Contents

Chapter 1

Vector Calculus

1.1 Introduction

Vector Calculus is an important branch of mathematics that deals with differentiating and integrating vector fields, mainly in two or three dimensions. It provides powerful tools to analyze physical quantities such as fluid flow, heat flow, electric and magnetic fields, gravitational forces, and many other phenomena in engineering and science.

The chapter begins with the study of the gradient, which measures the rate and direction of change of a scalar function. It is a fundamental concept used to determine maximum or minimum points and steepest ascent in physical systems. The divergence of a vector field measures how much a field spreads out from a point, playing a crucial role in fluid mechanics and electromagnetism. The curl describes the rotational tendency or swirling strength of a vector field.

Beyond these differential operators, the chapter introduces three important integral theorems that connect line, surface, and volume integrals. Green's Theorem relates a line integral around a closed curve in the plane to a double integral over the enclosed region. Stokes' Theorem generalizes this idea to three dimensions by linking the circulation of a vector field around a closed curve to the curl of the field over a surface. Finally, the Gauss Divergence Theorem establishes a relationship between the flux of a vector field across a closed surface and the divergence of the field inside the volume.

Together, these concepts form the foundation for understanding and solving complex engineering and physical problems, making vector calculus a central part of higher mathematics for B.Tech students.

1.2 Engineering Application

- **Mechanical Engineering**

  - Analysis of fluid flow over aircraft and automobile bodies using vector fields.
  - Calculation of torque, work, and forces in machines using line and surface integrals.
  - Heat conduction analysis in engines using gradient and divergence.
  - Stress and strain distribution in beams and mechanical components.

- **Civil Engineering**
  - Determining water flow in open channels and pipelines using vector fields.
  - Analyzing stress distribution in buildings, bridges, and dams.
  - Modeling traffic flow as a vector field for urban planning.
  - Soil pressure distribution using gradient and divergence.

- **Electrical & Electronics Engineering**
  - Understanding electric and magnetic fields using vector calculus (Maxwell's equations).
  - Designing antennas, transformers, and motors through flux calculations.
  - Signal propagation and energy transfer in transmission lines.
  - Electromagnetic field simulation in circuits and microdevices.

- **Computer Science & Engineering**
  - Vector fields used in computer graphics for motion simulation.
  - Gradient descent in machine learning optimization.
  - Path planning in robotics using vector fields.
  - Image processing applications such as edge detection (using divergence and curl).

- **Electronics & Communication Engineering**
  - Propagation of electromagnetic waves in air and waveguides.
  - Radar and satellite communication system design using flux and divergence.
  - Antenna radiation pattern analysis.
  - Signal flow analysis in wireless communication networks.

- **Chemical Engineering**
  - Modeling concentration gradients during diffusion processes.
  - Fluid dynamics inside reactors and pipelines.
  - Heat and mass transfer calculations using vector operators.

  - Chemical potential and equilibrium modeling with gradient fields.

- **Biotechnology**
  - Modeling blood flow through arteries using vector calculus.
  - Diffusion of nutrients and drugs in biological tissues.
  - Analyzing electrical activity in the heart and brain (ECG/EEG modeling).
  - Image reconstruction in MRI and CT using vector calculus principles.

- **Information Technology**
  - Data visualization using vector fields.
  - Optimization algorithms based on gradients.
  - Motion tracking in augmented and virtual reality applications.

1.3 Vector Point Function

A vector point function is a mathematical rule that assigns a unique vector to every point (x, y, z) within a region of space, forming a vector field. Think of it as giving a direction and magnitude (like a velocity or force) at every single location. For example, the electric intensity vector at each point in space is a vector point function, as is the gravitational force field. To study the calculus of vector-valued functions, we follow a similar path to the one we took in studying real-valued functions. First, we define the derivative, then we examine applications of the derivative. However, we will find some interesting new ideas along the way as a result of the vector nature of these functions and the properties of space curves.

1.3.1 Definition

If to each point $P(x, y, z)$ in a region R of space, there corresponds a unique vector $\overrightarrow{f}(P)$ i.e. $\overrightarrow{f}(x, y, z)$. Then $\overrightarrow{f}$ is called a vector point function. This function together with R forms a vector field. Next we give an example.
(i) $\overrightarrow{f}(x, y, z) = xy^2\hat{i} + 3x\hat{j} - 2z^2\hat{k}$ defines a vector field.

1.4 Derivatives of Vector-valued Function

1.4.1 Definition

The derivative of a vector-valued function $\overrightarrow{r}(t)$ is defined as

$$\overrightarrow{r'}(t) = \lim_{\Delta t \to 0} \frac{\overrightarrow{r}(t+\Delta t) - \overrightarrow{r}(t)}{\Delta t}$$

provided the limit exists. If $\overrightarrow{r'}(t)$ exists, then $\overrightarrow{r}(t)$ is differentiable at t. If $\overrightarrow{r'}(t)$ exists for all t in the interval (a, b) then $\overrightarrow{r}(t)$ is differentiable over the interval (a, b). . For the function to be differentiable over the closed interval $[a, b]$, the following two limits must exist as well:

$$\overrightarrow{r'}(a) = \lim_{\Delta t \to 0^+} \frac{\overrightarrow{r}(t+\Delta t) - \overrightarrow{r}(t)}{\Delta t}$$

$$\overrightarrow{r'}(b) = \lim_{\Delta t \to 0^-} \frac{\overrightarrow{r}(t+\Delta t) - \overrightarrow{r}(t)}{\Delta t}$$

Many of the rules for calculating derivatives of real-valued functions can be applied to calculating the derivatives of vector-valued functions as well. Recall that the derivative of a real-valued function can be interpreted as the slope of a tangent line or the instantaneous rate of change of the function. The derivative of a vector-valued function can be understood to be an instantaneous rate of change as well; for example, when the function represents the position of an object at a given point in time, the derivative represents its velocity at that same point in time.

Example 1. *Find the Derivative of a Vector-Valued Function* $\overrightarrow{f}(t) = (3t + 4)\hat{i} + (t^2 - 4t + 3)\hat{j}$.
Solution : We know

$$\begin{aligned}
\overrightarrow{f}'(t) &= \lim_{\Delta t \to 0} \frac{\overrightarrow{f}(t+\Delta t) - \overrightarrow{f}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{[(3(t+t)+4)\,\hat{\mathbf{i}} + ((t+t)^2 4(t+t)+3)\,\hat{\mathbf{j}}][(3t+4)\,\hat{\mathbf{i}} + (t^2 4t+3)\,\hat{\mathbf{j}}]}{t} \\
&= \lim_{\Delta t \to 0} \frac{(3t+3t+4)\,\hat{\mathbf{i}}(3t+4)\,\hat{\mathbf{i}} + (t^2+2tt+(t)^2 4t4t+3)\,\hat{\mathbf{j}}(t^2 4t+3)\,\hat{\mathbf{j}}}{t} \\
&= \lim_{\Delta t \to 0} \frac{(3t)\,\hat{\mathbf{i}} + (2tt+(t)^2 4t)\,\hat{\mathbf{j}}}{t} \\
&= \lim_{\Delta t \to 0} (3\,\hat{\mathbf{i}} + (2t+t4)\,\hat{\mathbf{j}}) \\
&= 3\,\hat{\mathbf{i}} + (2t4)\,\hat{\mathbf{j}}
\end{aligned}$$

Example 2. *Use the definition above to find the derivative of the function given below*

$$\overrightarrow{f}(t) = (2t^2 + 3)\,\hat{\mathbf{i}} + (5t - 6)\,\hat{\mathbf{j}}$$

.

Answer : $\overrightarrow{f}'(t) = 4t\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}}$
Notice that in the calculations in the Example above, we could also obtain the answer by first calculating the derivative of each component function, then putting these derivatives back into the vector-valued function. This is always true for calculating the derivative of a vector-valued function, whether it is in two or three dimensions. We state this in the following theorem. The proof of this theorem follows directly from the definitions of the limit of a vector-valued function and the derivative of a vector-valued function.

Theorem 1. *Differentiation of Vector-Valued Functions:*

Let f, g and h be differentiable function of t. Then,

1. *If* $\overrightarrow{r}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$

then,. $\qquad \overrightarrow{r'}(t) = f'(t)\,\hat{\mathbf{i}} + g'(t)\,\hat{\mathbf{j}}$

2.If, $\overrightarrow{r}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$

then,. $\qquad \overrightarrow{r'}(t) = f'(t)\,\hat{\mathbf{i}} + g'(t)\,\hat{\mathbf{j}} + h'(t)\,\hat{\mathbf{k}}$

Example 3. *Using above theorem calculate*

i.) $\overrightarrow{r}(t) = (6t + 8)\,\hat{\mathbf{i}} + (4t^2 + 2t - 3)\,\hat{\mathbf{j}}$

ii.) $\overrightarrow{r}(t) = e^t \sin t\,\hat{\mathbf{i}} + e^t \cos t\,\hat{\mathbf{j}} - e^{2t}\,\hat{\mathbf{k}}$

Solution:- i.) The first component of $\overrightarrow{r}(t) = (6t + 8)\,\hat{\mathbf{i}} + (4t^2 + 2t3)\,\hat{\mathbf{j}}$is $6t + 8$. The second component is $g(t) = 4t^2 + 2t - 3$. We have $f'(t) = 6$ and $g'(t) = 8t + 2$. Then using the above theorem we get $\overrightarrow{r'}(t) = 6\,\hat{\mathbf{i}} + (8t + 2)\,\hat{\mathbf{j}}$.
ii.) The first component of $\overrightarrow{r'}(t) = e^t \sin t\,\hat{\mathbf{i}} + e^t \cos t\,\hat{\mathbf{j}} - e^{2t}\,\hat{\mathbf{k}}$ is $f(t) = e^t \sin t$, the second component is $g(t) = e^t \cos t$ and the third component is $h(t) = e^{2t}$, we have $f(t) = e^t(\sin t + \cos t)$, $g(t) = e^t(\cos t \sin t)$ and $h(t) = 2e^{2t}$ so from the above theorem we get $\overrightarrow{r'}(t) = e^t(\sin t + \cos t)\,\hat{\mathbf{i}} + e^t(\cos t \sin t)\,\hat{\mathbf{j}} 2e^{2t}\,\hat{\mathbf{k}}$

Example 4. *Calculate the derivative of the function*

$$\overrightarrow{r}(t) = (t \ln t)\,\hat{\mathbf{i}} + (5e^t)\,\hat{\mathbf{j}} + (\cos t \sin t)\,\hat{\mathbf{k}}.$$

Solution:- Left as an exercise.
We can extend to vector-valued functions the properties of the derivative that we presented previously. In particular, the constant multiple rule, the sum and difference rules, the product rule, and the chain rule all extend to vector-valued functions. However, in the case of the product rule, there are actually three extensions.

