Thurston's fibered faces for non-orientable 3-manifolds and an application to minimal stretch factors

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Abstract

We prove that for non-orientable surface the minimal stretch factor of a pseudo-Anosov homeomorphism of a surface of genus g with a fixed number of punctures is asympotototically on the order of $\frac{1}{g}$. Our result adapts the work of Yazdi to non-orientable surfaces. We include the details of Thurston's theory of fibered faces for non-orientable 3-manifolds.

1 Introduction

Let $S_{g,n}$ be a surface of genus g with n punctures. The mapping class group of $S_{g,n}$ consists of homotopy classes of orientation preserving homeomorphisms of $S_{g,n}$. The Nielsen–Thurston classification of mapping classes (elements of the mapping class group) says that each mapping class is periodic, preserves some multicurve, or has a representative that is pseudo-Anosov. For each pseudo-Anosov homeomorphism $\varphi : S_{g,n} \to S_{g,n}$, the stretch factor $\lambda(\varphi)$ is an algebraic integer that describes the amount by which φ changes the length of curves. Arnoux–Yaccoz [4] and Ivanov [15] prove that the set

$$\mathrm{Spec}(S_{g,n}) = \{\log(\lambda(\varphi)) \mid \varphi \text{ is a pseudo-Anosov homeomorphism of } S_{g,n}\}$$

is a closed discrete subset of $(0, \infty)$. The minimum of $\mathrm{Spec}(S_{g,n})$:

$$\ell_{g,n} = \min\{\log(\lambda(\varphi)) \mid \varphi \text{ is a pseudo-Anosov homeomorphism of } S_{g,n}\}$$

quantitatively describes both the dynamics of the mapping class group of $S_{g,n}$ and the geometry of the moduli space of $S_{g,n}$.

Penner [26] showed that for orientable surfaces,

$$\ell_{g,0} \asymp \frac{1}{g}.$$

Penner conjectured that $\ell_{g,n}$ will have the same asymptotic behavior for $n \geq 0$ punctures. Following Penner, substantial attention has been given to finding bounds for $\ell_{g,n}$ [1, 5, 12, 13, 14, 17, 21, 24], calculating $\ell_{g,n}$ for specific values of (g, n) [7, 11, 18, 27], and finding asymptotic behavior of $\ell_{g,n}$ for *orientable* surfaces with $n \geq 0$ [17, 30, 31, 33]. We adapt a result of Yazdi [33] to non-orientable surfaces.

Theorem 1.1. *Let $\mathcal{N}_{g,n}$ be a non-orientable surface of genus g with n punctures, and let $\ell'_{g,n}$ be the logarithm of the minimum stretch factor of the pseudo-Anosov mapping classes acting on $\mathcal{N}_{g,n}$. Then for any fixed $n \in \mathbb{N}$, there is a positive constant $B'_1 = B'_1(n)$ and $B'_2 = B'_2(n)$ such that for any $g \geq 3$, the quantity $\ell'_{g,n}$ satisfies the following inequalities:*

$$\frac{B'_1}{g} \leq \ell'_{g,n} \leq \frac{B'_2}{g}.$$

Pseudo-Anosov homeomorphisms. Let S be a (possibly non-orientable) surface of finite type. A homeomorphism $\varphi : S \to S$ is said to be *pseudo-Anosov* if there exist a pair of transverse measured singular foliations $\mathcal{F}_s$ and $\mathcal{F}_u$ and a real number λ such that

$$\varphi(\mathcal{F}_s) = \lambda^{-1}(\mathcal{F}_s) \text{ and } \varphi(\mathcal{F}_u) = \lambda(\mathcal{F}_u).$$

The *stretch factor* of φ is the algebraic integer $\lambda = \lambda(\varphi)$.

Endow S with a Riemannian metric. The stretch factor $\lambda(\varphi)$ measures the growth rate of the length of geodesic representatives of a simple closed curve S under iteration of φ [8, Proposition 9.21]. Moreover, $\log(\lambda(\varphi))$ is the minimal topological entropy of any homeomorphism of S that is istopic to φ [8, Exposé 10].

Geometry of moduli space. Let $\mathcal{T}_{g,n}$ denote the Teichmüller space of $S_{g,n}$, that is: the space of isotopy classes of hyperbolic metrics on $S_{g,n}$. When endowed with the Teichmüller metric, the mapping class group of a surface $S_{g,n}$ acts properly discontinuously on $\mathcal{T}_{g,n}$ by isometries. The quotient of this action is the *moduli space* of $S_{g,n}$. The set $\mathrm{Spec}(S_{g,n})$ is the length spectrum of geodesics in the moduli space of $S_{g,n}$. Therefore the quantity $\ell_{g,n}$ is the length of the shortest geodesic in the moduli space of $S_{g,n}$.

Explicit bounds. In his foundational work, Penner found $\frac{\log 2}{12g-12+4n}$ to be a lower bound for $\ell_{g,n}$ for orientable surfaces [26]. He also determined $\frac{\log 11}{g}$ to be an upper bound for $\ell_{g,0}$. Penner's work proves that $\ell_{g,0} \asymp \frac{1}{g}$. McMullen [23] later asked:

Question (McMullen)**.** To what value does $\lim\limits_{g\to\infty} g \cdot \ell_{g,0}$ converge?

To this end, Bauer [5] strengthened the upper bound for $g \cdot \ell_{g,0}$ to $\log 6$, and Minakawa [24] and Hironaka–Kin [13] further sharpened the upper bounds for $g \cdot \ell_{g,0}$ and $g \cdot \ell_{0,2g+1}$ to $\log(2+\sqrt{3})$. Later Aaber–Dunfield [1], Hironaka [12], and Kin-Takasawa [16] determined that $\log\left(\frac{3+\sqrt{5}}{2}\right)$ is an upper bound for $g \cdot \ell_{g,0}$ and conjectured it is the supremum of $g \cdot \ell_{g,0}$.

Asymptotic behavior of punctured surfaces. Tsai initiated the study of asymptotic behavior of $\ell_{g,n}$ along lines in the (g,n)-plane [30]. In particular, Tsai determined that for orientable surfaces of fixed genus $g \geq 2$, the asymptotic behavior in n is:

$$\ell_{g,n} \asymp \frac{\log n}{n}.$$

Further, he showed that $\ell_{0,n} \asymp \frac{1}{n}$. Later, Yazdi [33] determined that for an orientable surface with a fixed number of punctures $n \geq 0$, the asymptotic behavior in g is:

$$\ell_{g,n} \asymp \frac{1}{g},$$

confirming the conjecture of Penner.

Non-orientable surfaces. Let $\mathcal{N}_{g,n}$ be a non-orientable surface of genus g with n punctures. As above, let $l'_{g,n}$ denote the minimum stretch factor of pseudo-Anosov homeomorphisms of $\mathcal{N}_{g,n}$. For any $n \geq 0$ and $g \geq 1$, $\ell_{g-1,2n}$ is a lower bound for $\ell'_{g,n}$, which can be seen by passing to the orientation double cover of $\mathcal{N}_{g,n}$ (note that the definition of genus is different for orientable an non-orientable surfaces). Because the upper bounds for $\ell_{g,n}$ are constructed by example, upper bounds for $\ell'_{g,n}$ do not follow immediately from passing to the orientation double cover. Recently Liechti–Strenner determined $\ell'_{g,0}$ for $g \in \{4,5,6,7,8,10,12,14,16,18,20\}$ [20]. Our work captures the asymptotic behavior for the punctured case.

Techniques. To prove Theorem 1.1, we adapt the strategy of Yazdi [33] to non-orientable surfaces with punctures. The lower bound of $\ell'_{g,n}$ is found by lifting to the orientation double cover of $\mathcal{N}_{g,n}$. The upper bound (in all prior work) is constructive. Fix $n \geq 0$: the desired number of punctures. Yazdi's construction is as follows. For a sequence of genera $g_{n,k}$ (dependent on n), use the Penner construction [25] to obtain a homeomorphism $f_{n,k}$ of $S_{g,n}$ that is pseudo-Anosov and has low stretch factor. In order to find pseudo-Anosov homeomorphisms of $S_{g,n}$ with small stretch factor for all g (not just those in the sequence above), construct a mapping torus for each $f_{n,k}$. To do this Yazdi's appeals to a technique of McMullen, using Thurston's theory of fibered faces.

Thurston norm for non-orientable 3-manifolds. In Thurston's development of what is now called the Thurston norm for 3-manifolds [28], his definitions and theorems required that all surfaces were orientable. Thurston said that the theorems should still be true for non-orientable surfaces, but there are some subtleties that have not been addressed elsewhere in the literature. In this paper, we write the details of Thurston's theory of fibered faces to non-orientable 3-manifolds. In particular, the Thurston norm is a norm on the second homology of a 3-manifold, that measures the minimum complexity of an embedded (orientable) surface. However the Thurston norm does not recognize embedded non-orientable surfaces in the second homology of a non-orientable 3-manifold. To address this limitation, we instead calculate the Thurston norm on the first cohomology of a non-orientable manifold. We develop a (weak) version of Poincaré duality in Theorem 2.8 that suffices to define a Thurston norm on $H^1(M;R)$ for a non-orientable 3-manifold M.

Fibered faces. A special case of Thurston's hyperbolization theorem says that the monodromy of any fibration of a hyperbolic 3-manifold over S^1 is a pseudo-Anosov homeomorphism. Therefore by finding other fibrations of the same 3-manifold, one obtains additional pseudo-Anosov homeomorphism. Work of Fried [9, 10], Matsumoto [22], and Agol–Leininger–Margalit [2] can be used to find a bound on the stretch factors of certain pseudo-Anosov homeomorphisms obtained in this way.

In Section 2 we state Thurston's theory of fibered faces and adapt it to the non-orientable setting. In Section 3.1 we show how Thurston's theory of fibered faces can be used to construct pseudo-Anosov homeomorphisms of low stretch factor for non-orientable surfaces. Specifically, we state and prove the Nielsen–Thurston classification for non-orientable surfaces. Then we adapt the results of Fried [9, 10], Matsumoto [22], and Agol–Leininger–Margalit [2] used to construct pseudo-Anosov homeomorphisms with low stretch factor of orientable surfaces to the non-orientable setting. In Section 4, we prove Theorem 1.1, following the strategy of Yazdi.

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2 Thurston norm for non-orientable 3-manifolds

In his original manuscript defining what is now called the Thurston norm, Thurston defined a norm on $H_2(M;\mathbb{R})$ where M is an orientable 3-manifold [28]. He wrote "Most of this paper works also for non-orientable manifolds but for simplicity we only deal with the orientable case." However, the details are not explained in Thurston's work or in subsequent literature. Therefore the goal of this section is to write the details of the Thurston norm for non-orientable 3-manifolds. We recall the Thurston norm for orientable manifolds in Section 2.1 and then adapt it to the non-orientable setting. In Section 2.2 we describe the challenge of defining the Thurston norm on $H_2(M;\mathbb{R})$ if M is non-orientable and present the solution of defining the Thurston norm instead on $H^1(M;\mathbb{R})$. However, Poincaré duality does not hold for non-orientable manifolds. We therefore define a condition – *relative orientability* on a pair consisting of a manifold and an embedded surface. A surface that is relatively orientable in a non-orientable 3-manifold M will have a corresponding cohomology class in $H^1(M;\mathbb{Z})$, giving a version of Poincaré duality for non-orientable 3-manifolds as stated in Theorem 2.8. Finally in Section 2.4, we define the oriented sum for relatively oriented embedded surfaces in non-orientable manifolds.

2.1 Thurston norm and mapping tori

In this section we recall the Thurston norm for orientable surfaces and how it detects when a 3-manifold fibers over a circle.

Mapping tori. Let S be a surface, and let $\varphi : S \to S$ be a homeomorphism. A *mapping torus* of S by φ is a 3-manifold M_φ given by the identification:

$$M_\varphi := \frac{S \times [0,1]}{(x,1) \sim (\varphi(x),0)}.$$

A mapping torus is *fibration over* S^1, denoted $S \to M_\varphi \to S^1$. A fibration defines a flow on M, called the *suspension flow*, where for any $x_0 \in S$ and $t_0 \in S^1$ the pair (x_0, t_0) is sent to $(x_0, t_0 + t)$. The fiber of a fibration is the preimage of any point $\theta \in S^1$ under the projection map from $M_\varphi \to S^1$. The fiber as a subset of M_φ is only well defined up to isotopy, since we don't specify the choice of θ, but the homology class of the fiber in $H_2(M_\varphi;\mathbb{R})$ is well defined.

