Module 2

1. Find $D^n\{\sin(ax+b)\}$. Show that if $u = \sin ax + \cos ax$, $D^n u = a^n\{1 + (-1)^n \sin 2ax\}^{\frac{1}{2}}$, where $D \equiv \frac{d}{dx}$.
2. If $y = x^{2n}$ where n is a positive integer, show that $y_n = 2^n\{1.3.5 \ldots (2n-1)\}x^n$
3. $If\ y = \sin(m\sin^{-1}x)$ prove that
   (i) $(1-x^2)y_2 - xy_1 + m^2 y = 0$
   (ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$
4. 3. If $u = sinax + cosax$, show that $u_n = a^n\{1 + (-1)^n sin2ax\}^{\frac{1}{2}}$
5. If $y = \cos(m\sin^{-1}x)$, then prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$ Find y_n for $x = 0$.
6. If $y = x^{n-1}\log x$, prove that $y_n = \frac{(n-1)!}{x}$.
7. If f(x)= tanx, then prove that $f^n(0) - n_{C_2}f^{n-2}(0) + n_{C_4}f^{n-4}(0) - \cdots = sin\frac{n\pi}{2}$
8. If $y = \frac{1}{\sqrt{1+2x}}$, then prove that $(1+2x)y_{n+1} + (2n+1)y_n = 0$. Find y_n at $x = 0$.
9. If $y = a\cos(\log x) + b\sin(\log x)$, prove that
   (i) $x^2y_2 + xy_1 + y = 0$
   (ii) $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$.
10. If $y = \tan^{-1}x$, then prove that
    $(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$. Also find the value of $y_n\ at\ x = 0$
11. Find $y_n\ where\ y = \frac{x^2+1}{(x-1)(x-2)(x-3)}$
12. Verify Rolle's theorem for
    a) $f(x) = tanx\ in\ [0, \pi]$
    b) $f(x) = |x|,\ -1 \le x \le 1$
    c) $f(x) = x^3 - 6x^2 + 11x - 6$ in $1 \le x \le 3$
13. Verify Lagrange's Mean Value theorem for
    a) $f(x) = cosx\ where\ 0 \le x \le \frac{\pi}{2}$
    b) $f(x) = x^2 + 3x + 2$
    c) $f(x) = 4 - (6-x)^{\frac{2}{3}}$ in the interval $[5, 7]$.
    d) $f(x) = 4 - (4-x)^{\frac{2}{3}}$ in the interval $[2, 7]$
14. Verify Cauchy Mean value theorem for $f(x) = \sqrt{x}$ & $g(x) = \frac{1}{\sqrt{x}}\ in\ [1, 2]$
15. If $f(x) = e^x\ and\ g(x) = e^{-x}$, using Cauchy Mean value theorem show that θ is independent of both $x\ and\ h$ and is equal to $\frac{1}{2}$.
16. Apply Lagrange's Mean value theorem to prove that $\frac{x}{1+x} < \log(1+x) < x, \forall x > 0$.
17. Use mean value theorem to prove that $\frac{(b-a)}{\sqrt{1-a^2}} < \sin^{-1}b - \sin^{-1}a < \frac{(b-a)}{\sqrt{1-b^2}}, 0 < a < b < 1$.
18. Prove the inequality by Lagrange's Mean value theorem $\frac{\mathrm{b-a}}{1+\mathrm{b}^2} < \tan^{-1}b - \tan^{-1}a < \frac{\mathrm{b-a}}{1+\mathrm{a}^2},\quad 0 < a < b$

19. Prove the inequality by Lagrange's Mean value theorem $\frac{x}{1+x^2} < \tan^{-1} x < x, \quad 0 < x < \frac{\pi}{2}$

20. Use Mean value theorem to prove $1 + \frac{x}{2\sqrt{1+x}} < \sqrt{1+x} < 1 + \frac{x}{2}, \quad -1 < x < 0$

21. Use Mean value theorem to prove $x < \sin^{-1} x < \frac{x}{\sqrt{1-x^2}}, 0 < x < 1$

22. Prove the inequality by Lagrange's Mean value theorem $0 < \frac{1}{x} log \frac{e^x - 1}{x} < 1, \quad x > 0$

23. Find $\int_0^{\infty} \sqrt{x} e^{-x^3} \, dx$

24. Show that $\int_0^{\frac{\pi}{2}} sin^p x dx \times \int_0^{\frac{\pi}{2}} sin^{p+1} x \, dx = \frac{\pi}{2(p+1)}$

25. Show that $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}$

26. Prove that $\int_0^{\pi/2} \sqrt{tan\theta} d\theta = \frac{\pi}{\sqrt{2}}$ using Beta and Gamma function.

27. Evaluate $\int_0^{\infty} e^{-x} x^{\frac{3}{2}} \, dx$ using Beta and Gamma function.

28. Find the value of $\int_0^1 x^2 (1 - x^2)^{\frac{7}{2}} \, dx$ using Beta and Gamma function.

29. Evaluate $\int_0^{\infty} e^{-x^2} dx$

30. Evaluate $\int_0^{\frac{\pi}{2}} sin^4 x \, cos^5 x \, dx$ by using Beta and Gamma function.

31. Evaluate $\int_0^{\frac{\pi}{2}} \sin^7 \theta \cos^4 \theta d\theta$

32. Evaluate $\int_0^{\frac{\pi}{2}} \cos^6 \theta d\theta$

33. Prove that $\boldsymbol{\Gamma}(n+1) = n\boldsymbol{\Gamma}(n)$

34. Prove that $\int_0^1 \sqrt{1 - x^4} \, dx = \frac{\left\{\Gamma\left(\frac{1}{4}\right)\right\}^2}{6\sqrt{2\pi}}$

35. Prove that $\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$