Theorem 2. *Properties of the derivative of vector valued functions.*

$$
\begin{array}{llcll}
\text{i.} & \dfrac{d}{dt}[c\overrightarrow{r}(t)] & = & c\overrightarrow{r}'(t) & \textit{Scalar multiple} \\[2em]
\text{ii.} & \dfrac{d}{dt}[r(t)+u(t)] & = & \overrightarrow{r}'(t)+\overrightarrow{u}'(t) & \textit{Sum and difference} \\[2em]
\text{iii.} & \dfrac{d}{dt}[f(t)\overrightarrow{u}(t)] & = & f'(t)\overrightarrow{u}(t)+f(t)\overrightarrow{u}'(t) & \textit{Scalar product} \\[2em]
\text{iv.} & \dfrac{d}{dt}[\overrightarrow{r}(t)u(t)] & = & \overrightarrow{r}'(t)\overrightarrow{u}(t)+\overrightarrow{r}(t)\overrightarrow{u}'(t) & \textit{Dot product} \\[2em]
\text{v.} & \dfrac{d}{dt}[\overrightarrow{r}(t)\overrightarrow{u}(t)] & = & \overrightarrow{r}'(t)\overrightarrow{u}(t)+\overrightarrow{r}(t)\overrightarrow{u}'(t) & \textit{Cross product} \\[2em]
\text{vi.} & \dfrac{d}{dt}[\overrightarrow{r}(f(t))] & = & \overrightarrow{r}(f(t))f'(t) & \textit{Chain rule} \\[2em]
\text{vii.} & \textit{If } \overrightarrow{r}(t)\overrightarrow{r}(t) & = & c, \textit{then } \overrightarrow{r}(t)\overrightarrow{r}'(t) \ = 0\ . &
\end{array}
$$

Example 5. *Given the vector-valued functions*

$$\overrightarrow{r}(t) = (6t+8)\,\hat{\mathbf{i}} + (4t^2+2t3)\,\hat{\mathbf{j}} + 5t\,\hat{\mathbf{k}}$$

and

$$\overrightarrow{u}(t) = (t^2 3)\,\hat{\mathbf{i}} + (2t+4)\,\hat{\mathbf{j}} + (t^3 3t)\,\hat{\mathbf{k}},$$

calculate each of the following derivatives using the properties of the derivative of vector-valued functions.

$$a.)\frac{d}{dt}[\overrightarrow{r}(t)\overrightarrow{u}(t)]$$

$$b.)\frac{d}{dt}[\overrightarrow{u}(t)\times\overrightarrow{u}(t)]$$

Solution: We have $\overrightarrow{r}'(t) = 6\,\hat{\mathbf{i}} + (8t+2)\,\hat{\mathbf{j}} + 5\,\hat{\mathbf{k}}$ and $\overrightarrow{u}'(t) = 2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (3t^2 3)\,\hat{\mathbf{k}}$. Therefore using property (iv). above

$$
\begin{aligned}
(a.) \qquad \frac{d}{dt}[r(t)u(t)] &= r(t)u(t) + r(t)u(t) \\
&= (6\,\hat{\mathbf{i}} + (8t+2)\,\hat{\mathbf{j}} + 5\,\hat{\mathbf{k}})((t^2 3)\,\hat{\mathbf{i}} + (2t+4)\,\hat{\mathbf{j}} + (t^3 3t)\,\hat{\mathbf{k}}) \\
&\quad + ((6t+8)\,\hat{\mathbf{i}} + (4t^2 + 2t3)\,\hat{\mathbf{j}} + 5t\,\hat{\mathbf{k}})(2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (3t^2 3)\,\hat{\mathbf{k}}) \\
&= 6(t^2 3) + (8t+2)(2t+4) + 5(t^3 3t) \\
&\quad + 2t(6t+8) + 2(4t^2 + 2t3) + 5t(3t^2 3) \\
&= 20t^3 + 42t^2 + 26t16.
\end{aligned}
$$

We can similarly evaluate (b). Using Property (v).

Note:-

1. Vector valued function and Vector point function" are not the same, although they are often used interchangeably. A vector-valued function is a broad term for any function whose output is a vector, regardless of its input or how it's used. A vector field (a type of vector point function) is more specific, assigning a vector to each point in a region of space, like the gravitational force at every location around a planet.
2. However the derivative of a vector point function and a vector-valued function are the same concept. To find the derivative of a vector-valued function, we take the derivative of each of its component functions with respect to its input parameter, resulting in a new vector that is tangent to the curve defined by the original function.

1.5 Scalar and Vector Point Functions

1.5.1 Scalar Point Function

A **scalar point function** is a function that associates a *single scalar value* with every point in a region of space. Mathematically, if $\phi = \phi(x, y, z)$, then for each point (x, y, z) in the region, ϕ gives a scalar value.

$$\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

Example:

- Temperature distribution in a room: $T = T(x, y, z)$ gives the temperature (a scalar) at every point.
- Pressure in a fluid: $P = P(x, y, z)$ assigns a scalar pressure at each spatial point.

1.5.2 Vector Point Function

A **vector point function** assigns a *vector quantity* to every point in a region of space. Mathematically, if $\mathbf{F} = \mathbf{F}(x, y, z)$, then for each point (x, y, z), $\mathbf{F}$ gives a vector:

$$\mathbf{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

Example:

- Velocity field of a fluid: $\mathbf{V} = \mathbf{V}(x, y, z) = u\hat{i} + v\hat{j} + w\hat{k}$ gives the velocity vector at every point.
- Electric field: $\mathbf{E} = \mathbf{E}(x, y, z)$ assigns an electric field vector at each spatial point.

1.6 Gradient

1.6.1 Definition

The gradient of a scalar point function $f(x, y, z)$ is a vector that points in the direction of the greatest rate of increase of f. Mathematically,

$$grad\ f = \vec{\nabla} f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

1.6.2 Explanation

- The gradient gives the slope of the surface at a point.
- If $f(x, y, z)$ represents temperature, $\vec{\nabla} f$ points towards the direction where temperature rises fastest.
- The magnitude $|\vec{\nabla} f|$ tells us how steep the change is.

1.6.3 Examples

Illustration. 1
If $f(x, y, z) = x^2 y + y^2 x + z^2$, then

$$\text{grad } f = \vec{\nabla} f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= \hat{i}(2xy + y^2) + \hat{j}(x^2 + 2xy) + \hat{k}(2z)$$

At the point $(1,1,1)$,

$$\vec{\nabla} f = \left(2\times 1\times 1+1^2\right)\hat{\imath} + \left(1^2+2\times 1\times 1\right)\hat{\jmath} + (2\times 1)\,\hat{k}$$

i.e.,

$$\vec{\nabla} f = 3\hat{\imath} + 3\hat{\jmath} + 2\hat{k}.$$

Illustration. 2
If $f(x,y,z) = x^2 + y^2 + z^2$, then

$$\text{grad } f = \vec{\nabla} f = \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= \hat{\imath}(2x) + \hat{\jmath}(2y) + \hat{k}(2z)$$

At the point $(1,1,1)$,

$$\vec{\nabla} f = (2\times 1)\hat{\imath} + (2\times 1)\hat{\jmath} + (2\times 1)\hat{k}$$

i.e.,

$$\vec{\nabla} f = 2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}.$$

Illustration. 3
If $f(x,y) = x^2 y + y^3$, then

$$\text{grad } f = \vec{\nabla} f = \hat{\imath}\,\frac{\partial f}{\partial x} + \hat{\jmath}\,\frac{\partial f}{\partial y}$$

$$= \hat{\imath}\,(2xy) + \hat{\jmath}\,(x^2 + 3y^2).$$

At the point $(1,1)$,

$$\vec{\nabla} f = \left(2\times 1\times 1\right)\hat{\imath} + \left(1^2 + 3\times 1^2\right)\hat{\jmath}$$

i.e.,

$$\vec{\nabla} f = 2\hat{\imath} + 4\hat{\jmath}.$$

1.6.4 Exercise Problems

(i) Find $\vec{\nabla} f$ if $f(x,y,z) = x^3 + y^3 + z^3$.

(ii) If $f = xy + yz + zx$, find $\vec{\nabla} f$.

(iii) For $f = r^2 = x^2 + y^2 + z^2$, prove that $\vec{\nabla} f = 2\vec{r}$.

(iv) If $f(x,y) = e^{xy}$, compute $\vec{\nabla} f$.

1.7 Divergence

1.7.1 Definition

For a vector field $\mathbf{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$, divergence is:

$$div.\mathbf{F} = \vec{\nabla} \cdot \mathbf{F} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

1.7.2 Explanation

- Divergence measures the **rate of expansion** or **compression** of a vector field.
- If $\vec{\nabla} \cdot \mathbf{F} > 0$: source point (outflow).
- If $\vec{\nabla} \cdot \mathbf{F} < 0$: sink point (inflow).
- If $\vec{\nabla} \cdot \mathbf{F} = 0$: incompressible field and vector function $\mathbf{F}$ is said to be *solenoidal.*

1.7.3 Examples

Illustration 1 If $\vec{v} = x^2y\hat{i} - 2xz\hat{j} + 2yz\hat{k}$, then

$$\begin{aligned} \operatorname{div} \vec{v} &= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(-2xz) + \frac{\partial}{\partial z}(2yz) \\ &= 2xy + 0 + 2y = 2y(x+1) \end{aligned}$$

Illustration 2 The vector $\vec{f} = (y^2 + z^2)\hat{i} + (z^2 + x^2)\hat{j} + (x^2 + y^2)\hat{k}$
is solenoidal, for,

$$\begin{aligned} \operatorname{div} \vec{f} &= \frac{\partial}{\partial x}(y^2 + z^2) + \frac{\partial}{\partial y}(z^2 + x^2) + \frac{\partial}{\partial z}(x^2 + y^2) \\ &= 0 + 0 + 0 = 0. \end{aligned}$$

1.7.4 Exercise Problems

(i) Find divergence of $\mathbf{F} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$.

(ii) Show that $\nabla \cdot (r^n\mathbf{r}) = (n+3)r^n$.

(iii) For $\mathbf{F} = yz\hat{i} + xz\hat{j} + xy\hat{k}$, prove $\nabla \cdot \mathbf{F} = 0$.

(iv) Find $\nabla \cdot \mathbf{F}$ if $\mathbf{F} = e^x\hat{i} + e^y\hat{j} + e^z\hat{k}$.

(v) Check whether $\mathbf{F} = y\hat{i} - x\hat{j}$ is solenoidal.

1.8 Curl

1.8.1 Definition

In vector calculus, the **curl** of a vector field measures the **rotation** or **circulation** of the field at a given point. It is widely used in physics, fluid dynamics, and electromagnetism to determine rotational behavior.

For a vector field $\mathbf{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$:

$$curl.\mathbf{F} = \vec{\nabla} \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Thus,

$$\vec{\nabla} \times \vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{k}$$

1.8.2 Explanation

- Curl measures the **rotation** of a vector field.
- If $\vec{\nabla} \times \mathbf{F} = 0$, the field is **irrotational**.
- Non-zero curl means the field has a rotational component.

1.8.3 Properties of Curl

(i) $\vec{\nabla} \times (\nabla\phi) = 0$ (Curl of a gradient is zero)

(ii) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$ (Divergence of a curl is zero)

(iii) $\vec{\nabla} \times (\vec{F} + \vec{G}) = \vec{\nabla} \times \vec{F} + \vec{\nabla} \times \vec{G}$ (Linearity)

(iv) $\vec{\nabla} \times (f\vec{F}) = f(\vec{\nabla} \times \vec{F}) + (\vec{\nabla} f) \times \vec{F}$

1.9 Worked-Out Examples

Example 1: Non-zero Curl

Problem: Find $\vec{\nabla} \times \vec{F}$ for $\vec{F} = (y, -x, 0)$.

Solution:

$$\vec{F} = y\hat{i} - x\hat{j} + 0\hat{k}$$

Using the formula:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & -x & 0 \end{vmatrix}$$

$$= \left(\frac{\partial 0}{\partial y} - \frac{\partial(-x)}{\partial z} \right) \hat{i} - \left(\frac{\partial 0}{\partial x} - \frac{\partial y}{\partial z} \right) \hat{j} + \left(\frac{\partial(-x)}{\partial x} - \frac{\partial y}{\partial y} \right) \hat{k}$$

$$= 0\hat{i} - 0\hat{j} + (-1 - 1)\hat{k} = -2\hat{k}$$

$$\boxed{\vec{\nabla} \times \vec{F} = -2\hat{k}}$$

Example 2: Zero Curl (Irrotational Field)

Problem: Find $\vec{\nabla} \times \vec{F}$ for $\vec{F} = (x^2, y^2, z^2)$.

Solution:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \left(\frac{\partial z^2}{\partial y} - \frac{\partial y^2}{\partial z} \right) \hat{i} + \left(\frac{\partial x^2}{\partial z} - \frac{\partial z^2}{\partial x} \right) \hat{j} + \left(\frac{\partial y^2}{\partial x} - \frac{\partial x^2}{\partial y} \right) \hat{k}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

$$\boxed{\vec{\nabla} \times \vec{F} = \vec{0}}$$

Example 3: Another Non-zero Curl

Problem: Find $\vec{\nabla} \times \vec{F}$ for $\vec{F} = (yz, zx, xy)$.

