Thurston's fibered faces for non-orientable 3-manifolds and an application to minimal stretch factors

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Abstract

We generalize Thurston norm and the related theory of fibered faces to the setting of non-orientable 3-manifolds. This lets us construct examples of pseudo-Anosov maps on non-orientable surfaces with small stretch factors. Using this, we prove that for a fixed number of punctures, the minimal stretch factor of a genus g non-orientable surface behaves like $\frac{1}{g}$, generalizing the techniques and a result of Yazdi.

1 Introduction

Let $S_{g,n}$ be a surface of genus g with n punctures. The mapping class group of $S_{g,n}$ consists of homotopy classes of orientation preserving homeomorphisms of $S_{g,n}$. The Nielsen–Thurston classification of mapping classes (elements of the mapping class group) says that each mapping class is periodic, preserves some multicurve, or has a representative that is pseudo-Anosov. For each pseudo-Anosov homeomorphism $\varphi : S_{g,n} \to S_{g,n}$, the stretch factor $\lambda(\varphi)$ is an algebraic integer that describes the amount by which φ changes the length of curves. Arnoux–Yaccoz [3] and Ivanov [14] prove that the set

$$\mathrm{Spec}(S_{g,n}) = \{\log(\lambda(\varphi)) | \varphi \text{ is a pseudo-Anosov homeomorphism of } S_{g,n}\}$$

is a closed discrete subset of $(0, \infty)$. The minimum of $\mathrm{Spec}(S_{g,n})$:

$$l_{g,n} = \min\{\log(\lambda(\varphi)) | \varphi \text{ is a pseudo-Anosov homeomorphism of } S_{g,n}\}$$

quantitatively describes both the dynamics of the mapping class group of $S_{g,n}$ and the geometry of the moduli space of $S_{g,n}$.

Penner [25] showed that for orientable surfaces,

$$l_{g,0} \asymp \frac{1}{g}.$$

Penner conjectured that $l_{g,n}$ will have the same asymptotic behavior for $n \geq 0$ punctures. Following Penner, substantial attention has been given to finding bounds for $l_{g,n}$ [1, 4, 11, 12, 13, 16, 20, 23], calculating $l_{g,n}$ for specific values of (g, n) [6, 10, 17, 26], and finding asymptotic behavior of $l_{g,n}$ for *orientable* surfaces with $n \geq 0$ [16, 29, 30, 31]. We adapt a result of Yazdi [31] to non-orientable surfaces.

Theorem 1.1. *Let $\mathcal{N}_{g,n}$ be a non-orientable surface of genus g with n punctures, and let $l'_{g,n}$ be the minimum stretch factor of the pseudo-Anosov mapping classes acting on $\mathcal{N}_{g,n}$. Then for any fixed $n \in \mathbb{N}$, there is a positive constant $B'_1 = B'_1(n)$ and $B'_2 = B'_2(n)$ such that for any $g \geq 2$, the quantity $l'_{g,n}$ satisfies the following inequalities:*

$$\frac{B'_1}{g} \leq l'_{g,n} \leq \frac{B'_2}{g}.$$

Pseudo-Anosov homeomorphisms. Let S be a (possibly non-orientable) surface of finite type. A homeomorphism $\varphi : S \to S$ is said to be *pseudo-Anosov* if there exist a pair of transverse measured singular foliations $\mathcal{F}_s$ and $\mathcal{F}_u$ and a real number λ such that

$$\varphi(\mathcal{F}_s) = \lambda^{-1}(\mathcal{F}_s) \text{ and } \varphi(\mathcal{F}_u) = \lambda(\mathcal{F}_u).$$

The *stretch factor* of φ is the algebraic integer $\lambda = \lambda(\varphi)$.

Endow S with a Riemannian metric. The stretch factor $\lambda(\varphi)$ measures the growth rate of a geodesic S under iteration of φ [7, Proposition 9.21]. Moreover, $\log(\lambda(\varphi))$ is the minimal topological entropy of any homeomorphism of S that is istopic to φ [7, Exposé 10].

Geometry of moduli space. Let $\mathcal{T}_{g,n}$ denote the Teichmüller space of $S_{g,n}$, that is, the space of isotopy classes of hyperbolic metrics on $S_{g,n}$. When endowed with the Teichmüller metric, the mapping class group of a surface $S_{g,n}$ acts properly discontinuously on $\mathcal{T}_{g,n}$ by isometries. The quotient of this action is the *moduli space* of $S_{g,n}$. The set $\mathrm{Spec}(S_{g,n})$ is the length spectrum of geodesics in the moduli space of $S_{g,n}$. Therefore the quantity $l_{g,n}$ is the length of the shortest geodesic in the moduli space of $S_{g,n}$.

Explicit bounds. In his foundational work, Penner found $\frac{\log 2}{12g-12+4n}$ to be a lower bound for $l_{g,n}$ for orientable surfaces [25]. He also determined $\frac{\log 11}{g}$ to be an upper bound for $l_{g,0}$. Penner's work proves that $l_{g,0} \asymp \frac{1}{g}$. McMullen [22] later asked:

Question (McMullen)**.** To what value does $\lim\limits_{g\to\infty} g \cdot l_{g,0}$ converge?

To this end, Bauer [4] strengthened the upper bound for $g \cdot l_{g,0}$ to $\log 6$, and Minakawa [23] and Hironaka–Kin [12] further sharpened the upper bounds for $g \cdot l_{g,0}$ and $g \cdot l_{0,2g+1}$ to $\log(2 + \sqrt{3})$. Later Aaber–Dunfield [1], Hironaka [11], and Kin-Takasawa [15] determined that $\log\left(\frac{3+\sqrt{5}}{2}\right)$ is an upper bound for $g \cdot l_{g,0}$ and conjectured it is the supremum of $g \cdot l_{g,0}$.

Asymptotic behavior of punctured surfaces. Tsai initiated the study of asymptotic behavior of $l_{g,n}$ along lines in the (g, n)-plane [29]. In particular, Tsai determined that for orientable surfaces of fixed genus $g \geq 2$, the asymptotic behavior in n is:

$$l_{g,n} \asymp \frac{\log n}{n}.$$

Further, he showed that $l_{0,n} \asymp \frac{1}{n}$. Later, Yazdi [31] determined that for an orientable surface with a fixed number of punctures $n \geq 0$, the asymptotic behavior in g is:

$$l_{g,n} \asymp \frac{1}{g},$$

confirming the conjecture of Penner.

Non-orientable surfaces. Let $N_{g,n}$ be a non-orientable surface of genus g with n punctures. As above, let $l'_{g,n}$ denote the minimum stretch factor of pseudo-Anosov homeomorphisms of $N_{g,n}$. For any $n \geq 0$ and $g \geq 1$, $l_{g-1,2n}$ is a lower bound for $l'_{g,n}$, which can be seen by passing to the orientation double cover of $N_{g,n}$. Because the upper bounds for $l_{g,n}$ are constructed by example, upper bounds for $l'_{g,n}$ do not follow immediately from passing to the orientation double cover. Recently Liechti–Strenner determined $l'_{g,0}$ for $g \in \{4, 5, 6, 7, 8, 10, 12, 14, 16, 18, 20\}$ [19]. Our work captures the asymptotic behavior for the punctured case.

Techniques. To prove Theorem 1.1, we adapt the strategy of Yazdi [31] to non-orientable surfaces with punctures. The lower bound of $l'_{g,n}$ is found by lifting to the orientation double cover of $\mathcal{N}_{g,n}$. The upper bound (in all prior work) is constructive. Fix $n \geq 0$: the desired number of punctures. Yazdi's construction is as follows. For a sequence of genera $g_{n,k}$ (dependent on n), use the Penner construction [24] to obtain a homeomorphism $f_{n,k}$ of $S_{g,n}$ that is pseudo-Anosov and has low stretch factor. In order to find pseudo-Anosov homeomorphisms of $S_{g,n}$ with small stretch factor for all g (not just those in the sequence above), Yazdi constructs a mapping torus for each $f_{n,k}$. To do this Yazdi's appeals to a technique of McMullen, using Thurston's theory of fibered faces.

Thurston norm for non-orientable 3-manifolds. In Thurston's development of what is now called the Thurston norm for 3-manifolds [27], his definitions and theorems required that all surfaces were orientable. Thurston said that the theorems should still be true for non-orientable surfaces, but there are some subtleties that have not been addressed elsewhere in the literature. In this paper, we write the details of Thurston's theory of fibered faces to non-orientable 3-manifolds. In particular, the Thurston norm is a norm on the second homology of a 3-manifold, that measures the minimum complexity of an embedded (orientable) surface. However the Thurston norm does not recognize embedded non-orientable surfaces in the second homology of a non-orientable 3-manifold. To address this limitation, we instead calculate the Thurston norm on the first cohomology of a non-orientable manifold. We develop a (weak) version of Poincaré duality in Theorem 2.8 that suffices to define a Thurston norm on $H^1(M;R)$ for a non-orientable 3-manifold M.

Outline of paper. In Section 2.1 we recall Thurston's theory of fibered faces for orientable 3-manifolds. Then in Section 2.2 we define the Thurston norm on the first cohomology of a non-orientable 3-manifold. In order to use the Thurston norm to detect non-orientable surfaces, we will need a version of Poincaré duality for a pair consisting of a non-orientable 3-manifold and an embedded non-orientable surface, which we state and prove in Section 2.3.

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2 Thurston norm for non-orientable 3-manifolds

In his original manuscript defining what is now called the Thurston norm, Thurston defined a norm on $H_2(M;\mathbb{R})$ where M is an orientable 3-manifold [27]. He wrote "Most of this paper works also for non-orientable manifolds but for simplicity we only deal with the orientable case." However, the details are not explained in Thurston's work or in subsequent literature. Therefore the goal of this section is to write the details of the Thurston norm for non-orientable 3-manifolds. We recall the Thurston norm for orientable manifolds in Section 2.1 and then adapt it to the non-orientable setting. In Section 2.2 we describe the challenge of defining the Thurston norm on $H_2(M;\mathbb{R})$ if M is non-orientable and present the solution of defining the Thurston norm instead on $H^1(M;\mathbb{R})$. However, Poincaré duality does not hold for non-orientable manifolds. We therefore define a condition – *relative orientability* on a pair consisting of a manifold and an embedded surface. A surface that is relatively orientable in a non-orientable 3-manifold M will have a corresponding cohomology class in $H^1(M;\mathbb{Z})$, giving a version of Poincaré duality for non-orientable 3-manifolds as stated in Theorem 2.8. Finally in Section 2.4, we define the oriented sum for relatively oriented embedded surfaces in non-orientable manifolds.

2.1 Thurston norm and mapping tori

In this section we recall the Thurston norm for orientable surfaces and how it detects when a 3-manifold fibers over a circle.

Mapping tori. Let S be a surface, and let $\varphi : S \to S$ be a homeomorphism. A *mapping torus* of S by φ is a 3-manifold M_φ given by the identification:

$$M_\varphi := \frac{S \times [0,1]}{(x,1) \sim (\varphi(x),0)}.$$

A mapping torus is a surface bundle over S^1, or *fibrations over* S^1, denoted $S \to M_\varphi \to S^1$. A fibration defines a flow on M, called the *suspension flow*, where for any $x_0 \in S$ and $t_0 \in S^1$ the pair (x_0, t_0) is sent to $(x_0, t_0 + t)$. The fiber of a fibration is the preimage of any point $\theta \in S^1$ under the projection map from $M_\varphi \to S^1$. The fiber as a subset of M_φ is only well defined up to isotopy, since we don't specify the choice of θ, but the homology class of the fiber in $H_2(M_\varphi; \mathbb{R})$ is well defined.

A natural inverse question is to determine when a 3-manifold fibers over a circle, and the possible fibers. To this end, Thurston established a correspondence between second cohomology of 3-manifolds and surfaces embedded in 3-manifolds.

Complexity of an embedded surface. Let M be an compact orientable closed 3-manifold. Let S be a connected surface embedded in M. The complexity of S is $\chi_-(S) = \max\{-\chi(S), 0\}$. If the surface S has multiple components $S_1, \dots, S_m$ then $\chi_-(S) = \sum_{i=1}^{m} \chi_-(S_i)$. The elements in $H_2(M;\mathbb{Z})$ can be represented by embedded surfaces inside of M [27, Lemma 1].

Thurston norm. Let a be a homology class in $H_2(M;\mathbb{Z})$. Define the integer valued norm $x : H_2(M;\mathbb{Z}) \to \mathbb{Z}$ as the following:

$$x(a) = \min\{\chi_-(S) \mid [S] = a \text{ and } S \text{ is compact, properly embedded and oriented}\}.$$

We then linearly extend x to $H_2(M;\mathbb{Q})$. The *Thurston norm* is the unique continuous $\mathbb{R}$ valued function that is an extension of x to $H_2(M;\mathbb{R})$. The unit ball for the Thurston norm is a convex polyhedron in $H_2(M;\mathbb{R})$.

