THE LIMIT SET OF NON-ORIENTABLE MAPPING CLASS GROUPS

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Abstract. We provide evidence both for and against a conjectural analogy between geometrically finite infinite covolume Fuchsian groups and the mapping class group of compact non-orientable surfaces. In the positive direction, we show the complement of the limit set is open and dense. Moreover, we show that the limit set of the mapping class group contains the set of uniquely ergodic foliations and is contained in the set of all projective measured foliations not containing any one-sided leaves, establishing large parts of a conjecture of Gendulphe. In the negative direction, we show that the action of the mapping class group on the boundary has an empty wandering set, and the conjectured convex core is not even quasi-convex, in contrast with the geometrically finite setting.

1. Introduction

The moduli space $\mathcal{M}(\mathcal{N}_g)$ of compact *non-orientable* hyperbolic surfaces of genus g is conjectured to have similarities to infinite volume geometrically finite manifolds (in a manner similar to how moduli spaces of compact orientable surfaces have properties similar to finite volume hyperbolic manifolds). The main results suggesting the analogy between moduli spaces of non-orientable surfaces and infinite volume geometrically finite manifolds are due to Norbury and Gendulphe.

- The $\mathcal{M}(\mathcal{N}_g)$ has infinite Teichmüller volume (Theorem 17.1 of [Gen17]). While the associated Teichmüller space does not have a Weil-Peterson volume form, it has an analogous volume form with respect to which the moduli space has infinite volume as well (see [Nor08]).
- The action of the mapping class group $\mathrm{MCG}(\mathcal{N}_g)$ on the Thurston boundary is not minimal (Proposition 8.9 in [Gen17]).
- The Teichmüller geodesic flow is not topologically transitive, and thus not ergodic with respect to any Borel measure with full support (Proposition 17.5 of [Gen17]).
- There exists an $\mathrm{MCG}(\mathcal{N}_g)$-equivariant finite volume deformation retract of $\mathcal{T}(\mathcal{N}_g)$.

We extend this analogy further, by showing that the limit set of $\mathrm{MCG}(\mathcal{N}_g)$ is contained in the complement of a full measure dense open set.

Theorem 4.4. *The limit set of* $\mathrm{MCG}(\mathcal{N}_g)$ *is contained in the complement of* $\mathbb{P}\mathcal{MF}^-(\mathcal{N}_g)$.

Here $\mathbb{P}\mathcal{MF}^-(\mathcal{N}_g)$ is the set of all projective measured foliations that have one-sided compact leaf. The fact that such foliations form a full measure dense open subset is classical, due to Danthony-Nogueira (see [DN90]). This is analogous to limit sets of infinite volume geometrically finite groups, where the complement of the limit set is a full measure open set as well.

Date: Monday 16th August, 2021.
2010 *Mathematics Subject Classification.* 57K20.
Key words and phrases. mapping class group, Teichmüller space, Thurston boundary.

In [Gen17], Gendulphe constructed a retract of $\mathcal{T}(\mathcal{N}_g)$ to $\mathcal{T}_\varepsilon^-$, the set of points in the Teichmüller space that have no one-sided curves shorter than ε, and showed that it has finite volume. They also asked the following question about $\mathcal{T}_\varepsilon^-$.

Question (Question 19.1 of [Gen17])**.** Is $\mathcal{T}_\varepsilon^-$ quasi-convex with respect to the Teichmüller metric?

We show that $\mathcal{T}_\varepsilon^-$ is not quasi-convex, answering the above question.

Theorem 6.2. *For all $\varepsilon > 0$, and all $D > 0$, there exists a Teichmüller geodesic segment whose endpoints lie in $\mathcal{T}_\varepsilon^-$ such that some point in the interior of the geodesic is more than distance D from $\mathcal{T}_\varepsilon^-$.*

We exhibit another failure of the analogy, by showing that $\mathrm{MCG}(\mathcal{N}_g)$ does not act properly discontinuously on the complement of the limit set.

Theorem 5.1. *The wandering set for the $\mathrm{MCG}(\mathcal{N}_g)$ action on $\mathbb{P}\mathcal{MF}(\mathcal{N}_g)$ is empty.*

Contrast this with the action of geometrically finite groups on the complement of the limit set, where the action is properly discontinuous.

The key idea involved in the proof of Theorem 5.1 and Theorem 6.2 is determining which projective measured foliations are in the limit set. Conjecture 9.1 of [Gen17] states that the limit set should exactly be the complement of $\mathbb{P}\mathcal{MF}^-(\mathcal{N}_g)$, the set of projective measured foliations that do not contain any one-sided leaves (denoted $\mathbb{P}\mathcal{MF}^+(\mathcal{N}_g)$). We prove a result that is slightly weaker than the conjecture.

Theorem 3.3. *For $g \geq 3$, a foliation $\lambda \in \mathbb{P}\mathcal{MF}^+(\mathcal{N}_g)$ is in the limit set of $\mathrm{MCG}(\mathcal{N}_g)$ if all the minimal components λ_j of λ satisfy one of the following criteria.*

(i) λ_j is periodic.
(ii) λ_j is orientable, i.e. all leaves exiting one side of a transverse arc always come back from the other side.
(iii) λ_j is uniquely ergodic.

Furthermore, if λ_j is minimal, but not uniquely ergodic, there exists some other foliation λ_j' supported on the same topological foliation as λ_j which is in the limit set.

We now exhibit two contexts in which one is naturally led to consider the limit set of $\mathrm{MCG}(\mathcal{N}_g)$.

Counting simple closed curves. In the orientable setting, the number of simple closed geodesics of length less than L grows like a polynomial of degree $6g-6$, which is precisely the dimension of the limit set of the mapping class group: in this case, that happens to be all of $\mathbb{P}\mathcal{MF}$. In the non-orientable setting, Gendulphe showed that the growth rate of the simple closed geodesics of length less than L is smaller than $L^{\dim(\mathbb{P}\mathcal{MF}(\mathcal{N}_g))}$, and one might conjecture that the growth rate is L^h, where h is the Hausdorff dimension of the limit set. Mirzakhani, in [Mir08], obtained precise asymptotics for the counting function in the orientable case by essentially proving equidistribution (with respect to Thurston measure) of $\mathrm{MCG}(\mathcal{S}_g)$-orbits in $\mathbb{P}\mathcal{MF}(\mathcal{S}_g)$. To do the same for the non-orientable case, we need an ergodic measure supported on sets minimal with respect to the MCG action, e.g. $\overline{\mathbb{P}\mathcal{MF}^+(\mathcal{N}_d)}$. One way to construct such a measure would be to replicate the construction of Patterson-Sullivan measures for geometrically finite manifolds, which brings us back to the analogy between $\mathcal{M}(\mathcal{N}_d)$ and infinite volume geometrically finite manifolds.

Interval exchange transformations with flips. Teichmüller spaces of non-orientable surfaces also show up in the context of *interval exchange transformations with flips*. The dynamics of interval exchange transformations are closely related to the dynamics of horizontal/vertical flow on an associated quadratic differential, which is related to the geodesic flow on the Teichmüller surface via Masur's criterion (a version of which holds in the non-orientable setting as well). IETs with flips do not have very good recurrence properties: in fact, almost all of them (with respect to the Lebesgue measure) have a periodic point (see [Nog89]) and the set of minimal IETs with flips have a lower Hausdorff dimension (see [ST18]). To understand the IETs which are uniquely ergodic, one is naturally led to determine which "quadratic differentials" on non-orientable surfaces are recurrent. A necessary but not sufficient condition for recurrence of a Teichmüller geodesic is that its forward and backward limit points lie in the limit set. From this perspective, Theorems 3.3 and 4.4 can be seen as a statement about the closure of the recurrent set. Constructing a measure supported on the closure of the recurrent set can be then used to answer questions about uniquely ergodic IETs with flips.

Organization of the paper. Section 2.1 contains the background on non-orientable surfaces and measured foliations, and section 2.2 contains the background on limit sets of mapping class subgroups. These sections can be skipped and later referred to if some notation or definition is unclear. Section 3 contains the proof of Theorem 3.3, section 4 contains the proof of Theorem 4.4, section 5 contains the proof of Theorem 5.1, and section 6 contains the proof of Theorem 6.2. Sections 3, 4, and 6 are independent of each other, and can be read in any order, but section 5 relies on the notation and results from section 3.

2. Background

2.1. Non-orientable surfaces and measured foliations.

For the purposes of this paper, the most convenient way to think about non-orientable surfaces will be to attach *crosscaps* to orientable surfaces. Given a surface S, attaching a crosscap is the operation of deleting the interior of a small embedded disc, and gluing the boundary S^1 via the antipodal map. Attaching k crosscaps to a genus g surface results in a genus $2g+k$ non-orientable surface $\mathcal{N}_{2g+k}$ (i.e. the non-orientable surface obtained by taking the connect sum of $2g+k$ copies of $\mathbb{RP}^2$). Associated to each cross cap is a one-sided curve, which is the image of the boundary under the quotient map. We say that a curve intersects the crosscap if it intersects the associated one-sided curve.

Consider the set $\mathscr{S}$ of simple closed curves on a non-orientable surface $\mathcal{N}$. The elements of $\mathscr{S}$ can be classified into two types.

Two sided curves: Tubular neighbourhoods are cylinders.
One sided curves: Tubular neighbourhoods are Möbius bands.

The subset of two sided curves in denoted by $\mathscr{S}^+$ and one sided curves by $\mathscr{S}^-$. Since these two types are topologically distinct, they form invariant subspaces with respect to the mapping class group action. If we think of our non-orientable surface as an orientable subsurface with crosscaps attached, a two-sided curve is one that intersects an even number of crosscaps, and a one-sided curve is one that intersects an odd number of crosscaps.

The orientable double cover of $\mathcal{N}_g$ is the orientable surface $\mathcal{S}_{g-1}$, and comes with an orientation reversing involution ι. Since this is an orientation double cover, the subgroup of

$\pi_1(\mathcal{N}_g)$ corresponding to this cover is characteristic, i.e. left invariant by every homeomorphism induced automorphism of the fundamental group. A useful consequence of this fact is that one can lift mapping classes uniquely.

Fact. Any self homeomorphism of $\mathcal{N}_g$ lifts to a unique orientation preserving self homeomorphism of $\mathcal{S}_{g-1}$, and as a consequence, one has the injective homomorphism induced by the covering map p.

$$p^* : \mathrm{MCG}(\mathcal{N}_d) \hookrightarrow \mathrm{MCG}^+(\mathcal{S}_{d-1})$$

Furthermore, this inclusion preserves the mapping class type, i.e. finite order, reducible and pseudo-Anosov maps in $\mathrm{MCG}(\mathcal{N}_g)$ stay finite order, reducible, and pseudo-Anosov in $\mathrm{MCG}(\mathcal{S}_{g-1})$.

One also obtains a map from $\mathcal{T}(\mathcal{N}_g)$ to $\mathcal{T}(\mathcal{S}_{g-1})$ using the fact that mapping classes can be lifted canonically. Given a point (p, φ) in $\mathcal{T}(\mathcal{N}_g)$, where p is a hyperbolic surface homeomorphic to $\mathcal{N}_g$, and φ is an isotopy class of homeomorphism from $\mathcal{N}_g$ to p, we define the image of (p, φ) in $\mathcal{T}(\mathcal{S}_{g-1})$ to be $(\widetilde{p}, \widetilde{\varphi})$, where $\widetilde{p}$ is the orientation double cover of p, and $\widetilde{\varphi}$ is the orientation preserving lift of the homeomorphism φ. One can also explicitly describe the image of this map. To do so, we consider the extended Teichmüller space of $\mathcal{S}_{g-1}$, i.e. also allowing orientation reversing markings. This space has two connected components, one for each orientation, and there is a canonical involution, given by reversing the orientation, that exchanges the two connected components. We denote this conjugation map by $\overline{\cdot}$. There is another involution, induced by the orientation reversing deck transformation of $\mathcal{S}_{g-1}$, which we denote by ι^*. This map also exchanges the two components of the extended Teichmüller space. The image of $\mathcal{T}(\mathcal{N}_g)$ is precisely the set of points fixed by the composition of these two maps, i.e. $\overline{\iota^*}$. We skip the proof of these two facts, since they follow by relatively elementary covering space arguments, and summarize the result in the following theorem.