Solution:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & zx & xy \end{vmatrix}$$

$$= \left(\frac{\partial(xy)}{\partial y} - \frac{\partial(zx)}{\partial z} \right) \hat{i} + \left(\frac{\partial(yz)}{\partial z} - \frac{\partial(xy)}{\partial x} \right) \hat{j} + \left(\frac{\partial(zx)}{\partial x} - \frac{\partial(yz)}{\partial y} \right) \hat{k}$$

$$= (x - x)\hat{i} + (y - y)\hat{j} + (z - z)\hat{k} = \vec{0}$$

$$\boxed{\vec{\nabla} \times \vec{F} = \vec{0}}$$

1.9.1 Exercise Problems

(i) Find $\nabla \times \mathbf{F}$ for $\mathbf{F} = xz\hat{\imath} + yz\hat{\jmath} + xy\hat{k}$.

(ii) Show that $\nabla \times (\nabla f) = 0$ for any scalar function f.

(iii) Prove that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

(iv) If $\mathbf{F} = r^2\mathbf{r}$, find $\nabla \times \mathbf{F}$.

(v) Determine whether $\mathbf{F} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ is irrotational.

1.10 Directional Derivative

1.10.1 Definition

The **directional derivative** of a scalar function measures the rate of change of the function at a given point in the direction of a specified vector. It generalizes the concept of a partial derivative to any direction in space.

Let $f(x, y, z)$ be a scalar function defined in $\mathbb{R}^3$. At a point $P(x_0, y_0, z_0)$, the **directional derivative** of f in the direction of a unit vector

$$\mathbf{u} = u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}$$

is defined as:

$$D_{\mathbf{u}}f = \lim_{h \to 0} \frac{f(x_0 + hu_1,\ y_0 + hu_2,\ z_0 + hu_3) - f(x_0, y_0, z_0)}{h}.$$

1.10.2 Formula Using Gradient

The directional derivative can be calculated using the gradient of f:

$$D_{\mathbf{u}}f = \vec{\nabla} f \cdot \mathbf{u},$$

where

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}.$$

If $\mathbf{u}$ is not a unit vector, we normalize it:

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|}.$$

Step-by-Step Procedure

(i) Find the gradient vector $\vec{\nabla} f$.

(ii) Find the unit vector $\mathbf{u}$ in the given direction.

(iii) Use the formula $D_{\mathbf{u}} f = \vec{\nabla} f \cdot \mathbf{u}$.

1.10.3 Worked-out Examples

Problem 1: Find the directional derivative of

$$f(x, y) = x^2 y + y^3$$

at the point $(1, 2)$ in the direction of the vector $\mathbf{v} = 3\hat{\imath} + 4\hat{\jmath}$.

Solution:

Step 1: Gradient of f:

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} = (2xy)\hat{\imath} + (x^2 + 3y^2)\hat{\jmath}.$$

At $(1, 2)$:

$$\vec{\nabla} f = (2 \cdot 1 \cdot 2)\hat{\imath} + (1^2 + 3 \cdot 2^2)\hat{\jmath} = 4\hat{\imath} + 13\hat{\jmath}.$$

Step 2: Unit vector in the direction of $\mathbf{v}$:

$$\|\mathbf{v}\| = \sqrt{3^2 + 4^2} = 5, \quad \mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\hat{\imath} + \frac{4}{5}\hat{\jmath}.$$

Step 3: Directional derivative:

$$D_{\mathbf{u}} f = \vec{\nabla} f \cdot \mathbf{u} = (4\hat{\imath} + 13\hat{\jmath}) \cdot \left(\frac{3}{5}\hat{\imath} + \frac{4}{5}\hat{\jmath}\right) = \frac{12}{5} + \frac{52}{5} = \frac{64}{5}.$$

$$\boxed{D_{\mathbf{u}} f = \frac{64}{5}}$$

Problem 2: Find the directional derivative of

$$f(x, y, z) = x^2 + y^2 + z^2$$

at the point $(1, 1, 1)$ in the direction of $\mathbf{v} = 2\hat{\imath} + 2\hat{\jmath} + 1\hat{k}$.

Solution:

Step 1: Gradient of f:

$$\vec{\nabla} f = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}.$$

At $(1,1,1)$:

$$\vec{\nabla} f = 2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}.$$

Step 2: Unit vector in the direction of $\mathbf{v}$:

$$\|\mathbf{v}\| = \sqrt{2^2 + 2^2 + 1^2} = 3, \quad \mathbf{u} = \frac{2}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} + \frac{1}{3}\hat{k}.$$

Step 3: Directional derivative:

$$D_{\mathbf{u}} f = \vec{\nabla} f \cdot \mathbf{u} = (2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}) \cdot \left(\frac{2}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} + \frac{1}{3}\hat{k}\right) = \frac{4}{3} + \frac{4}{3} + \frac{2}{3} = \frac{10}{3}.$$

$$\boxed{D_{\mathbf{u}} f = \frac{10}{3}}$$

1.10.4 Explanation

- The directional derivative measures the rate of change of a function in any given direction.
- It is computed using the dot product of the gradient vector and a unit direction vector.
- The gradient $\vec{\nabla} f$ points in the direction of the **maximum** rate of increase of the function.

1.10.5 Exercise Problems

(i) Find $D_{\mathbf{u}} f$ for $f = x^2yz$ along $\mathbf{u} = \frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath})$.

(ii) Show that the maximum directional derivative of f equals $|\nabla f|$.

(iii) If $f = x^3 + xy^2$, find $D_{\mathbf{u}} f$ along $\mathbf{u} = \frac{1}{\sqrt{5}}(2\hat{\imath} + \hat{\jmath})$.

(iv) For $f = e^{x+y}$, find $D_{\mathbf{u}} f$ when $\mathbf{u} = \frac{1}{\sqrt{2}}(\hat{\imath} - \hat{\jmath})$.

(v) Find $D_{\mathbf{u}} f$ if $f = xy + z^2$ and $\mathbf{u} = \frac{1}{3}(2\hat{\imath} + \hat{\jmath} + 2\hat{k})$.

1.11 Green's theorem

Green's Theorem is a fundamental result in vector calculus that establishes a relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C. Suppose C be a simple closed curve in $\mathbb{R}^2$ that is piecewise smooth and positively oriented (counterclockwise). Let D be the region bounded by C. Suppose $P(x, y)$ and $Q(x, y)$ are functions defined on an open region containing D with continuous partial derivatives. Then:

$$\oint_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy \tag{1.0}$$

Equivalently, in vector form:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \left(\nabla \times \vec{F} \right) \cdot \hat{k}\,dA \tag{1.0}$$

where $\vec{F} = P\,\hat{\imath} + Q\,\hat{\jmath}$.

Problem

(i) Verify Green's theorem in the plane for

$$\oint_C (xy + y^2)\,dx + x^2\,dy$$

where C is the closed curve bounded by $y = x$ and $y = x^2$.

Solution

Green's Theorem states:

$$\oint_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

where C is the positively oriented (counterclockwise) boundary of region D.

Given:

$$P = xy + y^2, \quad Q = x^2$$

$$\frac{\partial Q}{\partial x} = 2x, \quad \frac{\partial P}{\partial y} = x + 2y$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - (x + 2y) = x - 2y$$

Thus Green's theorem gives:

$$\oint_C (xy + y^2)dx + x^2 dy = \iint_D (x - 2y)\, dA$$

We will compute both sides and verify equality.

Curves: $y = x$ (line) and $y = x^2$ (parabola).

Intersection: $x = x^2 \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0 \Rightarrow x = 0, 1$

Points: $(0, 0)$ and $(1, 1)$.

Region D: For $0 \le x \le 1$, $x^2 \le y \le x$.

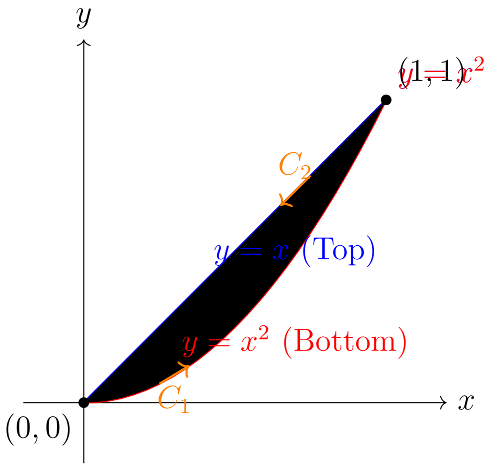


Figure 1.1: Region D bounded by $y = x$ and $y = x^2$. Arrows show counterclockwise orientation.

$$\iint_D (x - 2y)\, dA = \int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y)\, dy\, dx$$

Integrate with respect to y:

$$\int_{y=x^2}^{x} (x - 2y)\, dy = \left[xy - y^2\right]_{y=x^2}^{y=x}$$

At $y = x$: $x \cdot x - x^2 = x^2 - x^2 = 0$
At $y = x^2$: $x \cdot x^2 - (x^2)^2 = x^3 - x^4$

So:

$$\left[xy - y^2\right]_{x^2}^{x} = 0 - (x^3 - x^4) = -x^3 + x^4$$

Integrate with respect to x:

$$\int_0^1 (-x^3 + x^4)\,dx = \left[-\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 = -\frac{1}{4} + \frac{1}{5} = \frac{-5+4}{20} = -\frac{1}{20}$$

Thus:

$$\text{RHS} = -\frac{1}{20}$$

The closed curve C has two parts with counterclockwise orientation:

(a) C_1: from $(0,0)$ to $(1,1)$ along $y = x^2$ (bottom curve)

(b) C_2: from $(1,1)$ to $(0,0)$ along $y = x$ (top curve, reverse direction)

Part C_1: $y = x^2$, $dy = 2x\,dx$, $x : 0 \to 1$

$$P = xy + y^2 = x \cdot x^2 + (x^2)^2 = x^3 + x^4$$
$$Q = x^2$$
$$P\,dx + Q\,dy = (x^3 + x^4)dx + x^2(2x\,dx) = (3x^3 + x^4)dx$$
$$\int_{C_1} = \int_0^1 (3x^3 + x^4)dx = \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$

Part C_2: $y = x$, $dy = dx$, $x : 1 \to 0$

$$P = xy + y^2 = x \cdot x + x^2 = 2x^2$$
$$Q = x^2$$
$$P\,dx + Q\,dy = 2x^2dx + x^2dx = 3x^2dx$$
$$\int_{C_2} = \int_1^0 3x^2dx = -\int_0^1 3x^2dx = -[x^3]_0^1 = -1$$

Total Line Integral

$$\oint_C = \int_{C_1} + \int_{C_2} = \frac{19}{20} + (-1) = \frac{19}{20} - \frac{20}{20} = -\frac{1}{20}$$

Both sides yield the same result:

$$\oint_C (xy + y^2)dx + x^2dy = \iint_D (x - 2y)\,dA = -\frac{1}{20}$$

Green's theorem is verified.

(ii) Evaluate by Green's theorem:

$$\oint_C \left(x^2 - \cosh y\right) dx + (y + \sin x)dy,$$

where C is the rectangle with vertices $(0,0), (\pi,0), (\pi,1), (0,1)$.