The following remarkable theorem of Thurston [27] determines all possible fibrations of an oriented 3-manifold over the circle. We use the restatement of Yazdi [31].

Theorem 2.1 (Thurston)**.** *Let M be an orientable 3-manifold. Let $\mathcal{F}$ be the set of homology classes in $H_2(M;\mathbb{R})$ that are representable by fibers of fibrations of M over the circle.*

(i) Elements of $\mathcal{F}$ are in one-to-one correspondence with (non-zero) lattice points inside some union of cones over open faces of the unit ball in the Thurston norm.

(ii) If a surface F is transverse to the suspension flow associated to some fibration of $M \to S^1$ then $[F]$ lies in the closure of the corresponding cone in $H_2(M;\mathbb{R})$.

The class $[F]$ has orientation such that the positive flow direction is pointing outwards relative to the surface. The open faces of the unit ball whose cones contain the fibers of fibrations are called the *fibered faces.*

The goal for the rest of this section is to prove a version of Theorem 2.1 for compact non-orientable 3-manifolds. Most of the work in the proof will involve reducing the version for non-orientable 3-manifolds to the orientable version by passing to the double cover.

2.2 Thurston norm on cohomology of non-orientable mapping tori

A naïve first attempt at defining the Thurston norm would be to define it on the second homology group, like in the orientable case. However, if the norm is defined in that fashion, the non-orientable version of Theorem 2.1 will not be true. Consider a compact non-orientable surface $\mathcal{N}$, a homeomorphism

$\varphi : \mathcal{N} \to \mathcal{N}$, and the associated mapping torus N_φ. Clearly, N_φ fibers over S^1, and $\mathcal{N}$ is the fiber of this fibration. However, the homology class associated to $\mathcal{N}$ is the zero homology class, since the top-dimensional homology of non-orientable compact surfaces is 0-dimensional.

Our workaround for this problem will be to deal with the first cohomology $H^1(N_\varphi)$ rather than the second homology $H_2(N_\varphi)$. By Poincaré duality they are the same for orientable 3-manifolds, but that is not true for non-orientable 3-manifolds.

Poincaré Duality. To see why Poincaré duality for non-orientable 3-manifolds fails, we first consider the orientable case. Let M be an orientable manifold. To define the Poincare dual of $H^1(M;\mathbb{Z})$, we first define a homotopy class of maps $M \to S^1$. Then we construct an element of $H^2(M;\mathbb{Z})$. Let α be a 1-form on M and $[\alpha]$ its class in $H^1(M;\mathbb{Z})$. Fix a basepoint $y_0 \in M$. The associated map $f_\alpha : M \to S^1$ is given by:

$$f_\alpha(y) \coloneqq \int_{y_0}^{y} \alpha \mod \mathbb{Z}. \tag{1}$$

The choice of basepoint does not affect the homotopy class of f_α (see Section 5.1 of [5] for the details).

Now let $\theta \in S^1$ be a regular value so that $S = f^{-1}(\theta)$ is a surface. To construct the associated element of $H_2(M;\mathbb{Z})$, we choose an orientation on S by assigning positive values of α to the outward pointing normal vectors on S. Then S inherits an orientation from the orientation on M, and we have defined a fundamental class $[S] \in H_2(M;\mathbb{Z})$. We claim that $[S]$ is the Poincaré dual to α.

Lemma 2.2. *Let θ and θ' be two regular values of the function f_α and let $S = f_\alpha^{-1}(\theta)$ and $S' = f_\alpha^{-1}(\theta')$. Then for any closed 2-form ω on M, the following identity holds:*

$$\int_S \omega = \int_{S'} \omega.$$

Furthermore, the following identity also holds:

$$\int_S \omega = \int_M \alpha \wedge \omega.$$

In particular, the homology class of S is Poincaré dual to α.

Proof. The first part of the lemma follows from the fact that S and S' are homologous, i.e. $f_\alpha^{-1}([\theta,\theta'])$ is a singular 3-chain that has S and S' as boundaries. From Stokes' theorem, we get the following:

$$\begin{aligned}\int_{S-S'} \omega &= \int_{f_\alpha^{-1}([\theta,\theta'])} d\omega \\ &= 0.\end{aligned}$$

To prove the second claim, observe that we can break up the second integral as a product integral:

$$\int_M \alpha \wedge \omega = \int_{S^1} \left(\int_{f_\alpha^{-1}(\theta)} \omega \right) d\theta.$$

The above equation is true because α is the pullback of $d\theta$ along the map f_α. Observe that the inner integral only makes sense when θ is a regular value, but by Sard's theorem, almost every $\theta \in [0,1]$ is a regular value, so the right hand side is well-defined. By the first claim, the inner integral is a constant function, as we vary over the θ which are regular values of f_α. Then the integral of $d\theta$ over S^1 is 1, giving us the identity we want:

$$\int_M \alpha \wedge \omega = \int_S \omega.$$

$\square$

What we have here is an explicit formula for the Poincaré duality map from $H^1(M;\mathbb{R})$ to $H_2(M;\mathbb{R})$. For orientable 3-manifolds, this is an isomorphism, and more specifically the following theorem is true.

Theorem 2.3 (Poincaré duality for orientable 3-manifolds)**.** *Let M be an orientable 3-manifold, and let S be an oriented embedded surface. Then there exists a 1-form α and a regular value $\theta \in S^1$ such that S and $f_\alpha^{-1}(\theta)$ are homologous surfaces.*

Note that the map from the space of 1-forms to homology classes of an embedded surface still makes sense for a non-orientable 3-manifold N. However in that case the map from $H^1(N;\mathbb{Z})$ to $H_2(N;\mathbb{Z})$ has a nontrivial kernel. For example, when N_φ is the mapping torus of a non-orientable surface $\mathcal{N}$, as above, the fiber is trivial in $H_2(N;\mathbb{Z})$.

Non-orientable manifolds. Let N be a non-orientable 3-manifold. Let $\widetilde{N}$ and the covering map $p : \widetilde{N} \to N$ be the orientation double covering space of N. Let ι be the orientation reversing deck transformation of $\widetilde{N}$. If $N = N_\varphi$ is the mapping torus of the non-orientable surface $\mathcal{N}$ and a self-homeomorphism $\varphi : \mathcal{N} \to \mathcal{N}$, then $\widetilde{N}$ is the mapping torus of $(\mathcal{S}, \widetilde{\varphi})$, where $\mathcal{S}$ is the orientable double cover of $\mathcal{N}$, and $\widetilde{\varphi}$ is the orientation preserving lift of φ.

Defining the Thurston norm on cohomology. In order to define the Thurston norm on $H^1(N;\mathbb{Z})$, we first need to relate $H^1(N;\mathbb{R})$ and $H^1(\widetilde{N};\mathbb{R})$. We do so by pulling back $H^1(N;\mathbb{R})$ to $H^1(\widetilde{N};\mathbb{R})$ via p.

Lemma 2.4. *The pullback $p^* : H^1(N;\mathbb{R}) \to H^1(\widetilde{N};\mathbb{R})$ maps $H^1(N;\mathbb{R})$ bijectively to the ι^*-invariant subspace of $H^1(\widetilde{N};\mathbb{R})$.*

Proof. For any 1-form α on N, $p^*(\alpha)$ will be ι^*-invariant. To check that p^* is injective, consider a 1-form α on N such that $p^*(\alpha)$ is exact. Then there exists a smooth function $g : \widetilde{N} \to \mathbb{R}$ such that the following relation holds:

$$dg = p^*(\alpha).$$

But since $p^*(\alpha)$ is ι^*-invariant, we must have $dg = \iota^* dg$. Because ι^* commutes with the exterior derivative, we have $dg = d(\iota^* g)$. That means g and $\iota^* g$ differ by a constant, but that constant must be 0 since ι^2 is the identity map. Thus g is ι-equivariant and descends to a function $N \to \mathbb{R}$. Therefore α must be exact, which proves injectivity of p^*.

To show surjectivity, let $[\alpha]$ be an element in $H^1(\widetilde{N};\mathbb{R})$ that is ι^*-invariant and let α be a representative. Since the cohomology class $[\alpha]$ is ι^*-invariant, α and $\iota^*(\alpha)$ must differ by an exact form.

$$\alpha - \iota^*(\alpha) = dg$$

Applying ι^* to both sides of the equality, we have $\iota^* dg = -dg$. This means that the 1-form $\beta = \alpha - \frac{dg}{2}$ is an ι^*-invariant representative of the cohomology class $[\alpha]$. Since β is ι^*-invariant, the pushforward of β under p is a well-defined 1-form on N. $\square$

Note that if we change the coefficients in the statement of this lemma from $\mathbb{R}$ to $\mathbb{Z}$, the proof of injectivity follows through, but the proof of surjectivity does not.

Lemma 2.4 tells us that $H^1(N;\mathbb{R})$ is a subspace of $H^1(\widetilde{N};\mathbb{R})$, so we define the Thurston norm on $H^1(N;\mathbb{R})$ by restricting the Thurston norm to the subspace $p^*(H^1(N;\mathbb{R}))$ of $H^1(\widetilde{N};\mathbb{R})$.

Thurston norm for non-orientable 3-manifolds. Let N be a non-orientable 3-manifold and $\widetilde{N}$ its orientation double cover. Let $\widetilde{x}$ be the Thurston norm on $H^1(\widetilde{N};\mathbb{R})$ and let $\alpha \in H^1(N;\mathbb{R})$. The *Thurston norm on* $H^1(N;\mathbb{R})$, is the norm $x : H^1(N;\mathbb{R}) \to \mathbb{R}$ defined:

$$x(\alpha) \coloneqq \widetilde{x}(p^*\alpha).$$

We next see that the unit ball for the Thurston norm on $H^1(N;\mathbb{R})$ can be defined analogously to the orientable case.

Theorem 2.5. *The unit ball with respect to the dual Thurston norm on* $\left(H^1(N;\mathbb{R})\right)^*$ *is a polyhedron whose vertices are lattice points* $\{\pm\beta_1, \dots \pm \beta_k\}$. *The unit ball* B_1 *with respect to Thurston norm is a polyhedron given by the following inequalities:*

$$B_1 = \left\{a \in H^1(N, \mathbb{R}) \mid |\beta_i(a)| \leq 1 \text{ for } 1 \leq i \leq k\right\}.$$

Proof. The proof is identical to the original proof of Thurston [27, Theorem 2]. For any $\alpha \in H^1(N;\mathbb{Z})$, the norm $x(\alpha)$ is $\widetilde{x}(p^*\alpha)$ where $p^*(\alpha) \in H^1(\widetilde{N};\mathbb{Z})$. Therefore $x(\alpha)$ is always an integer. The rest of the proof is identical to the orientable case. □

Note that defining the Thurston norm on $H^1(N;\mathbb{R})$ rather than $H_2(N;\mathbb{R})$ is not quite satisfactory. In particular, fibers of fibrations are embedded surfaces in N. In the orientable case, the embedded surfaces define the Thurston norm. In Section 2.3, we develop a version of Poincaré duality for non-orientable 3-manifolds that will let us translate a family of embedded surfaces into 1-forms, giving us a more concrete relationship between the Thurston norm and embedded surfaces for non-orientable 3-manifolds.

2.3 Weak inverse to the Poincaré duality map

Let M be a 3-manifold. Regardless of whether M is orientable or not, we can construct a dual map $f_\alpha : M \to S^1$, using equation (1). The preimage of a regular value $\theta \in S^1$, denoted by $f_\alpha^{-1}(\theta)$, will be an embedded surface in M. When M is orientable, Poincaré duality determines a cohomology class in $H^1(M;\mathbb{Z})$ corresponding any embedded surface S. In that case, let α be a 1-form representative α of a cohomology class and let θ of f_α. The surfaces $f_\alpha^{-1}(\theta)$ and S are homologous. In this section, we state a weaker version of Poincaré duality for non-orientable surfaces in Theorem 2.8, that lets us associate 1-forms to a certain class of embedded surfaces in non-orientable 3-manifolds. The condition we need to impose upon the embedded surfaces is *relative orientability*.

Relative orientability. Let M be a 3-manifold, and S an embedded surface in M. The surface S is said to be *relatively oriented with respect to* M if there is a nowhere vanishing vector field on S that is transverse to the tangent plane of S. Two such vector fields are said to induce the same orientation if locally they induce the same orientation after choosing a local frame for S. A surface S is *relatively oriented* if both S and the choice of positive normal vector field are specified.

Note that relative orientability is a strictly weaker notion than orientability. If S and M are orientable, then S is relatively orientable with respect to M. But even if M is non-orientable, a non-orientable embedded surface S may be relatively orientable with respect to M. For instance, let S be the fiber of a non-orientable mapping torus N_φ. The preimage under the projection map to S^1 of a non-vanishing vector field is a non-vanishing vector field on M that is always transverse to the fiber.