A natural inverse question is to determine when a 3-manifold fibers over a circle, and the possible fibers. To this end, Thurston established a correspondence between second cohomology of 3-manifolds and surfaces embedded in 3-manifolds.

Complexity of an embedded surface. Let M be an compact orientable closed 3-manifold. Let S be a connected surface embedded in M. The complexity of S is $\chi_-(S) = \max\{-\chi(S), 0\}$. If the surface S has multiple components $S_1, \ldots, S_m$ then $\chi_-(S) = \sum_{i=1}^{m} \chi_-(S_i)$. The elements in $H_2(M;\mathbb{Z})$ can be represented by embedded surfaces inside of M [28, Lemma 1].

Thurston norm. Let a be a homology class in $H_2(M;\mathbb{Z})$. Define the integer valued norm $x : H_2(M;\mathbb{Z}) \to \mathbb{Z}$ as the following:

$$x(a) = \min\{\chi_-(S) \mid [S] = a \text{ and } S \text{ is compact, properly embedded and oriented}\}.$$

We then linearly extend x to $H_2(M;\mathbb{Q})$. The *Thurston norm* is the unique continuous $\mathbb{R}$ valued function that is an extension of x to $H_2(M;\mathbb{R})$. The unit ball for the Thurston norm is a convex polyhedron in $H_2(M;\mathbb{R})$.

The following remarkable theorem of Thurston [28] determines all possible fibrations of an oriented 3-manifold over the circle. We use the restatement of Yazdi [33].

Theorem 2.1 (Thurston)**.** *Let M be an orientable 3-manifold. Let $\mathcal{F}$ be the set of homology classes in $H_2(M;\mathbb{R})$ that are representable by fibers of fibrations of M over the circle.*

(i) Elements of $\mathcal{F}$ are in one-to-one correspondence with (non-zero) lattice points inside some union of cones over open faces of the unit ball in the Thurston norm.

(ii) If a surface F is transverse to the suspension flow associated to some fibration of $M \to S^1$ then $[F]$ lies in the closure of the corresponding cone in $H_2(M;\mathbb{R})$.

The class $[F]$ has orientation such that the positive flow direction is pointing outwards relative to the surface. An open face of the unit ball is said to be a *fibered face* if the cone over the face contains the fibers of the corresponding fibration.

The goal for the rest of this section is to prove a version of Theorem 2.1 for compact non-orientable 3-manifolds. Most of the work in the proof will involve reducing the version for non-orientable 3-manifolds to the orientable version by passing to the double cover.

2.2 Thurston norm on cohomology of non-orientable mapping tori

A naïve first attempt at defining the Thurston norm would be to define it on the second homology group, like in the orientable case. However, if the norm is defined in that fashion, the non-orientable version of Theorem 2.1 will not be true. Consider a compact non-orientable surface $\mathcal{N}$, a homeomorphism $\varphi : \mathcal{N} \to \mathcal{N}$, and the associated mapping torus N_φ. Clearly, N_φ fibers over S^1, and $\mathcal{N}$ is the fiber of this fibration. However, the homology class associated to $\mathcal{N}$ is the zero homology class, since the top-dimensional homology of non-orientable compact surfaces is 0-dimensional.

Our workaround for this problem will be to define a norm on the first cohomology $H^1(N_\varphi)$ rather than the second homology $H_2(N_\varphi)$. By Poincaré duality they are the same for orientable 3-manifolds, but that is not true for non-orientable 3-manifolds.

Poincaré Duality. To see why Poincaré duality fails for non-orientable 3-manifolds, we will work through the construction of the isomorphism between first homology and second cohomology for 3-manifolds. Let M be a 3-manifold. To define the Poincare dual of $H^1(M;\mathbb{Z})$, we first define a homotopy class of maps $M \to S^1$. Then we construct an element of $H^2(M;\mathbb{Z})$. Let α be a 1-form on M and $[\alpha]$ its class in $H^1(M;\mathbb{Z})$. Fix a basepoint $y_0 \in M$. The associated map $f_\alpha : M \to S^1$ is given by:

$$f_\alpha(y) \coloneqq \int_{y_0}^{y} \alpha \mod \mathbb{Z}. \tag{1}$$

The choice of basepoint does not affect the homotopy class of f_α (see Section 5.1 of [6] for the details).

Now let $\theta \in S^1$ be a regular value so that $S = f^{-1}(\theta)$ is a surface. To construct the associated element of $H_2(M;\mathbb{Z})$, we choose an orientation on S by assigning positive values of α to the outward pointing normal vectors on S. Then S inherits an orientation from the orientation on M, and we have defined a fundamental class $[S] \in H_2(M;\mathbb{Z})$. We claim that $[S]$ is the Poincaré dual to α.

Lemma 2.2. *Let θ and θ' be two regular values of the function f_α and let $S = f_\alpha^{-1}(\theta)$ and $S' = f_\alpha^{-1}(\theta')$. Then for any closed 2-form ω on M, the following identity holds:*

$$\int_S \omega = \int_{S'} \omega.$$

Furthermore, the following identity also holds:

$$\int_S \omega = \int_M \alpha \wedge \omega.$$

In particular, the homology class of S is Poincaré dual to α.

Proof. The first part of the lemma follows from the fact that S and S' are homologous, i.e. $f_\alpha^{-1}([\theta,\theta'])$ is a singular 3-chain that has S and S' as boundaries. From Stokes' theorem, we have the following:

$$\int_{S-S'} \omega = \int_{f_\alpha^{-1}([\theta,\theta'])} d\omega = 0.$$

To prove the second claim, observe that we can write the second integral as a double integral:

$$\int_M \alpha \wedge \omega = \int_{S^1} \left(\int_{f_\alpha^{-1}(\xi)} \omega \right) d\xi$$

because α is the pullback of $d\xi$ along the map f_α. Observe that the inner integral only makes sense when ξ is a regular value, but by Sard's theorem, almost every $\xi \in [0,1]$ is a regular value. Therefore the right hand side is well-defined. By the first claim, the inner integral is a constant function, as we vary over the ξ which are regular values of f_α. Then the integral of $d\xi$ over S^1 is 1, giving us the identity we want:

$$\int_M \alpha \wedge \omega = \int_S \omega.$$

□

What we have here is an explicit formula for the Poincaré duality map from $H^1(M;\mathbb{R})$ to $H_2(M;\mathbb{R})$. For orientable 3-manifolds, this is an isomorphism, and more specifically the following theorem is true.

Theorem 2.3 (Poincaré duality for orientable 3-manifolds)**.** *Let M be an orientable 3-manifold, and let S be an oriented embedded surface. Then there exists a 1-form α and a regular value $\theta \in S^1$ such that S and $f_\alpha^{-1}(\theta)$ are homologous surfaces.*

Note that the map from the space of 1-forms to homology classes of an embedded surface still makes sense for a non-orientable 3-manifold N. However in that case the map from $H^1(N;\mathbb{Z})$ to $H_2(N;\mathbb{Z})$ has a nontrivial kernel. For example, when N_φ is the mapping torus of a non-orientable surface $\mathcal{N}$, as above, the fiber is trivial in $H_2(N;\mathbb{Z})$.

Non-orientable manifolds. Let N be a non-orientable 3-manifold. Let $\widetilde{N}$ and the covering map $p : \widetilde{N} \to N$ be the orientation double covering space of N. Let ι be the orientation reversing deck transformation of $\widetilde{N}$. If $N = N_\varphi$ is the mapping torus of the non-orientable surface $\mathcal{N}$ and a self-homeomorphism $\varphi : \mathcal{N} \to \mathcal{N}$, then $\widetilde{N}$ is the mapping torus of $(\mathcal{S}, \widetilde{\varphi})$, where $\mathcal{S}$ is the orientation double cover of $\mathcal{N}$, and $\widetilde{\varphi}$ is the orientation preserving lift of φ.

Defining the Thurston norm on cohomology. In order to define the Thurston norm on $H^1(N;\mathbb{Z})$, we first need to relate $H^1(N;\mathbb{R})$ and $H^1(\widetilde{N};\mathbb{R})$. We do so by pulling back $H^1(N;\mathbb{R})$ to $H^1(\widetilde{N};\mathbb{R})$ via p.

Lemma 2.4. *The pullback $p^* : H^1(N;\mathbb{R}) \to H^1(\widetilde{N};\mathbb{R})$ maps $H^1(N;\mathbb{R})$ bijectively to the ι^*-invariant subspace of $H^1(\widetilde{N};\mathbb{R})$.*

Proof. For any 1-form α on N, $p^*(\alpha)$ will be ι^*-invariant. To check that p^* is injective, consider a 1-form α on N such that $p^*(\alpha)$ is exact. Then there exists a smooth function $g : \widetilde{N} \to \mathbb{R}$ such that:

$$dg = p^*(\alpha).$$

Since $p^*(\alpha)$ is ι^*-invariant, we must have $dg = \iota^* dg$. Because ι^* commutes with the exterior derivative, we have $dg = d(\iota^* g)$. That means g and $\iota^* g$ differ by a constant, but that constant must be 0 since ι^2 is the identity map. Thus g is ι-equivariant and descends to a function $N \to \mathbb{R}$. Therefore α must be exact, which proves injectivity of p^*.

To show surjectivity, let $[\widetilde{\alpha}]$ be an element in $H^1(\widetilde{N};\mathbb{R})$ that is ι^*-invariant. We need to find a representative that is ι^* invariant that we can pushforward to $H^1(N;\mathbb{R})$. To do so, start with a representative $\widetilde{\alpha}$ of $[\widetilde{\alpha}]$. Since $[\widetilde{\alpha}]$ is ι^*-invariant, $\widetilde{\alpha}$ and $\iota^*(\widetilde{\alpha})$ must differ by an exact form.

$$\widetilde{\alpha} - \iota^*(\widetilde{\alpha}) = dg$$

Applying ι^* to both sides of the equality, we have $\iota^* dg = -dg$. This means that the 1-form $\beta = \widetilde{\alpha} - \frac{dg}{2}$ is an ι^*-invariant representative of the cohomology class $[\widetilde{\alpha}]$. Since β is ι^*-invariant, the pushforward of β under p is a well-defined 1-form on N. □

Note that if we change the coefficients in the statement of this lemma from $\mathbb{R}$ to $\mathbb{Z}$, the proof of injectivity follows through, but the proof of surjectivity does not.

Lemma 2.4 tells us that $H^1(N;\mathbb{R})$ is a subspace of $H^1(\widetilde{N};\mathbb{R})$, so we define the Thurston norm on $H^1(N;\mathbb{R})$ by restricting the Thurston norm to the subspace $p^*(H^1(N;\mathbb{R}))$ of $H^1(\widetilde{N};\mathbb{R})$.

Thurston norm for non-orientable 3-manifolds. Let N be a non-orientable 3-manifold and $\widetilde{N}$ its orientation double cover. Let $\widetilde{x}$ be the Thurston norm on $H^1(\widetilde{N};\mathbb{R})$ and let $\alpha \in H^1(N;\mathbb{R})$. The *Thurston norm on* $H^1(N;\mathbb{R})$, is the norm $x : H^1(N;\mathbb{R}) \to \mathbb{R}$ defined:

$$x(\alpha) \coloneqq \widetilde{x}(p^*\alpha).$$

Theorem 2.5. *Let B_1 be the unit ball with respect to the dual Thurston norm on $\left(H^1(N;\mathbb{R})\right)^*$. Then B_1 is a polyhedron whose vertices are lattice points $\{\pm\beta_1, \dots \pm \beta_k\}$ that satisfy the following inequalities:*

$$B_1 = \left\{a \in H^1(N,\mathbb{R}) \mid |\beta_i(a)| \leq 1 \text{ for } 1 \leq i \leq k\right\}.$$

Proof. For any $\alpha \in H^1(N;\mathbb{Z})$, the norm $x(\alpha)$ is $\widetilde{x}(p^*\alpha)$ where $p^*(\alpha) \in H^1(\widetilde{N};\mathbb{Z})$. Therefore $x(\alpha)$ is always an integer. The rest of the proof is identical to the original proof of Thurston [28, Theorem 2]. □

Note that defining the Thurston norm on $H^1(N;\mathbb{R})$ rather than $H_2(N;\mathbb{R})$ is not quite satisfactory. In particular, fibers of fibrations are embedded surfaces in N. In the orientable case, the embedded surfaces define the Thurston norm. In Section 2.3, we develop a (weak) version of Poincaré duality for non-orientable 3-manifolds.

