Basic Riemannian geometry

g is a $(0,2)$ tensor that is symmetric & positive definite

(M, g) Riemannian manifold

Let $v, w \in T_pM$, then $\langle v, w \rangle = g_{ij} v^i w^j$ $\qquad g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$

or $g = g_{ij}\, dx^i \otimes dx^j$

uses of metrics

- gives us length of vector.

$\gamma : [0,1] \to M$ piecewise smooth

$$l(\gamma) = \int_0^1 \langle \gamma'(t), \gamma'(t) \rangle^{1/2} dt \qquad \gamma'(t) \in T_{\gamma(t)}M.$$

(independent of parametrization)

- g gives distance on manifold.

$$d(p,q) = \inf_{\substack{\gamma \text{ curv} \\ \text{fr } p \text{ to } q}} L(\gamma)$$

- volume form. Assume M is oriented. Then g gives $\omega \in \Omega^n(M)$

In local coords, $\omega = \sqrt{\det(g_{ij})}\, dx_1 \wedge \cdots \wedge dx_m$

Laplace Beltrami operator:

generalises the usual Laplace operator on euclidean space.

It is defined by $\Delta_M f = \operatorname{div}(\operatorname{grad} f)$

where $\operatorname{grad} f$ is the dual of the differential df i.e

$$\langle \operatorname{grad} f, X \rangle = df(X) = Xf \quad \forall\, X \in \Gamma(TM)$$

In local coords $x = \{x^i\}$,

$$\nabla f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}$$

using $g_{ij} = \langle \partial_i, \partial_j \rangle$

$\operatorname{div}(X)$ is the contraction of the $(1,1)$ tensor ∇X.

write $X = a^i \frac{\partial}{\partial x^i}$ then in local coords $x = (x^i)$

$$\operatorname{div} X = \frac{1}{\sqrt{G}} \frac{\partial(\sqrt{G}\, a^i)}{\partial x^i}$$

So Δ_M in local coords is

$$\Delta_M f = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}\left(\sqrt{G}\, g^{ij} \frac{\partial}{\partial x^j}\right)$$

Equivalent formulation: For any orthonormal basis $\{X_i\}$ of $T_x M$ can write $\Delta_M f = \operatorname{trace} \nabla^2 f = \sum_{i=1}^{d} \nabla^2 f(X_i, X_i)$

Connections:

Answer 2 questions:

Q. How to take directional derivatives of vector field?

Q. What are straight lines in manifold?

Def: An affine connection ∇ on M is a map

$$\nabla : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$$
$$(X, Y) \longmapsto \nabla_X Y \qquad \text{satisfying}$$

① $\nabla_{X_1 + X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$, $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$

② $\nabla_{fX} = f \nabla_X Y$ $\quad f \in C^\infty(M)$

③ $\nabla_X(fY) = f\nabla_X Y + X(f)Y$

In coordinates :

Let $x^1, \dots, x^n$ be coordinates, then $T_pM = \langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \rangle$

$T_p^*M = \langle dx^1, \dots, dx^n \rangle$

Vector field $X = a^i \partial_i$, 1-form $\eta = b_i dx^i$

for $f \in C^\infty(U)$, $Xf = a^i \partial_i f$

Christoffel symbol: Let $x^1, \dots, x^n$ be coordinates. Then n^3 smooth function $\Gamma_{ij}^k \in C^\infty(U)$, $1 \le i, j, k \le n$ is defined by,

$$\nabla_{\partial_i} \partial_j = \sum_k \Gamma_{ij}^k \partial_k = \Gamma_{ij}^k \partial_k$$

Eienstien notation

- Connections are local operators.
- Christoffel symbols determine connections on coordinate charts

$X = x^i \partial_i \qquad Y = Y^j \partial_j$

then $\nabla_X Y = \nabla_{x^i \partial_i} Y^j \partial_j = (XY^k + x^i Y^j \Gamma_{ij}^k)\partial_k$

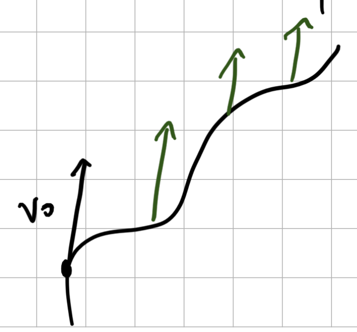
Parallel Translation: Given a sm curve $\gamma : I \to M$, $t_0 \in I$

& $V_0 \in T_{\gamma(t_0)}M$, $\exists$ parallel vector field V along γ

such that $\gamma(t_0) = V_0$

$D_{\dot\gamma(t)} V = 0$

We call V the parallel translation of V_0 along γ.