Theorem 2.1 (Embedding Teichmüller spaces)**.** *Given a point (p, φ) in $\mathcal{T}(\mathcal{N}_g)$, there is a unique point $(\widetilde{p}, \widetilde{\varphi})$ in $\mathcal{T}(\mathcal{S}_{g-1})$, where $\widetilde{p}$ is the pullback of the metric, and $\widetilde{\varphi}$ is the unique orientation preserving lift of the marking. The image of the inclusion map is the intersection of the invariant set of $\overline{\iota^*}$ with the connected component of the extended Teichmüller space corresponding to orientation preserving maps.*

It turns out that the image of $\mathcal{T}(\mathcal{N}_g)$ in $\mathcal{T}(\mathcal{S}_{g-1})$ is an isometrically embedded submanifold, and the geodesic flow can be represented by the action of the diagonal subgroup of $SL(2, \mathbb{R})$.

To understand the Teichmüller geodesic flow on $\mathcal{T}(\mathcal{N}_g)$, we need to determine what the cotangent vectors look like: let X be a point in $\mathcal{T}(\mathcal{N}_g)$ and let $\widetilde{X}$ be the corresponding point in $\mathcal{T}(\mathcal{S}_{g-1})$. Then the map on the extended Teichmüller space induced by the orientation reversing deck transformation maps $\widetilde{X}$ to $\overline{\widetilde{X}}$, i.e. the conjugate Riemann surface. Following that with the canonical conjugation map brings us back to $\widetilde{X}$. Let q be a cotangent vector at $\overline{\widetilde{X}}$, i.e. an anti-holomorphic quadratic differential on the Riemann surface $\overline{\widetilde{X}}$. Pulling back q along the canonical conjugation map gives a holomorphic quadratic differential on $\underline{X}$. In local coordinate chart on $\widetilde{X}$, this looks like $q(z)dz^2$ if on the corresponding chart on $\overline{\widetilde{X}}$ it looked like $q(\overline{z})dz^2$. We want this to equal $\iota^* q$, which will also be a holomorphic quadratic differential on $\widetilde{X}$. If that happens, then $\iota^* q$ is a cotangent vector to the point X in $\mathcal{T}(\mathcal{N}_g)$.

Example 2.2 (A cotangent vector to a point in $\mathcal{T}(\mathcal{N}_3)$)**.** Consider the quadratic differential q on a genus two Riemann surface pictured in Figure 1.

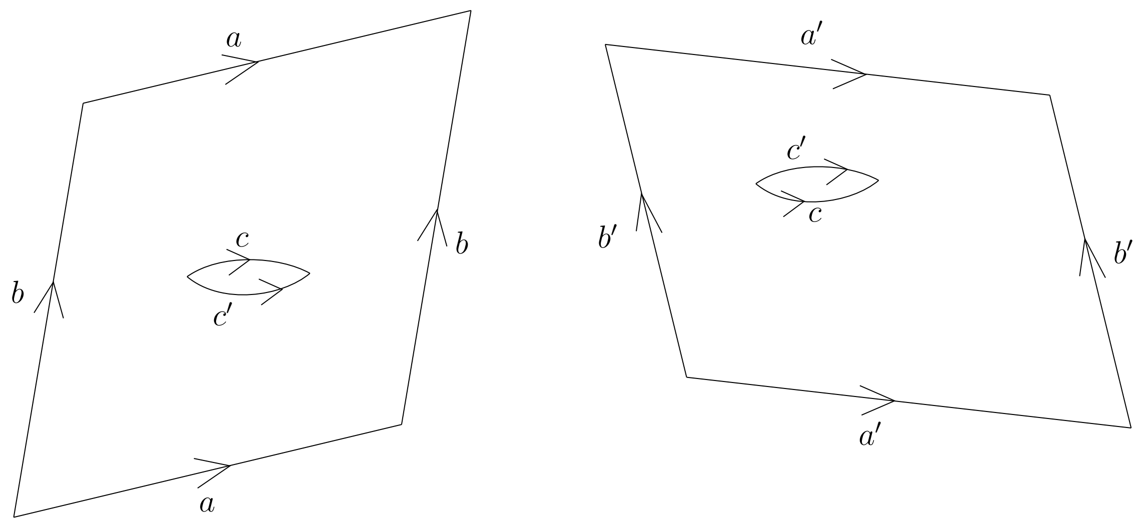


FIGURE 1. A quadratic differential q on $\mathcal{S}_2$ given by the slit torus construction.

Observe that this particular quadratic differential is the global square of an abelian differential, so it makes sense to talk about the pairing between $\sqrt{q}$ and the homology classes $\{a, a', b, b', c, c'\}$. Recall that the action of a mapping class like ι is merely relabelling homology classes: in this case ι swaps a with $-a'$, b with b', and c with $-c'$. That gives us the following expressions involving $\sqrt{q}$.

$$\langle \iota^* \sqrt{q}, a \rangle = \langle \sqrt{q}, -a' \rangle \tag{1}$$

$$\langle \iota^* \sqrt{q}, b \rangle = \langle \sqrt{q}, b' \rangle \tag{2}$$

$$\langle \iota^* \sqrt{q}, c \rangle = \langle \sqrt{q}, -c' \rangle \tag{3}$$

On the other hand, the conjugation action conjugates the complex value of each pairing.

$$\langle \overline{\sqrt{q}}, a \rangle = \overline{\langle \sqrt{q}, a \rangle} \tag{4}$$

$$\langle \overline{\sqrt{q}}, b \rangle = \overline{\langle \sqrt{q}, b \rangle} \tag{5}$$

$$\langle \overline{\sqrt{q}}, c \rangle = \overline{\langle \sqrt{q}, c \rangle} \tag{6}$$

For q to be invariant under $\bar{\iota}$, both of the above set of equations must be satisfied, which imposes certain conditions on q. For instance, if the complex lengths of a and a' to be conjugates of each other, the complex lengths of b and b' to be negative conjugates of each other, and forces the complex length of c and c' to be real. Only the quadratic differentials satisfying these constraints will be the cotangent vectors to points in the image of $\mathcal{T}(\mathcal{N}_3)$.

To realize the quadratic differential directly as an object on $\mathcal{N}_3$, we can quotient out the flat surface given by q by the orientation reversing deck transformation. Doing that for our example gives the non-orientable flat surface gives the picture seen in Figure 2.

This example suggests what the right definition of a quadratic differential on a non-orientable surface ought to be: in the flat picture, rather than allowing gluing via just the maps $z \mapsto \pm z + c$, we also allow $z \mapsto \pm \overline{z} + c$. This leads to the definition of *dianalytic quadratic differentials* (which we'll abbreviate to DQDs).

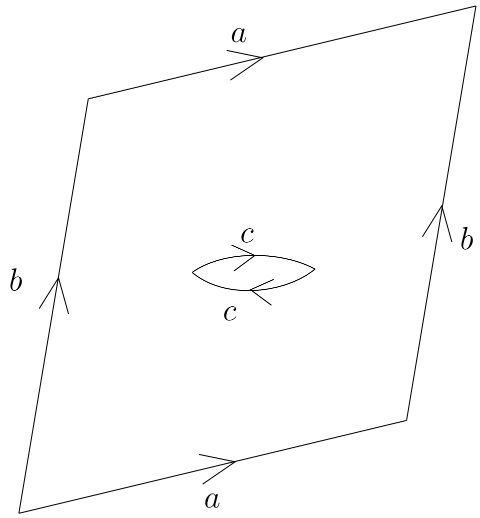


FIGURE 2. A quadratic differential on $\mathcal{N}_3$.

Definition 2.3 (Dianalytic quadratic differential (adapted from [Wri15]))**.** A dianalytic quadratic differential is the quotient of a collection of polygons in $\mathbb{C}$, modulo certain equivalences. The quotienting satisfies the following conditions.

(1) The interiors of the polygons are disjoint.
(2) Each edge is identified with exactly one other edge, and the mapping must be of one of the following four forms: $z \mapsto z + c$, $z \mapsto -z + c$, $z \mapsto \overline{z} + c$, or $z \mapsto -\overline{z} + c$.
(3) Extending the edge identification map to a small enough open neighbourhood of a point on the edge should not map it to an open neighbourhood of the image of the point: in other words, it should get mapped to the "other side" of the edge.

Two such quotiented collections of polygons are considered the same if they differ by a composition of the following moves.

(1) A polygon may be translated, rotated by π radians, or reflected across the real or imaginary axis.
(2) A polygon may be cut along a straight line to form two polygons, or two polygons sharing an edge may be glued together to form a single polygon.

Given a DQD, we can pull it back to the orientation double cover, getting an actual quadratic differential: this operation corresponds to identifying a cotangent vector to a point in $\mathcal{T}(\mathcal{N}_g)$ to the corresponding cotangent vector in $\mathcal{T}(\mathcal{S}_{g-1})$.

To verify that $\mathcal{T}(\mathcal{N}_g)$ is isometrically embedded, all we need to do is verify that the Teichmüller geodesic flow takes the quadratic differentials satisfying the symmetry condition $\iota^*(q) = \overline{q}$ to quadratic differentials that satisfy the symmetry conditions.

Lemma 2.4. *If q satisfies $\iota^*(q) = \overline{q}$, then for any t, $\iota^*(g_t q) = \overline{g_t q}$.*

Proof. Recall that if q satisfies the given condition, we must have the following hold for any homology class a.

$$\langle \sqrt{q}, \iota(a) \rangle = \overline{\langle \sqrt{q}, a \rangle} \tag{7}$$

If q is not the global square of an abelian differential, we may have to pass to the holonomy double cover. Observe now what g_t does to q.

$$\langle \sqrt{g_t q}, \iota(a) \rangle = e^t \mathrm{Re} \langle \sqrt{q}, \iota(a) \rangle + i e^{-t} \mathrm{Im} \langle \sqrt{q}, \iota(a) \rangle \tag{8}$$

Using (7), we simplify (8) to the following.

$$\langle \sqrt{g_t q}, \iota(a) \rangle = e^t \mathrm{Re} \langle \sqrt{q}, a \rangle - i e^{-t} \mathrm{Im} \langle \sqrt{q}, a \rangle \tag{9}$$

$$= \overline{\langle \sqrt{g_t q}, a \rangle} \tag{10}$$

This proves the lemma. □

Remark. The key idea that diagonal matrices commute: the conjugation action is really multiplication by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ which happens to commute with the diagonal matrices of determinant 1, which are exactly the matrices corresponding to geodesic flow. On the other hand, the conjugation matrix does not commute with the horocycle flow matrices, and that shows that the horocycle flow is not well defined on the cotangent bundle of $\mathcal{T}(\mathcal{N}_g)$.

Lemma 2.4 shows that the Teichmüller geodesic flow for the cotangent bundle of $\mathcal{T}(\mathcal{N}_g)$ is the restriction of the geodesic flow for the ambient space $\mathcal{T}(\mathcal{S}_{g-1})$.

