Regularized Bayesian Best Response Learning in Finite Games

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Learning, Evolution and Games 2021

September 2, 2021

Model

- Consider a population with unit mass.
- Let $S := \{1, \ldots, n\}$ denote the strategy space of the population.
- Let Δ denote the space of probability distributions on S.
- A ***population game*** is defined as a mapping $\omega : \Delta \longrightarrow \mathbb{R}^n$, that is, the payoff to a player exercising strategy $i \in S$ at population state $\boldsymbol{x} \in \Delta$ is given by $\omega_i(\boldsymbol{x})$.
- Let $\mathscr{C}(\Delta; \mathbb{R}^n)$ be the collection of continuous population games, equipped with the topology of uniform convergence.

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- Endow $\mathscr{C}(\Delta;\mathbb{R}^n)$ with the Borel sigma-algebra and a probability measure ξ and call $(\mathscr{C}(\Delta;\mathbb{R}^n), \mathscr{B}(\mathscr{C}(\Delta;\mathbb{R}^n)), \xi)$ the type space of the population.
- A ***Bayesian strategy*** is a Böchner measurable map $\sigma : \mathscr{C}(\Delta;\mathbb{R}^n) \longrightarrow \Delta$ and let Σ be the collection of all Bayesian strategies.
- The ***aggregate distribution*** is a map $\mathscr{E} : \Sigma \longrightarrow \Delta$ such that $i = 1, \ldots, n$,

$$\mathscr{E}^i(\sigma) := \int_{\mathscr{C}(\Delta;\mathbb{R}^n)} \sigma^i(\omega)\xi(d\omega); \quad \text{for all } \sigma \in \Sigma.$$

- A mapping $\mathscr{G} : \Sigma \times \mathscr{C}(\Delta;\mathbb{R}^n) \longrightarrow \mathbb{R}^n$ such that for every $\sigma \in \Sigma$,

$$\mathscr{G}(\sigma, \omega) = \omega(\mathscr{E}(\sigma)); \quad \text{for all } \omega \in \mathscr{C}(\Delta;\mathbb{R}^n)$$

is called a ***Bayesian aggregate population game***.

Bayesian best response

- Consider the type space $(\mathscr{C}(\Delta;\mathbb{R}^n),\xi)$. The ***Bayesian best response correspondence*** is a mapping $\beta:\Sigma \implies \Sigma$ is such that for all $\sigma\in\Sigma$,

$$\beta[\sigma](\omega) := \arg\max_{y\in\Delta(S)} \langle y, \mathscr{G}(\sigma,\omega)\rangle; \quad \xi - \text{a.s.}$$

- To ensure a unique best response, we resort to regularization, in view of which we have the following definition.

Definition (*Regularizer (Coucheney et al. (2015))*)

A mapping $v:\Delta \longrightarrow \mathbb{R}\cup\{\infty\}$ is called regularizer if it is finite except possibly on the relative boundary $\partial\Delta$ of Δ, continuous on Δ, smooth on Δ° with $\|\nabla v(x_n)\| \longrightarrow \infty$ as $x_n \longrightarrow \partial\Delta$, and convex on Δ, strongly convex on Δ°.

Regularized Bayesian best response

- Some examples of regularizers which have been widely used in theory of evolution and learning in finite games include:
  1. ***Shannon-Gibbs entropy***: $v(\boldsymbol{x}) := -\sum_j x_j \log x_j$.
  2. ***Tsallis entropy***: $v(\boldsymbol{x}) := (1-\gamma)^{-1}\sum_j (x_j - x_j^{\gamma})$, for $0 < \gamma < 1$.
  3. ***Burg entropy***: $v(\boldsymbol{x}) := -\sum_j \log x_j$.

Definition (*Regularized Bayesian best response*)

For $\varepsilon > 0$, the regularized Bayesian best response is a mapping $\beta_\varepsilon : \Sigma \longrightarrow \Sigma$ such that for every $\boldsymbol{x} \in \Delta$,

$$\beta_\varepsilon[\sigma](\omega) := \arg\max_{\boldsymbol{y}\in\Delta} \langle \boldsymbol{y}, \mathscr{G}(\sigma,\omega)\rangle - \varepsilon v(\boldsymbol{y}); \quad \textit{for all } \omega \in \mathscr{C}(\Delta;\mathbb{R}^n).$$

- The fact that the regularizer v is strongly convex, implies that for every $\sigma \in \Sigma$, the mapping $\omega \longmapsto \beta_\varepsilon[\sigma](\omega)$ is measurable.