2.3 Weak inverse to the Poincaré duality map

Let M be a 3-manifold and $[\alpha]$ an element of $H^1(M;\mathbb{R})$. Regardless of whether M is orientable or not, we can construct a dual map $f_\alpha : M \to S^1$, using equation (1). The preimage of a regular value θ is an embedded surface $f_\alpha^{-1}(\theta)$ in M. If we assume that M is orientable Poincareé duality allows us to reverse this correspondence. Indeed, let S is an embedded surface in (the orientable manifold) M. Poincaré duality determines a cohomology class in $H^1(M;\mathbb{Z})$ corresponding to S. Now let α be a 1-form representative of an element of $H^1(M;\mathbb{R})$ and let θ be a regular of f_α. The surfaces $f_\alpha^{-1}(\theta)$ and S are homologous. In this section, we state a weak version of Poincaré duality for non-orientable surfaces in Theorem 2.8. This version of Poincaré duality will allow us to associate 1-forms to a certain subset of surfaces embedded in non-orientable 3-manifolds. We will require that the embedded surfaces are *relative oriented* in the (non-orientable) 3-manifold.

Relative oriented surfaces. Let M be a 3-manifold, and S an embedded surface in M. The surface S is said to be *relatively oriented with respect to M* if there is a nowhere vanishing vector field on S that is transverse to the tangent plane of S. Two such vector fields are said to induce the same orientation if they induce the same local orientation after choosing a local frame for S. A surface S is *relatively oriented* in M if both S and the choice of positive normal vector field are specified.

If S and M are orientable, then S is relatively oriented with respect to M. But even if M is non-orientable, a non-orientable embedded surface S may be relatively oriented in M. For instance, let S be the fiber of a non-orientable mapping torus N_φ. The preimage under the projection map to S^1 of a non-vanishing vector field is a non-vanishing vector field on M that is always transverse to the fiber.

On the other hand, for orientable M and S, if S is relatively oriented with respect to S, then a choice of orientation on S determines an orientation on M and vice versa.

An orientable surface that is not relatively oriented in a 3-manifold. Let S be the standard torus $\mathbb{R}^2/\mathbb{Z}^2$, and let φ map (x, y) to $(-x, y)$. Then φ is an orientation-reversing homeomorphism. The mapping torus M_φ is non-orientable. Consider a vertical line γ in S preserved by φ, i.e. the line $x = 0$.

The image of γ in S under the suspension flow in M is a subsurface of M, which we'll call S'. The normal direction to S' when restricted to S is $\frac{\partial}{\partial x}$. Because the suspension flow reverses the direction of γ, the normal vector field cannot be continuously extended to all of M. This means that the surface S' cannot be relatively oriented in M (despite being orientable itself).

However, if both M and an embedded surface are non-orientable, the surface will be relatively oriented.

Proposition 2.6. *Let N be a non-orientable 3-manifold and let $\mathcal{N}$ is a connected embedded non-orientable surface in N. Then $\mathcal{N}$ is relatively oriented in N.*

Proof. Let $\widetilde{N}$ and the covering map $p : \widetilde{N} \to N$ be the orientation double covering space of N. Finally, let $\widetilde{\mathcal{N}}$ be the preimage of $\mathcal{N}$ under p. Let $\iota : \widetilde{N} \to \widetilde{N}$ be the orientation-reversing deck transformation. The restriction of $\iota : \widetilde{N} \to \widetilde{N}$ to $\widetilde{\mathcal{N}}$ is an orientation reversing homeomorphism of $\widetilde{\mathcal{N}}$. Let (v_1, v_2) be a positively oriented local frame for the tangent space to $\widetilde{\mathcal{N}}$. Let n be an outward pointing transverse vector to $\widetilde{\mathcal{N}}$ so the local frame (v_1, v_2, n) is positively oriented. Since ι reverses the orientation of both $\widetilde{\mathcal{N}}$ and $\widetilde{N}$, $(\iota(v_1), \iota(v_2))$ and $(\iota(v_1), \iota(v_2), \iota(n))$ are both negatively oriented. Thus $\iota(n)$ is outward pointing. Therefore the outward pointing transverse direction on $\widetilde{\mathcal{N}}$ descends to an outward pointing transverse direction on $\mathcal{N}$, and $\mathcal{N}$ is relatively oriented in N. ☐

We also need to define the notion of *incompressible surfaces* to state our version of Poincaré duality.

Incompressible surfaces. Let S be a surface with positive genus embedded in a 3-manifold M. The surface S is said to be *incompressible* if there does not exist an embedded disc D in M such that D intersects S transversely and $D \cap S = \partial D$. The following result of Thurston demonstrates the link between incompressible surfaces and fibers of fibrations.

Theorem 2.7 (Theorem 4 of [28])**.** *Let M be an oriented 3-manifold that fibers over S^1. Let S be an incompressible surface embedded in M. If S is homologous to a fiber, then S is isotopic to the fiber.*

In the remainder of the section, we will be working with a non-orientable 3-manifold N and an embedded non-orientable surface $\mathcal{N}$. Let $\widetilde{N}$ and the covering map $p : \widetilde{N} \to N$ be the orientation double covering space of N. Let $\widetilde{\mathcal{N}}$ be the preimage of $\mathcal{N}$ under p. Let $\iota : \widetilde{N} \to \widetilde{N}$ be the orientation-reversing deck transformation of p. We will initiate N and $\mathcal{N}$ in each result below, but we surpress the initiation of the orientation double cover.

Theorem 2.8 (Poincaré Duality for non-orientable 3-manifolds)**.** *Let N be a compact non-orientable 3-manifold, and let $\mathcal{N}$ be a relatively oriented incompressible surface embedded in N. Then there exists $[\alpha] \in H^1(N;\mathbb{Z})$ such that the pullback of $[\alpha]$ to $\widetilde{N}$ is the Poincaré dual of $\widetilde{\mathcal{N}}$ in $\widetilde{N}$. If $[\alpha]$ has a 1-form representative α that vanishes nowhere on N, then $\mathcal{N}$ is homeomorphic to $f_\alpha^{-1}(\theta)$ for all $\theta \in S^1$.*

We will refer to the 1-form α given in Theorem 2.8 as the *Poincaré dual* of the non-orientable surface $\mathcal{N}$.

Lemma 2.9. *Let N be a non-orientable 3-manifold. Let $\mathcal{N}$ be a relatively oriented embedded surface in N, and let $\widetilde{\mathcal{N}} = p^{-1}(\mathcal{N})$ in $\widetilde{N}$. Then the Poincaré dual to $[\widetilde{\mathcal{N}}]$ is ι^*-invariant.*

Proof. The relative orientation of $\mathcal{N}$ in N lifts to a relative orientation of $\widetilde{\mathcal{N}}$ in $\widetilde{N}$. Since $\widetilde{N}$ is orientable, the relative orientation of $\widetilde{\mathcal{N}}$ defines an orientation of $\widetilde{\mathcal{N}}$, and thus the homology class $[\widetilde{\mathcal{N}}]$ is well-defined.

Next we show that ι reverses the orientation of $\widetilde{\mathcal{N}}$. To do so, we first observe that because $\mathcal{N}$ is relatively oriented in N, the outward pointing transverse vector field on $\mathcal{N}$ must lift to an outward pointing transverse vector field on $\widetilde{\mathcal{N}}$. Therefore ι preserves outward pointing vector fields on $\widetilde{\mathcal{N}}$.

Let (v_1, v_2, v_3) be a local frame for some point in $\widetilde{\mathcal{N}}$ such that v_3 is the outward pointing transverse vector field. Since ι reverses the orientation of $\widetilde{N}$ but preserves the direction of $\iota(v_3)$, ι must reverse the orientation of the pair (v_1, v_2). In particular, that means ι reverses the orientation on $\widetilde{\mathcal{N}}$.

This implies $[\widetilde{\mathcal{N}}]$ is in the -1-eigenspace of the ι_* action on $H_2(\widetilde{N};\mathbb{R})$. Let the cohomology class $[\widetilde{\alpha}]$ be the the Poincaré dual to $[\widetilde{\mathcal{N}}]$. Pick some representative 1-form $\widetilde{\alpha}$ of this cohomology class (the 1-form need not be ι^*-invariant). Then the following chain of equalities hold for all closed 2-forms ω. We use the fact that $\iota^2 = \mathrm{id}$ in the first and third equalities:

$$\begin{aligned}
\int_{\iota_*\widetilde{\mathcal{N}}} \omega &= \int_{\widetilde{\mathcal{N}}} \iota^*\omega && \text{(By a change of variables)} \\
&= \int_{\widetilde{N}} \widetilde{\alpha} \wedge \iota^*\omega && \text{(Poincaré duality)} \\
&= \int_{\widetilde{N}} \iota^*\left(\iota^*\widetilde{\alpha} \wedge \omega\right) \\
&= \int_{\widetilde{N}} -\left(\iota^*\widetilde{\alpha} \wedge \omega\right) && (\iota \text{ is orientation reversing})
\end{aligned}$$

On the other hand, the following equalities follow from the fact that $\iota_*[\widetilde{\mathcal{N}}] = -[\widetilde{\mathcal{N}}]$.

$$\begin{aligned}
\int_{\iota_*\widetilde{\mathcal{N}}} \omega &= -\int_{\widetilde{\mathcal{N}}} \omega \\
&= -\int_{\widetilde{N}} \widetilde{\alpha} \wedge \omega
\end{aligned}$$

Because

$$\int_{\widetilde{N}} \widetilde{\alpha} \wedge \omega = \int_{\widetilde{N}} \iota^*\widetilde{\alpha} \wedge \omega$$

for all ω, it follows that $\widetilde{\alpha}$ and $\iota^*\widetilde{\alpha}$ differ by an exact form, and therefore the cohomology class $[\widetilde{\alpha}]$ is ι^*-invariant. ☐

As above, we will denote the Poincaré dual to $[\widetilde{\mathcal{N}}]$ by $[\widetilde{\alpha}]$. The class $[\widetilde{\alpha}]$ is an ι^*-invariant element of $H^1(\widetilde{N};\mathbb{Z})$, but it is not clear that $[\widetilde{\alpha}]$ is the pullback of an element of $H^1(N;\mathbb{Z})$ under p. In the next lemma, we show that is indeed the case, i.e. $[\widetilde{\alpha}]$ is the pullback of an element in $H^1(N;\mathbb{Z})$.

Lemma 2.10. *Let N be a non-orientable 3-manifold. Let $[\widetilde{\alpha}] \in H^1(\widetilde{N}, Z)$ and let $\widetilde{S}$ be the Poincaré dual of $[\widetilde{\alpha}]$ in $\widetilde{N}$. There exists $[\alpha] \in H^1(N;\mathbb{Z})$ such that $\widetilde{\alpha} = p^*\alpha$.*

Proof. It will suffice to show that for any simple closed curve γ in N, the integral of $\widetilde{\alpha}$ along any path lift of γ is an integer. Let $x_0 \in N$ be a base point of γ. Note that γ has two (path) lifts $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ under p in $\widetilde{N}$, one based at each element of $p^{-1}(x_0)$. Either $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ are both simple closed curves based at the each of the two preimages $p^{-1}(x_0)$ or $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ are both arcs between the two points of $p^{-1}(x_0)$. If each lift $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ of γ is a closed curve in $\widetilde{N}$, the integral $\displaystyle\int_{\widetilde{\gamma}_i} \widetilde{\alpha}$ will be an integer since $\widetilde{\alpha} \in H^1(\widetilde{N};\mathbb{Z})$.

If each lift $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ of γ is an arc between the two preimages of $p^{-1}(x_0)$, we consider the simple closed curve $\widetilde{\gamma} = \widetilde{\gamma}_1 \cup \widetilde{\gamma}_2$. We note that $\iota(\widetilde{\gamma}) = \widetilde{\gamma}$. Because $\widetilde{\alpha}$ is ι^*-invariant, we have that $\displaystyle\int_{\widetilde{\gamma}_1} \widetilde{\alpha} = \int_{\widetilde{\gamma}_2} \widetilde{\alpha}$. Therefore

$$\int_{\widetilde{\gamma}} \widetilde{\alpha} = 2\int_{\widetilde{\gamma}_1} \widetilde{\alpha}.$$

It will suffice to show that $\displaystyle\int_{\widetilde{\gamma}} \widetilde{\alpha}$ is an even integer. Without loss of generality, we can assume all intersections of the simple closed curve $\widetilde{\gamma}$ with the surface $\widetilde{\mathcal{N}}$ are transverse. Since $\widetilde{\alpha}$ is a representative of the Poincaré dual to $[\widetilde{\mathcal{N}}]$, the integral of $\widetilde{\alpha}$ along $\widetilde{\gamma}$ is the signed intersection number of $\widetilde{\gamma}$ with $\widetilde{\mathcal{N}}$. The intersection number must be even, for if $\widetilde{\gamma}$ and $\widetilde{\mathcal{N}}$ intersect at a point y, then they also intersect at $\iota(y)$. This proves the lemma. ☐

The last lemma we need is the claim that lifts of incompressible surfaces are incompressible.

