi} \tag{29}$$

or,
$$\vec{\nabla}\psi = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\right)\psi$$

So, the del operator ($\vec{\nabla}$) in spherical polar coordinate system can be written as

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \tag{30}$$

We have the expressions of unit vectors $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ as follows:

$$\hat{r} = \hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta,$$
$$\hat{\theta} = \hat{i}\cos\theta\cos\phi + \hat{j}\cos\theta\sin\phi - \hat{k}\sin\theta$$
$$\hat{\phi} = -\hat{i}\sin\phi + \hat{j}\cos\phi$$

Let us now differentiate $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ with respect to various coordinates; we are left with the following relations after differentiation,

$$\left.\begin{aligned} &\frac{\partial \hat{r}}{\partial r} = 0,\ \frac{\partial \hat{r}}{\partial \theta} = \hat{\theta},\ \frac{\partial \hat{r}}{\partial \varphi} = \hat{\phi}\sin\theta \\ &\frac{\partial \hat{\theta}}{\partial r} = 0,\ \frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r},\ \frac{\partial \hat{\theta}}{\partial \varphi} = \hat{\phi}\cos\theta \\ &\frac{\partial \hat{\varphi}}{\partial r} = 0,\ \frac{\partial \hat{\varphi}}{\partial \theta} = 0,\ \frac{\partial \hat{\varphi}}{\partial \varphi} = -\hat{r}\sin\theta - \cos\phi\hat{\theta} \end{aligned}\right\} \tag{31}$$

(ii) Divergence of $\vec{A}$

$$\vec{\nabla}\cdot\vec{A} = \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\right)\cdot\left(\hat{r}A_r + \hat{\theta}A_\theta + \hat{\varphi}A_\varphi\right)$$

Now, using the Eqs (24) and (31) and expanding the dot product, we get

$$\vec{\nabla}\cdot\vec{A} = \hat{r}\cdot\frac{\partial}{\partial r}(\hat{r}A_r + \hat{\theta}A_\theta + \hat{\varphi}A_\varphi) + \hat{\theta}\cdot\frac{1}{r}\frac{\partial}{\partial \theta}(\hat{r}A_r + \hat{\theta}A_\theta + \hat{\varphi}A_\varphi)$$
$$+\hat{\varphi}\cdot\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}(\hat{r}A_r + \hat{\theta}A_\theta + \hat{\varphi}A_\varphi)$$

or,
$$\nabla \cdot \bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi} \tag{32}$$

(iii) Curl of $\bar{A}$

In order to find the expression of (curl $\bar{A}$) one may use the Eqs (23) and (31).

$$\bar{\nabla}\times\bar{A} = \hat{r}\times\frac{\partial}{\partial r}(\hat{r}A_r + \hat{\theta}A_\theta + \hat{\varphi}A_\varphi) + \hat{\theta}\times\frac{1}{r}\frac{\partial}{\partial\theta}(\hat{r}A_r + \hat{\theta}A_\theta + \hat{\varphi}A_\varphi) + \hat{\varphi}\times\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(\hat{r}A_r + \hat{\theta}A_\theta + \hat{\varphi}A_\varphi)$$

or,
$$\bar{\nabla}\times\bar{A} = \hat{r}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\varphi) - \frac{\partial A_\theta}{\partial\varphi}\right] - \hat{\theta}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\varphi) - \frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi}\right] + \frac{\hat{\varphi}}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

or,
$$\bar{\nabla}\times\bar{A} = \frac{1}{r^2\sin\theta}\left[\hat{r}\left\{\frac{\partial}{\partial\theta}(r\sin\theta A_\varphi) - \frac{\partial}{\partial\varphi}(rA_\theta)\right\} - r\sin\theta\,\hat{\theta}\left\{\frac{\partial}{\partial r}(rA_\varphi) - \frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi}\right\} + r\sin\theta\,\hat{\varphi}\left\{\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right\}\right]$$

or,
$$\bar{\nabla}\times\bar{A} = \frac{1}{r^2\sin\theta}\left[\hat{r}\left\{\frac{\partial}{\partial\theta}(r\sin\theta A_\varphi) - \frac{\partial}{\partial\varphi}(rA_\theta)\right\} - r\,\hat{\theta}\left\{\frac{\partial}{\partial r}(r\sin\theta A_\varphi) - \frac{\partial A_r}{\partial\varphi}\right\} + r\sin\theta\,\hat{\varphi}\left\{\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right\}\right]$$

or,
$$\bar{\nabla}\times\bar{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ rA_r & rA_\theta & r\sin\theta A_\varphi \end{vmatrix} \tag{33}$$

(iv) The laplacian (∇^2) and $\nabla^2\psi$

Laplacian of $\quad \psi = \bar{\nabla}^2\psi = \bar{\nabla}\cdot\bar{\nabla}\psi$

or,
$$\nabla^2\psi = \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\theta}}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\cdot\left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\psi$$

Now, by expanding using dot product rules, one can get

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial^2\psi}{\partial\phi^2}$$

or,
$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{\partial^2}{\partial\varphi^2} \quad (34)$$

Problem 1 Show that in spherical polar coordinates

$$\frac{\partial\hat{\varphi}}{\partial\varphi} = -\hat{r}\sin\theta - \hat{\theta}\cos\theta$$

where $\hat{r}$, $\hat{\theta}$ and $\hat{\varphi}$ are unit vectors

Ans. We know that
$$\hat{r} = \hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta$$
$$\hat{\theta} = \hat{i}\cos\theta\cos\phi + \hat{j}\cos\theta\sin\phi - \hat{k}\sin\theta$$
$$\hat{\phi} = -\hat{i}\sin\phi + \hat{j}\cos\phi$$
$$\frac{\partial\hat{\varphi}}{\partial\varphi} = -\hat{i}\cos\phi - \hat{j}\sin\phi \quad \text{(i)}$$

or,
$$\sin\theta\frac{\partial\hat{\varphi}}{\partial\varphi} = -\hat{i}\sin\theta\cos\phi - \hat{j}\sin\theta\sin\phi$$

or,
$$\sin\theta\frac{\partial\hat{\varphi}}{\partial\varphi} = -(\hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta) + \hat{k}\cos\theta$$

or,
$$\sin\theta\frac{\partial\hat{\varphi}}{\partial\varphi} = -\hat{r} + \hat{k}\cos\theta$$

or,
$$\sin^2\theta\frac{\partial\hat{\varphi}}{\partial\varphi} = -\hat{r}\sin\theta + \hat{k}\sin\theta\cos\theta \quad \text{(ii)}$$

Multiplying Eq. (i) by cos θ, we get
$$\cos\theta\frac{\partial\hat{\varphi}}{\partial\varphi} = -\hat{i}\cos\theta\cos\phi - \hat{j}\cos\theta\sin\phi$$

or,
$$\cos\theta\frac{\partial\hat{\varphi}}{\partial\varphi} = -(\hat{i}\cos\theta\cos\phi + \hat{j}\cos\theta\sin\phi - k\sin\theta) - \hat{k}\sin\theta$$

or,
$$\cos\theta\frac{\partial\hat{\varphi}}{\partial\varphi} = \hat{\theta} - \hat{k}\sin\theta$$

or,
$$\cos^2\theta\frac{\partial\hat{\varphi}}{\partial\varphi} = -\hat{\theta}\cos\theta - \hat{k}\sin\theta\cos\theta \quad \text{(iii)}$$

Adding Eqs (ii) and (iii), we get
$$\frac{\partial\hat{\varphi}}{\partial\varphi} = -\hat{r}\sin\theta - \hat{\theta}\cos\theta$$

Problem 2 Derive the expression of $\overline{\nabla} \times \overline{A}$ where

$$\overline{A} = \overline{A}(\overline{r}) = \overline{A}(r, \theta, \varphi)$$

Ans. curl $\overline{A} = \overline{\nabla} \times \overline{A}$

or,
$$\overline{\nabla} \times \overline{A} = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\varphi}\frac{1}{r \sin\theta}\frac{\partial}{\partial \varphi}\right) \times (\hat{r}A_r + \hat{\theta}A_\theta + \hat{\varphi}A_\varphi)$$

Now, using Eqs (23) and (31), we get

$$\overline{\nabla} \times \overline{A} = \hat{r} \times \frac{\partial}{\partial r}(\hat{r}A_r + \hat{\theta}A_\theta + \hat{\varphi}A_\varphi) + \hat{\theta} \times \frac{1}{r}\frac{\partial}{\partial \theta}(\hat{r}A_r + \hat{\theta}A_\theta + \hat{\varphi}A_\varphi) + \hat{\varphi} \times \frac{1}{r \sin\theta}\frac{\partial}{\partial \varphi}(\hat{r}A_r + \hat{\theta}A_\theta + \hat{\varphi}A_\varphi)$$

$$= \hat{r}A_r \times \frac{\partial \hat{r}}{\partial r} + \hat{r} \times \hat{\theta}\frac{\partial A_\theta}{\partial r} + \hat{r}A_\theta \times \frac{\partial \hat{\theta}}{\partial r} + \hat{r} \times \hat{\varphi}\frac{\partial A_\varphi}{\partial r} + \hat{r}A_\varphi \times \frac{\partial \hat{\varphi}}{\partial r} + \hat{\theta} \times \hat{r}\frac{1}{r}\frac{\partial A_r}{\partial \theta}$$
$$+ \hat{\theta} \times \frac{A_r}{r}\frac{\partial \hat{r}}{\partial \theta} + \hat{\theta} \times \frac{A_\theta}{r}\frac{\partial \hat{\theta}}{\partial \theta} + \hat{\theta} \times \hat{\varphi}\frac{1}{r}\frac{\partial A_\varphi}{\partial \theta} + \hat{\theta} \times \frac{A_\varphi}{r}\frac{\partial \hat{\varphi}}{\partial \theta} + \hat{\varphi} \times \hat{r}\frac{1}{r \sin\theta}\frac{\partial A_r}{\partial \varphi}$$
$$+ \frac{A_r}{r \sin\theta}\hat{\varphi} \times \frac{\partial \hat{r}}{\partial \varphi} + \hat{\varphi} \times \hat{\theta}\frac{1}{r \sin\theta}\frac{\partial A_\theta}{\partial \varphi} + \hat{\varphi} \times \frac{A_\theta}{r \sin\theta}\frac{\partial \hat{\theta}}{\partial \varphi} + \hat{\varphi} \times \frac{A_\varphi}{r \sin\theta}\frac{\partial \hat{\varphi}}{\partial \varphi}$$

or,
$$\overline{\nabla} \times \overline{A} = \hat{\varphi}\frac{\partial A_\theta}{\partial r} - \hat{\theta}\frac{\partial A_\varphi}{\partial r} - \hat{\varphi}\frac{1}{r}\frac{\partial A_r}{\partial \theta} + \hat{\varphi}\frac{A_\theta}{r} + \hat{r}\frac{1}{r}\frac{\partial A_\varphi}{\partial \theta} + \frac{\hat{\theta}}{r \sin\theta}\frac{\partial A_r}{\partial \varphi} - \hat{r}\frac{1}{\sin\theta}\frac{\partial A_\theta}{\partial \varphi} + \hat{r}\frac{A_\phi \cos\theta}{r \sin\theta} - \hat{\theta}\frac{A_\varphi}{r}$$

or,
$$\overline{\nabla} \times \overline{A} = \hat{r}\left(\frac{1}{r}\frac{\partial A\varphi}{\partial \theta} - \frac{1}{r \sin\theta}\frac{\partial A_\theta}{\partial \varphi} + \frac{A_\varphi \cos\phi}{r \sin\theta}\right) + \hat{\theta}\left(\frac{1}{r \sin\theta}\frac{\partial A_r}{\partial \varphi} - \frac{A_\varphi}{r} - \frac{\partial A_\varphi}{\partial r}\right) + \hat{\varphi}\left(\frac{\partial A_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta} + \frac{A_\theta}{r}\right)$$

or,
$$\overline{\nabla} \times \overline{A} = \hat{r}\frac{1}{r \sin\theta}\left(\sin\theta\frac{\partial A_\varphi}{\partial \theta} - \frac{\partial A_\theta}{\partial \varphi} + A_\varphi \cos\theta\right) + \hat{\theta}\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \varphi} - A_\varphi - r\frac{\partial A_\varphi}{\partial r}\right) + \hat{\varphi}\frac{1}{r}\left(r\frac{\partial A_\theta}{\partial r} - \frac{\partial A_r}{\partial \theta} + A_\theta\right)$$

or,
$$\overline{\nabla} \times \overline{A} = \hat{r}\frac{1}{r \sin\theta}\left(\frac{\partial(\sin\theta A_\varphi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \varphi}\right) + \hat{\theta}\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right) + \hat{\varphi}\frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)$$

or, $$\overline{\nabla}\times\overline{A}=\frac{1}{r^2\sin\theta}\hat{r}\left[\left\{\frac{\partial(r\sin\theta A_\varphi)}{\partial\theta}-\frac{\partial(rA_\theta)}{\partial\varphi}\right\}-r\hat{\theta}\left\{\frac{\partial(r\sin\theta A_\varphi)}{\partial r}-\frac{\partial A_r}{\partial\varphi}\right\}+r\sin\theta\hat{\varphi}\left\{\frac{\partial(rA_\theta)}{\partial r}-\frac{\partial A_r}{\partial\theta}\right\}\right]$$

or, $$\overline{\nabla}\times\overline{A}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\hat{\varphi}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi}\\ A_r & rA_\theta & r\sin\theta A_\varphi\end{vmatrix}$$

Problem 3 Derive the expression for laplacian in spherical coordinates.

Ans. Laplacian of ψ is given by

$$\nabla^2\psi=\nabla\cdot\nabla\psi$$

or, $$\nabla^2\psi=\left(\hat{r}\frac{\partial}{\partial r}+\frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta}+\frac{\hat{\varphi}}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\cdot\left(\hat{r}\frac{\partial}{\partial r}+\frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta}+\frac{\hat{\varphi}}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\psi$$

or, $$\nabla^2\psi=\hat{r}\cdot\frac{\partial}{\partial r}\left(\hat{r}\frac{\partial}{\partial r}+\frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta}+\hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\psi+\hat{\theta}\cdot\frac{1}{r}\frac{\partial}{\partial\theta}\left(\hat{r}\frac{\partial}{\partial r}+\frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta}+\hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\psi+\hat{\varphi}\cdot\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(\hat{r}\frac{\partial}{\partial r}+\frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta}+\frac{\hat{\varphi}}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\psi$$

or, $$\nabla^2\psi=\hat{r}\cdot\frac{\partial\hat{r}}{\partial r}\frac{\partial\psi}{\partial r}+\hat{r}\cdot\hat{r}\frac{\partial^2\psi}{\partial r^2}+\frac{\hat{r}\cdot\hat{\theta}}{r}\frac{\partial^2\psi}{\partial r\partial\theta}+\frac{\hat{r}}{r}\cdot\frac{\partial\hat{\theta}}{\partial r}-\frac{1}{r}\hat{r}\cdot\hat{\theta}\frac{\partial\psi}{\partial\theta}-\frac{\hat{r}\cdot\hat{\varphi}}{r^2\sin\theta}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial\psi}{\partial\theta}+\frac{1}{r^2\sin\theta}\frac{\partial^2\psi}{\partial\varphi^2}+\frac{\hat{\varphi}}{r^2\sin^2\theta}(-\hat{r}\sin\theta-\hat{\theta}\cos\theta)\frac{\partial\psi}{\partial\varphi}$$

$$=\frac{\partial^2\psi}{\partial r^2}+\frac{2}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}+\frac{1}{r^2\sin\theta}\frac{\partial^2\psi}{\partial\varphi^2}$$

$$=\left(\frac{\partial^2\psi}{\partial r^2}+\frac{2}{r}\frac{\partial\psi}{\partial r}\right)+\left(\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\right)+\left(\frac{1}{r^2\sin\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right)$$

$\therefore$ $$\nabla^2\psi=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\psi\right)+\frac{\partial^2\psi}{\partial\varphi^2}$$

or, $$\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{\partial^2}{\partial\varphi^2}$$

APPENDIX E

Set of Experiments (For Practical Classes)

EXPERIMENT NO. 1

Aim: Determination of Planck's constant (h) by measuring consisting filament radiation in a fixed spectral range.

Apparatus: A Planck's constant measurement kit which consists of a filament bulb with series resistance to control its supply current, an ammeter, a voltmeter, a small optical bench, a photovoltaic cell and a microammeter to measure the photocurrent.

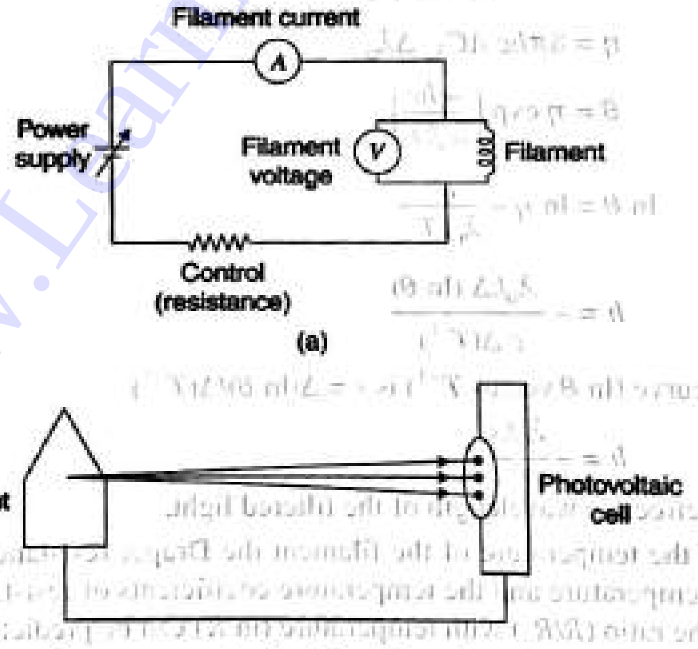


Fig. E.1 (a) Filament and power supply circuit with resistance, ammeter and voltmeter. (b) Filament and photovoltaic cell arrangement.

Theory: For a black body at absolute temperature T, the total radiation and the spectral distribution for this radiation are functions of the temperature alone, the spectral distribution is given by

$$E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\left[\exp\left(\frac{hc}{\lambda kT}\right) - 1\right]^{-1} d\lambda \qquad ...(1)$$

While working with visible light and temperature up to 2500 K, we have

$$\frac{hc}{\lambda kT} >> 1$$

Hence Eq. (1) gets reduced to

$$E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \times \left[\exp\left(\frac{-hc}{\lambda kT}\right) d\lambda\right] \qquad ...(2)$$

If the radiation is received through a filter on a photovoltaic cell, and the galvanometer response θ is measured, we have

$$\theta = 8\pi hc\, A \int \frac{B_\lambda}{\lambda^5} \exp\left(-\frac{hc}{\lambda kT}\right) d\lambda \qquad ...(3)$$

where, A is a factor depending on the geometry of the arrangement and the sensitivity of the galvanometer and the factor B_λ is a function of λ which includes (i) transmission characteristics of the filter and (ii) the frequency response characteristic of the photovoltaic cell.

For a very narrow transmission band of the filter, Eq. (3) reduces to the form

$$\theta = 8\pi hc\, AC_{\lambda_0} \exp\left(\frac{-hc}{\lambda_0 kT}\right) \Delta\lambda_0$$

$$= \eta \exp\left(-\frac{hc}{\lambda_0 kT}\right)$$

where $\quad \eta = 8\pi hc\, AC_{\lambda_0} \Delta\lambda_0$

or, $$\theta = \eta \exp\left(\frac{-hc}{\lambda_0 kT}\right) \qquad ...(4)$$

or, $$\ln\theta = \ln\eta - \frac{hc}{\lambda_0 kT}$$

Hence, $$h = -\frac{\lambda_0 k \Delta(\ln\theta)}{c\,\Delta(T^{-1})} \qquad ...(5)$$

where the slope of the curve ($\ln\theta$ versus T^{-1}) is $s = \Delta(\ln\theta)/\Delta(T^{-1})$

$$h = -\frac{\lambda_0 k s}{c} \qquad ...(6)$$

Here λ_0 is the mean effective wavelength of the filtered light.

For measurement of the temperature of the filament the Draper resistance of the given tungsten wire is measured. The Draper temperature and the temperature coefficients of resistance of tungsten being known a priori, the variation of the ratio (R/R_g) with temperature (in K) can be predicted and therefore supplied in the manual. (R_g is the Draper resistance)

The temperature of the bulb can be measured by measuring its resistance and using the graph of T versus R/R_g.

Procedure:

1. Turn on the system. Keep the voltage control knob in minimum position. Initially vary the voltage very slowly and observe carefully, when the bulb exactly starts glowing. A dark room will help this part of the experiment because stray light will produce an erroneous result. Note down the current and voltage and hence the resistance of the bulb wire. This is the Draper resistance R_g.
2. Increase the voltage in small steps and record the current and hence calculate the resistance. These are the values of the resistance R. Note down also the photocurrent θ. Take at least ten readings in the full range of the voltmeter.
3. From the supplied table of R/R_g and T draw a graph and refer it as graph 1. Use the above graph to determine the value of the temperature T of the filament from the experimentally observed value of R/R_g obtained in steps 1 and 2. Hence, calculate $\frac{1}{T}$.
4. Draw a graph of $\ln\theta$ versus $\frac{1}{T}$ and determine the value of Planck's constant h from the slope of this curve from Eq. (5).

Observations:

Table 1 Supplied data for variation of R/R_g with T.

S. No.	*Ratio of R and R_g $\left(\text{i.e., } \frac{R}{R_g}\right)$*	*Absolute Temperature T*
1.		
2.		
3.		
4.		
5.		

Table 2 To find Draper resistance.

S. No.	*Voltage, V (V)*	*Current, I (A)*	*Resistance R_g (Ω)*	*Average, R_g (Ω)*
1.				
2.				
3.				

Table 3 To find the resistance and hence the temperature of the filament wire.

S. No.	*Voltage, V (V)*	*Current I (A)*	*Resistance, R (Ω)*	$\frac{R}{R_g}$	*Temperature, T (K)*	$\frac{1}{T} \times 10^{-4}$ (K^{-1})	*Photocurrent, θ (A)*	*ln θ*
1.								
2.								
-								
-								
-								
-								
15.								

Calculations:

Results:

Percentage error:

Discussion:

Aids to Viva Voce:

1. What is Planck's constant?

Ans. It is a fundamental physical constant. It gives the order of energy exchange in case of quantum mechanical action. It is the ratio of the energy of a photon to its frequency.

2. What is the value of Planck's constant? State its unit.

Ans. The value of Planck's constant is given by $h = 6.6262 \times 10^{-34}$ J-s. Its unit is joule-second.

3. Write down Planck's radiation law.

Ans. Planck's radiation law is given by $u_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \cdot \frac{d\lambda}{\exp\left(\frac{hc}{\lambda kT}\right) - 1}$

where u_λ = the energy density at work-length λ,
h = Planck's constant,
c = velocity of light in the vacuum,
k = Boltzmann's constant,
and T = Absolute temperature.

4. What is quantum theory of light?

Ans. It is one of the dual theories of light. It states that energy can be exchanged between two elementary particles in discrete quantum only.

5. From which statistical distribution function Planck's radiation law is derived?

Ans. It is derived from Bose-Einstein statistical distribution function.

6. How can you classify quantum particles?

Ans. Quantum particles can be classified into two categories depending on their spin angular momentum. Namely, bosons and fermions.

7. What are the values of spin angular momentum for bosons and fermions?

Ans. The spin angular momentum is given by $L_s = m_s \hbar$ for bosons $m_s = n$ and for fermions $m_s = \left(n + \frac{1}{2}\right)$ where n is an integer.

8. What are bosons and fermions?

Ans. Bosons are the particles which have integral values for their spin angular momentum while in case of fermions, spin angular momentum values are half-integral.

9. Cite some examples of bosons and fermion?

Ans. Examples of bosons are phonon, photon, α-particles, etc., while examples of fermions are electron, proton, neutron, ${}_2He^3$ nuclei, etc.

10. **What are the values of spin in case of photons and electrons?**

Ans. For photons it is unity and for electrons it is $\pm \frac{1}{2}$.

11. **Why do we take help of B–E statistical distribution function to determine the value of Planck's constant?**

Ans. In the present experiment, we use a filament which emits light photons. As photons are bosons, we take help of B–E statistics to determine Planck's constant.

12. **What is photovoltaic cell?**

Ans. It is a *pn*-junction which can convert light energy into electrical energy.

EXPERIMENT NO. 2

Aim: Determination of Stefan's constant by using a vacuum tube diode.

Apparatus: Stefan's constant apparatus with a diode value.

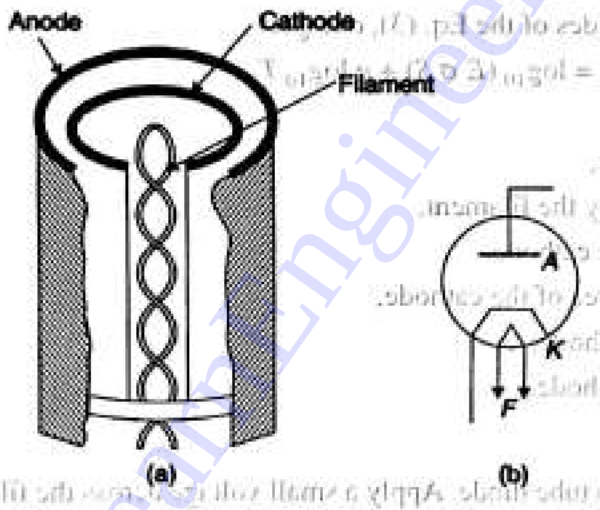


Fig. E.2 (a) Diagram of a vacuum tube diode. (b) Circuit diagram of a vacuum tube diode.

Theory: In the experiment of Stefan's constant one usually uses a commercially available vacuum tube diode which has a cylindrical cathode made of nickel (Fig. E.2). Inside the cathode, a tungsten filament is used as a heater. The cathode is heated by passing electric current through the tungsten filament. The temperature of the filament can be determined by using a known resistance (R_T) and temperature (T) relationship for tungsten as given below:

$$\frac{R_T}{R_{273}} = \left(\frac{T}{273}\right)^{1.2} \quad \text{...(1)}$$

where

R_T is the resistance of the tungsten filament at temperature T K, and

R_{273} is the resistance of the filament at temperature 273 K.

The above Eq. (1) can be approximated as

$$\frac{R_T}{R_{300}} = \left(\frac{T}{300}\right)^{1.2} \quad \text{...(2)}$$

where, R_{300} is the resistance of the filament at 300 K, a value very close to the room temperature. It is possible to estimate the value of the filament resistance at room temperature from a plot of the resistance of the wire for very small voltage range following an extrapolation to zero volt.

From a current, voltage data for the tungsten filament resistance R_T (and hence R_T/R_{300}) can be estimated.

From a supplied data table, temperature T versus (R_T/R_{300}) graph can be drawn which can be used to estimate the temperature T from the measured value of (R_T/R_{300}). On the other hand, the power generated by the filament is given by VI, where V is the voltage across the filament and I is the current through it. Neglecting other modes of heat loss, this is equal to the power radiated by the filament.

From Stefan's law, one has,

$$P = E\,\sigma\,ST^n \quad \text{...(3)}$$

Taking logarithm of both sides of the Eq. (3), one gets

$$\log_{10} P = \log_{10}(E\,\sigma\,S) + n\log_{10} T \quad \text{...(4)}$$

where,

σ = Stefan's constant,

P = Power radiated by the filament,

E = Emissivity of the cathode,

$S = 2\pi rl$ = surface area of the cathode,

r = Radius of the cathode,

and l = Length of the cathode.

Procedure:

1. Switch on the vacuum tube diode. Apply a small voltage across the filament. Note down the current.
2. Increase the voltage in small steps up to the maximum possible value. Note down the voltage in each step. Take at least ten readings within the range of the voltage.
3. Determine the resistance in each case.
4. Plot a graph of resistance R, versus voltage V only for small values of V. Extrapolate the curve to V = 0 volt. This will be a good approximation of R_{300}.
5. Calculate the values of (R_T/R_{300}).
6. Plot the supplied data of the variation of (T) versus (R_T/R_{300}) in a graph.
7. Use the above graph to estimate the values of (T) corresponding to the values of calculated (R_T/R_{300}) in a graph.
8. Calculate the values of the power by multiplying the current and the voltage obtained in Step 2.
9. Plot a graph of $\log_{10}^{P}$ versus $\log_{10}^{T}$. The slope of the curve (in this case a straight line) gives the estimate of the power n of temperature T in Stefan's law.
10. Choose a suitable point on the curve and use Eq. (4) to estimate the Stefan's constant.

Observations:

Table 1 Resistance-Temperature (*K*) data (supplied).

S. No.	R_T/R_{300}	*T(K)*
–	–	–
–	–	–
–	–	–

Table 2 I-V (current-voltage) data of the filament wire.

S. No.	$V_f(V)$	$I_f(A)$	$P = V_f I_f (W)$	$R_T(\Omega)$	R_T/R_{300}	*TK*	log_{10}^T	log_{10}^P
–	–	–	–	–	–	–	–	–
–	–	–	–	–	–	–	–	–
–	–	–	–	–	–	–	–	–

Calculations:

Results:

Percentage error:

Discussion:

*Aids to Viva Voce:

1. What is Stefan's law of radiation?

Ans. This law states that the total heat radiated per second per unit surface area of a perfectly blackbody is directly proportional to the fourth power of its absolute temperature.

2. What is Stefan's constant?

Ans. If the total energy radiated per second per unit surface area of a perfectly blackbody be E and its absolute temperature be T, then from Stefan's law, we get, $E \propto T^4$ or, $E = \sigma T^4$, where σ is the constant of proportionality. This constant σ is called Stefan's constant.

3. Write down Stefan's law for any general radiating body.

Ans. Stefan's law can in general be written as $P = E\sigma ST^n$

where

P = Power radiated by the hot body,

E = Emissivity of the radiator,

σ = Stefan's constant,

S = Surface area of the radiator,

T = The absolute temperature of the radiator,

and n = a constant.

4. What will be the value of n in the equation $P = E\sigma ST^n$ for a perfectly blackbody?

Ans. The value of n in the above equation will be exactly 4 (four) for a perfectly blackbody radiator and $n < 4$ for all real bodies.

5. Does Stefan's constant depend on the wavelength of radiation?

Ans. No. It is applicable to whole range of wavelength.

6. What is the relation between the temperature of a filament and its resistance?

Ans. If R_T be the resistance of a filament at absolute temperature T, then

$$R_T = C\,T^{1.2}$$

where C is a constant of proportionality.

7. What wavelengths are present in blackbody radiation?

Ans. All wavelengths from zero to infinity.

8. What is the standard value of σ?

Ans. It lies between 5.32×10^{-8} to 6.15×10^{-8} $Wm^{-2}K^{-4}$.

9. What is the application of the value of the Stefan's constant σ?

Ans. It is used to determine the temperature of heavenly bodies such as the sun and other stars.

10. What has been used in the present experiment as black body radiator?

Ans. In this experiment, a vacuum tube diode has been used as a blackbody radiator.

11. What is a diode?

Ans. It is an electronic device which allows current to flow in one direction only in a circuit.

12. How many technologies are available for making a diode? Name them.

Ans. There are two technologies to build a diode, namely, vacuum tube technology and semiconductor technology.

13. How can you build a vacuum tube diode?

Ans. A vacuum tube diode is built by using two coaxial glass cylinders. A tungsten filament in kept along the common axis. The space between the two coaxial cylinders is evaculated. The outer surface of the inner cylinder is coated with a mixture of barium and strontium oxide which can emit electrons easily on being heated. The inner surface of the outer cylinder is coated with some metal to make it conductive. When the cylinders are connected externally though an ammeter and the circuit of the filament is closed, the filament heats up the inner cylinder (the cathode). It then starts emitting thermions. The device behaves as a diode. If the cathode is kept at a lower potential it conducts and if the anode (outer cylinder) is kept at lower potential does not conduct.

EXPERIMENT NO. 3

Aim: Determination of dielectric constant (i.e., relative permittivity) of a solid.

Apparatus: An electrical signal generator, an oscilloscope and a parallel plate capacitor with circular plates.

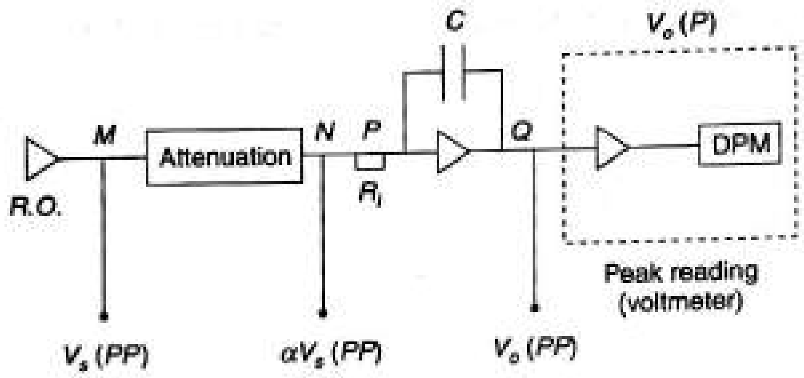


Fig. E.3 Circuit diagram for measurement of relative permittivity.

C : Capacitor
$V_s(PP)$: Peak-to-peak voltage of the signal
$V_0(PP)$: Peak-to-peak voltage of the output signal
$V_0(P)$: Voltage of the output signal from reference (zero) level to peak
α : Attenuation factor

Theory: The capacitance of a parallel-plate capacitor having air as dielectric medium between the plates is given by

$$C_0 = \frac{\epsilon_0 A}{d} \text{ (in farad)}$$

where ϵ_0 = Permittivity of air = $10^9/(36\pi)$
A = Area of each of the plates of the parallel-plate capacitor
$= \pi r^2$ (for round disc)
d = Distance between the parallel plates

When a dielectric material is introduced between two plates of a parallel-plate capacitor, it is found that the capacitance increases by a factor 'ϵ_r' which is the relative dielectric constant of the material concerned. It is also called relative permittivity which is the ratio of absolute permittivity of the medium to that of air.

Now we get,
$$C_0 = \frac{r^2}{36\,d} \times \frac{1}{10^9} F = \frac{r^2}{36\,d} \text{ nF}$$

where, 'r' represents the radius of the gold-plated brass disc and 'd' represents the thickness of the sample dielectric material in meter.

Then,
$$\epsilon_r = \frac{C_d}{C_0}$$

where C_0 represents the capacitance of the capacitor with the plates separated by air whose thickness is the same as the thickness of the sample dielectric material and C_d is the capacitance of the capacitor when a sample dielectric is introduced between its plates (e.g., glass, mica, P&T, plywood, etc.).

Procedure:

1. Measure the diameter and hence radius 'r' of the discs (plates) of standard capacitor.
2. Measure the thickness 'd' of the dielectric slab (i.e., disc).
3. Put the disc of the dielectric material whose relative permittivity is to be measured in the dielectric cell.

4. Connect the standard capacitor, dielectric cell and the resistance (R_i) in series with the oscillator (i.e., signal generator).
5. Turn on the oscillator, set a frequency and an input voltage. Note down the voltage across the capacitor and the dielectric cell.
6. Change the input voltage to few other values, note down the voltage across the standard capacitor and the dielectric cell.
7. Change the input frequency, and repeat Step 6.

Observations:

Radius r of the disc is given.

The thickness d of the disc is given.

$$\therefore \qquad C_0 = \frac{r^2}{36\,d} = \ldots \text{ farad}$$

Table 1 For recording data.

S. No.	Time period, T (ms)	Frequency, f (kH$_z$)	$V_s\,(PP)\,(V)$	$\alpha \cdot V_s\,(PP)\,(V)$	$V_0\,(PP)\,(V)$	$V_0\,(P)\,(V)$
1.						
2.						
3.						

Attenuation factor α is given.

The series resistance (connected) R_i is given.

The peak reading of the voltmeter given $V_0(P)$ with $V_0(PP) = 2\,V_0(P)$.

$$C_d = \frac{V_s(PP) \times \alpha \times T}{8\,R \times V_0\,(P)}$$

Results:

$$\epsilon_r = \frac{C_d}{C_0}$$

Percentage error:

Discussion:

Aids to Viva Voce:

1. What is a dielectric?

Ans. A dielectric is an insulating material in which all electrons are lightly bound to the nuclei of the atoms and there is no electron available for conduction of current.

2. What is dielectric constant?

Ans. It is the ratio of the permittivity of a dielectric to that of the vacuum. It is also known as relative permittivity. It is given by

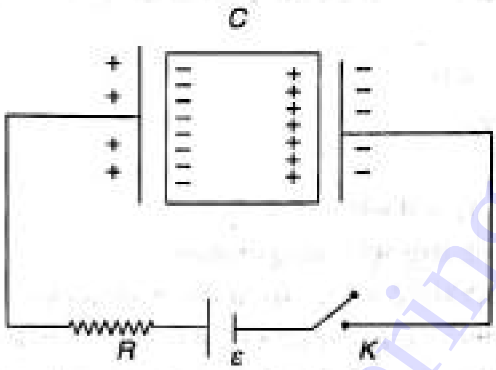
$$k = \epsilon_r = \frac{\epsilon}{\epsilon_0}$$

where k is the dielectric constant, ϵ_r is relative permittivity and the two quantities ϵ and ϵ_0 which are permittivities of the dielectric and vacuum respectively.

3. Show that relative permittivity is given by $\epsilon_r = \dfrac{C}{C_0}$

where C and C_0 are the capacitances of a capacitor in presence and in absence of a dielectric material?

Ans. Let us consider a parallel plate capacitor in an electrical circuit as shown below:



In absence of the dielectric between the plates the capacitance of the capacitor is given by

$$C_0 = \frac{\epsilon_0 A}{d} \tag{1}$$

where A is the area of each plate of the capacitor and d is separation of the plates.

And in the presence of a dielectric its capacitance is given by

$$C = \frac{\epsilon A}{d} \tag{2}$$

So, from Eqs (1) and (2), we get

$$\frac{C}{C_0} = \frac{\epsilon}{\epsilon_0} = \epsilon_r$$

Hence, $$\epsilon_r = \frac{C}{C_0}.$$

4. What is attenuation?

Ans. It is the reduction in the level of a physical quantity, such as the intensity of a wave, over an interval of a variable, such as the distance from a source.