On the other hand, for orientable M and S, if S is relatively oriented with respect to S, then a choice of orientation on S determines an orientation on M and vice versa.

A surface that is not relatively orientable in a 3-manifold. Let S be the standard torus $\mathbb{R}^2/\mathbb{Z}^2$, and let φ map (x, y) to $(-x, y)$. Then φ is an orientation-reversing homeomorphism. The mapping torus M_φ is non-orientable. Consider a vertical line γ in S preserved by φ, i.e. the line $x = 0$. The image of γ in S under the suspension flow in M is a subsurface of M, which we'll call S'. The normal direction to S' when restricted to S is $\frac{\partial}{\partial x}$. Because the suspension flow reverses the direction of γ, the normal vector field cannot be continuously extended to all of M. This means that the surface S' is not relatively orientable in M (despite being orientable itself.)

However, if both M and an embedded surface are non-orientable, the surface will be relatively orientable.

Proposition 2.6. *Let* N *be a non-orientable* 3*-manifold, and let* $\mathcal{N}$ *be an embedded connected non-orientable surface in* N*. Then* $\mathcal{N}$ *is relatively orientable with respect to* N*.*

Proof. Let $\widetilde{N}$ be the orientation double covering space of N, and $\widetilde{\mathcal{N}}$ be the preimage of $\mathcal{N}$ under the covering map. The restriction of $\iota : \widetilde{N} \to \widetilde{N}$ to $\widetilde{\mathcal{N}}$ is an orientation reversing homeomorphism of $\widetilde{\mathcal{N}}$. Let (v_1, v_2) be a positively oriented local frame for the tangent space to $\widetilde{\mathcal{N}}$. Let n be an outward pointing transverse vector to $\widetilde{\mathcal{N}}$ so the local frame (v_1, v_2, n) is positively oriented. Since ι reverses the orientation of both $\widetilde{\mathcal{N}}$ and $\widetilde{N}$, $(\iota(v_1), \iota(v_2))$ and $(\iota(v_1), \iota(v_2), \iota(n))$ are both negatively oriented. Then $\iota(n)$ is outward pointing. Therefore the outward pointing transverse direction on $\widetilde{\mathcal{N}}$ descends to an outward pointing transverse direction on $\mathcal{N}$, and $\mathcal{N}$ is relatively orientable in N. $\square$

We also need to define the notion of *incompressible surfaces* to state our version of Poincaré duality.

Incompressible surfaces. Let S be a surface with positive genus embedded in a 3-manifold M. The surface S is said to be *incompressible* if there does not exist an embedded disc D in M such that $D \cap S = \partial D$, and D intersects S transversally. The following result of Thurston demonstrates the link between incompressible surfaces and fibers of fibrations.

Theorem 2.7 (Theorem 4 of [27])**.** *Let M be an oriented 3-manifold that fibers over S^1. Let S be an incompressible surface embedded in M. If S is homologous to a fiber, then S is isotopic to the fiber.*

Relatively orientable incompressible surfaces are those for which our version of Poincaré duality holds.

Theorem 2.8 (Poincaré Duality for non-orientable 3-manifolds)**.** *Let N be a compact non-orientable 3-manifold, and let $\mathcal{N}$ be a relatively oriented incompressible surface. Then there exists $[\alpha] \in H^1(N;\mathbb{Z})$ such that the pullback of $[\alpha]$ to $\widetilde{N}$ is dual to $\widetilde{\mathcal{N}}$ in $\widetilde{N}$. If $[\alpha]$ has a 1-form representative α that vanishes nowhere on N, then $\mathcal{N}$ is homeomorphic to $f_\alpha^{-1}(\theta)$ for all $\theta \in S^1$.*

We will refer to the 1-form α given in Theorem 2.8 as the *Poincaré dual* of the non-orientable surface $\mathcal{N}$.

Lemma 2.9. *Let N be a non-orientable 3-manifold and let $\widetilde{N}$ and the map $p : \widetilde{N} \to N$ be its orientation double cover. Let $\mathcal{N}$ be a relatively oriented embedded surface in N, and let $\widetilde{\mathcal{N}} = p^{-1}(\mathcal{N})$ in $\widetilde{N}$. Then the Poincaré dual to $[\widetilde{\mathcal{N}}]$ is ι^*-invariant, where ι is the non-trivial deck transformation.*

Proof. If $\mathcal{N}$ is relatively oriented with respect to N, then the relative orientation lifts to a relative orientation of $\widetilde{\mathcal{N}}$ with respect to $\widetilde{N}$. Since $\widetilde{N}$ is orientable, the relative orientation of $\widetilde{\mathcal{N}}$ defines an orientation of $\widetilde{\mathcal{N}}$, and thus the homology class $[\widetilde{\mathcal{N}}]$ is well-defined.

We show first that ι reverses the orientation on $\widetilde{\mathcal{N}}$. Let (v_1, v_2, v_3) be a local frame for some point in $\widetilde{\mathcal{N}}$ such that v_3 is the outward pointing transverse vector field. Because $\mathcal{N}$ is relatively oriented in N, the outward pointing transverse vector field on $\mathcal{N}$ must lift to an outward pointing transverse vector field on $\widetilde{\mathcal{N}}$.

This means ι takes outward pointing vector fields to outward pointing vector fields, and $\iota(v_3)$ must be outward pointing. Since ι reverses the orientation on $\widetilde{N}$ but preserves the direction of $\iota(v_3)$, ι must reverse the orientation of the pair (v_1, v_2). In particular, that means ι reverses the orientation on $\widetilde{\mathcal{N}}$.

This implies $[\widetilde{\mathcal{N}}]$ is in the -1-eigenspace of the ι_* action on $H_2(\widetilde{N};\mathbb{R})$. Let the cohomology class $[\widetilde{\alpha}]$ be the the Poincaré dual to $[\widetilde{\mathcal{N}}]$. Then there exists a representative 1-form $\widetilde{\alpha}$ that is ι^*-invariant. This follows from the following chain of equalities which hold for all closed 2-forms ω. We use the fact that $\iota^2 = \mathrm{id}$ in the first and third equalities:

$$\begin{aligned}
\int_{\iota_*\widetilde{\mathcal{N}}} \omega &= \int_{\widetilde{\mathcal{N}}} \iota^*\omega && \text{(By a change of variables)} \\
&= \int_{\widetilde{N}} \widetilde{\alpha} \wedge \iota^*\omega && \text{(Poincaré duality)} \\
&= \int_{\widetilde{N}} \iota^* \left(\iota^*\widetilde{\alpha} \wedge \omega\right) \\
&= \int_{\widetilde{N}} - \left(\iota^*\widetilde{\alpha} \wedge \omega\right) && \text{(ι is orientation reversing)}
\end{aligned}$$

On the other hand, the following equalities follow from the fact that $\iota_*[\widetilde{\mathcal{N}}] = -[\widetilde{\mathcal{N}}]$.

$$\begin{aligned}\int_{\iota_*\widetilde{\mathcal{N}}} \omega &= -\int_{\widetilde{\mathcal{N}}} \omega \\ &= -\int_{\widetilde{N}} \widetilde{\alpha} \wedge \omega\end{aligned}$$

Because

$$\int_{\widetilde{N}} \widetilde{\alpha} \wedge \omega = \int_{\widetilde{N}} \iota^*\widetilde{\alpha} \wedge \omega$$

for all ω, it follows that $\widetilde{\alpha}$ is ι^*-invariant. □

As above, we will denote the Poincaré dual to $[\widetilde{\mathcal{N}}]$ by $[\widetilde{\alpha}]$. The class $[\widetilde{\alpha}]$ is an ι^*-invariant element of $H^1(\widetilde{N};\mathbb{Z})$, but it is not clear that $[\widetilde{\alpha}]$ is the pullback of an element of $H^1(N;\mathbb{Z})$ under p. In the next lemma, we show that is indeed the case, i.e. $[\widetilde{\alpha}]$ is the pullback of an element in $H^1(N;\mathbb{Z})$.

Lemma 2.10. *Let N be a non-orientable 2-manifold. Let $[\widetilde{\alpha}] \in H^1(\widetilde{N}, Z)$ and let $\widetilde{\mathrm{N}}$ be its Poincaré dual. There exists $[\alpha] \in H^1(N;\mathbb{Z})$ such that $\widetilde{\alpha} = p^*\alpha$.*

Proof. It will suffice to show that for any simple closed curve γ in N, the integral of $\widetilde{\alpha}$ along any path lift of γ is an integer. Let $x_0 \in N$ be a base point of γ. Note that γ has two (path) lifts $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ under p in $\widetilde{N}$. Either $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ are both simple closed curves based at the each of the two preimages $p^{-1}(x_0)$ or $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ are both arcs between the two points of $p^{-1}(x_0)$. If each lift $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ of γ is a closed curve in $\widetilde{N}$, the integral $\int_{\widetilde{\gamma}_i} \widetilde{\alpha}$ will be an integer since $\widetilde{\alpha} \in H^1(\widetilde{N};\mathbb{Z})$.

If each lift $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ of γ is an arc between the two preimages of $p^{-1}(x_0)$, we consider the simple closed curve $\widetilde{\gamma} = \widetilde{\gamma}_1 \cup \widetilde{\gamma}_2$. We note that $\iota(\widetilde{\gamma}) = \widetilde{\gamma}$. Because $\widetilde{\alpha}$ is ι^*-invariant, we have that $\int_{\widetilde{\gamma}_1} \widetilde{\alpha} = \int_{\widetilde{\gamma}_2} \widetilde{\alpha}$. Therefore

$$\int_{\widetilde{\gamma}} \widetilde{\alpha} = 2\int_{\widetilde{\gamma}_1} \widetilde{\alpha}.$$

It will suffice to show that $\int_{\widetilde{\gamma}} \widetilde{\alpha}$ is an even integer. Without loss of generality, we can assume that the curve that all intersections of $\widetilde{\gamma}$ with the surface $\widetilde{\mathcal{N}}$ are transverse. Since $\widetilde{\alpha}$ is a representative of the Poincaré dual to $[\widetilde{\mathcal{N}}]$, the integral of $\widetilde{\alpha}$ along $\widetilde{\gamma}$ is the signed intersection number of $\widetilde{\gamma}$ with $\widetilde{\mathcal{N}}$. The intersection number must be even, for if $\widetilde{\gamma}$ and $\widetilde{\mathcal{N}}$ intersect at a point y, then they also intersect at $\iota(y)$, by ι-invariance of $\widetilde{\mathcal{N}}$ and $\widetilde{\gamma}$. This proves the lemma. □

The last lemma we need is the claim that lifts of incompressible surfaces are incompressible.

Lemma 2.11. *Let N be a non-orientable 3-manifold and let $\widetilde{N}$ be the orientation covering space of N with covering map p. If $\mathcal{N}$ is a relatively oriented incompressible surface in N, then $p^{-1}(\mathcal{N})$ is also incompressible.*

Proof. Because $\mathcal{N}$ is incompressible in N, the map on fundamental groups induced by inclusion $\mathcal{N} \to N$ is injective. Since $p_* : \pi_1(\widetilde{N}) \to \pi_1(N)$ is injective, the induced map $\pi_1(\widetilde{\mathcal{N}}) \to \pi_1(\widetilde{N})$ must also be injective. An injective induced map on fundamental groups is equivalent to the orientable surface N being incompressible. □

We now have everything we need to finish proving Theorem 2.8.

Proof of Theorem 2.8. Let $\widetilde{\mathcal{N}} = p^{-1}(\mathcal{N})$. The relative orientation of $\widetilde{\mathcal{N}}$ determines a homology class $[\widetilde{\mathcal{N}}] \in H_2(N;\mathbb{Z})$. Let the 1-form $\widetilde{\alpha}$ be the Poincaré dual to $[\widetilde{\mathcal{N}}]$. By Lemma 2.10, there exists a 1-form $\alpha \in H^1(N;\mathbb{Z})$ such that $\widetilde{\alpha} = p^*\alpha$.

By equation (1), we define the map f_α. Because α is non-vanishing, the map $f_\alpha : N \to S^1$ has full rank everywhere. Therefore f_α is a fibration. The map $f_\alpha \circ p$ is the lift of f_α to $\widetilde{N}$ under p, and is therefore also a fibration. By Lemma 2.11, $\widetilde{\mathcal{N}}$ is incompressible. It follows from the orientable version of Poincaré duality that $\widetilde{\mathcal{N}}$ and $p^{-1}(f_\alpha^{-1}(\theta))$ are homologous surfaces in $\widetilde{N}$. Theorem 2.7 then tells us $\widetilde{\mathcal{N}}$ must be isotopic to a fiber of $f_\alpha \circ p$. Because $\widetilde{\mathcal{N}}$ and $p^{-1}(f_\alpha^{-1})(\theta)$ are homeomorphic and $p : \widetilde{\mathcal{N}} \to \mathcal{N}$ and $p : p^{-1}(f_\alpha^{-1}(\theta)$ are 2-fold covering maps, the surfaces $\mathcal{N}$ and $f_\alpha^{-1}(\theta)$ must also be homeomorphic. □

Note that the above proof doesn't tell us if $\mathcal{N}$ and $f_\alpha^{-1}(\theta)$ are isotopic. To have that, we would require the isotopy on the orientation double cover to be ι^*-invariant, but the theorem is sufficient for our application.

