Dynamics on the Moduli Space of Non-Orientable Surfaces

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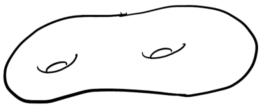
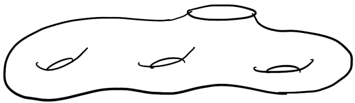
Orientable surfaces





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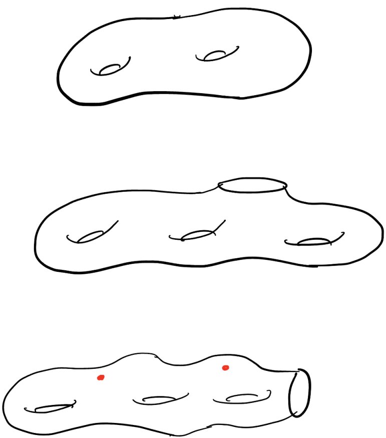

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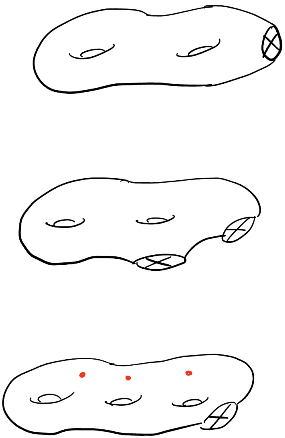

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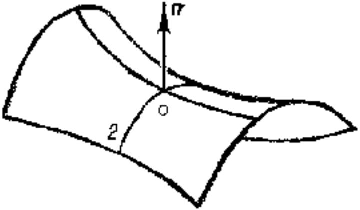
In particular, understand the set of metrics on a surface with curvature -1.



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Guiding principle: Dynamics on (co)tangent bundle should have analogies with dynamics of the $SL(2, \mathbb{R})$ action on $SL(2, \mathbb{R})/SL(2, \mathbb{Z})$.

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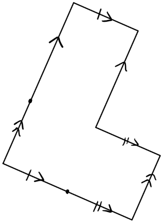

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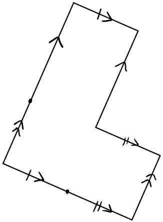

Theorem (Masur's criterion)

If the vertical flow on translation surface is not uniquely ergodic, then the geodesic ray in $\mathcal{M}(\mathcal{S}_g)$ escapes to infinity.

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Guiding principle?

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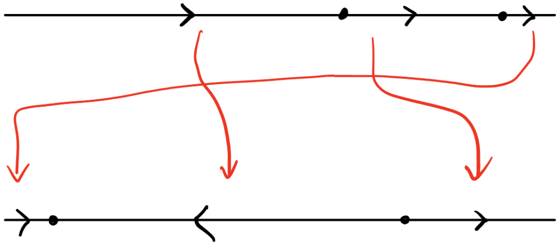

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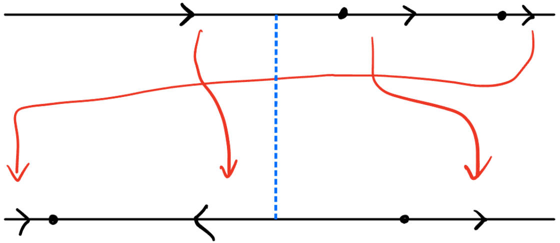

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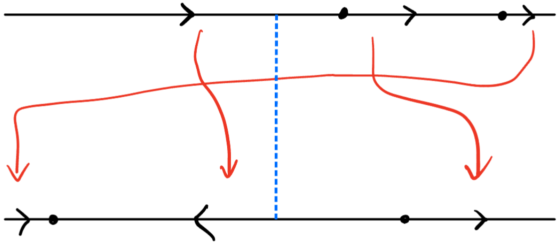

Theorem

Almost every geodesic in $S^1\mathcal{M}(\mathcal{N}_g)$ escapes to infinity.