5. What is attenuation constant?

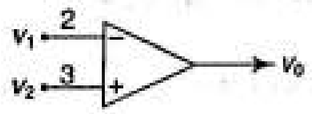
Ans. It is a rating for a line or medium through which a plane wave is being transmitted, equal to the relative rate of decrease of the amplitude of a field component, voltage or current in the direction of propagation.

6. What is an operational amplifier?

Ans. An operational amplifier (OP AMP) is a high gain direct coupled amplifier which can be used to implement a wide variety of linear and non-linear operations by merely changing a few external circuit elements such as resistors, capacitors and diodes.

7. Write down the circuit symbol of an operational amplifier and state the characteristics of an ideal OP AMP.

Ans. The following is the circuit symbol of and OP AMP:



An ideal OP AMP has the following characteristics:

(i) Input resistance $R_i = \infty$

(ii) Output resistance $R_o = 0$

(iii) Voltage gain $|A_v| = \infty$

(iv) Band width $\Delta w = \infty$

(v) Perfect balance, i.e., $v_0 = 0$ when $v_1 = v_2$

(vi) Characteristics do not drift with temperature.

However, a practical OP AMP deviates from these ideal characteristics.

8. Why is the capacitor connected in parallel with OP AMP in the circuit?

Ans. It is connected in parallel with an OP AMP because it will ensure a continuous current in the circuit and maintain a constant potential difference between the plates of the capacitor.

9. What is polarization?

Ans. Polarization is the process of creating or inducing dipoles in a dielectric material by an external electric field.

10. What is dielectric strength?

Ans. The dielectric strength of a dielectric is defined as the maximum value of the electric field that can be applied to the dielectric without its electric breakdown.

11. What is polarizability?

Ans. It is the ability of an atom or a molecule to become polarized in the presence of an electric field.

EXPERIMENT NO. 4

Aim: Determination of the specific charge (e/m) of an electron by using Thomson's (i.e., bar magnet) method.

Apparatus: Two bar magnets, a cathode ray tube, a magnetic compass box, a wooden frame with scales to hold magnetic compass and cathode ray tube.

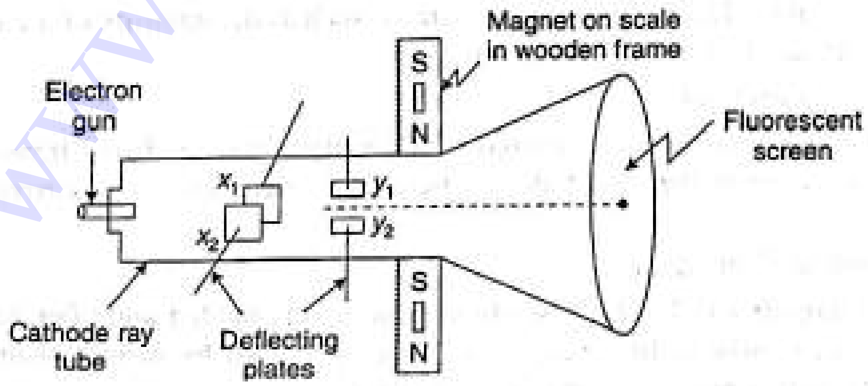


Fig. E.4 Experimental set-up for measurement of e/m of electron.

Theory: In the cathode ray tube, electrons ejected from the electron gun are allowed to pass through horizontally and vertically deflecting plates X_1-X_2 and Y_1-Y_2 respectively. The deflection of electrons due to electric field (applied by deflecting plates) is nullified by the magnetic field due to placement of two bar magnets on the scales of the wooden frame. And hence specific charge (e/m) is determined. The working formula is given by

$$e/m = V \times 10^7 \times Y/(ILH^2d) \text{ (emu of charge/gm)}$$

[The standard value of $e/m = 1.7592 \times 10^7$ emu of charge/gm]

where

l = Length of the deflecting plate,

L = Distance of the screen from the edges of the deflecting plates,

V = Voltage applied between the plates,

Y = Deflection of the spot (due to striking of electrons) on screen,

and d = Separation between the deflecting plates.

[The values of d, l and L are given and the horizontal magnetic field component of earth $H_e = 0.345$ oersted.]

$$H = H_e \tan \theta$$

Procedure:

1. Put the compass box at the center of the two armed wooden frame. Rotate the wooden frame till the arms are parallel to the east-west direction. For your convenience you may mark the boundary of the apparatus on the table with a chalk so that if the east-west alignment is disturbed any time during performance of the experiment it can easily be restored.
2. Take out the compass box and insert the cathode ray tube. Due to the alignment as said in Step 1, the electron beam coming out of the electron gun will lie along north-south direction.
3. Turn on the voltage supply unit. Then adjust the brightness and sharpness of the spot. Move the spot horizontally until it reaches this middle of the scale.
4. Put the Forward/Reverse knob in the forward position. Note down the position of the spot on the vertical scale. This is the initial position.
5. Apply a voltage to cause a deflection of the spot of about 10 mm. Note down the position of the final position of the spot.
6. Place two bar magnets on the two scales of the wooden frame. North pole of one of the magnets will face the south pole of the other one.
7. Adjust the position of the magnets in such a way that they remain equidistant from the cathode ray tube and the deflection produced in Step 5 is nullified and the spot returns back to its initial position. Note the position of the magnets on the arms.
8. Repeat the steps 5 to 7 several times for different values of the deflecting voltages. Reverse the direction of deflection and repeat the steps 5 to 7 as done earlier.
9. Take out the cathode ray tube and place the compass box again. Adjust the dial to read zero in absence of the bar magnets. Bring the two bar magnets on the positions recorded in Step 7. Note down the deflection of both the ends of the indicator. Interchange the position of the magnets on the arms of the wooden frame and also the polarities of the magnets so that the resultant magnetic field is in opposite direction. Note down the deflections. The average of there four readings will give the value of the angle θ (The deflection angle of the compass needle).

10. Draw a graph H^2/Y versus V. The slope of the curve when used in the working formula will give you the estimated value of specific charge (e/m).

Observations:

Table 1 For calculation of specific charge (e/m).

S. No.	Applied voltage, V (Volt)	Spot deflection, Y (cm)	Position of magnet to nullify the spot-deflection, x(cm)	Average compass deflection, θ (degree)	Applied magnetic field, $H = H_e \tan\theta$	H^2/Y
1.						
2.						
3.						
4.						
5.						

Results:

Supplied values

$L =$

$l =$

$d =$

$H_e = 0.345$ oersted

From the graph:

$$\text{Slope } S = \frac{H^2/Y}{V} = \frac{H^2}{VY} \Rightarrow \frac{1}{S} = \frac{VY}{H^2}$$

$$\therefore \quad e/m = \frac{10^7}{lLd} \cdot \frac{1}{S} \text{ emu of charge/gm}$$

Percentage error:

Discussion:

Aids to Viva Voce:

1. What is specific charge?

Ans. It is the charge to mass ratio of a charged fundamental particle like electron, proton, etc.

2. What type of electrons do you require to determine their specific charge?

Ans. Free electrons are required to determine the specific charge of electrons, protons, etc.

3. How can you generate free electrons?

Ans. One can generate free electrons in the process of thermionic emission, in the process of photoelectric effect and through operation of a cathode-ray tube.

4. What method of free electron generation is used in the present experiment?

Ans. In the present experiment, a beam of freely moving electrons is generated by using a cathode-ray tube.

5. What is a cathode-ray tube? Why is it named so? How does it work?

Ans. It is an evacuated hard glass tube fitted with two electrodes (namely, the cathode and the anode) which are externally connected in an electrical circuit which is capable of developing a high voltage between the electrodes. The cathode is kept at lower potential and the anode is kept at higher potential. When the potential difference crosses almost 4000 volt, a beam of free electrons starts moving from the cathode towards the anode. If the potential difference between the electrodes increases, the electrons in the beam are speeded up. The pressure in the tube usually remains at a fraction of one mm of mercury column.

It is known as cathode-ray tube because the ray was discovered before formal discovery of electrons. As the ray emerges from the cathode and moves towards the anode it was maned cathode ray and the tube in which it is generated is known as cathode ray tube (CRT). After formal discovery of electrons, it was realized that the cathode ray is nothing but a beam of moving electrons with high speed. But the ray is still called cathode ray.

6. How is the cathode ray treated to measure the specific charge of electron?

Ans. To perform the experiment the cathode ray is passed through two uniform (as well as variable) vertical and horizontal electric fields and a horizontal uniform (as well as variable) magnetic field. During the experiment the aforesaid fields are varied and the corresponding deflection of the ray is measured.

7. What is Lorentz force?

Ans. It is the exerted force on a charged particle moving in electric and magnetic fields. It is given by

$$\overline{F}_l = \overline{F}_e + \overline{F}_m$$

or,

$$\overline{F}_l = q\overline{E} + q(\overline{v} \times \overline{B})$$

where F_l = Lorentz force,
F_e = electric force,
F_m = magnetic force,
E = electric field,
B = magnetic field,
q = charge of the particle,
and v = velocity of the particle.

8. How is the effect of interference of the earth's horizontal magnetic field overcome in this experiment?

Ans. By using a magnetometer the CRT is held in such a way that the electrons move from south to north or from north to south direction. In the first case, the angle between the horizontal component of earth's magnetic field and the velocity of the moving electrons is π^c and in the second case it is 0^c. So, the value of F_m becomes zero, since

$$\overline{F}_m = q(\overline{v} \times \overline{B})$$

$$\therefore \quad F_m = qvB \sin\theta,\ F_m = 0 \text{ for } \theta = 0^c \text{ or } \pi^c$$

9. Electrons are not visible particles, then how can one sense the presence of the electrons?

Ans. In this experiment the cathode ray is deflected using one horizontal and one vertical electric field. The deflected ray is allowed to fall on a fluorescent screen (a screen coated with zinc sulfide). The kinetic energy of electrons of the cathode ray is transferred to the molecules of the fluorescent material. As a result of this, they get excited and immediately get deexcited (within 10^{-8} to 10^{-4} second) by emitting visible light making a bright coloured spot on the screen. The deflection of the spot gives the deflection of the cathode ray.

10. What is fluorescence?

Ans. When a beam of light is incident on certain substances (e.g., zinc sulfide) they emit visible light or radiations. This phenomenon is known as fluorescence and the substances showing this phenomenon are known as fluorescent substances.

The phenomenon of fluorescence is instantaneous and starts immediately after the absorption of light and stops as soon as the incident light is cut off.

11. What is phosphorescence?

Ans. When light radiation is incident on certain substances, they emit light continuously even after the incident light is cut off. This type of delayed fluorescence is called phosphorescence and the substances responsible for this phenomenon are called phosphorescent substances.

12. Compare the lifetime of fluorescence to that of phosphorescence.

Ans. Materials exhibiting fluorescence generally reemit excess radiation within 10^{-8} to 10^{-4} second of absorption. On the other hand, materials exhibiting phosphorescence reemit excess radiation within 10^{-4} to 20 seconds or longer. Thus the lifetime of phosphorescence is much longer than that of fluorescence.

13. What is the value of specific charge of an electron?

Ans. The specific charge of an electron is given by $\frac{e}{m} = 1.7592 \times 10^7$ emu of charge/g.

EXPERIMENT NO. 5

Aim: Determination of Lande's g factor (or spectroscopic splitting factor) by using electron spin resonance spectrometer.

Apparatus: Lande's g-factor set up which consists of a CRO, the main oscillator, detector unit, Helmholtz coil, sample tank containing the sample, and an external RF oscillator

Theory: Spin, the intrinsic angular momentum, S, couples with the orbital angular-momentum, L to give a resultant angular momentum J. When it is kept in an external magnetic field, H_0, $2J + 1$ magnetic sublevels are generated with equal energy difference.

$$\Delta E = g\mu_0 H_0 \qquad ...(1)$$

where, μ_0 is the Bohr magneton and g is a factor known as Lande's g factor, which is given by

$$g = 1 + \frac{J(J+1) + S(S+1) + L(L+1)}{2J(J+1)} \qquad ...(2)$$

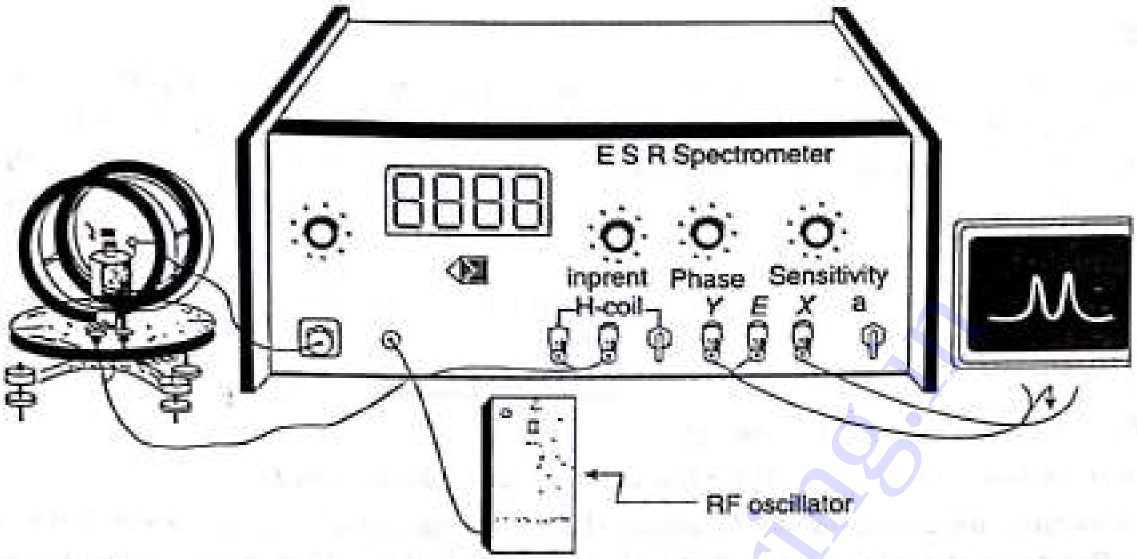
(g is 2 for free electrons).

Now, if the energy level is perturbed by an alternating magnetic field with frequency ν_1 such that the quantum of energy $h\nu_1$ is exactly equal to ΔE as given in Eq. (1) and if the direction of the alternating magnetic field is normal to the static magnetic field then there will be transitions between neighboring sublevels according to selection rule $\Delta m = \pm 1$. Therefore, at resonance,

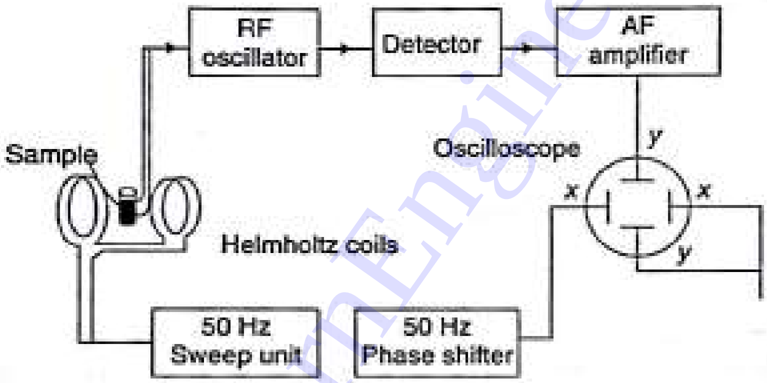
$$\Delta E = g\mu_0 H_0 = h\nu_1 \qquad ...(3)$$

Now, if I be the current flowing through the Helmholtz coil with a number of turns n in each coil of radius 'a' then the peak-to-peak magnetic field so produced is given by

$$H = 2\sqrt{2}\,\frac{32\pi \times n}{10\sqrt{125}\,a}\,I \qquad ...(4)$$



(a)



(b)

Fig. E.5 **(a) Diagram of the set up. (b) Block diagram of the set up.**

which produces P division of deflection in the x axis on the CRO screen and if the distance between the two absorption peaks is $2Q$ divisions on the said screen, then

$$H_0 = \frac{Q}{P} H$$

Using Eqs (3) and (4), we get

$$g = \frac{h\nu_1}{\mu_0 H_0} = \frac{h\nu_1 P}{\mu_0 k I Q} \qquad \text{...(5)}$$

where
$$k = 2\sqrt{2}\,\frac{32\pi n}{10\sqrt{125}\,a}$$

The product IQ may be obtained from the slope (s) of the curve Q versus $1/I$, whereas the RF frequency ν_1 can be estimated by formation of beat by an external RF oscillator.

Procedure:

1. Connect the X and Y ports of the spectrometer unit to the corresponding ports of the cathode ray oscilloscope (CRO). Apply a current through the Helmholtz coil. Note down the current in the circuit. Maximise the horizontal sensitivity of the CRO and with this setting vary the frequency, detection level and vertical sensitivity of the spectrometer until the four peaks superpose into two distinct very well-separated peaks.
2. Now measure the maximum X-deflection (P) and the separation between the absorption peaks (dips) $2Q$ in terms of the CRO screen divisions.
3. Change the value of current and record it. Adjust the X-sensitivity in order to produce the same value of P as in case of step 2. Measure $2Q$.
4. Repeat the last step (i.e., step 3) for few other values of the current I.
5. Now, connect the external radio frequency (RF) oscillator in the relevant socket of the spectrometer unit. Rotate the dial until a bit is observed on the screen of the CRO. Read out the dial reading.
6. Now a graph of Q versus $1/I$. Find its slope and hence determine the product of QI. Use Eq. (5) to calculate the value of g.

Observations:

Given parameters:

$n =$

$a =$

$\mu_0 =$

$h =$

$$k = 2\sqrt{2}\,\frac{32\pi n}{10\sqrt{125}\,a}$$

Table 1 Containing I-Q data.

S. No.	I (A)	P (div.)	2Q (div)	Q (div)

Calculations:

From the graph, slope $S = IQ = \dfrac{\Delta Q}{\Delta\left(\frac{1}{I}\right)} =$

Lande's g factor $g = \dfrac{h\nu_1 p}{\mu_0 k s}$

Results:

Percentage error:

Discussion:

Aid to Viva Voce:

1. What is Lande's g factor?

Ans. It is the negative ratio of the magnetic moment of an electron or atom (in units of Bohr magneton) to its angular momentum (in units of Planck's constant divided by 2π). It is also known as Lande's splitting factor or spectroscopic splitting factor.

2. What is Bohr magneton?

Ans. It is given by $\mu_e = \dfrac{e\hbar}{2m_e}$

where e = charge of an electron,

m_e = mass of an electron,

and $\hbar$ = Planck's constant divided by 2π (i.e., $\hbar = h/2\pi$).

In fact, it is the magnetic dipole moment of an electron.

3. What is resonance?

Ans. It is a phenomenon exhibited by a physical system when acted upon by an external periodic driving force in which the resulting amplitude of oscillation of the system becomes large when the frequency of the driving force approaches the natural frequency of vibration of the system.

4. What is electron spin resonance?

Ans. It is the magnetic resonance arising from the magnetic moment of unpaired electrons in a paramagnetic substance when the substance is kept in a variable external magnetic field.

5. What is a magnetic moment?

Ans. It is a vector associated with a magnet (or current loop or a particle) whose cross-product with the magnetic induction (or field strength) of a magnetic field is equal to the torque exerted on the system by the field.

i.e., $\overline{\tau} = \overline{\mu} \times \overline{B}$

where $\overline{\mu}$ is the magnetic moment of the system and $\overline{B}$ is the field strength of the magnetic field.

If the system is a particle its magnetic moment is given by

$$\mu = \frac{e\hbar}{2mc}$$

where m is the mass of the particle, e is its charge and c is the speed of light in free space.

6. What is paramagnetism?

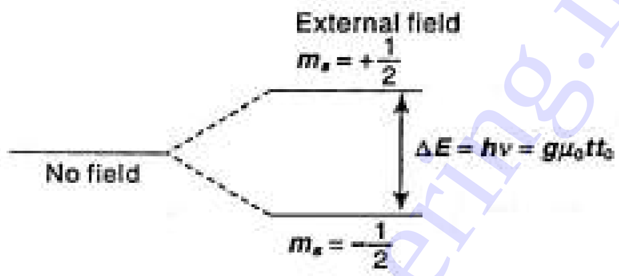
Ans. It is a property exhibited by substances which, when placed in a magnetic field, are magnetized parallel to the field to an extent proportional to the field.

7. What is paramagnetic substance?

Ans. It is a substance within which an applied magnetic field is increased by the alignment of electron orbits.

8. How does electron-spin comes under resonance in some materials in the presence of an external magnetic field? Explain.

Ans. ESR is observed in paramagnetic substances in the presence of an external magnetic field. These substances have unpaired electrons. The interaction of the unpaired electron spin with an external magnetic field depends on the magnetic moment associated with the said electron spin. And the nature of an unpaired electron spin is such that two or only two orientations are possible. The application of an external magnetic field then provides a magnetic potential energy which splits the spin states by an amount proportional to the magnetic field, and then a radio frequency radiation of the appropriate frequency can cause a transition from one spin-state to the other. When the frequency of oscillation of the radio oscillator becomes equal or an integral multiple of the natural frequency of transition of the electron-spin resonance takes place. The splitting electron spin in the presence of an external magnetic field is shown below:

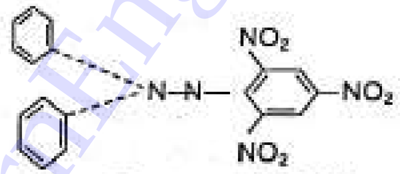


9. Name a paramagnetic substance which is usually used as a sample in ESR spectrometer.

Ans. 1, 1, diphenyl-2-picrylhydrazyl free radical (DPPH).

10. What is its chemical structure?

Ans. Its structure is given below:



Structure of DPPH

DPPH is a chemically stable material having the spectroscopic splitting factor $g = 2.0036$. DPPH contains 1.53×10^{21} unpaired electrons per gram.

11. How can you use ESR spectroscopy to calculate the number of unpaired electrons in a sample?

Ans. In order to calculate the number of unpaired electrons in an unknown sample, a comparison can be made with a standard sample (e.g., DPPH) having a known number of unpaired electrons and possessing the same line shape as the unknown (gaussian or lorentzian).

EXPERIMENT NO. 6

Aim: To observe discrete energy levels of an atom and hence to determine the first excitation potential of an atom by using Frank–Hertz experimental kit.

Apparatus: Frank–Hertz experimental kit with a (vacuum tube) tetrode filled with some inert gas vapour.

Theory: It was known from Bohr's postulates that the internal energies of an atom are quantized. This can be proved directly by this experiment.

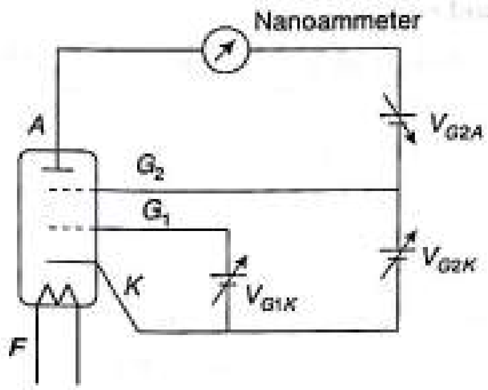


Fig. E.6 Frank–Hertz experimental kit.

In the set-up for experiment (Fig. E.6) there is a tetrode filled with argon vapour. Electrons emitted by the heated filament F are accelerated by the potential V_{G2K} (applied between the cathode and the grid G_2). Initially some electrons reach the anode A provided that their kinetic energy is sufficient to overcome the returding potential V_{G1K} (applied between the cathode K and the grid G_1). So, in the beginning it is seen that the anode current increases with the increase of voltage V_{G2K}. But as the voltage increases further, the energy of the electrons reaches the threshold value to excite the atoms to their first excited state and in the process many thermions lose their kinetic energy, for this reason the current in the circuit falls abruptly. And then the voltage V_{G2K} is increased further. Again the current starts to increase and when the voltage V_{G2K} reaches to a value twice that of its first excitation potential again, the current drops abruptly. In this way the current drops a few times in a certain voltage range. This observation proves that the energies of the atom are quantized.

Procedure:

1. First of all confirm that all the control knobs are in their minimum position.
2. Then turn on the experimental kit. Now turn on the "manual/auto" switch to manual mode.
3. Turn on the voltage display selector to V_{G1K} and adjust the V_{G1K} control knob to 1.5 V.
4. Select the voltage display of V_{G2A} and adjust it to 7.5 V.
5. Now change the value of V_{G2K} in small steps (say, 0.5 V) and record the corresponding current reading.
6. Now draw a graph showing the variation of current as a function of the accelerating voltage.
7. Turn on the "manual/auto" switch to auto mode.
8. Connect the Y, G, K sockets of the instrument to the corresponding ports of the oscilloscope. Set the oscilloscope to X-Y mode and the trigger to external X.
9. Now adjust the 'shift' and the 'gain' switches to obtain a clear waveform. Apply the maximum scan range through the instrument.
10. Get the average horizontal distance measured between the peaks. This would give the value of gas atom's first excitation potential in eV.
11. The excitation potential can also be obtained from the current versus voltage graph drawn in Step 6. And they may be compared.

Observations:

$V_{G1K} = 1.5$ V, $V_{G2A} = 7.5$ V

Table 1 For recording of voltage and current.

S. No.	*Gride voltage (V) × 2 V*	*Current (I) × 0.02 × 10^{-8} mA*
–	–	–
–	–	–
–	–	–
–	–	–

Calculations:

Results:

Discussions:

Aids to Viva Voce:

1. What is the structure of an atom? Explain.

Ans. An atom consists of three types of fundamental particles namely, proton, neutron and electrons. The atom has a solid central core consisting of protons and neutrons. It is called the nucleus of the atom. The electrons are distributed among different spherical shells (called energy levels) according to Pauli's exclusion principle. The energy levels of an atom are discrete and well defined. According to Pauli, the maximum number of electron allowed in an energy level is given by

$$N_n = 2n^2$$

where n is the serial/order number of the shell and N_n is the number of maximum allowed electrons in the nth shell. The protons are positively charged particles, the electrons are negatively charged particles and the neutrons are neutral particles. So, the nucleus of an atom is positively charged. As the electrons are negatively charged, they experience a strong electrostatic attractive force towards the nucleus. Because their force the electrons keep on revolving around the nucleus to maintain a dynamic balance. This gives the idea of discrete energy levels. Other than the first energy level, all other energy levels are divided into a number of sublevels of energy. An electron belonging to an energy level can jump to a higher energy level if it is supplied with a definite amount of energy given by

$$E = E_2 - E_1 = h\nu$$

where E_1 and E_2 are the energies of the first and second energy levels and ν is the frequency of the radiation and h is Planck's constant.

2. What do you mean by excitation of an atom?

Ans. When an assembly of atoms of a matter is exposed to a radiation, some of the atoms may absorb a quantum of energy ($h\nu$) and sends one of its electrons from a lower energy level to a higher energy level which satisfies the following condition:

$$h\nu = E_2 - E_1$$

This phenomenon is known as excitation of an atom.

3. What is electron volt?

Ans. Electron volt (eV) is a unit of energy. It is defined as follows:

When an electron moves from a point of lower potential to a point of higher potential it gains energy. If the potential difference between the two points is one volt, then energy gained by the electron is called one electron volt (eV).

$$1 \text{ eV} = 1.6 \times 10^{-19} \text{ joule}$$

4. What is a vacuum tube used in electronics?

Ans. It is an electronic device like diode, triode, tetrode, etc., which are used in an electronic circuit. These devices can be built by using vacuum tube technology.

5. What is a tetrode? What is its counterpart in semiconductor technology?

Ans. It is a vacuum tube which has four electrodes called cathode, anode, grid one and grid two. It does not have any counterpart in semiconductor technology.

6. Which vacuum tube is used in this experiment?

Ans. In this experiment a tetrode is used.

7. Why are the energy levels of an atom discrete?

Ans. The atoms have permanent atomic structure which does not collapse. An atom has positive charge at the central nucleus and negatively charged electrons revolve around the nucleus. In this process they are supposed to lose energy through radiation. But practically it does not happen. To support this situation Neils Bohr suggested one atomic model called the quantum model of atom. According to this model, the electrons of an atom are distributed in different discrete energy levels having well-defined value of energy.

8. What is excitation potential?

Ans. It is the energy required to shift one electron from the ground state to higher energy state.

9. What do you mean by first excitation potential?

Ans. It is the energy required to shift one electron from the valance level to the first unfilled level.

10. What is a grid?

Ans. It is a mesh cylinder which is placed between the cathode and the anode in a vacuum tube (e.g., a triode). It allows the thermions to pass through it and controls the flow of thermions.

EXPERIMENT NO. 7

Aim: Determination of the energy of the band gap of a semiconductor by using the four probe method.

Apparatus: A four probe device, an oven, a thermometer, a sample semiconductor crystal, a voltmeter, an ammeter and connecting leads.

Theory: The highest filled energy band including electrons shared in covalent bonds or transferred in ionic bonds in a semiconductor is known as valence band. The energy level which corresponds to the top of the valence band in an intrinsic semiconductor is denoted by E_v.

The energy band which remains fully unfilled or includes some free electrons is called the conduction band. The energy level which corresponds to the bottom of the conduction band in an intrinsic semiconductor is denoted by E_c.

The energy gap between the top of the valence band and the bottom of the conduction band is called band gap. So, the energy of the band gap (E_g) is given by

$$E_g = E_c - E_v \quad \text{...(1)}$$

The band gap of a semiconductor can be calculated from the following formula:

$$\ln \rho = \frac{E_g}{2kT} - \ln A$$

i.e.,

$$\log_e \rho = \frac{E_g}{2kT} - \log_e A \quad \text{...(2)}$$

where

ρ is the resistivity of the semiconductor,

E_g is the band gap energy of a semiconductor,

k is the Boltzmann constant ($k = 1.38 \times 10^{-23}$ J/K), and

T is the absolute temperature and A is a constant.

The resistivity of the given semiconductor is given by $\rho = \dfrac{\rho_0}{G_7(W/s)}$

where $G_7(W/S)$ is the correction factor. And it is expressed as

$$G_7(W/s) = \frac{2S}{W} \log_e{}^2$$

Again

$$\rho_0 = \frac{V}{I}(2\pi S)$$

where

V is the applied voltage,

I is the current,

W is the thickness of the crystal,

whose value is given (/ supplied) and S is the distance between the probes, the value of which is also supplied.

Procedure:

1. Insert one sensitive thermometer through the hole of the oven.
2. Turn on the apparatus and set a current of the order of 5 mA.
3. Note down the temperature in the thermometer and the voltage in the voltmeter.
4. Now turn on the oven. Thus record the temperature and the corresponding voltage at an interval of 5°C up to at least 120°C.
5. Turn off the oven. Record the temperatures and the corresponding voltages at intervals of 5°C while the temperature decreases.
6. Calculate average voltage at each temperature. Calculate ρ. Convert the temperatures into absolute scale.
7. Plot a graph of $\log_e \rho$ versus $\frac{1}{T}$. Determine the slope from the linear part of the graph.

8. Calculate the value of $G_7(W/S)$ from the given value of W and S.

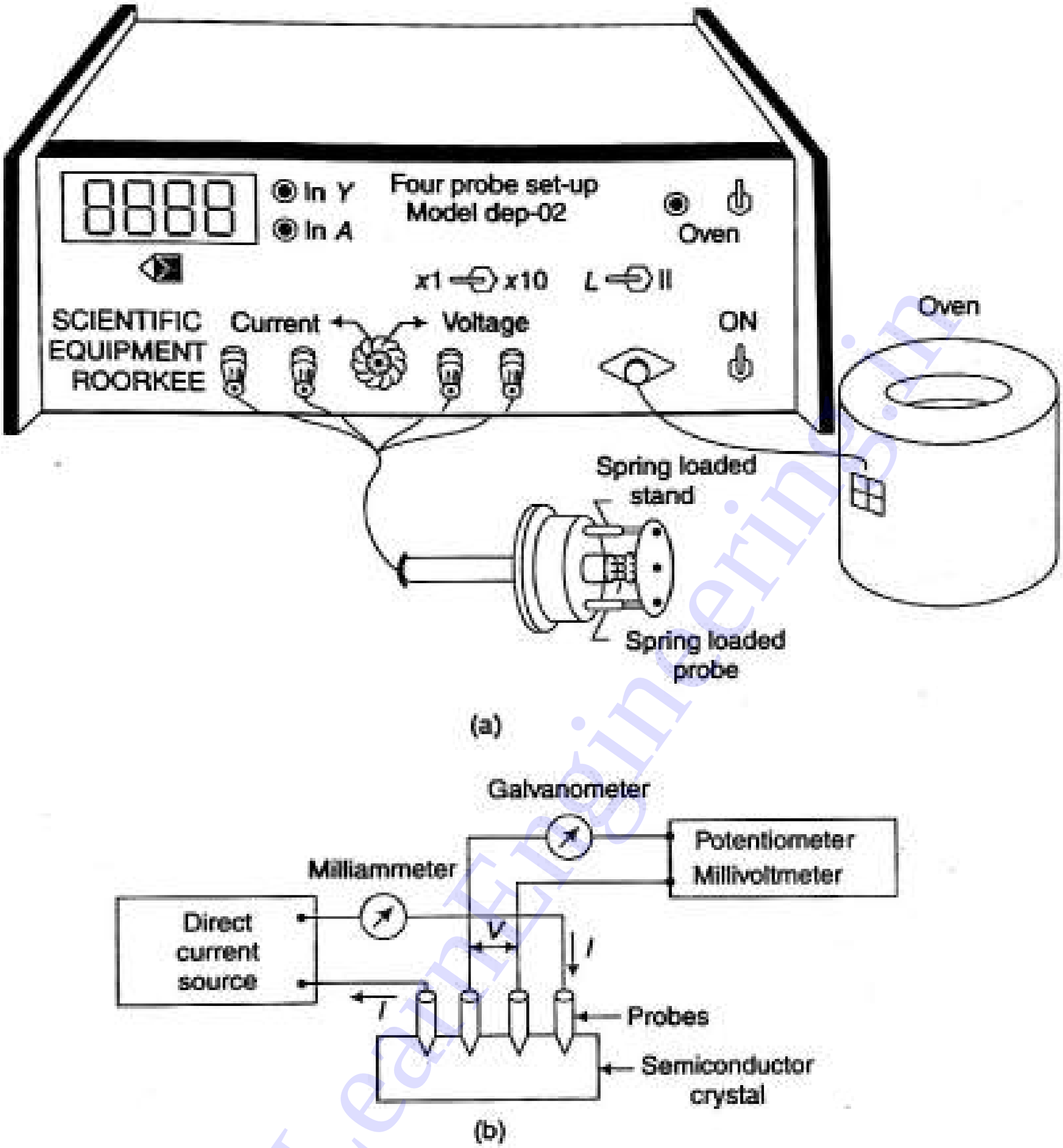


Fig. E.7 (a) Experimental set-up of band gap energy measurement. (b) Circuit diagram of four probes.

Observations:

Current I = mA (constant), W =

S = , $G_7(W/S)$ =

Table 1 Records of voltage and temperature.

S. No.	*Voltage (V)*	*Temperature (°C) while increasing*	*while decreasing*	*Average temperature T (K)*	$T^{-1} \times 10^{-3}$ (K^{-1})	ρ_0 (Ω-cm)	ρ (Ω-cm)	$\log_e \rho$
–	–	–	–	–	–	–	–	–
–	–	–	–	–	–	–	–	–

Calculations:

Results:

Percentage error:

Discussion:

Aids to Viva Voce:

1. What is a semiconductor?

Ans. It is a solid crystalline material whose electrical conductivity is intermediate between that of a conductor and an insulator, ranging from about 10^5 mho to 10^{-7} mho per meter. It is usually strongly temperature dependent.

2. What is an intrinsic semiconductor?

Ans. If the electric conductivity of a semiconductor is entirely determined by the carriers which are generated by thermal excitations from the valence band to the conduction band, the semiconductor is referred to as a pure or intrinsic semiconductor.

3. How energy bands are generated in a semiconductor?

Ans. A semiconductor (like silicon or germanium) remains in crystalline form. In such a crystal, the constituent atoms are orderly arranged, so the unfilled energy levels of the crystal atoms merge together to form an energy band called the conduction band and the filled and partially filled energy levels merge together to form another energy band called valence band. In a semiconductor there remains a gap between conduction band and the valence band.

4. What is a band gap?

Ans. The difference between the lowest energy level of the conduction band and uppermost energy level of the valence band is called band gap of the semiconductor.

5. How much is the value of band gap in case of germanium and silicon?

Ans. In case of germanium, the band is 0.785 eV and in case of silicon it is 1.21 eV at 0 K.

6. Comment on the band gap of insulator and metal.

Ans. The band gap of insulator is larger than that of a semiconductor, while in case of metal (i.e., conductor) the band gap is zero because the conduction bond and the valence band of a metal overlap each other.

7. What is a probe?

Ans. It is a small device which can be brought into contact with or inserted into a system in order to make measurements on the system, ordinarily it is designed so that it does not significantly disturb the system.

8. Why is the present method of determination of band gap of semiconductor called a four-probe method?

Ans. The present method of determination of band-gap energy is called four probe method because four probes are required to determine the band-gap energy in this method.

9. On the basis of what parameter can you compare one curve and one straight line? Explain.

Ans. We can compare a curve with a straight line on the basis of the radius of curvature. A straight line is a special case of a curve whose radius of curvature is infinity.

10. What is the slope of a curve? What is it in case of a straight line?

Ans. The slope of a curve is the tangent of the angle of inclination of the line, which touches the curve at the point of intersect, with the x-axis. In case of a straight line, it is tangent of the angle of inclination of the line itself with x-axis because a line and its tangent coincide with each other.

EXPERIMENT NO. 8

Aim: Determination of the thermoelectric power of a given thermocouple at a certain temperature.

Apparatus: A copper-constantan thermocouple, a potentiometer, a heater, thermometers, an beaker, a resistance box and a galvanometer.