Theorem 2.1 gives us an alternative perspective into the action of $\mathrm{MCG}(\mathcal{N}_g)$ on $\mathcal{T}(\mathcal{N}_g)$. $\mathrm{MCG}(\mathcal{N}_g)$ can be thought of as the subgroup of $\mathrm{MCG}(\mathcal{S}_{g-1})$ that stabilizes a totally real isometrically embedded submanifold $\mathcal{T}(\mathcal{N}_g)$. With this perspective, $\mathrm{MCG}(\mathcal{N}_g)$ can be thought of as the higher dimensional generalization of the subgroups obtained by stabilizing Teichmüller discs, i.e. Veech groups.

We now state a few classical results about measured foliations on non-orientable surfaces that show why the theory diverges significantly from the orientable case.

A measured foliation on a non-orientable surface $\mathcal{N}_g$ is singular foliation along with an associated transverse measure, up to equivalence by Whitehead moves. Any leaf of a measured foliation can either be non-compact or compact: in the former case, the closure of the non-compact leaf fills out a subsurface. Restricted to the subsurface given by the closure of a non-compact leaf, the foliation is minimal, i.e. the orbit of every point under the flow given by the foliation is dense. For a compact leaf, there are two possibilities for the topology of the subsurface containing it: if the closed leaf is the core curve or the boundary curve of an embedded Möbius strip, then the subsurface is the maximal neighbourhood of the periodic leaf that is foliated by periodic leaves as well, and this turns out to be an embedded Möbius strip. If the compact leaf is not the core curve or the boundary curve of an embedded Möbius strip, then it is the core curve of an embedded cylinder, and the maximal neighbourhood of the periodic leaf foliated by periodic leaf is an embedded cylinder. The identification of leaves with associated subsurfaces lets us decompose a measured foliation into its minimal components. Note the slightly confusing terminology: when the minimal component is a Möbius strip or a cylinder, then the foliation restricted to the component is not minimal, but when the minimal component has higher genus, then the foliation restricted to that component indeed is minimal.

We denote the set of measured foliations on $\mathcal{N}_g$ by $\mathcal{MF}(\mathcal{N}_g)$, the set of foliations whose minimal components do not contain a Möbius strip by $\mathcal{MF}^+(\mathcal{N}_g)$, and the set of foliations whose minimal components contain at least one Möbius strip by $\mathcal{MF}^-(\mathcal{N}_g)$. Via the standard identification between simple closed curves and measured foliations, we can associate $\mathbb{Q}$-weighted two-sided multicurves on $\mathcal{N}_g$ to a subset of $\mathcal{MF}^+(\mathcal{N}_g)$, denoted by $\mathcal{MF}^+(\mathcal{N}_g, \mathbb{Q})$.

Quotienting out $\mathcal{MF}(\mathcal{N}_g)$ by the $\mathbb{R}_+$-action given by scaling the transverse measure gives us the set of projective measured foliations $\mathbb{P}\mathcal{MF}(\mathcal{N}_g)$. The subsets $\mathcal{MF}^-(\mathcal{N}_g)$, $\mathcal{MF}^+(\mathcal{N}_g)$, and $\mathcal{MF}^+(\mathcal{N}_g, \mathbb{Q})$ are $\mathbb{R}$-invariant, and thus descend to their projective versions $\mathbb{P}\mathcal{MF}^-(\mathcal{N}_g)$, $\mathbb{P}\mathcal{MF}^+(\mathcal{N}_g)$, and $\mathbb{P}\mathcal{MF}^+(\mathcal{N}_g, \mathbb{Q})$. The set $\mathbb{P}\mathcal{MF}(\mathcal{N}_g)$ is the boundary of the Teichmüller space of $\mathcal{N}_g$, and admits a continuous mapping class group action. It is when considering the mapping class group action that we see differences between the orientable and the non-orientable case.

Theorem 2.5 (Proposition 8.9 of [Gen17])**.** *The action of* $\mathrm{MCG}(\mathcal{N}_g)$ *(for* $g \geq 2$*) on* $\mathcal{MF}(\mathcal{N}_g)$ *is not minimal. In fact, the action is not even topologically transitive.*

Remark. The proof of non minimality and topological non-transitivity follow from the fact that one can construct a $\mathrm{MCG}(\mathcal{N}_g)$-invariant continuous function on $\mathcal{MF}(\mathcal{N}_g)$. That is because starting with a foliation in $\mathcal{MF}^+(\mathcal{N}_g)$, it is impossible to approximate an element of $\mathcal{MF}^-(\mathcal{N}_g)$ since one does not have Dehn twists about one-sided curves.

One can now consider subspaces of $\mathcal{MF}(\mathcal{N}_g)$ where the $\mathrm{MCG}(\mathcal{N}_g)$ action might be nicer. There are two natural subspaces: $\mathcal{MF}^+(\mathcal{N}_g)$, and $\mathcal{MF}^-(\mathcal{N}_g)$. Danthony-Nogueira proved the following theorem about $\mathcal{MF}^-(\mathcal{N}_g)$ in [DN90].

Theorem 2.6 (Theorem II of [DN90])**.** $\mathcal{MF}^-(\mathcal{N}_g)$ *is an open dense subset of* $\mathcal{MF}(\mathcal{N}_g)$ *of full Thurston measure.*

Theorem 2.5 and Theorem 2.6 suggest that studying the $\mathrm{MCG}(\mathcal{N}_g)$ dynamics restricted to $\mathcal{MF}^-(\mathcal{N}_g)$ will be hard since one will not have minimality, or ergodicity with respect to any measure with full support. In Section 3, we get a lower bound for the set on which $\mathrm{MCG}(\mathcal{N}_g)$ acts minimally.

2.2. **Limit sets of mapping class groups.** The first results on limit sets of subgroups of mapping class groups were obtained by Masur for handlebody subgroups [Mas86], and McCarthy-Papadopoulos for general mapping class subgroups [MP89]. They defined two distinct notions of limit sets; while they did not give distinct names to the two different definitions, we will do so, for the sake of clarity.

Definition 2.7 (Dynamical limit set)**.** Given a subgroup Γ of the mapping class group, the dynamical limit set $\Lambda_{\mathrm{dyn}}(\Gamma)$ is the minimal closed invariant subset of $\mathbb{P}\mathcal{MF}$ under the action of Γ.

Definition 2.8 (Geometric limit set)**.** Given a subgroup Γ of the mapping class group, and a point x in the Teichmüller space, its boundary orbit closure $\Lambda_{\mathrm{geo},x}(\Gamma)$ is intersection of its orbit closure with the Thurston boundary, i.e. $\overline{\Gamma x} \cap \mathbb{P}\mathcal{MF}$. The geometric limit set is the union of all boundary orbit closures, as we vary x in the Teichmüller space, i.e. $\Lambda_{\mathrm{geo}}(\Gamma) = \bigcup_{x \in \mathcal{T}} \Lambda_{\mathrm{geo},x}(\Gamma)$.

Remark. The specific family of subgroups considered by McCarthy-Papadopoulos were subgroups containing at least two non-commuting pseudo-Anosov mapping classes, in which case, the dynamical limit set is unique. Unless otherwise specified, we will only talk about mapping class subgroups which contain at least two non-commuting pseudo-Anosovs.

Both of these definitions are natural generalizations of the limit sets of Fuchsian groups acting on $\mathbb{H}^2$. In the hyperbolic setting, the two notions coincide, but for mapping class subgroups, the dynamical limit set may be a proper subset of the geometric limit set.

For simple enough subgroups, one can explicitly work out $\Lambda_{\text{dyn}}(\Gamma)$ and $\Lambda_{\text{geo}}(\Gamma)$: for instance, when Γ is the stabilizer of the Teichmüller disc associated to a Veech surface, $\Lambda_{\text{dyn}}(\Gamma)$ is the visual boundary of the Teichmüller disc, which by Veech dichotomy, only consists of either uniquely ergodic directions on the Veech surface, or the cylinder directions, where the coefficients on the cylinders are their moduli in the surface. On the other hand, $\Lambda_{\text{geo}}(\Gamma)$ consists of all the points in $\Lambda_{\text{dyn}}(\Gamma)$, but it additionally contains all possible convex combinations of the cylinders appearing in $\Lambda_{\text{dyn}}(\Gamma)$ (see Section 2.1 of [KL07]).

The gap between Λ_{geo} and Λ_{dyn} suggests the following saturation operation on subsets of $\mathbb{P}\mathcal{MF}$

Definition 2.9 (Saturation)**.** Given a projective measured foliation λ, we define its saturation $\text{Sat}(\lambda)$ to be the image in $\mathbb{P}\mathcal{MF}$ of set of all non-zero measures invariant measures on the topological foliation associated to λ. Given a subset Λ, we define its saturation $\text{Sat}(\Lambda)$ to be the union of saturations of the projective measured laminations contained in Λ.

Observe that for a uniquely ergodic foliation λ, $\text{Sat}(\lambda) = \{\lambda\}$, for a minimal but not uniquely ergodic λ, $\text{Sat}(\lambda)$ is the convex hull of all the ergodic measures supported on the topological lamination associated to λ, and for a foliation with all periodic leaves, $\text{Sat}(\lambda)$ consists of all foliations that can be obtained by assigning various weights to the core curves of the cylinders.

Going back to the example of the stabilizer of the Teichmüller disc of a Veech surface, we see that $\Lambda_{\text{geo}}(\Gamma) = \text{Sat}(\Lambda_{\text{dyn}}(\Gamma))$. One may ask if this is always the case.

Question 2.10. Is $\Lambda_{\text{geo}}(\Gamma) = \text{Sat}(\Lambda_{\text{dyn}}(\Gamma))$ for all Γ?

McCarthy-Papadopoulos also formulated an equivalent definition of $\Lambda_{\text{dyn}}(\Gamma)$, which is easier to work with in practice.

Theorem (Theorem 4.1 of [MP89])**.** *$\Lambda_{\text{dyn}}(\Gamma)$ is the closure in $\mathbb{P}\mathcal{MF}$ of the stable and unstable foliations of all the pseudo-Anosov mapping classes in Γ.*

A corollary of this theorem is that Γ does not act properly discontinuously on $\Lambda_{\text{dyn}}(\Gamma)$. In fact, one can construct a wandering set for the Γ action using $\Lambda_{\text{dyn}}(\Gamma)$.

Definition 2.11 (Zero intersection set)**.** Given a subset Λ of $\mathbb{P}\mathcal{MF}$, its zero intersection set $Z(\Lambda)$ is the following subset of $\mathbb{P}\mathcal{MF}$

$$Z(\Lambda) \coloneqq \{\lambda' \mid i(\lambda, \lambda') = 0 \text{ for some } \lambda \in \Lambda\}$$

Theorem (Theorem 6.16 of [MP89])**.** *The action of Γ on $\mathbb{P}\mathcal{MF} \setminus Z(\Lambda_{\text{dyn}}(\Gamma))$ is properly discontinuous.*

It is not obvious that the action of Γ on $Z(\Lambda_{\text{dyn}}(\Gamma))$ is not properly discontinuous. We prove that is indeed the case when $\Gamma = \text{MCG}(\mathcal{N}_g)$ in Section 5, but the general case is unknown.

List of notation. Here we describe some of the more commonly used symbols in the paper.

- $\mathcal{S}_g$: The compact orientable surface of genus g.
- $\mathcal{N}_g$: The compact non-orientable surface of genus g.
- ι: The deck transformation of the orientation double cover of a non-orientable surface.
- $\mathcal{T}(S)$: The Teichmüller space of S.
- $\mathcal{T}_\varepsilon^-(\mathcal{N}_d)$: The set of points in $\mathcal{T}(\mathcal{N}_d)$ where no one-sided curve is shorter than ε.