Lemma 2.11. *Let N be a non-orientable 3-manifold. If $\mathcal{N}$ is a relatively oriented incompressible surface in N, then $\widetilde{\mathcal{N}} = p^{-1}(\mathcal{N})$ is also incompressible in $\widetilde{N}$.*

Proof. Because $\mathcal{N}$ is incompressible in N, the map on fundamental groups induced by inclusion $\mathcal{N} \to N$ is injective. Since $p_* : \pi_1(\widetilde{N}) \to \pi_1(N)$ is injective, the induced map $\pi_1(\widetilde{\mathcal{N}}) \to \pi_1(\widetilde{N})$ must also be injective. An injective induced map on fundamental groups is equivalent to the orientable surface $\widetilde{\mathcal{N}}$ being incompressible. □

We now have everything we need to finish proving Theorem 2.8.

Proof of Theorem 2.8. Let $\widetilde{\mathcal{N}} = p^{-1}(\mathcal{N})$. The relative orientation of $\widetilde{\mathcal{N}}$ determines a homology class $[\widetilde{\mathcal{N}}] \in H_2(N;\mathbb{Z})$. Let the 1-form $\widetilde{\alpha}$ be the Poincaré dual to $[\widetilde{\mathcal{N}}]$ in $\widetilde{N}$. By Lemma 2.10, there exists a 1-form $\alpha \in H^1(N;\mathbb{Z})$ such that $\widetilde{\alpha} = p^*\alpha$.

We define the map $f_\alpha : N \to S^1$ according to equation (1). Because α is non-vanishing, f_α has full rank everywhere. Therefore f_α is a fibration. The map $f_\alpha \circ p$ is a lift of f_α to $\widetilde{N}$ under p, and is therefore also a fibration. By Lemma 2.11, $\widetilde{\mathcal{N}}$ is incompressible. It follows from the orientable version of Poincaré duality that $\widetilde{\mathcal{N}}$ and $p^{-1}(f_\alpha^{-1}(\theta))$ are homologous surfaces in $\widetilde{N}$. Theorem 2.7 then tells us $\widetilde{\mathcal{N}}$ must be isotopic to a fiber of $f_\alpha \circ p$. Because $\widetilde{\mathcal{N}}$ and $p^{-1}(f_\alpha^{-1})(\theta)$ are homeomorphic and $p : \widetilde{\mathcal{N}} \to \mathcal{N}$ and $p : p^{-1}(f_\alpha^{-1}(\theta)) \to f_\alpha^{-1}(\theta)$ are 2-fold covering maps, the surfaces $\mathcal{N}$ and $f_\alpha^{-1}(\theta)$ must also be homeomorphic. □

Note that the above proof does not tell us that $\mathcal{N}$ and $f_\alpha^{-1}(\theta)$ are isotopic. Isotopy of the fibers of N requires the isotopy between $\widetilde{\mathcal{N}}$ and $p^{-1}(f_\alpha^{-1}(\theta))$ to be ι^*-invariant. However, the theorem is sufficient for our application.

We conclude the section with a non-orientable version of Theorem 2.1.

Theorem 2.12. *Let N be a compact non-orientable* 3*-manifold, and let $\mathcal{F}$ be the elements of $H^1(N;\mathbb{Z})$ corresponding to fibrations of N over S^1.*

(i) Elements of $\mathcal{F}$ are in one-to-one correspondence with (non-zero) lattice points (i.e. points of $H^1(N;\mathbb{Z})$) inside some union of cones over open faces of the unit ball in the Thurston norm.

(ii) Let $\mathcal{N}$ be relatively oriented surface in N that transverse to the suspension flow associated to some fibration $f : N \to S^1$. Let $[\alpha]$ be the Poincaré dual $[\alpha]$ to $\mathcal{N}$. Then $[\alpha]$ lies in the closure of the cone in $H^1(N;\mathbb{R})$ containing the 1*-form corresponding to f.*

Proof. For (i), we observe that by Theorem 2.1 the fibrations of $\widetilde{N}$ are in one-to-one correspondence with lattice points inside a union of cones over open faces of the unit ball in $H_2(\widetilde{N};\mathbb{R})$. Let $\widetilde{\mathcal{K}}$ be the union of cones in $H_2(\widetilde{N};\mathbb{R})$. By Poincaré duality, $\widetilde{\mathcal{K}}$ is in one-to-one correspondence to a union of cones in $H^1(\widetilde{N};\mathbb{R})$, which we will call $\widetilde{\mathcal{K}}^*$

Because $H^1(N;\mathbb{R})$ is isomorphic to a subspace of $H^1(\widetilde{N};\mathbb{R})$, we can construct a union of cones in $H^1(N;\mathbb{R})$ by intersecting $p^*(H^1(N;\mathbb{R}))$ with $\widetilde{\mathcal{K}}^*$. Indeed, every lattice point in $\widetilde{\mathcal{K}}^*$ corresponds to a fibration $f : N \to S^1$, since the pullback of f to $H^1(\widetilde{N};\mathbb{Z})$ corresponds to a fibration of $\widetilde{N}$. Conversely, every fibration of $f : N \to S^1$ must lie in $\widetilde{\mathcal{K}}^*$, since the composition $f \circ p$ is a fibration of $\widetilde{N} \to S^1$.

For (ii), assume that the surface $\mathcal{N}$ is transverse to the suspension flow of a fibration $f : N \to S^1$. Then $\widetilde{\mathcal{N}}$ is transverse to the suspension flow $p \circ f : \widetilde{N} \to S^1$. Let $\widetilde{\alpha}$ be the pullback of α under p. Then $\widetilde{\alpha}$ is the Poincaré dual of $\widetilde{\mathcal{N}}$. By Theorem 2.1, the 1-form $\widetilde{\alpha}$ lies in the closure of a component of $\widetilde{\mathcal{K}}^*$ that contains the 1-form corresponding to $f \circ p$. Let $\widetilde{K}$ be this component. Since the $\widetilde{\alpha}$ is the pullback of α, the 1-form α lies in $p^*(\widetilde{K})$, which contains the 1-form corresponding to f. □

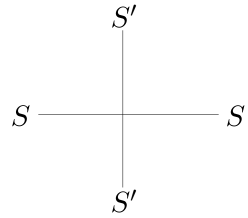

Figure 1: Cross section of intersection of S and S'.

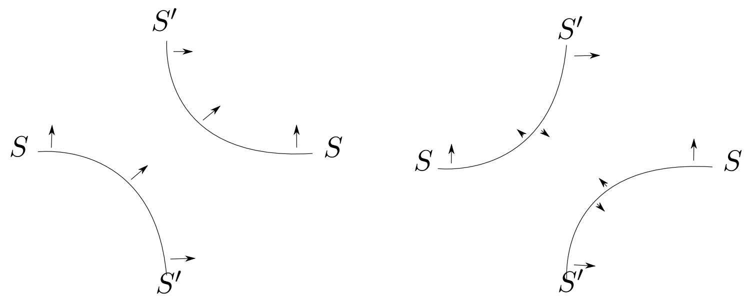

Figure 2: On the left, the normal vectors on S and S' are consistent. On the right, they are not.

2.4 Oriented sums

The next step in studying embedded non-orientable surfaces will be to describe *oriented sums*. The oriented sum of two surfaces embedded in a manifold M is additive in both the Euler characteristic and $H^1(M;\mathbb{R})$. This operation is well-known in the case of orientable 3-manifolds (along with orientable embedded surfaces), but we will sketch the relevant details. We then extend the construction to relatively oriented embedded surfaces.

Oriented sum for oriented manifolds Let M be an orientable manifold. Let S and S' be orientable embedded surfaces in M. Assume that S and S' intersect transversally. Thus $S \cap S'$ is a disjoint union of copies of S^1. For each component of $S \cap S'$, take a tubular neighborhood that has cross section as in Figure 1.

Figure 1.

We then perform a surgery on the leaves of S and S' so that the outward pointing normal vector fields match as in Figure 2.

By performing this surgery at all the intersections, we get a new submanifold S'' of M (which may have multiple components). This new submanifold S'' is called the *oriented sum* of S and S'. The operation of taking oriented sums is additive on Euler characteristic, as well as the homology classes (and thus the cohomology classes of their Poincaré duals):

$$\chi(S'') = \chi(S) + \chi(S')$$
$$[S''] = [S] + [S'].$$

Oriented sum for non-orientable manifolds Let N be a non-orientable manifold and let $\mathcal{N}$ and $\mathcal{N}'$ be embedded surfaces in N that are relatively oriented. We define the oriented sum on $\mathcal{N}$ and $\mathcal{N}'$ as follows. As above, let $p : \widetilde{N} \to N$ be the orientation double cover and let ι be the orientation reversing deck transformation of $\widetilde{N}$. Let $\widetilde{\mathcal{N}} = p^{-1}(\mathcal{N})$ and $\widetilde{\mathcal{N}}' = p^{-1}(\mathcal{N}')$, which are embedded oriented surfaces in $\widetilde{N}$. The oriented sum of $\mathcal{N}$ and $\mathcal{N}'$ is the image under p of the oriented sum of $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{N}}'$ (as defined above for oriented surfaces in oriented manifolds).

To see that the operation is well defined, we recall that ι preserves the relative orientation of $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{N}}'$. Therefore ι leaves the outward normal vector fields on $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{N}}'$ invariant (see the proof of Lemma

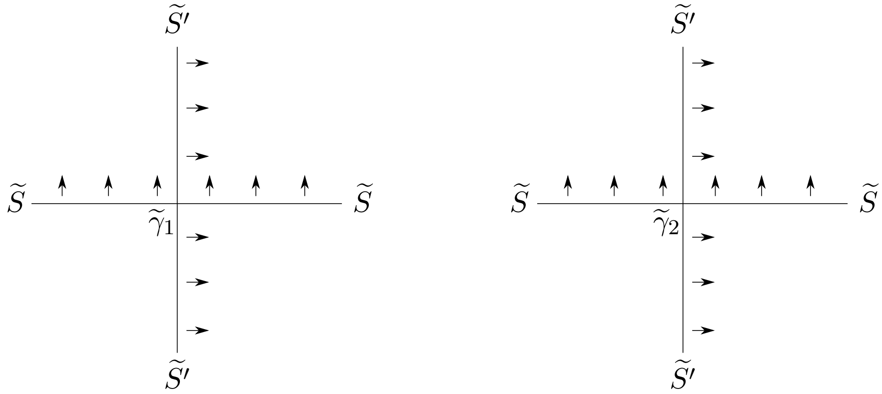

Figure 3: Neighborhoods of $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$, with the outward pointing normal vector field.

2.9). Thus a leaf L of $\widetilde{\mathcal{N}}$ is surgered with a leaf of L' of $\widetilde{\mathcal{N}}'$ if and only if $\iota(L)$ and $\iota(L')$ are surgered. Therefore surgery factors through p and $[\mathcal{N}] + [\mathcal{N}']$ is well-defined for non-orientable surfaces.

Example 2.13. Let γ be a component of $\mathcal{N} \cap \mathcal{N}'$ and $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ be the path lifts of γ. One possible orientation of $\widetilde{S}$ and $\widetilde{S}'$ is given in Figure 3. The outward pointing normal vectors to $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{N}'}$ determine which leaves are glued together along $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$.

To preserve the normal vector field, glue the left $\widetilde{\mathcal{N}}$ leaf to the bottom $\widetilde{\mathcal{N}'}$ leaf near $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$. Since $\iota(\widetilde{\gamma}_1) = \widetilde{\gamma}_2$, the outward pointing normal vector fields point the same (relative) directions.

Additivity By the consistency of the oriented sum in N and $\widetilde{N}$, it easily follows that the oriented sum is additive in Euler characteristic, as well as in terms of Poincaré dual, since the Poincaré dual was also defined by passing to the orientation double cover.

3 Mapping classes with small stretch factors

In this section, we construct mapping classes with small stretch factor on non-orientable surfaces.