We close the section with a non-orientable version of Theorem 2.1.

Theorem 2.12. *Let N be a compact non-orientable 3-manifold, and let $\mathcal{F}$ be the elements of $H^1(N;\mathbb{Z})$ corresponding to fibrations of N over S^1.*

(i) Elements of $\mathcal{F}$ are in one-to-one correspondence with (non-zero) lattice points (i.e. points of $H^1(N;\mathbb{Z})$) inside some union of cones over open faces of the unit ball in the Thurston norm.

(ii) Let $\mathcal{N}$ be relatively oriented surface in N that transverse to the suspension flow associated to some fibration $f : N \to S^1$. Let $[\alpha]$ be the Poincaré dual $[\alpha]$ to $\mathcal{N}$. Then $[\alpha]$ lies in the closure of the cone in $H^1(N;\mathbb{R})$ containing the 1-form corresponding to f.

Proof. Let $\widetilde{N}$ be the orientation double cover of N with covering map $p : \widetilde{N} \to N$.

For (i), we observe that by Theorem 2.1 the fibrations of $\widetilde{N}$ are in one-to-one correspondence with lattice points inside a union of cones over open faces of the unit ball in $H_2(\widetilde{N};\mathbb{R})$. Let $\widetilde{\mathcal{K}}$ be the union of cones in $H^1(\widetilde{N};\mathbb{R})$. By Poincaré duality, $\widetilde{\mathcal{K}}$ maps to a union of cones in $H^1(\widetilde{N};\mathbb{R})$, which we will call $\widetilde{K}^*$

The union of cones $\widetilde{K}^*$ in $H^1(N;\mathbb{R})$ is determined by intersecting the pullback of $H^1(N;\mathbb{R})$ to $H^1(\widetilde{N};\mathbb{R})$ with $\widetilde{\mathcal{K}}$. Every lattice point in $\widetilde{K}^*$ corresponds to a fibration $f : N \to S^1$, since the pullback of f to $H^1(\widetilde{N};\mathbb{Z})$ corresponds to a fibration of $\widetilde{N}$. Conversely, every fibration of $f : N \to S^1$ must lie in $\widetilde{K}^*$, since the composition $f \circ p$ is a fibration of $\widetilde{N} \to S^1$.

For (ii), assume that the surface $\mathcal{N}$ is transverse to the suspension flow of a fibration $f : N \to S^1$. Then $\widetilde{\mathcal{N}}$ is transverse to the suspension flow $p \circ f : \widetilde{N} \to S^1$. Let $\widetilde{\alpha}$ be the pullback of α under p. Then $\widetilde{\alpha}$ is the Poincaré dual of $\widetilde{\mathcal{N}}$. By Theorem 2.1, the 1-form $\widetilde{\alpha}$ lies in the closure of a component of $\widetilde{\mathcal{K}}^*$ that contains the 1-form corresponding to $f \circ p$. Let $\widetilde{K}$ be this component. Since the $\widetilde{\alpha}$ is the pullback of α, the 1-form α lies in $p^*(\widetilde{K})$, which contains the 1-form corresponding to f. □

2.4 Oriented sums

The next step in studying embedded non-orientable surfaces will be to describe *oriented sums*. The oriented sum of two surfaces embedded in a manifold M is additive in both the Euler characteristic and $H^1(M;\mathbb{R})$. This operation is well-known in the case of orientable 3-manifolds (along with orientable embedded surfaces), but we will sketch the relevant details. We then extend the construction to relatively orientable embedded surfaces.

Oriented sum for oriented manifolds Let M be an orientable manifold. Let S and S' be oriented embedded surfaces in M. Assume that S and S' intersect transversally. Thus $S \cap S'$ is a disjoint union of copies of S^1. For each component of $S \cap S'$, take a tubular neighborhood that has cross section as in Figure 1.

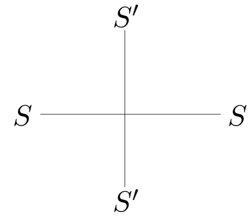

Figure 1: Cross section of intersection of S and S'.

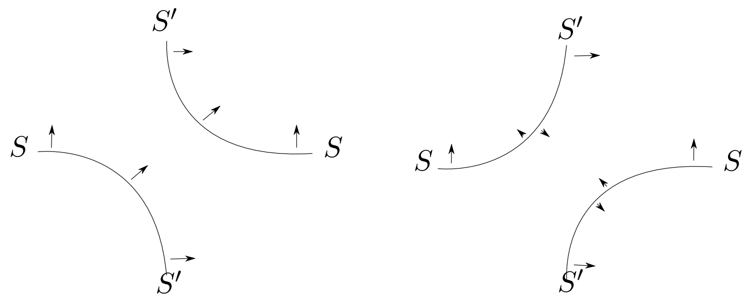

Figure 2: On the left, the normal vectors on S and S' are consistent. On the right, they are not.

We then perform a surgery on the leaves of S and S' so that the outward pointing normal vector fields match as in Figure 2.

By performing this surgery at all the intersections, we get a new submanifold S'' of M (which may have multiple components). This new submanifold S'' is called the *oriented sum* of S and S'. The operation of taking oriented sums is additive on Euler characteristic, as well as the homology classes (and thus the cohomology classes of their Poincaré duals):

$$\chi(S'') = \chi(S) + \chi(S')$$
$$[S''] = [S] + [S'].$$

Oriented sum for non-orientable manifolds Let N be a non-orientable manifold and let $\mathcal{N}$ and $\mathcal{N}'$ be embedded surfaces in N that are relatively oriented. We define the oriented sum on $\mathcal{N}$ and $\mathcal{N}'$ as follows. As above, let $p : \widetilde{N} \to N$ be the orientation double cover and let ι be the orientation reversing deck transformation of $\widetilde{N}$. Let $\widetilde{\mathcal{N}} = p^{-1}(\mathcal{N})$ and $\widetilde{\mathcal{N}'} = p^{-1}(\mathcal{N}')$, which are embedded oriented surfaces in $\widetilde{N}$. The oriented sum of $\mathcal{N}$ and $\mathcal{N}'$ is the image under p of the oriented sum of $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{N}'}$ (as defined above for oriented surfaces in oriented manifolds).

To see that the operation is well defined, we recall that ι preserves the relative orientation of $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{N}'}$. Therefore ι leaves the outward normal vector fields on $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{N}'}$ invariant (see the proof of Lemma 2.9). Thus a leaf ℓ of $\widetilde{\mathcal{N}}$ is surgered with a leaf of ℓ' of $\widetilde{\mathcal{N}'}$ if and only if $\iota(\ell)$ and $\iota(\ell')$ are surgered. Therefore surgery factors through p and $[\mathcal{N}] + [\mathcal{N}']$ is well-defined for non-orientable surfaces.

Example 2.13. Let γ be a component of $\mathcal{N} \cap \mathcal{N}'$ and $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ be the path lifts of γ. One possible orientation of $\widetilde{S}$ and $\widetilde{S}'$ is given in Figure 3. The outward pointing normal vectors to $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{N}'}$ determine which leaves are glued together along $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$.

To preserve the normal vector field, glue the left $\widetilde{\mathcal{N}}$ leaf to the bottom $\widetilde{\mathcal{N}'}$ leaf near $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$. Since $\iota(\widetilde{\gamma}_1) = \widetilde{\gamma}_2$, the outward pointing normal vector fields point the same (relative) directions.

Additivity By the consistency of the oriented sum in N and $\widetilde{N}$, it easily follows that the oriented sum is additive in Euler characteristic, as well as in terms of Poincaré dual, since the Poincaré dual was also defined by passing to the orientation double cover.

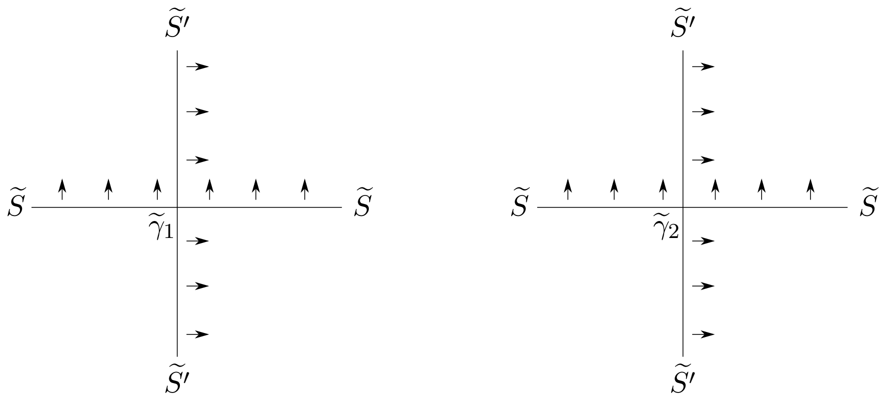

Figure 3: Neighborhoods of $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$, with the outward pointing normal vector field.

3 Mapping classes with small stretch factors

In this section, we construct mapping classes with small stretch factor on non-orientable surfaces.

3.1 Mapping class groups of non-orientable surfaces

Let $\mathcal{N}$ be a non-orientable surface and let $\widetilde{\mathcal{N}}$ and the covering map $p : \widetilde{\mathcal{N}} \to \mathcal{N}$ be its orientation double covering space. Every homeomorphism $\varphi : \mathcal{N} \to \mathcal{N}$, has a unique orientation preserving lift $\widetilde{\varphi} : \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}$.

A consequence is that lifting homeomorphisms induces a monomorphism between orientation preserving homeomorphisms of $\mathcal{N}$ and (orientation preserving) homeomorphisms of $\widetilde{\mathcal{N}}$. Every homotopy of $\mathcal{N}$ lifts to a homotopy of $\widetilde{\mathcal{N}}$. In particular, if $f, g : \mathcal{N} \to \mathcal{N}$ are homeomorphisms such that their orientation preserving lifts are homotopic, then f and g are homotopic. Therefore there is an inclusion from the mapping class group of $\mathcal{N}$ to the (orientation preserving) mapping class group of $\widetilde{\mathcal{N}}$. This inclusion also respects the Nielsen-Thurston classification of mapping classes, both qualitatively, and quantitatively, as the following proposition shows.

Proposition 3.1. *Let $\varphi : \mathcal{N} \to \mathcal{N}$ be a homeomorphism and let $\widetilde{\varphi} : \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}$ be the orientation preserving lift of φ. Then:*

(i) φ is periodic if and only if $\widetilde{\varphi}$ is periodic,

(ii) φ is reducible if and only if $\widetilde{\varphi}$, and

(iii) φ is pseudo-Anosov if and only if $\widetilde{\varphi}$ is pseudo-Anosov. Moreover if φ has stretch factor λ, then $\widetilde{\varphi}$ also has stretch factor λ.

Proof. The fact that the map from $\mathrm{Mod}(\mathcal{N})$ to $\mathrm{Mod}(\widetilde{\mathcal{N}})$ is type-preserving follows from Lemma 10 of [2] (while the statement of the Lemma is for orientable surfaces, the argument, which we will skip, is identical for non-orientable surfaces).

Suppose now that φ is a psuedo-Anosov on $\mathcal{N}$ with stretch factor λ and stable and unstable foliations $\mathcal{F}_s$ and $\mathcal{F}_u$ respectively. Let $\widetilde{\mathcal{F}_s}$ and $\widetilde{\mathcal{F}_u}$ denote the lifts of the stable and unstable foliations to the orientation double cover. We need to show that the following identities hold for all simple closed curves γ on $\widetilde{\mathcal{N}}$.

$$i(\gamma, \widetilde{\varphi}(\widetilde{\mathcal{F}_u})) = \lambda \cdot i(\gamma, \widetilde{\mathcal{F}_u}) \tag{2}$$

$$i(\gamma, \widetilde{\varphi}(\widetilde{\mathcal{F}_s})) = \frac{1}{\lambda} \cdot i(\gamma, \widetilde{\mathcal{F}_s}) \tag{3}$$

To see that these identities hold, we partition γ into short arcs $\{\gamma_i\}$ such that the restriction of the covering map p to a neighbourhood of each arc is a homeomorphism. The local homeomorphism lets us compute the intersection number for each arc γ_i by instead computing the intersection number on the surface $\mathcal{N}$.

$$i(\gamma_i, \widetilde{\mathcal{F}_u}) = i(p(\gamma_i), \mathcal{F}_u) \tag{4}$$

$$i(\gamma_i, \widetilde{\varphi}(\widetilde{\mathcal{F}_u})) = i(p(\gamma_i), \varphi(\mathcal{F}_u)) \tag{5}$$

Since we know that $\mathcal{F}_u$ is the unstable foliation for φ with stretch factor λ, we can compute the ratio of the right hand side of (4) and (5).

$$i(p(\gamma_i), \varphi(\mathcal{F}_u)) = \lambda \cdot i(p(\gamma_i), \mathcal{F}_u) \tag{6}$$

Combining (4), (5), and (6), and summing up over all γ_i gives us (2). A similar argument also proves (3). $\square$

In the case of orientable surfaces, the Penner construction is used to construct pseudo-Anosov maps, as well compute their stretch factors [24]. The Penner construction also works in the non-orientable setting, with some minor modifications. We outline the Penner construction for orientable surfaces below, and provide the necessary details to modify the Penner construction to orientable surfaces. Liechti–Strenner [19, Section 2] provide more complete details of the Penner construction for non-orientable surfaces.