(Pf follows from existence & uniqueness of 1st order system of ODE)

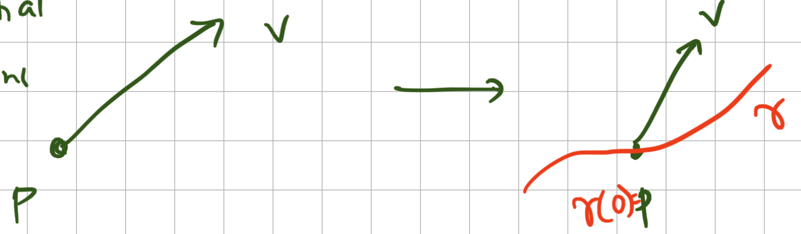
Geodesics: curves of zero acceleration i.e $\nabla_{\dot\gamma}\dot\gamma = 0$.

geodesic eqn $\ddot\gamma^k(t) + \Gamma^k_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t) = 0$

2^{nd} order ODE system $\Rightarrow$ no global existence & uniqueness.

If $v \in T_pM$ $\exists$ I about 0 & $!$ geodesic $\gamma: [0,1] \to M$ such that $\gamma'(0) = v$.

Initial conditions



Riemannian / Levi Civita Connection: unique connection on (M, g) that is g-compatible. i.e $\nabla g \equiv 0$.

$\nabla g(X,Y,Z) = Zg(X,Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y)$

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right) \qquad g^{kl} = (g_{kl})^{-1}$$

eg. $M = \mathbb{R}^n$, g usual metric $\Rightarrow \Gamma^k_{ij} = 0$

Curvature Gauss defined "Gaussian curvature" for surfaces & can be extended to 2D slices of a mfld called "sectional curvature."

Curvature tensor: $X, Y, Z \in \Gamma(TM)$; R is a $(1,3)$ tensor

$$R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(M) \longrightarrow \Gamma(M)$$

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

First Bianchi identity: $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

We can think of the $(1,3)$ tensor $R(X,Y)Z$ as $(0,4)$ tensor by using the metric:

$$R(X,Y,Z,W) := \langle R(X,Y)Z, W \rangle$$

In coordinates write $R_{ijkl} = \langle R(\partial_i, \partial_j)\partial_k, \partial_l \rangle$

& $R(\partial_i, \partial_j)\partial_k = R_{ijk}^{\ l} \partial_l$

so $R_{ijkl} = R_{ijk}^{\ m} g_{ml}$

First Bianchi identity $\Rightarrow R_{ijkl} + R_{jkil} + R_{kijl} = 0$

Can find formula for $R_{ijk}^{\ l}$ using christoffel symbols:

$$R_{ijk}^{\ m} = \Gamma_{ik}^{l}\Gamma_{jl}^{m} - \Gamma_{jk}^{l}\Gamma_{il}^{k} + \partial_j \Gamma_{ik}^{m} - \partial_i \Gamma_{jk}^{m}$$

eg. $M = \mathbb{R}^n$, $R_{ijk}^{\ m} = 0$

Idea: If $n=2$, curvature is determined by one number R_{1212}. So slice by a plane to get a 2d mfld.

Sectional curvature: $\sigma \subset T_pM$ 2 dimensional subspace

$$K(\sigma) := \frac{R(X,Y,X,Y)}{|X \wedge Y|^2} \quad ; \quad |X \wedge Y|^2 = \det \begin{bmatrix} g(X,X) & g(X,Y) \\ g(Y,X) & g(Y,Y) \end{bmatrix}$$

where X, Y is any basis of σ.

Proposition: The sectional curvature at a point determines the curvature tensor.