A digression to Böchner spaces

- Let $(\widehat{\Omega}, \widehat{\mathscr{A}}, \widehat{\mu})$ be a probability space and let $\mathscr{X}$ be a Banach space.
- Let $\mathscr{L}^1(\widehat{\Omega}, \widehat{\mu}, \mathscr{X})$ denote the linear space of all ($\widehat{\mu}$-a.s) equivalence classes Böchner integrable functions with norm

$$\|f\|_{\mathscr{L}^1(\widehat{\Omega},\widehat{\mu},\mathscr{X})} := \int_{\widehat{\Omega}} \|f(\widehat{\omega})\| \widehat{\mu}(d\widehat{\omega}); \quad \textit{for all } f \in \mathscr{L}^1(\widehat{\Omega}, \widehat{\mu}, \mathscr{X}).$$

  This norm induces the ***strong topology*** on $\mathscr{L}^1(\widehat{\Omega}, \widehat{\mu}, \mathscr{X})$.
- For $\widehat{\omega} \in \widehat{\Omega}$, let $\langle \cdot, \cdot \rangle(\widehat{\omega}) : \mathscr{L}^1(\widehat{\Omega}, \widehat{\mu}, \mathscr{X}) \times \mathscr{L}^\infty(\widehat{\Omega}, \widehat{\mu}, \mathscr{X}^*) \longrightarrow \mathbb{R}$ be the bilinear pairing defined as

$$\langle f, g \rangle(\widehat{\omega}) := g(\widehat{\omega})(f(\widehat{\omega})); \; \textit{for all } f \in \mathscr{L}^1(\widehat{\Omega}, \widehat{\mu}, \mathscr{X}), g \in \mathscr{L}^\infty(\widehat{\Omega}, \widehat{\mu}, \mathscr{X}^*),$$

  where $\mathscr{X}^*$ denotes the dual space of $\mathscr{X}$.

Contd...

- Let $(\widehat{\Omega}, \widehat{\mathscr{A}}, \widehat{\mu})$ be a probability space and let $\mathscr{X}$ be a Banach space. Then the ***weak topology*** on $\mathscr{L}^1(\widehat{\Omega}, \widehat{\mu}, \mathscr{X})$ is the topology induced by the convergence:

$$f_n \longrightarrow^{\mathbf{w}} f \text{ if and only if } \int_{\widehat{\Omega}} \langle f_n, g\rangle(\widehat{\omega})\widehat{\mu}(d\omega) \longrightarrow \int_{\widehat{\Omega}} \langle f, g\rangle(\widehat{\omega})\widehat{\mu}(d\omega);$$

  for all $g \in \mathscr{L}^\infty(\widehat{\Omega}, \widehat{\mu}, \mathscr{X}^*)$.
- In our case, as we will just see, that $\mathscr{X} = \mathbb{R}^n$, a reflexive Banach space. Hence, as a matter of fact, it follows that the space $\mathscr{L}^1(\widehat{\Omega}, \widehat{\mu}, \mathscr{X})^*$ is isometrically isomorphic to the space $\mathscr{L}^\infty(\widehat{\Omega}, \widehat{\mu}, \mathscr{X}^*)$. This result is due to Ralph S. Phillips.

Regularized Bayesian best response dynamic

- In our paper, we set $\widehat{\Omega} = \mathscr{C}(\Delta;\mathbb{R}^n)$, $\widehat{\mu} = \xi$, and $\mathscr{X} = \mathbb{R}^n$. We call this Böchner space, the space of ***integrable signed Bayesian strategies***. In other words,

$$\widehat{\Sigma} := \left\{ \widehat{\sigma} : \mathscr{C}(\Delta;\mathbb{R}^n) \longrightarrow \mathbb{R}^n : \|\widehat{\sigma}\|_{\mathscr{L}^1(\mathscr{C}_n,\xi,\mathbb{R}^n)} < \infty \right\}.$$

Definition (*Regularized Bayesian best response dynamic*)

Consider the type space $(\mathscr{C}(\Delta;\mathbb{R}^n),\xi)$. Let $\varepsilon > 0$ be a noise parameter and let $\beta_\varepsilon : \Sigma \longrightarrow \Sigma$ be the RBBR map. Then the regularized Bayesian best response learning dynamic is defined as

$$\dot{\sigma} = \beta_\varepsilon(\sigma) - \sigma.$$

- A rest point of the RBBR learning dynamic (or a fixed point of the map β_ε), is a Bayesian strategy σ° satisfying $\dot{\sigma}^\circ \equiv 0$.

Existence of Regularized Bayesian Equilibrium (RBE)

- In view of showing existence of a fixed point of the RBBR map, it is desirable to consider the following subsets of the space of continuous population games.
- For a compact set $K \subseteq \mathbb{R}^n$, let $\mathscr{C}(\Delta; K)$ denote the collection of population games with range contained in the compact subset K.
- Let $\mathscr{C}_{\mathsf{eq}}(\Delta; \mathbb{R}^n)$ denote an equicontinuous family of population games.