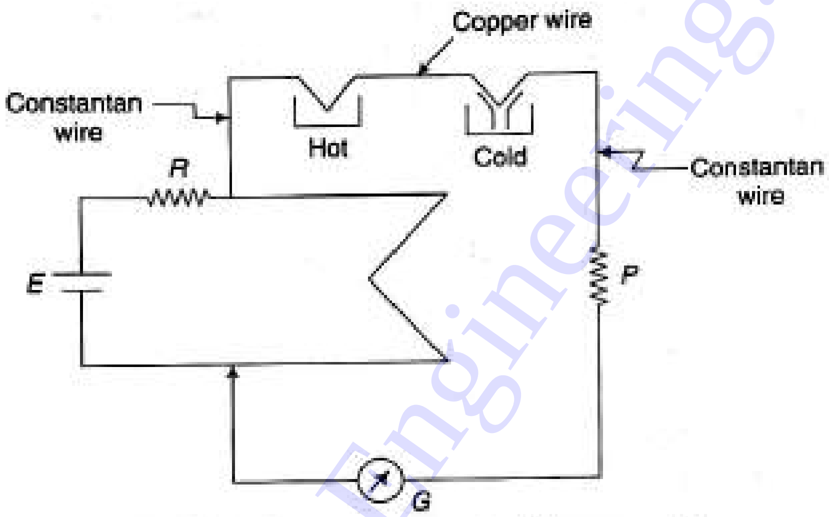


Fig. E.8 A copper-constantan thermocouple.

Theory: When two wires of two different materials are soldered together, having put them in contact, a thermocouple is prepared. A small voltage in the millivolt range is generated when the two junctions of the wires are kept at two different temperatures. The induced emf is dependent on temperature. A small current flows around the circuit made of the wire loop.

In fact, a temperature difference produces a potential difference and the phenomenon is known as thermoelectric effect.

When one junction of the thermocouple is kept at 0°C and the other junction at some higher temperature (t), a thermo emf (e) develops in the thermocouple. Measuring the thermo emf at different temperatures of the hot junction a graph of emf (e) versus temperature (t) can be drawn.

The thermoelectric power is the slope of the aforesaid graph.

It is defined by the following relation:

$$P = \frac{de}{dt} \text{ (in } \mu v/^{\circ}\text{C)} \quad \ldots(1)$$

The thermo emf (e) is measured with the help of a potentiometer, which has a dc source of emf ε and a resistance R is the driving circuit. If l be the position of the null point from the start of the first wire, and L be the total length of the potentiometer wire (usually 10 meter) of resistance r, then one can have.

$$e = \frac{Erl}{(R + r)\,L} \qquad \text{...(2)}$$

So, the potential drop per unit length of the wire is given by

$$\rho = \frac{e}{l} = \frac{Er}{(R + r)\,L} \qquad \text{...(3)}$$

Procedure:

1. Measure the resistance of the potentiometer wire (r) with the help of a multimeter calculate the value of the series resistance R so that you can find out the potential drop per unit length of the wire (ρ) by using the formula of Eq. (3).
2. Connect the circuit as shown in Fig. E.8. Apply a value of R calculated from the formula of Eqs (2) or (3). Fill the funnel of the cold junction with crushed ice. Care should be taken to avoid any air pocket surrounding the junction.
3. Dip the hot junction of the thermocouple in a beaker containing water at room temperature. Measure the position of the null point. Take care so that a high current does not pass through the galvanometer/nanoammeter. It can be a good idea to keep a high resistance in series to delimit the current which is then removed when the position of the null point is roughly known.
4. Turn on the spirit lamp (or use a hot plate) and note the null point at an interval of 5°C up to at least 80°C.
5. Draw a graph having plotted e versus t. Then find out the slope of the graph.

Observations:

Resistance of the potentiometer r = (ohm)

Value the series resistance (from Eq. (3)), R = (ohm)

Actual value of the series resistance

R_a = (ohm)

Table 1 emf of the driving cell.

	E (volt)
Before expt.	
After expt.	
Average	

Table 2 Null-point-temperature data.

S. No.	*Temperature, t (°C)*	*Position of the null point*			
		Wire No.	*Length*	*Total, l (cm)*	$e = \frac{1000\,Erl}{(R + r)\,L}$ *(mV)*

Calculations:

Percentage error:

Results:

Discussion:

Aids to Viva Voce:

1. What is a thermocouple?

Ans. When two wires of different metals are joined at their ends and a temperature difference is maintained between the junctions, an electric current flows in the circuit. This device is called thermocouple. The emf which is generated between the two junctions of a thermocouple is known as thermo-emf.

2. What is Seeback effect?

Ans. When two wires of different metals are joined at their ends and a temperature difference is maintained between the two junctions, a current flows in the circuit. This effect is called Seeback effect.

3. What is Peltier effect?

Ans. When a battery is inserted in a thermocouple circuit whose two junctions are initially at the same temperature, one of the junctions becomes hot and the other becomes cold. This phenomenon is known as Peltier effect.

4. What is Thomson effect?

Ans. When the two points of a conductor are kept at different temperatures, a potential difference develops between the said two points. This phenomenon is called Thomson effect.

5. Explain the existence of an emf at the junction of two metals?

Ans. When two different metals are joined at their ends, the free electrons of one metal will flow to the other because the pressure of electrons of the metals is different. As a result of flow of the electrons, one metal becomes positive as compared to the other and a potential difference is created at the junction. The magnitude of this potential difference, referred to as the contact potential difference, increases with increase of temperature.

6. How is a thermocouple constructed? Name some pairs of metals which are generally used for the construction of thermocouple.

Ans. To construct a thermocouple, say, copper-constantan thermocouple, one piece of constantan wire and one piece of copper wire are taken. After cleaning their ends with sand paper, one end of the copper wire is spot-welded with one end of the constantan wire. Similarly other end of each one is joined. Thus, the junctions of a thermocouple are formed, copper-constantan, copper-iron, platinum-rodium, etc., are the pairs of metals which are usually used for construction of thermocouple.

7. What is the neutral temperature of thermocouple?

Ans. The neutral temperature is the temperature of the hot junction at which the generated thermo emf is a maximum, when the cold junction remains at 0°C.

8. Mention two applications of thermocouple?

Ans. (i) It is used to measure the temperature of a point, and (ii) to measure the radiant heat.

9. Can one use an ordinary voltmeter to measure the thermo emf?

Ans. No, one cannot use a voltmeter because the emf developed in thermocouple is in millivolt range. Also, a voltmeter actually gives the potential difference and not the emf.

10. What is the value of the thermoelectric power at the neutral temperature?

Ans. The thermoelectric power at the neutral temperature is zero.

11. Explain the difference between joule effect and Peltier effect?

Ans. In the case of joule effect the heat generated is proportional to the square of the current and hence is independent of the direction of current. On the other hand, the Peltier effect produces heating or cooling at the junction that is proportional to the current. Thus, a junction which is heated by a current will be cooled when the direction of the current is reversed.

12. What is the general nature of the thermo emf versus temperature curve? What is the nature of the curve you have used?

Ans. The general nature of the thermo emf versus temperature curve is parabolic. We get a straight line, because the temperature of the hot junction is much less than the neutral temperature of the couple. That is we get the straight portion of the parabola.

EXPERIMENT NO. 9

Aim: Determination of Rydberg constant by studying the spectrum of hydrogen gas.

Apparatus: A spectrometer, a spirit level, a hydrogen discharge tube, a power supply for the discharge tube and a magnifying lens.

Theory: Balmer series is the emission spectrum related to the transition of electrons from states with principal quantum number $n > 2$ to the state with $n = 2$. The wavelengths of the corresponding lines are expressed by a formula

$$\frac{1}{\lambda} = R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

where $n_f = 2$ and $n_i > 2$.

In the spectrum of hydrogen gas there are a few bright lines, namely,

H_α: $n_i = 3$: color = red

H_β: $n_i = 4$: color = blue-green

H_γ: $n_i = 5$: color = violet

H_δ: $n_i = 6$: color = violet

In this experiment wavelengths of these lines are determined by measuring the angle of diffraction by a transmission grating using the following formula:

$$(a + b)\sin\theta = n\lambda$$

where n is an integer.

Here 'a' is the width of the opaque line in the grating and b is the with of the transparent region lying between two opaque lines and consequently $(a + b) = \frac{1}{N}$, where N is the number of rulings per unit length of the grating, n is the order of the spectrum and θ is the angle of diffraction.

Observations:

Vernier constant $VC = \dfrac{1 \text{ SDMS}}{\text{No. of divisions in } VS}$

[SDMS is the smallest division in the main scale and VS is the vernier scale]

No. of rulings $N = \dots$ per cm.

Table 1 Measurement of angle of diffraction.

Order of the spectrum	*Color*	*Vernier (V)*	*Left spectrum*			*Right spectrum*			*Difference 2θ*	*Avg. θ*
			MSR	*VSR*	*Total*	*MSR*	*VSR*	*Total*		
First	Red	V_1								
		V_2								
	Blue-green	V_1								
		V_2								
	Violet$_1$	V_1								
		V_2								
	Violet$_2$	V_1								
		V_2								

Calculations:

$$\lambda_{\text{red}} =$$
$$\lambda_{\text{blue-green}} =$$
$$\lambda_{\text{violet 1}} =$$
$$\lambda_{\text{violet 2}} =$$

Rydberg constant for red line $R_1 =$

Rydberg constant for blue-green line $R_2 =$

Rydberg constant for violet (1) line $R_3 =$

Rydberg constant for violet (2) line $R_4 =$

Average Rydberg constant $R =$

Standard value of Rydberg constant $R_s =$

Percentage error:

Results:

Discussion:

Aids to Viva Voce:

1. What is a Rydberg constant (R)?

Ans. It is a physical constant relating to atomic spectra in the science of spectroscopy. This constant R is named after the Swedish physicist Johannes Rydberg. The value of Rydberg constant is given by

$$R = 1.097 \times 10^7 \text{ m}^{-1}$$

i.e., $$R = 0.01097 \text{ nm}^{-1}$$

2. What is a wave number?

Ans. It is inverse of the wavelength of a photon. Sometimes inverse of wavelength is multiplied by 2π to the wave number.

3. What does Rydberg constant represents?

Ans. Rydberg constant represents the limiting value of the highest wave number of any photon that can be emitted from the hydrogen atom or alternatively the wave number of the photon (having lowest energy) capable of ionizing the hydrogen atom from its ground state.

4. What is Rydberg unit of energy?

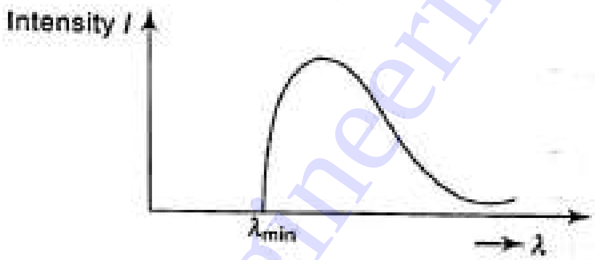
Ans. Rydberg unit of energy (symbol *Ry*) is closely related to the Rydberg constant. It corresponds to the energy of the photon whose wave-number is Rydberg constant, i.e., the ionization energy of the hydrogen atom.

5. Comment about the definition of the wave number. Is it unique?

Ans. Definition of wave number is not unique. It is defined as $k = \frac{1}{\lambda}$ which means the number of waves present per unit length. But sometimes it is expressed as $k = \frac{2\pi}{\lambda}$.

6. What is spectrum? Explain.

Ans. A spectrum is a condition that is not limited to a specific set of values but can vary infinitely within a continuum. As for example we see the graphical representation of the spectrum of continuous X-ray given below.



7. What is a spectrometer?

Ans. It is a device which is used to study the spectrum. It is also called a spectroscope.

8. Write down Rydberg formula for any hydrogen-like atom and explain the symbols used?

Ans. The Rydberg formula for hydrogen-like atom is given by

$$\frac{1}{\lambda_v} = Rz^2\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

where

λ_v is the wavelength of light emitted in the vacuum,

R is the Rydberg constant for the concerned element,

z is the atomic number of the element,

and n_f and n_i are integers such that $n_f < n_i$

9. Express Rydberg constant in terms of other physical constants for an atom of hydrogen?

Ans. The Rydberg constant for hydrogen when expressed in terms of other constants is given by

$$R = \frac{m_e e^4}{8\varepsilon_0^2 h^3 C} = 1.0973 \times 10^7 \text{ m}^{-1}$$

10. Express Rydberg unit of energy in terms of electron volt. Comment on it.

Ans. 1 Ry = 13.605 eV

$$[1\ Ry = hcR] \text{ where } R = 1.097 \times 10^7 \text{ m}^{-1}$$

one Rydbergy energy is the excitation potential of a hydrogen potential.

EXPERIMENT NO. 10

Aim: To study current-voltage characteristics, load response, areal characteristics and spectral response of a photovoltaic (or solar) cell.

Apparatus: A solar panel, a voltmeter, a milliammeter, a decade resistance box, a 100 W lamp fitted with an intensity control, area choppers of different sizes and color filters.

Theory: The photovoltaic cell is basically a *p-n* junction diode which converts solar energy into electrical energy. It is also known as solar cell. As its name is photovoltaic cell, the phenomenon is known as photovoltaic effect. When this is illuminated, the photons incident on the cell generate moving electrons and accordingly holes are also generated. At the junction, the barrier field separates the positive and negative charge carriers. Under the action of the electric field, the electrons (i.e., minority carriers) from the *p*-region are swept into the *n*-region. Similarly the holes from the *n*-region are swept into the *p*-region. It leads to an increase of holes on the *p*-side and elections on the *n*-side of the junction. The accumulation of charges on the two sides of the junction produces an emf which is called photo-emf. It is also known as open circuit voltage. It is proportional to the illuminated area. When an external circuit is connected across the solar cell terminals, the minority carriers return to their original side through the said circuit. The solar cell behaves as a battery with the *n*-side as negative terminal of the circuit and the *p*-side as the positive terminal.

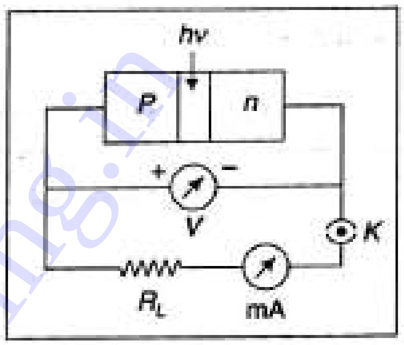


Fig. E.9 Circuit diagram of a solar cell.

The photo-emf or voltage can be measured with a voltmeter.

Procedure:

1. *Illumination characteristics:* Set the load resistance (R_L) to zero. Then vary the voltage across the light bulb and record the photocurrent. Draw a graph by plotting photocurrent (I_p) versus lamp voltage (V_l).
2. *Current-voltage characteristics:*
   - Keep the intensity of the lamp at a moderate level. Note the open circuit voltage of the photocell (V_{open}). Short the output of the solar cell and note down the short circuit current (I_s).
   - Change the load resistance in steps of 100 ohm and record the corresponding photocurrent and voltage. Draw a graph with voltage V along the y axis and current I along the x axis.
3. *Power load characteristics:* Calculate power from the current-voltage data obtained in Step 2 and plot a graph of power against load resistance. Find out the optimum value of load for which the power dissipation is the maximum.
4. *Areal characteristics:* Set the load at the optimum value obtained in Step 3. Set the chopper in the slot provided in front of the solar cell in different settings and measure the current and voltage. Calculate the power. Draw a graph of power versus area of the chopper settings.
5. *Frequency characteristics:* Keeping the load at its optimum value, put several fillers and take the current-voltage data to get the power. Plot a graph of power versus frequency of the used filter.

6. To obtain the load line on the current-voltage characteristic curve refer to the power versus load resistance curve and find the value of maximum power (P_{max}) corresponding to the optimum load (R_{op}). Then calculate the corresponding value of voltage (V) from the following relation:

$$P = \frac{V^2}{R}$$

Now, find the point corresponding to V on the I-V characteristic curve. Draw a straight line passing through the origin and the point referred to above and extend it this is the required load line.

Observations:

Table 1 Illumination characteristics.

No. of obs.	*Filament voltage, V_f (V)*	*Current through solar cell, I (A)*
–	–	–
–	–	–

Table 2 Current-voltage characteristics

No. of obs.	*Filament voltage, V_f (V)*	*Load resistance, R (Ω)*	*Load voltage, V (V)*	*Current through solar cell, I (A)*	*Power, P = VI (W)*
–	–	–	–	–	–
–	–	–	–	–	–

Optimum resistance, R_{op} =

Table 3 Areal characteristics.

No. of obs.	*Chopper area, A (cm²)*	*Voltage, V (V)*	*Current, I (A)*	*Power, P (W)*
–	–	–	–	–
–	–	–	–	–

Table 4 Frequency characteristics.

No. of obs.	*Color of the filter*	*Frequency of the filter*	*Voltage, V (V)*	*Current, I (A)*	*Power, P (W)*
–	–	–	–	–	–
–	–	–	–	–	–

Calculations:

From the graphs:

Voltage $V =$, Current $I =$, Power $P = VI =$

Results:

Percentage error:

Discussion:

Aids to Viva Voce:

1. What is a solar cell?

Ans. A solar cell (also called photovoltaic cell or photoelectric cell) is a solid state electrical device that converts light energy directly into electricity by photovoltaic effect.

2. What is photovoltaics?

Ans. It is the field of technology and research related to the practical application of photovoltaic cells in producing electricity from light, though it is often used specifically to refer to the generation of electricity from sunlight.

3. What is optimum load resistance?

Ans. It is the load resistance used in the circuit of a solar cell for which the power output of the circuit is maximum.

4. How can one determine the optimum load resistance?

Ans. By plotting a graph of output power versus the load resistance in the circuit, one can determine the optimum resistance. The resistance in the aforesaid graph for which the output power is maximum in the optimum load resistance.

5. How can you build a solar cell?

Ans. One can prepare a photovoltaic cell by doping a semiconductor chip. One half of the chip is doped positively and the other half is doped negatively: In fact, a photovoltaic cell is a *p-n* junction.

6. What is the difference between excitation and ionization? Which one is responsible for the generation of photovoltaic effect?

Ans. In case of excitation an electron which absorbs the light photon does not leave the atom. It jumps from a lower energy level to a higher energy level of the atom, while in case of ionization, the electron, which absorbs the light photon, becomes completely free by separating itself from the atom. Ionization is responsible for generation of photovoltaic effect.

7. What are open circuit voltage and short circuit current?

Ans. Open circuit voltage is the voltage across the source when the circuit remains open. And short-circuit current is the current in circuit when there is no external resistance present in the circuit.

8. What is the difference between a *p-n* junction diode and a solar cell?

Ans. So far as construction is concerned a *p-n* junction diode and a solar cell are same. But when it is used as diode it is operated with the help of electrical energy and when it is used as solar cell it is operated with the help of light energy.

9. What is a hole?

Ans. The absence of an electron in the valence band of *p*-type region is referred to as a hole.

10. Can one establish the presence of a hole practically?

Ans. Yes, by performing one experiment, one can practically establish the presence of holes.

EXPERIMENT NO. 11

Aim: To determine Hall coefficient of a semiconductor.

Apparatus: Electromagnet, constant power supply, semiconductor, digital millivoltmeter, gaussimeter, etc.

Theory: A static magnetic field does not have any effect on charged particles if they are at rest. But the moving charged particles are affected by a static magnetic field. When a charged particle moves in a magnetic field which is perpendicular to the direction of the moving charge, a force acts on the charge which is perpendicular to both, the direction of the magnetic field and the direction of the velocity of the charged particle. When a semiconductor chip is placed in a constant magnetic field and kept its surface perpendicular to the magnetic field and a current is allowed to flow through its surface in a direction which is perpendicular to the said field, then the electrons and the holes of the semiconductor are separated by opposite forces. They will in turn produce an electric field $(\bar{E}_h)$ which depends on the cross product of the magnetic field intensity $(\bar{H})$ and the current density $(\bar{J})$. Mathematically it can be expressed as follows:

$$\bar{E}_h = R(\bar{J} \times \bar{H}) \tag{1}$$

where R is known as Hall coefficient or Hall constant. Let us consider a bar of semiconductor having dimensions x, y and z. Let $\bar{J}$ be directed along x-axis and $\bar{H}$ along the z-axis, then $\bar{E}_h$ will be directed along the y-axis.

In general, the Hall voltage is not a linear function of the magnetic field applied, i.e., the Hall coefficient is not generally constant, but it is a function of the applied magnetic field. The working formula is given by

$$R = \frac{V_h t}{I_h H} \tag{2}$$

where,

I_h = the current flowing through the semiconductor due to Hall effect.

t = thickness of the semiconductor.

V_h = Hall voltage

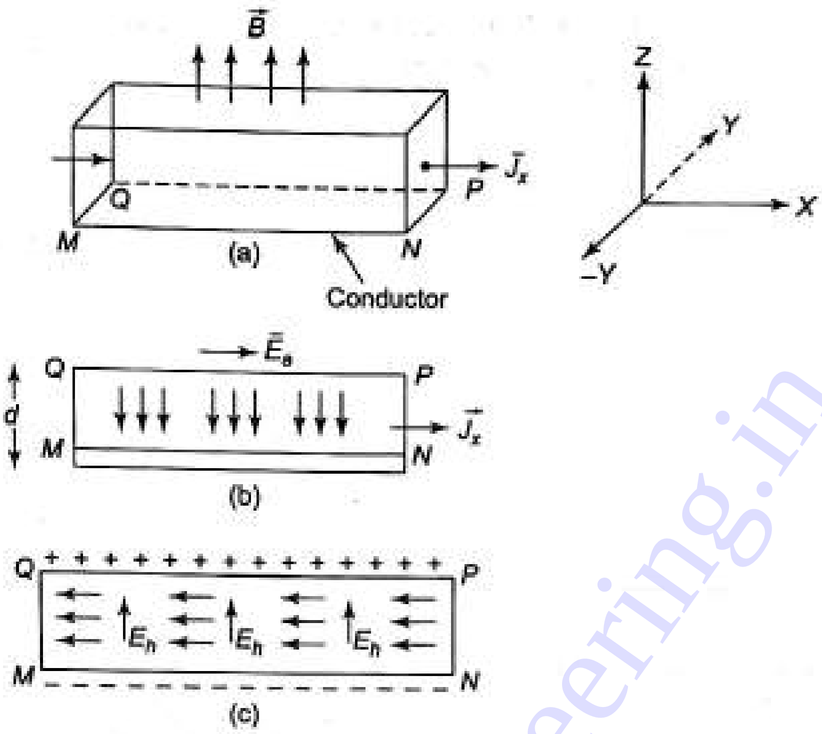


Fig. E.10 **Hall effect is shown here. (a) $\vec{B}$: magnetic field, $\vec{J}_x$ current density, (b) $\vec{E}_a$: applied electric field related to $\vec{J}_x$ and ↓: direction of electron shift, (c) ↑: direction of Hall field E_h, ←: direction of electron flow.**

To measure the magnetic field intensity H, a standard Hall prove is used and calibrated against the magnetizing current passed through the coil.

Procedure:

1. Place the Hall probe in the central region of the gap between the poles of the electromagnet. Turn on the current source and the gaussimeter. Turn on and adjust the knob so that the current from this source is zero. The Hall probe should read zero magnetic field.
2. Increase the current in small steps and note down the magnetic field in each case. The current should be increased up to a level so as to produce 3000 gauss magnetic field.
3. Remove the Hall probe and insert the semiconductor chip for which you wish to measure the Hall constant.
4. Apply a magnetizing current, which can produce a magnetic field of 1000 gauss.
5. Increase the current through the semiconductor (Hall current) in small steps and note down the voltage (Hall voltage) in each case.
6. Take three sets of reading for Hall current (I_h) and Hall voltage (V_h) by fixing the value of the magnetizing current (I_m) at three values respectively (say 1 ampere, 2 amperes and 3 amperes).
7. Draw a graph of magnetic field (H) against the magnetizing current (I_m) (Table 1).
8. Draw three more curves of Hall voltage (V_h) against Hall current (I_h) and take slopes of them $\left(S = \dfrac{\Delta V_h}{\Delta I_h}\right)$.

Model Question Paper 1

Group A

Multiple Choice Questions

1. (i) The value of $\overline{\nabla} \cdot \bar{r}$ is given by

(a) 1 (b) 0 (c) 2 (d) 3

(ii) If $\overline{A}$ and $\overline{B}$ are two rotational vectors, then $\overline{A} \times \overline{B}$ is

(a) solenoidal (b) irrotational (c) rotational (d) None of these

(iii) In free space, Poisson's equation is given by

(a) $\nabla^2 V = -\frac{\rho}{\epsilon_0}$ (b) $\nabla^2 V = \frac{\rho}{\epsilon_0}$ (c) $\nabla^2 V = 0$ (d) $\nabla^2 V = \rho \epsilon_0$

(iv) If $\overline{\nabla} \cdot \overline{B} = 0$, then

(a) $\overline{B} = \overline{\nabla} \cdot \overline{A}$ (b) $\overline{B} = \overline{\nabla} \times \overline{A}$ (c) $\overline{B} = \nabla^2 \overline{A}$ (d) None of these

(v) In free space, modified Ampere's circuital law is given by

(a) $\overline{\nabla} \times \overline{E} = 0$ (b) $\overline{\nabla} \times \overline{B} = \mu_0 \epsilon_0 \frac{\partial \overline{E}}{\partial t}$

(c) $\overline{\nabla} \cdot \overline{E} = 0$ (d) $\overline{\nabla} \cdot \overline{B} = 0$

(vi) Electromagnetic wave is propagated through a region of vacuum which does not contain any charge or current. If the electric vector is given by $\overline{E} = \hat{j}\, E_0 \exp i(kx - wt)$. Then magnetic vector is

(a) in the x direction (b) in the y direction

(c) in the z direction (d) rotating uniformly in the xy plane

(vii) If $\psi(x, t)$ be a normalized wave function, it must have

(a) $\int_{-\infty}^{+\infty} \psi \psi \, dx = 1$ (b) $\int_{-\infty}^{+\infty} \psi^* \psi \, dx = 1$ (c) $\int_{-\infty}^{+\infty} \psi^* \psi \, dx = 0$ (d) $\int_{-\infty}^{+\infty} \psi^* \psi \, dx = \frac{1}{2}$

(viii) For $T > OK$, The probability of occupancy of an electron at Fermi energy level is

(a) 1 (b) 0 (c) $\frac{1}{2}$ (d) 2

(ix) The one-dimensional time-independent Schrödinger wave equation for a free particle is given by

(a) $\frac{d^2\psi(x)}{dx^2} + \frac{2m}{h^2} E\psi(x) = 0$ (b) $\frac{d^2\psi}{dx^2} + \frac{2m}{h^2} [E - V]\ \psi(x) = 0$

(c) $ih \frac{\partial\psi}{\partial t} = -\frac{h^2}{2m}\frac{\partial^2\psi}{\partial x^2}$ (d) $\frac{d^2\psi}{dx^2} + \frac{2mh}{E}\psi(x) = 0$

(x) If the vectors $\overline{A}$ and $\overline{B}$ are conservative, then

(a) $\overline{A} \times \overline{B}$ is solenoidal (b) $\overline{A} \times \overline{B}$ is conservative

(c) $\overline{A} + \overline{B}$ is solenoidal (d) $\overline{A} - \overline{B}$ is solenoidal

(xi) The value of $\oint_c d\vec{l}$ along a circle of radius 2 unites is

(a) zero (b) 2π (c) 4π (d) π

(xii) Which one of the following operators is associated with energy?

(a) $-\frac{h^2}{2m}\nabla^2 + V$ (b) $-\frac{h}{2m}\nabla^2$ (c) $\frac{h}{i}\overline{\nabla}$ (d) $ih\frac{\partial}{\partial t}$

Group B

Short Answer Questions

Answer any three of the following questions:

2. (a) Sketch the Fermi distribution function for $T = 0$ and $T > OK$.
   (b) Assume that in tungsten (at. wt. = 183.8, density = 19.3 g/cc) There are two free electrons per atom. Calculate the Fermi energy and electron density.
   [9.2 eV, 1.264×10^{23} g/cc]
3. Which of the following functions are eigen functions of the operator $\frac{d^2}{dx^2}$? What are the eigen values?
   (a) $A \sin kx$ (b) $A \ln x$ (c) Be^{ikx}
4. What do you mean by a scalar field? If $\phi(x, y, z) = 3x^2y - y^3z^2$, then find $\overline{\nabla}\phi$ at the point (1, –2, 1).
5. (a) What is an electric line of force? What is its importance?
   (b) Sketch the electric lines of force due to point charger (i) $q > 0$ and (ii) $q < 0$.
6. (a) What is electric field intensity?
   (b) Prove that newtons per coulomb is dimensionally same as volt per meter.
7. State Ampere's law in magnetostatics in integral form and from that deduce its differential form.

Group C

Long Answer Questions

Answer any three questions from the following:

8. (a) State Gauss' law in electrostatics and hence obtain its differential form.
   (b) What is scalar field? Give example.
   (c) Show that the vector field

   $\overline{V} = -\left(\dfrac{\hat{i}x + \hat{j}y}{\sqrt{x^2 + y^2}}\right)$ is a sink field.

   (d) Three charges: 1 μc, 1 μc and -4 μc are placed at the three corners of an equilateral triangle with side 10 cm. Calculate electric potential energy of the system.
9. (a) State Biot–Savart law. Calculate the magnetic field at a point on the axis of a circular coil carrying current by using Biot–Savart law.
   (b) What is motional emf? Write down its expression.
   (c) Write down Maxwell's equation for vacuum and derive wave equation for electric field in vacuum.
10. (a) Explain the terms surface integral and volume integral.
    (b) Name the theorem which relates surface and volume integrals of a system. Give its mathematical expression in terms of unit normal vector.
    (c) If $\overline{H} = \overline{\nabla} \times \overline{A}$, then prove that

    $\iint_s \overline{H} \cdot d\overline{s} = 0$ for any closed surface S.
11. (a) Define (i) phase space and (ii) thermodynamical probability.
    (b) Give an expression of BE statistics and hence obtain Planck's formula for black-body radiation.
    (c) Determine the wavelength λm corresponding to the maximum emissivity of a black body at temperature T equal to 300 K.

Model Question Paper 2

Group A

Multiple Choice Questions

1. (i) The value of $\overline{\nabla} \cdot \bar{r}$ is

(a) 1 (b) 0 (c) 2 (d) 3

(ii) If $\overline{A}$ and $\overline{B}$ are irrotational, then $\overline{A} \times \overline{B}$ is

(a) solenoidal (b) irrotational (c) rotational (d) None of these

(iii) In free space, Poisson's equation is

(a) $\nabla^2 V = -\dfrac{\rho}{\epsilon_0}$ (b) $\nabla^2 V = \dfrac{\rho}{\epsilon_0}$ (c) $\nabla^2 V = 0$ (d) $\nabla^2 V = \rho\epsilon_0$

(iv) The magnetic field at the origin due to a current element $i\,d\bar{l}$ placed at a position $\bar{r}$ is

(a) $\dfrac{\mu_0}{4\pi}\dfrac{i d\bar{l} \times \bar{r}}{r^2}$ (b) $\dfrac{\mu_0}{4\pi}\dfrac{i d\bar{l} \times \bar{r}}{r^3}$ (c) $\dfrac{\mu_0}{4\pi}\dfrac{i d\bar{l} \times \hat{r}}{r^3}$ (d) zero

(v) Displacement current arises due to

(a) positive charge only

(b) negative charge only

(c) time-varying electric field

(d) None of these

(vi) If $\overline{\nabla} \cdot \overline{B} = 0$, then

(a) $\overline{B} = \overline{\nabla} \cdot \overline{A}$ (b) $\overline{B} = \overline{\nabla} \times \overline{A}$ (c) $\overline{B} = \nabla^2 \overline{A}$ (d) None of these

(vii) In free space, modified Ampere's circuital law

(a) $\overline{\nabla} \times \overline{E} = 0$ (b) $\overline{\nabla} \times \overline{B} = \mu_0 \epsilon_0 \dfrac{\partial \overline{E}}{\partial t}$

(c) $\overline{\nabla} \cdot \overline{E} = 0$ (d) $\overline{\nabla} \cdot \overline{B} = 0$

(viii) The angle between $\overline{\nabla}\phi$ and the normal to the surface $\phi =$ constant is

(a) $\dfrac{\pi}{2}$ (b) 0 (c) π (d) $\dfrac{\pi}{4}$

(ix) Condition of coplanarity of three vectors $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$ is

(a) $\alpha \cdot (\bar{\beta} + \bar{\gamma}) = 0$ (b) $\bar{\alpha} \cdot (\bar{\beta} + \bar{\gamma}) = 0$

(c) $\bar{\alpha} \times (\bar{\beta} + \bar{\gamma})$ (d) $\bar{\alpha} \cdot (\bar{\beta} - \bar{\gamma}) = 0$

(x) The magnitude of electric field $\overline{E}$ in the annular region of a charged cylindrical capacitor

(a) is same anywhere (b) varies as $\frac{1}{r}$ (c) varies as $\frac{1}{r^2}$ (d) None of these

(xi) For a particle trapped in a box of length l, the value of the expected average is

(a) $\frac{1}{l}$ (b) $\frac{2}{l}$ (c) $\frac{l}{2}$ (d) None of these

(xii) Which one of the following is not an acceptable wave function of a quantum particle?

(a) $\psi = e^{x}$ (b) $\psi = e^{-x}$ (c) $\psi = x^{n}$ (d) $\psi = \sin x$

(xiii) A non-polar molecule is the one in which the center of gravity of positive and negative charges

(a) coincides (b) gets separated by 10^{-8} m

(c) gets separated by 1 Å (d) None of these

GROUP B

Short Answer Questions

Answer any three of the following questions:

2. Show that the wave equation in free space of electric field $\overline{E}$ is given by

$$\nabla^2 \overline{E} = \mu_0 \epsilon_0 \frac{\partial^2 \overline{E}}{\partial^2 t}.$$

3. Distribute two particles in three different states according to (i) *MB* statistics, (ii) *BE* statistics, and (iii) *FD* statistics.

4. The wave function of a particle in a one-dimensional box of length L is given by

$$\phi = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

Find the expectation value of x and x^2.

$$\left[\frac{L}{2}, \frac{L^2}{3} - \frac{L^2}{2\pi^2 n^2}\right]$$

5. State Stokes theorem. Find the vector perpendicular to $x^2 + y^2 - z^2 = 100$ at the point (1, 2, 3).

6. Derive the differential form of the Gauss' law from its integral form.

GROUP C

Long Answer Questions

Answer any three questions from the following:

7. (a) Define electric flux.

(b) A spherical gaussian surface encloses a charge 8.55×10^{-8} C. Calculate the electric flux passing through the surface.

(c) State Ampere's circuital law.

(d) In case of a long solenoid, prove that the intensity of the magnetic field at a point inside the solenoid is double of that at the end.

(e) What is Lorentz force?

8. (a) Two straight conductors, each 10 cm long, are parallel to each other and separated by a distance of 2 cm. They carry current of 30 A and 40 A, respectively. Calculate the force on conductor.

(b) What are scalar and vector potentials?

(c) If the vector potential $\overline{A} = (x^2 + y^2 + z^2)\,\hat{j}$ at the position (x, y, z), then find the magnetic field at (1, 1, 1).

(d) Write down Poisson's and Laplace's equations.

(e) Show that $\overline{\nabla} \cdot \overline{B} = 0$

9. (a) What is a dielectric? What is dielectric constant?

(b) Discuss different polarisation mechanisms in dielectrics.

(c) Define and explain the three electric vectors $\overline{P}$, $\overline{E}$ and $\overline{D}$. Why electric field inside a dielectric decreases due to polarization? Show that $\overline{D} = \epsilon_0 \overline{E} + \overline{P}$. Also give their units.

10. (a) State a comparative study among *MB*, *BE* and *FD* statistics.

(b) Define macrostate and microstate with suitable example.

(c) Draw the Fermi distribution function for $T = OK$ and $T > OK$.

(d) Calculate the total member of particles in a Fermionic gas in terms of the Fermi level at absolute zero.

(e) Indicate which statistics will be applicable for

(i) e^- (ii) photon (iii) ${}_2He^4$

11. (a) Write down Schrödinger's equation for one-dimensional motion of a free particle in one-dimensional potential box. Find its eigen function and eigen energy.

(b) Show that $\psi = Ax + B$ (where *A* and *B* are constants) is a solution to the Schrödinger equation for $E = 0$ energy of a particle in a box.

(c) Show that $\psi(x, t) = \psi(x)\,e^{-i\omega t}$ is a wave function of a stationary state.

(d) Show that $[x, p_x] = i\hbar$.

Model Question Paper 3

GROUP A

Multiple Choice Questions

1. (i) A vector field is said to be sodenoidal only when
   (a) its divergence is zero (b) its curl is zero
   (c) both of its curl and divergence are zero (d) None of these

   (ii) The angle between $\hat{i}$ and $2\hat{i} + \hat{j}$ is
   (a) $\cos^{-1}\frac{2}{5}$ (b) $\cos^{-1}\frac{1}{2}$ (c) $\cos^{-1}\frac{\sqrt{2}}{5}$ (d) $\cos^{-1}\frac{1}{3}$

   (iii) For incompressible fluid of velocity $\overline{v}$ satisfies
   (a) $\overline{\nabla} \times \overline{v} = 0$ (b) $\overline{\nabla} \cdot (\overline{\nabla} \times \overline{v}) = 0$ (c) $\overline{\nabla} \cdot \overline{v} = 0$ (d) $\overline{\nabla} \times (\overline{\nabla} \times \overline{v}) = 0$

   (iv) In a gaussian surface, the electric field intensity at energy point on the surface is
   (a) constant (b) not constant (c) zero (d) None of these

   (v) Flux of the electric field for a point charge $(+q)$ at the origin through a spherical surface centered at the same origin is
   (a) $\frac{q}{\epsilon_0}$ (b) $\frac{q}{4\pi\epsilon_0}$ (c) $\frac{2q}{\epsilon_0}$ (d) $\frac{4q}{\epsilon_0}$

   (vi) The direction of propagation of electromagnetic wave is given by
   (a) $\overline{\nabla} \cdot \overline{B} = 0$ (b) $\overline{\nabla} \times \overline{B} = \mu_0 \overline{J}$ (c) $\overline{\nabla} \times \overline{B} = \overline{J}$ (d) $\overline{\nabla} \cdot \overline{B} = \mu_0 \overline{J}$

   (vii) The direction of propagation of electromagnetic wave is given by
   (a) along the direction of $\overline{E}$ (b) along $\overline{B}$
   (c) along $\overline{E} \times \overline{B}$ (d) None of these

   (viii) The relation between scalar potential ϕ and vector potential $\overline{A}$ is
   (a) $\overline{\nabla} \times \overline{A} = \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t}$ (b) $\overline{\nabla} \times \overline{A} = -\mu_0 \epsilon_0 \frac{\partial \phi}{\partial t}$
   (c) $\overline{\nabla} \cdot \overline{A} = \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t}$ (d) $\overline{\nabla} \cdot \overline{A} = -\mu_0 \epsilon_0 \frac{\partial \phi}{\partial t}$

(ix) The *FD* distribution function is given by

(a) $f(\in) = \dfrac{1}{e^{(\in - \in_F)/(K_B T)} + 1}$ (b) $f(\in) = \dfrac{1}{e^{(\in - \in_F)/(K_B T)} - 1}$

(c) $f(\in) = \dfrac{1}{e^{(\in - \in_F)/(K_B T)}}$ (d) None of these

(x) Choose the function which can be a physically acceptable wave function:

(a) ax (b) $e^{-a^2x^2}$ (c) $\sqrt{x}$ (d) e^{a^2x}

(xi) Which one of the following is a Fermion?