Solution

From the given line integral:

$$\oint_C (Pdx + Qdy) = \oint_C \left[(x^2 - \cosh y)dx + (y + \sin x)dy\right]$$

We have:

$$P(x,y) = x^2 - \cosh y, \quad Q(x,y) = y + \sin x$$

Green's theorem states:

$$\oint_C (Pdx + Qdy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(y + \sin x) = \cos x$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(x^2 - \cosh y) = -\sinh y$$

Therefore:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \cos x - (-\sinh y) = \cos x + \sinh y$$

The region D is the rectangle: $0 \le x \le \pi$, $0 \le y \le 1$

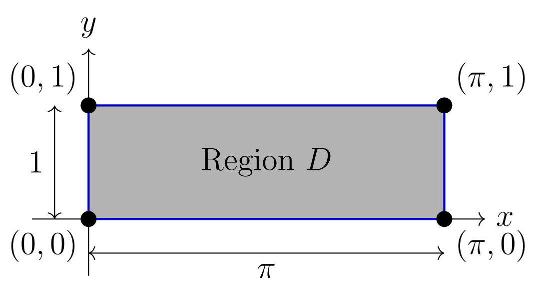


Figure 1.2: Rectangular region D with vertices $(0,0), (\pi,0), (\pi,1), (0,1)$

The double integral becomes:

$$\iint_D (\cos x + \sinh y)dA = \int_{x=0}^{\pi} \int_{y=0}^{1} (\cos x + \sinh y)dydx$$

First, integrate with respect to y:

$$\begin{aligned}\int_{y=0}^{1} (\cos x + \sinh y)dy &= [y\cos x + \cosh y]_{y=0}^{y=1} \\ &= (1 \cdot \cos x + \cosh 1) - (0 \cdot \cos x + \cosh 0) \\ &= \cos x + \cosh 1 - 1 \quad (\text{since } \cosh 0 = 1)\end{aligned}$$

Now integrate with respect to x:

$$\begin{aligned}\int_{x=0}^{\pi} (\cos x + \cosh 1 - 1)dx &= [\sin x + x(\cosh 1 - 1)]_0^{\pi} \\ &= [\sin \pi + \pi(\cosh 1 - 1)] - [\sin 0 + 0(\cosh 1 - 1)] \\ &= (0 + \pi(\cosh 1 - 1)) - (0 + 0) = \pi(\cosh 1 - 1)\end{aligned}$$

$$\oint_C \left(x^2 - \cosh y\right) dx + (y + \sin x)dy = \pi(\cosh 1 - 1)$$

Exercise

(i) Verify Green's theorem for

$$\oint_C (x^2 - y^2)dx + 2xydy$$

where C is the boundary of the rectangle $0 \le x \le 2$, $0 \le y \le 3$.

(ii) Use Green's theorem to find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(iii) Evaluate

$$\oint_C \frac{-y}{x^2 + y^2}dx + \frac{x}{x^2 + y^2}dy$$

where C is the positively oriented boundary of the annular region $1 \le x^2 + y^2 \le 4$.

(iv) Use Green's theorem to evaluate

$$\oint_C (e^x \sin y - y)dx + (e^x \cos y - 1)dy$$

where C is the circle $x^2 + y^2 = 9$.

(v) Evaluate

$$\oint_C (3x^2y + y^3)dx + (x^3 + 3xy^2)dy$$

where C is the boundary of the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$.

1.12 Introduction to Stokes' Theorem

Stokes' Theorem, also known as the Kelvin-Stokes Theorem or the curl theorem, is a fundamental theorem in vector calculus that relates the line integral of a vector field around a closed curve to the surface integral of the curl of the vector field over any surface bounded by that curve. It is a powerful generalization of Green's Theorem to three dimensions and is a crucial tool in electromagnetism, fluid dynamics, and other fields of physics and engineering.

1.13 Statement of Stokes' Theorem

Let S be an oriented, piecewise smooth surface in $\mathbb{R}^3$ with a boundary C that is a simple, closed, piecewise smooth curve. The orientation of C must be consistent with the orientation of S (following the right-hand rule: if you curl the fingers of your right hand in the direction of the curve C, your thumb points in the direction of the normal vector $\hat{n}$ of the surface S).

Let $\vec{F} = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$ be a vector field whose component functions have continuous first partial derivatives on an open region in $\mathbb{R}^3$ containing S.

Then Stokes' Theorem states:

$$\oint\limits_C \vec{F} \cdot d\vec{r} = \iint\limits_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$

where:

- $\oint\limits_C \vec{F} \cdot d\vec{r}$ is the line integral of $\vec{F}$ around the boundary curve C. It represents the circulation of $\vec{F}$ along C.
- $\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$ is the surface integral of the curl of $\vec{F}$ over the surface S. It represents the flux of the curl of $\vec{F}$ through S.
- $\vec{\nabla} \times \vec{F}$ is the curl of the vector field $\vec{F}$, defined as:

$$\vec{\nabla} \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

- $d\vec{S} = \hat{n}\, dS$, where $\hat{n}$ is the unit normal vector to the surface S, and dS is the scalar area element.

1.13.1 Intuitive Understanding

Stokes' Theorem essentially states that the total "circulation" of a vector field around a closed loop is equal to the total "flux" of the "rotational tendency" (curl) of the field through any surface bounded by that loop. Imagine a fluid flow: the line integral measures how much the fluid is swirling around the loop, while the surface integral measures the sum of microscopic swirls (vorticity) passing through the surface. The theorem says these two quantities are equal.

1.13.2 Key Concepts

- **Orientation:** The consistency between the orientation of the boundary curve C and the surface S is crucial. The right-hand rule provides this link.

- **Choice of Surface:** One of the powerful aspects of Stokes' Theorem is that the surface S can be *any* surface that has C as its boundary. This flexibility often allows us to choose a simpler surface for integration.

- **Conservative Vector Fields:** If $\vec{\nabla} \times \vec{F} = \vec{0}$ (i.e., $\vec{F}$ is a conservative vector field), then $\oint_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve C. This is consistent with the path independence of line integrals for conservative fields.

1.14 Examples

1.14.1 Example 1: Verification on a Simple Surface

Let $\vec{F}(x, y, z) = -y\hat{i} + x\hat{j} + z\hat{k}$. Let S be the upper hemisphere $x^2 + y^2 + z^2 = 1$ with $z \geq 0$, oriented upwards. The boundary curve C is the unit circle $x^2 + y^2 = 1$ in the xy-plane, oriented counterclockwise.

Method 1: Line Integral $\oint_C \vec{F} \cdot d\vec{r}$

Parametrize C: $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + 0\,\hat{k}$ for $0 \leq t \leq 2\pi$. Then $d\vec{r} = (-\sin t\,\hat{i} + \cos t\,\hat{j})\,dt$. $\vec{F}(\vec{r}(t)) = -\sin t\,\hat{i} + \cos t\,\hat{j} + 0\,\hat{k}$. $\vec{F} \cdot d\vec{r} = ((-\sin t)(-\sin t) + (\cos t)(\cos t) + (0)(0))\,dt = (\sin^2 t + \cos^2 t)\,dt = 1\,dt$. $\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} 1\,dt = 2\pi$.

Method 2: Surface Integral $\iint\limits_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$

First, calculate $\vec{\nabla} \times \vec{F}$:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & z \end{vmatrix} = \left(\frac{\partial z}{\partial y} - \frac{\partial x}{\partial z}\right)\hat{\imath} - \left(\frac{\partial z}{\partial x} - \frac{\partial(-y)}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right)\hat{k}$$

$$= (0-0)\hat{\imath} - (0-0)\hat{\jmath} + (1-(-1))\hat{k} = 2\hat{k}$$

Now, parametrize the surface S. For the upper hemisphere, use spherical coordinates: $\vec{r}(\phi,\theta) = \sin\phi\cos\theta\,\hat{\imath} + \sin\phi\sin\theta\,\hat{\jmath} + \cos\phi\,\hat{k}$, where $0 \le \phi \le \pi/2$ and $0 \le \theta \le 2\pi$. The normal vector $d\vec{S} = \left(\frac{\partial\vec{r}}{\partial\phi} \times \frac{\partial\vec{r}}{\partial\theta}\right) d\phi\, d\theta$. Calculate the cross product:

$$\frac{\partial\vec{r}}{\partial\phi} \times \frac{\partial\vec{r}}{\partial\theta} = \sin^2\phi\cos\theta\,\hat{\imath} + \sin^2\phi\sin\theta\,\hat{\jmath} + \cos\phi\sin\phi\,\hat{k}$$

This vector points upwards (positive z component for $\phi \in [0, \pi/2]$), which matches the desired orientation. $(\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = (2\hat{k}) \cdot (\sin^2\phi\cos\theta\,\hat{\imath} + \sin^2\phi\sin\theta\,\hat{\jmath} + \cos\phi\sin\phi\,\hat{k})\, d\phi\, d\theta$ $= 2\cos\phi\sin\phi\, d\phi\, d\theta$.

$$\iint\limits_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi/2} 2\cos\phi\sin\phi\, d\phi\, d\theta$$

Let $u = \sin\phi$, $du = \cos\phi\, d\phi$. When $\phi = 0, u = 0$; when $\phi = \pi/2, u = 1$.

$$= \int_0^{2\pi}\left[\int_0^1 2u\, du\right] d\theta = \int_0^{2\pi}\left[u^2\right]_0^1 d\theta = \int_0^{2\pi} 1\, d\theta = 2\pi$$

Both methods yield 2π, verifying Stokes' Theorem.

1.14.2 Example 2: Choosing a Simpler Surface

Evaluate $\oint\limits_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y,z) = (x^2+y-4)\hat{\imath} + (3xy)\hat{\jmath} + (2xz+z^2)\hat{k}$ and C is the curve of intersection of the plane $z = y$ and the cylinder $x^2+y^2 = 1$, oriented counterclockwise when viewed from above.

Directly calculating the line integral would be complicated. We use Stokes' Theorem. The curve C is the boundary of the surface S, which can be chosen as the portion of the plane $z = y$ inside the cylinder $x^2 + y^2 = 1$.

First, calculate $\vec{\nabla} \times \vec{F}$:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix}$$

$$= (0-0)\hat{i} - (2z-0)\hat{j} + (3y-1)\hat{k} = -2z\hat{j} + (3y-1)\hat{k}$$

Now, parametrize the surface S. Since S is a portion of the plane $z = y$, we can project it onto the xy-plane. The projection D is the disk $x^2 + y^2 \leq 1$. The surface can be parametrized as $\vec{r}(x, y) = x\hat{i}+y\hat{j}+y\hat{k}$ for $(x, y) \in D$. The normal vector $\vec{r}_x \times \vec{r}_y = -\hat{j}+\hat{k}$. This normal vector points slightly backwards (negative y) and upwards (positive z). The curve is oriented counterclockwise when viewed from above, which means the surface normal should point generally upwards. Our normal vector has a positive k component, so it's consistent.

Now, evaluate $(\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$: Substitute $z = y$ into $\vec{\nabla} \times \vec{F}$: $\vec{\nabla} \times \vec{F} = -2y\hat{j} + (3y-1)\hat{k}$. $(\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = (-2y\hat{j} + (3y-1)\hat{k}) \cdot (-\hat{j} + \hat{k})\, dA = (-2y)(-1) + (3y-1)(1)\, dA = (2y + 3y - 1)\, dA = (5y - 1)\, dA$.

Finally, perform the surface integral over the disk $D : x^2 + y^2 \leq 1$. Use polar coordinates: $x = r\cos\theta, y = r\sin\theta$, $dA = r\, dr\, d\theta$.

$$\iint\limits_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \iint\limits_D (5y-1)\, dA = \int_0^{2\pi}\int_0^1 (5r\sin\theta - 1)r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (5r^2\sin\theta - r)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{5}{3}r^3\sin\theta - \frac{1}{2}r^2\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{5}{3}\sin\theta - \frac{1}{2}\right) d\theta$$

$$= \left[-\frac{5}{3}\cos\theta - \frac{1}{2}\theta\right]_0^{2\pi}$$

$$= \left(-\frac{5}{3}\cos(2\pi) - \frac{1}{2}(2\pi)\right) - \left(-\frac{5}{3}\cos(0) - \frac{1}{2}(0)\right)$$

$$= \left(-\frac{5}{3} - \pi\right) - \left(-\frac{5}{3} - 0\right) = -\pi$$

So, $\oint\limits_C \vec{F} \cdot d\vec{r} = -\pi$.

1.14.3 Example 3: Curl is Zero

Evaluate $\oint\limits_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y, z) = (2xy)\hat{i} + (x^2 + z)\hat{j} + y\hat{k}$ and C is any closed curve.

First, calculate $\vec{\nabla} \times \vec{F}$:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2 + z & y \end{vmatrix}$$

$$= \left(\frac{\partial y}{\partial y} - \frac{\partial(x^2 + z)}{\partial z}\right)\hat{i} - \left(\frac{\partial y}{\partial x} - \frac{\partial(2xy)}{\partial z}\right)\hat{j} + \left(\frac{\partial(x^2 + z)}{\partial x} - \frac{\partial(2xy)}{\partial y}\right)\hat{k}$$

$$= (1-1)\hat{i} - (0-0)\hat{j} + (2x - 2x)\hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

Since $\vec{\nabla} \times \vec{F} = \vec{0}$, $\vec{F}$ is a conservative vector field. By Stokes' Theorem, $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \iint_S \vec{0} \cdot d\vec{S} = 0$. This confirms that the line integral of a conservative vector field over any closed curve is zero.

