Horizontal lift - stochastic formulation -

Given M with a connection there are invariantly defined horizontal vector fields H_t on the frame bundle $F(M)$.

The results on curves translate to semimartingales.

Start with the SDE on the frame bundle $F(M)$

$$dU_t = H(U_t) \circ dW_t$$

W is an $\mathbb{R}^d$-valued semimartingale

$$dU_t = \sum_{i=1}^{d} H_i(U_i) \circ dW_i^i$$

M has a connection and $\{H_i\}$ are the horizontal vector fields on $F(M)$.

Def. (i) An $F(M)$-valued semimartingale is horizontal if there exists an $\mathbb{R}^d$-valued semimartingale W s.t. ✱ holds. The unique W is called the anti-development of U, $X = \pi U$.

(ii) Let W be a $\mathbb{R}^d$-valued semimartingale and U_0 an $F(M)$-valued, $\mathcal{F}_0$-measurable random variable. The solution U of the SDE ✱ is called a stochastic development W in $F(M)$. The projection $X = \pi U$ is the stochastic development of W in M.

(iii) Let X be an M-valued semimartingale. An $F(M)$-

valued horizontal semimartingale U s.t. $\pi U = X$ is called a stochastic horizontal lift of X.

The EEM construction relies on the existence of the following maps

$$W \longleftrightarrow U \longleftrightarrow X.$$

We now need to show

$$X \mapsto U$$
$$U \mapsto W$$

$X \longmapsto U$ exists and is unique

$U \longmapsto W$

We will look at evolution of frame bundle $F(M)$ driven by X.

M is a closed submanifold of $\mathbb{R}^N$ and $X = \{X^\alpha\}$ as a $\mathbb{R}^N$-valued semimartingale.

For each $x \in M$ let $P(x): \mathbb{R}^N \to T_x M$ be an orthogonal projection from $\mathbb{R}^N$ onto the subspace $T_x M \subseteq \mathbb{R}^N$, we want to prove

$$X_t = X_0 + \int_0^t P(X_s) \circ dX_t$$

or

$$dX_t = P_\alpha(X_t) \circ dX_t^\alpha.$$

The horizontal lift U of X is the solution of of the following process on $F(M)$

$$dU_t = \sum_{\alpha=1}^{N} P_\alpha^*(U_t) \circ dX_t^\alpha \quad \bigstar$$

$P_\alpha^*(u)$ is the horizontal lift of $P_\alpha(\pi u)$

Proof of $\bigstar$:

$f = \{f^\alpha\} : M \to \mathbb{R}^N$ is the coordinate function.

The lift $\tilde{f} : F(M) \to \mathbb{R}^N$

$$\tilde{f}(u) = f(\pi u) = \pi u \in M \subseteq \mathbb{R}^N$$

is $\pi : F(M) \to M$ written as a $\mathbb{R}^N$-valued function on $F(M)$.

Let e_i be the i-th coordinate vector of $\mathbb{R}^d$.

Lemma: 2.3.2

Let $\hat{f} : F(M) \to M \subseteq \mathbb{R}^N$ be π. The following identities hold on $F(M)$:

$$P_\alpha^{*} \hat{f}(u) = P_\alpha(\pi u)$$

$$\sum_{\alpha=1}^{N} P_\alpha(\pi u) H_i \hat{f}^\alpha(u) = u e_i$$

Every submartingale on a manifold is a solution of a Stratonovich type SDE on the manifold.

Lemma 2.3.3: Suppose M is a closed submanifold of $\mathbb{R}^N$. For each $x \in M$ let $P(x) : \mathbb{R}^N \to T_x M$ be the orthogonal projection from $\mathbb{R}^N \to T_x M$. If X is a M-valued semimartingale then

$$X_t = X_0 + \int_0^t P(X_s) \circ dX_s .$$

2.3.4

A horizontal semimartingale U on the frame bundle $F(M)$ has a unique anti-development

$$W_t = \int_0^t U_s^{-1} P_\alpha(X_s) \circ dX_s^\alpha$$

$X_s = \pi U_s$.

Theorem 2.3.5:

$X = \{X_t, 0 \leq t \leq \tau\}$ is a semimartingale on M upto τ, U_0 is an $F(M)$-valued $\mathcal{F}_0$-r.v. s.t. $\pi U_0 = X_0$. There is a unique horizontal lift $\{U_t, 0 \leq t < \tau\}$ of X starting at U_0.