Dynamics on the Moduli Space of Non-Orientable Surfaces

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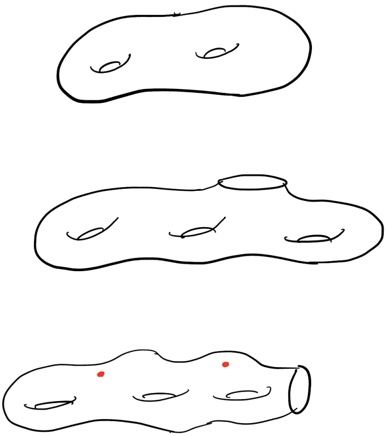
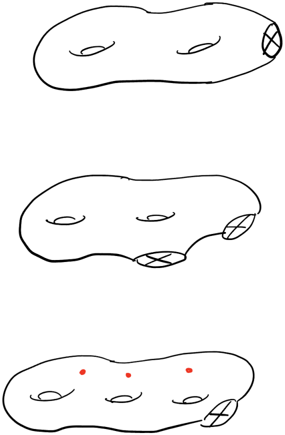
Orientable surfaces



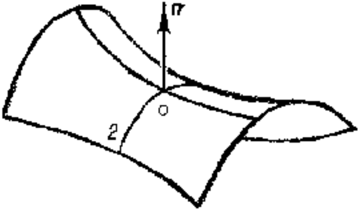
Non-orientable surfaces



Goal

Understand the collection of geometric structures we can put on a topological surface.

In particular, understand the set of metrics on a surface with curvature -1.

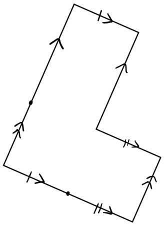


$$\mathcal{M}(\mathcal{S}_g) = \{\text{Curvature } -1 \text{ metrics on } \mathcal{S}_g\}$$

- $\mathcal{M}(\mathcal{S}_g)$ is $6g-6$ dimensional.
- $\mathcal{M}(\mathcal{S}_g)$ is a metric space: x and y in $\mathcal{M}(\mathcal{N}_g)$ are close if the derivative of some map $x \to y$ maps circles to "almost" circles.
- Unit (co)tangent bundle of $\mathcal{M}(\mathcal{S}_g)$ is non-compact, but finite volume, and admits a geodesic flow (and an $SL(2,\mathbb{R})$ action) with good dynamical properties.

Guiding principle: Dynamics on (co)tangent bundle should have analogies with dynamics of the $SL(2,\mathbb{R})$ action on $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$.

$$S^1\mathcal{M}(\mathcal{S}_g) = \{\text{Area 1 half-translation surface structures on } \mathcal{S}_g\}$$



Theorem (Masur's criterion)

If the vertical flow on translation surface is not uniquely ergodic, then the geodesic ray in $\mathcal{M}(\mathcal{S}_g)$ escapes to infinity.

$$\mathcal{M}(\mathcal{N}_g) = \{\text{Curvature } -1 \text{ metrics on } \mathcal{N}_g\}$$

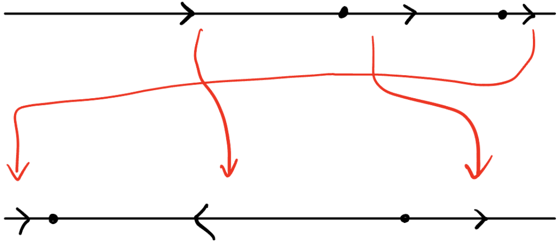
- $\mathcal{M}(\mathcal{N}_g)$ is $3g - 6$ dimensional.
- $\mathcal{M}(\mathcal{N}_g)$ is a metric space, with a similarly defined metric.
- Unit (co)tangent bundle of $\mathcal{M}(\mathcal{N}_g)$ is non-compact, and also infinite volume (with respect to the canonical volume form).
- $S^1\mathcal{M}(\mathcal{N}_g)$ admits a geodesic flow but not an $SL(2, \mathbb{R})$ action: however, the dynamical properties are not great.

Guiding principle?

Goal

Analyze generic tangent vector in $S^1\mathcal{M}(\mathcal{N}_g)$.

- Analyze vertical flow on translation surface.
- Will suffice to look at first return map to a horizontal arc.



Theorem

Almost every geodesic in $S^1\mathcal{M}(\mathcal{N}_g)$ escapes to infinity.

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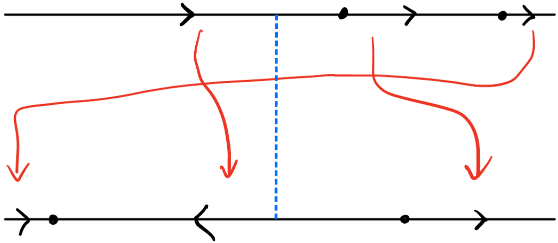


Theorem

Almost every geodesic in $S^1\mathcal{M}(\mathcal{N}_g)$ escapes to infinity.

MCG($\mathcal{N}_g$)-action on $\mathcal{T}(\mathcal{N}_g)$

- MCG($\mathcal{N}_g$) is finitely generated.
- The action is infinite covolume.
- Almost no geodesics recur.
- **Question**: What is the limit set for the MCG($\mathcal{N}_g$) action?
- **Question**: Can we construct a Patterson-Sullivan measure on the limit set with “good” dynamical properties?