1.14.4 Example 4: Calculating Flux of Curl Directly

Evaluate $\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$ where $\vec{F}(x, y, z) = (z^2 - y^2)\hat{i} + (x^2 - z^2)\hat{j} + (y^2 - x^2)\hat{k}$ and S is the surface of the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane, oriented upward.

Instead of directly computing the surface integral, we can use Stokes' Theorem and evaluate the line integral around the boundary curve C. The surface S is bounded by the curve where $z = 0$, so $4 - x^2 - y^2 = 0$, which means $x^2 + y^2 = 4$. This is a circle of radius 2 in the xy-plane, oriented counterclockwise.

Parametrize C: $\vec{r}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} + 0\,\hat{k}$ for $0 \le t \le 2\pi$. Then $d\vec{r} = (-2\sin t\,\hat{i} + 2\cos t\,\hat{j})\,dt$. $\vec{F}(\vec{r}(t)) = (0^2 - (2\sin t)^2)\hat{i} + ((2\cos t)^2 - 0^2)\hat{j} + ((2\sin t)^2 - (2\cos t)^2)\hat{k} = -4\sin^2 t\,\hat{i} + 4\cos^2 t\,\hat{j} + (4\sin^2 t - 4\cos^2 t)\hat{k}$. $\vec{F} \cdot d\vec{r} = (-4\sin^2 t)(-2\sin t) + (4\cos^2 t)(2\cos t) + (4\sin^2 t - 4\cos^2 t)(0)\,dt = (8\sin^3 t + 8\cos^3 t)\,dt$.

Now, integrate:

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (8\sin^3 t + 8\cos^3 t)\,dt$$

Recall that $\int_0^{2\pi} \sin^3 t\,dt = 0$ and $\int_0^{2\pi} \cos^3 t\,dt = 0$ (due to symmetry over the full period for odd powers). Therefore, $\oint_C \vec{F} \cdot d\hat{r} = 0$. By Stokes' Theorem, $\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = 0$.

1.14.5 Example 5: Surface on a Cylinder

Evaluate $\oint_C \vec{F} \cdot d\hat{r}$ where $\vec{F}(x, y, z) = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$ and C is the curve of intersection of the cylinder $x^2 + y^2 = 9$ and the plane $z = 1$, oriented counterclockwise when viewed from above.

The curve C is a circle of radius 3 in the plane $z = 1$. We can choose S to be the flat disk $x^2 + y^2 \leq 9$ at $z = 1$, oriented upward.

First, calculate $\vec{\nabla} \times \vec{F}$:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix}$$

$$= \left(\frac{\partial z^2}{\partial y} - \frac{\partial x}{\partial z}\right)\hat{i} - \left(\frac{\partial z^2}{\partial x} - \frac{\partial(-y^2)}{\partial z}\right)\hat{j} + \left(\frac{\partial x}{\partial x} - \frac{\partial(-y^2)}{\partial y}\right)\hat{k}$$

$$= (0-0)\hat{i} - (0-0)\hat{j} + (1-(-2y))\hat{k} = (1+2y)\hat{k}$$

The surface S is the disk $x^2+y^2 \leq 9$ in the plane $z = 1$. The normal vector for an upward orientation is $\hat{n} = \hat{k}$, so $d\vec{S} = \hat{k}\,dA$. $(\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = ((1+2y)\hat{k}) \cdot (\hat{k}\,dA) = (1+2y)\,dA$.

Now, perform the surface integral over the disk $D : x^2+y^2 \leq 9$. Use polar coordinates: $y = r\sin\theta$, $dA = r\,dr\,d\theta$.

$$\iint\limits_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \iint_D (1+2y)\,dA = \int\limits_0^{2\pi}\int\limits_0^3 (1+2r\sin\theta)r\,dr\,d\theta$$

$$= \int\limits_0^{2\pi}\int\limits_0^3 (r + 2r^2\sin\theta)\,dr\,d\theta$$

$$= \int\limits_0^{2\pi} \left[\frac{1}{2}r^2 + \frac{2}{3}r^3\sin\theta\right]_0^3 d\theta$$

$$= \int\limits_0^{2\pi} \left(\frac{1}{2}(3^2) + \frac{2}{3}(3^3)\sin\theta\right) d\theta$$

$$= \int\limits_0^{2\pi} \left(\frac{9}{2} + 18\sin\theta\right) d\theta$$

$$= \left[\frac{9}{2}\theta - 18\cos\theta\right]_0^{2\pi}$$

$$= \left(\frac{9}{2}(2\pi) - 18\cos(2\pi)\right) - \left(\frac{9}{2}(0) - 18\cos(0)\right)$$

$$= (9\pi - 18) - (0 - 18) = 9\pi - 18 + 18 = 9\pi$$

Thus, $\oint\limits_C \vec{F} \cdot d\hat{r} = 9\pi$.

1.15 Engineering Applications of Stokes' Theorem

Stokes' Theorem is not just a mathematical curiosity; it has profound applications in various fields of engineering and physics, particularly where vector fields describe physical quantities.

1.15.1 Electromagnetism (Faraday's Law of Induction)

One of the most direct and famous applications is in Maxwell's equations. Faraday's Law of Induction, in its integral form, states that the electromotive force (EMF) induced in a closed loop is equal to the negative rate of change of magnetic flux through any surface bounded by the loop. Mathematically, $\mathcal{E} = \oint\limits_C \hat{E} \cdot d\hat{r} = -\frac{d\Phi_B}{dt}$, where $\hat{E}$ is the electric field and $\Phi_B = \iint\limits_S \hat{B} \cdot d\vec{S}$ is the magnetic flux. Using Stokes' Theorem on the left side:

$$\oint\limits_C \hat{E} \cdot d\hat{r} = \iint\limits_S (\vec{\nabla} \times \hat{E}) \cdot d\vec{S}$$

So, $\iint\limits_S (\vec{\nabla} \times \hat{E}) \cdot d\vec{S} = -\frac{d}{dt} \iint\limits_S \hat{B} \cdot d\vec{S}$. If the surface S is fixed in time, we can bring the time derivative inside the integral:

$$\iint\limits_S (\vec{\nabla} \times \hat{E}) \cdot d\vec{S} = \iint\limits_S \left(-\frac{\partial \hat{B}}{\partial t}\right) \cdot d\vec{S}$$

For this to hold for any arbitrary surface S, the integrands must be equal, leading to the differential form of Faraday's Law:

$$\vec{\nabla} \times \hat{E} = -\frac{\partial \hat{B}}{\partial t}$$

This shows how a changing magnetic field creates a circulating electric field, a fundamental principle behind generators and transformers.

1.15.2 Fluid Dynamics (Vorticity and Circulation)

In fluid dynamics, the vector field $\vec{F}$ can represent the velocity field $\hat{v}$ of a fluid. The line integral $\oint\limits_C \hat{v} \cdot d\hat{r}$ is called the *circulation* of the fluid around the curve C. It measures the tendency of the fluid to rotate about the loop. The curl of the velocity field, $\vec{\nabla} \times \hat{v}$, is known as the *vorticity* of the fluid, often denoted by $\vec{\omega}$. Vorticity is a measure of the local rotation of the fluid. Stokes' Theorem then states:

$$\text{Circulation} = \oint\limits_C \hat{v} \cdot d\hat{r} = \iint\limits_S (\vec{\nabla} \times \hat{v}) \cdot d\vec{S} = \iint\limits_S \boldsymbol{\omega} \cdot d\vec{S}$$

This means the total circulation around a closed loop is equal to the total flux of vorticity through any surface bounded by that loop. This is critical for understanding phenomena like turbulence, lift on an airfoil, and the dynamics of hurricanes. For example, if a fluid flow is irrotational ($\vec{\nabla} \times \hat{v} = \hat{0}$), then the circulation around any closed loop is zero.

1.15.3 Aerodynamics (Lift on an Airfoil)

The Kutta-Joukowski theorem, which determines the lift generated by an airfoil in a two-dimensional flow, is derived using concepts related to circulation. While not a direct application of 3D Stokes' Theorem, the underlying principle of relating circulation to the rotational properties of the fluid (vorticity) is directly from the concept that Stokes' theorem formalizes. The circulation around an airfoil is directly proportional to the lift force it generates. Understanding the curl of the velocity field around the airfoil section allows engineers to predict and design for lift.

1.15.4 Stress Analysis in Materials

In continuum mechanics, the stress tensor can be represented by a vector field. Stokes' Theorem can be applied to relate the work done by surface forces around a boundary to the internal rotational tendencies (e.g., shear stresses) within the material. For instance, in analyzing the twisting of a shaft, the line integral of the tangential stress around a cross-section can be related to the internal angular deformation (twist per unit length) and the properties of the material.

1.15.5 Magnetic Fields Generated by Currents (Ampere's Law)

Ampere's Law, another of Maxwell's equations, relates the circulation of the magnetic field $\hat{B}$ around a closed loop to the electric current I passing through any surface bounded by the loop.

$$\oint_C \hat{B} \cdot d\hat{r} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

Where I_{enc} is the enclosed current and Φ_E is the electric flux. Applying Stokes' Theorem to the left side:

$$\iint_S (\vec{\nabla} \times \hat{B}) \cdot d\vec{S} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

The enclosed current I_{enc} can also be written as a surface integral of the current density $\hat{j}$: $I_{\text{enc}} = \iint_S \hat{j} \cdot d\vec{S}$. Thus,

$$\iint_S (\vec{\nabla} \times \hat{B}) \cdot d\vec{S} = \iint_S \mu_0 \hat{j} \cdot d\vec{S} + \mu_0 \epsilon_0 \frac{d}{dt} \iint_S \hat{E} \cdot d\vec{S}$$

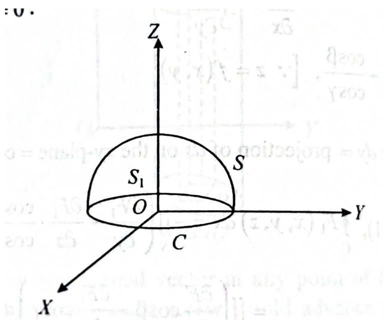
For a fixed surface, this leads to the differential form:

$$\vec{\nabla} \times \hat{B} = \mu_0 \hat{j} + \mu_0 \epsilon_0 \frac{\partial \hat{E}}{\partial t}$$

This equation is fundamental to understanding how electric currents and changing electric fields generate magnetic fields, essential in designing motors, inductors, and communication systems.

1.16 Illustrated Examples

EX.1 Verify Stoke's theorem for $\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$, where S is the upper half surface of the sphere $x^2+y^2+z^2 = 1$ and C is its boundary.
Sol. The boundary C of S is a circle in the $xy-$plane whose equation is $x^2+y^2 = 1$, $z = 0$. So, Let the parametric euqation of C be $x = \cos t$, $y = \sin t$, $z = 0$, $0 \le t \le 2\pi$.



$$\therefore \oint_C \vec{F} \cdot d\vec{r} = \oint_C \left[(2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}\right].(dx\hat{i} + dy\hat{j} + dz\hat{k})$$
$$= \oint_C \left[(2x-y)dx - yz^2dy - y^2zdz\right]$$
$$= \oint_C (2x-y)dx, [\because on \quad C, z=0, \therefore dz = 0]$$
$$= \int_0^{2\pi} (2\cos t - \sin t)(-1)\sin t dt$$
$$= \int_0^{2\pi} \left(-2\sin 2t + \frac{1-\cos 2t}{2}\right) dt$$
$$= \left[\frac{\cos 2t}{2} + \frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} = \pi \tag{1.-4}$$

Again

$$\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \hat{k}$$

$$\therefore \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}ds = \iint_{S_1} \operatorname{curl}\vec{F}\cdot\hat{n}\frac{dx\,dy}{\hat{n}\cdot\hat{k}}$$

Where S_1 is the plane region bonded by the circle C

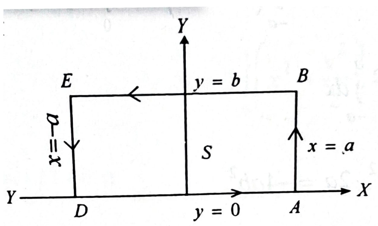
$$= \iint_{S_1}\left(\hat{k}\cdot\hat{n}\right)\frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \iint_{S_1} dx\,dy = \text{ area of } S_1 = \pi \tag{1.-6}$$

[$\because$ area of the region S_1 = area of the circle $(x^2+y^2=1)$= $\pi(1^2)$] $\therefore$ From (1.16) and (1.16) we get

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}ds$$

Hence Stoke's theorem is verified.