The Penner construction. Let S be an orientable surface. Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$ be multicurves in S. A Penner construction is mapping class given as a composition of Dehn twists that satisfy the following:

1. the complement of $A \cup B$ in N consists of disks with at most one puncture or marked point,
2. a Dehn twist about each curve in $A \cup B$ is included in the composition,
3. each Dehn twist about a curve in A is a left-handed Dehn twist, and
4. each Dehn twist about a rurve in B is a right-handed Dehn twist.

A set of curves that satisfies the first condition is said to *fill* S. Penner proves that this construction is pseudo-Anosov [24].

However, for non-orientable surfaces, there is not a well-defined notion of a left or right Dehn twist. Therefore we use the notion of an inconsistent marking, as follows.

Inconsistent markings. Each two-sided curve c on a non-orientable surface $\mathcal{N}$ has a neighborhood homeomorphic to an annulus $\mathcal{A}_c$. The homeomorphism $\phi : \mathcal{A}_c \to \mathcal{N}$ is called a *marking* of c. A pair consisting of a curve c and the homeomorphism $\phi : \mathcal{A}_c \to N$ is called a *marked curve*. We define the Dehn twist $T_{c,\phi}(x)$ around a marked curve (c, ϕ) as:

$$T_{c,\phi}(x) = \begin{cases} \phi \circ T \circ \phi^{-1}(x) & \text{for } x \in \phi(\mathcal{A}_c) \\ x & \text{for } x \in \mathcal{N} - \phi(\mathcal{A}_c) \end{cases}.$$

Here T is the left-handed Dehn twist on $\mathcal{A}_c$, i.e. $T(\theta, t) = (\theta + 2\pi t, t)$. If we fix an orientation of $\mathcal{A}_c$, then we can pushforward this orientation to N. Let (c, ϕ_c) and (d, ϕ_d) be two marked curves that intersect in a point p. We say that (c, ϕ_c) and (d, ϕ_d) are *marked inconsistently* if the pushforward of the orientation of $\mathcal{A}_c$ and disagrees with the pushforward of the orientation of $\mathcal{A}_d$ in a neighborhood of p.

The Penner construction for non-orientable surfaces. Let $\mathcal{N}$ be a non-orientable surface and let $\mathcal{C}$ be a set of marked curves in N that fill N. A Penner construction on $\mathcal{N}$ is a composition of Dehn twists about the marked curves in $\mathcal{C}$ such that:

1. the complement of curves in $\mathcal{C}$ in $\mathcal{N}$ consists of disks with at most one puncture or marked point,
2. for any $(c_i, \phi_i), (c_j, \phi_j) \in \mathcal{C}$, the marked curves $(c_i, \phi_i), (c_j, \phi_j)$ for $i \neq j$ are marked inconsistently,
3. a Dehn twist about each curve in $\mathcal{C}$ is included in the composition, and
4. all powers of Dehn twists are positive (alternatively, all powers are negative).

As above, if the set $\mathcal{C}$ satisfies the first condition, it is said to *fill* $\mathcal{N}$.

Train tracks. Penner not only proved that mapping classes constructed by the Penner construction are pseudo-Anosov, he also determines their stretch factor (see [24]). Let φ be a pseudo-Anosov homeomorphism of $\mathcal{N}$. A *train track* is an embedded graph in $\mathcal{N}$ such that for every vertex v of valence three or greater, all edges adjacent to v have the same tangent vector at v. An *invariant train track for* φ is a train track track τ such that $\varphi(\tau)$ is homotopic to τ. Let $\mathcal{C}$ be a collection of curves in $\mathcal{N}$. For every curve $\gamma \in \mathcal{C}$, there is an associated transverse measure μ_γ for τ that assigns 1 to all edges lying in γ and 0 to everything else. Let V_τ be the cone of transverse measures on τ, and H the subspace of V_τ spanned by the transverse measure associated to curves in $\mathcal{C}$. The measures μ_γ are linearly independent and form the *standard basis* for H. The subspace H is invariant under the action of φ on V_τ, thus φ has a linear action on H. If we let M be the matrix representing this action in the standard basis, then the stretch factor of φ, $\lambda(\varphi)$, is the Perron-Frobenius eigenvalue of φ.

3.2 Constructing pseudo-Anosov maps on nearby surfaces using oriented sums

The goal of this section is to obtain an asymptotic upper bound on the minimum stretch factor of a pseudo-Anosov homeomorphism. We do this in Lemma 3.2.

Lemma 3.2. *Let $\mathcal{N}_g$ be a non-orientable surface of genus g and let $\varphi : \mathcal{N}_g \to \mathcal{N}_g$ be a pseudo-Anosov homeomorphism with stretch factor λ. Let N_φ be the mapping torus of $\mathcal{N}_g$ by φ. Let $\mathcal{N}_{g'}$ be an incompressible surface embedded in N_φ that is transverse to the supension flow associated to φ. Then for all $k \in \mathbb{Z}^+$, there is a pseudo-Anosov homeomorphism $\mathcal{N}_{g+kg'} \to \mathcal{N}_{g+kg'}$ with stretch factor at most λ.*

Our strategy for doing this is to find fiber bundles of N_φ over S^1 that have fiber $\mathcal{N}_g + k\mathcal{N}_{g'}$. We then apply a special case of Thurston's hyperbolization theorem, which says that the mapping torus of an orientable surface S by a homeomorphism φ is hyperbolic if and only if φ is pseudo-Anosov [28, Theorem 0.1]. In particular, Thurston's theorem implies that if M fibers over S^1 in two ways, either both fiber maps are pseudo-Anosov or neither fiber map is pseudo-Anosov. Finally, we adapt theorems of Fried and Matsumoto (Theorem 3.4) and Agol–Leininger–Margalit (Theorem 3.5) to work for mapping tori with non-orientable fibers.

We will repeatedly use the following two facts for orientable surfaces and 3-manifolds:

1. An orientable surface minimizes the Thurston norm in its homology class if and only if it is incompressible.
2. If an orientable 3-manifold M fibers over S^1, then the fiber is incompressible.

Proposition 3.3. *Let $\mathcal{N}'$ be an incompressible surface embedded in N that is transverse to the suspension flow direction associated to φ. Let α be the Poincaré dual of $\mathcal{N}$ and α' the Poincaré dual of $\mathcal{N}'$. If the oriented sum of $\mathcal{N}$ and $\mathcal{N}'$ is connected, then $\mathcal{N} + \mathcal{N}'$ is homeomorphic to the fiber of the fibration given by $\alpha + \alpha'$.*

Proof. Let $p : \widetilde{N} \to N$ be the orientation double cover of N. The surface $\mathcal{N}$ is incompressible because it is a fiber of f; therefore its pre-image under p is also incompressible. Therefore the Thurston norm of $\mathcal{N}$ of α is $2\chi_-(\mathcal{N})$. Likewise, the Thurston norm of α' is $2\chi_-(\mathcal{N}')$.

Both α and α' lie in a cone over a fibered face in $H^1(N;\mathbb{Z})$. Therefore the Thurston norm x on $H^1(N;\mathbb{Z})$ is linear function on that cone. Since the Thurston norm is linear on oriented sums of $\mathcal{N}$ and $\mathcal{N}'$, we have:

$$\begin{aligned} x(\alpha+\alpha') &= x(\alpha)+x(\alpha') \\ &= 2\chi_-(\mathcal{N})+2\chi_-(\mathcal{N}') \\ &= 2\chi_-(\mathcal{N}+\mathcal{N}'). \end{aligned}$$

Because $2\chi_(\mathcal{N}+\mathcal{N}')$ achieves the Thurston norm of $\alpha+\alpha'$, the preimage $p^{-1}(\mathcal{N}+\mathcal{N}')$ achieves the Thurston norm of the pullback of $\alpha+\alpha'$ under p. Therefore $p^{-1}(\mathcal{N}+\mathcal{N}')$ is incompressible. Then $\mathcal{N}+\mathcal{N}'$ is also incompressible.

By Theorem 2.12, we have that $\alpha+\alpha'$ corresponds to some other fibration $f'': N \to S^1$. By Theorem 2.8, the fiber of f'' must be homeomorphic to $\mathcal{N}+\mathcal{N}'$. □

In the proof of Lemma 3.2, we will use Proposition 3.3 along with a theorem of Thurston to obtain a pseudo-Anosov homemorphism φ_k of the surface of genus of genus $g+kg'$. We the use Theorems 3.4 and 3.5 to obtain a upper bound on the stretch factor of φ_k.

The next step is to show that the stretch factor of the pseudo-Anosov on the new surfaces is less than the stretch factor of the original surface. We use the following two theorems that hold for orientable 3-manifolds.

Theorem 3.4 (Fried [8],[9],Matsumoto[21])**.** *Let M be an orientable hyperbolic 3-manifold and let $\mathcal{K}$ be the union of cones in $H^1(M;\mathbb{R})$ whose lattice points correspond to fibrations over S^1. There exists a strictly convex function $h: \mathcal{K} \to \mathbb{R}$ satisfying the following properties.*

(i) For all $c > 0$ and $u \in \mathcal{K}$, $h(cu) = \frac{1}{c}h(u)$.

(ii) For every primitive lattice point $u \in \mathcal{K}$, $h(u) = \log(\lambda)$, where λ is the stretch factor of the pseudo-Anosov map associated to this lattice point.

(iii) $h(u)$ goes to ∞ as u approaches $\partial\mathcal{K}$.

Theorem 3.5 (Agol-Leininger-Margalit)**.** *Let $\mathcal{K}$ be a fibered cone for a mapping torus M and let $\overline{\mathcal{K}}$ be its closure in $H^1(M;\mathbb{R})$. If $u \in \mathcal{K}$ and $v \in \overline{\mathcal{K}}$, then $h(u+v) < h(u)$.*

Proof of Lemma 3.2. The oriented sum $\mathcal{S} = \mathcal{N}_g + k\mathcal{N}_{g'}$ constructed in Proposition 3.3 is a surface of genus $g+kg'$, and is $\mathcal{S}$ is homeomorphic to a fiber of N_φ given by $\alpha+k\alpha'$. Let $\varphi_{\mathcal{S}}: \mathcal{S} \to \mathcal{S}$ be the fiber map of N_φ over $\mathcal{S}$. By Thurston's theorem, $\varphi_{\mathcal{S}}$ is pseudo-Anosov. We claim that $\varphi_{\mathcal{S}}$ has stretch factor at most λ.

Let $p: \widetilde{N} \to N_\varphi$ be the orientation double cover of N_φ. Let $h\mid_N$ be the restriction of h to the pullback of $H^1(N_\varphi;\mathbb{R})$ to $H^1(\widetilde{N};\mathbb{R})$. The restriction $h\mid_N$ satisfies all the properties of Theorems 3.4 and 3.5.