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Theorem (K., Erlandsson-Gendulphe-Pasquinelli-Souto)

The limit set of the MCG($\mathcal{N}_g$) action on $\mathcal{T}(\mathcal{N}_g)$ is $\mathbb{P}\mathcal{MF}^+(\mathcal{N}_g)$.

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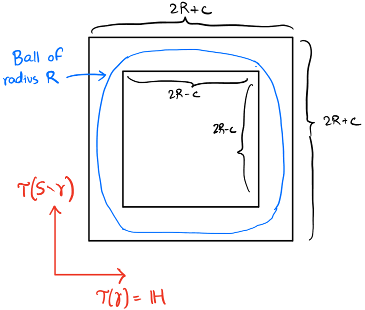

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- Obstruction 2: It's not obvious that the volume growth entropy of $\mathcal{T}^-_{\varepsilon_t}(\mathcal{N}_g)$ is equal to the lattice point growth entropy for the $\mathrm{MCG}(\mathcal{N}_g)$ action.

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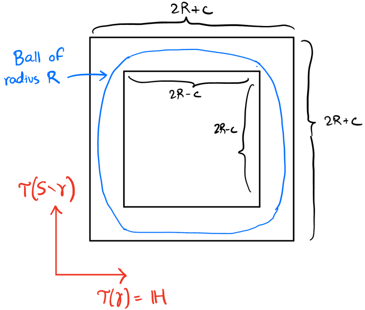

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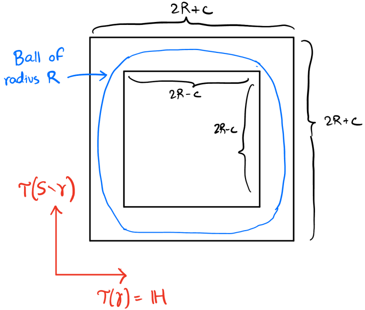

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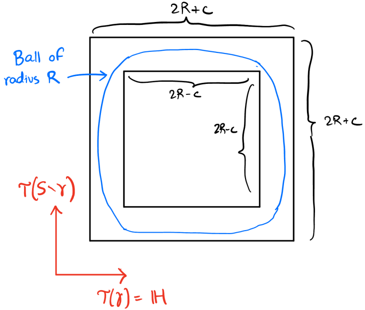

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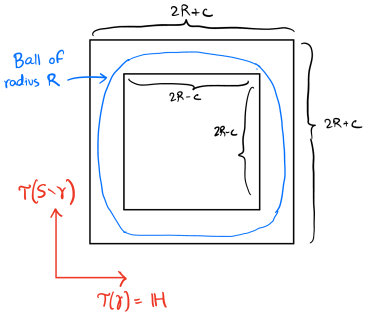

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- For one-sided thin regions, we get a symmetric random walk on $\mathbb{Z}$, which is not strongly recurrent.

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- Technical tool to prove a statement like this was invented recently by Dowdall and Masur: *complexity length*.

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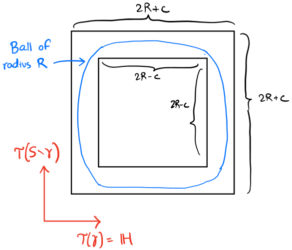

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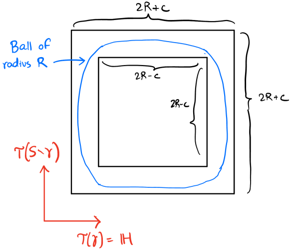

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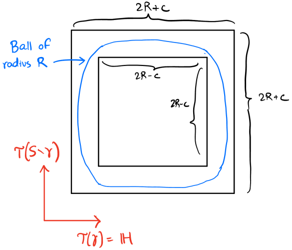

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- Complexity length is based on this idea, but accounts for the fact that the geodesic joining p and n could travel through several product regions while staying in the thin part.

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