Geometrically finite Γ action on $\mathbb{H}$

- Γ is finitely generated.
- Γ action is infinite covolume (with respect to the Liouville measure).
- Almost no geodesics recur.
- Limit set of Γ is a subset of the boundary with smaller Hausdorff dimension.
- Have a Patterson-Sullivan measure on the limit set with “good” dynamical properties.

Theorem (K., Erlandsson-Gendulphe-Pasquinelli-Souto)

The limit set of the MCG($\mathcal{N}_g$) action on $\mathcal{T}(\mathcal{N}_g)$ is $\mathbb{P}\mathcal{MF}^+(\mathcal{N}_g)$.

What do we need for "good" Patterson-Sullivan measures?

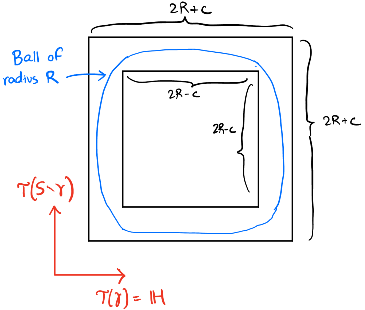
- Classically, Patterson-Sullivan measures were constructed for hyperbolic spaces.
- Then variable negative curvature, Gromov hyperbolic, $\mathrm{CAT}(k)$ for $k \leq 0$, etc.
- Teichmüller spaces are not hyperbolic on the nose, but somewhat hyperbolic.

Thick part

- Region where all curves on underlying surface are longer than ε.
- Geometry of thick region mostly governed by curve complex of surface, which is hyperbolic.

Thin part

- Region where some curve γ on underlying surface is shorter than ε.
- Metric in this region is L^∞ product of metrics on subsurfaces (Minsky's product region theorem).



Statistical convex-cocompactness

- Suffices to show geodesics enter the thin part with exponentially low probabilities. This is the notion of *statistical convex-cocompactness*, and Coulon and Yang showed that this is enough for "good" Patterson-Sullivan theory.
- For Teichmüller spaces of orientable surfaces, Eskin and Mirzakhani proved statistical convex-cocompactness.
- They did this by studying a random walk on Teichmüller space instead, and show good recurrence properties for the random walk.
- There are two obstructions in adapting their random walk based techniques to the non-orientable setting.
- Obstruction 1: Thin region where some *one-sided curve* gets short does not have good recurrence properties for the random walk.

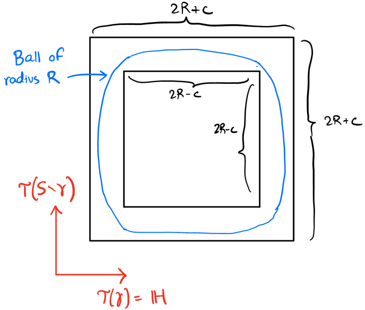
Theorem (K.)

The map $(\mathcal{T}^{-}_{\varepsilon_t}(\mathcal{N}_g), \textit{induced path metric}) \to (\mathcal{T}^{-}_{\varepsilon_t}(\mathcal{N}_g), \textit{Teichmüller metric})$ *is* $(1+\varepsilon_d)$ *bi-Lipschitz.*

- Obstruction 2: It's not obvious that the volume growth entropy of $\mathcal{T}^{-}_{\varepsilon_t}(\mathcal{N}_g)$ is equal to the lattice point growth entropy for the $\mathrm{MCG}(\mathcal{N}_g)$ action.

How to understand random walks on Teichmüller space

- Random walk is with respect to a net in $\mathcal{T}(\mathcal{N}_g)$: the Teichmüller metric controls the law of the random walk.
- In the thin part, we understand the Teichmüller metric really well, thanks to Minsky.



- Since the ball is approximately an L^∞ product, picking a point uniformly at random is equivalent to picking a points *independently* on the two coordinates.
- We can show strong recurrence for random walks on a horoball in $\mathbb{H}$.
- For one-sided thin regions, we get a symmetric random walk on $\mathbb{Z}$, which is not strongly recurrent.

Volume entropy vs lattice point entropy

- The random walk argument gives us an estimate in terms of the volume growth entropy. Statistical convex cocompactness requires an estimate in terms of lattice point entropy.
- Hope: the entropies might be equal since $\mathcal{T}^{-}_{\varepsilon_t}(\mathcal{N}_g)/\mathsf{MCG}(\mathcal{N}_g)$ has finite volume.
- Technical tool to prove a statement like this was invented recently by Dowdall and Masur: *complexity length*.
- Key idea is based off of Minsky's theorem again: we need to understand how volume grows in product region, and therefore, a horoball.

$$\text{Volume growth for ball of radius } R \text{ in } \mathbb{H} \sim \exp(2R)$$

$$\text{Volume growth for ball of radius } R \text{ intersected with a horoball} \sim \exp(R)$$

- Gap in exponential growth rates tells us that the thin part does not contribute a significant fraction of total volume.

Theorem (K.)

For all surfaces S, and any $\varepsilon_t > 0$, we have the following equality of entropies.

$$h_V(\mathcal{T}^{-}_{\varepsilon_t}(S)) = h_{LP}(\mathcal{T}^{-}_{\varepsilon_t}(S))$$

Theorem (K.)

The action of $MCG(\mathcal{N}_g)$ on $\mathcal{T}^{-}_{\varepsilon_t}(\mathcal{N}_g)$ is statistically convex-cocompact.