Let $\mathcal{K}$ be the cone in $H^1(N_\varphi;\mathbb{R})$ containing α. Since $\mathcal{N}_{g'}$ is transverse to the suspension flow in the direction of φ, we have that α' is in the closure of $\mathcal{K}$ in $H^1(N;\mathbb{R})$. Let $\widetilde{\alpha}$ be the pullback of α under p and let $\widetilde{\alpha'}$ be the pullback of α' under p. Then $h\mid_N(\widetilde{\alpha}+\widetilde{\alpha'}) < h\mid_N(\widetilde{\alpha})$. By Theorem 3.4, $h(\widetilde{\alpha})$ is equal to the stretch factor of the pseudo-Anosov homeomorphism associated to $\widetilde{\alpha}$. Let $\widetilde{\varphi}$ be the orientation preserving lift of φ to $p^{-1}(\mathcal{N})$. Since $\widetilde{\alpha}$ is the pullback of α, the $\widetilde{\varphi}$ is the pseudo-Anosov homeomorphism associated to $\widetilde{\alpha}$. By Proposition 3.1, the stretch factor of $\widetilde{\varphi} = \lambda$. □

4 Minimal stretch factors for non-orientable surfaces with marked points

In this section we will use Theorems 2.12 and 3.3 to adapt the methods of [31] to non-orientable surfaces. In particular, we prove the following theorem on the asymptotic behavior of the minimal stretch factor of non-orientable surfaces:

Theorem 1.1. For any fixed $n \in \mathbb{N}$, there are positive constants $B'_1 = B'_1(n)$ and $B'_2 = B'_2(n)$ such that for any $g \geq 2$, the stretch factor satisfies the following inequalities.

$$\frac{B'_1}{g} \leq l'_{g,n} \leq \frac{B'_2}{g}$$

Observe that the lower bound for the non-orientable case follows easily from the lower bound for the orientable case. Let φ be a pseudo-Anosov map with the minimal stretch factor on $\mathcal{N}_{g,n}$. Then, by Proposition 3.1, φ lifts to a map $\widetilde{\varphi} : \mathcal{S}_{g-1,2n} \to \mathcal{S}_{g-1,2n}$ (possibly after squaring). Furthermore, $\widetilde{\varphi}$ has the same stretch factor as φ. The former is bounded below by $\frac{B_1}{g}$, and thus the stretch factor of φ is bounded below as well. The more challenging part of the proof is showing the upper bound holds. This will be done by explicitly constructing pseudo-Anosov maps with small stretch factors, adapting Yazdi's techniques to the non-orientable setting.

We will closely follow Yazdi's construction, which proceeds in five steps. In steps 1 and 2, a family of small dilatation psuedo-Anosov maps is constructed on $\mathcal{S}_{g_i,n}$, where $\{g_i\}$ is a sequence of genera going off to infinity, but not containing every element of $\mathbb{N}$, i.e. there are plenty of gaps. Steps 3 through 5 deal with constructing small dilatation pseudo-Anosov maps on the missing surfaces to fill in the gaps. This is where Thurston's fibered face theory enters the picture. In this section, we will adapt the steps to work for non-orientable surfaces.

Step 1: Constructing the surfaces The first step in the construction is defining a family of surfaces that exhibit a specific rotational symmetry. Using this symmetry, if one shows that a power of some homeomorphism is pseudo-Ansov, then so is the original homeomorphism [25].

We begin by defining a family of surfaces $P_{n,k}$. Let S be an orientable surface of genus 5 with 3 boundary components c, d and e. Choose an orientation for S and let c, d and e inherit the induced orientations. We obtain a non-orientable surface T from S as follows. Add two cross caps to S (but retain the orientation of the boundary components of S). Remove a point p from the boundary component e and let q be a marked point in e. Let r and s be the components of $e \setminus p, q$. The resulting surface T is given in Figure 4.

Let $T_{i,j}$ be copies of the surface T, where $i, j \in \mathbb{Z}$. Let $c_{i,j}, d_{i,j}$ and $e_{i,j}$ be the (oriented) boundary components of $T_{i,j}$ and let $r_{i,j}$ and $s_{i,j}$ be the copies of the arcs r and s in $T_{i,j}$. Define a connected infinite surface T_∞ as the following quotient:

$$T_\infty \coloneqq \left(\bigcup T_{i,j}\right) \Big/ \sim$$

for all i and j are integers. The gluing $\sim$ is given by the following two families of (orientation-reversing) identifications:

$$c_{i,j} \sim d_{i+1,j} \tag{7}$$
$$r_{i,j} \sim s_{i,j+1}. \tag{8}$$

Furthermore, the boundary components are glued by an orientation-reversing homeomorphism.

We have two natural shift maps $\overline{\rho_1}, \overline{\rho_2} : T_\infty \to T_\infty$ that act in the following manner:

$$\overline{\rho_1} : T_{i,j} \mapsto T_{i+1,j}$$
$$\overline{\rho_2} : T_{i,j} \mapsto T_{i,j+1}.$$

Note that these maps commute. Define the surface $P_{n,k}$ as the quotient of the surface T_∞ by the covering action of the group generated by $(\overline{\rho_1})^n$ and $(\overline{\rho_2})^k$. Therefore, $\overline{\rho_1}$ and $\overline{\rho_2}$ induce maps on the surface $P_{n,k}$, which we denote by ρ_1 and ρ_2.

A natural question at this point is why we chose the surface T for our building block? It has two advantages:

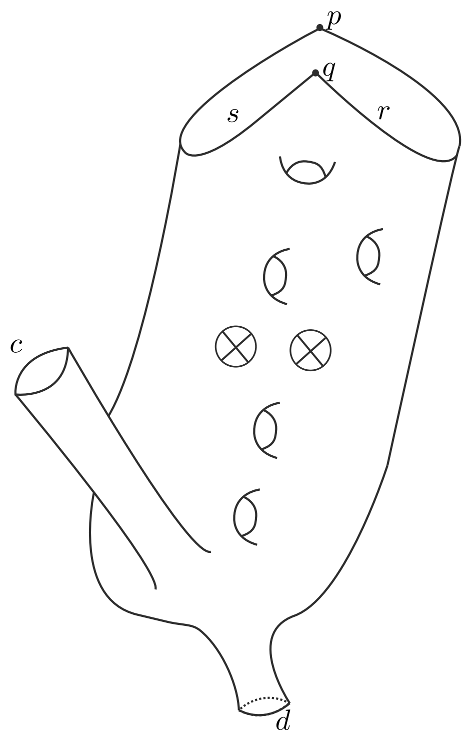

Figure 4: The surface T, which will be the building block of the construction.

- The combinatorics of the curves in Figure 5 make the associated matrix from the Penner construction satisfy the conditions of Lemma 4.3. This is used to prove our family of pseudo-Anosov maps have stretch factors bounded above by the quantity we desire.
- Having a curve γ such that it and its image under a given pseudo-Anosov we construct form the boundary of an embedded $\mathbb{R}P^2$ with two boundary components in the mapping torus, which will come into play when extending our family of surfaces in Step 3.

Lemma 4.1. *Let*

$$g_{n,k} = (14k-2)n + 2$$

for $n \geq 1$ and $k \geq 3$. The genus of $P_{n,k}$ is $g_{n,k}$.

Proof. Let $U \subset P_{n,k}$ be the subsurface

$$U = \left(\bigcup_{i=0}^{k-1} T_{0,i} \right) \Big/ \sim$$

where $\sim$ is given by (7). Then U is a compact, non-orientable surface of genus $12k$ with $2k$ boundary components, and forms a fundamental domain for the covering action of $\overline{\rho_1}$ on T_∞. We can compute the Euler characteristic of U in order to determine the Euler characteristic of $P_{n,k}$:

$$\begin{aligned} \chi(U) &= 2 - 12k - 2k \\ &= 2 - 14k. \end{aligned}$$

Thus

$$\begin{aligned} \chi(P_{n,k}) &= n \cdot \chi(U) \\ &= -n(14k-2), \end{aligned}$$

since $P_{n,k}$ is formed by gluing n copies of U together along circle boundary components. By the relation between genus and Euler characteristic, we have the claimed formula for the genus of $P_{n,k}$:

$$g_{n,k} = n(14k - 2) + 2$$

$\square$

Step 2: Constructing the maps.

We now construct homeomorphisms $f_{n,k} : P_{n,k} \to P_{n,k}$ that are defined as a composition of specific Dehn twists followed by a finite order mapping class. The key insight is that a power of this map will be a composition of Dehn twists that satisfy the criteria to be a Penner construction and are therefore themselves pseudo-Anosov. This is how we take advantage of the rotational symmetry of the $P_{n,k}$.

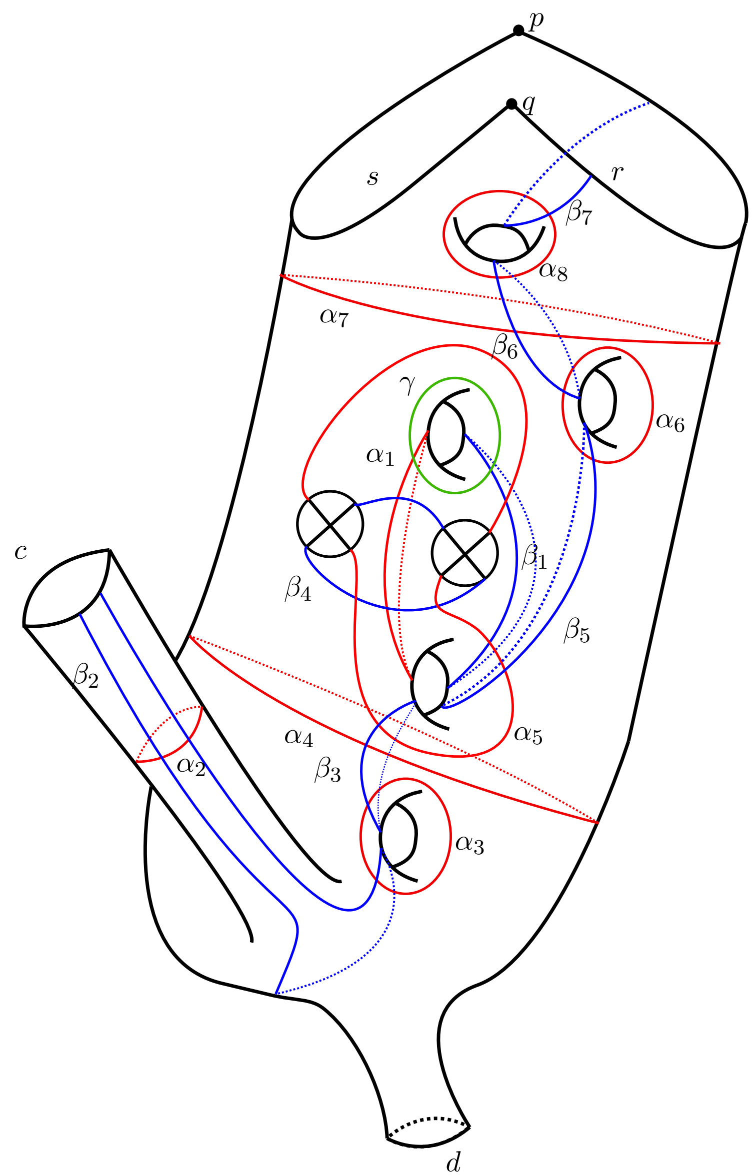

Figure 5: The curves α_i, β_j, and γ in $T_{0,0}$

Recall that for non-orientable surfaces, we did not initially have a well-defined notion of a positive or negative Dehn twist. As we saw in Section 3.1, in order to use the Penner construction to construct

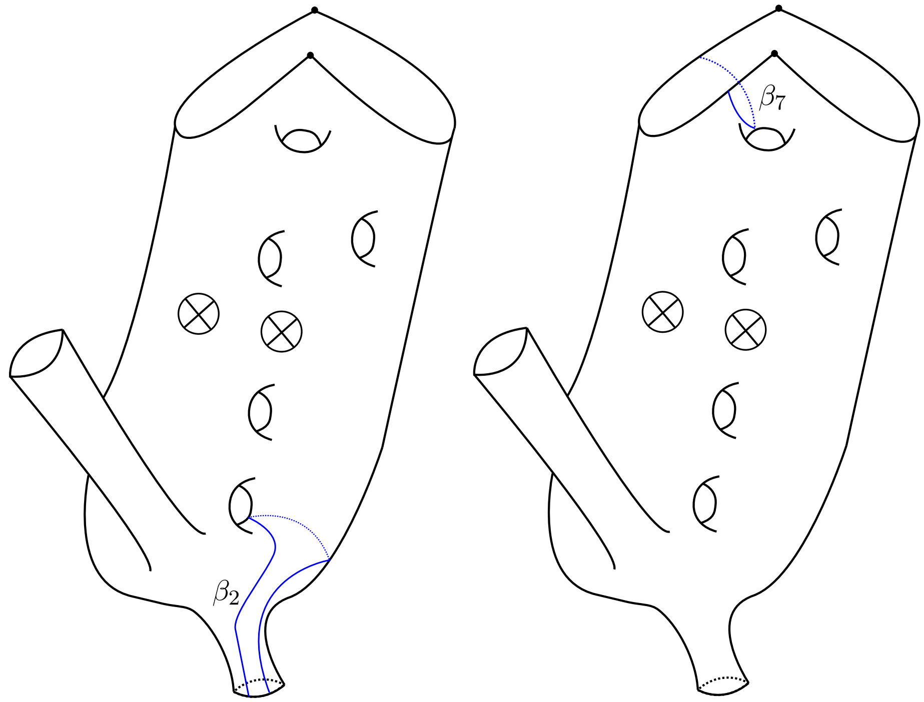

Figure 6: The parts of curves β_2 and β_7 on $T_{0,1}$ and $T_{1,0}$

pseudo-Anosov mapping classes, we need to ensure that the curves about which we twist are marked inconsistently. Construct the multi-curves $\{\alpha_1, \cdots, \alpha_8\}$ in $T_{0,0}$ as shown in Figure 5.

