

Dynamics on the Moduli Space of Non-Orientable Surfaces

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Orientable surfaces



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Non-orientable surfaces



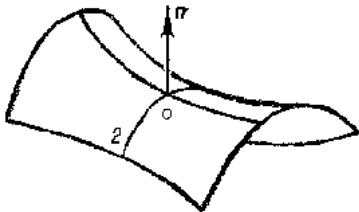
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In particular, understand the set of metrics on a surface with curvature -1 .



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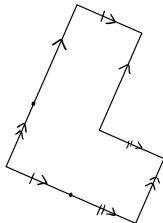
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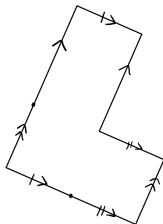
Guiding principle: Dynamics on (co)tangent bundle should have analogies with dynamics of the $SL(2, \mathbb{R})$ action on $SL(2, \mathbb{R})/SL(2, \mathbb{Z})$.

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Theorem (Masur's criterion)

If the vertical flow on translation surface is not uniquely ergodic, then the geodesic ray in $\mathcal{M}(\mathcal{S}_g)$ escapes to infinity.

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Guiding principle?

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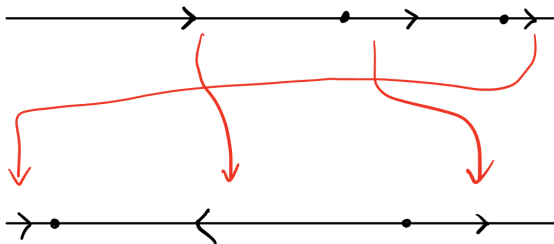
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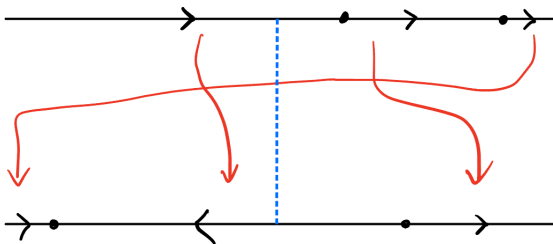
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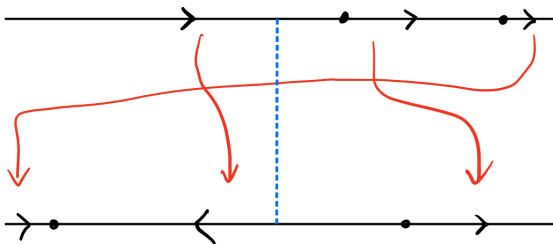
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Theorem

Almost every geodesic in $S^1\mathcal{M}(\mathcal{N}_g)$ escapes to infinity.

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Theorem (K., Erlandsson-Gendulph-Pasquinelli-Souto)

The limit set of the $MCG(\mathcal{N}_g)$ action on $\mathcal{T}(\mathcal{N}_g)$ is $\mathbb{P}\mathcal{MF}^+(\mathcal{N}_g)$.

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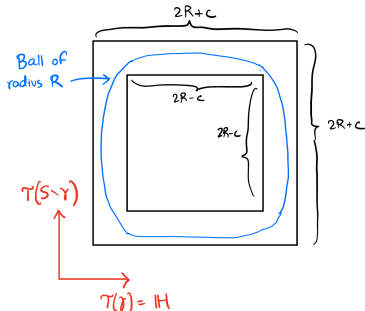
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- ▶ Obstruction 2: It's not obvious that the volume growth entropy of $\mathcal{T}_{\varepsilon_t}^-(\mathcal{N}_g)$ is equal to the lattice point growth entropy for the $\text{MCG}(\mathcal{N}_g)$ action.

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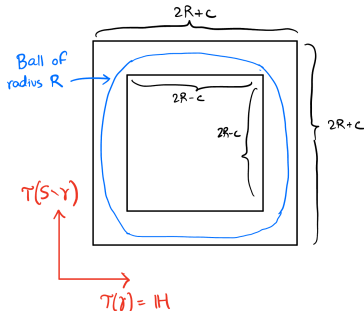
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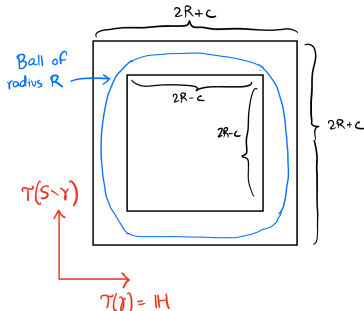
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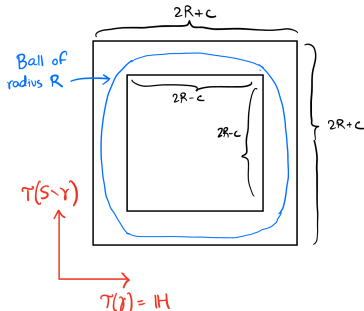
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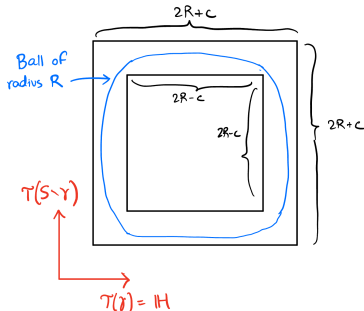
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- ▶ Since the ball is approximately an L^∞ product, picking a point uniformly at random is equivalent to picking a points *independently* on the two coordinates.
- ▶ We can show strong recurrence for random walks on a horoball in \mathbb{H} .
- ▶ For one-sided thin regions, we get a symmetric random walk on \mathbb{Z} , which is not strongly recurrent.

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- ▶ Key idea is based off of Minsky's theorem again: we need to understand how volume grows in product region, and therefore, a horoball.

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