MCG(S): The mapping class group of S.
$\mathcal{MF}(S)$: The space of measured foliations on S.
$\mathbb{P}\mathcal{MF}(S)$: The space of projective measured foliations on S.
$\mathcal{MF}^+(\mathcal{N}_d)$, $\mathbb{P}\mathcal{MF}^+(\mathcal{N}_d)$: The set of (projective) measured foliations on $\mathcal{N}_d$ containing no one-sided leaves.
$\mathcal{MF}^-(\mathcal{N}_d)$, $\mathbb{P}\mathcal{MF}^-(\mathcal{N}_d)$: The set of (projective) measured foliations on $\mathcal{N}_d$ containing some one-sided leaf.
$\mathcal{MF}(S;\mathbb{Q})$, $\mathbb{P}\mathcal{MF}(S;\mathbb{Q})$: The set of all (projective) weighted rational multicurves on S.
$\Lambda_{\mathrm{geo}}(\Lambda)$: The geometric limit set of the discrete group Λ.
$\Lambda_{\mathrm{dyn}}(\Lambda)$: The dynamical limit set of the discrete group Λ.
$\ell_{\mathrm{hyp}}(M,\gamma)$: The hyperbolic length of γ with respect to the hyperbolic structure on $M \in \mathcal{T}(S)$. We will suppress M when it is clear from context.
$\ell_{\mathrm{flat}}(q,\gamma)$: The flat length of γ with respect to the flat structure given by the DQD q. We will suppress q when it is clear from context.
$o(f)$: A function g is $o(f)$ if $\frac{g}{f}$ goes to 0 as the value of f goes to ∞.
μ_c: The probability measure on a transverse arc given by the closed curve c.

3. Lower bound for the limit set

A natural lower bound for $\Lambda_{\mathrm{dyn}}(\mathcal{N}_g)$ is the closure of the set of rational two-sided multicurves $\mathbb{P}\mathcal{MF}^+(\mathcal{N}_g,\mathbb{Q})$. For any $\lambda \in \mathbb{P}\mathcal{MF}^+(\mathcal{N}_g,\mathbb{Q})$, and any psuedo-Anosov γ, conjugating γ with large enough powers of the Dehn multi-twist given by λ gives us a sequence of pseudo-Anosov maps whose stable foliation approaches λ, which shows that $\Lambda_{\mathrm{dyn}}(\mathcal{N}_g)$ must contain λ. Note that the same argument does not work if $\lambda \in \mathbb{P}\mathcal{MF}^-(\mathcal{N}_g,\mathbb{Q})$, since one cannot Dehn twist about one-sided curves. In Section 4, we show that the geometric limit set is indeed contained in the complement of $\mathbb{P}\mathcal{MF}^-(\mathcal{N}_g)$.

In [Gen17], Gendulphe made the following conjecture about $\overline{\mathbb{P}\mathcal{MF}^+(\mathcal{N}_g,\mathbb{Q})}$.

Conjecture 3.1 (Conjecture 9.1 of [Gen17])**.** *For $g \geq 4$, $\mathbb{P}\mathcal{MF}^+(\mathcal{N}_g) = \overline{\mathbb{P}\mathcal{MF}^+(\mathcal{N}_g,\mathbb{Q})}$.*

We prove a slightly weaker version of the above conjecture, by describing a subset of the foliations that can be approximated by multicurves in $\mathbb{P}\mathcal{MF}^+(\mathcal{N}_g,\mathbb{Q})$. To state the theorem, we need to define what it means for a minimal foliation to be orientable.

Definition 3.2 (Orientable foliation)**.** A local orientation on a foliation is the choice of a constant tangent direction on the leaves in a small open set. If the local orientation can be extended to an entire minimal foliation, the foliation is said to be orientable.

In the setting of orientable surfaces, the vertical foliations of translation surfaces are orientable, while there are some directions in half-translation surfaces where the foliation is non-orientable. There exist similar examples of orientable and non-orientable foliations on non-orientable surfaces.

Having defined the notion of orientable foliations, we can state the main theorem of this section.

Theorem 3.3. *For $g \geq 3$, a foliation $\lambda \in \mathbb{P}\mathcal{MF}^+(\mathcal{N}_g)$ can be approximated by foliations in $\mathbb{P}\mathcal{MF}^+(\mathcal{N}_g,\mathbb{Q})$ if all the minimal components λ_j of λ satisfy one of the following criteria.*

(i) λ_j is periodic.

(ii) λ_j *is orientable.*
(iii) λ_j *is uniquely ergodic.*

Furthermore, if λ_j *is minimal, but not uniquely ergodic, there exists some other foliation* λ'_j *supported on the same topological foliation as* λ_j *that can be approximated by elements of* $\mathbb{P}\mathcal{MF}^+(\mathcal{N}_g, \mathbb{Q})$*.*

To prove Theorem 3.3, we will first need to describe a method of resolving intersections of curves. Given two two-sided simple curves γ_1 and γ_2, their union will not be a simple curve if $i(\gamma_1, \gamma_2) > 0$. However, at each point of intersection, there exist two possible surgeries that resolve the intersection. Resolving all the intersections by choosing surgeries at each of the points results in a simple multicurve (i.e. it may have more than one component). It is not necessarily the case that the resulting multicurve has two-sided components either. However, if it is the case that the resulting multicurve is a curve, then it will be a two-sided curve: that is because both γ_1 and γ_2 pass through an even number of crosscaps, and if the resolution of $\gamma_1 \cup \gamma_2$ has only one component, it must also pass through an even number of crosscaps (see Section 2 to recall what it means for a curve to pass through a crosscap). We prove that there always exists a choice of surgery that results in a simple curve (we skip the proof of this fact since it is fairly straightforward).

Lemma 3.4. *Let* γ_1 *and* γ_2 *be two-sided curves on* $\mathcal{N}_g$ *with* $i(\gamma_1, \gamma_2) > 0$*. There exists a resolution of all the intersections of* γ_1 *and* γ_2 *such that the resulting simple multicurve contains only one component.*

Call the resulting simple curve κ: for all that follows, we will pick an arbitrary hyperbolic metric on our surface. We want the geodesic representative of κ to have approximately the same intersection numbers with all curves as the $\gamma_1 \cup \gamma_2$ did. For that to happen, it is important that the geodesic tightening of κ stays within a bounded distance of long sub-arcs of γ_1 and γ_2. We formally state this as a lemma, and sketch a proof.

Lemma 3.5. *Let* $\{\gamma_i\}$ *be a sequence of geodesic arcs on a compact hyperbolic surface* S *such that* $\dfrac{\gamma_i}{\ell_{\mathrm{hyp}}(\gamma_i)}$ *converges to a minimal measured foliation* $\lambda \in \mathcal{MF}(S)$*. Let* $\{\gamma'_i\}$ *be another sequence of geodesic arcs such that for all* i*,* γ_i *and* γ'_i *are within a bounded distance of each other, i.e. the distance is bounded independent of* i*. Then the new sequence* $\dfrac{\gamma'_i}{\ell_{\mathrm{hyp}}(\gamma'_i)}$ *also converges to* λ*.*

Sketch of proof. Let δ be any simple closed curve, and let $\widehat{\delta}$ be the pre-image of δ in the universal cover $\mathbb{H}^2$. Let $\widehat{\gamma_i}$ and $\widehat{\gamma'_i}$ be single lifts of γ_i and γ'_i to $\mathbb{H}^2$. The intersection number of δ and $\dfrac{\gamma_i}{\ell_{\mathrm{hyp}}(\gamma_i)}$ is given by the number of times $\widehat{\gamma_i}$ intersects $\widehat{\delta}$ divided by the length of $\widehat{\gamma_i}$, and $i\left(\delta, \dfrac{\gamma'_i}{\ell_{\mathrm{hyp}}(\gamma'_i)}\right)$ can be computed similarly.

We denote the constant speed parameterization of $\widehat{\gamma_i}$ and $\widehat{\gamma'_i}$ by the same symbols as well: recall that a these are constant speed maps from $[0, 1]$ to $\mathbb{H}^2$. Pick any small $\varepsilon \in (0, 1)$, and consider the parameterized geodesic arcs restricted to $(\varepsilon, 1-\varepsilon)$. For large enough i, if $\widehat{\gamma_i(x)}$ is also in $\widehat{\delta}$, then for some other point x'_i in the neighbourhood of x, $\widehat{\gamma'_i(x')}$ is also in $\widehat{\delta}$ and vice versa (checking this fact requires some elementary hyperbolic geometry which we skip).

Since the lengths of $\widehat{\gamma_i}$ and $\widehat{\gamma_i'}$ go to ∞, and we can pick ε to be arbitrarily small, we have convergence to the same foliation in $\mathcal{MF}(S)$. □

We now state and prove a very general proposition that lets us approximate any convex combination of two foliations supported on the same underlying topological foliation, subject to some technical conditions.

Proposition 3.6. *Let γ_1 and γ_2 be two measured foliations on a compact surface S satisfying the following conditions.*

- *The intersection of γ_1 and γ_2 with the Liouville current is 1.*
- *The intersection number $i(\gamma_1, \gamma_2)$ is equal to 0.*
- *γ_1 and γ_2 are mutually singular.*
- *There exist a sequence of curves converging to γ_1 and γ_2 in $\mathbb{P}\mathcal{MF}(S)$.*

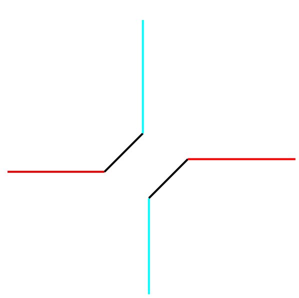
Then for any $c \in (0, \infty)$, there exists a sequence of curves converging to $c\gamma_1 + \gamma_2$ in $\mathcal{MF}(S)$.

Proof. Let $\{\gamma_{1j}\}$ and $\{\gamma_{2j}\}$ be sequences of simple two-sided curves converging to γ_1 and γ_2 in $\mathbb{P}\mathcal{MF}(S)$. Since the intersection number of γ_1 and γ_2 with the Liouville current is 1, we have that $\frac{\gamma_{1j}}{\ell_{\text{hyp}}(\gamma_{1j})}$ approaches γ_1 in $\mathcal{MF}(S)$, and $\frac{\gamma_{2j}}{\ell_{\text{hyp}}(\gamma_{2j})}$ approaches γ_2 in $\mathcal{MF}(S)$. We can assume without loss of generality that $i(\gamma_{1i}, \gamma_{2j'}) > 0$ for all j and j', for if the intersection number is 0 for infinitely many j, the union of the two curves with the appropriate weights will approximate the combination $c\gamma_1 + \gamma_2$. We pass to a subsequence of γ_{2j} such that after passing to the subsequence, the following holds.

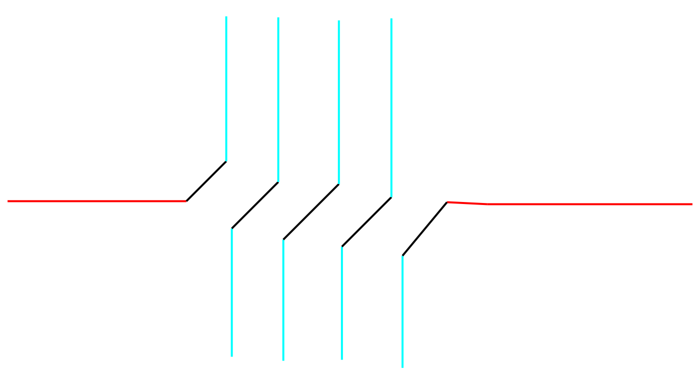
$$\lim_{i\to\infty} \frac{\ell_{\text{hyp}}(\gamma_{2j})}{\ell_{\text{hyp}}(\gamma_{1j})} = \infty \tag{11}$$

We take the union of m_j parallel copies of γ_{1j} and 1 copy of γ_{2j}, where we pick m_j such the fraction $\dfrac{m_j \cdot \ell_{\text{hyp}}(\gamma_{1j})}{\ell_{\text{hyp}}(\gamma_{2j})}$ is as close to c as possible. Equation (11) will ensure that the ratio of the lengths (with multiplicities) will approach c as i goes to ∞.

