Stochastic Differential Equations on euclidean space:

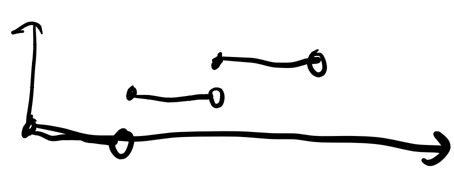
Some required definitions:

Càdlàg: "continue à droite, limite à gauche"
Right continuous with left limits.

Let (M,d) be a metric space and $E \subseteq \mathbb{R}$. A function $f: E \to M$ is càdlàg if for every $t \in E$,

1) $f(t_-) := \lim_{s \uparrow t} f(s)$ exists

2) $f(t_+) := \lim_{s \downarrow t} f(s)$ exists & equals $f(t)$



Bounded variation: Given $u \in L^1(\Omega)$ the total variation of $u \in \Omega$ is

$$V(u,\Omega) := \sup \left\{ \int_\Omega u(x) \operatorname{div} \phi(x)\, dx : \phi \in C_c^1(\Omega, \mathbb{R}^n), \|\phi\|_{L^\infty(\Omega)} \leq 1 \right\}$$

$$BV(\Omega) = \{ u \in L^1(\Omega) : V(u,\Omega) < \infty \}.$$

Local Martingale: Given a probability space (Ω, F, P) and a filtration $F_* = \{F_t \mid t \geq 0\}$ of F. Let $X: [0,\infty) \times \Omega \to S$ be a F_* adapted stochastic process on S. Then X is called an F_*-local martingale if $\exists$ a sequence of F_*-stopping times $\tau_k : \Omega \to [0,\infty)$ such that

(1) τ_k a.s. increasing — $P(\tau_k < \tau_{k+1}) = 1$

(2) τ_k a.s. diverge : $P(\lim_{k\to\infty} \tau_k = \infty) = 1$

(3) the stopped process

$$X_t^{\tau_k} := X_{\min(t,\tau_k)}$$

is an F_* martingale $\forall k$

Reminder for a martingale:

$$\mathbb{E}(X_{n+1} \mid X_1, \dots, X_n) = X_n.$$

Now start (almost): Given a driving semimartingale $Z = \{Z_t,\ t \geq 0\}$ is $\mathbb{R}^l$ valued and Z is an F_*-semimartingale on (Ω, F_*, P) and a diffusion coefficient matrix σ

$$\sigma = \{\sigma^i_\alpha\} : \mathbb{R}^N \to M(N, l).$$

The process is locally Lipschitz if for $R > 0$ there is

$$|\sigma(x) - \sigma(y)| \leq C(R)\,|x - y|,$$
$$\forall x, y \in B(R).$$

If $C(R) = C$ then globally Lipschitz.

Semimartingale: X is a semimartingale on $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ if it can be decomposed as

$$X_t = M_t + A_t$$

M_t is a local martingale

A_t is a càdlàg adapted process with locally bounded variation.

An $\mathbb{R}^n$ process $X = (X^1, \ldots, X^n)$ is a semimartingale if each coordinate is semimartingale.

Ito + Stratonovich -

$W : [0,T] \times \Omega \to \mathbb{R}$ is BM and

$X : [0,T] \times \Omega \to R$ is a semimartingale adapted to $\mathcal{F}_*$ of W.

1) The stratonovich integral

$$\int_0^T X_t \circ dW_t$$

is a r.v. $\Omega \to \mathbb{R}$ that is the limit

$$\sum_{i=0}^{K-1} \frac{X_{t_{i+1}} + X_{t_i}}{2} \left(W_{t_{i+1}} - W_{t_i} \right)$$

as the intervals $0 = t_0 < t_1 \dots < t_K = T$ of $[0, T]$ tends to zero.

If X_t, Y_t, Z_t are s.p. with

$$X_T - X_0 = \int_0^T Y_t \circ dW_t + \int_0^T Z_t \, dt$$

then $\forall\, t > 0$

$$dX = Y \circ dW + Z \, dt$$

2) Itô integral

$$\int_0^T X_t \, dW_t$$

$$\sum_{i=0}^{K-1} X_{t_i} (W_{t_{i+1}} - W_{t_i})$$

as the mesh tends to zero.