(a) Photon (b) Electron (c) Phonon (d) Alpha-particle

(xii) The normalized wave function for a particle moving in a one-dimensional potential box of length L is

(a) $\psi = \sqrt{\frac{2}{L}} \sin \frac{n\pi}{xL}$ (b) $\psi = \sqrt{\frac{L}{2}} \sin \frac{n\pi x}{L}$ (c) $\psi = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$ (d) $\psi = \sqrt{\frac{2}{L}} \sin \frac{\pi x}{nL}$

(xiii) The net charge inside a dielectric before and after polarization remains

(a) negative (b) positive (c) same (d) None of these

GROUP B

Short Answer Questions

Answer any three of the following:

2. Find the displacement current within a parallel-plate capacitor in series with a resistor which carries current I. Area of the capacitor plates is A and dielectric is vacuum.
3. Show that $\vec{F} = (2xy + z^3)\,\hat{i} + x^2\hat{j} + 3zx^2\hat{k}$ is a conservative force field. Hence obtain its scalar potential.
4. A particle is located in a two-dimensional square potential well with absolutely impenetrable walls ($0 < x < L$, $0 < y < L$). Find the probability of finding the particle within a region $0 < x < \frac{L}{3}$, $0 < y < \frac{L}{3}$.
5. What is a dielectric substance? Discuss the importance of dielectrics.
6. Distribute two particles in three different states according to (a) *MB* statistics (b) *BE* statistics, and (iii) *FD* statistics.

GROUP C

Long Answer Questions

Answer any three questions.

7. (a) Evaluate the expectation value of x for a one-dimensional potential box of length L in the ground state.

(b) State the basic postulates of quantum mechanics.

(c) Write down the Schrödinger wave equation for a particle of mass m in a rectangular box with infinitely rigid walls with edges a, b and c. What are the boundary conditions? What conditions are responsible for the quantization for energy of the particle?

(d) Prove that the lowest state of a free particle in a cubical box is not degenerate.

(e) Compute the lowest energy of a neutron confined in the nucleus where the nucleus in considered as a box with a size of 10^{-14} m ($h = 6.2 \times 10^{-34}$ Js, $m_n = 1.6 \times 10^{-24}$ kg).

8. (a) State Gauss' law in [6.43 MeV] electrostatics and extend the same to Poisson's equations.

(b) A spherical symmetric charge distribution is given by

$$\rho(r) = \rho_0 \left(1 - \frac{r^2}{a^2}\right) \quad \text{for} \quad 0 \le r \le a$$

$$= 0 \qquad \text{for} \quad u > a$$

(c) Calculate (i) the total charge and (ii) the electric field intensity both inside and outside the charge distribution.

(d) What is vector potential? Verify that the vector potential $\overline{A}$ is given by

$$\overline{A} = \frac{1}{2}(\overline{r} \times \overline{B})$$

9. (a) State Ampere's circuital law. Show that the magnetic field inside a toroid having n member of turns per unit length and carrying a current I is $\mu_0 nI$.

(b) What is Lorentz force? Show that Lorentz force does not work on a charged particle.

(c) Show that $\overline{\nabla} \cdot \overline{J} + \frac{\partial \rho}{\partial t} = 0$, where the symbols have their usual meanings.

10. (a) Write down Maxwell's field equations, explaining the terms used.

(b) Prove that electromagnetic waves are transverse in nature.

(c) What is displacement current?

(d) Show that $\overline{E} = -\overline{\nabla}\phi - \frac{\partial \overline{A}}{\partial t}$.

11. (a) What are the three electric vectors in dielectrics? Name and find the relation between them.

(b) Explain why the introduction of a dielectric slab between the plates of a capacitor changes its capacitance?

(c) State and prove Gauss' Law in case of dielectrics.

(d) Deduce an expression for energy stored in dielectric in electrostatic field.

Model Question Paper 4

GROUP A

Multiple Choice Questions

1. (i) If $\hat{n}$ be the unit vector in the direction $\overline{A}$, then

(a) $\hat{n} = \dfrac{\overline{A}}{|\vec{A}|}$ (b) $\hat{n} = \overline{A}\,|\vec{A}|$ (c) $\hat{n} = \dfrac{|\vec{A}|}{\overline{A}}$ (d) None of these

(ii) $\overline{\nabla} \cdot \vec{r}$ is equal to

(a) 2 (b) 0 (c) 1 (d) 3

(iii) Flux of the electric field for a point charge (q) at origin through a spherical surface centered at the origin is

(a) $\dfrac{2q}{\epsilon_0}$ (b) $\dfrac{q}{\epsilon_0}$ (c) $\dfrac{q}{4\pi\epsilon_0}$ (d) 0

(iv) In free space, Poisson's equation is

(a) $\nabla^2 V = 0$ (b) $\nabla^2 V = \dfrac{\epsilon_0}{\rho}$ (c) $\nabla^2 V = \infty$ (d) None of these

(v) A moving charge produces

(a) electric field only (b) magnetic field only

(c) both of these (d) None of these

(vi) A current-carrying straight wire cannot move, but a current-carrying square loop adjacent to it can move under the influence of a magnetic force. The square loop will remain

(a) remain stationary (b) move towards the wire

(c) move away from the wire (d) None of these

(vii) The magnetic flux linked with a coil at any instant t is given by $\phi(t) = 5t^3 - 100t + 200$, the emf induced in the coil at $t = 2$ seconds is

(a) 200 V (b) 40 V (c) 20 V (d) –20 V

(viii) The differential form of Faraday's law of electromagnetic induction is

(a) $\overline{\nabla} \times \overline{E} = -\dfrac{\partial \overline{B}}{\partial t}$ (b) $\overline{\nabla} \times \overline{E} = \dfrac{\partial \overline{B}}{\partial t}$ (c) $\overline{\nabla} \times \overline{B} = \dfrac{\partial \overline{E}}{\partial t}$ (d) $\overline{\nabla} \times \overline{B} = -\dfrac{\partial \overline{E}}{\partial t}$

(ix) A non-polar molecule is the one in which the center of gravity of positive and negative charges

(a) coincide
(b) get separated by 10^{-8} m
(c) get separated by 1Å
(d) None of these

(x) The ground state energy of a particle moving in a one-dimensional potential box is given in terms of length l of the box by

(a) $\frac{2h^2}{8ml^2}$ (b) $\frac{h^2}{8ml^2}$ (c) $\frac{h}{8ml^2}$ (d) 0

(xi) The waves representing a free particle in three dimensions are
(a) standing waves
(b) progressive waves
(c) transverse waves
(d) polarized waves

(xii) The spin angular momentum of photon is

(a) h (b) $\frac{h}{8}$ (c) 0 (d) $2h$

Group B

Short Answer Questions

Answer any three questions:

2. State Stokes theorem. Find the vector perpendicular to $x^2 + y^2 - z^2 = 100$ at the point (1, 2, 3).
3. Derive Poisson's and Laplace's law from Gauss' law of electrostatics.
4. Show that the wave equation in free space of electric field $\overline{E}$ is given by $\nabla^2\overline{E} = \mu_0\epsilon_0 \frac{\partial^2 \overline{E}}{\partial t^2}$.
5. Show that the equation of continuity is given by $\overline{\nabla} \cdot \overline{J} + \frac{\partial \rho}{\partial t} = 0$, where $\overline{J}$ and ρ have their usual meaning.
6. The wave function of a particle in one-dimensional box of length L is given by $\phi_n = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$. Find the expectation value of x and x^2.

GROUP C

Long Answer Questions

Answer any three questions out of the following:

7. (a) Find if the work done in moving an object in the field

$$F = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$$

from the point (1, – 2, 1) to (3, 1, 4) is independent of the path chosen.

(b) If the vector potential $\overline{A} = (x^2 + y^2 - z^2)\hat{j}$ at position (x, y, z), find the magnetic field at (1, 1, 1).

(c) Find a unit vector perpendicular to $x^2 + y^2 - z^2 = 100$ at point (1, 2, 3).

(d) Find the magnetic induction $\overline{B}$, at a point on the axis of an infinitely long solenoid carrying current 1, number of turns per unit length being n.

8. (a) Write down Laplace's equation in spherical coordinate system and hence find the solution.
   (b) Show that the potential $V = V_0 (x^2 - 2y^2 + z^2)$ satisfies Laplace's function where V_0 in a constant.
   (c) Charge is distributed along the x axis from $x = 0$ to $x = L = 500$ cm in such a way that its linear charge density is given by $\lambda = ax^2$, where $a = 18.0\,\mu$ cm^{-3}. Calculate the total charge in the region $0 \le x \le 1$.

9. (a) Write down the Schrödinger's equation for a particle confined in a one-dimensional box.
   (b) If the wave function $\psi(x)$ of a quantum mechanical particle is given by

$$\psi(x) = a \sin \frac{\pi x}{L}, \quad \text{for} \quad 0 \le x \le L$$
$$= 0, \quad \text{for} \quad 0 \ge x \ge L$$

   Then determine the value of a. Also determine the value of x where the probability of finding the particle is maximum.
   (c) Show that $[\hat{P}x, \hat{x}_n] = -i\hbar\, nx^{n-1}$ and $[\hat{P}_x, f(x)] - i\hbar \frac{\partial f}{\partial x}$.
   (d) Show that $\frac{\partial}{\partial x}$ and $\frac{\partial^2}{\partial x^2}$ are commutative.

10. (a) What are three electric vectors in dielectrics? Name them and find a relation among them.
   (b) Explain the phenomenon of polarization of dielectric medium and show that $K = 1 + \chi_e$. Hence the symbols have their usual meaning.
   (c) Explain why the introduction of a dielectric slab between the plates of a capacitor changes its capacitance?
   (d) Determine the electrical susceptibility at 0°C for a gas whose dielectric constant at 0°C is 1.000041.

11. (a) Sketch the Fermi distribution for $T = OK$ and $T > OK$ and explain.
   (b) What is the occupation probability at $\in\, = \,\in_F$?
   (c) Calculate how the degeneracy function $g(\in)$ depends on $\in$ for a Fermionic gas.
   (d) Express the Fermi level in a metal in terms of free electron density.

Model Question Paper 5

GROUP A

Multiple Choice Questions

1. (i) Two vectors $\overline{A}$ and $\overline{B}$ are parallel where

(a) $\overline{A} \times \overline{B} = 0$ (b) $\overline{A} \cdot \overline{B} = 0$ (c) $\overline{A} \cdot \overline{B} = 1$ (d) None of these

(ii) The angle between $\overline{\nabla}\phi$ and surface of ϕ = constant is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) π (d) 0

(iii) The electric flux through each of the four faces of a cube of 1 m side if a charge q coulomb is placed at its center is

(a) $\frac{q}{4\epsilon_0}$ (b) $4\epsilon_0 q$ (c) $\frac{q}{6\epsilon_0}$ (d) $\frac{\epsilon_0}{6q}$

(iv) Laplace's equation for an electrostatic field is

(a) $\nabla^2 V = \frac{\rho}{\epsilon_0}$ (b) $\nabla^2 V = 0$ (c) $\nabla^2 V = -\frac{\rho}{\epsilon_0}$ (d) $\overline{\nabla} V = \frac{\rho}{\epsilon_0}\hat{r}$

(v) The equation of continuity in a steady charged distribution is

(a) $\overline{\nabla} \cdot \overline{J} = 0$ (b) $\overline{\nabla} \times \overline{J} = 0$ (c) $\overline{\nabla} \cdot \overline{J} = \rho$ (d) $\overline{\nabla} \cdot \overline{J} = \frac{\rho}{\epsilon_0}$

(vi) The work done by the Lorentz force $\overline{F}$ on a charge particle is

(a) $\overline{F} \cdot d\overline{r}$ (b) zero (c) $\frac{q}{\epsilon_0}$ (d) qF

(vii) Displacement current arises due to

(a) positive charge only (b) negative charge only
(c) time-varying electric field (d) None of these

(viii) The solution of a plane electromagnetic wave $\nabla^2 \overline{B} = \mu_0 \epsilon_0 \frac{\partial^2 \overline{B}}{\partial t^2}$ is

(a) $\overline{B} = \overline{B}_0 e^{i(wt - \overline{k} \cdot \overline{r})}$ (b) $\overline{B} = \overline{B}_0 e^{i(wt + \overline{k} \cdot \overline{r})}$ (c) $\overline{B} = \overline{B}_0 e^{i(\overline{k} \cdot \overline{r} - wt)}$ (d) None of these

(ix) The relation between three electric vectors $\overline{E}$, $\overline{D}$ and $\overline{P}$ is

(a) $\overline{D} = \epsilon_0 \overline{E} + \overline{P}$ (b) $\overline{D} = \epsilon_0 (\overline{E} + \overline{P})$ (c) $\overline{D} = \overline{E} + \epsilon_0 \overline{P}$ (d) $\overline{D} = \frac{1}{\epsilon_0} \overline{E} + \overline{P}$

(x) The spacing between the n^{th} energy state and the next energy state in a one-dimensional potential box increases by

(a) $(2n-1)$ (b) $(2n+1)$ (c) $(n-1)$ (d) $(n+1)$

(xi) The wave function $\psi_m(x)$ and $\psi_n(x)$ are orthogonal to each other. Which of the following relations must hold for them?

(a) $\int_{-\infty}^{+\infty} \psi_m^* \psi_n \, dx = 1$ (b) $\int_{-\infty}^{+\infty} \psi_m \psi_n \, dx = 0$ (c) $\int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx = 0$ (d) $\int_{-\infty}^{+\infty} \psi_m \psi_n \, dx = 1$

(xii) Average energy ϵ of an electron in a metal at $T = OK$ is

(a) E_F (b) $\frac{1}{2} \epsilon_F$ (c) $\frac{3}{5} \epsilon_F$ (d) $\frac{5}{3} \epsilon_F$

GROUP B

Short Answer Questions

Answer any three of the following questions:

2. Prove that $\int_V \overline{\nabla} \psi dV = \int_s \psi \hat{n} \, ds$.
3. What is an electrostatic field? Why is it called a conservative field?
4. What are polar and non-polar molecules? Discuss different polarization methods in dielectrics.
5. What is a wave function? Mention four points on its physical significance.
6. Sketch the Fermi distribution function for $T = OK$ and $T > OK$ and explain. What is the occupation probability at $\epsilon = \epsilon_F$?

GROUP C

Long Answer Questions

Answer any three questions from the following:

7. (a) The gradient of a scalar quantity is a vector quantity. Explain.

(b) Prove that $\nabla^2 f(r) = \frac{d^2 f(r)}{dr^2} + \frac{2}{r} \frac{df(r)}{dr}$.

(c) Find the work done to move an object along a vector $\bar{r} = 3\hat{i} + 2\hat{j} - 5\hat{k}$ if the applied force is $\overline{F} = 2\hat{i} - \hat{j} - \hat{k}$.

(d) For what value of 'a' is the vector $\overline{A} = (x+3y)\,\hat{i} + (y-2z)\,\hat{j} + (x+9z)\,\hat{k}$ solenoidal?

8. (a) The gradient of a scalar quantity is a vector quantity explain.

   (b) Show that $\nabla^2(\log_e^l r) = \frac{1}{r^2}$.

   (c) If a rigid body rotates about an axis passing through the origin with angular velocity $\overline{\omega}$ and with linear velocity $\vec{v} = \overline{\omega} \times \vec{r}$, then prove that $\overline{\omega} = \frac{1}{2}(\overline{\nabla} \times \vec{v})$.

   (d) Find the torque about the point 0(3, – 1, 3) of a force $\overline{F}(4, 2, 1)$ passing through the point $\overline{A}$ (5, 2, 4).

9. (a) Write down Schrödinger's equation for one-dimensional motion of a free particle in one dimensional potential box. Find the eigen function and eign energy.

   (b) What are the values of $\left[\hat{x}, \frac{\partial}{\partial x}\right]$, $[\hat{L}_x, \hat{x}]$ and $[\hat{P}_x, \hat{P}_y]$?

   (c) Prove that the wave function $\psi(x, t) = A\cos(kx - \omega t)$ does not satisfy equation for a free particle.

   (d) Calculate the normalization constant for a wave function (at $t = 0$) given by $\psi(x) = Ae^{-6^2x^2/2} \cdot e^{ikx}$.

10. (a) What is a dielectric? What are dielectric losses?

    (b) What are polar and non-polar molecules? Discuss the effect of electric field on polar dielectrics. What is meant by polarization of a dielectric?

    (c) State and prove Gauss' law in case of dielectrics.

    (d) Dielectric constant of a gas at NTP is 1.00074. Calculate dipole moment of each atom of the gas when it is held in an external field of 3×10^4 V/m.

11. (a) What are fermions and bosom? Given two examples of each.

    (b) Three distinguishable particle's, each of which can be in one of the ϵ, 2ϵ, 3ϵ energy states, have total energy 6ϵ. Find all possible distributions of particles is energy states. Find the number of microstate in each case.

    (c) Indicate which statistics will be applicable for

    (i) ${}_1H^1$ (ii) e^- (iii) π° (iv) photon

    (d) The Fermi energy for sodium at $T = OK$ is 3.1 eV. Find its value for aluminum given that the free electron density in aluminum is approximately 7 times that of sodium.

Solved WBUT Question Paper (2008)

Group A

Multiple Choice Questions

1. (i) If the quantum mechanical state of a particle is described by

$$\psi(x) = \psi_0 e^{ik(x-a)}, \quad \text{for} \quad |x| \le L$$
$$= 0 \qquad \text{otherwise}$$

Then

(a) The particle has a definite position but uncertainly in momentum.
(b) The particle has a definite momentum but uncertainly in position.
(c) The particle has uncertainties both in position and momentum.
(d) The particle has no uncertainties in either momentum or in position.

Ans. (c)

(ii) If $\psi(x, t)$ in a normalized wave function, we must have

(a) $\int \psi^* \psi \, dx = 1$, (b) $\int \psi^* \psi \, dx = 0$ (c) $\int \psi^* \psi \, dx = \frac{1}{2}$, (d) $\int \psi \psi \, dx = 1$

Ans. (a)

(iii) Which one of the following functions is an eigen function of the operator $\frac{d^2}{dx^2}$?

(a) x, (b) x^2, (c) e^{-x^2}, (d) $\cos x$

Ans. (d)

(iv) The member of possible arrangement of two Fermions is 3 cells is

(a) 9, (b) 6, (c) 3, (d) 1

Ans. (c)

(v) If n_i be the number of identical and indistinguishable particles in the i^{th} energy state with degeneracy g_i, then classical statistics can be applied if

(a) $\frac{n_i}{g_i} = 1$, (b) $\frac{n_i}{g_i} << 1$, (c) $\frac{n_i}{g_i} >> 1$ (d) $g_i = 0$

Ans. (b)

(vi) For arbitrary scalar and vector fields ϕ and $\overline{A}$ which of the following is always correct?

(a) $\overline{\nabla} \times (\overline{\nabla} \times \overline{A}) = 0$ and $\overline{\nabla} \times (\overline{\nabla}\phi) = 0$

(b) $\overline{\nabla} (\overline{\nabla} \cdot \overline{A}) = 0$ and $\overline{\nabla} \cdot (\overline{\nabla}\phi) = 0$

(c) $\overline{\nabla} \cdot (\overline{\nabla} \times \overline{A}) = 0$ and $\overline{\nabla} \times (\overline{\nabla}\phi) = 0$

(d) $\overline{\nabla} \cdot (\overline{\nabla} \times \overline{A}) = 0$ and $\overline{\nabla} \cdot (\overline{\nabla}\phi) = 0$

Ans. (c)

(vii) When the magnitude of $\overline{A}$ is constant which one of the following is true?

(a) $\frac{d\overline{A}}{dt} = 0$, (b) $\overline{A} \cdot \frac{d\overline{A}}{dt} = 0$ (c) $\overline{A} \times \frac{d\overline{A}}{dt} = 0$, (d) $\left|\frac{d\overline{A}}{dt}\right| = 0$

Ans. (b)

(viii) In an electromagnetic wave in free space, the electric and magnetic fields are

(a) parallel to each other (b) perpendicular to each other

(c) inclined at acute angle (d) inclined at obtuse angle

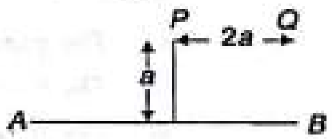
Ans. (b)

(ix) In the following figure, AB is an infinite line of charge distribution. P and Q are points as shown below. The ratio of electric field at P and Q is

(a) 1:1, (b) 1:2,

(c) 2:1, (d) 1:4

Ans. (a)



(x) The direction of magnetic induction due to a straight infinitely long current carrying wire is

(a) perpendicular to the wire (b) parallel to the wire

(c) at an inclination of 30° to the wire (d) None of these

Ans. (a)

(xi) Which of the following statements is not correct regarding electrostatic field vector, $\overline{E}$?

(a) $\oint_c \overline{E} \cdot d\overline{r} = 0$, where c is a simple closed curve.

(b) $\int_a^b \overline{E} \cdot d\overline{r}$ is independent of the path for given end points a and b.

(c) $\overline{E} = \overline{\nabla} \times \overline{A}$ for some vector potential $\overline{A}$.

(d) $\overline{E} = \overline{\nabla}\phi$ for some scalar field ϕ.

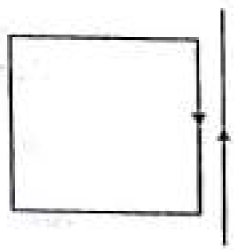
Ans. (c)

(xii) A current-carrying straight wire cannot move, but a current-carrying square loop adjacent to it can move under the influence of magnetic force. The square loop will

(a) remain stationary (b) move towards the wire

(c) move away from the wire (d) None of these

Ans. (c)



Group B

Short Answer Questions

1. Given $\phi = \phi(r)$ and $\overline{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$, find $\nabla^2\phi = \overline{\nabla}\cdot(\overline{\nabla}\phi)$ without using directly the explicit form of ∇^2 in the spherical polar coordinates.

Ans. Refer to Example 1.43.

2. If ϕ is the electrostatic potential in a region V, free of charge, then show that

$$\int_v |\overline{\nabla}\phi|^2 dv = \oint_\Sigma \phi\overline{\nabla}\phi \cdot d\bar{s}.$$

where Σ is the surface enclosing V. Hence show that if either the potential or the electric field vanishes identically on the surface Σ, the potential has to be constant inside V.

Ans. Here $\rho = 0$, so $\nabla^2\phi = 0$

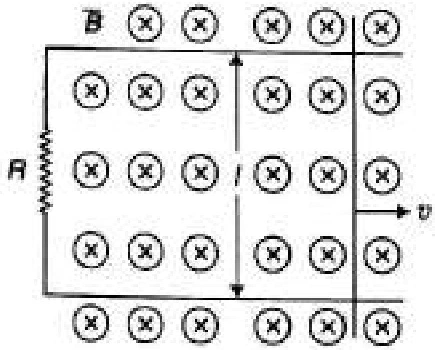
Now applying divergence theorem, on the right-hand side, we get

$$\begin{aligned}\text{RHS} &= \oint_\Sigma \phi\,\nabla\phi\cdot d\bar{s}\\ &= \int_V [\overline{\nabla}\cdot(\phi\overline{\nabla}\phi)]\,dV\\ &= \int_V [(\nabla\phi\cdot\nabla\phi) + \phi\nabla^2\phi]\,dV\\ &= \int_V |\nabla\phi|^2\,dV + \int_v \nabla^2\phi\,dV\end{aligned}$$

But $\qquad \nabla^2\phi = 0$

$$\therefore \quad \oint_\Sigma \phi\overline{\nabla}\phi\cdot d\bar{s} = \int_V |\overline{\nabla}\phi|^2\,dV \text{ (Proved)}$$

3. A metal bar slides without friction on two parallel conducting rails at distance l apart. A resistor R is connected across the rails and a uniform magnetic field B, pointing into this plane fills the entire region. If the bar moves to the right with a constant speed v, then what is the current in the resistor?



Ans. The rate of change of flux, $\frac{d\phi}{dt} = Blv$

and the induced *emf*, $e = \frac{d\phi}{dt} = Blv$

Again, the induced current is given by

$$i = \frac{e}{R} = \frac{Blv}{R}$$

4. A system has non-degenerate single-particle states with 0, 1, 2, 3 energy units. Three particles are to be distributed in these states such that the total energy of the system is 3 units. Find the number of microstates if particles obey

(i) *MB* statistics (ii) *BE* statistics (iii) *FD* statistics.

Ans. Refer to Example 8.3.

Group C

Long Answer Questions

1. (a) A particle of mass m is confined within $x = 0$ to $x = L$.

(i) Write down the Schrödinger equations for such a system.

Ans. Refer to Example 7.23.

(ii) Solve the equation to find out the normalized eigen functions.

Ans. Refer to Example 7.23.

(iii) Show that the eigen functions corresponding to two different eigen values are orthogonal.

Ans. Refer to Example 7.23.

(iv) If p denotes momentum then find $\langle p \rangle$ as well as $\langle p^2 \rangle$ in the ground state.

Ans. Refer to Example 7.23.

(b) Using the principle of operator correspondence write down the operators $\hat{L}_x, \hat{L}_y, \hat{L}_z$ (components of $\hat{L}$) in terms of position and momentum operators. Then show that $[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$

Ans. Refer to Example 7.12.

2. (a) Give two examples of Bosons.

Ans. Photon and atom of $_2He^4$.

(b) State Gauss' law in electrostatics and hence obtain Poisson's equation.

Ans. Refer to Section 2.13.

(c) Derive Coulomb's law from Gauss' law.

Ans. Refer to Section 2.12.2.

(d) The potential in a medium is given by $\varphi(r) = \dfrac{qe^{-r/\lambda}}{4\pi\epsilon_0 r}$.

Obtain corresponding electric field.

Ans. We know that for a function,

$$f = f(r, \theta, \phi)$$

$$\vec{\nabla} f = \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

But in the present case the given function $\phi = \phi(r)$. So it is independent of θ and ϕ.

Given $$\phi(r) = \frac{qe^{-r/\lambda}}{4\pi\epsilon_0 r}$$

$$\therefore \quad \overline{\nabla}\phi(r) = \hat{r}\frac{\partial\phi}{\partial r} + \hat{\theta}(0) + \hat{\phi}(0) = \hat{r}\frac{\partial\phi}{\partial r}$$

$$\therefore \quad \overline{E} = -\nabla\phi(r) = -\hat{r}\left[\frac{q}{4\pi\epsilon_0 r}\frac{d}{dr}(e^{-r/\lambda}) + \frac{qe^{-r/\lambda}}{4\pi\epsilon_0}\frac{d}{dr}\left(\frac{1}{r}\right)\right]$$

or,
$$\overline{E} = -\hat{r}\left[\frac{qe^{-r/\lambda}}{4\pi\epsilon_0 r\lambda} - \frac{qe^{-r/\lambda}}{4\pi\epsilon_0 r^2}\right]$$

or,
$$\overline{E} = \frac{\hat{r}e^{-r/\lambda}}{4\pi\epsilon_0 r}\left(\frac{1}{\lambda} + \frac{1}{r}\right)$$

$$\therefore \quad E(r) = \frac{e^{-r/\lambda}}{4\pi\epsilon_0 r}\left(\frac{1}{\lambda} + \frac{1}{r}\right)q$$

(e) Find the charge density that may produce the potential mentioned above.

Ans. Refer to Example 2.26.

3. (a) Write down the energy eigen function of a particle in a three-dimensional box of length L(do not derive it). Hence find out the energy as well as degeneracy in the ground and first excited state.

Ans. Refer to Section 7.8.1 (b).

(b) Sketch the Fermi distribution for $T = 0$ and $T > 0$ and explain.

Ans. Refer to Section 8.13.

(c) What is the occupation probability at $E = F_E$?

Ans. Refer to Section 8.13.

(d) Calculate how the degeneracy functions $g(\epsilon)$ depends on E for a Fermionic gas.

Ans. Refer to Section 8.12.

(e) Express the Fermi level in a metal in terms of free electron density.

Ans. Refer to Section 8.13.3.

4. (a) State Biot–Savart law in magnetostatics. Find the magnetic field of an infinitely long straight wire at a transverse distance of d from the expression of $\overline{B}$ found in Biot–Savart law.

Ans. Refer to Sections 4.7 and 4.8 (i).

(b) Express Biot–Savart law in terms of current density and hence show that the magnetic field is solenoidal.

Ans. Refer to Section 4.11.

(c) Express Ampere's circuital law in terms of vector potential.

Ans. We know that $\overline{\nabla}\cdot\overline{B} = 0$ Ampere's circuital law in differential form is given by

$$\overline{\nabla}\times\overline{B} = \mu_0\overline{J}$$

Again
$$\overline{B} = \overline{\nabla}\times\overline{A}$$

or,
$$\overline{\nabla}\times\overline{B} = \overline{\nabla}\times\overline{\nabla}\times\overline{A}2$$

$$\therefore \quad \overline{\nabla}\times\overline{\nabla}\times\overline{A} = \mu_0\overline{J} \qquad [\because \overline{\nabla}\times\overline{B} = \mu_0\overline{J}]$$

or,
$$\overline{\nabla}(\overline{\nabla}\cdot\overline{A}) - \nabla^2\overline{A} = \mu_0\overline{J}$$

But $\qquad \nabla \cdot \overline{A} = 0$

So. $\qquad \nabla^2 \overline{A} = -\mu_0 \overline{J}$

This is the Ampere's circuital law in terms of vector potential $\overline{A}$.

5. (a) State Maxwell's equations. From these equations derive the wave equations for an electromagnetic wave. What is the velocity of this wave?

Ans. Refer to Sections 5.9.1 and 5.10.

(b) Assuming a plane wave solution, establish the relation between the propagation vector $(\overline{k})$, electric field $(\overline{E})$ and magnetic field $(\overline{B})$

Ans. Refer to Section 5.11

6. Given $\overline{F} = f(r)\,\overline{r}$, show that $\overline{\nabla} \times \overline{F} = 0$ and hence show that

$$\oint_c \overline{F} \cdot d\overline{r} = 0, \text{ where } C \text{ is a simple closed curve.}$$

Ans. Given $\overline{F} = \overline{r} f(r)$

Now, $\qquad \overline{\nabla} \times \overline{F} = \overline{\nabla} \times [\hat{i}x + \hat{j}y + \hat{k}z] f(r)$

or, $$\overline{\nabla} \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix}$$

or, $$\overline{\nabla} \times \overline{F} = \hat{i}\left[z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}\right] + \hat{j}\left[x\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial x}\right] + \hat{k}\left[y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\right]$$

or, $$\overline{\nabla} \times \overline{E} = \hat{i}\left[z\frac{y}{r}\frac{\partial f}{\partial r} - y\frac{z}{r}\frac{\partial f}{\partial r}\right] + \hat{j}\left[x\frac{z}{r}\frac{\partial f}{\partial r} - z\frac{x}{r}\frac{\partial f}{\partial r}\right] + \hat{k}\left[y\frac{x}{r}\frac{\partial f}{\partial r} - x\frac{y}{r}\frac{\partial f}{\partial r}\right]$$

or, $\qquad \overline{\nabla} \times \overline{F} = 0$

$\left[\text{Here } \dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial r}\dfrac{\partial r}{\partial x} = \dfrac{x}{r}\dfrac{\partial f}{\partial r} \text{ and } r^2 = x^2 + y^2 + z^2\right.$

So. $\qquad 2r\dfrac{\partial r}{\partial x} = 2x$

or, $\qquad r\dfrac{\partial r}{\partial x} = x$

Similarly $\qquad \dfrac{\partial f}{\partial y} = \dfrac{y}{r}\dfrac{\partial f}{\partial r}$

and $\qquad \left.\dfrac{\partial f}{\partial z} = \dfrac{z}{r}\dfrac{\partial f}{\partial r}\right]$

For $\oint_c \overline{F} \cdot d\overline{r}$, by applying Stokes theorem, we get

$$\oint_c \overline{F} \cdot d\overline{r} = \iint_s (\overline{\nabla} \times \overline{F}) \cdot \hat{n}\,ds. \text{ But } \overline{\nabla} \times \overline{F} = 0$$

$\therefore \qquad \oint_c \overline{F} \cdot \overline{dr} = 0$

Solved WBUT Question Paper (2008)

(Answers to Questions on Classical Mechanics)

Group A

Multiple Choice Questions

1. Choose the correct alternative for the following:

(i) A system consists of three point masses. If the mutual distance between the point masses is fixed with time then the degree of freedom of such a system is

(a) zero (b) three (c) nine (d) six

Ans. (d)

(ii) The lagrangian of a system is given in terms of generalized coordinates r, θ, φ and associated velocities as

$$L = \frac{1}{2}\dot{\theta}^2 + \frac{1}{2}r^2\dot{\phi}^2 + \frac{1}{2}\alpha\,\theta^2$$

The cyclic coordinate/s/is/are

(a) θ (b) θ and ϕ (c) r and θ (d) ϕ

Ans. (d)

(iii) The lagrangian of a harmonic oscillator is given by $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k\dot{x}^2$.

The hamiltonian is given by

(a) $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ (b) $\frac{p^2}{2m} + \frac{1}{2}kx^2$, where $p = \frac{\partial L}{\partial \dot{x}}$

(c) $\frac{1}{2}m\dot{x}^2$ (d) $\frac{1}{2}kx^2$

Ans. (a and b)

Group B

Short Answer Questions

1. A particle is executing one-dimensional Simple Harmonic Motion under the action of a potential $V = \frac{1}{2}kx^2$. Write down the Lagrangian. Derive the hamiltonian and Hamilton's equations. (1 + 2 + 2)

Ans. Refer to Solution of Semester-1 of 2008.

Ans. As the particle is executing simple harmonic motion, we can express its displacement as

$$x = A \sin(\omega t - \varphi)$$

$\therefore$ the velocity, $v = \dot{x} = A\omega \cos(\omega t - \varphi)$

Its kinetic energy is given by

$$T = \frac{1}{2} mv^2$$

and its potential energy is given by

$$V = \frac{1}{2} kx^2$$

$\therefore$ the lagrangian is given by

$$L = T - V = \frac{1}{2} mv^2 - \frac{1}{2} kx^2$$

The momentum, $p_x = mv = m\dot{x}$

Its hamiltonian H is given by

$$H = p_x \dot{x} - L \quad \left[\because H = \sum_j p_j \dot{q}_j - L\right]$$

or, $$H = m\dot{x}^2 - \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2$$

or, $$H = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2 \quad ...(1)$$

The hamiltonian canonical equations are:

$$\dot{p}_x = \frac{\partial H}{\partial x} \quad ...(2)$$

and $$\dot{x} = -\frac{\partial H}{\partial p_x} \quad ...(3)$$

From Eqs (1) and (2), we get

$$m\ddot{x} = -kx$$

or, $$\ddot{x} + \frac{k}{m}x = 0 \quad ...(4)$$

From Eqs (1) and (3), we get

$$\dot{x} = \frac{1}{2}m.\, 2\, m\dot{x} = \dot{x}$$

or, $$\dot{x} = \dot{x} \quad ...(5)$$

Equation (5) is an identity, so Eq. (4) is the required equation.

Group C

Long Answer Questions

1.(a) What are the advantages of lagrangian formulation over newtonian formulation?