3.1 Mapping class groups of non-orientable surfaces

Let $\mathcal{N}$ be a non-orientable surface and let $\widetilde{\mathcal{N}}$ and the covering map $p : \widetilde{\mathcal{N}} \to \mathcal{N}$ be its orientation double covering space. Every homeomorphism $\varphi : \mathcal{N} \to \mathcal{N}$, has a unique orientation preserving lift $\widetilde{\varphi} : \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}$.

A consequence is that lifting homeomorphisms induces a monomorphism between orientation preserving homeomorphisms of $\mathcal{N}$ and (orientation preserving) homeomorphisms of $\widetilde{\mathcal{N}}$. Every homotopy of $\mathcal{N}$ lifts to a homotopy of $\widetilde{\mathcal{N}}$. Therefore there is an inclusion from the mapping class group of $\mathcal{N}$ to the (orientation preserving) mapping class group of $\widetilde{\mathcal{N}}$. This inclusion also respects the Nielsen-Thurston classification of mapping classes, both qualitatively, and quantitatively, as the following proposition shows.

Proposition 3.1. *Let $\varphi : \mathcal{N} \to \mathcal{N}$ be a homeomorphism and let $\widetilde{\varphi} : \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}$ be the orientation preserving lift of φ. Then:*

(i) φ is periodic if and only if $\widetilde{\varphi}$ is periodic,

(ii) φ is reducible if and only if $\widetilde{\varphi}$, and

(iii) φ is pseudo-Anosov if and only if $\widetilde{\varphi}$ is pseudo-Anosov. Moreover if φ has stretch factor λ, then $\widetilde{\varphi}$ also has stretch factor λ.

Proof. The fact that the map from $\mathrm{Mod}(\mathcal{N})$ to $\mathrm{Mod}(\widetilde{\mathcal{N}})$ is type-preserving follows from Aramayona–Leininger–Souto [3, Lemma 10] (while the statement of the Lemma is for orientable surfaces, the argument, which we will skip, is identical for non-orientable surfaces).

Suppose now that $\varphi : \mathcal{N} \to \mathcal{N}$ is a psuedo-Anosov homeomorphism with stretch factor λ and stable and unstable foliations $\mathcal{F}_s$ and $\mathcal{F}_u$ respectively. Let $\widetilde{\mathcal{F}_s}$ and $\widetilde{\mathcal{F}_u}$ denote the lifts of the stable and unstable foliations to the orientation double cover. We need to show that the following identities hold for all simple closed curves γ in $\widetilde{\mathcal{N}}$ (see [8, Exposé 5] for the definition of intersection number with measured foliations; the fact that these identities suffice follows from [8, Lemma 9.15]):

$$i(\gamma, \widetilde{\varphi}(\widetilde{\mathcal{F}_u})) = \lambda \cdot i(\gamma, \widetilde{\mathcal{F}_u}) \tag{2}$$

$$i(\gamma, \widetilde{\varphi}(\widetilde{\mathcal{F}_s})) = \frac{1}{\lambda} \cdot i(\gamma, \widetilde{\mathcal{F}_s}). \tag{3}$$

To see that these identities hold, we partition γ into short arcs $\{\gamma_i\}$ such that the restriction of the covering map p to a neighbourhood of each arc is a homeomorphism. Then we have:

$$i(\gamma_i, \widetilde{\mathcal{F}_u}) = i(p(\gamma_i), \mathcal{F}_u) \tag{4}$$

$$i(\gamma_i, \widetilde{\varphi}(\widetilde{\mathcal{F}_u})) = i(p(\gamma_i), \varphi(\mathcal{F}_u)). \tag{5}$$

Since we know that $\mathcal{F}_u$ is the unstable foliation for φ with stretch factor λ, we can compute the ratio of the right hand side of (4) and (5):

$$i(p(\gamma_i), \varphi(\mathcal{F}_u)) = \lambda \cdot i(p(\gamma_i), \mathcal{F}_u). \tag{6}$$

Combining (4), (5), and (6), and summing over all γ_i gives us (2). A similar argument also proves (3). $\square$

3.2 Constructing pseudo-Anosov maps on nearby surfaces using oriented sums

The goal of this section is to obtain an asymptotic upper bound on the minimum stretch factor of a pseudo-Anosov homeomorphism. We do this in Proposition 3.2.

Proposition 3.2. *Let $\mathcal{N}_g$ be a non-orientable surface of genus g and let $\varphi : \mathcal{N}_g \to \mathcal{N}_g$ be a pseudo-Anosov homeomorphism with stretch factor λ. Let N_φ be the mapping torus of $\mathcal{N}_g$ by φ. Let $\mathcal{N}_{g'}$ be an incompressible surface embedded in N_φ that is transverse to the supension flow associated to φ. Then for all $k \in \mathbb{Z}^+$, there is a pseudo-Anosov homeomorphism of the oriented sum $\mathcal{N}_g + k\mathcal{N}_{g'} \to \mathcal{N}_g + k\mathcal{N}_{g'}$ with stretch factor at most λ.*

Our strategy for proving Proposition 3.2 is to find fiber bundles of N_φ over S^1 that have fiber $\mathcal{N}_g + k\mathcal{N}_{g'}$. We then apply a special case of Thurston's hyperbolization theorem, which says that the mapping torus of an orientable surface S by a homeomorphism φ is hyperbolic if and only if φ is pseudo-Anosov [29, Theorem 0.1]. In particular, Thurston's theorem implies that if $M = M_\varphi$ fibers over S^1 in two ways, either both mondromies are pseudo-Anosov or neither monodromy is pseudo-Anosov. Finally, we adapt theorems of Fried and Matsumoto (Theorem 3.4) and Agol–Leininger–Margalit (Theorem 3.5) to work for mapping tori with non-orientable fibers.

We will repeatedly use the following two facts for orientable surfaces and 3-manifolds:

1. An orientable surface minimizes the Thurston norm in its homology class if and only if it is incompressible.

2. If an orientable 3-manifold M fibers over S^1, then the fiber is incompressible.

Proposition 3.3. *Let $\mathcal{N}'$ be an incompressible surface embedded in N that is transverse to the suspension flow direction associated to φ. Let α be the Poincaré dual of $\mathcal{N}$ and α' the Poincaré dual of $\mathcal{N}'$. If the oriented sum of $\mathcal{N}$ and $\mathcal{N}'$ is connected, then $\mathcal{N} + \mathcal{N}'$ is homeomorphic to the fiber of the fibration given by $\alpha + \alpha'$.*

Proof. Let $p : \widetilde{N} \to N$ be the orientation double cover of N. The surface $\mathcal{N}$ is incompressible because it is a fiber of f; therefore $p^{-1}(\mathcal{N})$ is also incompressible. Then the Thurston norm of $\mathcal{N}$ of α is $2\chi_-(\mathcal{N})$. Likewise, the Thurston norm of α' is $2\chi_-(\mathcal{N}')$.

Both α and α' lie in a cone over a fibered face in $H^1(N;\mathbb{Z})$. Therefore the Thurston norm x on $H^1(N;\mathbb{Z})$ is linear function on that cone. Since the Thurston norm is linear on oriented sums of $\mathcal{N}$ and $\mathcal{N}'$, we have:

$$\begin{aligned} x(\alpha+\alpha') &= x(\alpha)+x(\alpha') \\ &= 2\chi_-(\mathcal{N})+2\chi_-(\mathcal{N}') \\ &= 2\chi_-(\mathcal{N}+\mathcal{N}'). \end{aligned}$$

Because $2\chi_-(\mathcal{N}+\mathcal{N}')$ achieves the Thurston norm of $\alpha+\alpha'$, the preimage $p^{-1}(\mathcal{N}+\mathcal{N}')$ achieves the Thurston norm of the pullback of $\alpha+\alpha'$ under p. Therefore $p^{-1}(\mathcal{N}+\mathcal{N}')$ is incompressible. Then $\mathcal{N}+\mathcal{N}'$ is also incompressible.

By Theorem 2.12, we have that $\alpha+\alpha'$ corresponds to some other fibration $f'' : N \to S^1$. By Theorem 2.8, the fiber of f'' must be homeomorphic to $\mathcal{N}+\mathcal{N}'$. $\square$

In the proof of Proposition 3.2, we will use Proposition 3.3 along with a theorem of Thurston to obtain a pseudo-Anosov homemorphism φ_k of the surface of genus of genus $g+kg'$. We the use Theorems 3.4 and 3.5 to obtain a upper bound on the stretch factor of φ_k.

Theorem 3.4 (Fried [9],[10],Matsumoto[22])**.** *Let M be an orientable hyperbolic 3-manifold and let $\mathcal{K}$ be the union of cones in $H^1(M;\mathbb{R})$ whose lattice points correspond to fibrations over S^1. There exists a strictly convex function $h : \mathcal{K} \to \mathbb{R}$ satisfying the following properties.*

(i) For all $c > 0$ and $u \in \mathcal{K}$, $h(cu) = \frac{1}{c}h(u)$.

(ii) For every primitive lattice point $u \in \mathcal{K}$, $h(u) = \log(\lambda)$, where λ is the stretch factor of the pseudo-Anosov map associated to this lattice point.

(iii) $h(u)$ goes to ∞ as u approaches $\partial\mathcal{K}$.

Theorem 3.5 (Agol-Leininger-Margalit)**.** *Let $\mathcal{K}$ be a fibered cone for a mapping torus M and let $\overline{\mathcal{K}}$ be its closure in $H^1(M;\mathbb{R})$. If $u \in \mathcal{K}$ and $v \in \overline{\mathcal{K}}$, then $h(u+v) < h(u)$.*

Proof of Proposition 3.2. The oriented sum $\mathcal{S} = \mathcal{N}_g + k\mathcal{N}_{g'}$ constructed in Proposition 3.3 is a surface of genus $g+kg'$, and is $\mathcal{S}$ is homeomorphic to a fiber of N_φ given by $\alpha+k\alpha'$. Let $\varphi_k : \mathcal{S} \to \mathcal{S}$ be the monodromy of N_φ over $\mathcal{S}$. By Thurston's theorem, φ_k is pseudo-Anosov. We claim that φ_k has stretch factor at most λ.

Let $p : \widetilde{N} \to N_\varphi$ be the orientation double cover of N_φ. Let $h\mid_N$ be the restriction of h to the pullback $p^*(H^1(N_\varphi;\mathbb{R}))$ in $H^1(\widetilde{N};\mathbb{R})$. The restriction $h\mid_N$ satisfies all the properties of Theorems 3.4 and 3.5.

Let $\widetilde{\varphi}$ be the orientation preserving lift of φ to $p^{-1}(\mathcal{N})$. Since $\widetilde{\alpha}$ is the pullback of α, the $\widetilde{\varphi}$ is the pseudo-Anosov homeomorphism associated to $\widetilde{\alpha}$. By Proposition 3.1, the stretch factor of $\widetilde{\varphi}$ is λ.

Let $\mathcal{K}$ be the cone in $H^1(N_\varphi;\mathbb{R})$ containing α. Since $\mathcal{N}_{g'}$ is transverse to the suspension flow in the direction of φ, we have that α' is in the closure of $\mathcal{K}$ in $H^1(N;\mathbb{R})$. Let $\widetilde{\alpha}$ be the pullback of α under p and let $\widetilde{\alpha}'$ be the pullback of α' under p. Then $h\mid_N(\widetilde{\alpha}+\widetilde{\alpha}') < h\mid_N(\widetilde{\alpha})$. By Theorem 3.4, $h(\widetilde{\alpha})$ is equal to the stretch factor of the pseudo-Anosov homeomorphism associated to $\widetilde{\alpha}$. Therefore we have $h\mid_N(\widetilde{\alpha}+\widetilde{\alpha}') < \log(\lambda)$. It follows that the stretch factor of φ_k is less than λ. $\square$

4 Minimal stretch factors for non-orientable surfaces with marked points

In this section we will use Theorem 2.12 and Proposition 3.2 to adapt the methods of Yazdi [33] to non-orientable surfaces. We recall the statement of the main theorem:

Theorem 1.1. Let $\mathcal{N}_{g,n}$ be a non-orientable surface of genus g with n punctures, and let $\ell'_{g,n}$ be the logarithm of the minimum stretch factor of the pseudo-Anosov mapping classes acting on $\mathcal{N}_{g,n}$. Then for any fixed $n \in \mathbb{N}$, there is a positive constant $B'_1 = B'_1(n)$ and $B'_2 = B'_2(n)$ such that for any $g \geq 3$, the quantity $\ell'_{g,n}$ satisfies the following inequalities:

$$\frac{B'_1}{g} \leq \ell'_{g,n} \leq \frac{B'_2}{g}.$$

Observe that the lower bound for the non-orientable case follows easily from the lower bound for the orientable case. Indeed, let φ be a pseudo-Anosov map with the minimal stretch factor on $\mathcal{N}_{g,n}$. The orientation double cover of $\mathcal{N}_{g,n}$ is $\mathcal{S}_{G-1,2n}$ where $G = 2g$. Note that in the non-orientable case we measure genus as the number of copies of the projective plane attached to S^2 via a connect sum and in the orientable case we measure genus as the number of copies of the torus attached to S^2 via a connected sum. Let $\widetilde{\varphi} : \mathcal{S}_{G-1,2n} \to \mathcal{S}_{G-1,2n}$ be the orientation preserving lift of φ. By Proposition 3.1, $\widetilde{\varphi}$ has the same stretch factor as φ. The former is bounded below by $\frac{B_1}{G-1}$, and thus the stretch factor of φ is bounded below as well. The more challenging part of the proof is showing that the upper bound holds.