Note that our labeling of the curves in Figure 5 already gives us an inconsistent marking. For any alpha curve α_i, we let the marking ϕ_{α_i} be orientation preserving and for beta curves β_j let ϕ_{β_j} be orientation reversing. Since alpha curves only intersect beta curves (and vice versa), we have an inconsistent marking at each point of intersection.

Let $\mathcal{B}$ be the union of all β curves except β_1 in $T_{0,0} \cup T_{0,1} \cup T_{1,0}$ (see Figure 6 above). Let $\rho_1(\mathcal{B})$ be the image of $\mathcal{B}$ under ρ_1. Define ϕ_b as the composition of Dehn twists along all the curves in the set $\overline{\mathcal{B}} \coloneqq \mathcal{B} \cup \rho_1(\mathcal{B}) \cup \cdots \cup \rho_1^{n-1}(\mathcal{B})$. Since the curves in $\overline{\mathcal{B}}$ are disjoint, Dehn twists along them commute and therefore it is not necessary to specify the order in which we compose these Dehn twists in ϕ_b. Let $\mathcal{R}$ be the union of all α curves except α_1 in $T_{0,0}$. Define $\overline{\mathcal{R}}$ and ϕ_r in the same manner as above.

Let $\alpha_1, \beta_1 \subset T_{0,0}$ be the curves in Figure 5. Let ϕ be the composition of Dehn twists along all the curves $\alpha_1, \rho_1(\alpha_1), \ldots, \rho_1^{n-1}(\alpha_1)$ followed by Dehn twists along all the curves $\beta_1, \rho_1(\beta_1), \ldots, \rho_1^{n-1}(\beta_1)$. Define the map $f_{n,k}$ in the following manner.

$$f_{n,k} \coloneqq \rho_2 \circ \phi \circ \phi_b \circ \phi_r$$

It follows from the Penner construction that $(f_{n,k})^k$ is pseudo-Anosov. Hence $f_{n,k}$ itself is pseudo-Anosov and an invariant train track $\tau_{n,k}$ for $f_{n,k}$ can be obtained from Penner's construction that we described in Section 3.1.

Step 3: The Mapping Torus.

We have now constructed an infinite family of non-orientable surfaces and pseudo-Anosov maps, but this is not enough. By Lemma 4.1, the family does not contain a surface of every genus. In fact, the family does not include surfaces of infinitely many genera. We will use our extension of the Thurston's fibered face theory to fill in the gaps. For each $n \in \mathbb{N}$ we will find a fibration of a mapping torus of $f_{n,k}$ that has a fiber that is homeomorphic to $\mathcal{N}_3$, the hyperbolic non-orientable surface of lowest possible genus.

Let $M_{n,k}$ be the mapping torus of $f_{n,k}$. Likewise, let $\mathcal{K}_{n,k}$ denote the fibered cone of $H^1(M_{n,k};\mathbb{R})$ corresponding to the map $f_{n,k}$.

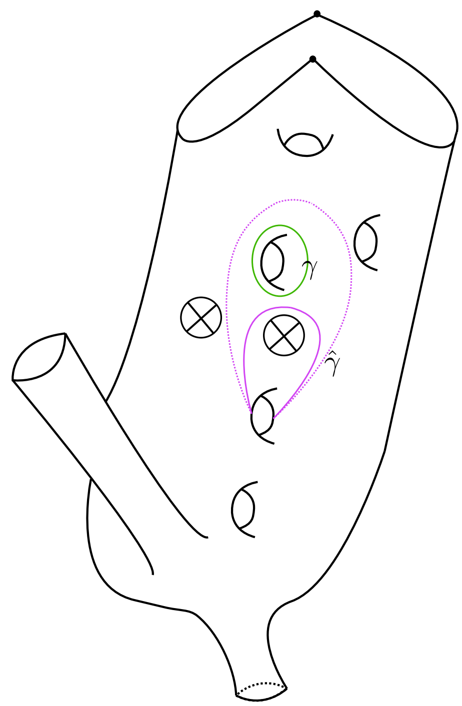

Figure 7: The curves γ and $\widehat{\gamma}$ bound an a non-orientable surface of genus 1.

Lemma 4.2. *Let $M_{n,k}$ be the mapping torus of $f_{n,k}$. Let $\mathcal{K}_{n,k}$ denote the fibered cone of $H^1(M_{n,k};\mathbb{R})$ corresponding to the map $f_{n,k}$. There is a relatively orientable incompressible surface $F_{n,k}$ in $M_{n,k}$ that is homeomorphic to $\mathcal{N}_3$. Moreover $F_{n,k}$ is transverse to the suspension flow direction given by $f_{n,k}$ and the Poincaré dual of $F_{n,k}$ is in the closure $\overline{\mathcal{K}_{n,k}}$.*

Proof. Let $\gamma \subset T_{0,0}$ be the curve shown in Figure 7. Note that γ and $\phi(\gamma)$ bound a non-orientable surface $\hat{F}$ of genus 1 with boundary. For convenience, we will denote $\phi(\gamma)$ by $\widehat{\gamma}$. We are going to follow the image of γ under iterations of the pseudo-Anosov map $f_{n,k}$. Doing so will allow us to attach annuli to the boundary of $\widehat{F}$ to get a closed $\mathcal{N}_3$. Using the facts that $f_{n,k} = \rho_2 \circ \phi \circ \phi_b \circ \phi_r$ and both ϕ_r and ϕ_b act trivially on γ, we have the following:

$$\begin{aligned} f_{n,k}(\gamma) &= \rho_2 \circ \phi \circ \phi_b \circ \phi_r(\gamma) \\ &= \rho_2 \circ \phi(\gamma) \\ &= \rho_2(\widehat{\gamma}) \\ f_{n,k}^2(\gamma) &= \rho_2^2(\widehat{\gamma}) \\ &\vdots \\ f_{n,k}^k(\gamma) &= \rho_2^k(\widehat{\gamma}) \\ &= \widehat{\gamma}. \end{aligned}$$

That is, for all $1 \le i \le k$, the curve $f_{n,k}^i(\gamma)$ is the copy of $\widehat{\gamma}$ in the copy of $T_{0,0}$ under rotation by ρ_2^i. For $1 \le i \le k$, let T_i be an annulus in $M_{n,k}$ that connects $f_{n,k}^{i-1}(\gamma)$ to $f_{n,k}^i(\gamma)$ obtained by following the suspension flow of $f_{n,k}$ around $M_{n,k}$. We can now construct our embedded surface $F_{n,k}$ by taking the union of $T_1, T_2, \ldots, T_k$ and $\hat{F}$. Since we are adding an orientable genus to a non-orientable surface of genus 1, we see $F_{n,k}$ is homeomorphic to $\mathcal{N}_3$.

The resulting surface is an embedded non-orientable surface in a non-orientable 3-manifold, so we have relative orientability by Proposition 2.6.

The proof that $F_{n,k}$ can be isotoped to be transverse to the suspension flow is the same as the proof in [31], which in turn follows the proof in [18]. Let $N(\gamma)$ be a tubular neighbrhood of γ in $\hat{F}$, and $\eta : \hat{F} \to [0,1]$ be a smooth function supported on $N(\gamma)$ with $\eta^{-1}(1) = \gamma$ and such that the derivative of η on γ vanishes. Denote the suspension flow of the map $f_{n,k}$ by $\phi_t : M_{n,k} \to M_{n,k}$, where $t \in \mathbb{R}$, and define the map $g : \hat{F} \to M_{n,k}$ as $g(x) = \phi_{k \cdot \eta(x)}(x)$. Then g restricted to the interior of $\hat{F}$ is an embedding, and satisfies $g(\gamma) = \hat{\gamma}$. The image of $g : \hat{F} \to M_{n,k}$ is an embedded non-orientable surface of genus three, that is isotopic to the natural embedding of $F_{n,k}$ in $M_{n,k}$, and is tranverse to the suspension flow. Therefore, its Poincaré dual is in $\overline{\mathcal{K}_{n,k}}$ by Theorem 2.12.

Finally, $F_{n,k}$ is incompressible in $M_{n,k}$ because $M_{n,k}$ is hyperbolic, and $F_{n,k}$ is genus 3, the lowest possible genus for a hyperbolic non-orientable surface. □

Step 4: Bounding the Stretch Factor.

Yazdi then finds an upper bound for the log of the stretch factors of the pseudo-Anosov homeomorphisms he constructed. Similarly, we want to find an upper bound for the homeomorphisms we constructed in Step 2. In order to to do this we use *train tracks*, embedded graphs in our surface with that property that for every vertex v of valence three or greater, all edges adjacent to v have the same tangent vector at v. All pseudo-Anosov homeomorphisms come equipped with an *invariant train track*, a train track whose image under the map is homotopic to itself. These invariant train tracks have an associated matrix whose Perron-Frobenius eigenvalue is the stretch factor of our pseudo-Anosov.

Yazdi uses the Lemma 4.3 to bound the spectral radius of the associated matrices.

Lemma 4.3 (Lemma 2.3 of [31]). *Let A be a non-negative integral matrix, Γ be the adjacency graph of A, and $V(\Gamma)$ the set of vertices of Γ. For each $v \in V(\Gamma)$, define v^+ to be the set of vertices $u \in V(\Gamma)$ such that there is an oriented edge from v to u. Let D and k be fixed natural numbers. Assume the following conditions hold for Γ:*

(i) For each $v \in V(\Gamma)$ we have $\deg_{out}(v) \leq D$,

(ii) There is a partition $V(\Gamma) = V_1 \cup \cdots \cup V_\ell$ such that for each $v \in V_i$ we have $v^+ \subset V_{i+1}$, for any $1 \leq i \leq \ell$ except possibly when $i = 1$ or 3 (indices are mod ℓ),

(iii) For each $v \in V_1$, we have $v^+ \subset V_2 \cup V_3$,

(iv) For each $v \in V_3$ we have $v^+ \subset V_3 \cup V_4$, and for $u \in v^+ \cap V_3$ we have $u^+ \subset V_4$, and

(v) For all $3 < j \leq k$ and each $v \in V_j$, the set v^+ consists of a single element.

Then the spectral radius of $A^{\ell-1}$ is at most $4D^4$.

With this result in hand, we can now show that the stretch factors for our main family of examples are all bounded above in the way we hope.

Lemma 4.4. *Let $\lambda_{n,k}$ be the stretch factor of $f_{n,k}$. Then there exists a universal positive constant C' such that for every $n \geq 1$ and $k \geq 3$, we have the following upper bound on $\log(\lambda_{n,k})$.*

$$\log(\lambda_{n,k}) \leq C' \frac{n}{g_{n,k}}$$

Proof. We deliberately constructed our examples so our curves are in the same general form as the ones in [31] such that all intersections between the curves happen inside the building block T, except for the intersections between building blocks given by the beta curves β_3 and β_8.

We define the following multi-curves:

$$\begin{aligned}\mathcal{A} &\coloneqq \mathcal{B} \cup \mathcal{R} \cup \{\alpha_1, \beta_1\} = \bigcup_{i=1}^{8} (\alpha_i \cup \beta_i) \\ \overline{\mathcal{A}} &\coloneqq \mathcal{A} \cup \rho_1(\mathcal{A}) \cup \cdots \cup \rho_1^{n-1}(\mathcal{A}) \\ \widehat{\mathcal{A}} &\coloneqq \overline{\mathcal{A}} \cup \rho_2\left(\overline{\mathcal{A}}\right) \cup \cdots \cup \rho_2^{k-1}\left(\overline{\mathcal{A}}\right).\end{aligned}$$

Thus, $\widehat{\mathcal{A}}$ is union of the curves about which we Dehn twist to obtain $f_{n,k}$.

Because $f_{n,k}$ is constructed via the Penner construction, it is pseudo-Anosov and therefore has a corresponding invariant train track τ. For each $\gamma \subset \widehat{\mathcal{A}}$ there is an associated transverse measure μ_γ on τ that assigns 1 to all edges in γ and 0 to everything else. Let V_τ be the cone of transverse measures on τ. Let H be the subspace of V_τ spanned by the transverse measures

$$\{\mu_\gamma | \gamma \subset \widehat{\mathcal{A}}\}.$$

The measures μ_γ form a basis for H. The subspace H is invariant under the action of φ on V_τ, thus φ has a linear action on H. Let M be the matrix representing this linear action on H. Let Γ be the adjacency graph for M. Work of Penner [24] tells us that the stretch factor of $f_{n,k}$, λ, is the Perron-Frobenius eigenvalue of M.

To bound the spectral radius of M, we need to show that Γ satisfies the criteria of Lemma 4.3.