[Refer to Article 8.1.1]

(b) A particle of mass m moves in a force field of potential V. The kinetic energy and the potential energy of the system is

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \dot{\phi}^2 \sin^2 \theta)$$

$$V = V(r, \theta, \phi)$$

Write the equation of motion using the lagrangian method and the hamiltonian method and compare them. [Refer to Solution of Semester-1 of 2008] [3 + (5 + 7)]

Ans. (b) Given,

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \dot{\phi}^2 \sin^2 \theta)$$

and $$V = V(r, \theta, \varphi)$$

$\therefore$ the Lagrangian is given by

$$L = T - V$$

or, $$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \dot{\phi}^2 \sin^2 \theta) - V(r, \theta, \varphi)$$

$\therefore$ $$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 + mr\dot{\phi}^2 \sin^2 \theta - \frac{\partial V}{\partial r},$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r},$$

$$\frac{\partial L}{\partial \theta} = mr^2 \dot{\phi}^2 \sin \theta \cos \theta - \frac{\partial V}{\partial \theta},$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2 \dot{\theta},$$

$$\frac{\partial L}{\partial \phi} = -\frac{\partial V}{\partial r},$$

and $$\frac{\partial L}{\partial \dot{\phi}} = m\dot{\phi}\, r^2 \sin^2 \theta$$

In general, Lagrange's equations are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \quad \text{for } j = 1, 2, \ldots, n$$

So, in the present case, Lagrange's equations are given by

(i) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

or, $$\frac{d}{dt}(m\dot{r}) - mr\,\dot{\theta}^2 - mr\,\dot{\phi}^2\sin^2\theta + \frac{\partial V}{\partial r} = 0$$

or, $$m\ddot{r} - mr\,\dot{\theta}^2 - mr\,\dot{\phi}^2\sin^2\theta + \frac{\partial V}{\partial r} = 0 \qquad ...(1)$$

(ii) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

or, $$\frac{d}{dt}(mr^2\dot{\theta}) - (mr^2\dot{\phi}^2\sin\theta\cos\theta) + \frac{\partial V}{\partial \theta} = 0$$

or, $$2mr\,\dot{r}\,\dot{\theta} + mr^2\ddot{\theta} - mr^2\dot{\phi}^2\sin\theta\cos\theta + \frac{\partial V}{\partial \theta} = 0$$

or, $$mr^2\ddot{\theta} + 2\,mr\,\dot{r}\,\dot{\theta} - mr^2\,\dot{\phi}^2\sin\theta\cos\theta + \frac{\partial V}{\partial \theta} = 0 \qquad ...(2)$$

(iii) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \varphi} = 0$$

or, $$\frac{d}{dt}(m\dot{\phi}r^2\sin^2\theta) + \frac{\partial V}{\partial \varphi} = 0$$

or, $$m\ddot{\phi}r^2\sin^2\theta + 2\,m\dot{\phi}\,r\,\dot{r}\sin^2\theta + 2\,mr^2\,\dot{\phi}\sin\theta\cos\theta\dot{\theta} + \frac{\partial V}{\partial \varphi} = 0$$

or, $$(mr^2\sin^2\theta)\,\ddot{\phi} + 2\,mr\,\dot{r}\dot{\phi}\sin^2\theta + mr^2\,\dot{\phi}\,\dot{\theta}\sin 2\theta + \frac{\partial V}{\partial \varphi} = 0 \qquad ...(3)$$

The hamiltonian is given by

$$H = \sum_j p_j\,\dot{q}_j - L$$

The generalized momenta are given by

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

So, momenta are given by

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r},$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\,\dot{\theta}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = (mr^2\sin^2\theta)\,\dot{\phi}$$

So, the hamiltonian is given by

$$H = p_r\,\dot{r} + p_\theta\,\dot{\theta} + p_\varphi\,\dot{\phi} - \frac{1}{2}m(\dot{r}^2 + r^2\,\dot{\theta}^2 + r^2\,\dot{\phi}^2\sin^2\theta) + V$$

or, $$H = m\dot{r}^2 + mr^2\dot{\theta}^2 + mr^2\dot{\phi}^2\sin^2\theta - \frac{1}{2}m\dot{r}^2 - \frac{1}{2}mr^2\dot{\theta}^2 - \frac{1}{2}mr^2\dot{\phi}^2\sin^2\theta + V$$

or, $$H = \frac{1}{2}m\dot{r}^2 + mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\dot{\phi}^2\sin^2\theta + V$$

or, $$H = \frac{1}{2}m.(m\dot{r})^2 + \frac{1}{2mr^2}.(mr^2\dot{\theta})^2 + \frac{1}{2mr^2\sin^2\theta}(mr^2\dot{\phi}\sin^2\theta)^2 + V$$

or, $$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\varphi^2}{2mr^2\sin^2\theta} + V \qquad ...(4)$$

Hamilton's general equations of motion are given by

$$\left.\begin{aligned} \dot{q}_j &= \frac{\partial H}{\partial p_j} \\ \text{and} \quad \dot{p}_j &= -\frac{\partial H}{\partial q_j} \end{aligned}\right\} \qquad ...(5)$$

So, the Hamilton's equations of motion related to the generalized coordinates r, θ and φ are given by

$$\dot{r} = \frac{\partial H}{\partial p_r}$$

or, $$\dot{r} = \frac{p_r}{m} \qquad ...(6)$$

$$\dot{p}_r = -\frac{\partial H}{\partial r}$$

or, $$\dot{p}_r = \frac{p_\theta^2}{mr^3} + \frac{p_\varphi^2}{mr^3\sin^2\theta} - \frac{\partial V}{\partial r} \qquad ...(7)$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} \quad \text{or} \quad \dot{\theta} = \frac{p_\theta}{mr^2} \qquad ...(8)$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta}$$

or, $$\dot{p}_\theta = \frac{\partial}{\partial\theta}\left(\frac{p_\theta^2}{2mr^2\sin^2\theta}\right) - \frac{\partial V}{\partial\theta}$$

$$\dot{p}_\theta = \frac{p_\phi^2\cos\theta}{mr^2\sin^3\theta} - \frac{\partial V}{\partial\theta} \qquad ...(9)$$

or, $$\dot{\phi} = \frac{\partial H}{\partial p_\phi} \quad \text{or,} \quad \dot{\phi} = \frac{p_\phi}{mr^2\sin^2\theta} \qquad ...(10)$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial\phi} \quad \text{or,} \quad \dot{p}_\phi = -\frac{\partial V}{\partial\phi} \qquad ...(11)$$

Now, differentiating Eq. (6) with respect to t, we get

$$m\ddot{r} = \dot{p}_r$$

Replacing $\dot{p}_r$ in Eq. (7) by $m\ddot{r}$, we get

$$m\ddot{r} = \frac{p_\theta^2}{mr^3} + \frac{p_\varphi^2}{mr^3\sin^2\theta} - \frac{\partial V}{\partial r}$$

Now, putting the values of p_θ and p_φ, we get

$$m\ddot{r} = \frac{(mr^2\dot{\theta})^2}{mr^3} + \frac{(mr^2\dot{\phi}\sin^2\theta)^2}{mr^3\sin^2\theta} - \frac{\partial V}{\partial r}$$

or,
$$m\ddot{r} = mr\,\dot{\theta}^2 + mr\,\dot{\phi}\sin^2\theta - \frac{\partial V}{\partial r}$$

or,
$$m\ddot{r} - mr\,\dot{\theta}^2 - mr\,\dot{\phi}\sin^2\theta + \frac{\partial V}{\partial r} = 0 \quad \text{...(12)}$$

Again, differentiating Eq. (8) with respect to t, we get

$$\ddot{\theta} = \frac{\dot{p}_\theta}{mr^2} - \frac{p_\theta}{m}(2r^{-3}\dot{r})$$

or,
$$mr^2\,\ddot{\theta} = \dot{p}_\theta - 2\,p_\theta\frac{\dot{r}}{r}$$

Now, replacing $\dot{p}_\theta$ by using Eq. (9), we get

$$mr^2\ddot{\theta} = \frac{p_\phi^2\cos\theta}{mr^2\sin^3\theta} - \frac{\partial V}{\partial\theta} - \frac{2\,p_\theta}{r}\dot{r}$$

or,
$$mr^2\ddot{\theta} = \frac{(mr^2\dot{\phi}\sin^2\theta)^2\cos\theta}{mr^2\sin^3\theta} - \frac{\partial V}{\partial\theta} - \frac{2\,mr^2\dot{\theta}}{r}$$

or,
$$mr^2\ddot{\theta} = mr^2\dot{\phi}^2\sin\theta\cos\theta - \frac{\partial V}{\partial\theta} - 2\,mr\,\dot{r}\,\dot{\theta}$$

or,
$$mr^2\ddot{\theta} - mr^2\dot{\phi}^2\sin\theta\cos\theta + 2mr\,\dot{r}\,\dot{\theta} + \frac{\partial V}{\partial\theta} = 0 \quad \text{...(13)}$$

Equation (10) can be written as $mr^2\dot{\phi}\sin^2\theta = p_\phi$

Now, differentiating with respect to t, we get

$$mr^2\,\ddot{\phi}\sin^2\theta + 2mr\,\dot{r}\dot{\phi}\sin^2\theta + mr^2\,\dot{\phi}\,\dot{\theta}\,(2\sin\theta\cos\theta) = \dot{p}_\theta$$

or,
$$mr^2\ddot{\phi}\sin^2\theta + 2\,mr\dot{r}\dot{\phi}\sin^2\theta + mr^2\,\dot{\phi}\,\dot{\theta}\sin 2\theta = -\frac{\partial V}{\partial\phi} \quad \text{[by Equation (11)]}$$

or,
$$(mr^2\sin^2\theta)\,\ddot{\phi} + 2\,mr\dot{r}\dot{\phi}\sin^2\theta + mr^2\,\dot{\phi}\,\dot{\theta}\sin 2\theta + \frac{\partial V}{\partial\phi} = 0 \quad \text{...(14)}$$

Equations (1), (2) and (3) which were derived with the help of the lagrangian method are same as Eqs (12), (13) and (14) respectively, which have been derived by using the hamiltonian method.

Solved WBUT Question Paper (2009)

GROUP A

Multiple Choice Questions

1. (i) The eigen value of the eigen function e^{ix} for the operator $\frac{d^2}{dx^2}$ is

(a) 1 (b) 0 (c) –1 (d) None of these

Ans. (c)

(ii) One milligram of matter converted into energy will give

(a) 90 joule (b) 9×10^{10} joule (c) 1.5×10^{11} joule (d) None of these

Ans. (b)

(iii) The value of $\overline{\nabla} \times \overline{r}$ is equal to

(a) 1 (b) 0 (c) –1 (d) 2

Ans. (b)

(iv) The electric flux through the surface vector $\overline{s}$ equal to $6\hat{j}$ in a region of electric field $3\hat{i} + \hat{j}$ is

(a) $10\ Nm^2C^{-1}$ (b) $6\ Nm^2C^{-1}$ (c) $15\ Nm^2C^{-1}$ (d) None of these

Ans. (b)

(v) In free space, Poisson's equation reduces to

(a) $\nabla^2 V = 0$ (b) $\nabla^2 V = \rho/\epsilon_0$ (c) $\nabla^2 V = \infty$ (d) $\nabla^2 V = -\rho/\nabla\epsilon_0$

Ans. (a)

(vi) The divergence of a magnetic flux density $(\overline{B})$ is

(a) 0 (b) 1 (c) –1 (d) None of these

Ans. (a)

(vii) The differential form of Faraday's law electromagnetic induction is

(a) $\overline{\nabla} \times \overline{E} = \frac{\partial \overline{B}}{\partial t}$ (b) $\nabla \cdot \overline{E} = 2\overline{B}$ (c) $\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}$ (d) None of these

Ans. (c)

Group B

Short Answer Questions

1. Find the possible arrangements of two particles in three cells for
   (i) Bose–Einstein Statistics
   (ii) Fermi–Dirac Statistics

Ans. Refer to Example 8.1.

2. (a) Prove that $\overline{E} = \cos(y-t)\,\hat{k}$ and $\overline{B} = \cos(y-t)\,\hat{i}$ constitute possible electromagnetic wave.

Ans. Given $\overline{E} = \hat{k}\cos(y-t)$ and $\overline{B} = \hat{i}\cos(y-t)$ If $\overline{E}$ and $\overline{B}$ are to constitute an electromagnetic wave, then they must satisfy. Faraday's law as given below:

$$\overline{\nabla} \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \quad (1)$$

Now, $$\overline{\nabla} \times \overline{E} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial y}\right) \times \hat{k}\cos(y-t)$$

or, $$\overline{\nabla} \times \overline{E} = \hat{i}\frac{\partial}{\partial y}\cos(y-t)$$

or, $$\overline{\nabla} \times \overline{E} = -\hat{i}\sin(y-t) \quad (2)$$

Again, $$-\frac{\partial \overline{B}}{\partial t} = -\hat{i}\frac{\partial}{\partial t}\cos(y-t)$$

or, $$-\frac{\partial \overline{B}}{\partial t} = \hat{i}\sin(y-t) \quad (3)$$

Now, from Eqs (2) and (3), we get

$$\overline{\nabla} \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \quad (4)$$

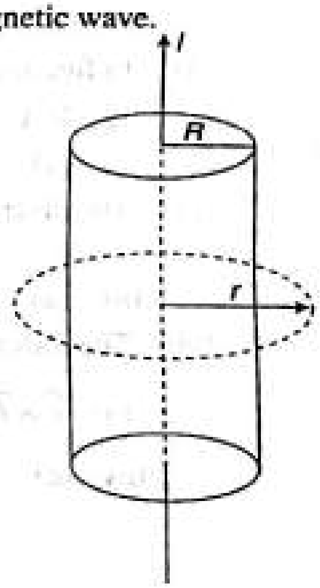
As can be seen, Eqs (1) and (4) are identical.

So, $\overline{E} = \hat{k}\cos(y-t)$ and $\overline{B} = \hat{i}\cos(y-t)$ constitute a possible electromagnetic wave.

(b) Define displacement current.

Ans. Refer to Section 5.6.

3. (a) Calculate the magnetic field intensity just outside and inside of a hollow cylinder of radius 4 cm carrying 50 A current.



Ans. Let us consider that the hollow cylinder is long enough so that $l >> R$, where l is its length and R is its radius. Let it carry current I (as shown in the diagram) which is uniformly distributed.

Now, applying Ampere's law

$$\oint \overline{B} \cdot d\overline{l} = \mu_0 I$$

or, $$B \times 2\pi r = \mu_0 I$$

when $r > R$, the current flowing through the cylinder of radius r is I and when $r < R$ the current flowing through the cylinder of radius r is zero.

So, the magnetic field intensity B just outside the cylinder is given by

$$B = \frac{\mu_0 I}{2\pi R} \quad [\because r = R]$$

$$= \frac{4\pi \times 10^{-7} \times 50}{2\pi \times 4 \times 10^{-2}} \text{ Tesla}$$

or, $$B = 2.5 \times 10^{-4} \text{ Tesla}$$

And B inside the cylinder is given by

$$B = 0 \text{ Tesla } [\because I = 0]$$

(b) Differentiate between electrostatic field and magnetic field.

Ans. Electrostatic field is the force experienced by unit charge due to the presence of one or more charges nearby the space, while magnetic field is the force experienced by unit magnetic pole due to the presence of any magnet or electric current present (nearby the said unit pole) in the space.

4. Prove that $\overline{A} \times (\overline{B} \times \overline{C}) = \overline{B} \times (\overline{A} \cdot \overline{C}) - \overline{C} \times (\overline{A} \cdot \overline{B})$

Ans. Let the three vectors be

$$\overline{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}, \quad \overline{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

and $$\overline{C} = C_x\hat{i} + C_y\hat{j} + C_z\hat{k}$$

Now, $$\overline{A} \times (\overline{B} \times \overline{C}) = \overline{A} \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

or, $$\overline{A} \times (\overline{B} \times \overline{C}) = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \times \left[(B_yC_z - B_zC_y)\,\hat{i} + (B_zC_x - B_xC_z)\,\hat{j} + (B_xC_y - B_yC_x)\,\hat{k}\right]$$

or, $$\overline{A} \times (\overline{B} \times \overline{C}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ (B_yC_z - B_zC_y) & (B_zC_x - B_xC_z) & (B_xC_y - B_yC_x) \end{vmatrix}$$

or, $$\overline{A} \times (\overline{B} \times \overline{C}) = (A_yB_xC_y - A_yB_yC_x - A_zB_zC_x + A_zB_xC_z)\,\hat{i} + (A_zB_yC_z - A_zB_zC_y - A_xB_xC_y + A_xB_yC_x)\,\hat{j} + (A_xB_zC_x - A_xB_xC_z - A_yB_yC_z + A_yB_zC_y)\,\hat{k}$$

Again, $$\overline{B}(\overline{A} \cdot \overline{C}) - \overline{C}(\overline{A} \cdot \overline{B}) = (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})\,(A_xC_x + A_yC_y + A_zC_z) - (C_x\hat{i} + C_y\hat{j} + C_z\hat{k})\,(A_xB_x + A_yB_y + A_zB_z) \quad (1)$$

or, $$\bar{B}(\bar{A}\cdot\bar{C})-\bar{C}(\bar{A}\cdot\bar{B})=(A_yB_xC_y-A_yB_yC_x-A_zB_zC_x+A_zB_xC_z)\hat{i}+(A_zB_yC_z-A_zB_zC_y-A_xB_xC_y+A_xB_yC_x)\hat{j}+(A_xB_zC_x-A_xB_xC_z-A_yB_yC_z+A_yB_zC_y)\hat{k} \quad (2)$$

Now, having compared Eqs (1) and (2), we can write

$$\bar{A}\times(\bar{B}\times\bar{C})=\bar{B}(\bar{A}\cdot\bar{C})-\bar{C}(\bar{A}\cdot\bar{B})$$

(Proved).

Group C

Long Answer Questions

1. (a) Write down the postulates of Fermi–Dirac statistics.

Ans. Refer to Section 8.10.1.

(b) Plot electron distribution function governed by Fermi–Dirac statistics in material at $T = OK$ and $T > OK$. Explain their physical significance.

Ans. Refer to Section 8.13.

2. (a) Write down Schrödinger equation for one-dimensional motion of a free particle in a one-dimensional potential box. Find its eigen function and eigen energy.

Ans. The Schrödinger equation for a particle in one-dimensional motion in a potential box is given by

$$\frac{d^2}{dx^2}\psi(x)+\frac{2m}{\hbar^2}[E-V(x)]\,\psi(x)=0$$

For a free particle, the potential $V(x) = 0$.

$\therefore$ the equation (for a free particle) reduces to

$$\frac{d^2}{dx^2}\psi(x)+\frac{2mE}{\hbar^2}\psi(x)=0$$

or, $$\frac{d^2\psi(x)}{dx^2}+k^2\psi(x)=0 \quad \text{where} \quad k^2=\frac{2mE}{\hbar^2}$$

the general solution of this equation is given by

$$\psi(x) = A\cos kx + B\sin kx$$

When $x = 0$, $\psi(x) = 0$ and when $x = L$, $\psi(x) = 0$ where L is the length of the box.

Thus, $\quad x = 0 \Rightarrow A = 0$

$\therefore \quad \psi(x) = B\sin kx$

Again at $\quad x = L, \quad \psi(x) = 0$

$\therefore \quad 0 = B\sin kL$

But $B \neq 0$, otherwise trial solution $\psi(x)$

$\therefore \quad \sin kL = 0 \Rightarrow \sin kL = \sin n\pi$

$\therefore \qquad kL = n\pi \quad \Rightarrow \quad k = \frac{n\pi}{L}$

where $\qquad n = 1, 2, 3, \ldots$

$\therefore$ the eigen function becomes

$$\psi_n(x) = B \sin \frac{n\pi}{L}$$

The eigen energy of the particle is given by

$$E_n = \frac{\hbar^2 k^2}{2m}$$

or, $$E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2$$

$\therefore$ $$E_n = \frac{n^2 h^2}{8mL^2}$$

(b) Prove that the first excited energy state of a free particle in a cubical box has three-fold degeneracy.

Ans. The expression for eigen energy for a free particle in a three-dimensional box is given by

$$E_{n_x n_y n_z} = \frac{h^2}{8m}\left[\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2}\right]$$

where a, b and c are the dimensions of the box; for a cubical box

$$a = b = c$$

$\therefore$ $$E_{n_x n_y n_z} = \frac{h^2}{8ma^2}\left[n_x^2 + n_y^2 + n_z^2\right]$$

Now, for the first excited state, there are the quantum states with quantum number sets (2, 1, 1), (1, 2, 1) and (1, 1, 2) for which the eigen energies are the same

$\therefore$ $$E_{211} = E_{121} = E_{112} = \frac{6h^2}{8ma^2} = \frac{3h^2}{4ma^2}$$

but there are different energy states. So, the first excited state has three-fold degeneracy.

3. (a) Distinguish between scalar and vector fields with examples.

Ans. In electromagnetic field, there are two potential functions: electric potential (V) and the magnetic potential ($\overline{A}$). V is a scalar potential while $\overline{A}$ is a vector potential.

The electric potential (V) is defined as the work done in bringing a unit positive test charge to point concerned (nearby a charge distribution) from infinity while magnetic vector potential $\overline{A}$ is defined as a vector, the curl of which gives the magnetic induction produced at any point by a closed loop carrying current.

V and $\overline{A}$ are given by the following equations:

$$\overline{E} = -\overline{\nabla} V \text{ and } \overline{B} = \overline{\nabla} \times \overline{A}.$$

(b) If the potential of a field is given by $V(x, y, z) = (4x^2 + 2y^2 + z^2)^{\frac{1}{2}}$ find the field intensity at the point (1, 1, 1).

Ans. We know that the field intensity $\overline{E}$ is given by

$$\overline{E} = -\overline{\nabla} V$$

or,
$$\overline{E} = -\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(4x^2 + 2y^2 + z^2)^{\frac{1}{2}}$$

or,
$$-\overline{E} = \hat{i}\frac{\partial}{\partial x}(4x^2 + 2y^2 + z^2)^{\frac{1}{2}} + \hat{j}\frac{\partial}{\partial y}(4x^2 + 2y^2 + z^2)^{\frac{1}{2}} + \hat{k}\frac{\partial}{\partial z}(4x^2 + 2y^2 + z^2)^{\frac{1}{2}}$$

or,
$$-\overline{E} = \frac{\hat{i}(8x)}{2(4x^2 + 2y^2 + z^2)^{\frac{1}{2}}} + \frac{\hat{j}(4y)}{2(4x^2 + 2y^2 + z^2)^{\frac{1}{2}}} + \frac{\hat{k}(2z)}{2(4x^2 + 2y^2 + z^2)^{\frac{1}{2}}}$$

or,
$$-\overline{E} = \frac{4x\hat{i}}{(4x^2 + 2y^2 + z^2)^{\frac{1}{2}}} + \frac{2y\hat{j}}{(4x^2 + 2y^2 + z^2)^{\frac{1}{2}}} + \frac{z\hat{k}}{(4x^2 + 2y^2 + z^2)^{\frac{1}{2}}}$$

or,
$$\overline{E} = -\left[\left(\frac{4\hat{i}}{7^{\frac{1}{2}}}\right) + \left(\frac{2\hat{j}}{7^{\frac{1}{2}}}\right) + \left(\frac{\hat{k}}{7^{\frac{1}{2}}}\right)\right]$$

$\therefore$
$$\overline{E} = \frac{-1}{\sqrt{7}}(4\hat{i} + 2\hat{j} + \hat{k})$$

(c) Prove that curl grad $\phi = 0$.

Ans.
$$\overline{\nabla} \times (\overline{\nabla}\phi) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

or,
$$\overline{\nabla} \times (\overline{\nabla}\phi) = \hat{k}\frac{\partial^2\phi}{\partial x\partial y} - \hat{j}\frac{\partial^2\phi}{\partial x\partial z} - \hat{k}\frac{\partial^2\phi}{\partial x\partial y} + \hat{i}\frac{\partial^2\phi}{\partial y\partial z} + \hat{j}\frac{\partial^2\phi}{\partial x\partial z} - \hat{i}\frac{\partial^2\phi}{\partial y\partial z}$$

or,
$$\overline{\nabla} \times (\overline{\nabla}\phi) = 0$$

i.e., curl grad $\phi = 0$ (Proved)

(d) Show that the potential function $x^2 - y^2 + z$ satisfies Laplace's equation.

Ans. Laplace's equation is given by

$$\nabla^2 V = 0$$

Given
$$V = x^2 - y^2 + z$$

$\therefore$
$$\nabla^2 V = \left(\frac{\partial^2}{x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(x^2 - y^2 + z)$$

$$= \frac{\partial}{\partial x^2}(x^2 - y^2 + z) + \frac{\partial^2}{\partial y^2}(x^2 - y^2 + z) + \frac{\partial^2}{\partial z}(x^2 - y^2 + z)$$

$$= 2 - 0 + 0 + 0 - 2 + 0 + 0 - 0 + 1$$

$$= 0$$

Hence, $\nabla^2 V = 0$ which is nothing but Laplace's equation.

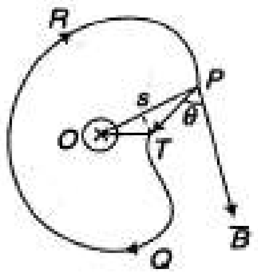
So, the given potential function satisfies the Laplace's equation.

(e) Find the potential of a uniformly charged sphere of radius R having a constant charge density ρ at a distance r from the center of the sphere where $r > R$.

Ans. Refer to Section 2.12.3 (i).

4. (a) State and prove Ampere's circuital law.

Ans. Ampere's circuital law states that the line integral of the magnetic field $\overline{B}$ around any closed path is equal to μ_0 times the net current enclosed by the path, where μ_0 is the magnetic permeability of free space or any non-magnetic conducting material, i.e., $\oint \overline{B} \cdot d\overline{l} = \mu_0 I$.



Proof of Ampere's Circuital Law: Let us consider a closed path PQR around any current-carrying wire placed at O pointing to a direction perpendicular to the plane of the paper. According to the figure given, the field B at any point P is in the plane of paper and is perpendicular to the radius vector $\overline{r}$ from the wire at O. The vector $\overline{B}$ makes an angle θ with the line element of the closed path PQR encircling the steady current-carrying wire.

The component of $\overline{B}$ along $d\overline{l}$ is given by

$$B_1 = B\cos\theta$$

If TS be the normal to OP from T and $TP = l$, then from the ΔPST, we get

$$TS = dl\cos\theta \text{ and from } \Delta OST\ TS = r\,d\phi \qquad \text{[if } d\phi \text{ is very small]}$$

$$\therefore \quad r\,d\phi = dl\cos\theta$$

or, $$dl = \frac{r\,d\phi}{\cos\theta}$$

Therefore, $$\overline{B}\cdot d\overline{l} = B\,dl\cos\theta = B\cos\theta\frac{r\,d\phi}{\cos\theta}$$

or, $$\overline{B}\cdot d\overline{l} = Br\,d\phi$$

According to Biot–Savart's law, the magnetic induction at a distance r from a current-carrying wire with steady current I is given by

$$B = \frac{\mu_0 I}{2\pi r}$$

$$\therefore \quad \overline{B}\cdot d\overline{l} = \frac{\mu_0 I}{2\pi r}r\,d\phi = \frac{\mu_0}{2\pi}I\,d\phi$$

$$\therefore \quad \oint \overline{B}\cdot d\overline{l} = \frac{\mu_0 I}{2\pi}\oint d\phi = \frac{\mu_0 I(2\pi)}{2\pi} = \mu_0 I$$

i.e., $$\oint \overline{B}\cdot d\overline{l} = \mu_0 I \qquad \text{(Proved)}$$

(b) Define magnetic vector potential.

If the vector potential $\overline{A} = (10x^2 + y^2 - z^2)\,\hat{j}$ at any position, then find the magnetic field at the point (1, 1, 2).

Ans. For the definition refer to Section 4.14.2. We know that magnetic field intensity B is given by

$$\overline{B} = \nabla \times \overline{A}$$

or, $$\overline{B} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (10x^2 + y^2 - z^2)\,\hat{j}$$

or, $$\overline{B} = \hat{k}\,(20x) + 0 - \hat{i}\,(-2z)$$

or, $$\overline{B} = 2z\hat{i} + 20x\,\hat{k}$$

(c) Write down the condition of steady-state current. Show that Ampere's law implies that the current is in the steady state.

Ans. The condition for a steady-state current is that the flow rate of charge through the conductor must be independent of time, i.e., the current will not vary with time.

(d) Calculate the magnetic field intensity just onside of a hollow cylinder of radius 4 cm.

Ans. The magnetic field intensity just outside the cylinder is given by

$$B = \frac{\mu_0 I}{2\pi R}$$

Here magnetic permeability in the free space (or non-magnetic material) is μ_0.

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m}$$

$$R = 4 \text{ cm} = 4 \times 10^{-2} \text{ m}$$

$$\therefore \quad B = \frac{4\pi \times 10^{-7} \times I}{2\pi \times 4 \times 10^{-2}}$$

$$\therefore \quad B = 0.5 \times 10^{-5}\, I \text{ Tesla}$$

If we are given numerical value of I, then we can calculate the numerical value of B. If I is given in ampere, then B will be obtained in Tesla.

5. (a) Write down Maxwell's equations for free space.

Ans. Maxwell's equations for free space are given by

(i) $\overline{\nabla} \cdot \overline{E} = 0$ (ii) $\overline{\nabla} \cdot \overline{B} = 0$

(iii) $\overline{\nabla} \times \overline{E} = -\dfrac{\partial \overline{B}}{\partial t}$ (iv) $\nabla \times \overline{B} = \mu_0 \dfrac{\partial \overline{D}}{\partial t}$

(b) Show that for free space, the electromagnetic wave equation for $\overline{E}$ is $\nabla^2 \overline{E} = \mu_0 \epsilon_0 \dfrac{\partial^2 \overline{E}}{\partial t^2}$, where symbols have their usual significance. Prove that electromagnetic wave moves with the velocity of light in free space.

Ans. Refer to Sections 5.10 and 5.11.

(c) Establish the integral form of Faraday's law of electromagnetic induction.

Ans. Refer to Section 5.5.

Solved WBUT Question Paper (2010)

Group A

Multiple Choice Questions

1. (i) The average energy of a free electron in a metal at absolute zero is

(a) zero

(b) more than $\frac{1}{2} E_f$

(c) less than or equal to $\frac{1}{2} E_f$

(d) equal to E_f when E_f is the Fermi level

Ans. (b)

(ii) If $f(x)$ denotes the wave function of a particle in a one-dimensional box then the dimension of $f(x)$ is of

(a) length

(b) $\frac{1}{\sqrt{\text{length}}}$

(c) $\frac{1}{\text{length}}$

(d) None of these ($f(x)$ is dimensionless)

Ans. (b)

(iii) If x component of the momentum operator, $\hat{p}_x$, is represented by $-i\hbar \frac{\partial}{\partial x}$, then the commutator of $\hat{x}$ and $\hat{p}_x^2$ (i.e., $[\hat{x}, \hat{p}_x^2]$) is represented by

(a) $-\hbar^2 \frac{\partial^2}{\partial x^2}$

(b) $i\hbar$

(c) $2\hbar^2 \frac{\partial}{\partial x}$

(d) 0

Ans. (b)

(iv) Example of a Boson is

(a) neutron

(b) deuterium

(c) proton

(d) positron

Ans. (b)

Group B

Short Answer Questions

1. If a system has two eigen states ψ_1 and ψ_2 with eigen values E_1 and E_2, under what condition, will linear combination ($\psi = a\psi_1 + b\psi_2$) be also an eigen state.

Ans. ψ_1 and ψ_2 are two eigen states with eigen values E_1 and E_2 respectively. The linear combination of then $\psi = a\psi_1 + b\psi_2$ will also be an eigen state of the system

if $$\int_{-\infty}^{+\infty} (a\psi_1 + b\psi_2)^* (a\psi_1 + b\psi_2)\, dV = 1$$

or, if $$\int_{-\infty}^{+\infty} |a|^2 \psi_1^* \psi_1\, dV + \int_{-\infty}^{+\infty} a^* b\, \psi_1^* \psi_2\, dV + \int_{-\infty}^{+\infty} ab^* \psi_1 \psi_2^*\, dV + \int_{-\infty}^{+\infty} |b|^2 \psi_2^* \psi_2\, dV = 1$$

or, if $$\int_{-\infty}^{+\infty} |a|^2 \psi_1^* \psi_1\, dV + \int_{-\infty}^{+\infty} |b|^2 \psi_2^* \psi_2\, dV = 1$$

or, if $$a^2 + b^2 = 1$$

So, if $a^2 + b^2 = 1$, then $\psi = a\psi_1 + b\psi_2$ will be an eigen state. This is the required condition.

2. If the wave function $\psi(x)$ of quantum mechanical particle is given by

$$\psi(x) = a \sin\left(\frac{\pi x}{L}\right) \quad \text{for} \quad 0 \le x \le L$$

$$= 0 \qquad \text{otherwise}$$

Then determine the value of a.

Also determine the value of x where probability of finding the particle is maximum.

Ans. $$\int_{-\infty}^{\infty} \psi^*(x)\, \psi(x)\, dx = 1$$

or, $$\int_0^L \psi^*(x)\, \psi(x)\, dx = 1 \qquad \text{[because of the given condition]}$$

or, $$\int_0^L a^* \sin\left(\frac{\pi x}{L}\right) a \sin\left(\frac{\pi x}{L}\right) dx = 1$$

or, $$|a|^2 \int_0^L \sin^2 \frac{\pi x}{L}\, dx = 1$$

or, $$a^2 \int_0^L 2\sin^2 \frac{\pi x}{L}\, dx = 2$$

or, $$a^2 \int_0^L \left(1 - \cos\frac{2\pi x}{L}\right) dx = 2$$

or, $$a^2\left[\int_0^L dx - \int_0^L \cos\frac{2\pi x}{L}\,dx\right] = 2$$

or, $$a^2\left[L - \frac{L}{2\pi}\left(\sin\frac{2\pi x}{L}\right)\Big|_0^L\right] = 2$$

or, $$La^2 = 2$$

or, $$a^2 = \frac{2}{L} \Rightarrow a = \sqrt{\frac{2}{L}}$$

Now, the wave function is given by

$$\psi(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)$$

The probability density P of the particle is given by the following equation:

$$P = |\psi(x)|^2 = \psi^*(x)\,\psi(x)$$

or, $$P = \frac{2}{L}\sin^2\left(\frac{\pi x}{L}\right)$$

$\therefore$ $$P_{max} = \frac{2}{L} \text{ and } \sin^2\left(\frac{\pi x}{L}\right) = 1$$

Since $$\sin^2\left(\frac{\pi x}{L}\right) = 1, \frac{\pi x}{L} = (2n+1)\frac{\pi}{2}$$

where $$n = 0, 1, 2, 3, \ldots$$

Hence, $$x = (2n+1)\frac{L}{2}$$

$\therefore$ the probability will be maximum at $x = \frac{L}{2}$.

3. A system has non-degenerate single particle states with energy levels 0, E, $2E$ and $3E$. Three particles are to be distributed in these states such that the total energy is $6E$. Write down in each case all possible microstates assuming that the particles obey *MB* statistics or *FD* statistics.

Ans. (i) *MB* statistics

Since the particles are distinguishable, let the particles be A, B and C. The possible microstates will be

Microstate	*0E*	*1E*	*2E*	*3E*	*Total energy*	*Microstates*
(0, 0, 3, 0)	0	0	ABC	0	6E	1
(0, 1, 1, 1)	0	A	B	C	6E	6
	0	A	C	B	6E	
	0	B	A	C	6E	
	0	B	C	A	6E	
	0	C	A	B	6E	
	0	C	B	A	6E	
(1, 0, 0, 2)	A	0	0	BC	6E	3
	B	0	0	AC	6E	
	C	0	0	AB	6E	

So, total number of microstates = 10

(ii) *FD* statistics

Since the particles are indistinguishable and according to the Pauli exclusion principle, each state can accommodate only one particle, so the total number of states (microstates) will be as follows:

Macrostate	*0E*	*1E*	*2E*	*3E*	*Totalenergy*	*Microstates*
(0, 1, 1, 1)	0	*A*	*A*	*A*	6*E*	1

So total member of microstates = 1

Group C

Long Answer Questions

1. (a) The ground state and the excited state normalized wave functions of an atom are ψ_0 and ψ_1 respectively, and the corresponding energy is E_0 and E_1. If the probability of finding the atom in the ground state is 90% and that for the excited state is 10%, then find the average energy of the atom. Also determine the normalized wave function of the atom.

Ans. In this case the state of the particle is generated by superpositioning of two states ψ_0 and ψ_1, so the wave function should be a linear combination of the two wave functions ψ_0 and ψ_1.

$\therefore$ the wave function could be written as

$$\psi = C_0\psi_0 + C_1\psi_1$$

The wave functions ψ_0, ψ_1 and ψ are normalized and the functions ψ_0 and ψ_1 are orthogonal to each other

So, one can write

$$\int_V \psi^*\psi\, dV = 1 \qquad \text{[where } V \text{ represents entire space]}$$

or, $$\int_V (C_0\psi_0 + C_1\psi_1)^* (C_0\psi_0 + C_1\psi_1)\, dV = 1$$

or, $$C_0^*C_0 \int_V \psi_0^*\psi_0\, dV + C_1^*C_1 \int_V \psi_1^*\psi_1\, dV + C_0^*C_1 \int_V \psi_0^*\psi_1\, dV + C_1^*C_0 \int_V \psi_1^*\psi_0\, dV = 1$$

Since ψ_0 and ψ_1 are orthogonal wave functions, the third and fourth terms on the left side of the equation are zero.

$\therefore$ $$C_0^*C_0 \int_V \psi_0^*\psi_0\, dV + C_1^*C_1 \int_V \psi_1^*\psi_1\, dV = 1$$

Because of the normalization property of the wave functions ψ_0 and ψ_1, the integrals are unity.

$\therefore$ $$C_0^*C_0 + C_1^*C_1 = 1$$

or, $$C_1^2 + C_0^2 = 1 \qquad [\because\ C_0 \text{ and } C_1 \text{ are real constants.}]$$

The sum of probabilities of all possible states must be unity. So, C_0^2 and C_1^2 may be regarded as probabilities of finding the atom in the ground and first excited states respectively.

$\therefore$ we have $C_0 = \sqrt{0.9}$ and $C_1 = \sqrt{0.1}$. Then the wave function of the atom is given by

$$\psi = C_0\psi_0 + C_1\psi_1 = \sqrt{0.9}\ \psi_0 + \sqrt{0.1}\ \psi_1$$

If P_n be the probability of finding the atom at the energy level E_n, then the average energy is given by

$$E_{av} = \sum_n P_n E_n = P_0E_0 + P_1E_1$$

or, $$E_{av} = 0.9\ E_0 + 0.1\ E_1 \qquad [\because P_0 = C_0^2 \text{ and } P_1 = C_1^2]$$

(b) A particle, of mass m, moving in three dimensions is confined within a box $0 < x < a$, $0 < y < b$, $0 < z < c$. (The potential is zero inside and infinite outside.) Write down Schrödinger's equation for the particle. By considering a stationary state wave function of the form $u(x, y, z) = f_1(x) f_2(y) f_3(z)$, show that the allowed energies are $E = \frac{\pi^2\hbar^2}{2m}\left(\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2} + \frac{n_3^2}{c^2}\right)$. What is the degeneracy of the first excited energy level when $a = b = c$? Explain. What is ground state energy?

Ans. As the potential inside the box is zero, it is independent of time. So, the required Schrödinger equation is given by

$$\frac{-\hbar^2}{2m}\nabla^2 u(x, y, z) = Eu(x, y, z) \qquad (1)$$

where $$\psi(x, y, z, t) = u(x, y, z) f(t)$$
$$u(x, y, z) = f_1(x) f_2(y) f_3(z)$$

Equation (1) can be written as

$$\left(\nabla^2 + \frac{2mE}{\hbar^2}\right) u(x, y, z) = 0$$

or, $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{2mE}{\hbar^2}\right) f_1(x) f_2(y) f_3(z) = 0$$

or, $$f_2 f_3 \frac{\partial^2 f_1}{\partial x^2} + f_1 f_3 \frac{\partial^2 f_2}{\partial y^2} + f_1 f_2 \frac{\partial^2 f_3}{\partial z^2} + \frac{2mE}{\hbar^2} f_1 f_2 f_3 = 0$$

Dividing this equation by $f_1 f_2 f_3$ and rearranging the terms, we get

$$\frac{1}{f_1}\frac{\partial^2 f_1}{\partial x^2} + \frac{1}{f_2}\frac{\partial^2 f_2}{\partial y^2} + \frac{1}{f_3}\frac{\partial^2 f_3}{\partial z^2} = \frac{-2mE}{\hbar^2} \qquad (2)$$

Since f_1 is a function of x only, we have

$$\frac{\partial^2 f_1}{\partial x^2} = \frac{d^2 f_1}{dx^2}.$$

For similar reasons we can write

$$\frac{\partial^2 f_2}{\partial y^2} = \frac{d^2 f_2}{dy^2} \quad \text{and} \quad \frac{\partial^2 f_3}{\partial z^2} = \frac{d^2 f_3}{dz^2}.$$

Then Eq. (2) can be written as

$$\frac{1}{f_1}\frac{d^2 f_1}{dx^2}+\frac{1}{f_2}\frac{d^2 f_2}{dy^2}+\frac{1}{f_3}\frac{d^2 f_3}{dz^2}=\frac{-2mE}{\hbar^2} \tag{3}$$

In Eq. (3), the three terms on the left-hand side are completely independent of each other. There are respectively functions of x, y and z and their sum is independent of x, y and z. This is possible only when each term is separately equal to a constant.