Ricci Tensor

Ricci tensor $\mathrm{Ric}_p(X,Y) = \frac{1}{n-1} \sum_{i=1}^{n} R(X, e_i, Y, e_i)$ $(0,2)$ tensor

Ricci curvature $\mathrm{Ric}_p(X) = \mathrm{Ric}_p(X,X)$

If X = unit vector in T_pM then $\mathrm{Ric}_p(X)$ = avg sectional curvature of planes through X.

scalar curvature $\quad s(p) = \frac{1}{n} \sum_{i=1}^{n} Ric_p(e_i) = \frac{1}{n-1} \sum_{i,j} R(e_i, e_j, e_i, e_j)$

These are independent of the choice of $e_1, e_2, \dots, e_n$.

In local coords, $\quad S = \frac{1}{n(n-1)} R_{ijkl}\, g^{ij} g^{kl}$

Orthonormal Frame Bundle.

From a theoretical pt of view, the most satisfactory construction of brownian motion on a manifold is EEM construction.

For that we need to understand orthonormal frame bundles.

An orthonormal frame at the point $x \in M$ is an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_x M$.

Let $O_x M = \{$ orthonormal frames of $T_x M \}$

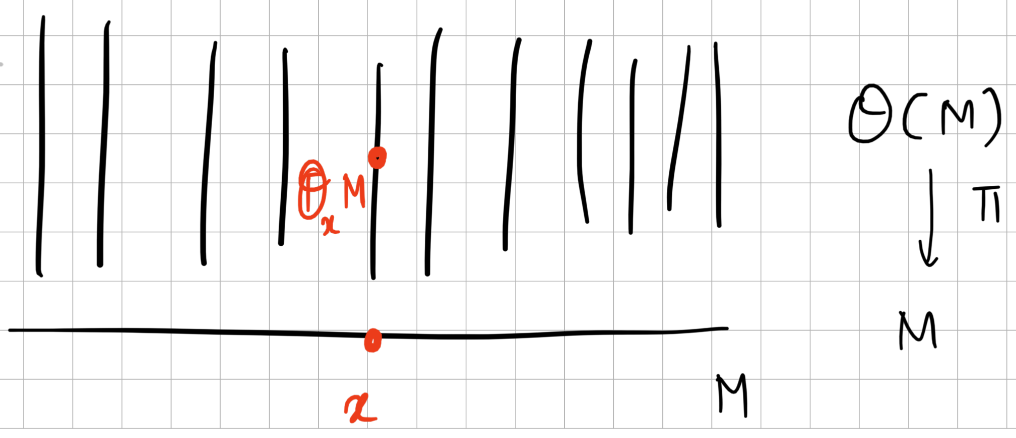
$$O_x M = \{ u : \mathbb{R}^n \xrightarrow{iso} (T_x M, g_x) \text{ orthonormal} \}$$

$u : \mathbb{R}^n \longrightarrow T_x M$ is orthonormal if $\langle u e_i, u e_j \rangle = \delta_{ij}$
where e_i, e_j are standard basis vectors of $\mathbb{R}^n$.

* u is a choice of orthonormal basis on $T_p M$.

$$O(M) = \bigsqcup_{x \in M} O_x(M) \quad \text{orthonormal frame bundle}$$

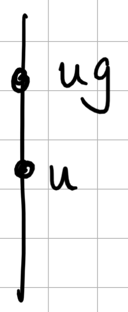
smooth mfld of $\frac{n(n+1)}{2}$ dim.



Note: The set of orthonormal frames at each point is isomorphic to the Lie group $O(n)$. (but not canonically)

$\Rightarrow$ each fiber of the orthonormal frame bundle has a group structure.

Also can view $\mathcal{O}(M)$ as the principal bundle over M with fiber $O(n)$.



$$\mathbb{R}^n \xrightarrow{g} \mathbb{R}^n \xrightarrow{u} T_x M$$

$g \in O(n)$.