EX.2 Verify Stoke's theorem for $\vec{F} = (x^2+y^2)\hat{\imath} - 2xy\hat{\jmath}$ taken round the rectangle bounded by the lines $x=\pm a$, $y=0\,y=b$.
Sol. The given region is shown in the following figure



Here

$$\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = -4y\hat{k}$$

For the surface S lies in $X-Y$ plane, $\hat{n}=\hat{k}$.

$$\therefore \operatorname{curl}\vec{F}\cdot\hat{n} = -4y\hat{k}.\hat{k} = -4y$$

$$\therefore \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}ds = \int\limits_{y=0}^{b}\int\limits_{x=-a}^{a} -4ydxdy = -4\left[\frac{y^2}{2}\right]_0^b [x]_{-a}^{a} = 4ab^2 \tag{1.-7}$$

Here C represents the curve $DABED$. Now,

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C \left[(x^2+y^2)\hat{\imath} - 2xy\hat{\jmath}\right]\cdot\left[dx\hat{\imath} + dy\hat{\jmath}\right] = \oint_C \left[\left(x^2+y^2\right)dx - 2xydy\right]$$

$$= \int_{DA}\left[\left(x^2+y^2\right)dx - 2xydy\right] + \int_{AB}\left[\left(x^2+y^2\right)dx - 2xydy\right]$$

$$+ \int_{BE}\left[\left(x^2+y^2\right)dx - 2xydy\right] + \int_{ED}\left[\left(x^2+y^2\right)dx - 2xydy\right]$$

$$= \int_{-a}^{a} x^2 dx \quad [\because \text{ on } DA,\ y=0,\ dy=0, x \text{ ranges from } -a \text{ to } a]$$

$$+ \int_{0}^{b} (-2ay)dy \quad [\because \text{ on } AB,\ x=a,\ dx=0, y \text{ ranges from } 0 \text{ to } b]$$

$$+ \int_{a}^{-a} (x^2+b^2)dx \quad [\because \text{ on } BE,\ y=b,\ dy=0, x \text{ ranges from } a \text{ to } -a]$$

$$+ \int_{b}^{0} (2ay)dy \quad [\because \text{ on } ED,\ x=-a,\ dx=0, y \text{ ranges from } b \text{ to } 0]$$

$$= \int_{-a}^{a} x^2 dx - \int_{0}^{b}(-2ay)dy - \int_{-a}^{a}(x^2+b^2)dx - \int_{0}^{b}(2ay)dy$$

$$= -4a\int_{0}^{b} ydy - b^2\int_{-a}^{a} dx = -4a\cdot\frac{1}{2}b^2 - b^2\cdot 2a = -4ab^2 \tag{1.-14}$$

From (1.16) and (1.16), we have

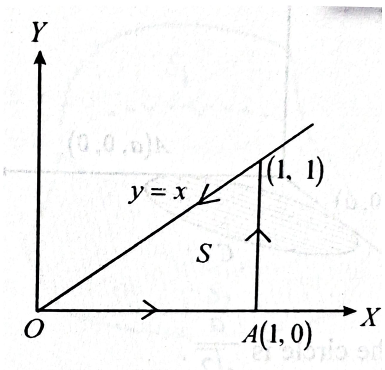
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}ds$$

Hence Stoke's theorem is verified.

EX.3 Evaluate $\oint_C \vec{F}\cdot d\vec{r}$ by Stoke's theorem, where $\vec{F} = y^2\hat{\imath} + x^2\hat{\jmath} - (x+z)\hat{k}$ and C is the boundary of the triangle with vertices at $(0,0,0)$, $(1,0,0)$ and $(1,1,0)$.

Sol. Here the given region is a triangle in XY plane shown in the following figure. Now

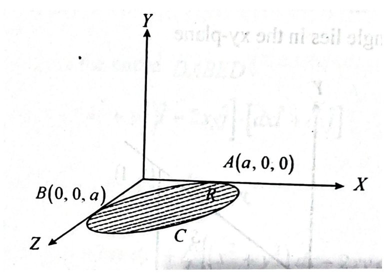
$$\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix} = \hat{\jmath} + 2(x-y)\hat{k}$$



Since, $\hat{n} = \hat{k}$ in $XY-$plane.
$\therefore$ curl $\vec{F} \cdot \hat{n} = 2(x-y)$, since the triangle is the boundary of the surface S.
$\therefore$ By Stoke's theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{F} \cdot \hat{n} ds$$
$$= \int_0^1 \int_0^x 2(x-y) dy dx = 2\int_0^1 \left(x^2 - \frac{x^2}{2}\right) dx = \int_0^1 x^2 dx = \frac{1}{3}$$

EX.4 Apply Stoke's theorem to evaluate $\int_C (ydx + zdy + xdz)$ where C is the curve of intersection of $x^2 + y^2 + z^2 = a^2$ and $x + z = a$.
Sol. The following is the given region as shown in the figure Here the curve C is a circle



with the diameter AB where $A(a, 0, 0$ and $B(0, 0, a)$.
So, the radius of the circle is $\frac{a}{\sqrt{2}}$.
$\therefore \hat{n} = \frac{1 \cdot \hat{i} + 1 \cdot \hat{k}}{\sqrt{2}} = \frac{\hat{i}}{\sqrt{2}} + \frac{\hat{k}}{\sqrt{2}}$

$$\therefore \int_C (ydx + zdy + xdz) = \int_C \left(y\hat{i} + z\hat{j} + x\hat{k}\right) \cdot d\vec{r}$$

$$= \iint_S \text{curl}(dx\hat{i} + dy\hat{j} + dz\hat{k}).\hat{n}ds, [\text{ by Stoke's theorem}]$$

$$= \iint_S -(\hat{i} + \hat{j} + \hat{k}) \cdot \left(\frac{\hat{i}}{\sqrt{2}} + \frac{\hat{k}}{\sqrt{2}}\right) ds$$

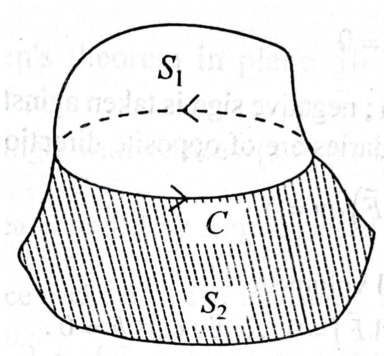
$$[\because \text{curl}\left(y\hat{i} + z\hat{j} + x\hat{k}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x) \end{vmatrix} = -\hat{i} - \hat{j} - \hat{k}]$$

$$= -\frac{2}{\sqrt{2}} \iint_S ds = -\frac{2}{\sqrt{2}}\pi\left(\frac{a}{\sqrt{2}}\right)^2 \quad [\because \text{ area of a circle bounded by } C \text{ is } \pi\left(\frac{a}{\sqrt{2}}\right)^2]$$

$$= -frac\pi a^2\sqrt{2}.$$

EX.5. By Stoke's theorem, prove that curl grad $\phi = \vec{0}$

Sol. Let S be any surface, whose boundary is simple closed curve C. As shown in the following figure. Then by Stoke's theorem, we have



$$\iint_S (\text{curl grad}\,\phi) \cdot \hat{n}ds = \oint_C \text{grad}\,\phi \cdot d\vec{r}$$

Now

$$\text{grad}\,\phi \cdot d\vec{r} = \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right) \cdot \left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$= \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right) = d\phi$$

$$\therefore \oint_C \text{grad}\,\phi \cdot d\vec{r} = \oint_C d\phi = 0 \quad [\because C \text{ is closed curve}]$$

Hence $\iint\limits_S (\operatorname{curl} \operatorname{grad} \phi) \cdot \hat{n} ds = \vec{0}$ which holds for all surface S. $\therefore \operatorname{curl} \operatorname{grad} \phi = \vec{0}$.

EX. 6. By Stoke's theorem, prove that $\operatorname{div} \operatorname{curl} \vec{F} = 0$.
Sol. Let V be an arbitrary volume enclosed by a surface S. Then by divergence theorem

$$\iiint\limits_V \vec{\nabla} \cdot (\operatorname{curl} \vec{F}) dv = \iint\limits_S \vec{F} \cdot \hat{n} ds$$

Devide S by a closed curve C into two surfaces S_1 and S_2. Then

$$\iint\limits_S (\operatorname{curl} \vec{F}.\hat{n} ds = \iint\limits_{S_1} (\operatorname{curl} \vec{F}.\hat{n} ds_1 + \iint\limits_{S_2} (\operatorname{curl} \vec{F}.\hat{n} ds_2$$
$$= \oint\limits_C \vec{F} \cdot d\vec{r} - \oint\limits_C \vec{F} \cdot d\vec{r} = 0$$

[by Stoke's theorem; negative sign is taken against the second integral because the two boundaries are of opposite direction]
Hence $\iiint\limits_V \vec{\nabla} \cdot (\operatorname{curl} \vec{F}) dv = 0$. Which holds for all volume. Therefore $\vec{\nabla} \cdot (\operatorname{curl} \vec{F}) = 0$, $\therefore \operatorname{div} \operatorname{curl} \vec{F} = 0$

EX.7 Prove that $\oint\limits_C \vec{F} \cdot d\vec{r} = 0$, for every closed curve C if and only if $\vec{\nabla} \times \vec{F} = \vec{0}$ every where
Sol. If Part: Let $\vec{\nabla} \times \vec{F} = \vec{0}$, by Stoke's theorem

$$\oint\limits_C \vec{F} \cdot d\vec{r} = \iint\limits_S (\vec{\nabla} \times \vec{F}) \cdot d\hat{n} ds$$
$$\iint\limits_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n} ds = \iint\limits_S \vec{0} \cdot \hat{n} ds = 0.$$

Only if part: Let $\oint\limits_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve C.
Let if possible , $\vec{\nabla} \times \vec{F} \neq 0$ at a point P. Since $\vec{\nabla} \times \vec{F}$ is continuous, so we get a region surrounding P where $\vec{\nabla} \times \vec{F} \neq 0$. We take a surface S within this region whose boundary is C and whose normal has same direction as $\vec{\nabla} \times \vec{F}$ at each point (on the surface). So $\vec{\nabla} \times \vec{F} = k\hat{n}$, where k is a positive constant.
Then by Stoke's theorem

$$\oint\limits_C \vec{F} \cdot d\vec{r} = \iint\limits_S \left(\vec{\nabla} \times \vec{F}\right) \cdot \hat{n} ds = \iint\limits_S k\hat{n} \cdot \hat{n} ds = k \iint\limits_S |\hat{n}|^2 = k \iint\limits_S ds$$
$$= k \times \text{ area of } S \neq 0.$$

This contradicts our hypothesis. So,$\vec{\nabla} \times \vec{F} = \vec{0}$

1.17 Exercises

1. Verify Stokes' Theorem for $\vec{F}(x, y, z) = y\hat{\imath} - x\hat{\jmath} + z\hat{k}$ and S being the surface of the cone $z = \sqrt{x^2 + y^2}$ below $z = 1$, oriented such that the normal points outwards.

2. Use Stokes' Theorem to evaluate $\oint\limits_C \vec{F} \cdot d\hat{r}$ where $\vec{F}(x, y, z) = z\hat{\imath} + x\hat{\jmath} + y\hat{k}$ and C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, oriented counterclockwise when viewed from the positive z-axis.

3. Evaluate $\iint\limits_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$ where $\vec{F}(x, y, z) = (x^2 + y - z)\hat{\imath} + (x + y^2)\hat{\jmath} + (x - y + z^2)\hat{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the plane $z = 1$, oriented upwards.