Let K be a constant so that $K^2=\frac{2mE}{\hbar^2}$.

So, Eq. (3) takes the following form:

$$\frac{1}{f_1}\frac{d^2 f_1}{dx^2}+\frac{1}{f_2}\frac{d^2 f_2}{dy^2}+\frac{1}{f_3}\frac{d^2 f_3}{dz^2}=-K^2$$

Let $K^2=K_1^2+K_2^2+K_3^2$, such that

$$\frac{1}{f_1}\frac{d^2 f_1}{dx^2}=-K_1^2 \tag{4}$$

$$\frac{1}{f_2}\frac{d^2 f_2}{dy^2}=-K_2^2 \tag{5}$$

$$\frac{1}{f_3}\frac{d^2 f_3}{dz^3}=-K_3^2 \tag{6}$$

Now, we have

$$K^2=K_1^2+K_2^2+K_3^2=\frac{2mE}{\hbar^2}$$

Equation (4) can be written as

$$\frac{d^2 f_1}{dx^2}+K_1^2 f_1=0$$

The general solution of this equation is

$$f_1=A_1\sin K_1x+B_1\cos K_1x \tag{7}$$

Similarly, the general solutions of Eqs (5) and (6) are:

$$f_2=A_2\sin K_2y+B_2\cos K_2y \tag{8}$$

and

$$f_3=A_3\sin K_3z+B_3\cos K_3z \tag{9}$$

$$f_1=0 \quad \text{at} \quad x=0 \quad \text{and} \quad x=a \tag{10}$$

$$f_2=0 \quad \text{at} \quad y=0 \quad \text{and} \quad y=b \tag{11}$$

and

$$f_3=0 \quad \text{at} \quad z=0 \quad \text{and} \quad z=c \tag{12}$$

Applying the condition (10) upon Eq. (7), we get

$$B_1=0 \quad \text{and} \quad K_1=\frac{n_1\pi}{a} \tag{13}$$

where $n_1 = 1, 2, 3, \ldots$

Similarly, we can write

$$B_2 = 0 \quad \text{and} \quad K_2 = \frac{n_2\pi}{b} \tag{14}$$

where $n_2 = 1, 2, 3, \ldots$

and

$$B_3 = 0 \quad \text{and} \quad K_3 = \frac{n_3\pi}{c} \tag{15}$$

where $n_3 = 1, 2, 3, \ldots$

The expressions for f_1, f_2 and f_3 can be written as

$$f_1 = A_1 \sin\left(\frac{n_1\pi x}{a}\right) \tag{16}$$

$$f_2 = A_2 \sin\left(\frac{n_2\pi y}{b}\right) \tag{17}$$

$$f_3 = A_3 \sin\left(\frac{n_3\pi z}{c}\right) \tag{18}$$

Now, substituting these expressions in equation,

$$u(x, y, z) = f_1(x) f_2(y) f_3(z)$$

we get

$$u(x, y, z) = A_1A_2A_3 \sin\left(\frac{n_1\pi x}{a}\right) \sin\left(\frac{n_2\pi y}{b}\right) \times \sin\left(\frac{n_3\pi z}{c}\right)$$

Due to the dependence of u on the quantum numbers n_1, n_2, n_3, the expression for $u(x, y, z)$ is written as

$$u_{n_1n_2n_3} = A_1A_2A_3 \sin\left(\frac{n_1\pi x}{a}\right) \sin\left(\frac{n_2\pi y}{b}\right) \sin\left(\frac{n_3\pi z}{c}\right)$$

or,

$$u_{n_1n_2n_3} = A \sin\left(\frac{n_1\pi x}{a}\right) \sin\left(\frac{n_2\pi y}{b}\right) \sin\left(\frac{n_3\pi z}{c}\right) \tag{19}$$

where $A = A_1A_2A_3$

$n_1 = 1, 2, 3, \ldots$

$n_2 = 1, 2, 3, \ldots$

$n_3 = 1, 2, 3, \ldots$

using the equation $K^2 = K_1^2 + K_2^2 + K_3^2 = \dfrac{2mE}{\hbar^2}$

we get

$$E = \frac{K^2\hbar^2}{2m} = \frac{(K_1^2 + K_2^2 + K_3^2)}{2m}\hbar^2 \tag{20}$$

Substituting in it the expressions for K_1, K_2, and K_3 from Eqs (13), (14) and (15) we get

$$E = \frac{(K_1^2 + K_2^2 + K_3^2)}{2m} = \frac{\pi^2 \eta^2}{2m}\left(\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2} + \frac{n_3^2}{c^2}\right)$$

Due to its dependence on the quantum numbers n_1, n_2, n_3, the energy expression is written as,

$$E_{n_1 n_2 n_3} = \frac{\pi^2 h^2}{2m}\left(\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2} + \frac{n_3^2}{c^2}\right) \tag{21}$$

Degeneracy of Energy States: In a cubical box, the total energy is proportional to the sum of squares of the quantum numbers (n_1, n_2, n_3). The particle can have the same value of energy for more than one set of values of the quantum numbers (n_1, n_2, n_3). It means that the energy values corresponding to more than one wave functions may be same. The states, represented by these wave functions, are called degenerate states. When $a = b = c = L$ (say), the energy value is given by

$$E_{n_1 n_2 n_3} = \frac{\pi^2 h^2}{2mL^2}(n_1^2 + n_2^2 + n_3^2)$$

In the first excited state the values of n_1, n_2 and n_3 can have either of the following three sets: (1, 1, 2), (1, 2, 1), (2, 1, 1).

So, the first excited state has three-fold degeneracy with energy as follows:

$$E_{112} = E_{121} = E_{211} = \frac{6h^2}{8mL^2}$$

The ground state energy is given by

$$E_{111} = \frac{3h^2}{8mL^2} \quad \text{with no degeneracy.}$$

(c) Define the follow:

(i) Microstate (ii) Macrostate (iii) Thermodynamic probability

Ans. Refer to Sections 8.4 and 8.5.

2. (a) Write down the time-dependent Schrödinger equation explaining the symbols used. Derive time-independent Schrödinger equation assuming that the potential is time independent.

Ans. The time-dependent Schrödinger equation is given by

$$\frac{-h^2}{2m}\nabla^2 \psi(x, y, z, t) + V(x, y, z, t)\, \psi(x, y, z, t)$$
$$= ih\frac{\partial}{\partial t}\psi(x, y, z, t) \tag{1}$$

where ∇^2 is the laplacian operator and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$\psi(x, y, z, t)$ is the wave function of the particle which represented by it.

$V(x, y, z, t)$ is the potential function.

If the potential is independent of time, then

$$V = V(x, y, z).$$

Now, Eq. (1) can be written as

$$\frac{-\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t} \tag{2}$$

As the potential energy V is a function of position only, the total energy is a constant. So, the equation is separable into the time-dependent and time-independent parts.

So, we can write

$$\psi(x, y, z, t) = \phi(x, y, z)\, f(t)$$

Now, putting this value of ψ in Eq. (2), one can get

$$\frac{-\hbar^2}{2m}\nabla^2\phi(\bar{r})f(t) + V(\bar{r})\,\phi(\bar{r})f(t)$$

$$= i\hbar\frac{\partial}{\partial t}\{\phi(\bar{r})f(t)\}$$

Now, dividing both sides by $\phi(\bar{r})f(t)$, one gets

$$\frac{-\hbar^2}{2m}\frac{1}{\phi(\bar{r})}\nabla^2\phi(\bar{r}) + V(\bar{r}) = i\hbar\frac{1}{f(t)}\frac{\partial f(t)}{\partial t}$$

The left-hand side of Eq. (3) is a function of $\bar{r}$ only while the right-hand side is a function of t only. this is possible only when both of the sides are equal to a constant separately. This constant is the total energy E of the v particle. Thus, one can write two equations, when one equation the left side to E, the following equation is obtained:

$$\frac{-\hbar^2}{2m}\nabla^2\phi(\bar{r}) + V(\bar{r})\,\phi(r) = E\,\phi(\bar{r})$$

or,

$$\nabla^2\phi(\bar{r}) + \frac{2m}{\hbar^2}[E - V(\bar{r})]\,\phi(\bar{r}) = 0 \tag{4}$$

Equation (4) is the required time-independent Schrödinger equation.

(b) What are Fermions? Give two examples.

Ans. Fermions are the particles which obey Fermi–Dirac statistics. Fermions are spin-half-integral particles.

Examples of Fermions are electrons and protons.

(c) Write the expression of *BE* distribution law and derive Planck's black-body radiation law from *BE* statistics.

Ans. Bose–Einstein distribution law is given by

$$f(\in) = \frac{1}{e^{(\in - \mu)/(K_B T)} - 1}$$

where μ is the chemical potential and K_B is the Boltzmann constant. T is the absolute temperature and ε is the energy level.

For the second part refer to Section 8.14.

(d) Sketch the Fermi distribution for $T = 0$ and $T > 0$ and explain.

Ans. Refer to Section 8.13.

Solved WBUT Question Paper (2011)

Group A

Multiple Choice Questions

1. Choose the correct alternatives for any *ten* of the following:

$10 \times 1 = 10$

(i) Which of the following functions is acceptable as a wave function of a one-dimensional quantum mechanical system?

(a) $\psi = \dfrac{A}{x}$ (b) $\psi = Ae^{-x^2}$ (c) $\psi = Ax^2$ (d) $\psi A \sec x$

Ans. (b)

(ii) A system with time independent potential is in an energy state E. The wave function of this state is

(a) independent of time
(b) an exponentially decaying function of time
(c) a periodic function of time with time period proportional to E
(d) a periodic function of time with time period inversely proportional to E

Ans. (b)

(iii) For a free particle in cubical box the degeneracy of the first excited state is

(a) two-fold (b) three-fold (c) six-fold (d) nine-fold

Ans. (b)

(iv) Wave function of a certain particle is given by

$$\psi = A\cos x \quad \text{for } |x| \le \frac{\pi}{2}$$
$$= 0 \quad \text{otherwise}$$

Then the value of A is

(a) $\dfrac{2}{\pi}$ (b) $\sqrt{\dfrac{2}{\pi}}$ (c) 1 (d) zero

Ans. (b)

(v) The physical interpretation of $\vec{\nabla} \cdot \vec{B} = 0$ is ($\vec{B}$ is the magnetic field)

(a) magnetic monopole cannot exist

(b) magnetic field is irrotational

(c) magnetic field is conservative

(d) magnetic lines of force are open curves

Ans. (a)

(vi) If a vector field, F is given by $\vec{F} = \dfrac{\sin h(r)}{1+2r^3}\vec{r}$ then $\vec{\nabla} \times \vec{F}$ is given by

(a) $3\vec{r}$

(b) zero

(c) $\left(\dfrac{\cos h(r)}{1+2r^3} - 3r\right)\hat{\theta}$, $\hat{\theta}$ is the unit vector along θ-line

(d) $\left(\dfrac{\cos h(r)}{1+2r^3} - 3r\right)\vec{F}$

Ans. (b)

(vii) A proton traveling vertically downwards experiences a southward force due to a magnetic field directed at right angles to its path. An electron traveling northward in the same magnetic field will experience a magnetic force directed.

(a) upwards (b) downwards (c) towards east (d) towards west

Ans. (a)

(viii) Ampere's circuital law is applicable when the current density is

(a) constant over space (b) time independent

(c) solenoidal (d) irrotational

Ans. (b)

(ix) Dimension of $\mu_0 \varepsilon_0$ (symbols with their usual significance) is

(a) $L^{-2}T^{-2}$ (b) LT^{-1} (c) $L^{-2}T^{-1}$ (d) $L^{-2}T^{2}$

Ans. (d)

(x) Electrostatic field may be obtained as the gradient of a scalar potential

(a) because it is a conservative field

(b) because it is always solenoidal

(c) when the field is linear in x, y, z

(d) where there is no magnetic field in the same location

Ans. (a)

(xi) If $\vec{P}$ represents the polarization vector then $\vec{\nabla} \cdot \vec{P} = -\rho$, where ρ is the density of

(a) free charge

(b) bound charge

(c) free charge at the boundary of the dielectric

(d) sum of free and bound charge

Ans. (b)

(xii) Two conducting spheres of radii r and $2r$ contain charges q and $2q$. Electric field just outside the surface of these two spheres are E_1 and E_2 respectively. Then

(a) $E_1 = E_2$ (b) $E_1 = 2E_2$ (c) $2E_1 = E_2$ (d) $E_1 = 4E_2$

Ans. (b)

(xiii) Which of the following particles is not a Fermion?

(a) Proton (b) Neutron (c) Alpha-particle (d) Electron

Ans. (c)

(xiv) For $T > 0$ the probability of occupancy of a state of a Fermion with energy equal to Fermi energy is

(a) 1 (b) decreases linearly with T

(c) $\frac{1}{2}$ (d) decreases exponentially with T

Ans. (c)

(xv) Number of ways a total of N distinguishable particles may be placed in different energy levels (N_i particles in energy level E_i with degeneracy g_i) is

(a) $\dfrac{N!}{\sum_i N_i g_i}$ (b) $N!\sum_i \dfrac{g_i^{N_i}}{N_i!}$ (c) $\dfrac{\sum_i (N_i!)}{\sum_i g_i^{N_i}}$ (d) $\dfrac{N!}{\sum_i (N_i + g_i - 1)!}$

Ans. (b)

Group B

Short Answer Questions

Answer any *three* of the following. $3 \times 5 = 15$

1. (a) Show that $\nabla^2\left(\frac{1}{r}\right) = 0$ when $r \neq 0$. 3

Ans. $$\nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\frac{\partial^2}{\partial x^2}(x^2+y^2+z^2)^{-\frac{1}{2}} = \frac{\partial}{\partial x}\left[-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x)\right]$$

$$= \frac{\partial}{\partial x}[x(x^2+y^2+z^2)^{-3/2}]$$

$$= -(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2}$$

$\therefore$ $$\frac{\partial^2}{\partial x^2}(x^2+y^2+z^2)^{-\frac{1}{2}} = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$$

Similarly, $$\frac{\partial^2}{\partial y^2}(x^2+y^2+z^2)^{-\frac{1}{2}} = \frac{2y^2-z^2-x^2}{(x^2+y^2+z^2)^{5/2}}$$

and $$\frac{\partial^2}{\partial z^2}(x^2+y^2+z^2)^{-\frac{1}{2}} = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$$

$\therefore$ $$\nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$= \frac{2x^2-y^2-z^2+2y^2-z^2-x^2+2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$$

Hence, $$\nabla^2\left(\frac{1}{r}\right) = 0$$ (Proved)

(b) Find the angle between the normal to the $x-y$ plane and the normal to surface $x^2+y^2=z^2$. 2

Ans. The unit vector normal to the $X-Y$ plane is $\hat{k}$. And the normal to the surface $x^2+y^2=z^2$ is given by

$$\vec{n} = \vec{\nabla}(x^2+y^2-z^2)$$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+-z^2)$$

or, $$\vec{n} = 2x\hat{i}+2y\hat{j}-2z\hat{k}$$

$\therefore$ The unit vector normal to the surface is given by

$$\hat{n} = \frac{2x\hat{i}+2y\hat{j}-2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}}$$

Let θ be the required angle

$\therefore$ $$\hat{x}\cdot\hat{k} = (1)\,(1)\cos\theta = \hat{k}\cdot\frac{(\hat{i}x+\hat{j}y-\hat{k}z)}{\sqrt{x^2+y^2+z^2}}$$

$\therefore$ $$\cos\theta = \frac{-z}{\sqrt{x^2+y^2+z^2}}$$

Now, the set of values of (x, y, z) like $(1, \sqrt{3}, 2)$ will be on the plane given,

$\therefore$ $$\cos\theta = \frac{-2}{\sqrt{1+3+4}} \Rightarrow \cos\theta = \frac{-2}{2\sqrt{2}}$$

or, $$\cos\theta = -\frac{1}{\sqrt{2}} \Rightarrow \cos\theta = 135°$$

Hence, $\theta = 135°$

2. Starting from Maxwell's equation in a charge free conducting media arrive at the concept of skin depth and express it in terms of signal frequency and conducting for a very good conductor.

Ans. Refer to Sections 5.13 and 5.14.

3. A capacitor is made with two infinitely long cylindrical conductors of radii a and b ($a > b$), with vacuum in the intervening space. If the internal cylinder is kept at ground and the outer cylinder has a charge density σ then solve Laplace equation to find the electrostatic potential in the space between the cylinders.

Ans. The diagram given in Fig. 1 explains the phenomenon.

The radii of the inner and outer cylinders are given by

$r_i = b$ and $r_o = a$.

And the relative permittivity in the space between the cylinders is ε_o.

Since the potential exists only along the radial direction, we get,

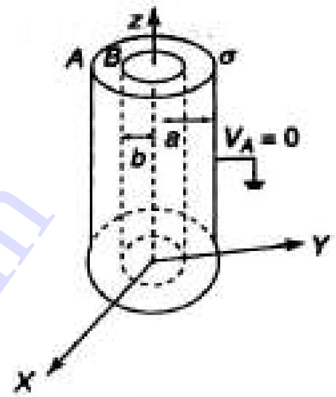


Fig. 1

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)=0 \tag{1}$$

$$\left[\begin{array}{l}\because V \text{ is independent of } \theta \text{ and } z \text{ here} \\ \text{we have } \dfrac{\partial^2 V}{\partial \theta^2}=\dfrac{\partial^2 V}{\partial z^2}=0\end{array}\right]$$

Now, integrating Eq. (1) with respect to r, we get,

$$r\frac{\partial V}{\partial r}=c \qquad \text{where } c \text{ is a constant of integration}$$

or,
$$\frac{\partial V}{\partial r}=\frac{c}{r}$$

Integrating again with respect to r, we get

$$V = c \ln r + D \tag{2}$$

where D is another constant of integration C and D both are arbitrary.

Now, applying appropriate boundary conditions (namely when $r = a$, $V = V_A$ and when $r = b$, $V = V_B = 0$) in Eq. (2), we have

Eq. (2)

$$V_A = c \ln a + D$$

$$V_B = c \ln b + D$$

$\therefore$
$$V_A - V_B = c \ln a - c \ln b$$

or,
$$V_A = c(\ln a - \ln b) \qquad [\because V_B = 0]$$

or,
$$V_A = c \ln\frac{a}{b} \Rightarrow c = \frac{V_A}{\ln\left(\dfrac{a}{b}\right)}$$

Again
$$V_B = c \ln b + D$$

or,
$$D + c \ln b = 0 \qquad [\because V_B = 0]$$

or,
$$D = -c \ln b$$

or, $$D = -\frac{V_A}{\ln\left(\frac{a}{b}\right)} \cdot \ln b$$

From Eq. (2), we get

$$V = \frac{V_A}{\ln\left(\frac{a}{b}\right)} \cdot \ln r - \frac{V_A}{\ln\left(\frac{a}{b}\right)} \times \ln b$$

or, $$V = \frac{V_A}{\ln\left(\frac{a}{b}\right)} (\ln r - \ln b)$$

or, $$V = V_A\left(\frac{\ln r/b}{\ln a/b}\right) \qquad (3)$$

Equation (3) is the solution of Laplace's equation in cylindrical coordinate, which gives the potential in the space between the cylinders of the cylindrical capacitor.

4. Write down the lagrangian of a simple harmonic oscillator moving in one dimension. Calculate the generalized momentum. Write down Lagrange's equation of motion. Find out the hamiltonian of this system. Write down Hamilton's equations for this system. 1 + 1 + 1 + 1 + 1

Ans. The lagrangian of a simple harmonic oscillator is given by

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

where $$T = \frac{1}{2}m\dot{x}^2 \text{ and } V = -\int_0^x F dx = \frac{1}{2}kx^2$$

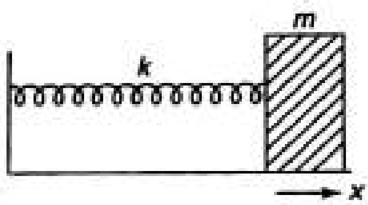


Fig. 2

The generalized momentum is given by

$$p_x = \frac{\partial T}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2\right)$$

$\Rightarrow$ $$p_x = m\dot{x}$$

The lagrange's equation of motion is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

or, $$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right)\right) - \frac{\partial}{\partial x}\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right)$$

or, $$\frac{d}{dt}(m\dot{x}) + kx = 0$$

or, $$m\ddot{x} + kx = 0$$

The hamiltonian is given by

$$H = p_x\dot{x} - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \qquad \left[\because H = \sum_i p_i\dot{q}_i - L\right]$$

or, $$H = m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

or, $$H = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{p_x^2}{2m} + \frac{1}{2}kx^2$$

The Hamilton's equations of motion given by

$$\dot{x} = \frac{\partial H}{\partial p_x} \Rightarrow \dot{x} = \frac{\partial}{\partial p_x}\left(\frac{p_x^2}{2m} + \frac{1}{2}kx^2\right)$$

or $$\dot{x} = \frac{p_x}{m}$$

and $$\dot{p}_x = -\frac{\partial H}{\partial x} \Rightarrow \dot{p}_x = -\frac{\partial}{\partial x}\left(\frac{p_x^2}{2m} + \frac{1}{2}kx^2\right) \Rightarrow \dot{p}_x = -kx$$

5. In how many ways 2 indistinguishable particles can be distributed in three distinct non-degenerate states, if the particles obey (i) *F-D* statistics, (ii) *B-E* statistics?
Write down the expression of occupation probability of photon in the frequency interval ν to $\nu + d\nu$. (2 + 2) + 1

Ans. Refer to Example 8.1.

The answer of the last part of the question is as follows:

The probability of a boson to occupy an energy state (ε) at absolute temperature T is given by

$$f(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/(kT)} - 1}$$

As in the present case the bosons are photons, the energy level corresponding to frequency ν is given by

$$\varepsilon = h\nu$$

and the energy level corresponding to frequency $\nu + d\nu$ is given by

$$h(\nu + d\nu) = \varepsilon + d\varepsilon$$

$\therefore$ the required occupation probability is given by

$$p(\varepsilon) = f(\varepsilon)\, d\varepsilon$$

or, $$p(\varepsilon) = \frac{d\epsilon}{e^{(\varepsilon-\mu)/(kT)} - 1}$$

or, $$p(\varepsilon) = \frac{h d\nu}{e^{(h\nu-\mu)/(kT)} - 1}$$

Group C

Long Answer Questions

1. (a) Prove that for any closed curve $\int_s \bar{B}\cdot d\bar{s} = 0$ where $\bar{B}$ is the magnetic field. 3

Ans. The differential form of Gauss' law in magnetostatics is given by

$$\bar{\nabla}\cdot\bar{B} = 0 \quad (1)$$

where $\bar{B}$ is the magnetic field.

The magnetic flux ϕ_B through a closed surface is given by

$$\phi_B = \int_s \bar{B}\cdot d\bar{s}$$

Now, by applying Gauss' divergence theorem we get,

$$\int_s \bar{B}\cdot d\bar{s} = \int_v (\bar{\nabla}\cdot\bar{B})dv$$

or, $$\int_s \bar{B}\cdot d\bar{s} = 0 \quad \text{[by Eq. (1)]}$$

(b) Given $\vec{v} = \vec{w}\times\vec{r}$, where $\vec{r}$ is the position vector and $\vec{w}$ is a constant angular velocity vector, then find out $\vec{\nabla}\times\vec{v}$. 4

Ans. Refer to Section 1.11.

(c) Use the expression of gradient in the spherical coordinate system to find the normal to the surface $r \sin\theta = 1$. 3

Ans. The equation of the surface is given by

$$r\sin\theta = 1$$

Let ψ represent the surface.

$\therefore$ $$\psi = r\sin\theta - 1 = 0$$

In spherical polar coordinate system

$$\bar{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$\therefore$ the normal to the surface is given by

$$\bar{\nabla}\psi = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\psi$$

or, $$\bar{\nabla}\psi = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)(r\sin\theta - 1)$$

or, $$\bar{\nabla}\psi = \hat{r}\sin\theta + \hat{\theta}\cos\theta$$

$\therefore$ the normal to the surface is

$$\hat{n} = \hat{r}\sin\theta + \hat{\theta}\cos\theta \quad [\because |\hat{r}\sin\theta + \hat{\theta}\cos\theta| = 1]$$

(d) If $\bar{A}$ is a constant vector, show that $\bar{\nabla}(\bar{r}\cdot\bar{A})=\bar{A}$, where $\bar{r}$ is the position vector. 3

Ans. As $\bar{A}$ is a constant vector, we can write

$$\bar{A}=\hat{i}A_1+\hat{j}A_2+\hat{k}A_3$$

where A_1, A_2 and A_3 are three constants.

$$\therefore \quad \bar{r}\cdot\bar{A}=(\hat{i}x+\hat{j}y+\hat{k}z)\cdot(\hat{i}A_1+\hat{j}A_2+\hat{k}A_3)$$

$$\text{or,}\quad \bar{r}\cdot\bar{A}=A_1x+A_2y+A_3z$$

$$\text{So,}\quad \bar{\nabla}(\bar{r}\cdot\bar{A})=\bar{\nabla}(A_1x+A_2y+A_3z)$$

$$\text{or,}\quad \bar{\nabla}(\bar{r}\cdot\bar{A})=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(A_1x+A_2y+A_3z)$$

$$\text{or,}\quad \bar{\nabla}(\bar{r}\cdot\bar{A})=\hat{i}A_1+\hat{j}A_2+\hat{k}A_3$$

$$\text{or,}\quad \bar{\nabla}(\bar{r}\cdot\bar{A})=\bar{A}$$ (proved)

(e) If the electric field is given by $E=\frac{1}{\epsilon_0}(x\hat{i}+y\hat{j}-2z\hat{k})$, then find the charge density. 2

Ans. The differential form of Gauss' law is given by

$$\bar{\nabla}\cdot\bar{E}=\frac{\rho}{\varepsilon_0}$$

$$\text{or,}\quad \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\frac{1}{\varepsilon_0}(x\hat{i}+y\hat{j}-2z\hat{k})=\frac{\rho}{\varepsilon_0}$$

$$\text{or,}\quad \frac{1}{\varepsilon_0}(1+1-2)=\frac{1}{\varepsilon_0}\rho$$

$$\text{or,}\quad \rho=0$$

2. (a) Define polarization vector and displacement vector. Rewrite differential form of Coulomb's law in terms of displacement vector. 1 + 1 + 2

Ans. Refer to Sections 3.3.4 and 3.4.

The answer of the last part of the question is as follows:

Coulomb's law is given by

$$\bar{F}=\frac{1}{4\pi\varepsilon_0}\frac{q_0q}{r^2}\hat{r}$$

where $\bar{F}$ is the electric force between the two changes q and q_0 and r is the distance between them.

$$\bar{F}=q_0\bar{E}=\frac{1}{4\pi\varepsilon_0}\frac{q_0q}{r^2}\hat{r}$$

Considering q as field generating charge and q_0 as test charge.

$$\therefore \quad \bar{E}=\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

or, $$\bar{\nabla}\cdot\bar{E}=\frac{q}{4\pi\varepsilon_0}\bar{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)$$

or, $$\bar{\nabla}\cdot(\varepsilon_0\bar{E})=\frac{q}{4\pi}\bar{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)$$

or, $$\bar{\nabla}\cdot\bar{D}=\frac{q}{4\pi}\bar{\nabla}\cdot\left(\frac{\bar{r}}{r^3}\right) \quad (1)$$

$[\because\ \bar{D}=\varepsilon_0\bar{E}]$

So, Eq. (1) is the differential form of Coulomb's law in terms of the displacement vector $\bar{D}$.

(b) Define electronic polarizibility. Discuss how monatomic gases can be polarized. 1 + 1

Ans. Refer to Sections 3.5 and 3.7.

(c) State Gauss' law in electrostatics and derive Poisson's equation from that. 2 + 2

Ans. Refer to Sections 2.12 and 2.13.

(d) Starting from the principle of charge conservation establish the equation of continuity. 5

Ans. Refer to Section 4.4.

3. (a) Find magnetic field at a point (1, 1, 1) if vector potential at that position is $\vec{A}=(10x^2+y^2=z^2)\hat{j}$. 2

Ans. Let $\bar{B}$ and $\bar{A}$ represent respectively the magnetic field and magnetic vector potential.

$\therefore$ $$\bar{B}=\bar{\nabla}\times\bar{A} \quad [\text{Given } \bar{A}=(10x^2+y^2-z^2)\hat{j}]$$

or, $$\bar{B}=\left(i\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times(10x^2+y^2-z^2)\hat{j}$$

or, $$\bar{B}=\hat{k}(20x)+0+(-\hat{i})(-2z)$$

or, $$\bar{B}=2z\hat{i}+20x\hat{k}$$

The value of $\bar{B}$ at the point (1, 1, 1) is given by

$$\bar{B}(1,1,1)=2\hat{i}+20\hat{k}$$

(b) A very long cylinder of radius a carries current I, which is uniformly distributed over the cross-section of the cylinder. If the current flows along the axis of the cylinder then find out the electric field in the conductor, the magnetic field just outside the conductor and the Poynting vector at the surface of the cylinder. You may assume that the conductivity of the material is σ. 2 + 2+ 2

Ans. Refer to Sections 2.12. 3(c), 4.13.1, and 5.15.

(c) Calculate the magnetic field at the center of a circular lamp carrying current I. 5

Ans. Refer to Section 4.8(B).

(d) If $\bar{E}=\bar{E}_0\, e^{i(kx-\omega t)}$ denotes the electric vector of an electromagnetic field in vacuum then find out the magnetic vector. 2

Ans. From the set of Maxwell's equations we know that

$$\bar{\nabla}\times\bar{E}=-\frac{\partial\bar{B}}{\partial t}$$

or, $\overline{B} = -\int \overline{\nabla} \times \overline{E}\, dt$

or, $\overline{B} = -\int \overline{\nabla} \times \overline{E}_0 e^{i(kx-\omega t)} dt$

$[\because \overline{E} = \overline{E}_0 e^{i(kx-\omega t)}]$

or, $\overline{B} = -\overline{\nabla} \times \overline{E}_0 \int e^{i(kx-\omega t)} dt$

or, $\overline{B} = -\overline{\nabla} \times \overline{E}_0 e^{ikx} \int e^{-i\omega t} dt$

or, $\overline{B} = -\overline{\nabla} \times \overline{E}_0 \left(\frac{1}{-iw}\right)(e^{ikx})(e^{-i\omega t})$

or, $\overline{B} = -\frac{i}{w} \overline{\nabla} \times \overline{E}_0 e^{i(kx-\omega t)}$

or, $\overline{B} = -\frac{i}{w} \overline{\nabla} \times \overline{E}$

$[\because \overline{E} = \overline{E}_o e^{i(kx-\omega t)}]$

4. (a) What do you mean by cyclic coordinate? Explain with an example. 2

Ans. If the lagrangian (L) of a system does not contain a generalized coordinate explicitly, then that coordinate is called a cyclic coordinate.

In case of a cyclic coordinate (q_j), we get

$$\frac{\partial L}{\partial q_j} = 0$$

Example: In case of a free particle (i.e. when the potential energy $V = 0$, the coordinate x, y and z are cyclic.

(b) Show that if generalized force for a conservative system is zero then the generalized momentum will be conserved. 3

Ans. The generalized force is given by

$$Q_j = \frac{\partial V}{\partial q_j}$$

As the given system is conservative and the generalized force (Q_j) is zero, we get

$$\frac{\partial V}{\partial q_j} = 0$$

$\therefore$ the potential energy V is not a function of q_j, i.e., $V \neq V(q_j)$

Hence the generalized coordinate q_j is cyclic $[\because T \neq T(q_j)$ and $V \neq V(q_j) \Rightarrow L \neq L(q_j)]$

For such a coordinate the Lagrange's equation $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$ reduces to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)=0$$

or, $$\frac{\partial L}{\partial \dot{q}_j} = \text{constant} \qquad (1)$$

Now, we can write,

$$\frac{\partial L}{\partial \dot{q}_j}=\frac{\partial}{\partial \dot{q}_j}(T-V)=\frac{\partial T}{\partial \dot{q}_j} \qquad [\because V \neq V(q_j)]$$

or, $$\frac{\partial L}{\partial \dot{q}_j} = p_j = \text{constant [by eqn. (1)]}$$

Hence, the generalized momentum p_j is conserved.

(c) The wave function in a one-dimensional potential box with rigid walls is given by

$$\psi_n(x) = \sqrt{\frac{2}{l}}\sin\frac{n\pi x}{l}, \quad |x| \le l,$$
$$= 0 \qquad \text{otherwise}$$

Find the expectation value of $\hat{x}$ and $\hat{p}$. 3 + 3

Ans. The expected value $\hat{x}$ is given by

$$<x> = \int_0^l \psi_n^*(x)\hat{x}\,\psi_n(x)dx$$

or, $$<x> = \int_0^l \sqrt{\frac{2}{l}}\sin\frac{n\pi x}{l}x\sqrt{\frac{2}{l}}\sin\frac{n\pi x}{l}dx$$

or, $$<x> = \frac{2}{l}\int_0^l x\sin^2\frac{n\pi x}{l}dx$$

or, $$<x> = \frac{1}{l}\int_0^l x\left(1-\cos\frac{2n\pi x}{l}\right)dx$$

or, $$<x> = \frac{1}{l}\int_0^l x\,dx - \int_0^l x\cos\frac{2n\pi x}{l}dx$$

or, $$<x> = \frac{l}{2} - 0$$

or, $$<x> = \frac{l}{2}$$

Now $$<p_x> = \int_0^l \psi_n^*(x)\hat{p}_x\psi_n(x)dx$$

[$\because$ the problem is one-dimensional $\hat{p} = \hat{p}_x$]

or, $$<p_x> = \int_0^l \sqrt{\frac{2}{l}} \sin\frac{n\pi x}{l}\left(-i\hbar\frac{\partial}{\partial x}\right)\sqrt{\frac{2}{l}}\sin\frac{n\pi x}{l}dt$$

or, $$<p_x> = -\frac{2i\hbar}{l}\int_0^l \sin\frac{n\pi x}{l}\frac{d}{dx}\left(\sin\frac{n\pi x}{l}\right)dx$$

or, $$<p_x> = -\frac{2i\hbar}{l}\times\frac{n\pi}{l}\int_0^l \sin\frac{n\pi x}{l}\cos\frac{n\pi x}{l}dx$$

or, $$<p_x> = -\frac{i\hbar\pi x}{l^2}\int_0^l \sin\left(\frac{2n\pi x}{l}\right)dx$$

or, $$<p_x> = -\frac{i\hbar n\pi}{l^2}\times\left[-\frac{l}{2n\pi}\cos\left(\frac{2n\pi x}{l}\right)\right]_0^l$$

or, $$<p_x> = \frac{i\hbar}{2l}\left[\cos\frac{2n\pi x}{l}\right]_0^l$$

or, $$<p_x> = \frac{i\hbar}{2l}(1-1)$$

or, $$<p_x> = 0$$

(d) Consider $\varphi = c_1\varphi_1 + c_2\varphi_2$, where φ_1 and φ_2 are orthonormal energy eigenstates of a system corresponding to energy E_1 and E_2 at $t = 0$. If φ is normalized and $c_1 = \frac{1}{\sqrt{3}}$ then what is the value of c_2?

Find the expectation value of E^2. Write the wave function at a subsequent time. 1 + 2 + 1

Ans. $\phi = c_1\phi_1 + c_2\phi_2$

As ϕ is normalized, we can write $\int_{-\infty}^{+\infty} \phi^*\phi dV = 1$

or, $$\int_{-\infty}^{+\infty} (c_1^*\phi_1^* + c_2^*\phi_2^*)\ (c_1\phi_1 + c_2\phi_2)dV = 1$$

or, $$\int_{-\infty}^{+\infty} c_1^*c_1\phi_1^*\phi_1 dV + \int_{-\infty}^{+\infty} c_1^*c_2\phi_1^*\phi_2 dV$$

$$+\int_{-\infty}^{+\infty} c_1c_2^*\phi_1\phi_2^* dV + \int_{-\infty}^{+\infty} c_2^*c_2\phi_2^*\phi_2 dV = 1$$

or, $$c_1^2 + 0 + 0 + c_2^2 = 1$$

or, $$c_1^2 + c_2^2 = 1$$

or, $$\frac{1}{3} + c_2^2 = 1$$

or, $c_2^2 = \frac{2}{3}$

$\therefore$ $c_2 = \sqrt{\frac{2}{3}}$

We know that $\hat{H}\phi = E\phi$

$$\hat{H} \cdot \hat{H}\phi = \hat{H}(E\phi) \Rightarrow \hat{H} \cdot \hat{H}\phi = E^2\phi$$

$\therefore$ $$<E^2> = \int_{-\infty}^{+\infty} (c_1^*\phi_1^* + c_2^*\phi_2^*)\hat{H}^2(c_1\phi_1 + c_2\phi_2)dV$$

or, $$<E^2> = \int_{-\infty}^{+\infty} (c_1^*\phi_1^* + c_2^*\phi_2^*)(E_1^2 c_1\phi_1 + E_2^2 c_2\phi_2)dV$$

or, $$<E^2> = \int_{-\infty}^{+\infty} (E_1^2 c_1^* c_1\phi_1^*\phi_1 + E_2^2 c_1^* c_2\phi_1^*\phi_2 + E_1^2 c_1 c_2^*\phi_1\phi_2^* + E_2^2 c_2^* c_2\phi_2^*\phi_2)dV$$

or, $$<E^2> = E_1^2 \int_{-\infty}^{+\infty} c_1^* c_1\phi_1^*\phi_1 dV + 0 + 0 + E_2^2 \int_{-\infty}^{+\infty} c_2^* c_2\phi_2^*\phi_2 dV$$

or, $$<E^2> = c_1^2 E_1^2 + c_2^2 E_2^2$$

or, $$<E^2> = \frac{E_1^2}{3} + \frac{2E_2^2}{3}$$

or, $$<E^2> = \frac{1}{3}(E_1^2 + 2E_2^2)$$

The wavefunction at a subsequent line t is given by

$$\phi(\overline{r}, t) = \phi(\overline{r})e^{-\frac{iEt}{\hbar}}$$

5. (a) Calculate the total number of particles in a fermionic gas in terms of Fermi level at absolute zero. 6

Ans. Refer to Section 8.13.3.