$\mathcal{O}_x M$

We can decompose the the tangent vectors of $\mathcal{O}(M)$ into vertical & horizontal tangent vectors

$$T_u \mathcal{O}(M) = H_u \mathcal{O}(M) \oplus V_u \mathcal{O}(M)$$

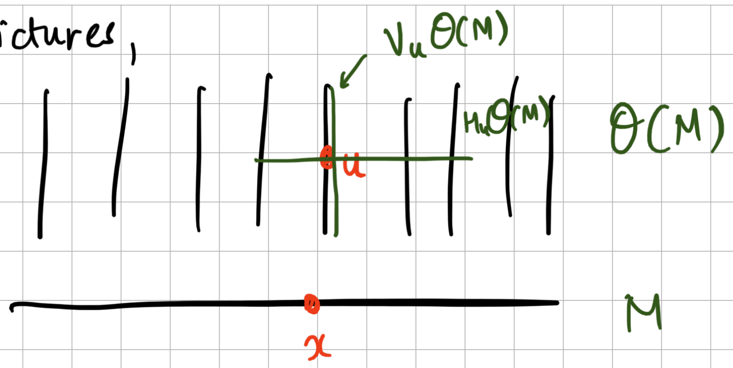
$V_u \mathcal{O}(M) = T_u\left(\mathcal{O}_{\pi(u)} M\right)$ i.e is the subspace of vectors that is tangent to the fibers.

Equivalently, the vertical bundle $V\mathcal{O}(M)$ is the kernel of tangent map: $d\pi : T\mathcal{O}(M) \longrightarrow TM$.

$H_u \mathcal{O}(M)$ = it is the choice of subspace of $T_u \mathcal{O}(M)$ such that $T_u \mathcal{O}(M)$ is the direct sum of $H_u \mathcal{O}(M)$ and $V_u \mathcal{O}(M)$.

The connection determines the horizontal subbundle.

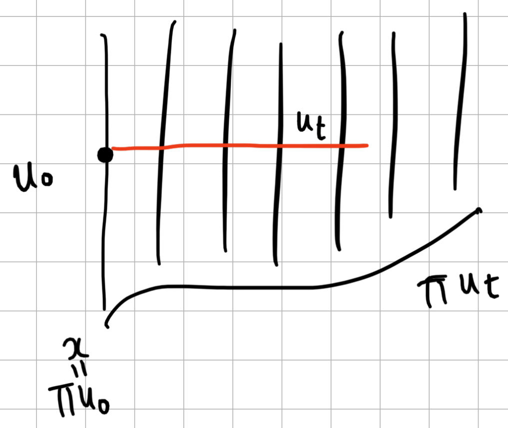
In pictures,

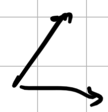


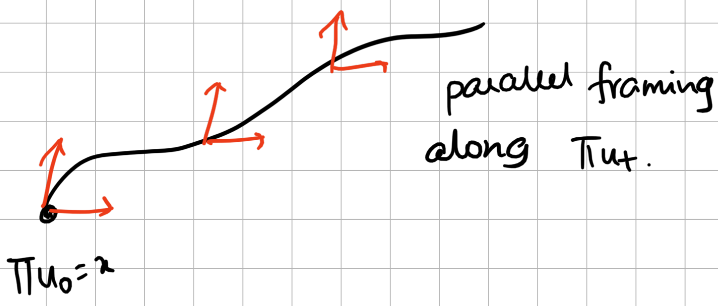
Understanding the horizontal subbundle:

A curve u_t in $\mathcal{O}(M)$ is said to be horizontal if u_t is the parallel transport of u_0 along the projection curve πu_t, i.e.
$$\nabla_{\dot{(\pi u_t)}} u_t e_i = 0 \qquad i = 1, 2, \dots, n$$

$$H_u\mathcal{O}(M) = \left\{ \frac{d}{dt}\Big|_{t=0} u_t \;:\; \{u_t\} \text{ is horizontal curve in } \mathcal{O}(M) \text{ with } u_0 = u \right\}$$



$u_0 =$  framing



Note that the projection $\pi : \mathcal{O}(M) \to M$ induces an isomorphism $\pi_* : H_u\mathcal{O}(M) \longrightarrow T_x M$ where $\pi u = x$.