(i) There exists a constant D', independent of n and k, such that for every (connected) curve $\gamma \in \widehat{\mathcal{A}}$, the geometric intersection number between γ and $\overline{\mathcal{A}}$ is at most D'. Recall that $f_{n,k} = \rho_2 \circ \phi \circ \phi_b \circ \phi_r$. Let M_1, M_2, M_3 and M_4 be the matrices describing the linear action of ϕ_r, ϕ_b, ϕ and ρ_2 on H, respectively. The matrix M can then be written as a product:

$$A = M_4 M_3 M_2 M_1.$$

For a (connected) curve $\delta \in \hat{\mathcal{A}}$, the L^1-norm of $A(\mu_\delta)$ is bounded above by the geometric intersection of $f_{n,k}(\delta)$ with the curves in $\overline{\mathcal{A}}$, thus each of M_1, M_2 and M_3 will change the norm by a factor of at most $(1 + D')$. Since ρ_2 will not change intersection numbers, M_4 will preserve the L^1-norm. If we let $D = (1 + D')^3$, then the outward degree of each vertex in Γ is at most D.

(ii) Next we partition the vertices of Γ. Recall

$$\widehat{\mathcal{A}} = \bigcup_{i=1}^{k} \rho_2^{i-2}(\overline{\mathcal{A}}).$$

Then define V_i for $1 \leq i \leq k$ as the vertices of Γ corresponding to elements in $\rho_2^{i-2}(\overline{\mathcal{A}})$. Suppose that $v \in V_i$, $i \neq 1, 3$, is a vertex that corresponds to μ_γ for a curve $\gamma \in \hat{\mathcal{A}}$. Then γ must be a curve in $\rho_2^{i-2}(\overline{\mathcal{A}})$, for $i \neq 1, 3$. The action of $\phi \circ \phi_b \circ \phi_r$ on $\widehat{\mathcal{A}}$ will preserve the set $\rho_2^{i-2}(\overline{\mathcal{A}})$ for each i. Then ρ_2 will rotate the curve $\phi \circ \phi_b \circ \phi_r(\gamma)$ to $\rho_2^{i-1}(\overline{\mathcal{A}})$. That is: $f_{n,k} = \rho_2 \circ \phi \circ \phi_b \circ \phi_r$ maps $\mu_\gamma \in H$ to

$$\sum_{\zeta \in \mathcal{Z}} \mu_\zeta$$

where $\mathcal{Z}$ is a subset of $\rho_2^{i-1}(\overline{\mathcal{A}})$. Therefore $f_{n,k}$ maps v to a subset of V_{i+1}.

(iii) We need to see which vertices in $v \in V_1$ have $v^+ \not\subset V_2$. These are the vertices of V_1 corresponding to curves that $\phi \circ \phi_b \circ \phi_r$ maps to curves that do not correspond to vertices in V_1. Recall that because ρ_1 and ρ_2 commute, each vertex of $v \in V_1$ corresponds to a curve in:

$$\rho_2^{-1}(\overline{\mathcal{A}}) = \rho_2^{-1}(\mathcal{A}) \cup \rho_1(\rho_2^{-1}(\mathcal{A})) \cup \cdots \cup \rho_1^{n-1}(\rho_2^{-1}(\mathcal{A})).$$

The elements of v^+ that are not in V_2 correspond to the images of curves in $\rho_2^{-1}(\overline{\mathcal{A}})$ under $f_{n,k}$ that are not in $\overline{\mathcal{A}}$. As in Yazdi, the only curves in $\rho_2^{-1}(\overline{\mathcal{A}})$ that intersect curves in $\overline{\mathcal{A}}$ are those in the set:

$$\mathcal{X} = \{\rho_1^i(\rho_2^{-1}(\beta_7)) \mid 0 \leq i \leq n-1\}.$$

Therefore $\phi \circ \phi_b \circ \phi_r$ maps curves in $\mathcal{X}$ to curves in $\rho_2^{-1}(\overline{\mathcal{A}}) \cup \overline{\mathcal{A}}$. Then $f_{n,k} = \rho_2 \circ \phi \circ \phi_b \circ \phi_r$ maps curves in $\mathcal{X}$ to curves in $\overline{\mathcal{A}} \cup \rho_2(\overline{\mathcal{A}})$.

Let $X = \{\mu_\eta | \eta \in \mathcal{X}\}$

Therefore the vertices of $v \in V_1$ corresponding to curves in X will have $v^+ \subset V_2 \cup V_3$. Moreover, $f_{n,k}$ maps the curves $\rho_2^{-1}(\overline{\mathcal{A}}) \setminus X$ to curves in $\overline{\mathcal{A}}$. Thus for any vertex $v \in V_1$ that does not correspond to an element of X, the set v^+ is contained in V_2.

(iv) Similarly, we need to see which vertices in $v \in V_3$ have $v^+ \not\subset V_4$. The vertices of V_3 correspond to elements of the form:

$$\rho_2(\overline{\mathcal{A}}) = \rho_2(\mathcal{A}) \cup \rho_1(\rho_2(\mathcal{A})) \cup \cdots \cup \rho_1^{n-1}(\rho_2(\mathcal{A})).$$

The set of curves

$$\mathcal{Y} = \{\rho_1^i(\rho_2(\alpha_8)) \mid 0 \leq i \leq n-1\}$$

are precisely the curves corresponding to vertices of V_3 that intersect with the curves in $\overline{\mathcal{A}}$, namely the curves $\rho_1^i(\beta_8)$.

Therefore elements $v \in V_3$ such that $v^+ \not\subset V_4$ are those that correspond to the elements of the following set:

$$Y = \{\mu_\eta \mid \exists\, \eta \in \mathcal{Y}\}.$$

Moreover, for any element $v \in V_3$ corresponding to Y and any $u \in v^+ \cap V_3$, the vertex u does not correspond to an element of Y and hence $u^+ \subset V_4$.

(v) All the curves corresponding to an element of V_j, $3 < j \leq k$ are disjoint from all the curves in $\overline{\mathcal{A}}$. Thus, $f_{n,k}$ just acts by rotation.

Set $\lambda = \lambda_{n,k}$. By Lemma 4.3, we have:

$$\lambda^{k-1} = \rho(M)^{k-1} = \rho(M^{k-1}) \leq 4D^4.$$

Then the logarithm of λ satisfies:

$$\log(\lambda^{k-1}) = (k-1) \cdot \log(\lambda) \leq \log(4D^4).$$

Then for $k \geq 2$

$$\frac{k}{2} \log(\lambda) \leq (k-1) \log(\lambda) \leq \log(4D^4).$$

On the other hand, we know $g_{n,k} = (14k-2)n + 2 \leq 14kn$. Therefore

$$\log(\lambda) \leq 2\log(4D^4) \cdot \frac{1}{k} \leq 2\log(4D^4) \cdot \frac{14n}{g_{n,k}}.$$

Let $C' := 28 \log(4D^4)$ to complete the result. $\square$

Step 5: Filling in the Gaps. Recall that the family of surfaces $P_{n,k}$ that we have constructed have genera in the set $\{(14k-2)n+2\}$. We now want to construct pseudo-Anosov maps with small stretch factors on surfaces of the genera not in the set $\{(14k-2)n+2\}$. To do this we use the mapping torus $M_{n,k} = (P_{n,k}, f_{n,k})$. Recall from Lemma 4.2 that there exists a relatively incompressible surface $F_{n,k}$ in $M_{n,k}$ that is homeomorphic to $\mathcal{N}_3$. Let $P^r_{n,k}$ be the oriented sum of $P_{n,k}$ and r times $F_{n,k}$, as defined in Theorem 3.3. The surfaces $P^r_{n,k}$ will be surfaces of the remaining genera.

Lemma 4.5. *The surfaces $P^r_{n,k}$ have genus $g^r_{n,k} = g_{n,k} + r$. In particular, as r varies between 0 and $14n$, the genera of $P^r_{n,k}$ cover the range between $g_{n,k}$ and $g_{n,k+1}$. Moreover, $P^r_{n,k}$ is isotopic to a fiber of a fibration of $M_{n,k}$ with pseudo-Anosov monodromy that fixes $2n$ of the singularities of its invariant foliation.*

Proof. The Euler characteristic of an oriented sum is the sum of the Euler characteristics of the summands:

$$\begin{aligned}\chi(P^r_{n,k}) &= \chi(P_{n,k}) + r\cdot\chi(F_{n,k})\\ &= (-2g_{n,k}+2) - 2r\\ &= -2(g_{n,k}+r) + 2.p\end{aligned}$$

This proves the identity for the genus of $P^r_{n,k}$.

By Lemma 4.2 we know that $F_{n,k}$ is incompressible and transverse to the suspension flow of given by $f_{n,k}$. Therefore Theorem 3.3 gives us that $P^r_{n,k}$ is isotopic to a fiber of a fibration of $M_{n,k}$. Let $f^r_{n,k}$ be the first return map of the new fibration of $M_{n,k}$ over $P^r_{n,k}$. Since $f_{n,k}$ is a pseudo-Anosov monodromy of $M_{n,k}$, we have that $M_{n,k}$ is hyperbolic. Therefore all monodromies of $M_{n,k}$ are pseudo-Anosov and in particular $f^r_{n,k}$ is a pseudo-Anosov map.

As in the proof of Lemma 3.5 of [31], the singularities of the stable foliation of $f_{n,k}$ that are fixed are the $2n$ intersection points of the axis of ρ_1 with $P_{n,k}$. By Lemma 4.2, the surface $F_{n,k}$ can be isotoped to be transverse to the suspension flow and disjoint from the orbit of the $2n$ singularities of $f_{n,k}$. Hence the monodromy $f^r_{n,k}$ still fixes the corresponding $2n$ singularities on $P^r_{n,k}$. ☐

We can now prove the non-orientable version of Lemma 3.6 of [31].

Lemma 4.6. *Let $\lambda^r_{n,k}$ be the stretch factor of $f^r_{n,k}$. Then there exists a constant $C>0$ such that for every $n\geq 1$, $k\geq 3$, and $0\leq r\leq 14n$ we have the following upper bound on $\log(\lambda^r_{n,k})$.*

$$\log(\lambda^r_{n,k}) \leq C\frac{n}{g^r_{n,k}}$$

Proof. Let $\mathcal{K} = \mathcal{K}_{n,k}$ be the fibered cone corresponding to $f_{n,k}$ and $h : \mathcal{K} \to \mathbb{R}$ the function described in Theorem 3.4. Note that $g_{n,k} \geq 42$, therefore we have the following bounds on $g^r_{n,k}$:

$$\begin{aligned}g^r_{n,k} &= g_{n,k} + r\\ &\leq g_{n,k} + 14n\\ &< 2g_{n,k}.\end{aligned}$$

Let ω be the Poincaré dual of $P^r_{n,k}$ and α the Poincaré dual of $P_{n,k}$. Then the following string of inequalities holds, the first inequality is by the convexity of h, the second inequality is the bound in Lemma 4.4, and the third inequality is from the bound on $g^r_{n,k}$ above:

$$\begin{aligned}h([\omega]) &< h([\alpha])\\ &\leq C'\frac{n}{g_{n,k}}\\ &\leq 2C'\frac{n}{g^r_{n,k}}.\end{aligned}$$

☐

So our surfaces $P^r_{n,k}$ are isotopic to fibers of fibrations of $M_{n,k}$ with pseudo-Anosov monodromies with bounded stretch factors.

Recall that all of the $P^r_{n,k}$ will have $2n$ singularities, and we can puncture n of them to view $f^r_{n,k}$ as a map on a non-orientable surface of genus $g^r_{n,k}$ with n punctures. Also note from above we know $g^r_{n,k}$ covers all natural numbers between $g_{n,k}$ and $g_{n,k+1}$, thus this set of genera for all r covers all natural numbers larger than $g_{n,3} = 40n + 2$.

We can now give a proof of Theorem 1.1.

Proof of Theorem 1.1. As above, the lower bound follows easily from the lower bound in the orientable setting. Let f be the pseudo-Anosov map with the minimal stretch factor on $\mathcal{N}_{g,n}$. Then, by Proposition 3.1, this map lifts to a map $\widetilde{f}$ on $\mathcal{S}_{g-1,2n}$ (possibly after squaring). Furthermore, $\widetilde{f}$ has the same stretch factor as f. The former is bounded below by $\frac{B}{g}$, and thus the stretch factor of f is bounded below as well.

To find the upper bound, let $C' = \frac{C}{2}$ be the value given in Lemma 4.6. Let $B_2'(n)$ be the following quantity:

$$B_2'(n) = \max\{2C'n, l'_{1,n}, 2l'_{2,n}, \ldots, (40n+1)l'_{40n+1,n}\}$$

By Lemma 4.6, $B_2'(n)$ is an upper bound for $g \cdot l'_{g,n}$. □

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