We now resolve the intersections of the m_j copies of γ_{1j} and γ_{2j} so that the resulting multicurve has only one component. If $m_j = 1$, Lemma 3.4 tells us that there is such a resolution of intersections. If $m_j > 1$, we resolve the intersections the same way as in the case of $m_j = 1$, but also glue together the strands of γ_{1j} as shown in Figure 3.



Resolution when $m_j = 1$



Resolution when $m_j > 1$

FIGURE 3. Modifying the resolution when $m_j > 1$.

Denote the resolved curve by κ_j (note that this is not a geodesic, but rather it follows the geodesics γ_{1j} and γ_{2j} performing a sharp turn every time γ_{1j} and γ_{2j} intersect). We claim that κ_j divided by $m_j \cdot \ell_{\text{hyp}}(\gamma_{1j}) + \ell_{\text{hyp}}(\gamma_{2j})$ converges to $c\gamma_1 + \gamma_2$ in $\mathcal{MF}(S)$. To prove the claim, we need to show that the following limit holds for all simple closed curves ξ.

$$\lim_{j\to\infty} \frac{i(\kappa_j, \xi)}{m_j \cdot \ell_{\text{hyp}}(\gamma_{1j}) + \ell_{\text{hyp}}(\gamma_{2j})} = \lim_{j\to\infty} \left(c\frac{i(\gamma_{1j}, \xi)}{\ell_{\text{hyp}}(\gamma_{1j})} + \frac{i(\gamma_{2j}, \xi)}{\ell_{\text{hyp}}(\gamma_{2j})} \right) \tag{12}$$

The equality replaced with an inequality in one direction is easy to see: namely, the left hand side is less than or equal to the right hand side. This is because resolving intersections can only reduce the intersection number with any given simple closed curve. Thus, we only need to prove the other inequality to prove the lemma.

$$\lim_{j\to\infty} \frac{i(\kappa_j, \xi)}{m_j \cdot \ell_{\text{hyp}}(\gamma_{1j}) + \ell_{\text{hyp}}(\gamma_{2j})} \geq \lim_{j\to\infty} \left(c\frac{i(\gamma_{1j}, \xi)}{\ell_{\text{hyp}}(\gamma_{1j})} + \frac{i(\gamma_{2j}, \xi)}{\ell_{\text{hyp}}(\gamma_{2j})} \right) \tag{13}$$

Let $\widehat{\kappa}_j$ be a lift of κ_j to the universal cover $\mathbb{H}^2$. It follows the lifts $\widehat{\gamma}_{1j}$ and $\widehat{\gamma}_{2j}$ of γ_{1j} and γ_{2j}, turning sharply whenever it switches from following a lift of γ_{1j} to following a lift of γ_{2j} or vice versa. Let $\widehat{\kappa}'_j$ denote the segments in $\widehat{\kappa}_j$ in the neighbourhood of the turning points where the arc before the turn is within 4δ distance (where δ is the Gromov hyperbolicity constant of $\mathbb{H}^2$) of the arc after the turn (see Figure 4 for an example). Denote the complement of $\widehat{\kappa}'_j$ in

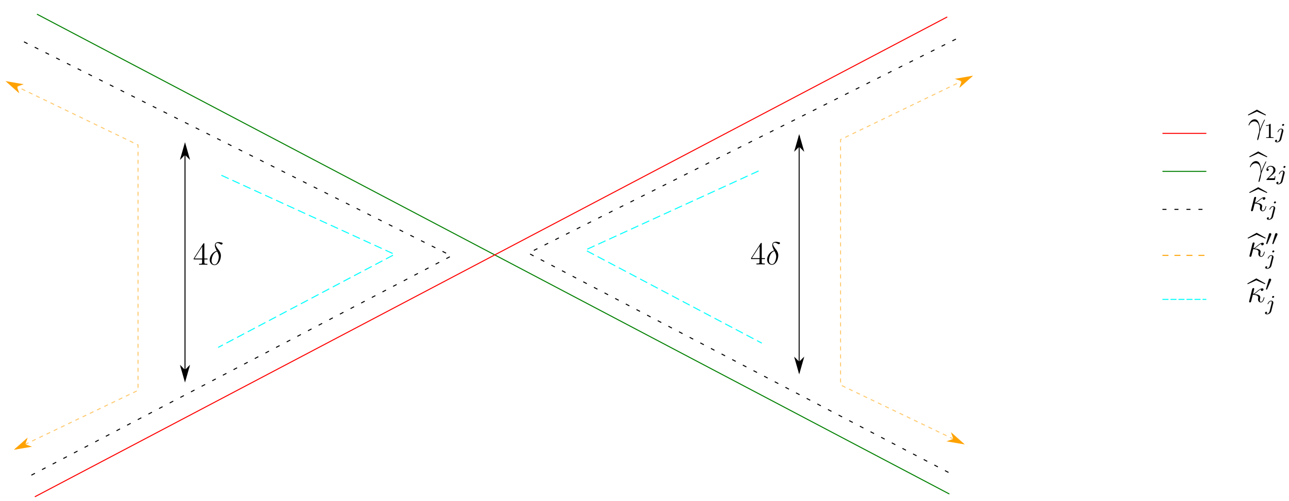


FIGURE 4. The segments $\widehat{\kappa}'_j$ and $\widehat{\kappa}''_j$ in $\widehat{\kappa}_j$.

$\widehat{\kappa}_j$ by $\widehat{\kappa}''_j$. We construct a new curve using $\widehat{\kappa}''_j$, connecting two successive disconnected pieces by a geodesic arc joining the endpoints (see Figure 4). We call this new arc in $\mathbb{H}^2$ $\widehat{\lambda}_j$, and its image in the surface λ_j. We claim that $\widehat{\lambda}_j$ is within 4δ distance of the geodesic tightening of $\widehat{\kappa}_j$. This follows from a standard thin triangles argument, so we will elide the details (see Figure 5). As j goes to ∞, the intersection number of ξ with the geodesic tightening of κ_j and with λ_j will approach the same value, by Lemma 3.5. Consider now the projection of $\widehat{\kappa}''_j$ to the surface S: we denote the segments of γ_{1j} which follow the projection of $\widehat{\kappa}''_j$ by γ''_{1j} and the corresponding segments of γ_{2j} by γ''_{2j}. We also denote the complements of γ''_{1j} and γ''_{2j} in γ_{1j} and γ_{2j} by γ'_{1j} and γ'_{2j}. Since the geodesic tightening of $\widehat{\kappa}_j$ is within a bounded

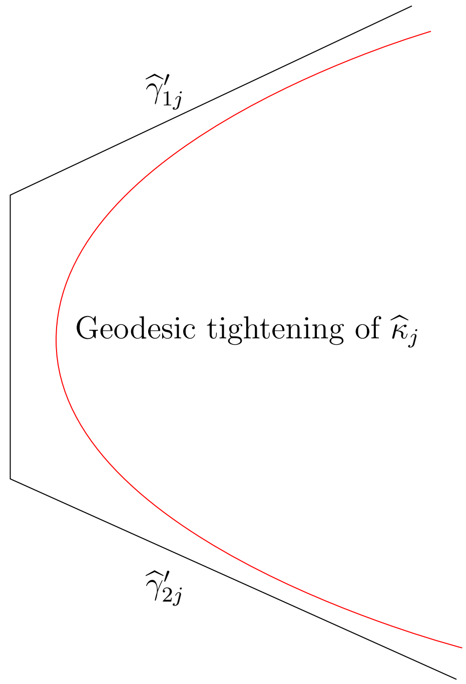


FIGURE 5. The geodesic tightening stays within a bounded distance of $\widehat{\lambda}_j$.

distance of $\widehat{\lambda}_j$, for large enough j, the intersection number with ξ will be the same, and we will get the following inequality between intersection numbers on the surface S.

$$i(\kappa_j, \xi) \geq m_j \cdot i(\gamma''_{1j}, \xi) + i(\gamma''_{2j}, \xi) \tag{14}$$

We express the right hand side as the intersection number with γ_{1j} and γ_{2j} minus an error term $\mathrm{err}_{\xi,j}$.

$$\begin{aligned} i(\kappa_j, \xi) &\geq (m_j \cdot i(\gamma_{1j}, \xi) + i(\gamma_{2j}, \xi)) - \big(m_j \cdot i(\gamma'_{1j}, \xi) + i(\gamma'_{2j}, \xi)\big) \\ &= (m_j \cdot i(\gamma_{1j}, \xi) + i(\gamma_{2j}, \xi)) - \mathrm{err}_{\xi,j} \end{aligned}$$

We claim that $\mathrm{err}_{\xi,j}$ is $o(m_j \cdot i(\gamma_{1j}, \xi) + i(\gamma_{2j}, \xi))$: this will prove inequality (13) and thus the lemma.

To begin with, we show that the lengths of γ'_{1j} and γ'_{2j} are $o(\ell_{\mathrm{hyp}}(\gamma_{1j}))$ and $o(\ell_{\mathrm{hyp}}(\gamma_{2j}))$. Suppose it was not the case, and $\dfrac{\ell_{\mathrm{hyp}}(\gamma'_{1j})}{\ell_{\mathrm{hyp}}(\gamma_{1j})} \geq k_1$, for some positive constant k_1. Then any short transverse arc that only intersected γ'_{1j} would get assigned a positive measure as j went to ∞. Since the lengths of $m_j \cdot \gamma_{1j}$ and γ_{2j} approach a fixed ratio, we would have that $\dfrac{\ell_{\mathrm{hyp}}(\gamma'_{2j})}{\ell_{\mathrm{hyp}}(\gamma_{2j})} \geq k_2$ for some other positive constant k_2. Thus the same short arc would also get assigned a positive measure by γ_{2j} as j went to ∞. This means that the transverse measure on ξ given by γ_1 decomposes as $\gamma_1\prime + \gamma''_1$, where γ'_1 is the limit of the transverse measures given by γ'_{1j} and γ''_1 is the limit of the transverse measures given by the limit of γ''_{1j}. We get a similar decomposition of γ_2 into $\gamma'_2 + \gamma''_2$. Our argument shows that γ'_1 and γ'_2 are absolutely continuous with respect to each other. But this violates mutual singularity of the measures γ_1 and γ_2, and therefore, the lengths of γ'_{1j} and γ'_{2j} must be $o(\ell_{\mathrm{hyp}}(\gamma_{1j}))$ and $o(\ell_{\mathrm{hyp}}(\gamma_{2j}))$

respectively. We sum up the result of this argument in the following inequalities.

$$\lim_{j\to\infty} \frac{\ell_{\mathrm{hyp}}(\gamma'_{1j})}{\ell_{\mathrm{hyp}}(\gamma_{1j})} = 0 \tag{15}$$

$$\lim_{j\to\infty} \frac{\ell_{\mathrm{hyp}}(\gamma'_{2j})}{\ell_{\mathrm{hyp}}(\gamma_{2j})} = 0 \tag{16}$$

$$\tag{17}$$

The bound on the length of γ'_{1j} and γ'_{2j} also gives a bound on $i(\xi, \gamma'_{1j})$ and $i(\xi, \gamma'_{2j})$. This follows from the fact that the intersection number of any arc with a fixed curve is bounded above by a constant times the length of the arc, where the constant depends on the curve.

$$i(\xi, \gamma'_{1j}) \leq c_\xi \ell_{\mathrm{hyp}}(\gamma'_{1j}) \tag{18}$$

$$i(\xi, \gamma'_{2j}) \leq c_\xi \ell_{\mathrm{hyp}}(\gamma'_{2j}) \tag{19}$$

$$\tag{20}$$

Using the fact that γ_1 and γ_2 have intersection number 1 with Liouville current, we get the following inequality for a large enough j, and for some finite k.

$$\frac{\ell_{\mathrm{hyp}}(\gamma_{1j})}{i(\xi, \gamma_{1j})} \leq k \tag{21}$$

$$\frac{\ell_{\mathrm{hyp}}(\gamma_{2j})}{i(\xi, \gamma_{2j})} \leq k \tag{22}$$

$$\tag{23}$$

Multiplying (15), (18), and (21), and (16), (19), and (22) proves the claim about the error term, and therefore the lemma. □

To state our next lemma, we need to define the *orbit measure* associated to simple curve, and define what it means for an orbit measure to be *almost invariant*. Consider an arc η transverse to a measured foliation λ. We assign one of the sides of η to be the "up" direction, and the other side to be the "down" direction. This lets us define the first return map to T.