We will closely follow Yazdi's construction, which proceeds in five steps that we will slightly reorder for clarity. In steps 1 and 2, we construct a family of psuedo-Anosov homeomorphisms of $\mathcal{N}_{g_i,n}$, where $\{g_i\}$ is an unbounded increasing sequence. However the sequence $\{g_i\}$ does not contain all natural numbers. In step 3 we gives an upper bound to the stretch factor of the previously constructed homeomorphisms. In steps 4 and 5, we construct pseudo-Anosov maps on surfaces of genera that do not belong to the sequence $\{g_i\}$. It is in steps 4 and 5 that we uses Thurston's fibered face theory. We have adapted each of Yazdi's five steps to work for non-orientable surfaces.

Step 1: Constructing the surfaces.

We begin by defining a family of surfaces $P_{n,k}$. Let S be an orientable surface of genus 5 with 3 boundary components c, d and e. Choose an orientation for S and let c, d and e inherit the induced orientations. We obtain a non-orientable surface T from S by adding two cross caps to S (but retaining the orientation of the boundary components of S). Let p and q be marked points in the boundary component e. (In Step 5 we will remove p and all copies). Let r and s be the components of $e \setminus p, q$. The resulting surface N is given in Figure 4.

Let $T_{i,j}$ be copies of the surface T, where $i, j \in \mathbb{Z}$. Let $c_{i,j}, d_{i,j}$ and $e_{i,j}$ be the (oriented) boundary components of $T_{i,j}$ and let $r_{i,j}$ and $s_{i,j}$ be the copies of the arcs r and s in $T_{i,j}$. Define a connected infinite surface T_∞ as the quotient:

$$T_\infty \coloneqq \left(\bigcup T_{i,j}\right) \Big/ \sim$$

for all integers i and j. The gluing $\sim$ is given by the orientation-reversing identifications:

$$c_{i,j} \sim d_{i+1,j} \tag{7}$$
$$r_{i,j} \sim s_{i,j+1}. \tag{8}$$

We have two natural shift maps $\overline{\rho_1}, \overline{\rho_2} : T_\infty \to T_\infty$ that act in the following manner:

$$\overline{\rho_1} : T_{i,j} \mapsto T_{i+1,j}$$
$$\overline{\rho_2} : T_{i,j} \mapsto T_{i,j+1}.$$

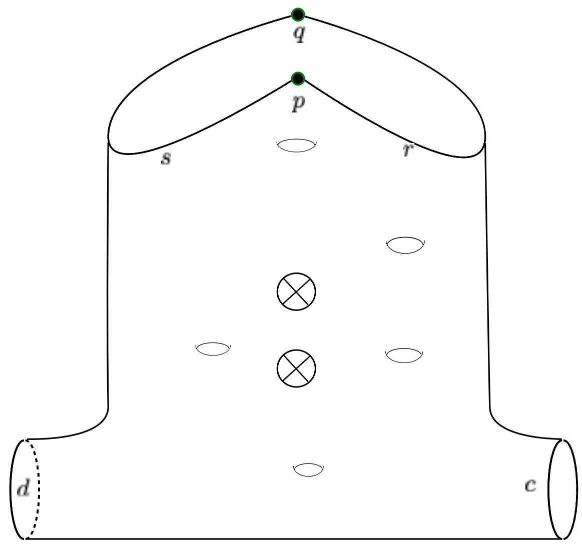

Figure 4: The surface T, which will be the building block of the construction.

Note that $\overline{\rho_1}$ and $\overline{\rho_2}$ commute. Define the surface $P_{n,k}$ as the quotient of the surface T_∞ by the covering action of the group generated by $(\overline{\rho_1})^n$ and $(\overline{\rho_2})^k$. Then $\overline{\rho_1}$ and $\overline{\rho_2}$ are equivariant with respect to the covering map. We denote the induced homeomorphisms of the quotient $P_{n,k}$ by ρ_1 and ρ_2. Note that later we will require that $k \geq 3$ and n is the number of punctures, given in Theorem 1.1.

Lemma 4.1. *Let*

$$g_{n,k} = (14k - 2)n + 2$$

for $n \geq 1$ and $k \geq 1$. The genus of $P_{n,k}$ is $g_{n,k}$.

Proof. Let $U \subset P_{n,k}$ be the subsurface

$$U = \left(\bigcup_{j=0}^{k-1} T_{0,j} \right) \Big/ \sim'$$

where $\sim'$ is given by (7) and by identifying $r_{i,k-1}$ and $s_{i,0}$. Then U is a compact, non-orientable surface of genus $12k$ with $2k$ boundary components. We can compute the Euler characteristic of U:

$$\begin{aligned} \chi(U) &= 2 - 12k - 2k \\ &= 2 - 14k. \end{aligned}$$

Since $P_{n,k}$ is formed by identifying $c_{i,j}$ with $d_{i+1,j}$ for $0 \leq i \leq n-1$ and $c_{n-1,j}$ with $d_{0,j}$ for all j, we calculate:

$$\begin{aligned} \chi(P_{n,k}) &= n \cdot \chi(U) \\ &= -n(14k - 2), \end{aligned}$$

Since $P_{n,k}$ is a non-orientable surface with empty boundary, we have that:

$$g_{n,k} = n(14k - 2) + 2.$$

□

Step 2: Constructing the maps. In what is now a classical paper, Penner gives a construction of pseudo-Anosov homeomorphisms both orientable and non-orientable surfaces [25]. Below we outline the Penner construction for non-orientable surfaces following the details of Liechti–Strenner [20, Section 2].

Inconsistent markings. Let $\mathcal{N}$ be a non-orientable surface and let c be a two-sided curve in $\mathcal{N}$. There exists a neighborhood of c that is homeomorphic to an annulus. Let $\mathcal{A}_c$ be an annulus and $\zeta_c : \mathcal{A}_c \to \mathcal{N}$ the homeomorphism that maps to a neighborhood of c. The homeomorphism ζ_c is called a *marking* of c. A pair consisting of a curve c and ζ_c is called a *marked curve.* If we fix an orientation of $\mathcal{A}_c$, then we can pushforward this orientation to $\mathcal{N}$. Let (c, ζ_c) and (d, ζ_d) be two marked curves that intersect at one point p. We say that (c, ζ_c) and (d, ζ_d) are *marked inconsistently* if the pushforward of the orientation of $\mathcal{A}_c$ and disagrees with the pushforward of the orientation of $\mathcal{A}_d$ in a neighborhood of p.

We define the Dehn twist $\phi_{c,\zeta_c}(x)$ around a marked curve (c, ζ_c) as:

$$\phi_{c,\zeta_c}(x) = \begin{cases} \zeta_c \circ \tau_c \circ \zeta_c^{-1}(x) & \text{for } x \in \zeta_c(\mathcal{A}_c) \\ x & \text{for } x \in \mathcal{N} - \zeta_c(\mathcal{A}_c) \end{cases}.$$

Here τ_c is the left-handed Dehn twist on $\mathcal{A}_c$, i.e. $\tau_c(\theta, t) = (\theta + 2\pi t, t)$.

The Penner construction for non-orientable surfaces. Let $\mathcal{C}$ be a set of marked curves in $\mathcal{N}$ such that no two curves in $\mathcal{C}$ are homotopic. A Penner construction on $\mathcal{N}$ is a composition of Dehn twists about the marked curves in $\mathcal{C}$ such that:

1. the complement of curves in $\mathcal{C}$ in $\mathcal{N}$ consists of disks with at most one puncture or marked point,
2. the marked curves $(c_i, \zeta_i), (c_j, \zeta_j) \in \mathcal{C}$ with $i \neq j$ are marked inconsistently,
3. a Dehn twist about each marked curve in $\mathcal{C}$ is included in the composition, and
4. all powers of Dehn twists are positive (alternatively, all powers are negative).

Construction of $f_{n,k}$. We now construct homeomorphisms $f_{n,k} : P_{n,k} \to P_{n,k}$ that are defined as a composition of specific Dehn twists followed by a finite order mapping class. The key insight is that a power of this map will be a composition of Dehn twists that satisfy the criteria to be a Penner construction. Therefore $f_{n,k}$ is pseudo-Anosov. Here we are using the rotational symmetry of the $P_{n,k}$.

Let $\{\alpha_1, \cdots, \alpha_8\}$ be the multi-curve in $T_{0,0}$ as shown in Figure 5. Let $\{\beta_1, \cdots, \beta_7\}$ be the multi-curve in $T_{0,0} \cup T_{0,1} \cup T_{1,0}$ shown in Figure 5. All intersections of curves in $\{\alpha_1, \cdots, \alpha_8\}$ and $\{\beta_1, \cdots, \beta_7\}$ are shown in Figure 5.

For any α_i, we choose a marking ζ_{α_i} to be orientation-preserving. For any β_j let ζ_{β_j} be orientation reversing. From here forward, we will think of α_i and β_j as (inconsistently) marked curves but we will suppress the marking maps. These choices give an inconsistent marking of $\{\alpha_1, \cdots, \alpha_8\} \cup \{\beta_1, \cdots, \beta_7\}$.

Let

$$\mathcal{R} = \bigcup_{i=2}^{8} \alpha_i.$$

Then $\mathcal{R}$ is a marked multi-curve that is disjoint from γ. Let

$$\overline{\mathcal{R}} = \mathcal{R} \cup \rho_1(\mathcal{R}) \cup \cdots \cup \rho_1^{n-1}(\mathcal{R}).$$

Let Φ_r be the composition of Dehn twists about the marked curves in $\overline{\mathcal{R}}$. Because the curves in $\overline{\mathcal{R}}$ are disjoint, the Dehn twists about the curves commute.

Let

$$\mathcal{B} = \bigcup_{j=2}^{7} \beta_j$$

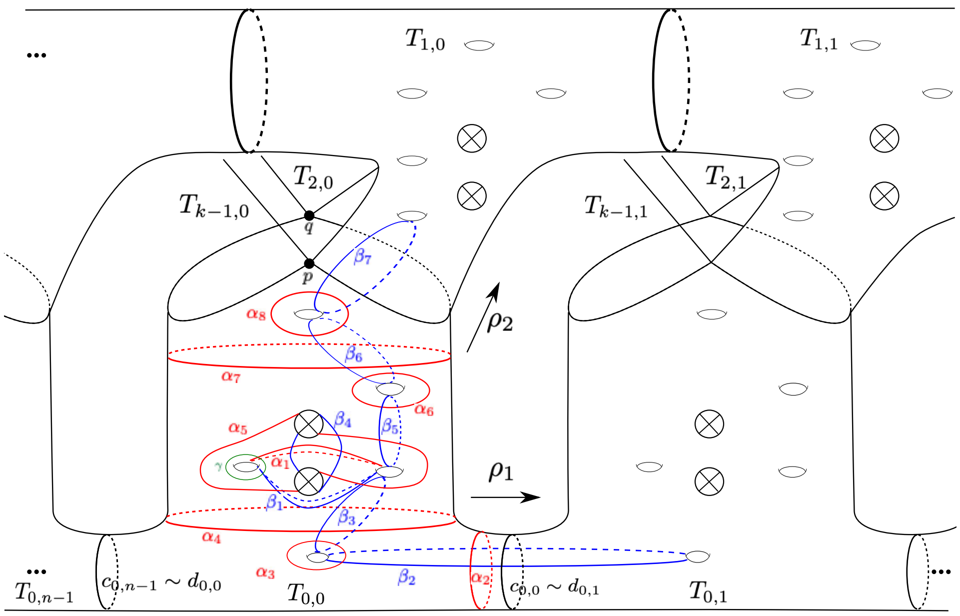

Figure 5: Part of surface $P_{n,k}$ that includes the subsurface $T_{0,0}$ and the curves α_i, β_j, and γ.

in $T_{0,0} \cup T_{0,1} \cup T_{1,0}$. As above, $\mathcal{B}$ is a marked multi-curve that is disjoint from γ. Let

$$\overline{\mathcal{B}} = \mathcal{B} \cup \rho_1(\mathcal{B}) \cup \cdots \cup \rho_1^{n-1}(\mathcal{B}).$$

Let Φ_b as the composition of Dehn twists about all of the marked curves in $\overline{\mathcal{B}}$. As with $\overline{\mathcal{R}}$, the Dehn twists about curves in $\overline{\mathcal{B}}$ commute.