Does not obey standard chain rule.

Relation between Itô & Stratonovich

$$\int_0^T f(W_t, t) \circ dW_t =$$

$$\frac{1}{2} \int_0^T \frac{\partial f}{\partial w} (W_t, t)\, dt + \int_0^T f(W_t, t)\, dW_t$$

f is continuously differentiable.

Itô integral does not look into the future.

SDE: $X_0 \in F_0$ is a $\mathbb{R}^w$-valued r.v. w.r. F_0.

τ is an F_t-stopping time and $X = \{X_t,\ 0 \le t \le \tau\}$ is a semimartingale

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dZ_s \quad 0 \le t \le \tau$$

In Itô sense.

$SDE(\sigma, Z, X_0)$.

Ex: $Z_t = (W_t, t)$ with $(\ell-1)$-dimensional BM W & $\sigma = (\sigma_1, b)$ $\sigma_1 : \mathbb{R}^N \to M(N, \ell-1)$

$b : \mathbb{R}^N \to \mathbb{R}^N$

$$dX_t = \sigma_1(X_t)\, dW_t + b(X_t)\, dt.$$

For $f \in C^2(\mathbb{R}^N)$ and f_{x_i}, f_{x_i, x_j} are the first & second partial derivatives. Let X be a solution of $SDE(\sigma, Z, X_0)$, then

$$f(X_t) = f(X_0) + \int_0^t f_{x_i}(X_s)\, \sigma_\alpha^i(X_s)\, dZ_s^\alpha + \frac{1}{2}\int_0^t f_{x_i, x_j}(X_s)\, \sigma_\alpha^i(X_s)\, \sigma_\beta^j(X_s)\, d\langle Z^\alpha, Z^\beta \rangle_s$$

$$= f(X_0) + \int_0^t f_{x_i}(X_s)\, \sigma_\alpha^i(X_s)\, dM_s^\alpha +$$

$$\int_0^t f_{x_i}(X_s)\, \sigma^i_\alpha (X_s)\, dA^\alpha_s +$$

$$\frac{1}{2}\int_0^t f_{x_i,x_j}(X_s)\, \sigma^i_\alpha (X_s)\, \sigma^j_\beta (X_s)\; d\langle M^\alpha, M^\beta\rangle_s$$

Some uniqueness results:

a) If σ is globally Lipschitz and X_0 square integrable then $SDE(\sigma, Z, X_0)$ has a unique solution $X = \{X_t, t \geq 0\}$

b) Explosion: $\frac{dX_t}{dt} = X_t^2$, $X_0 = 1$

$X_t = \frac{1}{1-t}$ explodes at $t=1$

$\hat{M} = M \cup \{\partial M\}$, M is a metric space (M, d)

c) An M-valued path x with explosion time $e = e(x) > 0$

is a continuous map
$x : [0, \infty) \to \hat{M}$ s.t.
$X_t \in M$ $\forall$ $0 \leq t < e$ & $X_t = \partial_M$ $\forall$ $t \geq e$.

The space of M-valued paths with explosion time is the path space of M, $W(M)$.

d) Weaker uniqueness results

i) Suppose σ is locally Lipschitz. Let X, Y be two solutions of $SDE(\sigma, Z, Y_0)$ upto stopping times τ and η. Then $X_t = Y_t$ for $0 \leq t < \tau \wedge \eta$.
In particular if X is the solution upto $e(X)$ then for $\eta < e(X)$, $X_t = Y_t$ $0 \leq t < \eta$.

ii) If (Z, X_0) and $(\hat{Z}, \hat{X}_0)$ have the same law then $SDE(\sigma, Z, X_0)$ and $SDE(\sigma, \hat{Z}, \hat{X}_0)$ have the same law.

e) Z is defined on $[0,\infty)$, σ is locally Lipschitz and there is a C s.t. $|\sigma(x)| \leq C(1+|x|)$ then $SDE(\sigma, Z, x_0)$ does not explode

Example: OU process

$$dX_t = dZ_t - X\,dt$$

$$X_t = e^{-t} X_0 + \int_0^t e^{-(t-s)}\, dZ_s$$

Z is 1-dimensional B.M.