This follows from the following lemma:

Lemma: Given a smooth curve $x : I \longrightarrow M$ & $t_0 \in I$ with $x(t_0) = x$ & initial point $u \in \pi^{-1}(x) \in \mathcal{O}_x M$, $\exists$ a unique horizontal curve $\{u_t\}$ in $\mathcal{O}M$ with $\pi u_t = x_t$.

In other words,
Given a smooth curve x_t and intial frame u at x, the horizontal lift of x_t is the unique curve u_t in $\mathcal{O}(M)$ such that for any $v \in \mathbb{R}^n$, $u_t v$ is parallel along x_t.

Any $X \in T_x M$ can be realised as $\dot{x}(t_0)$ for some curve $x : I \to M$ with $x(t_0) = x$.
Take the horizontal lift of the curve with initial point u, call it $\{u_t\}$.

So

$$T_x M \xrightarrow{\cong} H_u \mathcal{O}(M) \qquad \mathcal{O}(M)$$

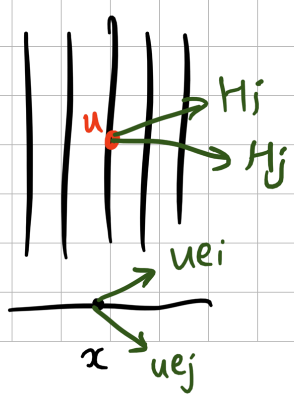
$$X = \frac{d}{dt} x_t \Big|_{t=t_0} \longmapsto X^* = \frac{d}{dt} u_t \Big|_{t=t_0} \qquad \downarrow \pi$$

$$\pi_* Y \longleftarrow Y \qquad M$$

call X_* horizontal lift of X.

As a result, we can define the "canonical" horizontal vector fields $H_1, \ldots, H_n$:

$$\pi_* H_i(u) = u e_i \quad ; \quad H_i(u) \in H_u \mathcal{O}(M)$$



$$H_u \mathcal{O}(M) \xrightarrow{\pi_*} T_x M$$

Define operator:

$$\Delta_{\mathcal{O}(M)} = \sum_{i=1}^{n} H_i^2$$

This is called the Bochner's horizontal Laplacian on $\mathcal{O}(M)$.

Proposition: For any smooth function f on M we have

$$\Delta_M f(x) = \Delta_{\mathcal{O}(M)} (f \circ \pi) u \quad \text{for any } u \in \mathcal{O}(M)$$

such that $\pi u = x$.

does not depend on choice of $u \in \pi^{-1} x$.

$$\begin{tikzcd} \mathcal{O}(M) \ni u \arrow[d, "\pi"] \arrow[dr, "f \circ \pi"] & \\ M \ni x \arrow[r, "f"'] & \mathbb{R} \end{tikzcd}$$

In this sense, the Bochners horizontal Laplacian $\Delta_{\mathcal{O}(M)}$ is the lift of the Laplace Beltrami operator Δ_M to the orthonormal frame bundle $\mathcal{O}(M)$.

Summary: $(M, \langle , \rangle)$ Riemannian mfld

$$\begin{tikzcd} \mathcal{O}(M) \arrow[d, "\pi"] \\ M \end{tikzcd}$$

is a orthonormal frame bundle

$$\mathcal{O}_p(M) = \{ u : \mathbb{R}^n \to T_pM \text{ orthogonal} \}$$

i.e if $\langle x, y \rangle = 0 \Rightarrow \langle u(x), u(y) \rangle = 0$

$$\begin{tikzcd} \left\{ \begin{array}{l} \text{horizontal curves} \\ \{u_t\} \text{ in } \mathcal{O}(M) \end{array} \right\} \arrow[r, leftrightarrow, "1-1"] & \left\{ \text{curves } \{x_t\} \text{ in } M \right\} \arrow[d, leftrightarrow, "1-1"] \\ & \left\{ \text{curves } \{w_t\} \text{ in } \mathbb{R}^n \right\} \end{tikzcd}$$

$u_t \longmapsto \pi u_t$

unique horizontal lift u_t with $\pi(u_t) = x_t$ $\longleftarrow$ x_t

development

Anti development

Anti-development: Given a smooth curve x_t on M, let u_t be the unique horizontal lift with initial frame u_0.