Definition 3.7 (First return map)**.** The first return map T maps a point $p \in \eta$ to the point obtained by flowing along the foliation in the "up" direction until the flow intersects η again. The point of intersection is defined to be $T(p)$. If the flow terminates at a singularity, $T(p)$ is left undefined: there are only countable many points in η such that this happens.

Since λ is a measured foliation, it defines a measure on η: we can scale it so that it is a probability measure. It follows from the definition of transverse measures that the measure is T-invariant. It is a classical result of Katok [Kat73] and Veech [Vee78] that the set of T-invariant probability measures is a finite dimensional simplex contained in the Banach space of bounded signed measures on η. Given an orbit of a point p under the T-action of length L, we can construct a probability measure on η, called the orbit measure of p.

Definition 3.8 (Orbit measure)**.** The orbit measure of length L associated to the point p is the following probability measure on η.

$$\mu_{p,L} \coloneqq \frac{1}{L}\sum_{i=0}^{L-1} \delta_{T^i(p)}$$

Here, δ_x is the Dirac delta measure at the point x.

One might expect that if a point p equidistributes, then a long orbit measure starting at p will be "close" to an invariant measure. We formalize this notion by metrizing the Banach space of signed finite measures on η.

Definition 3.9 (Lèvy-Prokhorov metric)**.** Define $\|\cdot\|_{\mathrm{BL}}$ denote the bounded Lipschitz norm on the space of continuous functions on η.

$$\|f\|_{\mathrm{BL}} \coloneqq \|f\|_\infty + \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|}$$

Then the Lèvy-Prokhorov distance d_{LP} between the probability measures μ_1 and μ_2 is defined to be the following.

$$d_{\mathrm{LP}}(\mu_1,\mu_2) \coloneqq \sup_{\|f\|_{\mathrm{BL}}\leq 1} \int f(\,\mathrm{d}\mu_1 - \mathrm{d}\mu_2)$$

Using the Lèvy-Prokhorov metric, we can define what it means for a probability measure to be ε-almost T-invariant.

Definition 3.10 (ε-almost T-invariance)**.** A measure μ is ε-almost T-invariant if $d_{\mathrm{LP}}(\mu, T\mu) \leq \varepsilon$. Here $T\mu$ is the pushforward of μ under T.

We state the following easy fact about orbit measures without proof.

Fact. An orbit measure of length L is $\frac{2}{L}$-almost T-invariant.

The following lemma shows that a long orbit measure is close to an invariant measure.

Lemma 3.11. *For any $\varepsilon > 0$, there exists an L large enough such that any orbit measure longer than L is within distance ε of an invariant measure.*

Proof. Let M_L denote the set of all orbit measures of length greater than or equal to L. The set M_L is compact, because it is a closed subset of a compact set, and we have that $\bigcap_{L=1}^\infty M_L$ is the set of invariant measures. By compactness, we have that for some large enough L, M_L must be in a ε-neighbourhood of the set of invariant measures. □

We now prove a lemma that gives us a criterion for deducing when a long orbit measure is close to an ergodic measure.

Lemma 3.12. *Let $\{n_i\}$ be a sequence of positive integers, and let $\{p_{ij}\}$ and $\{L_{ij}\}$ be points in η and positive integers respectively, where $1 \leq j \leq n_i$ and $\min_j L_{ij}$ goes to ∞ as i goes to ∞. Consider the following sequence of probability measures, indexed by i.*

$$\mu_i \coloneqq \frac{\sum_{j=1}^{n_i} L_{ij}\cdot \mu_{p_{ij},L_{ij}}}{\sum_{j=1}^{n_i} L_{ij}}$$

If the sequence $\{\mu_i\}$ converges to an ergodic measure ν, then there exists a subsequence of the orbit measures $\mu_{p_{ij},L_{ij}}$ also converging to ν.

Proof. Suppose for the sake of a contradiction that no subsequence of $\mu_{p_{ij},L_{ij}}$ converged to ν. That would mean there exists a small enough $\varepsilon > 0$ and a large enough i_0 such that for all $i > i_0$, the measures $\mu_{p_{ij},L_{ij}}$ are more than distance ε from ν. Since $\min_j L_{ij}$ goes to ∞, there exists some other large enough $i_1 > i_0$ such that for all $i > i_1$, $\mu_{p_{ij},L_{ij}}$ is within distance $\frac{\varepsilon}{k}$ of the simplex of invariant probability measures, where k is a large integer we will pick later: this is a consequence of Lemma 3.11. Using this, we decompose $\mu_{p_{ij},L_{ij}}$ as the sum of an invariant measure ι_{ij} and a signed measure e_{ij}, such that $d_{\mathrm{LP}}(0, e_{ij}) \leq \frac{\varepsilon}{k}$.

$$\mu_{p_{ij},L_{ij}} = \iota_{ij} + e_{ij}$$

Observe that the weighted average of $\mu_{p_{ij},L_{ij}}$ will differ from the weighted average of ι_{ij} by at most $\frac{\varepsilon}{k}$. Also note that all the invariant measures ι_{ij} are distance at least $\varepsilon - \frac{\varepsilon}{k}$ from ν. Since ν is the vertex of a finite-dimensional convex set, any weighted average of the ι_{ij} must be at least distance $\frac{\varepsilon}{k'}$ from ν, where k' is a positive number depending on the geometry of the convex set of invariant probability measures. By picking $k > 2k'$ we can ensure that any weighted average of the $\mu_{p_{ij},L_{ij}}$ must be at least distance $\frac{\varepsilon}{2k'}$ from ν. But this would contradict our hypothesis that the measures μ_i converge to ν. That would that there exists some subsequence of $\mu_{p_{ij},L_{ij}}$ that converges to ν, which proves the lemma. ☐

We now consider a tricky special case we are forced to reckon with when approximating ergodic foliations with simple closed curves. Let λ be a minimal foliation and η a transverse arc. Let q be a DQD on $\mathcal{N}_g$ such that λ is the vertical foliation, and η is a horizontal arc of flat length 1. Let a be a leaf starting at the left endpoint p of η and going up, and suppose it comes back from the top intersecting η a point q which is distance $\varepsilon \ll 1$ from the left endpoint (possibly after intersecting η several times). Let c be the simple closed curve obtained by concatenating a and the horizontal arc joining p and q. The curve c is clearly not close to the flat or hyperbolic geodesic representative of its homotopy class. One obvious way to make c shorter without changing its homotopy type is to flow the arc joining p and q along the foliation until it encounters a singularity, at which point we stop. We label the new endpoints p' and q', and define a new closed curve c' by starting at p', flowing along the vertical foliation until q' and then flowing horizontally back to p'. We consider the transverse probability measures on η given by intersection with c, c', and their hyperbolic geodesic representative, which we denote c''. We denote the associated transverse probability measures on η by μ_c, $\mu_{c'}$, and $\mu_{c''}$ respectively. The following lemma states that the $\mu_{c'}$ and $\mu_{c''}$ are approximately equal.

Lemma 3.13. *As ε approaches 0, $d_{\mathrm{LP}}(\mu_{c'}, \mu_{c''})$ approaches 0.*

Proof. Note that as ε goes to 0, the hyperbolic lengths of c' and c'' go to ∞. If we show that c' and c'' stay within a bounded distance of each other, independent of ε, the measures $\mu_{c'}$ and $\mu_{c''}$ converge to the same measures, since the intersection points in η with c' and c'' will get closer and closer, by elementary hyperbolic geometry.

Observe that the closed curve c' is almost a hyperbolic geodesic: namely the segment of it that follows the foliation λ: the only possibility is that the arc joining p' and q' results in the closed curve shortening significantly. However, note that the segment of c' that follows the foliation approaches the singularity from two different prongs which have a positive angle between them. Joining the ends with a short geodesic gives us a segment obtained by concatenating three geodesic arcs. Gromov hyperbolicity tells us that the geodesic representative of such a segment will be within a bounded distance of the segment: i.e. the geodesic

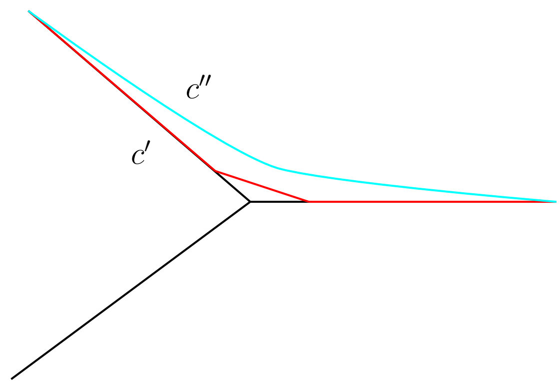


FIGURE 6. The curves c' and c'' are within bounded distance because all the prongs of the singularity have a positive angle between them.

representative c'' will be within a bounded distance of c' (see Figure 6). This proves the lemma. $\square$

The next lemma links the ratio of the flat lengths of c and c' to the measures μ_c and $\mu_{c'}$.

Lemma 3.14. *Consider a sequence of ε going to 0 such that $\frac{\ell_{\text{flat}}(c')}{\ell_{\text{flat}}(c)} \geq \nu > 0$ and both μ_c and $\mu_{c'}$ converge. Then $\lim_{\varepsilon\to 0} \mu_{c'}$ is absolutely continuous with respect to $\lim_{\varepsilon\to 0} \mu_c$.*

Proof. Consider a sub-interval η' of η such that $\lim_{\varepsilon\to 0} \mu_c(\eta') = 0$. We need to show that the same holds for the limit of the measures $\mu_{c'}$. Let $i(c, \eta)$ and $i(c, \eta')$ be the number of times c intersects η and η', and let $i(c', \eta)$ and $i(c', \eta')$ be defined similarly. Since we are working with a fixed DQD, the intersection number of a vertical arc with η is bounded above and below by the flat length of the vertical arc times a positive constant depending only on η (the positive constants are the maximum and minimum heights of the rectangles obtained when representing the flat structure as a zippered rectangle).

$$\alpha_\eta \ell_{\text{flat}}(c) \leq i(c, \eta) \leq \beta_\eta \ell_{\text{flat}}(c) \tag{24}$$

$$\alpha_\eta \ell_{\text{flat}}(c') \leq i(c', \eta) \leq \beta_\eta \ell_{\text{flat}}(c') \tag{25}$$

Using the inequality in the hypothesis, and (24) and (25), we get the following chain of inequalities.

$$\begin{aligned}\frac{i(c', \eta')}{i(c', \eta)} &\leq \frac{i(c, \eta')}{i(c', \eta)} \\ &\leq \frac{1}{\alpha_\eta} \cdot \frac{i(c, \eta')}{\ell_{\text{flat}}(c')} \\ &\leq \frac{1}{\alpha_\eta \nu} \cdot \frac{i(c, \eta')}{\ell_{\text{flat}}(c)} \\ &\leq \frac{\beta_\eta}{\alpha_\eta \nu} \cdot \frac{i(c, \eta')}{i(c, \eta)}\end{aligned}$$

The right hand side goes to 0 as ε goes to 0, proving absolute continuity of the limiting measure. $\square$

We now have everything we need to prove Theorem 3.3.