(b) Calculate the total energy of particles in a fermionic gas at absolute zero.

Ans. The total energy E_t of an electron gas at absolute zero temperature is given by

$$E_t = \int_0^{\varepsilon_F} \varepsilon N(\varepsilon)d\varepsilon = \int_0^{\varepsilon_F} \varepsilon g(\varepsilon)d\varepsilon$$

or, $$E_t = \int_0^{\varepsilon_F} \varepsilon\left(\frac{8\pi V}{h^3}\right)\sqrt{2m\varepsilon}\, md\varepsilon$$

$$\left[\because g(\varepsilon)d\varepsilon = \frac{8\pi V}{h^3}\sqrt{2m\varepsilon}\, md\varepsilon\right]$$

or, $$E_t = \frac{8\sqrt{2}\pi V m^{3/2}}{h^3}\int_0^{\varepsilon_F} \varepsilon^{3/2} d\varepsilon$$

or, $$E_t = \frac{8\sqrt{2}\pi V m^{3/2}}{h^3}\left(\frac{2}{5}\varepsilon_F^{3/2}\right)$$

or, $$E_t = \frac{16\sqrt{2}\pi V m^{3/2}}{3h^3} \times \frac{3}{5}\varepsilon_F^{5/2}$$

or, $$E_t = \frac{8\pi V}{3}\left(\frac{2m\,\varepsilon_F}{h^2}\right)^{3/2} \times \frac{3}{5}\varepsilon_F$$

or, $$E_t = N \times \frac{3}{5}\,\varepsilon_F \qquad \left[\because N = \frac{8\pi V}{3}\left(\frac{2m\,\varepsilon_F}{h^2}\right)^{3/2}\right]$$

or, $$E_t = \frac{3}{5}\,N\,\varepsilon_F$$

(c) Show that both *F-D* statistics and *B-E* statistics approach *MB* statistics at a certain limit. When does that happen? 3

Ans. Bose-Einstein's distribution function is given by

$$f(\varepsilon) = \frac{n}{g} = \frac{1}{e^{(\varepsilon-\mu)/(kT)}-1}$$

Hence $$n_i = \frac{g_i}{e^{(\varepsilon_i-\mu)/(kT)}-1}$$

where $$\beta = \frac{1}{kT} \text{ and } \alpha = -\frac{\mu}{kT}$$

If $\varepsilon_i >> kT$, then $e^{(\varepsilon_i-\mu)/(kT)} >> 1$

$\therefore$ $$n_i = \frac{g_i}{e^{(\varepsilon_i-\mu)/(kT)}} = \frac{g_i}{e^{(\varepsilon_i/(kT)-\mu/(kT))}}$$

or, $$n_i = \frac{g_i}{e^{\varepsilon_i\beta+\alpha}}$$

which is same as MB distribution function.

$\therefore$ if $\varepsilon_i >> kT$, *BE* statistics reduces to MB statistics.

The Fermi–Dirac distribution function is given by

$$f(\varepsilon) = \frac{n}{g} = \frac{1}{e^{(\varepsilon-\varepsilon_f)/(kT)}+1}$$

Hence $$n_i = \frac{g_i}{e^{(\varepsilon_i-\varepsilon_f)/(kT)}+1}$$

where $$\alpha = -\frac{\varepsilon_F}{kT} \text{ and } \beta = \frac{1}{kT}$$

If $\varepsilon_i >> kT$, then $e^{(\varepsilon_i-\varepsilon_f)/(kT)} >> 1$

$$\therefore \quad n_i = \frac{g_i}{e^{\varepsilon_i/(kT) - \varepsilon_f/(kT)}}$$

$$\text{or,} \quad n_i = \frac{g_i}{e^{\beta\varepsilon_i + \alpha}}$$

which is same as MB distribution function.

$\therefore$ if $\varepsilon_i >> kT$, FD statistics reduces to MB statistics.

(d) Two distinguishable particles are distributed in 2 non-degenerate states of energy 0 and ϵ. List all the microstates and find the total energy corresponding to the states with maximum number of microstates. 3

Ans. As the particles are distinguishable, let them be denoted by a and b.

The possible states will be as follows:

Macrostate	$\varepsilon = 0$	$\varepsilon = \varepsilon$	*Total energy*	*Microstates*	
(2, 0)	ab	0	0	1	
(1, 1)	a	b	ε	2	
	b	a	ε		The
(0, 2)	0	ab	2ε	1	The

The macrostate (1, 1) has maximum number of microstates with energy ε.

6. (a) Use separation of variables technique to calculate the eigenfunctions and eigenvalues of a rigid box of side L. Calculate the wavelength of the photon that must be absorbed for a transition from the lowest to the second excited state. 4 + 1

Ans. Refer to Sections 7.8.1 (ii) (3D box).

If the wavelength of the photon is given by λ, then its energy will be given by $E = h\nu$. This energy will be stansferred by the photon to the particle in the box to go to the second excited state from the ground state.

$$\therefore \quad E = E_{221} - E_{111}$$

$$\text{or,} \quad h\nu = \frac{6h^2}{8mL^2} - \frac{3h^2}{8mL^2}$$

$$\text{or,} \quad \frac{hc}{\lambda} = \frac{3h^2}{8mL^2}$$

$$\therefore \quad \lambda = \frac{hc}{3h^2} \times 8\,mL^2$$

$$\text{or,} \quad \lambda = \frac{8mcL^2}{3h} \text{ units}$$

where L is the side of the box and m is the mass of the particle

(b) Starting from Maxwell's equation show that the electric field can be written in terms of a scalar potential and the magnetic vector potential as

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$ 5

Ans. Maxwell's third equation is given by

$$\overline{\nabla} \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \quad (1)$$

But magnetic vector potential is given by

$$\overline{B} = \overline{\nabla} \times \overline{A} \quad (2)$$

where $\overline{A}$ is the magnetic vector potential

From Eq. (1) and Eq. (2), we get

$$\overline{\nabla} \times \overline{E} = -\frac{\partial}{\partial t}(\overline{\nabla} \times \overline{A})$$

or, $$\overline{\nabla} \times \overline{E} = \overline{\nabla} \times \left(-\frac{\partial \overline{A}}{\partial t}\right)$$

or, $$\overline{\nabla} \times \overline{E} = -\overline{\nabla} \times \overline{\nabla}\phi + \overline{\nabla} \times \left(-\frac{\partial \overline{A}}{\partial t}\right)$$

[$\because$ curl of gradient of any scalar point function is zero. i.e., $\overline{\nabla} \times \overline{\nabla}\phi = 0$]

So, $$\overline{\nabla} \times \overline{E} = \overline{\nabla} \times \left(-\overline{\nabla}\phi - \frac{\partial \overline{A}}{\partial t}\right)$$

Hence, $$\overline{E} = -\overline{\nabla}\phi - \frac{\partial \overline{A}}{\partial t} \qquad \text{(proved)}$$

(c) Using Gauss's law in integral form obtain the electric field due to the following charge distribution:

$$\rho = \rho_0\left(1 - \frac{r^2}{a^2}\right), 0 < r \le a$$
$$= 0 \qquad a < r < \alpha$$

at a point outside the distribution as well as at a point inside the distribution. Sketch the field as a function of r. 4 + 1

Ans. Refer to Example 2.17.

Solved WBUT Question Paper (2012)

Group A

Multiple Choice Questions

1. Choose the correct alternatives for any *ten* of the following:

$10 \times 1 = 10$

(i) The dimension of $\mu_0 t_0$ is

(a) $L^{-2} T^{-2}$ (b) $L^{-2} T^{2}$ (c) $L T^{-1}$ (d) $L^{-1} T^{-1}$

Ans. (i) none (Dimension of $\mu_0 t_0$ is $MLTQ^{-2}$)

(ii) The displacement current arises due to

(a) positive charge only (b) negative charge only

(c) time varying electric field (d) magnetic monopole

Ans. (c)

(iii) If the Fermi energy of metal (in 3*d*) at thermal equilibrium is 15 eV, then the average energy of the electron is

(a) 9 eV (b) 10 eV (c) 15 eV (d) 12 eV

Ans. (a)

(iv) Which of the following functions are eigen functions of the operator $\frac{d^2}{dx^2}$?

(a) $\psi\, c \log x$ (b) $\psi\, cx^2$ (c) $\psi\, \frac{c}{x}$ (d) $\psi\, ce^{-mx}$

(where c and m are arbitrary constants)

Ans. (d)

(v) BE statistics is applicable for

(a) ideal gas (b) electron (c) proton (d) photon

Ans. (d)

(vi) The degrees of freedom for a system of N particles with K constraint relation is given by

(a) $N - K$ (b) $N - 3K$ (c) $3N - K$ (d) $3(N - K)$

Ans. (c)

(vii) The number of ways in which 4 identical bosons can be distributed in 3 different energy states is

(a) 15 (b) 6 (c) 144 (d) 24

Ans. (a)

(viii) The equation of continuity essentially represents

(a) conservation of mass (b) conservation of charge

(c) conservation of potential (d) conservation of force

Ans. (b)

(ix) The ignorable co-ordinate corresponding to the motion of a particle under central force is given by

(a) r (b) θ (c) $\dot{r}$ (d) $\dot{\theta}$

Ans. (d)

(x) An electric field in a certain region has the components $E_x = ax - bz$, $E_y = -ay + bz$ and $E_z = b(y - x)$. Then which of the following statements is correct?

(a, b are positive constants)

(a) $\vec{E}$ is an electrostatic field (b) There is free charge in space

(c) $\vec{E}$ is irrotational (d) $\vec{E}$ is solenoidal

Ans. (d)

(xi) The vector potential $\vec{A}$ corresponding to a constant magnetic field $\vec{B}$ along z-axis can be represented by

(a) $-Bz\,\hat{k}$ (b) $\frac{B}{2}(\hat{i}x - \hat{j}y)$ (c) $B(\hat{j}x - \hat{i}y)$ (d) $\frac{B}{2}(\hat{j}x - \hat{i}y)$

Ans. (d)

(xii) Skin depth for a conductor in reference to electromagnetic wave varies

(a) inversely as frequency

(b) directly as frequency

(c) inversely as square root of frequency

(d) directly as square root of frequency

Ans. (c)

(xiii) The expectation value of the position of a particle in a one-dimensional potential box of length $L(V(x) = 0; 0 < x < L, V(x) = \infty$ at $x = \neq L$ is

(a) L (b) $\frac{L}{2}$ (c) $\frac{L}{3}$ (d) $\frac{L}{4}$

Ans. (b)

(xiv) The force experienced by a charged particle in a magnetic field is independent of

(a) velocity of the particle (b) strength of the field

(c) charge of the particle (d) mass of the particle

Ans. (d)

(xv) The electronic polarizability (α_e) of an atom is related to its radius (R) as

(a) $\alpha_e \propto R^3$ (b) $\alpha_e \propto R^2$ (c) $\alpha_e \propto R$ (d) $\alpha_e \propto R^0$

Ans. (a)

Group B

Short Answer Questions

Answer any three of the following. $3 \times 5 = 15$

1. Write down the Maxwell's equations of an electromagnetic field. Hence, obtain the wave equation for electric field in free space. $3 + 2$

Ans. The Maxwell's equations are given by

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \tag{1}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \tag{4}$$

In free space the equations become as follows:

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{5}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{6}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{7}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t}$$

or, $$\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{8}$$

For electric field, we can write

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} \qquad \text{[Eq. 7]}$$

or, $$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

But $$\vec{\nabla} \cdot \vec{E} = 0 \text{ and } \vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$\therefore$ $$-\vec{\nabla}^2 \vec{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

or, $$\nabla^2 \overline{E} = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} \tag{9}$$

Equation (9) is the required equation.

2. State Stoke's theorem in vector calculus. Find the unit vectors perpendicular to $x^2 + y^2 - z^2 = 100$ at the point (1, 2, 3). 2 + 3

Ans. The stoke's theorem states that the line integral of the tangential component of a vector taken around a closed path is equal to the surface integral of the normal component of the curl of the vector taken over any surface having the path as the boundary. Mathematically one can express it as follows:

$$\int_c \overline{F} \cdot d\overline{r} = \iint_s (\overline{\nabla} \times \overline{F}) \cdot d\overline{s}$$

where $\overline{F}$ is any vector.

In order to find a unit normal vector to the surface $x^2 + y^2 - z^2 = 100$ we can write

$$\psi = x^2 + y^2 - z^2 - 100$$

or, $$\overline{\nabla}\psi = \overline{\nabla}(x^2 + y^2 - z^2 - 100)$$

or, $$\overline{\nabla}\psi = 2x\hat{i} + 2y\hat{j} - 2z\hat{k}$$

or, $$\overline{\nabla}\psi|_{1,2,3} = 2\hat{i} + 4\hat{j} - 6\hat{k}$$

So, the unit normal vector is given by

$$\hat{n} = \frac{\overline{\nabla}\psi}{|\overline{\nabla}\psi|} = \frac{2\hat{i} + 4\hat{j} - 6\hat{k}}{\sqrt{2^2 + 4^2 + (-6)^2}}$$

$\therefore$ $$\hat{n} = \pm \frac{\hat{i} + 2\hat{j} - 3\hat{k}}{\sqrt{14}}$$

$\therefore$ $\hat{n}$ is the required unit normal vector.

3. State the Ampere's law of mangnetostatics. Obtain its differential form from the integral one. Apply Ampere's law of magnetostatics to deduce an expression of magnetic field B due to a straight conductor of infinite length carrying current I. 1 + 2 + 2

Ans. Ampere's law of magnetostatics states that the line integral of the magnetic field $\overline{B}$ around any closed path is equal to μ_0 times the net current enclosed by the path.

Mathematically, $$\int_c \overline{B} \cdot d\overline{l} = \mu_0 I \tag{1}$$

we know that

$$I = \mu_0 \iint_S \overline{J} \cdot d\overline{s} \tag{2}$$

where I is the current and $\overline{J}$ is current density.

So, combining Eqs (1) and (2), we get

$$\int_c \overline{B} \cdot dl = \mu_0 \iint_s \overline{J} \cdot d\overline{s}$$

Now applying Stoke's law, we get

i.e., $$\iint_s (\overline{\nabla} \times \overline{B}) \cdot d\overline{s} = \mu_0 \iint_s \overline{J} \cdot d\overline{s}$$

$$\left[\because \int_c \overline{A} \cdot d\overline{l} = \iint_s (\overline{\nabla} \times \overline{A}) \cdot d\overline{s}\right]$$

or, $$\iint_s [\overline{\nabla} \times \overline{B} - \mu_0 \overline{J}] \cdot d\overline{s} = 0$$

Since, the surface element $d\overline{s}$ is arbitrary,

so, $$\overline{\nabla} \times \overline{B} = \mu_0 \overline{J} \tag{3}$$

Equation (3) is the differential form of the Ampere's law.

For the answer of the last part, refer to Section 4.13.

4. (a) Four distinguishable particles each of which can be in one of the energy states ε, 2ε, 4ε and 6ε having total energy 6ε. Find all possible number of distributions of all the particles in the energy states. Write the number of microstates possible and the number of microstates corresponding to each macrostate.

Ans. Let the particles be a, b, c and d. The particles are distinguishable. The possible microstates will be

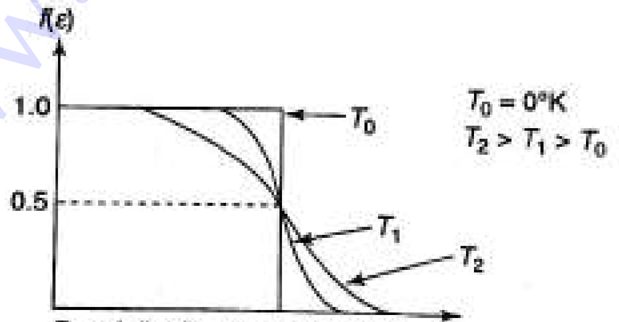
Macrostate	ε	2ε	4ε	6ε	*Total energy*	*Microstates*
(2, 2, 0, 0)	*ab*	*cd*	0	0	6ε	
	ac	*bd*	0	0	6ε	
	ad	*bc*	0	0	6ε	
	bc	*ad*	0	0	6ε	6
	bd	*ac*	0	0	6ε	
	cd	*ab*	0	0	6ε	

So, in the distribution, number of macrostate is one and the number of microstates is 6.

(b) Sketch the nature of Fermi-Dirac distribution function at $T = 0$ and $T > 0\ K$ in the same graph.

3 + 2

Ans. The following is the required graph:



Fermi distribution at zero and non-zero temperature

Fig. 1

5. Show that if the lagrangian does not depend on time, then the hamiltonian is a constant of motion. Write down the hamiltonian and obtain the equation of motion for a simple harmonic oscillator. 3 + 1 + 1

Ans. The Hamilton's equations of motion are given by

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \tag{1}$$

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} \tag{2}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \tag{3}$$

where $j = 1, 2, 3, \ldots, 2f$

Now in this case lagrangian L is independent of t. So from Eqn. (3), we find that the hamiltonian H is also independent of time t.

So, $$\frac{dH}{dt} = 0$$

Hence, H is constant of motion.

A simple harmonic oscillator has one-dimensional motion.

So, its kinetic and potential energies are given by $T = \frac{1}{2} m\dot{x}^2$

and $$V = \frac{1}{2} kx^2$$

$\therefore$ the lagrangian L is given by

$$L = T - V = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$$

$\therefore$ $$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

or, $$\dot{x} = \frac{p_x}{m}$$

The hamiltonian is given by

$$H = \sum_i p_i \dot{q}_i - L = p_x \dot{x} - \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2$$

or, $$H = m\dot{x}^2 - \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2 = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2$$

or, $$H = \frac{1}{2} \frac{p_x^2}{m} + \frac{1}{2} kx^2$$

$\therefore$ $$\frac{\partial H}{\partial p_x} = \frac{p_x}{m} \text{ and } \frac{\partial H}{\partial x} = kx$$

Now, the Hamilton's equations are given by,

$$\dot{x} = \frac{\partial H}{\partial p_x} \Rightarrow \dot{x} = \frac{p_x}{m}$$

and $$\dot{p}_x = -\frac{\partial H}{\partial x} \Rightarrow \dot{p}_x = -kx \Rightarrow m\ddot{x} + kx = 0$$

or $$\frac{md^2x}{dt^2} + kx = 0$$

6. (a) Find the value of $[\hat{L}_x, \hat{z}]$.

Ans. Refer to Example 7.11.

(b) Show that the eigenvalues of a hermitian operation are real. Give an example of a hermitian operator in quantum mechanics. 2 + 2 + 1

Ans. Refer to Example 7.25. Example of hermitian operators are hamiltonian operator $\left(\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(\hat{r})\right)$ and position operator $(\hat{r})$.

Group C

Long Answer Questions

Answer any *three* of the following. $3 \times 15 = 45$

1. (a) Distinguish between holonomic and non-holonomic constraints.

Ans. If the relation of the constraints can be expressed as an equation and if they are independent of velocity then such constraints are called holonomic constraints. On the other hand, if the constraints, cannot be expressed in equational form and they are dependent on velocity then such constraints are called non-holonomic constraints.

(b) Write down the equation of constraint, specify the nature of constraint and calculate the degrees of freedom in each case:

(i) A particle constrained to move on the surface of a sphere

(ii) A simple pendulum with a fixed support. 3 + 3

Ans. (i) It is a holonomic as well as scleronomic constraint.

The equation is $x^2 + y^2 + z^2 = a^2$

Degrees of freedom $f = 3 \times 1 - 1 = 2$

(ii) The constraint is holonomic and scleronomic. The equation is given by

$$x^2 + y^2 = l^2$$

Degrees of freedom $f = 2 \times 1 - 1 = 1$

(c) Show that if a generalized coordinate is cyclic in lagrangian, then the corresponding generalized momentum will be conserved. 3

Ans. Refer to Section 6.2.6.

(d) Find the equation of motion using Hamilton's canonical equation for a system comprising masses m_1 and m_2 connected by a massless string of length L through a frictionless pulley such that $m_1 < m_2$. 4

Ans. Figure 2 shows the device consisting of two masses m_1 and m_2 suspended over a frictionless pulley P of radius 'a' and connected by an inextensible string of length l.

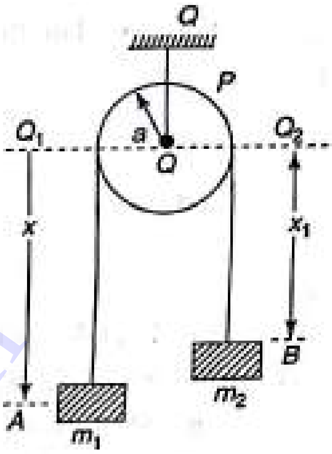


Fig. 2

Let $Q_1A = x$ and $Q_2B = x_1$

From the above diagram we can write

$$l = x + x_1 + \pi a \qquad [\because a \text{ is the radius}]$$

or,
$$x = l - x_1 - \pi a$$

$\therefore$
$$\dot{x} = -\dot{x}_1$$

Now the kinetic energy of the masses m_1 and m_2 are given by

$$T_1 = \frac{1}{2}m_1\dot{x}^2 \quad \text{and} \quad T_2 = \frac{1}{2}m_2\dot{x}_1^2$$

$\therefore$ the total kinetic energy is given by

$$T = T_1 + T_2 = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2x_1^2$$

or,
$$T = \frac{1}{2}(m_1 + m_2)\dot{x}^2 \qquad [\because \dot{x}_1 = -\dot{x}]$$

With reference to the horizontal line passing through Q, the potential energy of m_1 is given by

$$V_1 = -m_1gx$$

and that of m_2 is given by

$$V_2 = -m_2g(l - \pi a - x)$$

$\therefore$
$$V = V_1 + V_2 = -m_1gx - m_2g(l - \pi a - x)$$

$\therefore$ the lagrangian L is given by

$$L = T - V$$

or,
$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_1gx + m_2g(l - \pi a - x)$$

or,
$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 - m_2)gx + m_2(l - \pi a)g$$

$\therefore$
$$p_x = \frac{\partial L}{\partial \dot{x}} \quad \Rightarrow \quad p_x = (m_1 + m_2)\dot{x}$$

The hamiltonian is given by

$$H = \sum_j p_j\dot{q}_j - L$$

or,
$$H = p_x\dot{x} - L$$

or,
$$H = (m_1 + m_2)\dot{x}^2 - \frac{1}{2}(m_1 + m_2)\dot{x}^2 - (m_1 - m_2)gx - m_2(l - \pi a)g$$

or,
$$H = \frac{1}{2}(m_1 + m_2)\dot{x}^2 - (m_1 - m_2)gx - m_2g(l - \pi a)$$

or,
$$H = \frac{p_x^2}{2(m_1 + m_2)} - (m_1 - m_2)gx - m_2g(l - \pi a)$$

The Hamilton's equations are given by

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \quad \text{and} \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

From $\dot{q}_j = \frac{\partial H}{\partial p_j}$ we get

$$\dot{x} = \frac{p_x}{(m_1 + m_2)} \tag{1}$$

And from $\dot{p}_j = -\frac{\partial H}{\partial q_j}$, we get

or, $$(m_1 + m_2)\ddot{x} = -[-(m_1 - m_2)g]$$

or, $$\ddot{x} = \frac{m_1 - m_2}{m_1 + m_2} g \tag{2}$$

2. (a) What do you mean by μ and Γ-phase space? Find the area in the phase space of a one-dimensional harmonic oscillator of mass m whose total energy is E. 2 + 2

Ans. For N particle system, the mechanical state of the whole system can be determined completely in terms of $3N$ position coordinates $(q_1, q_2, ..., q_N)$ and $3N$ momentum coordinates $p_1, p_2, ... p_N$. The six-N dimensional space is called the phase space or Γ-space of the system. A point in the Γ-space represents a state of the entire system. For monoatomic gas it is also called molecular phase space or μ-space.

For a particle which is executing simple harmonic motion, the total energy E is given by

$$E = E_p + E_k$$

or, $$E = \frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2$$

where x is the displacement from the equilibrium position, k is force constant, m is the mass and the velocity $v = \dot{x}$.

∴ $$2E = k(x^2) + m(\dot{x}^2)$$

or, $$kx^2 + m\dot{x}^2 = 2E$$

or, $$\frac{kx^2}{2E} + \frac{m\dot{x}^2}{2E} = 1$$

or, $$\frac{x^2}{\left(\frac{2E}{k}\right)} + \frac{\dot{x}^2}{\left(\frac{2E}{m}\right)} = 1$$

or, $$\frac{x^2}{(2E/k)} + \frac{\dot{x}^2}{(2E/m)} = 1$$

or, $$\frac{x^2}{(2E/k)} + \frac{p_x^2}{(2mE)} = 1 \quad (1)$$

$[\because p_x = m\dot{x}]$

Equation (1) is an equation of an ellipse. So, the phase-space is two-dimensional and x–p_x plane forms a curve known as phase trajectory. The phase trajectory of a simple harmonic oscillator is an ellipse with semi-major axis $a = \sqrt{2E/k}$ and the semiminor axis $b = \sqrt{2mE}$.

So, the area in phase space (A) is given by

$$A = \pi ab$$

or, $$A = \pi\sqrt{\frac{2E}{k}} \times \sqrt{2mE}$$

or, $$A = 2\pi E\sqrt{\frac{m}{k}}$$

(b) Derive Planck's radiation law from *BE* statistics. State clearly the assumptions made in the theory. 3 + 2

Ans. Refer to the Section 8.14.

(c) What is Fermi energy? Calculate the degeneracy function $g(E)$ as a function of energy E for an ideal Fermi gas. 1 + 3

Ans. The Fermi energy ε_F is defined as the energy of the highest occupied level at absolute zero.

For calculation of the degeneracy function $g(E)$ refer to Section 8.12.

(d) Evaluate the temperature at which there is one per cent probability that a state with energy of 0.6 eV above the Fermi energy will be occupied by an electron. 2

Ans. Fermi–Dirac distribution function is given by

$$f(\varepsilon) = \frac{1}{e^{(\varepsilon-\varepsilon_f)/(kT)} + 1}$$

This function is the required probability function through which we can calculate the temperature.

Here $\Delta\varepsilon = \varepsilon - \varepsilon_f = 0.6$ eV.

or, $\Delta\varepsilon = 0.6 \times 1.6 \times 10^{-19}$ J

or, $\Delta\varepsilon = 0.96 \times 10^{-19}$ J

The probability is given by $p = \frac{1}{100}$

$\therefore$ $$p = f(\varepsilon) = \frac{1}{e^{\Delta\varepsilon/(kT)} + 1}$$

or, $$\frac{1}{100} = \frac{1}{e^{\Delta\varepsilon/(kT)} + 1}$$

or, $$e^{\Delta\varepsilon/(kT)} + 1 = 100$$

or, $$e^{\Delta\varepsilon/(kT)} = 99$$

or, $$\frac{\Delta\varepsilon}{kT} = \ln 99$$

or, $$T = \frac{\Delta\varepsilon}{k \ln 99} \Rightarrow T = \frac{0.96 \times 10^{-19}}{1.38 \times 10^{-23} \times 4.595}$$

Hence, $T = 1513.93$ K

3. (a) Give the physical interpretation of the wave function $\psi(x)$. 2

Ans. Refer to Section 7.4.

3. (b) Show that for a stationary state given by the wave function $\psi(x, t) = \psi(x)e^{-\frac{iE_n t}{\hbar}}$, the expectation value of energy is equal to the energy eigenvalue. 3

Ans. The given wave function is

$$\psi(x, t) = \psi(x)\, e^{-i\frac{E_n t}{\hbar}}$$

The energy operator is given by

$$\hat{E}_n = i\hbar\frac{\partial}{\partial t}$$

$\therefore$ $$\hat{E}_n\psi(x, t) = i\hbar\frac{\partial}{\partial t}\left\{\psi(x)e^{-\frac{iE_n t}{\hbar}}\right\}$$

or, $$\hat{E}_n\psi(x, t) = i\hbar\,\psi(x)\frac{\partial}{\partial t}\left(e^{-\frac{iE_n t}{\hbar}}\right)$$

or, $$\hat{E}_n\psi(x, t) = i\hbar\,\psi(x)\left(-\frac{iE_n}{\hbar}\right)e^{-\frac{iE_n t}{\hbar}}$$

or, $$\hat{E}_n\,\psi(x, t) = E_n\,\psi(x)\,e^{-\frac{iE_n t}{\hbar}}$$

or, $$\hat{E}_n\,\psi(x, t) = E_n\,\psi(x, t)$$

$\therefore$ E_n is the eigenvalue of $\hat{E}_n$.

The expectation value E_n is given by

$$<E_n> = \int_{-\infty}^{+\infty} \psi^*(x, t)\,\hat{E}_n\,\psi(x, t)dx$$

or, $$<E_n> = \int_{-\infty}^{+\infty} \psi^*(x)e^{\frac{iE_n t}{\hbar}}\, i\hbar\frac{\partial}{\partial t}\left\{\psi(x)e^{-\frac{iE_n t}{\hbar}}\right\}dx$$

or, $$<E_n> = \int_{-\infty}^{+\infty} \psi^*(x)(i\hbar)\left(-\frac{iE_n}{\hbar}\right)\psi(x)dx$$

or, $$<E_n> = E_n\int_{-\infty}^{+\infty} \psi^*(x)\,\psi(x)dx$$

or, $$<E_n> = E_n(1) \qquad \left[\because \int_{-\infty}^{+\infty}\psi^*\psi\,dx = 1\right]$$

or, $$<E_n> = E_n$$

$\therefore$ the expectation value $<E_n>$ is equal to the eigenvalue E_n.

(c) A particle is in a cubic box with infinitely hard walls whose edges are L units long. The wave function is given by

$$\psi(x, y, z) = A \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right)\sin\left(\frac{n\pi z}{L}\right)$$

Find the value of A. Find the ground state and first excited energy eigenvalues. Are they non-degenerate? Explain. $2+2+2$

Ans. The normalized wave function is given by

$$\psi(x, y, z) = A \sin\left(\frac{n_1\pi x}{L}\right) \sin\left(\frac{n_2\pi x}{L}\right) \times \sin\left(\frac{n_3\pi x}{L}\right)$$

As the wave function is normalized we have

$$\int_0^x\int_0^y\int_0^z \psi^* \psi \, dxdydz = 1$$

or, $$A^2 \int_0^x\int_0^y\int_0^z \sin^2\left(\frac{n_1\pi x}{L}\right)\sin^2\left(\frac{n_2\pi y}{L}\right)\sin^2\left(\frac{n_3\pi z}{L}\right) \times dxdydz = 1$$

or, $$A^2\left(\frac{L}{2}\right)\left(\frac{L}{2}\right)\left(\frac{L}{2}\right) = 1$$

or, $$A = \frac{2\sqrt{2}}{\sqrt{L^3}} \Rightarrow A = \left(\frac{2}{L}\right)^{3/2}$$

$\therefore$ the wave function can be written as

$$\psi(x, y, z) = \frac{2\sqrt{2}}{\sqrt{L^3}} \sin\left(\frac{n_1\pi x}{L}\right) \sin\left(\frac{n_2\pi y}{L}\right) \times \sin\left(\frac{n_3\pi z}{L}\right)$$

Now, $$\frac{n_1^2\pi^2}{L^2} + \frac{n_2^2\pi^2}{L^2} + \frac{n_3^2\pi^2}{L^2} = \frac{2mE}{\hbar^2}$$

[For details, refer to article 7.8.1(b)]

or, $$E = \frac{\hbar^2\pi^2}{2m}\left[\frac{n_1^2}{L^2} + \frac{n_2^2}{L^2} + \frac{n_3^2}{L^2}\right]$$

As E is dependent on n_1, n_2 and n_3, we can write

$$E_{n_1n_2n_3} = \frac{h^2}{8m}\left[\left(\frac{n_1}{L}\right)^2 + \left(\frac{n_2}{L}\right)^2 + \left(\frac{n_3}{L}\right)^2\right]$$

For ground state energy, $n_1 = n_2 = n_3 = 1$

$\therefore \qquad E_{111} = \dfrac{3h^2}{8mL^2}$

For first excited (n_1, n_2, n_3) will be either (1, 1, 2) or (1, 2, 1) or (2, 1, 1)

$\therefore \qquad E_{112} = E_{121} = E_{211} = \dfrac{6h^2}{8mL^2}$

The ground state eigenvalue E_{111} is non-degenerate. But first excited state is three fold degenerated.

(d) Show that the function $\psi(x) = Cx\, e^{-\frac{x^2}{2}}$ is an eigenfucntion of the operator $\left(x^2 - \dfrac{d^2}{dx^2}\right)$. Find the corresponding eigenvalue. 3 + 1

Ans. The given wave function is given by $\psi(x) = cx\, e^{-x^2/2}$ and the operator given is

$$\left(x^2 - \frac{d^2}{dx^2}\right)$$

Now, let us operate on the wave function by the operator.

$$\therefore \quad \left(x^2 - \frac{d^2}{dx^2}\right)\psi(x)$$

$$= \left(x^2 - \frac{d^2}{dx^2}\right)(cx\, e^{-x^2/2})$$

$$= cx^3 e^{-x^2/2} - \frac{d^2}{dx^2}(cx e^{-x^2/2})$$

$$= cx^3 e^{-x^2/2} - \frac{d}{dx}\left(ce^{-x^2/2} - cx^2 e^{-\frac{x^2}{2}}\right)$$

$$= (\cancel{cx^2 e^{-x^2/2}} + cx e^{-x^2/2} + 2cx e^{-x^2/2} - \cancel{cx^3 e^{-x^2/2}})$$

$$= 3c\, x\, e^{-x^2/2} = 3\psi(x)$$

i.e., $$\left(x^2 - \frac{d^2}{dx^2}\right)\psi(x) = 3\psi(x)$$

Hence, ψ is an eigen function of $\left(x^2 - \dfrac{d^2}{dx^2}\right)$ and the eigenvalue is 3.

4. (a) If $\hat{a}$ and $\hat{b}$ are unit vectors and θ is the angle between them, show that $2\sin\dfrac{\theta}{2} = |\hat{a} - \hat{b}|$. 2

Ans. $$|\hat{a} - \hat{b}| = \sqrt{(\hat{a} - \hat{b})^2}$$

or, $$|\hat{a} - \hat{b}| = \sqrt{(\hat{a} - \hat{b})\cdot(\hat{a} - \hat{b})}$$

or, $$|\hat{a} - \hat{b}| = \sqrt{\hat{a}\cdot\hat{a} - \hat{a}\cdot\hat{b} - \hat{b}\cdot\hat{a} + \hat{b}\cdot\hat{b}}$$

or, $$|\hat{a} - \hat{b}| = \sqrt{1 - \cos\theta - \cos\theta + 1}$$

or, $$|\hat{a}-\hat{b}| = \sqrt{2-2\cos\theta}$$

or, $$\sqrt{2}\sqrt{1-\cos\theta} = |\hat{a}-\hat{b}|$$

or, $$\sqrt{2}\times\sqrt{2\sin^2\frac{\theta}{2}} = |\hat{a}-\hat{b}|$$

or, $$2\sin\frac{\theta}{2} = |\hat{a}-\hat{b}|$$

(b) Show that the electric field is always perpendicular to the equipotential surface. 3

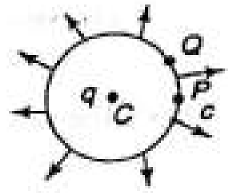


Fig. 3

Ans. Let us consider a sphere. And let charge q be placed at the center of it (vide Fig. 3):

As the surface of the sphere is equidistant from the centre C, the surface of the sphere is an equipotential surface. Now, if a charge q_1 be shifted from the point P to the point Q on the surface, the work done is given by

$$W = \int_P^Q q_1\bar{E}\cdot d\bar{l}$$

$$= \int_P^Q q_1 E\, dl\cos\theta$$

or, $$W = \int_P^Q q_1 E\cos\theta\, dl \qquad (1)$$

The potential energy possessed by the charge q_1 at point P is $E_p = Vq_1$ and the same at Q is $E'_p = Vq_1$.

$\therefore$ $$\Delta E_p = E_p - E'_p = Vq_1 - Vq_1 = 0$$

$\therefore$ the work done on the charge q_1 while moving it from P to Q is zero.

i.e., $$W = \Delta E_p = 0 \qquad (2)$$

or, $$\int_P^Q q_1 E\cos\theta\, dl = 0 \qquad \text{[from Eq. (1) and Eq. (2)]} \qquad (3)$$

The equation (3) can be true only if

$\theta = 90°$, i.e., if $\bar{E}$ and $d\bar{l}$ are normal to each other.

As $d\bar{l}$ is tangent to the surface, $\bar{E}$ is normal to the surface.

In this case $d\bar{l}$ is arbitrary, so the electric field is always perpendicular to the equipotential surface.

(c) Show that the vector

$\vec{A} = (6xy+z^3)\hat{i} + (3x^2-z)\hat{j} + (3xz^2-y)\hat{k}$ is irrotational. Find ϕ such that $\vec{A} = \vec{\nabla}\phi$. 2 + 3

Ans. A vector can be irrotational only if the curl of it is zero.