X is a diffusion with generator

$$L = \frac{1}{2}\left(\frac{d}{dx}\right)^2 - x\frac{d}{dx}$$

Multivariate Integrals

V_α, $\alpha = 1, \dots, l$ are smooth vector fields on $\mathbb{R}^n$

$V_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ + $V = (V_1, \dots, V_l)$ is a $M(n, l)$-valued fcn.

Given Z, X_0 the Stratonovitch stochastic differential equation

$$X_t = X_0 + \int_0^t V(X_s) \circ dZ_s$$

$$= X_0 + \int_0^t V_\alpha(X_s) \circ dZ_s^\alpha$$

the above integrals are in the Stratonovitch.

The Itô form is

$$X_t = X_0 + \int_0^t V_\alpha(X_s)\, dZ_s^\alpha + \frac{1}{2}\int_0^t \nabla_{V_\beta} V_\alpha (X_s)\, d\langle Z^\alpha, Z^\beta\rangle_s$$

$\nabla_{V_\beta} V_\alpha$ = derivative of V_α along V_β

SDE's on manifolds

Def. Let M be a differentiable manifold and $(\Omega, F_*, \mathbb{P})$ a filtered probability space. Let τ be an F_*-stopping time. A continuous M-valued process X defined on $[0, \tau)$ is called an M-valued semimartingale if $f(X)$ is a real-valued semimartingale on $[0, \tau)$ for all $f \in C^\infty(M)$.

A stochastic differential equation on a manifold M is defined by l vector fields $V_1, \ldots, V_l$ on M, an $\mathbb{R}^l$-valued driving semimartingale Z, and a M-valued random variable $X_0 \in \mathbb{F}_0$ as the initial value

$$dX_t = V_\alpha(X_t) \circ dZ_s^\alpha$$

is $SDE(V_1, \ldots, V_l; Z, X_0)$.

Def. An M-valued semimartingale X defined upto stopping time τ is a solation of $SDE(V_1, \dots, V_\ell; Z, X_0)$ upto τ if for all $f \in C^\infty(M)$

$$f(X_t) = f(X_0) + \int_0^t V_\alpha f(X_s) \circ dZ_s^\alpha$$

$$0 \leq t \leq \tau.$$

The reason to use the Stratanovich formulation is that the formulation is consistent under diffeomorphisms between manifolds.

$\Gamma(TM)$ is the space of vector fields on M (the space of sections of the tangent bundle TM).

A diffeomorphism $\Phi : M \to N$ induces a map $\Phi_* : \Gamma(TM) \to \Gamma(TN)$ between the respective vector fields

$$(\Phi_* V) f(y) = V(f \circ \Phi)_x,$$
$$y = \Phi(x), \quad f \in C^\infty(N).$$

Prop: 1.2.9

$\Phi : M \to N$ is a diffeomorphism and X a solution to $SDE(V_1, \dots, V_e ; Z, X_0)$ then $\Phi(X)$ is a solution to $SDE(\Phi_* V_1, \dots, \Phi_* V_e ; Z, \Phi(X_0))$ on N.

To prove $SDE(V_1, \dots, V_e ; Z, X_0)$ has a unique solution upto explosion use Whitney's embedding theorem to reduce to an equation on euclidean space.

Whitney's embedding: M is a differentiable manifold. There exists an embedding $i: M \to \mathbb{R}^N$ s.t. the image $i(M)$ is a closed subset of $\mathbb{R}^N$. $N = 2\dim M + 1$ will suffice.

Setting up a coordinate system to use as test functions.

M is a closed submanifold on $\mathbb{R}^N$. $x \in M$ has N coordinates $\{x^1, \dots, x^N\}$ we will use $f^i(x) = x^i$ in the Itô formulation. Set X as an M-valued continuous process and $f^1, \dots, f^N$ are coordinate fcns.

(i) X is a semimartingale if and only if it is an $\mathbb{R}^N$-valued semimartingale, $f^i(X)$ is a real-valued semimartingale.

for $i=1,\dots,N$

ii) X is the solution of $SDE(V_1,\dots,V_e; Z, X_0)$ up to stopping time σ if and only if $i=1,\dots,N$

$$f^i(X_t) = f^i(X_0) + \int_0^t V_\alpha f^i(X_s) \circ dZ^\alpha_s,$$

$$0 \leq t < \sigma$$