Theorem (*Existence of compact support of ξ*)

Let $K \subseteq \mathbb{R}^n$ be any compact set and let $\mathscr{C}_{eq}(\Delta; K)$ be an equicontinuous family of population games with range contained in K. Then subset $\mathscr{C}_{eq}(\Delta; K)$ has compact closure in $\mathscr{C}(\Delta; \mathbb{R}^n)$ under d_∞.

Contd...

Theorem (*Existence of RBE*)

Consider the type space $(\mathscr{C}(\Delta;\mathbb{R}^n),\xi)$. Let $\mathscr{C}_{eq}(\Delta;K)$ be as defined above. Suppose that $\xi(\text{closure}[\mathscr{C}_{eq}(\Delta;K)])=1$. Then for every $\varepsilon>0$, the RBBR mapping $\sigma\longmapsto\beta_\varepsilon(\sigma)$ admits a fixed point, that is, there exists $\sigma_\varepsilon^\circ\in\Sigma$ such that $\beta_\varepsilon(\sigma_\varepsilon^\circ)=\sigma_\varepsilon^\circ$.

- The first step to prove the equilibrium existence result is to show that Σ is a compact subset of $\widehat{\Sigma}$ under the weak topology. The fact that Σ is convex is trivial.
- Next, we show that the mapping $\sigma\longmapsto\beta_\varepsilon(\sigma)$ is continuous in the weak topology.
- Finally, the existence of regularized Bayesian equilibrium follows via the Brouwer-Schauder-Tychonoff fixed point theorem.

Matrix games

- For $F \subseteq \mathbb{R}$ compact, let $\mathscr{M}_n(F)$ be the collection of matrices with entries from F.
- Consider the case where the type measure ξ is concentrated on populations games which are obtained via random matching in matrix games.

Corollary (*Existence of RBE for matrix games*)

The following statements hold true:

1. *For $F \subseteq \mathbb{R}$ compact, the set $\mathscr{M}_n(F)$ has compact closure in $\mathscr{C}(\Delta;\mathbb{R}^n)$.*
2. *If $\xi(\text{closure}[\mathscr{M}_n(F)]) = 1$, then for every $\varepsilon > 0$, the RBBR map admits a fixed point.*

Existence of solutions to the RBBR dynamic

- In order to prove the existence of solutions to the RBBR dynamic, we consider the following definition.

Definition (*Lipschitz Population Game*)

A population game $\omega : \Delta \longrightarrow \mathbb{R}^n$ is Lipschitz if there exists a real number $\alpha > 0$ such that

$$\|\omega(\boldsymbol{x}) - \omega(\boldsymbol{y})\| \leq \alpha \|\boldsymbol{x} - \boldsymbol{y}\|, \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \Delta.$$

- For $\alpha > 0$, and $K \subseteq \mathbb{R}^n$ let $\mathrm{Lip}_\alpha(\Delta; K)$ be the collection of all α-Lipschitz population games with range contained in the set K.

Theorem (*Lipschitz support of ξ*)

For every $\alpha > 0$ and every $K \subseteq \mathbb{R}^n$ compact, the family $\mathrm{Lip}_\alpha(\Delta; K)$ is a compact subset of $\mathscr{C}(\Delta; \mathbb{R}^n)$ under d_∞.

Contd...

- The ***semiflow*** of the RBBR learning dynamic is the map $\zeta : \widehat{\Sigma} \times [0,\infty) \longrightarrow \widehat{\Sigma}$ defined as $\zeta(\sigma,t) := \Phi_\sigma(t)$ for all $\sigma \in \Sigma$ and $t \in [0,\infty)$, where $\Phi_\sigma(t)$ is the position of the trajectory at time $t \in [0,\infty)$ with initial condition $\sigma \in \Sigma$.

Theorem (*Existence of solution to RBBR dynamic*)

Let $(\mathscr{C}(\Delta;\mathbb{R}^n),\xi)$ be a type space, $K \subseteq \mathbb{R}^n$ be compact, and $\alpha > 0$. Suppose that $\xi(\text{closure}[\text{Lip}_\alpha(\Delta;K)]) = 1$. Then,

1. *For every initial condition $\sigma_0 \in \Sigma$, the RBBR dynamic admits a unique solution $(\sigma_t)_{t\geq 0}$.*
2. *The semiflow $\zeta : \widehat{\Sigma} \times [0,\infty) \to \widehat{\Sigma}$ of the RBBR dynamic is continuous in the initial conditions with respect to the strong topology on $\widehat{\Sigma}$.*

Bayesian potential games

Definition (*Bayesian Potential Games*)

A mapping $\mathscr{G} : \Sigma \times \mathscr{C}(\Delta;\mathbb{R}^n) \longrightarrow \mathbb{R}^n$ is called a Bayesian potential game if there exists a Fréchet differentiable map $\varphi : \widehat{\Sigma} \longrightarrow \mathbb{R}$ such that for every $\sigma \in \Sigma$, we have

$$\nabla^{\boldsymbol{F}}\varphi(\sigma)(\omega) = \mathscr{G}(\sigma,\omega), \quad \xi - a.s.$$

The map φ is called the Bayesian potential function of the Bayesian potential game $\mathscr{G}$.