Proof of Theorem 3.3. If a minimal component λ_j is periodic, then the proof is straightforward. Since λ contains no one-sided component, the core curve of λ_j must be two-sided, possibly with an irrational coefficient. Approximating the core curve with rational coefficients proves the result in case (i).

In case (ii), we have that λ_j is not periodic, but a minimal orientable foliation. If we know that all the ergodic measures supported on the underlying topological foliation can be approximated by simple two-sided curves, we can approximate any convex combination of those measures, using Proposition 3.6, since all the ergodic measures are mutually singular. It will therefore suffice to deal with the case that λ_j is ergodic.

Pick an arc η_0 transverse to λ_j such that the leaf passing through the left endpoint p_0 of η_0 equidistributes with respect to the ergodic transverse measure of λ_j. We can find such a leaf because almost every leaf equidistributes with respect to the ergodic measure. We now inductively define a sequence of points $\{p_i\}$, sequence of sub-intervals η_i, and a sequence of segments $\{a_i\}$ of the leaf passing through p_0. Let p_1 be the first return of the leaf going up through p_0 to the interval η_0. Define the sub-interval η_1 to be the sub-interval whose left endpoint is p_0 and right endpoint is p_1. Let a_1 be the segment of the leaf starting at p_0 and ending at p_1. Given a point p_i, define p_{i+1} to be the first return to the interval η_i, η_{i+1} to be the interval whose left endpoint is p_0 and right endpoint is p_{i+1}, and a_{i+1} to be the segment of the leaf starting at p_i and ending at p_{i+1}.

Since we have assumed λ_j is an orientable foliation, we have that the leaf we are working with always enters η_0 from the bottom, and exits from the top. If we pick η_0 to be small enough, we can pick a local orientation, and keep track of how a positively oriented frame returns to each p_i, i.e. with or without the orientation flipped (see Figure 7). If the flow returns

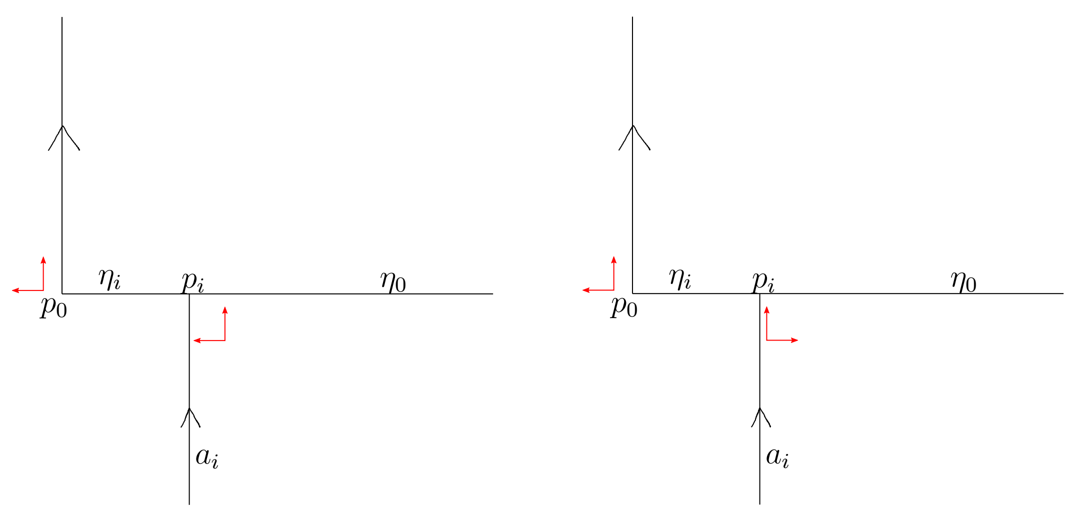


FIGURE 7. Two possibilities for first return to η_i: on the left, the arc returns without the local orientation flipping, and on the right, the arc returns with the local orientation flipped.

infinitely often without the orientation flipped, we join the endpoint p_i to p_0 by going left along η_i to get a simple closed curve that is two-sided. Furthermore, the geodesic tightening of the resulting curve is very close to the original curve, because the initial and final tangent vectors can be made arbitrarily close since they both face the "up" direction: the Anosov closing lemma then tells us that an orbit of the geodesic flow that approximately closes up can be perturbed by a small amount to exactly close up. This gives us a long geodesic that equidistributes with respect to the ergodic measure, and therefore an approximation by two-sided multicurves.

If the flow does not return without the orientation flipped infinitely often, it must always return with the orientation flipped after some large enough i_0. In that case, consider the simple two-sided curves c_i obtained by concatenating a_i with the arc on η_{i-1} joining p_{i-1} and p_i (see Figure 8). We have that as i goes to ∞, the length of c_i must go to ∞ as

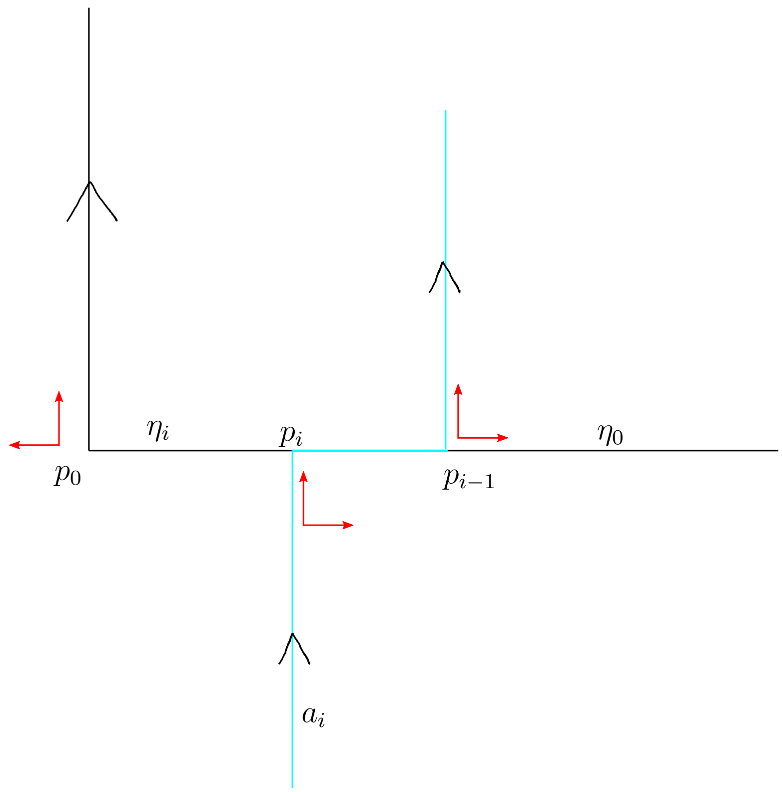


FIGURE 8. The curve c_i is colored blue. Since the leaf from p_0 returns with the local orientation flipped to both p_{i-1} and p_i, the curve c_i is two-sided.

well, otherwise a subsequence would converge to a closed vertical curve starting at p_0, which cannot happen since the leaf through p_0 equidistributes. Also, note that the average of the curves c_i weighted by their lengths for $i' < i < i''$ where $i'' \gg i'$ is close to the ergodic measure, since we assumed that the leaf through p_0 equidistributes. This lets us invoke Lemma 3.12 to claim that there is a subsequence of c_i whose orbit measures converge to the ergodic measure. Consequently, the geodesic representatives of c_i converge to λ_j, since the geodesic tightening is close to the original curve, by the virtue of the initial and final tangent vectors being arbitrarily close. This resolves the two cases that can appear in the case of an orientable foliation, proving the result for case (ii).

For case (iii), we define the points p_i, the nested intervals η_i, and the arcs a_i in a similar manner as to case (ii). The key difference is that we no longer have that the foliation is orientable, which means the leaf can approach p_i in one of four possible ways: from the "up" or the "down" direction, and with or without the orientation flipped.

In case that the leaf approaches p_i from the "down" direction without the orientation flipped infinitely often, the same closing argument as case (ii) works. Suppose now that the leaf approaches p_i from the "up" direction, but without the orientation flipping, infinitely often. We then construct simple two-sided curves by concatenating the flow with the arc joining p_i to p_0. While this curve does equidistribute with respect to the ergodic measure, it is not necessary that its geodesic tightening will do so. Denote the geodesic tightening by c_i'', and consider an intermediate closed curve c_i' in the same homotopy class as described in Lemma 3.13 and Lemma 3.14. We have by assumption that μ_{c_i} converges to an ergodic measure. If the ratio of the flat lengths of c_i and c_i' is bounded away from 0, possibly up to a subsequence, then Lemma 3.14 tells us that $\mu_{c_i'}$ converges to the ergodic measure, and Lemma 3.13 tells us the transverse measure associated to the hyperbolic geodesic representative is also ergodic. If $\ell_{\text{flat}}(c_i'')$ is $o(\ell_{\text{flat}}(c_i))$, we still have that $\mu_{c_i''}$ converges up to a subsequence, since $l(c_i'')$ goes to ∞, but it may converge to some other measure supported on the same underlying topological foliation. This proves the furthermore case of theorem. If λ_j is actually uniquely ergodic, there is only one measure in the simplex of invariant probability measures, namely the uniquely ergodic one, and therefore μ_{g_i} is forced to converge to it.

Suppose now that neither of the first two scenarios occur, i.e. the leaf returns to p_i from the "up" or "down" direction, but with the orientation always flipped. We deal with this case like we did with the second subcase of case (ii). See Figure 9 for the construction of the two-sided curves c_i. We have that the geodesic tightenings of the curves c_i are close to

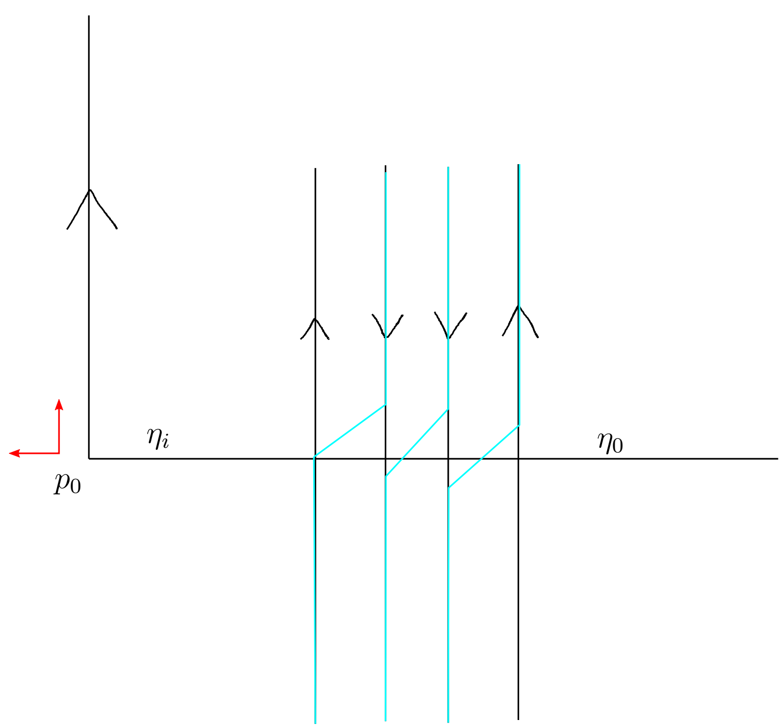


FIGURE 9. Construction of the blue curve c_i when the leaf always returns with orientation flipped from the "up" or "down" direction.

the original curve by the Anosov closing lemma, and that the weighted averages of the c_i converge the ergodic measure, which means by Lemma 3.12 we have a subsequence μ_{c_i} that converges to the ergodic measure. This proves the result for case (iii), and therefore the theorem. □

4. Upper bound for the limit set

In this section, we prove that $\Lambda_{\text{geo}}(\text{MCG}(\mathcal{N}_g))$ is contained in $\mathbb{P}\mathcal{MF}^+(\mathcal{N}_g)$. We do so by defining an $\text{MCG}(\mathcal{N}_g)$-invariant subset $\mathcal{T}_\varepsilon^-(\mathcal{N}_g)$, and showing that the intersection of its closure with $\mathbb{P}\mathcal{MF}(\mathcal{N}_g)$ is contained in $\mathbb{P}\mathcal{MF}^+(\mathcal{N}_g)$.