4. Let $\vec{F}(x, y, z) = (x - y)\hat{\imath} + (y - z)\hat{\jmath} + (z - x)\hat{k}$. Evaluate $\oint\limits_C \vec{F} \cdot d\hat{r}$ where C is the boundary of the surface S defined by $x^2 + y^2 = 9$ for $0 \leq z \leq 2$, oriented counterclockwise when viewed from the positive z-axis. (Hint: Choose a simpler surface or use direct line integral if easier for this specific boundary geometry.)

5. A vector field is given by $\hat{G}(x, y, z) = (yz)\hat{\imath} + (xz)\hat{\jmath} + (xy)\hat{k}$. Show that $\vec{\nabla} \times \hat{G} = \hat{0}$. What does this imply about $\oint\limits_C \hat{G} \cdot d\hat{r}$ for any closed curve C?

6. Evaluate $\oint\limits_C \vec{F} \cdot d\hat{r}$ for $\vec{F}(x, y, z) = (x^2 + y^2)\hat{\imath} + 2xz\hat{\jmath} + 2yz\hat{k}$ where C is the intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise when viewed from above.

7. Use Stokes' Theorem to calculate the circulation of $\vec{F}(x, y, z) = (y^2 + z^2)\hat{\imath} + (x^2 + z^2)\hat{\jmath} + (x^2 + y^2)\hat{k}$ around the boundary of the surface S given by $z = \sqrt{1 - x^2 - y^2}$ (upper hemisphere of radius 1), oriented upwards.

8. If $\vec{F} = \vec{\nabla} f$ for some scalar function f, what is $\vec{\nabla} \times \vec{F}$? Use this result and Stokes' Theorem to explain why line integrals of conservative vector fields are path-independent for closed curves.

9. Let S be the square with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(-1, 0, 0)$, and $(0, -1, 0)$, oriented with normal pointing away from the origin. Let $\vec{F}(x, y, z) = z\hat{\imath} - y\hat{\jmath} + x\hat{k}$. Calculate $\iint\limits_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$.

10. Consider the vector field $\hat{v}(x, y, z) = (-y)\hat{\imath} + (x)\hat{\jmath} + (0)\hat{k}$. Calculate the circulation of this field around a circle of radius R in the xy-plane centered at the origin, using both direct line integral calculation and Stokes' Theorem.

11. Use Stokes' theorem to prove that $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$.

12. State Stokes' Theorem.

13. Apply Stokes' Theorem to evaluate $\oint_C y\,dx + z\,dy + x\,dz$ where C is the curve of intersection of $x^2 + y^2 + z^2 = a^2$ and $x + z = a$.

14. Verify Stokes' Theorem for $\vec{F} = (2x - y)\hat{\imath} - yz^2\hat{\jmath} - y^2z\hat{k}$ where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and its boundary.

15. Verify Stokes' theorem for $\hat{A} = (y - z + 2)\hat{\imath} + (yz + 4)\hat{\jmath} - xz\hat{k}$ over the surface of the cube defined by $x = 0, y = 0, z = 0$ and $x = 2, y = 2, z = 2$ above the xy-plane.

Choose the single best answer for each question.

1. Stokes' Theorem relates:

   (a) A line integral over a closed curve to a volume integral.

   (b) A surface integral over a closed surface to a volume integral.

   (c) A line integral over a closed curve to a surface integral over a surface bounded by the curve.

   (d) A surface integral over a closed surface to a line integral over its boundary.

   Answer: (c)

2. The curl of a vector field $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$ is given by:

   (a) $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$

   (b) $\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$

   (c) $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

   (d) $\left(\frac{\partial R}{\partial y} + \frac{\partial Q}{\partial z}\right)\hat{\imath} + \left(\frac{\partial P}{\partial z} + \frac{\partial R}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right)\hat{k}$

   Answer: (b)

3. If a vector field $\vec{F}$ is conservative, then according to Stokes' Theorem:

   (a) $\vec{\nabla} \cdot \vec{F} = 0$

   (b) $\vec{\nabla} \times \vec{F} = \hat{0}$

   (c) $\oint_C \vec{F} \cdot d\hat{r} = 1$

   (d) $\iint_S \vec{F} \cdot d\vec{S} = 0$

Answer: (b)

4. In Stokes' Theorem, the orientation of the boundary curve C and the surface S must be consistent. This consistency is typically determined by the:
   (a) Left-hand rule
   (b) Right-hand rule
   (c) Perpendicularity rule
   (d) Parallelism rule

   Answer: (b)

5. Stokes' Theorem is a generalization of which theorem?
   (a) Divergence Theorem
   (b) Fundamental Theorem of Calculus
   (c) Green's Theorem
   (d) Gauss's Law

   Answer: (c)

6. The line integral $\oint_C \vec{F} \cdot d\hat{r}$ represents:
   (a) The flux of $\vec{F}$
   (b) The divergence of $\vec{F}$
   (c) The circulation of $\vec{F}$
   (d) The potential of $\vec{F}$

   Answer: (c)

7. Which of the following physical laws is a direct application or is closely related to Stokes' Theorem?
   (a) Newton's Law of Universal Gravitation
   (b) Coulomb's Law
   (c) Faraday's Law of Induction
   (d) Ohm's Law

Answer: (c)

8. If $\vec{F}(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k}$, what is $\vec{\nabla} \times \vec{F}$?
   (a) 1
   (b) 3
   (c) $\hat{0}$
   (d) $x\hat{i} + y\hat{j} + z\hat{k}$

   Answer: (c)

9. Given $\oint\limits_C \vec{F} \cdot d\hat{r} = 5$, and S_1 and S_2 are two different surfaces both bounded by C and consistently oriented. What can be said about $\iint_{S_1} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$ and $\iint_{S_2} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$?
   (a) They must be equal to 0.
   (b) They must be equal to 5.
   (c) Their sum must be 5.
   (d) They are unrelated.

   Answer: (b)

10. In fluid dynamics, the curl of the velocity field ($\vec{\nabla} \times \hat{v}$) is known as:
   (a) Divergence
   (b) Flow rate
   (c) Vorticity
   (d) Pressure

   Answer: (c)

1.18 Gauss Divergence Theorem

Theorem 3. *Let $V \subset \mathbb{R}^3$ be a bounded region with piecewise-smooth boundary surface $S = \partial V$, oriented by the outward unit normal $\mathbf{n}$. If $\mathbf{F} = (F_1, F_2, F_3)$ is a continuously differentiable vector field on an open set containing V, then*

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS \;=\; \iiint_V \nabla \cdot \mathbf{F}\, dV,$$

where $\nabla \cdot \mathbf{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$.

2. Short explanation

The divergence theorem is a 3D generalization of the fundamental theorem of calculus (and of Green's theorem). Intuitively, the divergence at a point measures the net rate at which the vector field "flows out" of an infinitesimal volume around that point. Integrating this local outflow over the whole volume equals the total flux across the boundary surface.

3. Applications of Gauss Divergence Theorem in Different Engineering Branches

The **Gauss Divergence Theorem** is a fundamental result in vector calculus that relates the flux of a vector field through a closed surface to the volume integral of the divergence of the field inside the surface. It is widely applied in different branches of engineering to simplify the analysis of physical systems involving fluid flow, heat transfer, electromagnetics, and stress analysis.

4. Problems and Solutions

Problem 1. *Compute the outward flux of* $\mathbf{F}(x, y, z) = \langle x, y, z\rangle$ *across the sphere* S *of radius* a *centered at the origin.*

Solution 1. *Compute divergence:*

$$\nabla \cdot \mathbf{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3.$$

By the divergence theorem,

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V 3\, dV = 3 \cdot \mathrm{Vol}(V) = 3 \cdot \frac{4}{3}\pi a^3 = 4\pi a^3.$$

So the outward flux is $4\pi a^3$.

Problem 2. *Let V be the region bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below by the plane $z = 0$. Find the flux of $\mathbf{F} = \langle x, y, z^3 \rangle$ outward across the closed surface ∂V.*

Solution 2. *Divergence:*

$$\nabla \cdot \mathbf{F} = 1 + 1 + 3z^2 = 2 + 3z^2.$$

Using cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$. The region has $0 \le r \le 1$ (since $z = 1 - r^2 \ge 0$), $0 \le \theta \le 2\pi$, and $0 \le z \le 1 - r^2$.

Apply divergence theorem:

$$\mathit{Flux} = \iiint_V (2 + 3z^2)\, dV = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} (2 + 3z^2)\, r\, dz\, dr\, d\theta.$$

Compute the inner z-integral:

$$\int_0^{1-r^2} (2 + 3z^2)\, dz = \left[2z + z^3\right]_0^{1-r^2} = 2(1 - r^2) + (1 - r^2)^3.$$

So the radial integral is

$$\int_0^1 \Big(2(1 - r^2) + (1 - r^2)^3\Big) r\, dr.$$

Expand $(1 - r^2)^3 = 1 - 3r^2 + 3r^4 - r^6$. Then

$$2(1 - r^2) + (1 - r^2)^3 = 3 - 5r^2 + 3r^4 - r^6.$$

Multiply by r and integrate from 0 to 1:

$$\int_0^1 (3r - 5r^3 + 3r^5 - r^7)\, dr = \left[\tfrac{3}{2}r^2 - \tfrac{5}{4}r^4 + \tfrac{3}{6}r^6 - \tfrac{1}{8}r^8\right]_0^1 = \tfrac{3}{2} - \tfrac{5}{4} + \tfrac{1}{2} - \tfrac{1}{8}.$$

Combine fractions: $\tfrac{3}{2} + \tfrac{1}{2} = 2$, so

$$2 - \tfrac{5}{4} - \tfrac{1}{8} = \frac{16 - 10 - 1}{8} = \frac{5}{8}.$$

Multiply by the angular integral 2π:

$$\mathit{Flux} = 2\pi \cdot \frac{5}{8} = \frac{5\pi}{4}.$$

So the outward flux is $\dfrac{5\pi}{4}$.

Problem 3. *Let* $\mathbf{F} = \langle -y, x, 0\rangle$*. Show that the flux of* $\mathbf{F}$ *across any closed surface is zero.*

Solution 3. *Compute divergence:*

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(0) = 0 + 0 + 0 = 0.$$

By the divergence theorem, for any region V *with boundary* S*,*

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V \nabla \cdot \mathbf{F}\, dV = \iiint_V 0\, dV = 0.$$

Thus the flux across any closed surface is 0*. (Fields with zero divergence are called* divergence-free *or* solenoidal*.)*

Problem 4. *Let* V *be the cube* $0 \le x, y, z \le a$*. Compute the outward flux of* $\mathbf{F} = \langle e^x, y^2, z^3\rangle$ *across the boundary of the cube.*

Solution 4. *Compute divergence:*

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}e^x + \frac{\partial}{\partial y}y^2 + \frac{\partial}{\partial z}z^3 = e^x + 2y + 3z^2.$$

By divergence theorem,

$$\text{Flux} = \iiint_V (e^x + 2y + 3z^2)\, dV.$$

Since the domain is a product, separate integrals:

$$\iiint_V e^x\, dV = \left(\int_0^a e^x dx\right)\left(\int_0^a dy\right)\left(\int_0^a dz\right) = (e^a - 1) \cdot a \cdot a = a^2(e^a - 1).$$

$$\iiint_V 2y\, dV = \left(\int_0^a dx\right)\left(\int_0^a dz\right)\left(\int_0^a 2y\, dy\right) = a \cdot a \cdot \left[y^2\right]_0^a = a^2 \cdot a^2 = a^4.$$

$$\iiint_V 3z^2\, dV = \left(\int_0^a dx\right)\left(\int_0^a dy\right)\left(\int_0^a 3z^2 dz\right) = a \cdot a \cdot \left[z^3\right]_0^a = a^2 \cdot a^3 = a^5.$$

Add them:

$$\text{Flux} = a^2(e^a - 1) + a^4 + a^5.$$

Problem 5. *Verify the divergence theorem for*

$$\mathbf{F} = 4xz\,\mathbf{i} - y^2\,\mathbf{j} + yz\,\mathbf{k}$$

taken over the unit cube $0 \le x, y, z \le 1$*.*

Solution 5. *Compute the divergence:*

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz) = 4z - 2y + y = 4z - y.$$

By the divergence theorem,

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V (4z - y)\, dV = \int_0^1 \int_0^1 \int_0^1 (4z - y)\, dx\, dy\, dz.$$

Integrate in x (gives factor 1):

$$\int_0^1 \int_0^1 (4z - y)\, dy\, dz.$$

Integrate in y:

$$\int_0^1 \left[4zy - \tfrac{1}{2}y^2\right]_{y=0}^1 dz = \int_0^1 (4z - \tfrac{1}{2})\, dz.$$

Integrate in z:

$$\left[2z^2 - \tfrac{1}{2}z\right]_0^1 = 2 - \tfrac{1}{2} = \boxed{\tfrac{3}{2}}.$$

So the total flux across the closed surface is $\dfrac{3}{2}$.