Let $\alpha_1, \beta_1 \subset T_{0,0}$ be the (marked) curves in Figure 5. Let Φ be the composition of Dehn twists along all the curves $\alpha_1, \rho_1(\alpha_1), \ldots, \rho_1^{n-1}(\alpha_1)$ followed by Dehn twists along all the curves $\beta_1, \rho_1(\beta_1), \ldots, \rho_1^{n-1}(\beta_1)$. Define the map $f_{n,k}$ as:

$$f_{n,k} \coloneqq \rho_2 \circ \Phi \circ \Phi_b \circ \Phi_r$$

Since the curves about which we twist to construct $f_{n,k}$ satisfy the conditions of Penner's construction, $f_{n,k}$ is a pseudo-Anosov homeomorphism.

Step 3: Bounding the Stretch Factor. Following Yazdi, our next goal is to find an upper bound for the stretch factor of the pseudo-Anosov homeomorphisms we found in Step 2.

Train tracks. Let S be a surface. A *train track* in S is graph embedded in S with that property that for every vertex v of valence three or greater, all edges adjacent to v have the same tangent vector at v. Let $\varphi : S \to S$ be a pseudo-Anosov homeomorphism. The map φ is equipped with a train track whose image under φ is homotopic to itself. Such a train track is an *invariant train track* associated to φ. Invariant train tracks have an associated matrix whose Perron-Frobenius eigenvalue is the stretch factor of φ.

Yazdi uses the Lemma 4.2 to bound the spectral radius of the associated matrices.

Lemma 4.2 (Lemma 2.3 of [33])**.** *Let A be a non-negative integral matrix, Γ be the adjacency graph of A, and $V(\Gamma)$ the set of vertices of Γ. For each $v \in V(\Gamma)$, define v^+ to be the set of vertices $u \in V(\Gamma)$ such that there is an oriented edge from v to u. Let D and k be fixed natural numbers. Assume the following conditions hold for Γ:*

(i) For each $v \in V(\Gamma)$ we have $\deg_{out}(v) \leq D$,

(ii) There is a partition $V(\Gamma) = V_1 \cup \cdots \cup V_\ell$ such that for each $v \in V_i$ we have $v^+ \subset V_{i+1}$, for any $1 \le i \le \ell$ except possibly when $i = 1$ or 3 (indices are mod ℓ),

(iii) For each $v \in V_1$, we have $v^+ \subset V_2 \cup V_3$,

(iv) For each $v \in V_3$ we have $v^+ \subset V_3 \cup V_4$, and for $u \in v^+ \cap V_3$ we have $u^+ \subset V_4$, and

(v) For all $3 < j \le k$ and each $v \in V_j$, the set v^+ consists of a single element.

Then the spectral radius of $A^{\ell-1}$ is at most $4D^4$.

With this result in hand, we can now show that the stretch factors for our main family of examples are all bounded above in the way we hope.

Lemma 4.3. *Let $\lambda_{n,k}$ be the stretch factor of $f_{n,k}$. Then there exists a universal positive constant C' such that for every $n \ge 1$ and $k \ge 3$, we have the following upper bound on $\log(\lambda_{n,k})$.*

$$\log(\lambda_{n,k}) \le C' \frac{n}{g_{n,k}}$$

Proof. We deliberately constructed our curves so that all intersections of the multi-curve $\{\alpha_1, \ldots, \alpha_8\}$ and $\{\beta_1, \ldots, \beta_7\}$ occur in the subsurface $T_{0,0}$. The curve β_3 intersects $\rho_2(\alpha_3)$ at one point in $T_{0,1}$ and β_7 intersects $\rho_1(\alpha_8)$ at one point in $T_{1,0}$.

We define the following unions of marked curves:

$$\begin{aligned}
\mathcal{A} &\coloneqq \mathcal{B} \cup \mathcal{R} \cup \{\alpha_1, \beta_1\} = \bigcup_{i=1}^{8} \alpha_i \cup \bigcup_{j=1}^{7} \beta_j \\
\overline{\mathcal{A}} &\coloneqq \mathcal{A} \cup \rho_1(\mathcal{A}) \cup \cdots \cup \rho_1^{n-1}(\mathcal{A}) \\
\widehat{\mathcal{A}} &\coloneqq \overline{\mathcal{A}} \cup \rho_2\left(\overline{\mathcal{A}}\right) \cup \cdots \cup \rho_2^{k-1}\left(\overline{\mathcal{A}}\right).
\end{aligned}$$

Because $f_{n,k}$ is pseudo-Anosov, it has a corresponding invariant train track τ. Let V_τ be the space of all measured foliations that can be obtained by varying the weights on the tracks of τ. This forms a finite dimensional cone of measures, all of which can be carried by the combinatorial train track τ. Furthermore, $f_{n,k}$ acts linearly on this cone, and leaves the cone invariant, since τ is an invariant track for $f_{n,k}$. Consider now the transverse measure μ_δ for any curve δ in $\widehat{\mathcal{A}}$. This transverse measure is carried by τ, and thus μ_δ belongs in the cone of measures V_τ. Let H be the subspace spanned by $\{\mu_\delta \mid \delta \subset \widehat{\mathcal{A}}\}$. This linear subspace is also left invariant by $f_{n,k}$, so we can restrict our attention to the matrix of $f_{n,k}$ with respect to the basis given by the $\{\mu_\delta \mid \delta \subset \widehat{\mathcal{A}}\}$. Let M be the matrix representing this linear action on H. Let Γ be the adjacency graph for M. Work of Penner [25] tells us that the Perron–Frobenius eigenvalue of M is the stretch factor of $f_{n,k}$.

To bound the spectral radius of M, we need to show that Γ satisfies the criteria of Lemma 4.2.

(i) There exists a constant D', independent of n and k, such that for every curve $\delta \in \widehat{\mathcal{A}}$, the geometric intersection number between δ and every curve in $\overline{\mathcal{A}}$ is at most D'. Recall that $f_{n,k} = \rho_2 \circ \Phi \circ \Phi_b \circ \Phi_r$. Let M_1, M_2, M_3 and M_4 be the matrices describing the linear action of Φ_r, Φ_b, Φ and ρ_2 on H, respectively. The matrix M can then be written as a product:

$$M = M_4 M_3 M_2 M_1.$$

For a curve $\delta \in \hat{\mathcal{A}}$, the L^1-norm of $M_i(\mu_\delta)$ is bounded above by the geometric intersection of $f_{n,k}(\delta)$ with the curves in $\overline{\mathcal{A}}$, thus each of M_1, M_2 and M_3 will change the norm by a factor of at most $(1 + D')$. Since ρ_2 will not change intersection numbers, M_4 will preserve the L^1-norm. If we let $D = (1 + D')^3$, then the outward degree of each vertex in Γ is at most D.

For the remaining conditions, we partition the vertices of Γ. Recall

$$\widehat{\mathcal{A}} = \bigcup_{i=1}^{k} \rho_2^{i-2}(\overline{\mathcal{A}}).$$

Then define V_i for $1 \leq i \leq k$ as the vertices of Γ corresponding to elements in $\rho_2^{i-2}(\overline{\mathcal{A}})$.

(ii) Suppose that $v \in V_i$, $i \neq 1, 3$, is a vertex that corresponds to μ_δ for a curve $\delta \in \hat{\mathcal{A}}$. Then δ must be a curve in $\rho_2^{i-2}(\overline{\mathcal{A}})$, for $i \neq 1, 3$. Then δ is disjoint from all curves in $\overline{\mathcal{A}}$. The action of $\Phi \circ \Phi_b \circ \Phi_r$ on $\widehat{\mathcal{A}}$ will preserve the set $\rho_2^{i-2}(\overline{\mathcal{A}})$ (and therefore δ) for each $i \neq 1, 3$. Then ρ_2 will rotate the curve $\Phi \circ \Phi_b \circ \Phi_r(\delta)$ to $\rho_2^{i-1}(\overline{\mathcal{A}})$. That is: $f_{n,k} = \rho_2 \circ \Phi \circ \Phi_b \circ \Phi_r$ maps $\mu_\delta \in H$ to

$$\sum_{\zeta \in \mathcal{Z}} \mu_\zeta$$

where $\mathcal{Z}$ is a subset of $\rho_2^{i-1}(\overline{\mathcal{A}})$. Therefore $f_{n,k}$ maps v to a subset of V_{i+1}.

(iii) To verify the third condition, we first look at the vertices $v \in V_1$ such that $v^+ \not\subset V_2$. Such vertices will correspond to the curves in $\rho_2^{-1}(\overline{\mathcal{A}})$ that $\Phi \circ \Phi_b \circ \Phi_r$ maps to curves that are not in $\rho_2(\overline{\mathcal{A}})$. Because ρ_1 and ρ_2 commute, we can write the curves of $\rho_2^{-1}(\overline{\mathcal{A}})$ as:

$$\rho_2^{-1}(\overline{\mathcal{A}}) = \rho_2^{-1}(\mathcal{A}) \cup \rho_1(\rho_2^{-1}(\mathcal{A})) \cup \cdots \cup \rho_1^{n-1}(\rho_2^{-1}(\mathcal{A})).$$

The elements of v^+ that are not in V_2 correspond to the images of curves in $\rho_2^{-1}(\overline{\mathcal{A}})$ under $f_{n,k}$ that are not in $\overline{\mathcal{A}}$. As in Yazdi, the only curves in $\rho_2^{-1}(\overline{\mathcal{A}})$ that intersect curves in $\overline{\mathcal{A}}$ are those in the set:

$$\mathcal{X} = \{\rho_1^i(\rho_2^{-1}(\beta_7)) \mid 0 \leq i \leq n-1\}.$$

Therefore $\Phi \circ \Phi_b \circ \Phi_r$ maps curves in $\mathcal{X}$ to curves in $\rho_2^{-1}(\overline{\mathcal{A}}) \cup \overline{\mathcal{A}}$. Then $f_{n,k} = \rho_2 \circ \Phi \circ \Phi_b \circ \Phi_r$ maps curves in $\mathcal{X}$ to curves in $\overline{\mathcal{A}} \cup \rho_2(\overline{\mathcal{A}})$. For any curve in $\mathcal{X}$, the corresponding vertex $v \in V_1$ will have $v^+ \subset V_2 \cup V_3$. Moreover, $f_{n,k}$ maps the curves $\rho_2^{-1}(\overline{\mathcal{A}}) \setminus \mathcal{X}$ to curves in $\overline{\mathcal{A}}$. Thus for any vertex $v \in V_1$ that does not correspond to an element of $\mathcal{X}$, the set v^+ is contained in V_2.

(iv) Similarly, we look for the $v \in V_3$ such that $v^+ \not\subset V_4$. Such vertices will correspond to the curves in $\rho_2(\overline{\mathcal{A}})$ that $\Phi \circ \Phi_b \circ \Phi_r$ maps to curves that are not in $\rho_2^2(\overline{\mathcal{A}})$. As above, we have:

$$\rho_2(\overline{\mathcal{A}}) = \rho_2(\mathcal{A}) \cup \rho_1(\rho_2(\mathcal{A})) \cup \cdots \cup \rho_1^{n-1}(\rho_2(\mathcal{A})).$$

The elements of v^+ that are not in V_4 correspond to the the images of $\rho_2(\overline{\mathcal{A}})$ that intersect the curves in $\overline{\mathcal{A}}$. The only vertices of V_4 that correspond to such curves are those in the set:

$$\mathcal{Y} = \{\rho_1^i(\rho_2(\alpha_8)) \mid 0 \leq i \leq n-1\}.$$

For any element $v \in V_3$ corresponding to a curve in $\mathcal{Y}$ and any $u \in v^+ \cap V_3$, the vertex u does not correspond to an element of $\mathcal{Y}$. Therefore $u^+ \subset V_4$.