- Let $M \subseteq \Delta^\circ$ be compact. Define

$$\Sigma_M := \{\sigma : \sigma(\omega) \in M \text{ for all } \omega \in \mathscr{C}(\Delta;\mathbb{R}^n)\}.$$

- Define the ***Bayesian counterpart of the regularizer*** as

$$\widetilde{v}(\sigma) := \int_{\mathscr{C}(\Delta;\mathbb{R}^n)} v(\sigma(\omega))\xi(d\omega); \quad \text{for all } \sigma \in \Sigma_M.$$

Contd...

Definition (*Entropy Adjusted Bayesian Potential Function*)

The entropy adjusted Bayesian potential function is a mapping $\widetilde{\varphi} : \Sigma_M \longrightarrow \mathbb{R}$ *defined as*

$$\widetilde{\varphi}_M(\sigma) = \varphi(\sigma) - \widetilde{v}(\sigma); \quad \textit{for all } \sigma \in \Sigma_M.$$

- We now define the notion of a Lipschitz Bayesian strategy.

Definition (*Lipschitz Bayesian strategy*)

A Bayesian strategy $\sigma : \mathscr{C}(\Delta; \mathbb{R}^n) \longrightarrow \Delta$ *is Lipschitz if there exists* $\alpha > 0$ *such that*

$$\|\sigma(\omega) - \sigma(\widetilde{\omega})\| \leq \alpha d_\infty(\omega, \widetilde{\omega}), \quad \textit{for all } \omega, \widetilde{\omega} \in \mathscr{C}(\Delta; \mathbb{R}^n).$$

Contd...

- For $K \subseteq \mathbb{R}^n$ be compact, let $\mathscr{C}_{\text{eq}}(\Delta; K)$ be an equicontinuous family of population games with range contained in K. Let Σ^α be the collection of Bayesian strategies which are α-Lipschitz on the set closure$[\mathscr{C}_{\text{eq}}(\Delta; K)]$.

Theorem (*Forward invariance of Σ_α under RBBRD*)

Let v be a θ-strongly convex regularizer for some $\theta > 0$, and let the type space $(\mathscr{C}(\Delta; \mathbb{R}^n), \xi)$ be such that $\xi(\text{closure}[\mathscr{C}_{eq}(\Delta; K)]) = 1$ for some compact $K \subseteq \mathbb{R}^n$. Suppose Σ^α is the collection of Bayesian strategies that are uniformly α-Lipschitz on closure$[\mathscr{C}_{eq}(\Delta; K)]$ for some $\alpha > 0$. Then the following statements hold true:

1. *The subset Σ^α is relatively norm compact in $\widehat{\Sigma}$.*
2. *If $\alpha \geq \dfrac{1}{\varepsilon\theta}$, then the subset Σ^α is forward invariant under the RBBR learning dynamic.*

Convergence in Bayesian potential games

Theorem (*The Convergence theorem*)

Suppose that the assumptions of the previous theorem are satisfied. Let $\mathscr{G} : \Sigma \times \mathscr{C}(\Delta; \mathbb{R}^n) \longrightarrow \mathbb{R}^n$ be a Bayesian potential game with Bayesian potential function $\varphi : \widehat{\Sigma} \longrightarrow \mathbb{R}$. For $M \subseteq \Delta^\circ$ compact, let $\widetilde{\varphi}_{M,\alpha} : \Sigma_M^\alpha \longrightarrow \mathbb{R}$ be the entropy adjusted Bayesian potential function restricted to Σ_M^α. Then the following statements hold true:

1. *$\widetilde{\varphi}_{M,\alpha}$ increases weakly along every solution trajectory to RBBR learning dynamic that originates in Σ_M^α and increases strictly across every non-stationary solution trajectory.*
2. *The set of omega–limit points (in the strong topology on Σ_M^α) of any trajectory to the RBBR learning dynamic is a non-empty connected compact set of regularized Bayesian equilibria. Moreover, such limit points are local maximizers of the entropy adjusted Bayesian potential function $\widetilde{\varphi}_{M,\alpha}$ on Σ_M^α.*

THANK YOU