$$u_t : \mathbb{R}^n \longrightarrow T_{x_t}M$$

$$\dot{x}_t \in T_{x_t}M.$$

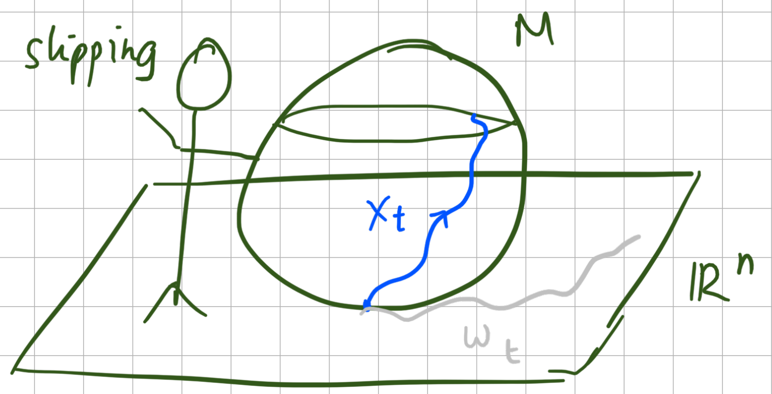
The anti development of the curve x_t is the curve W_t in $\mathbb{R}^n$

$$W_t := \int_0^t u_s^{-1}\dot{x}_s \, ds \qquad ✩$$

How to think of this?

① W_t is the curve that the joystick traces out to move the spaceship along the curve x_t

② rolling without slipping



The development of a curve W_t in $\mathbb{R}^n$ is the reverse process to construct X_t.

(spaceship movement based on the input joystick path)

$$u_t : \mathbb{R}^n \longrightarrow T_{x_t}M \qquad W_t \in \mathbb{R}^n \Rightarrow \dot{W}_t \in \mathbb{R}^n$$

$$u_t \dot{W}_t = \dot{x}_t \qquad (\text{differentiate } ✩)$$

$$\Rightarrow \dot{u}_t = \text{lift of } \dot{x}_t = \text{lift of } u_t \dot{w}_t$$
$$= \text{lift of } u_t \dot{w}_t^i e_i$$
$$= \text{lift of } \dot{w}_t^i u_t e_i$$
$$= \dot{w}_t^i H_i(u_t)$$

(lift of $u_t e_i$ is $H_i(u_t)$)

So we have ODE, $\dot{u}_t = H_i(u_t)\dot{w}_t^i$

EEM Construction:

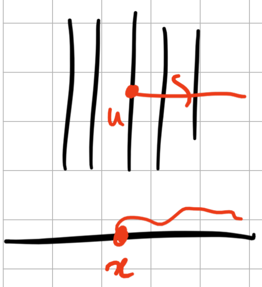
Let W_t be the standard BM on $\mathbb{R}^n$. (starting from 0)

Consider the stochastic differential equation on the frame bundle $\mathcal{O}(M)$.

SDE is development of W_t

$$dU_t = \sum_{i=1}^n H_i(U_t) \circ dW_t^i = H_i(U_t) \circ dW_t^i \longrightarrow (A)$$

Einstein summation convention



$U_0 = u$

$X_t := \pi(U_t)$ is BM on M. Can be verified by Levy's criterion.

M

$\pi U_0 = x$.

$$T_{X_t}M \xrightarrow{U_t^{-1}} \mathbb{R}^n \xrightarrow{U_0} T_x M$$

call $P_t := U_0 \circ U_t^{-1}$ as stochastic parallel transport.

Thm (HSU 3.2.1) The solution of $\textcircled{\ast}$ is the horizontal lift of W_t to $\Theta(M)$. "Horizontal Brownian motion."

Thm (Prop 3.22 HSU): TFAE

① U_t is a horizontal BM on $\Theta(M)$

② U_t is $\frac{1}{2}\Delta_{\Theta(M)}$ diffusion process

③ $X_t = \pi U_t$ is a BM on M.

④ The antidevelopment of U_t is the standard Euc BM.

Left to understand: ① proof of above thm

② Ito's formula for this SDE

③ Ito's formula in local coordinates

④ relation to Bochners Laplacian

⑤ heat kernel perspective

⑥ general construction for any frame bundle.