(v) All the curves corresponding to an element of V_j, $3 < j \leq k$ are disjoint from all the curves in $\overline{\mathcal{A}}$. Thus, $f_{n,k}$ just acts by rotation.

Let $\lambda = \lambda_{n,k}$ be the stretch factor of $f_{n,k}$. By Lemma 4.2, we have:

$$\lambda^{k-1} = \rho(M)^{k-1} = \rho(M^{k-1}) \leq 4D^4.$$

Then the logarithm of λ satisfies:

$$\log(\lambda^{k-1}) = (k-1) \cdot \log(\lambda) \leq \log(4D^4).$$

Then for $k \geq 2$

$$\frac{k}{2} \log(\lambda) \leq (k-1) \log(\lambda) \leq \log(4D^4).$$

On the other hand, we know $g_{n,k} = (14k-2)n + 2 \leq 14kn$ by Lemma 4.1. Therefore

$$\log(\lambda) \leq 2 \log(4D^4) \cdot \frac{1}{k} \leq 2 \log(4D^4) \cdot \frac{14n}{g_{n,k}}.$$

Let $C' \coloneqq 28 \log(4D^4)$ to complete the result. $\square$

Step 4: The Mapping Torus.

We have now constructed an infinite family of non-orientable surfaces and pseudo-Anosov maps, but this is not enough. By Lemma 4.1, the family does not contain a surface of every genus. In fact, the family does not include surfaces of infinitely many genera. We use the strategy of McMullen [23] and our extension of the Thurston's fibered face theory to fill in the gaps.

Next we follow the strategy of Leininger–Margalit [19] to find a surface embedded in the mapping torus of minimal genus. In our situation, this means that we will construct an embedded surface homeomorphic to $\mathcal{N}_3$.

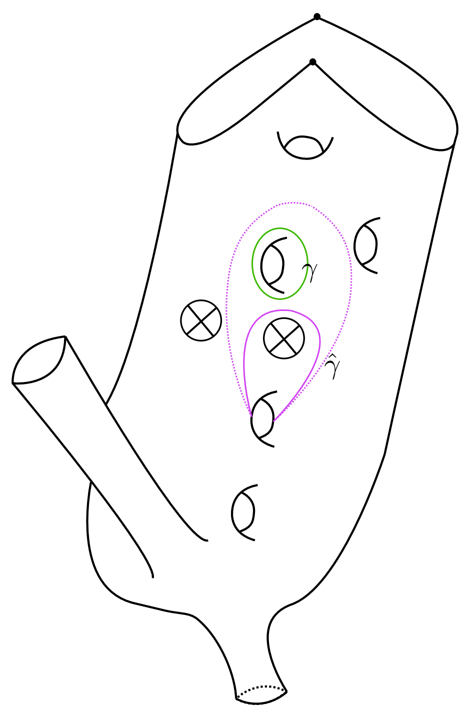

Figure 6: The curves γ and $\widehat{\gamma}$ bound an a non-orientable surface of genus 1.

Proposition 4.4. *Let $M_{n,k}$ be the mapping torus of $f_{n,k}$. Let $\mathcal{K}_{n,k}$ denote the fibered cone of $H^1(M_{n,k};\mathbb{R})$ corresponding to the map $f_{n,k}$. There is a relatively orientable incompressible surface $F_{n,k}$ embedded in $M_{n,k}$ that is homeomorphic to $\mathcal{N}_3$. Moreover $F_{n,k}$ is transverse to the suspension flow direction given by $f_{n,k}$ and the Poincaré dual of $F_{n,k}$ is in the closure $\overline{\mathcal{K}_{n,k}}$.*

Proof. Let $\gamma \subset T_{0,0}$ be the curve shown in Figure 6. Note that γ and $\Phi(\gamma)$ bound a non-orientable surface $\hat{F}$ of genus 1 with boundary. For convenience, we will denote $\Phi(\gamma)$ by $\widehat{\gamma}$. We are going to follow the image of γ under powers of $f_{n,k}$. Then we attach annuli to the boundary of $\widehat{F}$ to obtain $\mathcal{N}_3$. Since γ is disjoint from all curves in $\overline{\mathcal{R}}$ and $\overline{\mathcal{B}}$ (as seen in Figure 5), the maps Φ_r and Φ_b act trivially on γ. Recalling that $f_{n,k} = \rho_2 \circ \Phi \circ \Phi_b \circ \Phi_r$, we have the following:

$$\begin{aligned} f_{n,k}(\gamma) &= \rho_2 \circ \Phi \circ \Phi_b \circ \Phi_r(\gamma) \\ &= \rho_2 \circ \Phi(\gamma) \\ &= \rho_2(\widehat{\gamma}) \end{aligned}$$

It follows that for all $1 \leq i \leq k$, the curve $f^i_{n,k}(\gamma)$ is $\rho_2^i(\gamma)$. For $1 \leq i \leq k$, let A_i be an annulus in $M_{n,k}$ that connects $f^{i-1}_{n,k}(\gamma)$ to $f^i_{n,k}(\gamma)$ obtained by following the suspension flow of $f_{n,k}$ around $M_{n,k}$. We can now construct the embedded surface $F_{n,k}$ by taking the union of $A_1, A_2, \ldots, A_k$ and $\hat{F}$. Since each A_i has Euler characteristic 0 and the union of $\hat{F}$ with $A_1, A_2, \ldots, A_k$ has empty boundary we see $F_{n,k}$ is homeomorphic to $\mathcal{N}_3$.

Since $F_{n,k}$ is a non-orientable surface embedded in a non-orientable manifold $M_{n,k}$, Proposition 2.6 tells us that $F_{n,k}$ is relatively orientable (in $M_{n,k}$).

The proof that $F_{n,k}$ can be isotoped to be transverse to the suspension flow is the same as the proof Yazdi uses [33], which is a restatement of that of Leininger–Margalit [19]. We include it here for completeness.

Let $N(\gamma)$ be a tubular neighborhood of γ in $\hat{F}$. Let $\eta : \hat{F} \to [0, 1]$ be a smooth function supported on $N(\gamma)$ with the following properties:

- $\eta^{-1}(1) = \gamma$ and
- the derivative of η vanishes on γ.

Let $\pi : M_{n,k} \to S^1$ be the projection map and let t_0 be such that $\hat{F} \subset \pi^{-1}(t_0)$. Let $g : \hat{F} \to M_{n,k}$ be the suspension flow of $f_{n,k}$ defined as $g(x) = (x, t_0 + k \cdot \eta(x))$. Then the restriction of g to the interior of $\hat{F}$ is an embedding into $M_{n,k}$ and $g(\gamma) = \hat{\gamma}$. Therefore the image of $\hat{F}$ under g is an embedded non-orientable surface of genus three. Moreover, $g(\hat{F})$ is isotopic to the natural embedding of $F_{n,k}$ in $M_{n,k}$, and is transverse to the suspension flow. Therefore, the Poincaré dual of $F_{n,k}$ is in $\overline{\mathcal{K}_{n,k}}$ by Theorem 2.12.

Finally, $F_{n,k}$ is incompressible in $M_{n,k}$ because $M_{n,k}$ is hyperbolic, and $F_{n,k}$ is genus 3, the lowest possible genus for a hyperbolic non-orientable surface. ☐

Step 5: Filling in the Gaps. Recall that the family of surfaces $P_{n,k}$ that we have constructed have genera in the set $\{(14k-2)n+2\}$. We now want to construct surfaces of genera not in the set $\{(14k-2)n+2\}$ and pseudo-Anosov homeomorphisms of those surfaces that have small stretch factors. To do this we use the mapping torus $M_{n,k} = (P_{n,k}, f_{n,k})$. Recall from Proposition 4.4 that there exists a relatively incompressible surface $F_{n,k}$ in $M_{n,k}$ that is homeomorphic to $\mathcal{N}_3$. Let $P^r_{n,k}$ be the oriented sum of $P_{n,k}$ and $rF_{n,k}$, as defined in Proposition 3.3. The surfaces $P^r_{n,k}$ will be surfaces of the remaining genera.

Lemma 4.5. *The surfaces $P^r_{n,k}$ have genus $g^r_{n,k} = g_{n,k} + r$. In particular, as r varies between 0 and $14n$, the genera of $P^r_{n,k}$ span the range between $g_{n,k}$ and $g_{n,k+1}$. Moreover, $P^r_{n,k}$ is isotopic to a fiber of a fibration of $M_{n,k}$ with pseudo-Anosov monodromy that fixes $2n$ of the singularities of its invariant foliation.*

Proof. The Euler characteristic of an oriented sum is the sum of the Euler characteristics of the summands:

$$\begin{aligned} \chi(P^r_{n,k}) &= \chi(P_{n,k}) + r \cdot \chi(F_{n,k}) \\ &= (-2g_{n,k} + 2) - 2r \\ &= -2(g_{n,k} + r) + 2. \end{aligned}$$

Since $P^r_{n,k}$ has no boundary or punctures, we have that the genus of $P^r_{n,k}$ is $g_{n,k} + r$.

By Lemma 4.4 we know that $F_{n,k}$ is incompressible and transverse to the suspension flow of given by $f_{n,k}$. Therefore by Proposition 3.2, there is a pseudo-Anosov homeomorphism of $P^r_{n,k} = P_{n,k} + rF_{n,k}$.

As in Yazdi [32, Lemma 3.5], $f_{n,k}$ fixes the $2n$ singularities of the stable foliation that are the intersection points of the axis of ρ_1 with $P_{n,k}$. By Lemma 4.4, the surface $F_{n,k}$ can be isotoped to be transverse to the suspension flow and disjoint from the orbit of the $2n$ singularities of $f_{n,k}$. Hence the monodromy $f^r_{n,k}$ still fixes the corresponding $2n$ singularities on $P^r_{n,k}$. □

We now prove the non-orientable version of the final piece of Yazdi's proof [33, Lemma 3.6].

Lemma 4.6. *Let $\lambda^r_{n,k}$ be the stretch factor of $f^r_{n,k} : P^r_{n,k} \to P^r_{n,k}$. Then there exists a constant $C > 0$ such that for every $n \geq 1$, $k \geq 3$, and $0 \leq r \leq 14n$ we have the following upper bound on $\log(\lambda^r_{n,k})$:*

$$\log(\lambda^r_{n,k}) \leq C\frac{n}{g^r_{n,k}}.$$

Proof. Let $\mathcal{K} = \mathcal{K}_{n,k}$ be the fibered cone in $H^1(M_{n,k}; \mathbb{R})$ corresponding to $f_{n,k}$ and $h : \mathcal{K} \to \mathbb{R}$ the function described in Theorem 3.4. Note that $g_{n,k} \geq 42$, therefore we have the following bounds on $g^r_{n,k}$:

$$\begin{aligned} g^r_{n,k} &= g_{n,k} + r \\ &\leq g_{n,k} + 14n \\ &< 2g_{n,k}. \end{aligned}$$

Let ω be the Poincaré dual of $P^r_{n,k}$ and α the Poincaré dual of $P_{n,k}$. Then the following string of inequalities holds, the first inequality is by the convexity of h, the second inequality is the bound in Lemma 4.3, and the third inequality is from the bound on $g^r_{n,k}$ above:

$$\begin{aligned} h([\omega]) &< h([\alpha]) \\ &\leq C'\frac{n}{g_{n,k}} \\ &\leq 2C'\frac{n}{g^r_{n,k}}. \end{aligned}$$

□

Each surface $P^r_{n,k}$ is isotopic to a fiber of fibration of $M_{n,k}$ and has a pseudo-Anosov monodromy with a bounded stretch factor.

In the initial construction of $P_{n,k}$, there were $2n$ marked points, which were singularities of the map $f_{n,k}$. By the construction of $P^r_{n,k}$, these marked points are also singularities of $f^r_{n,k}$. Now we puncture $P^r_{n,k}$ at n of these marked points. We could think of this as removing all copies of the point p in the construction of $P_{n,k}$ in step 1.

We can now give a proof of Theorem 1.1.

Proof of Theorem 1.1. As above, the lower bound follows easily from the lower bound in the orientable setting.

To find the upper bound, let $C' = \frac{C}{2}$ be the value given in Lemma 4.6. Let $B_2'(n)$ be the following quantity:

$$B_2'(n) = \max\{2C'n, \ell'_{1,n}, 2\ell'_{2,n}, \dots, (40n+1)\ell'_{40n+1,n}\}$$

By Lemma 4.6, $B_2'(n)$ is an upper bound for $g \cdot \ell'_{g,n}$. □

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