Now, $\iint\limits_S \vec{F} \cdot \hat{n}\, ds = \iint\limits_{S_1} \vec{F} \cdot \hat{n}\, ds + \iint\limits_{S_2} \vec{F} \cdot \hat{n}\, ds + ... + \iint\limits_{S_6} \vec{F} \cdot \hat{n}\, ds,$

where S_1 is the face $DEFG$, S_2 the face $ABCO$, S_3 the face $ABEF$, S_4 the face $OGDC$, S_5 the face $BCDE$ and S_6 the face $AFGO$.

In the integral $\iint\limits_{S_1} \vec{F} \cdot \hat{n}\, ds$, the unit vector $\hat{n}$ is normal to face S_1.

So $\hat{n} = \hat{i}$

$$\therefore \iint\limits_{S_1} \vec{F} \cdot \hat{n}\, ds = \int_{z=0}^1 \int_{y=0}^1 \left(4 \cdot 1 \cdot z\hat{i} - y^2\hat{j} + yz\hat{k}\right) \cdot \hat{i}\, dy\, dz$$

$$= \int_0^1 \int_0^1 4z\, dy\, dz = 2$$

$$\iint\limits_{S_2} \vec{F} \cdot \hat{n}\, ds = \int_{z=0}^1 \int_{y=0}^1 (0 \cdot \hat{i} - y^2\hat{j} + yz\hat{k}) \cdot (-\hat{i})\, dy\, dz$$

$$[\because \; on \; S_2, \hat{n} = -\hat{i}, \; x = 0\,] = 0$$

$$\iint\limits_{S_3} \vec{F} \cdot \hat{n}\, ds = \int_{z=0}^{1} \int_{x=0}^{1} \left(4xz\hat{i} - 1^2\hat{j} + 1 \cdot z\hat{k}\right) \cdot \hat{j}\, dx\, dz$$

$$[\because \; on \; S_3, y = 1, \hat{n} = \hat{j}\,] = -\int_0^1 \int_0^1 dx\, dz = -1$$

$$\iint\limits_{S_4} \vec{F} \cdot \hat{n}\, ds = 0 \quad [\because \; on \; S_4, y = 0, \hat{n} = -\hat{j}\,]$$

$$\iint\limits_{S_5} \vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_0^1 y\, dy\, dz \quad [\because \; on \; S_5, z = 1, \hat{n} = \hat{k}\,]$$

$$= \tfrac{1}{2}$$

$$\iint\limits_{S_6} \vec{F} \cdot \hat{n}\, ds = 0 \quad [\because \; on \; S_6, z = 0, \hat{n} = -\hat{k}\,]$$

$$\therefore \; \iint\limits_{S} \vec{F} \cdot \hat{n}\, ds = 2 - 1 + \tfrac{1}{2} = \tfrac{3}{2} \tag{2}$$

From (1) and (2) the divergence theorem is verified.

Problem 6. *Using the divergence theorem, compute* $\iint\limits_{S} \vec{F} \cdot \hat{n}\, ds$ *for*

$$\mathbf{F} = 4x\,\mathbf{i} - 2y^2\,\mathbf{j} + z^2\,\mathbf{k}$$

over the region bounded by the cylinder $x^2 + y^2 = 4$ *and planes* $z = 0, \; z = 3$.

Solution 6. *Compute the divergence:*

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2) = 4 - 4y + 2z.$$

Let V be the cylinder $0 \le z \le 3$, $x^2+y^2 \le 4$. Use cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $0 \le r \le 2$, $0 \le \theta \le 2\pi$, $0 \le z \le 3$, and $dV = r\,dr\,d\theta\,dz$. Then

$$\iiint_V (4-4y+2z)\,dV = \int_0^{2\pi}\int_0^2\int_0^3 (4-4r\sin\theta+2z)\,r\,dz\,dr\,d\theta.$$

Observe the $-4r\sin\theta$ term: integrating $\sin\theta$ over 0 to 2π gives zero, so that contribution vanishes. Thus

$$\iiint_V (4-4y+2z)\,dV = \int_0^{2\pi}\int_0^2\int_0^3 (4+2z)\,r\,dz\,dr\,d\theta.$$

Integrate in z:

$$\int_0^3 (4+2z)\,dz = \big[4z+z^2\big]_0^3 = 12+9 = 21.$$

So

$$\iiint_V \cdots = \int_0^{2\pi}\int_0^2 21r\,dr\,d\theta = \int_0^{2\pi}\big[\tfrac{21}{2}r^2\big]_0^2\,d\theta = \int_0^{2\pi}\tfrac{21}{2}\cdot 4\,d\theta = \int_0^{2\pi} 42\,d\theta = 42\cdot 2\pi = \boxed{84\pi}.$$

Problem 7. *Use the divergence theorem to evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ where*

$$\mathbf{F} = 3xz\,\mathbf{i} + y^2\,\mathbf{j} - 3yz\,\mathbf{k}$$

and S is the surface of the unit cube $0 \le x, y, z \le 1$.

Solution 7. *Compute divergence:*

$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(3xz) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(-3yz) = 3z + 2y - 3y = 3z - y.$$

Use the cube $V = [0,1]^3$:

$$\iiint_V (3z-y)\,dV = \int_0^1\int_0^1\int_0^1 (3z-y)\,dx\,dy\,dz = \int_0^1\int_0^1 (3z-y)\,dy\,dz$$

(since integrating dx gives 1). Integrate in y:

$$\int_0^1 (3z-y)\,dy = 3z - \tfrac{1}{2}.$$

Integrate in z:

$$\int_0^1 (3z-\tfrac{1}{2})\,dz = \big[\tfrac{3}{2}z^2 - \tfrac{1}{2}z\big]_0^1 = \tfrac{3}{2} - \tfrac{1}{2} = 1.$$

Hence $\boxed{1}$ is the flux.

Problem 8. *Evaluate* $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ *over the entire closed surface of the region above the* xy*-plane bounded by the cone* $z^2 = x^2 + y^2$ *and the plane* $z = 4$*, where*

$$\mathbf{F} = 4xz\,\mathbf{i} + xyz^2\,\mathbf{j} + 3z\,\mathbf{k}.$$

Solution 8. *Compute divergence:*

$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(xyz^2) + \frac{\partial}{\partial z}(3z) = 4z + xz^2 + 3.$$

Let V *be the region* $0 \le z \le 4$*,* $x^2+y^2 \le z^2$*. Use cylindrical coordinates* $x = r\cos\theta$*,* $y = r\sin\theta$*,* $r\in[0,z]$*,* $\theta\in[0,2\pi]$*, with* $dV = r\,dr\,d\theta\,dz$*. Then*

$$\iiint_V (4z + xz^2 + 3)\,dV = \int_0^{2\pi}\int_0^4\int_0^z (4z + r\cos\theta\; z^2 + 3)\,r\,dr\,dz\,d\theta.$$

The term containing $\cos\theta$ *integrates to zero over* θ *(odd in* θ*), so drop it:*

$$= \int_0^{2\pi}\int_0^4\int_0^z (4z+3)\,r\,dr\,dz\,d\theta.$$

Integrate in r*:*

$$\int_0^z r\,dr = \tfrac{1}{2}z^2.$$

So the z*-integral becomes*

$$\int_0^4 (4z+3)\cdot\tfrac{1}{2}z^2\,dz = \tfrac{1}{2}\int_0^4 (4z^3 + 3z^2)\,dz = \tfrac{1}{2}\big[z^4 + z^3\big]_0^4 = \tfrac{1}{2}\big(4^4 + 4^3\big).$$

Thus the full triple integral is

$$\int_0^{2\pi} \tfrac{1}{2}\big(4^4+4^3\big)\,d\theta = 2\pi\cdot\tfrac{1}{2}\big(256+64\big) = \pi(256+64) = \pi\cdot 320 = \boxed{320\pi}.$$

Problem 9. *Evaluate*

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS, \qquad \mathbf{F} = y^2z\,\mathbf{i} + z^2x^2\,\mathbf{j} + z^2y^2\,\mathbf{k},$$

where S *is the part of the sphere* $x^2+y^2+z^2 = 1$ *above the* xy*-plane (the upper hemisphere), oriented outward.*

Solution 9. *Compute divergence:*

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(y^2 z) + \frac{\partial}{\partial y}(z^2 x^2) + \frac{\partial}{\partial z}(z^2 y^2) = 0 + 0 + 2zy^2 = 2zy^2.$$

Hence

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V 2zy^2\, dV,$$

where V is the upper hemisphere: $0 \le r \le 1,\ 0 \le \phi \le \pi/2,\ 0 \le \theta \le 2\pi$ in spherical coordinates

$$x = r\sin\phi\cos\theta, \quad y = r\sin\phi\sin\theta, \quad z = r\cos\phi, \quad dV = r^2 \sin\phi\, dr\, d\phi\, d\theta.$$

Compute the integrand in spherical coordinates:

$$2zy^2 = 2(r\cos\phi)\,(r^2\sin^2\phi\sin^2\theta) = 2r^3\cos\phi\sin^2\phi\sin^2\theta.$$

Thus integrand times dV is

$$2r^3\cos\phi\sin^2\phi\sin^2\theta \cdot r^2\sin\phi\; dr\; d\phi\; d\theta = 2r^5\cos\phi\sin^3\phi\sin^2\theta\; dr\; d\phi\; d\theta.$$

Separate integrals:

$$\iiint_V 2zy^2\, dV = \int_0^{2\pi} \sin^2\theta\, d\theta \cdot \int_0^{\pi/2} \cos\phi\sin^3\phi\, d\phi \cdot \int_0^1 2r^5\, dr.$$

Evaluate each factor:

$$\int_0^{2\pi} \sin^2\theta\, d\theta = \pi, \qquad \int_0^1 2r^5\, dr = 2 \cdot \tfrac{1}{6} = \tfrac{1}{3},$$

and with substitution $u = \sin\phi$ ($du = \cos\phi\, d\phi$),

$$\int_0^{\pi/2} \cos\phi\sin^3\phi\, d\phi = \int_0^1 u^3\, du = \tfrac{1}{4}.$$

Multiply them:

$$\pi \cdot \tfrac{1}{4} \cdot \tfrac{1}{3} = \boxed{\frac{\pi}{12}}.$$

Problem 10. *Evaluate $\iint_S \mathbf{r} \cdot \mathbf{n}\, dS$, where S is any closed surface and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.*

Solution 10. *Compute divergence of* $\mathbf{r}$*:*

$$\nabla \cdot \mathbf{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3.$$

By divergence theorem,

$$\iint_S \mathbf{r} \cdot \mathbf{n}\, dS = \iiint_V \nabla \cdot \mathbf{r}\, dV = \iiint_V 3\, dV = 3 \cdot \mathrm{Vol}(V).$$

So the flux equals $\boxed{3 \cdot \textit{Volume}(V)}$.

Practice Set

(i) Verify the divergence theorem for the vector function

$$\mathbf{F} = (x^2 - yz)\,\mathbf{i} + (y^2 - zx)\,\mathbf{j} + (z^2 - xy)\,\mathbf{k}$$

taken over the rectangular parallelopiped $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$.

(ii) Verify the divergence theorem for the vector function

$$\mathbf{F} = y\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$$

taken over the cylindrical region bounded by $x^2 + y^2 = 9,\ z = 0,\ z = 2$.

(iii) Verify the divergence theorem for the vector function

$$\mathbf{F} = 2xz\,\mathbf{i} + y^2\,\mathbf{j} + yz\,\mathbf{k}$$

taken over the surface of the cube bounded by $x = 0, x = 1, \quad y = 0, y = 1, \quad z = 0, z = 1$.