Now, $$\vec{\nabla}\times\vec{A} = \vec{\nabla}\times[(6xy+z^3)\hat{i} + (3x^2-z)\hat{j} + (3xz^2-y)\hat{k}]$$

or, $$\bar{\nabla}\times\bar{A}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 6xy+z^3 & 3x^2-z & 3xz^2-y\end{vmatrix}$$

or, $$\bar{\nabla}\times\bar{A}=\hat{i}\left\{\frac{\partial}{\partial y}(3xz^2-y)-\frac{\partial}{\partial z}(3x^2-z)\right\}$$
$$-\hat{j}\left\{\frac{\partial}{\partial x}(3xz^2-y)-\frac{\partial}{\partial z}(6xy+z^3)\right\}$$
$$+\hat{k}\left\{\frac{\partial}{\partial x}(3x^2-z)-\frac{\partial}{\partial y}(6xy+z^2)\right\}$$

or, $\bar{\nabla}\times\bar{A}=\hat{i}(-1+1)-\hat{j}(3z^2-3z^2)+\hat{k}(6x-6x)$

or, $\bar{\nabla}\times\bar{A}=0$

∴ $\bar{A}$ is an irrotational vector.

Now, $\bar{A}=\bar{\nabla}\phi$

or, $\hat{i}A_x+\hat{j}A_y+\hat{k}A_z=\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}$

∴ $\frac{\partial\phi}{\partial x}=A_x,\quad \frac{\partial\phi}{\partial y}=A_y,\quad \frac{\partial\phi}{\partial z}=A_z$

Hence, $\phi=\int A_x\,dx$ (1)

$\phi=\int A_y\,dy$ (2)

$\phi=\int A_z\,dz$ (3)

By using Eq. (1), we get

$\phi=\int(6xy+z^3)dx$

or, $\phi=3x^2y+z^3x$ (4)

By using Eq. (2), we get

$\phi=\int(3x^2-z)dy$

or, $\phi=3x^2y-yz$ (5)

By using Eq. (3), we get

$\phi=\int(3xz^2-y)dz$

or, $\phi=xz^3-yz$ (6)

Now, observing Eqs (4), (5) and (6), we can express ϕ as follows:

$$\phi=3x^2y-yz+z^3x+c$$

where c is an arbitrary constant.

(d) Calculate the work done in moving a particle in a force field given by $\bar{F}=3xy\hat{i}-4z\hat{j}+8y\hat{k}$ along the curve $x=t^2+1, y=t^2, z=t^3$ from $t=0$ to $t=1$. 3

Ans. $\overline{F} = 3xy\hat{i} - 4z\hat{j} + 8y\hat{k}$

$\therefore \quad F_x = 3xy,\ F_y = -4z,\ F_z = 8y$

The work done is given by

$$W = \int \overline{F} \cdot d\overline{r}$$

$$= \int (F_x dx + F_y dy + F_z dz)$$

or, $$W = \int (3xydx - 4zdy + 8ydz)$$

$$= \int_0^1 3(t^2+1) \cdot t^2 \cdot 2t\,dt - \int 4t^2 \cdot 2t\,dt + \int 8t^2 \cdot 3t^2\,dt$$

or, $$W = \int_0^1 \{6(t^5+t^3) - 4t^4 + 24t^4\}dt$$

or, $$W = \int_0^1 (6t^5 + 6t^3 + 16t^4)dt$$

or, $$W = \int_0^1 (6t^5 + 16t^4 + 6t^3)dt$$

or, $$W = \left[t^6 + \frac{16}{5}t^5 + \frac{6}{4}t^4\right]_0^1$$

or, $$W = \left[1 + \frac{16}{5} + \frac{3}{2}\right]$$

or, $$W = \frac{57}{10} = 5.7$$

(e) Show $\int_S (ax\hat{i} + by\hat{j} + cz\hat{k}) \cdot d\overline{S} = \frac{4\pi}{3}(a+b+c)$, where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$. 2

Ans. The equation of the sphere is given by

$$x^2 + y^2 + z^2 = 1$$

$\therefore$ its radius $r = 1$

$$I = \int_S (ax\hat{i} + by\hat{j} + cz\hat{k}) \cdot d\overline{s}$$

$$= \int_V \{\overline{\nabla} \cdot (ax\hat{i} + by\hat{j} + cz\hat{k})\}dV$$

[by using Gauss' divergence theorem]

$$= \int_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (ax\hat{i} + by\hat{j} + cz\hat{k})dV$$

or, $$I = \int_V (a+b+c)dV$$

or, $$I = (a+b+c)\int_V dV$$

or, $$I = (a+b+c)\left.\frac{4\pi r^3}{3}\right|_{r=1}$$ [∵ the volume is spherical]

or, $$I = \frac{4\pi}{3}(a+b+c)$$

∴ $$\int_S (ax\hat{i} + by\hat{j} + cz\hat{k})\cdot d\bar{S} = \frac{4\pi}{3}(a+b+c)$$

5. (a) A spherically symmetric charge distribution is given by $\rho(r) = \rho_0\left(1-\frac{r^2}{a^2}\right)$ for $0 \le r \le a$, where ρ_0 is a constant and $\rho(r) = 0$, for $r > a$.
Calculate the
(i) total charge
(ii) the electric field intensity $\bar{E}$ and potential V both inside $(r < a)$ and outside $(r > a)$ regimes.
1 + 2 + 2

Ans. (a) (i) $\rho(r) = \rho_0\left(1-\frac{r^2}{a^2}\right)$ for $0 \le r \le a$

$= 0$ for $r > a$

The total charge Q is given by

$$Q = \int \rho(r)dV$$

or, $$Q = \int_0^a \rho_0\left(1-\frac{r^2}{a^2}\right)dV$$

or, $$Q = \rho_0 \int_{r=0}^{a}\int_{\theta=\phi}^{\pi}\int_{\phi=0}^{2\pi}\left(1-\frac{r^2}{a^2}\right) r^2 \sin\theta\, d\theta d\phi dr$$

or, $$Q = 2\pi\rho_0 \int_{r=0}^{a}\int_{\theta=0}^{\pi}\left(1-\frac{r^2}{a^2}\right) r^2 \sin\theta\, d\theta dr$$

or, $$Q = 2\pi\rho_0 = \int_0^a\left(1-\frac{r^2}{a^2}\right) r^2 dr\,[-\cos\theta]_0^{\pi}$$

or, $$Q = 4\pi\rho_0 \int_0^a\left(r^2-\frac{r^4}{a^2}\right) dr$$

or, $$Q = 4\pi\rho_0\left[\frac{r^3}{3}-\frac{r^5}{5a^2}\right]_0^a$$

or, $$Q = 4\pi\rho_0\left(\frac{a^3}{3}-\frac{a^3}{5}\right)$$

or, $$Q = 4\pi\rho_0\left(\frac{2a^3}{15}\right)$$

∴ $$Q = \frac{8\pi}{15}\rho_0 a^3$$

(ii) Case I

Fig. 4

When $a < r < \infty$ charge $Q = \dfrac{8\pi}{15}\rho_0 a^3$

From Gauss' law, we have

$$\int_s \bar{E}\cdot d\bar{s} = \frac{Q}{\varepsilon_0}$$

or, $$E(4\pi r^2) = \frac{1}{\varepsilon_0}\times\frac{8\pi}{15}\rho_0 a^3$$

or, $$E = \frac{1}{4\pi\varepsilon_0 r^2}\times\frac{8\pi}{15}\rho_0 a^3$$

or, $$E = \frac{2\rho_0 a^3}{15\varepsilon_0 r^2}$$

Again, $$E = -\frac{dv}{dr}$$

or, $$\int dv = -\int E\, dr$$

or, $$V = -\int \frac{2\rho_0 a^3}{15\varepsilon_0 r^2}dr$$

$\therefore$ $$V = \frac{2\rho_0 a^3}{15\varepsilon_0 r}$$

Case II

When $r = a$

$\therefore$ $$E = \frac{2\rho_0 a^3}{15\varepsilon_0 a^2} \Rightarrow E = \frac{2\rho_0 a}{15\varepsilon_0}$$

And $$V = \frac{2\rho_0 a^3}{15\varepsilon_0 a} \Rightarrow V = \frac{2\rho_0 a^2}{15\varepsilon_0}$$

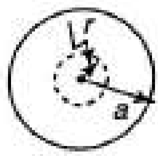


Fig. 5

Case III

$$\rho(r) = \rho_0\left(1 - \frac{r^2}{a^2}\right)$$

The charge within the gaussian surface of radius r is given by

$$Q' = \int_v \rho dv$$

or, $$Q' = \rho_0 \int_{r=0}^{r}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\left(1-\frac{r^2}{a^2}\right) r^2 \sin\theta\, d\theta d\phi dr$$

or, $$Q' = 2\pi\rho_0 \int_{r=0}^{r}\int_{\theta=0}^{\pi}(1 - r^2/a^2)r^2(\sin\theta d\theta)dr$$

or, $$Q' = 2\pi\rho_0 \int_0^r \left(1-\frac{r^2}{a^2}\right)[-\cos\theta]_0^\pi r^2 dr$$

or, $$Q' = 4\pi\rho_0 \int_0^r \left(r^2-\frac{r^4}{a^2}\right) dr$$

$$Q' = 4\pi\rho_0 \left[\frac{r^3}{3}-\frac{r^5}{5a^2}\right]_0^r \Rightarrow Q' = 4\pi\rho_0 \left(\frac{r^3}{3}-\frac{r^5}{5a^2}\right)$$

Again from Gauss' law, we can write

$$\int_s \bar{E}\cdot d\bar{s} = \frac{Q'}{\varepsilon_0}$$

or, $$E(4\pi r^2) = \frac{4\pi\rho_0}{\varepsilon_0}\left(\frac{r^3}{3}-\frac{r^5}{5a^2}\right)$$

$\therefore$ $$E = \frac{\rho_0}{\varepsilon_0}\left(\frac{r}{3}-\frac{r^3}{5a^2}\right)$$

or, $$V = -\int E\, dr$$

or, $$V = -\int \frac{\rho_0}{\varepsilon_0}\left(\frac{r}{3}-\frac{r^3}{5a^2}\right)dr$$

$\therefore$ $$V = -\frac{\rho_0}{\varepsilon_0}\left(\frac{r^2}{6}-\frac{r^4}{20a^3}\right)$$

(b) If ϕ is a scalar potential associated with the electric field $\bar{E}$ and $\bar{A}$ is the vector potential associated with the magnetic induction $\bar{B}$, show that they must satisfy the equation $\nabla^2\phi + \frac{\partial}{\partial t}(\bar{\nabla}\cdot\bar{A}) = -\frac{\rho}{\varepsilon_0}$. 5

Ans. Since the potential ϕ is related to the electric field $\bar{E}$, we can write

$$\bar{E} = -\bar{\nabla}\phi \qquad (1)$$

Since $\bar{A}$ is the vector potential related to the magnetic field $\bar{B}$, we get,

$$\bar{B} = \bar{\nabla}\times\bar{A} \qquad (2)$$

Given, $$\nabla^2\phi + \frac{\partial}{\partial t}(\bar{\nabla}\cdot\bar{A}) = -\rho/\varepsilon_0$$

or, $$\bar{\nabla}\cdot(\bar{\nabla}\phi) + \bar{\nabla}\cdot\left(\frac{\partial\bar{A}}{\partial t}\right) = -\rho/\varepsilon_0$$

or, $$\bar{\nabla}\cdot\left\{\bar{\nabla}\phi + \frac{\partial\bar{A}}{\partial t}\right\} = -\frac{\rho_0}{\varepsilon_0}$$

or, $$\bar{\nabla}\cdot\left(-\bar{E} + \frac{\partial\bar{A}}{\partial t}\right) = -\frac{\rho}{\varepsilon_0} \qquad \text{[by Eq. (1)]}$$

or, $$\bar{\nabla}\cdot\left(-\bar{E}+\frac{\partial\bar{A}}{\partial t}\right)=-\bar{\nabla}\cdot\bar{E}$$

or, $$\bar{\nabla}\cdot\left(-\bar{E}+\frac{\partial\bar{A}}{\partial t}+\bar{E}\right)=0$$

or, $$\bar{\nabla}\cdot\frac{\partial\bar{A}}{\partial t}=0$$

or, $$\frac{\partial}{\partial t}(\bar{\nabla}\cdot\bar{A})=0$$

or, $$0=0 \qquad [\because \bar{\nabla}\cdot\bar{A}=0]$$

∴ L.S. = R.S. Hence, $\nabla^2\phi+\frac{\partial}{\partial t}(\bar{\nabla}\cdot\bar{A})=-\frac{\rho}{\epsilon_0}$

(c) The intensity of sunlight reaching the earth's surface is about 1300 W/m^2. Calculate the strength of the electric and magnetic fields of the incoming sunlight. 3

Ans. The intensity of the sunlight reaching the earth surface is given by

$$I = 1300 \text{ W/m}^2$$

∴ the pointing vector P is given

$$P = I = 1300 \text{ W/m}^2$$

We know that,

$$\bar{P}=\bar{E}\times\bar{H}$$

or, $$|\bar{P}|=|\bar{E}\times\bar{H}|$$

or, $$P = E\,H\sin 90° \qquad [\because \bar{E}\perp\bar{H}]$$

or, $$P = EH$$

∴ $$EH = 1300 \text{ W/m}^2$$

Again, $$\frac{E}{H}=\sqrt{\frac{\mu_0}{\epsilon_0}}=\sqrt{\frac{4\pi\times10^{-7}}{8.85\times10^{-12}}}$$

or, $$\frac{E}{H}=376.82$$

$$EH\times\frac{E}{H}=1300\times376.82$$

or, $$E^2=489866 \Rightarrow E=699.90 \text{ Volt/m}$$

Again, $$EH\Big/\left(\frac{E}{H}\right)=1300/376.82$$

or, $$EH\times\frac{H}{E}=\frac{1300}{376.82}$$

or, $$H^2=345\Rightarrow H=1.86 \text{ amp/m}$$

(d) N charged spherical water drops, each having a radius r and charge q, coalesce into a single big drop. What is the potential of the big spherical drop? 2

Ans. Let the volume of a single water drop be represented by

$$V_s = \frac{4}{3}\pi r^2 \qquad [r = \text{radius}]$$

So, the volume of the coalesced drop of water is given

$$V_b = N\,V_s = \frac{4}{3}\pi R^3 \qquad [R = \text{radius}]$$

or, $$N\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi R^3$$

or, $$R^3 = Nr^3$$

$\therefore$ $$R = \sqrt[3]{N}\,r$$

The charge of the coalesced drop Q is given by

$$Q = Nq$$

Now from Gauss' law, we get

Fig. 6

$$\int_s \bar{E}\cdot d\bar{s} = \frac{Q}{\varepsilon_0}$$

or, $$(E)(4\pi r'^2) = \frac{Nq}{\varepsilon_0}$$

or, $$E = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Nq}{r'^2} \qquad (1)$$

But $$E = -\frac{dV'}{dr'} \qquad (2) \quad [V' = \text{potential}]$$

From Eqs (1) and (2), we get

$$-\frac{dV'}{dr'} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Nq}{r'^2}$$

or, $$\int dV' = \frac{1}{4\pi\varepsilon_0}\int (Nq)\frac{dr'}{r'^2}$$

or, $$V' = \frac{1}{4\pi\varepsilon_0}\frac{Nq}{r'}$$

$\therefore$ the potential, $V' = \dfrac{Nq}{4\pi\varepsilon_0 r'}$

The potential on the surface of the coalesced drop is given by

$$V'(R) = \frac{Nq}{4\pi\varepsilon_0 R}$$

or, $$V'(R) = \frac{Nq}{4\pi\varepsilon_0 N^{1/3} r}$$

or, $$V'(R) = \frac{N^{2/3} q}{4\pi\varepsilon_0 r}$$

Solved WBUT Question Paper (Dec. 2012)

Group A

Multiple Choice Questions

1. Choose the correct alternatives for any *ten* of the following:

$10 \times 1 = 10$

(i) The angle between the vectors $\hat{i}+\hat{j}$ and $\hat{i}-\hat{j}$ is

(a) 90° (b) 60° (c) 30° (d) 0°

Ans. (a)

(ii) The value of $\int_C \vec{r}\cdot d\vec{l}$ on any arbitrary closed curve C is

(a) 3 (b) 1 (c) –1 (d) 0

Ans. (d)

(iii) In free space Poisson's equation reduces to

(a) $\nabla^2 \upsilon = 0$ (b) $\nabla^2 \upsilon = \frac{\rho}{\varepsilon_0}$ (c) $\nabla^2 \upsilon = -\frac{\rho}{\varepsilon_0}$ (d) $\nabla^2 \upsilon = \infty$

Ans. (a)

(iv) The continuity equation for steady current is

(a) $\vec{\nabla}\cdot\vec{j}+\frac{\partial\rho}{\partial t}=0$ (b) $\vec{\nabla}\cdot\vec{j}=0$ (c) $\frac{\partial\rho}{\partial t}=0$ (d) $\vec{\nabla}\times\vec{j}=0$

Ans. (b)

(v) The electrostatic potential energy of a system of two charges q_1 and q_2 separated by distance r is

(a) $\frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r^2}$ (b) $\frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r}$ (c) $\frac{1}{4\pi\varepsilon_0}\frac{q_1^2q_2}{r^2}$ (d) $\frac{\varepsilon_0}{4\pi}\frac{q_1q_2}{r}$

Ans. (b)

(vi) If $\vec{B}=\vec{\nabla}\times\vec{A}$, $\vec{B}$ and $\vec{A}$ are any vectors then

(a) $\vec{\nabla}\cdot\vec{B}=0$ (b) $\vec{\nabla}\cdot\vec{B}=1$ (c) $\vec{\nabla}\cdot\vec{B}=-1$ (d) $\vec{\nabla}\cdot\vec{B}=|\vec{A}|$

Ans. (a)

(vii) The energy associated with a magnetic field $\vec{H}$ is

(a) $\frac{1}{2}H^2$ (b) $\mu_0 H^2$ (c) $\frac{1}{2}\mu_0 H^2$ (d) $\frac{1}{2\mu_0}H^2$

Ans. (c)

(viii) Skin depth for a conductor in reference to electromagnetic wave varies

(a) inversely as frequency (b) directly as frequency

(c) inversely as square root of frequency (d) directly as square root of frequency

Ans. (c)

(ix) Schroendinger time independent wave equation is

(a) $\hat{H}\,\psi = E\psi^2$ (b) $\hat{H}\,\psi^2 = E\psi^2$ (c) $\hat{H}\frac{1}{\psi} = E\frac{1}{\psi}$ (d) $\hat{H}\,\psi = E\psi$

Ans. (d)

(x) The ground state energy of a particle moving in a one-dimensional potential box is given in terms of length L of the box by

(a) $\frac{2\hbar^2}{8mL^2}$ (b) $\frac{\hbar^2}{8mL^2}$ (c) $\frac{h^2}{8mL^2}$ (d) 0

Ans. (c)

(xi) The communication bracket $[\hat{p}_y, \hat{y}]$ is equal to

(a) $i\hbar$ (b) $-i\hbar$ (c) $i\hbar^2$ (d) $i/\hbar$

Ans. (b)

(xii) The electric dipole moment of a particle (atom or molecule) per unit polarizing electric field is termed as

(a) polarization (b) polarizability

(c) net dipole moment (d) susceptibility

Ans. (b)

(xiii) A system is called strongly degenerate if

(a) $\frac{N_i}{g_i} = 1$ (b) $\frac{N_i}{g_i} >> 1$ (c) $\frac{N_i}{g_i} << 1$ (d) $g_i = 1$

Ans. (b)

(xiv) A coin and a six faced disc are thrown simultaneously. The probability that the coin shows head and the disc shows 2 is

(a) $\frac{1}{4}$ (b) $\frac{1}{12}$ (c) $\frac{1}{6}$ (d) $\frac{1}{8}$

Ans. (b)

(xv) The average energy of an electron in a metal at 0 K is

(a) E_F (b) $\frac{E_F}{2}$ (c) $\frac{3E_F}{5}$ (d) $\frac{5E_F}{3}$

where E_F is the Fermi energy.

Ans. (c)

Group B

Short Answer Questions

Answer any *three* of the following $3 \times 5 = 15$

1. (a) Use Gauss' law to calculate the electric field between infinite extend parallel plate capacitor carrying charge density σ and mutual separation d.

Ans. Take from solved WBUT Q. Paper (2006), Gr-C, Long Answer Questions 1(b). (page S1.3)

1. (b) Verify whether the potential function $V(x, y)$ satisfy Laplace's equation or not. Find also the charge density.

Ans. Refer to Section 2.13.

2. (a) A superposed state of a quantum particle is given by, $\psi(x) = C_1\psi_1(x) + C_2\psi_2(x)$ where $\psi_1(x)$ and $\psi_2(x)$ are orthonormal states. Show that $C_1^2 + C_2^2 = 1$.

Ans. From condition of normalization of the wave function $\int_{-\infty}^{+\infty} \psi^*(x)\psi(x)\,dx = 1$

Here $\psi(x) = C_1\psi_1(x) + C_2\psi_2(x)$

So, $$\int_{-\infty}^{+\infty} (C_1\psi_1 + C_2\psi_2)^*(C_1\psi_1 + C_2\psi_2)dx = 1$$

or, $$\int_{-\infty}^{+\infty} C_1^*C_1\,\psi_1^*\psi_1\,dx + \int_{-\infty}^{+\infty} C_2^*C_2\,\psi_2^*\psi_2\,dx + \int_{-\infty}^{+\infty} (C_1^*C_2\,\psi_1^*\psi_2 + C_1C_2^*\,\psi_2^*\psi_1)dx = 1$$

or, $$\int_{-\infty}^{+\infty} |C_1|^2\,\psi_1^*\psi_1\,dx + \int_{-\infty}^{+\infty} |C_2|^2\,\psi_2^*\psi_2 dx + 0 = 1$$

$$\left[\int_{-\infty}^{+\infty} \psi_1^*\psi_2\,dx \;\&\; \int_{-\infty}^{+\infty} \psi_2^*\psi_1\,dx \text{ are equals to zero}\right]$$

or, $$C_1^2 + C_2^2 = 1$$

2. (b) Show that $\psi(x) = Ae^{2ix}$ & $\psi(x) = Ae^{-2ix}$ are degenerate wave functions. Find out the energy eigenvalue. $3 + 2$

Ans. Here $\psi_1(x) = Ae^{2ix}$ and $\psi_2(x) = Ae^{-2ix}$ are linearly independent wave functions belonging to the same energy state. So they are degenerate wave functions.

For $\psi_1(x)$ $$-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} = E_1\psi_1 \quad \text{or,} \quad -\frac{\hbar^2}{2m}(2i)^2 Ae^{2ix} = E_1 Ae^{2ix}$$

or, $$E_1 = \frac{2\hbar^2}{m}$$

Similarly for $\psi_2(x)$, $E_2 = \dfrac{2\hbar^2}{m}$ So, $E_1 = E_2$

If ψ is the linear combination of the degenerate wave functions, $\psi = C_1\psi_1 + C_2\psi_2$ is also an eigen function belonging to the same energy eigen value.

We may get

$-\dfrac{\hbar^2}{2m}\dfrac{d^2}{dx^2}(C_1\psi_1 + C_2\psi_2) = E_1(C_1\psi_1 + C_2\psi_2)$ which shows that the linear combination $C_1\psi_1 + C_2\psi_2$ is also an eigenfunction belonging to the same eigen value E_1 $\left(\text{or } \dfrac{2\hbar^2}{m}\right)$

3. (a) Define displacement current.

Ans. Refer to Section 5.6.

(b) Find the displacement current within a parallel plate capacitor in series with a resistor which carries current I. Area of the capacitor plates are A and the dielectric is vacuum. 2 + 3

Ans. The discharge current i is equal to rate of change of charge on plates, i.e.,

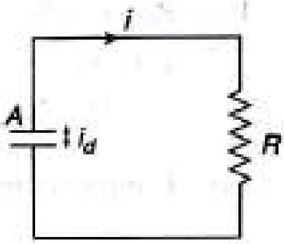


Fig. 1

$$i = \frac{\partial Q}{\partial t}$$

Again $Q = CV$ or, $i = C\dfrac{\partial V}{\partial t}$

For parallel plate capacitor $C = \varepsilon_o \dfrac{A}{d}$, or, $i = \varepsilon_o \dfrac{A}{d}\dfrac{\partial V}{\partial t}$

If E is the electric field $E = \dfrac{V}{d}$, so $i = \varepsilon_o A\dfrac{\partial E}{\partial t}$ or, $\dfrac{i}{A} = \varepsilon_o \dfrac{\partial E}{\partial t}$, or $J_d = \dfrac{\partial D}{\partial t}$

where J_d is the displacement current density and $D = \varepsilon_o E$, known as electric displacement.

4. (a) A proton moves with a velocity 0.6 c parallel to a straight current 1A at a distance of 10 cm from the current. What is the magnetic force on the proton?

Ans. The magnetic field due to the current I is

$$B = \frac{\mu_o}{4\pi}\frac{2I}{a} = 10^{-7}\frac{2\times 1}{0.1}$$
$$= 2\times 10^{-6} \text{ Tesla}$$

The magnetic force on the proton is

$$F = q\,v\,B \qquad \text{[Since } v \text{ and } B \text{ are mutually perpendicular]}$$
$$= 1.6\times 10^{-19}\times 0.6\times 3\times 10^{8}\times 2\times 10^{-6}$$
$$= 5.76\times 10^{-17}\text{ N}$$

The direction of the force is towards the wire or attractive.

(b) State the law you used in solving the above problem. 4 + 1

Ans. State Biot–Savart's Law.

5. (a) Construct the Hamiltonian and the Hamilton's equation of motion of a simple pendulum.

Ans. Refer to Worked out problems, Example 6.12 of Chapter 6.

(b) Give a comparative study of *BE* and *FD* statistics. 3 + 2

Ans. Refer to Section 8.15.

Group C

Long Answer Type Questions

Answer any *three* of the following. $3 \times 15 = 45$

1. (a) Define degree of freedom, generalized coordinates and hamiltonian of a system.

Ans. Refer to Sections 6.1.4, 6.1.5 and 6.3.

(b) Derive the lagrangian of a simple pendulum and obtain the equation of motion.

Ans. Refer to Worked out Problems, Example 6.3 of Chapter 6.

(c) Deduce the D'Alembert's principle from the principle of virtual work.

Ans. Refer to Section 6.2.3.

(d) Prove that for a conservative system, the hamiltonian represents the total energy of the system.

$3 + 4 + 4 + 4$

Ans. For a conservative system, V does not depend on $\dot{q}_j$, i.e., $\frac{\partial V}{\partial \dot{q}_j} = 0$

Again we know $H = \sum_j p_j \dot{q}_j - L = \sum \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L \quad \left[\because p_j = \frac{\partial L}{\partial \dot{q}_j}\right]$

$$= \sum_j \dot{q}_j \left[\frac{\partial}{\partial \dot{q}_j}(T - V)\right] - L$$

$$= \sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} - L \quad \left[\because \frac{\partial V}{\partial \dot{q}_j} = 0\right]$$

$$= \sum_j \dot{q}_j \frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2} m \dot{q}_j^2\right) - L \quad \left[\because T = \frac{1}{2} m \dot{q}_j^2\right]$$

$$= \sum_j m \dot{q}_j^2 - L = 2T - L$$

$$= 2T - (T - V) = T + V$$

$= E$, total energy of the system.

2. (a) What are the basic postulates of quantum mechanics?

Ans. Refer to Section 7.6.

(b) Apply Schrödinger equation for one dimension to a particle in one-dimensional box and find its total energy and normalized wave function. Plot the probability densities and explain.

Ans. Refer to Section 7.8.1.

(c) Find the energy difference between the ground state and first excited state of an electron moving in a one-dimensional potential box of length 1Å.

$3 + 8 + 4$

Ans. The energy of an electron is

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2} \quad \text{Here } L = 1\text{Å} = 10^{-10}\text{ m}$$

$$h = 6.6 \times 10^{-34}\ \text{Js}$$
$$m = 9.1 \times 10^{-31}\ \text{kg}$$

For ground state $n = 1$

$$E_1 = \frac{h^2}{8mL^2} = \frac{(6.6\times10^{-34})^2}{8\times9.1\times10^{-31}\times10^{-10}} = 6\times10^{-18}\ \text{J} = 38\ \text{eV}$$

$E_n = 38\,n^2$, for 1st excited state $n = 2$, $E_2 = 38 \times 4 = 152$ eV

So the energy difference between first excited state and ground state is (152 –38) or 114 eV.

3. (a) A fluid motion is given by $V = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$. Show that the motion is irrotational.

Ans. $\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$

For irrotational motion, $\vec{\nabla}\times\vec{V} = 0$

Here
$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(x+y) - \frac{\partial}{\partial z}(x+z)\right] - \hat{j}\left[\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial z}(y+z)\right] + \hat{k}\left[\frac{\partial}{\partial x}(z+x) - \frac{\partial}{\partial y}(y+z)\right]$$

$$= \hat{i}(1-1) - \hat{j}(1-1) + \hat{k}(1-1)$$
$$= 0$$

(b) Solve Laplace's equation to find the potential at a distance r from the axis of an infinitely long conducting cylinder of radius a charged with a surface charge density σ. Take the potential of the cylinder to be zero.

Ans. The variation of potential along radial direction

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0 \qquad \text{[From Laplace equation]}$$

or, $V = C\log_e r + D$ [where C & D are constants of integration]

Now applying boundary condition

at $r = a$, $V = 0$ so, $D = -C\log_e a$

$$\therefore \quad V = C\log_e\left(\frac{r}{a}\right)$$

From Coulomb's law $E = -\left(\frac{\partial V}{\partial r}\right)_{r=a} = \frac{\sigma}{\varepsilon_o}$

or, $$\left[\frac{c}{r/a}\times\frac{1}{a}\right]_{r=a} = -\frac{\sigma}{\varepsilon_o}$$

or, $$C = -\frac{a\sigma}{\varepsilon_o}$$

$\therefore$ $$V = \frac{a\sigma}{\varepsilon_o}\log_e\left(\frac{a}{r}\right)$$

(c) The electrostatic potential in free space is given by $\Phi = \alpha - \beta(x^2 + y^2) - \gamma \ln \sqrt{(x^2 + y^2)}$ where α, β and γ are constants. Find the charge density in the region. $5 + 5 + 5$

Ans. From Poisson's equation $\nabla^2\varphi = \frac{\rho}{\varepsilon_o}$

Here $$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

Now $$\frac{\partial\varphi}{\partial x} = \frac{\partial}{\partial x}\left[\alpha - \beta(x^2 + y^2) - \gamma\log_e\sqrt{x^2 + y^2}\right]$$

$$= -2\beta x - \gamma\frac{1}{\sqrt{x^2 + y^2}} \times \frac{1}{2}(x^2 + y^2)^{-1/2} \times 2x$$

$$= -2\beta x - \frac{\gamma x}{x^2 + y^2}$$

$$\frac{\partial^2\varphi}{\partial x^2} = \frac{\partial}{\partial x}\left[-2\beta x - \frac{\gamma x}{x^2 + y^2}\right] = -2\beta - \frac{\gamma}{x^2 + y^2} + \frac{2\gamma x^2}{(x^2 + y^2)^2}$$

Similarly, $$\frac{\partial^2\varphi}{\partial y^2} = -2\beta - \frac{\gamma}{x^2 + y^2} + \frac{2\gamma y^2}{(x^2 + y^2)^2}$$

and $$\frac{\partial^2\varphi}{\partial z^2} = 0$$

$\therefore$ $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -4\beta - \frac{2\gamma}{x^2 + y^2} + \frac{2\gamma(x^2 + y^2)}{(x^2 + y^2)^2}$$

or, $$\nabla^2\varphi = -4\beta = -\frac{\rho}{\varepsilon_o}$$

$\therefore$ charge density $\rho = 4\beta\varepsilon_o$

4. (a) The magnetic field in a region of free space is given by $B = B_0 \cos(\omega t - kz)\,\hat{y}$.

(i) What is the displacement current if there is no free charge?

(ii) Obtain an equation for E, neglecting the integration constant.

(iii) Verify that the differential form of Faraday's Law of electromagnetic induction is satisfied by E and B.

Ans. (i) Displacement current $\vec{J}_d = \frac{1}{\mu_0}(\vec{\nabla} \times \vec{B})$

or, $$\vec{J}_d = \frac{1}{\mu_o}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & B_0\cos(\omega t - kz) & 0 \end{vmatrix}$$

or, $$\vec{J}_d = \hat{x}\frac{B_o}{\mu_o}(-k)\sin(\omega t - kz)$$

Ans. (ii) Again $\vec{J}_d = \varepsilon_o \frac{\partial \vec{E}}{\partial t}$ or, $\vec{E} = \frac{1}{\varepsilon_o}\int \vec{J}_d\, dt$

or, $$\vec{E} = \frac{1}{\varepsilon_o}\frac{\hat{x}}{\mu_o}(-k)B_o\int \sin(\omega t - kz)\, dt$$

$$= \frac{k}{\omega}\frac{\hat{x}\, B_o}{\mu_o\varepsilon_o}\cos(\omega t - kz)$$

We know that $c^2 = \frac{1}{\mu_o\varepsilon_o}$ and $\frac{\omega}{k} = c$. So,

$$\vec{E} = c\;\hat{x}\;B_o\cos(\omega t - kz)$$

Ans. (iii) From Faraday law of electromagnetic induction

$$\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t}$$

Here $$\vec{\nabla}\times\vec{E} = CB_o\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos(\omega t - kz) & 0 & 0 \end{vmatrix} = CB_o\hat{y}\,k\sin(\omega t - kz)$$

and $$-\frac{\partial B}{\partial t} = -\frac{\partial}{\partial t}[B_o\cos(\omega t - kz)\,\hat{y}] = B_o\omega\;\hat{y}\;\sin(\omega t - kz)$$

$$= B_o ck\;\hat{y}\;\sin(\omega t - kz)$$

Hence Faraday's law of electromagnetic induction is satisfied.

4. (b) Find out Hamilton's equations of motion for a system comprising masses m_1 and m_2 connected by a massless string of length L through a frictionless pulley such that $m_1 > m_2$. (3 + 3 + 3) + 6

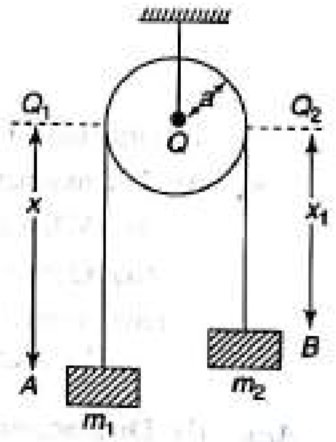


Fig. 2

Ans. From Fig. 2, Let $Q_1A = x$ and $Q_2B = x_1$

From the diagram we can write

$$L = x + x_1 + \pi a \qquad [a \text{ is the radius}]$$

$$x_1 = L - \pi a - x$$

or, $$\dot{x}_1 = -\dot{x} \quad \text{and} \quad \ddot{x}_1 = -\ddot{x}$$

The kinetic energy of m_2 is

$$T_2 = \frac{1}{2}m_2\dot{x}_1^2 = \frac{1}{2}m_2\dot{x}^2$$

Total kinetic energy $T = T_1 + T_2$

$$= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2$$

$$= \frac{1}{2}(m_1 + m_2)\dot{x}^2$$

With reference to the horizontal line passing through Q, the potential energy of m_1 is given by

$$V_1 = -m_1 g x$$

and that of m_2 is $V_2 = -m_2 g(L - \pi a - x)$

$\therefore$ total potential energy $V = V_1 + V_2 = -m_1 g x + m_2 g\ (L - \pi a - x)$

The lagrangian of the system is given by

$$L = T - V = \frac{1}{2}(m_1 + m_2)\,\dot{x}^2 \ + m_1 g x + m_2 g(L - \pi a - x)$$

The generalized momentum p_x can be written as

$$p_x = \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\,\dot{x} \quad \text{or,} \quad \dot{x} = \frac{p_x}{(m_1 + m_2)}$$

The hamiltonian of the system is given by

$$H = \dot{x}\,p_x - L \ = (m_1 + m_2)\,\dot{x}^2 \ - \frac{1}{2}(m_1 + m_2)\,\dot{x}^2 \ - m_1 g x - m_2 g(L - \pi a - x)$$

Again $\qquad \dot{x} = \dfrac{\partial H}{\partial p_x} = \dfrac{p_x}{(m_1 + m_2)}$

and $\qquad \dot{p}_x = -\dfrac{\partial H}{\partial x} = m_1 g - m_2 g = (m_1 - m_2)g$

$\ddot{x} = \dfrac{\dot{p}_x}{m_1 + m_2} = \left(\dfrac{m_1 - m_2}{m_1 + m_2}\right)g$, which is the hamiltonian equations of motion.

5. (a) State Gauss' law of electrostatics.

Ans. Refer to Section 2.12.

(b) Derive an expression for the electric field between two infinite extent parallel plate capacitors carrying charge density σ and mutual separation d. Draw the necessary diagram.

Ans. Same as Q. 1(a) of Short Answer Type Questions.

(c) Stating from the definition of current density derive the equation of continuity in current electricity. What is the condition of steady current?

Ans. Refer to Section 4.4.

(d) Two parallel wires carry equal current of 10A along with the same direction and are separated by a distance of 2.0 cm. Find the magnetic field at a point which is two cm away from any of these wires.

2 + 4 + (4 + 1) + 4

Ans. The magnetic field at a point which is 2 cm away from any of these wires is

$$B = \frac{\mu_o}{4\pi}\left(\frac{2i}{d}\right) + \frac{\mu_o}{4\pi}\left(\frac{2i}{2d}\right) \qquad \left[\text{Here } d = 2\text{ cm} = 0.02\text{ m}, I = 10\text{A and } \frac{\mu_0}{4\pi} = 10^{-7}\right]$$

$$= 10^{-7} \times \left(\frac{2i}{d}\right)\left(1 + \frac{1}{2}\right)$$

$$= 3 \times 10^{-7} \times \frac{10}{0.02} \text{ Tesla} = 1.5 \times 10^{-4} \text{ Tesla}$$

6. (a) Deduce density of states of free electrons having energy between E and $E + dE$ in the phase space.

Ans. Refer to Section 8.12.

(b) Write down the postulates of B-E statistics and write down the B-E distribution function explaining the symbols. At what condition BE-statistics will yield classical statistics?

Ans. Refer to Sections 8.9.1 & 8.9.2.

(c) A system has non-degenerate single particle states with 0, 1, 2, 3 energy units. Three particles are to be distributed in these states such that the total energy of the system is 3 units. Find the number of microstates if the particles obey

(i) MB statistics

(ii) BE statistics

(iii) FD statistics. 4 + (3 + 1 + 1) + (2 + 2 + 2)

Ans. Refer to Worked out examples, Example 8.3 of Chapter 8.