Definition 4.1 (One-sided systole superlevel set)**.** For any $\varepsilon > 0$, the set $\mathcal{T}_\varepsilon^-(\mathcal{N}_g)$ is the set of all points in $\mathcal{T}(\mathcal{N}_g)$ where the length of the shortest one-sided curve is greater than or equal to ε.

We can state the main theorem of this section.

Theorem 4.2. *For any $\varepsilon > 0$, $\overline{\mathcal{T}_\varepsilon^-(\mathcal{N}_g)} \cap \mathbb{P}\mathcal{MF}(\mathcal{N}_g)$ is contained in $\mathbb{P}\mathcal{MF}^+(\mathcal{N}_g)$.*

The key idea of the proof is proving a quantitative estimate on the Fenchel-Nielsen coordinates of points converging to points in $\mathbb{P}\mathcal{MF}^-(\mathcal{N}_g)$.

Proposition 4.3. *Let $\{m_i\}$ be a sequence of points in $\mathcal{T}(\mathcal{N}_g)$ converging to a projective measured foliation $[\lambda]$. If p is a one-sided atom of λ, for any Fenchel-Nielsen coordinate chart containing p as a cuff, the length coordinate of p goes to 0.*

Proof. Consider the following decomposition of the measured foliation λ.

$$\lambda = 1 \cdot p + \lambda_{\text{at}} + \lambda_{\text{Leb}}$$

Here, λ_{at} are the minimal components on periodic components other than p, i.e. cylinders and Möbius strips, and λ_{Leb} are non-periodic minimal components. In the above expression, p is the one-sided curve considered as a measured foliation (since we're picking a representative of $[\lambda]$, we can pick one such that p has weight 1)

Pick simple closed curves p_0, p_1, and p_2, where p_0 is the curve p, and $\{p_0, p_1, p_2\}$ bound a pair of pants. Furthermore, we impose the following conditions on p_1 and p_2.

$$i(p_1, \lambda_{\text{at}}) = 0$$
$$i(p_2, \lambda_{\text{at}}) = 0$$

Note that this can always be done, by deleting the support of λ_{at}, and looking at the resulting subsurfaces. Note that neither p_1 nor p_2 can be the same as p_0, since p_0 is one-sided. We focus our attention to this pair of pants, and consider the curves on it, labelled in Figure 10.

The curve labelled p_3 has intersection number at least 1 with λ which means its length should be going to ∞. We can bound the length of p_3 above and below via the lengths of the orthogeodesic p_4 and the length of p_0.

$$\ell_{\text{hyp}}(p_0) \leq \ell_{\text{hyp}}(p_3) \leq \ell_{\text{hyp}}(p_4) + \ell_{\text{hyp}}(p_0) \tag{26}$$

Observe that we get the upper bound by observing that the red and cyan arcs are isotopic to p_3 relative to their endpoints being fixed. The cyan arcs have length at most $\ell_{\text{hyp}}(p_0)$, in this setting; if one allowed a twist parameter, the length of the cyan arcs would be proportional to the twist parameters. The point of this inequality is that we can estimate $\ell_{\text{hyp}}(p_4)$ using $\ell_{\text{hyp}}(p_0)$, $\ell_{\text{hyp}}(p_1)$ and $\ell_{\text{hyp}}(p_2)$ via hyperbolic trigonometry. Cut the pair of pants along the seams, to get a hyperbolic right-angled hexagon, pictured in Figure 11.

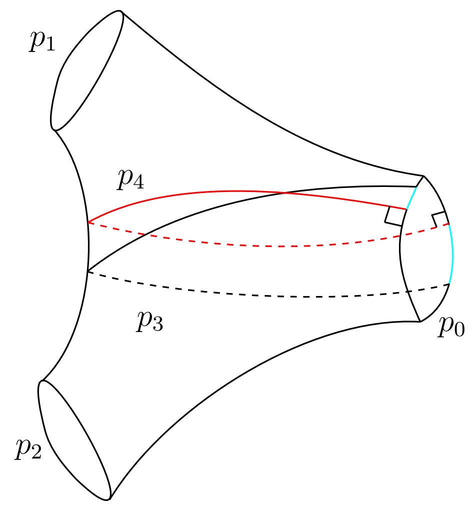


FIGURE 10. The curves restricted to a pair of pants.

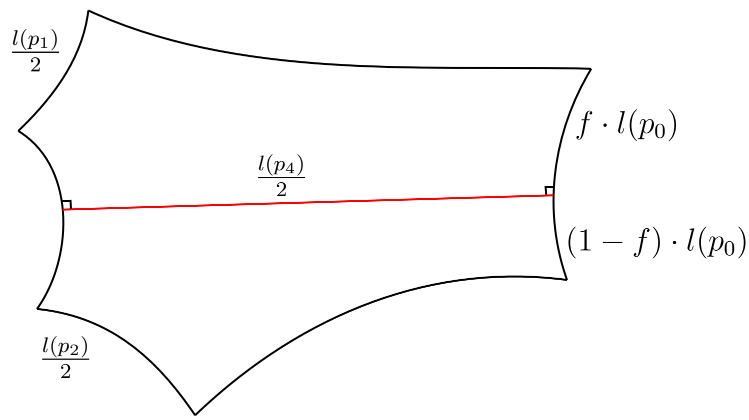


FIGURE 11. The right angled hexagon obtained by cutting the pants along the seams.

To get good estimates on $\ell_{\text{hyp}}(p_4)$, we need a universal lower bound on the fraction f as we move in the Teichmüller space. The analysis splits up into two cases, but it's not *a priori* clear that these two cases are exhaustive. We will deal with the two cases, and then show that any other case can be reduced to the second case by changing p_1 and p_2.

Case I. We're in this case if p_1 and p_2 don't intersect the foliation λ at all.

$$i(p_1, \lambda) = 0$$
$$i(p_2, \lambda) = 0$$

In this case, we can pass to a subsequence of $\{m_i\}$ such that the corresponding values of f are always greater than $\frac{1}{2}$ or less than $\frac{1}{2}$. In the former case, we focus on p_1, and in the latter case, we focus on p_2. Without loss of generality, we'll suppose $f \geq \frac{1}{2}$. In that case, we cut along the orthogeodesic p_4, and get a hyperbolic right-angled pentagon, which is the top half of Figure 11. We can relate the lengths of $\ell_{\text{hyp}}(p_0)$, $\ell_{\text{hyp}}(p_1)$, and $\ell_{\text{hyp}}(p_4)$ using the following identity for hyperbolic right-angled pentagons (see [Thu79] for the details).

$$\sinh\left(f \cdot \ell_{\text{hyp}}(p_0)\right) \cdot \sinh\left(\frac{\ell_{\text{hyp}}(p_4)}{2}\right) = \cosh\left(\frac{\ell_{\text{hyp}}(p_1)}{2}\right) \tag{27}$$

Now suppose that $\ell_{\text{hyp}}(p_0)$ does not go to 0. Then we must have that for all m_i, $\ell_{\text{hyp}}(p_0) \geq 2\varepsilon$ for some $\varepsilon > 0$. By the lower bound on f, we have that the first term on the left hand side of the above expression is bounded below by ε. Rearranging the terms gives us the following upper bound on $\ell_{\text{hyp}}(p_4)$.

$$\ell_{\text{hyp}}(p_4) \leq 2 \cdot \sinh^{-1}\left(\frac{\cosh\left(\frac{\ell_{\text{hyp}}(p_1)}{2}\right)}{\varepsilon}\right) \tag{28}$$

Using (26) and (28), we get an upper bound for $\ell_{\text{hyp}}(p_3)$.

$$\ell_{\text{hyp}}(p_3) \leq \ell_{\text{hyp}}(p_0) + 2\sinh^{-1}\left(\frac{\cosh\left(\frac{\ell_{\text{hyp}}(p_1)}{2}\right)}{\varepsilon}\right) \tag{29}$$

Since $\frac{i(p_0,\lambda)}{i(p_3,\lambda)} = 0$, as $\{m_i\}$ approaches λ, the ratio of lengths of p_0 and p_3 approach 0.

$$\lim_{i\to\infty} \frac{\ell_{\text{hyp}}(p_0)}{\ell_{\text{hyp}}(p_3)} = 0 \tag{30}$$

Using (29), we can find a lower bound for $\frac{\ell_{\text{hyp}}(p_0)}{\ell_{\text{hyp}}(p_3)}$, which goes to 0 by (30).

$$\lim_{i\to\infty} \frac{\ell_{\text{hyp}}(p_0)}{\ell_{\text{hyp}}(p_0) + 2\sinh^{-1}\left(\frac{\cosh\left(\frac{\ell_{\text{hyp}}(p_1)}{2}\right)}{\varepsilon}\right)} = 0 \tag{31}$$

Since we assumed that the length coordinate $\ell_{\text{hyp}}(p_0)$ is bounded away from 0, the only way the above expression can go to 0 if $\ell_{\text{hyp}}(p_1)$ goes to ∞. This is where the hypotheses of the Case I come in. Since $i(p_1, \lambda)$ is 0, the following equality must hold.

$$\lim_{i\to\infty} \frac{\ell_{\text{hyp}}(p_1)}{\ell_{\text{hyp}}(p_3)} = 0 \tag{32}$$

This means the lower bound for $\frac{\ell_{\text{hyp}}(p_1)}{\ell_{\text{hyp}}(p_3)}$ must go to 0.

$$\lim_{i\to\infty} \frac{\ell_{\text{hyp}}(p_1)}{\ell_{\text{hyp}}(p_0) + 2\sinh^{-1}\left(\frac{\cosh\left(\frac{\ell_{\text{hyp}}(p_1)}{2}\right)}{\varepsilon}\right)} = 0 \tag{33}$$

But this can't happen if $\ell_{\text{hyp}}(p_1)$ approaches $\pm\infty$. This contradiction means our assumption that both $\ell_{\text{hyp}}(p_0)$ and $\tau(p_0)$ were bounded away from 0 and $\pm\infty$ must be wrong, and thus proves the result in Case I.

Case II. We're in this case if the following inequality holds.

$$0 < i(p_1, \lambda) < 1 \tag{34}$$

The picture in this case looks similar to Figure 11. However, we can't necessarily pass to a subsequence where $f \geq \frac{1}{2}$ (and the trick of working with $1 - f$ won't work, since we know nothing about p_2). This is one of the points where the hypothesis on p_1 comes in. Since $\frac{i(p_2,\lambda)}{i(p_1,\lambda)}$ is finite, we must have that the ratio of lengths $\frac{\ell_{\text{hyp}}(p_2)}{\ell_{\text{hyp}}(p_1)}$ approaches some finite value as well. The fraction f is a continuous function of $\frac{\ell_{\text{hyp}}(p_2)}{\ell_{\text{hyp}}(p_1)}$, approaching 0 only as the ratio approaches ∞ (this follows from the same identity as (27)). Since the ratio approaches a finite value, we have a positive lower bound f_0 for f.

Assuming as before that $\ell_{\text{hyp}}(p_0)$ is bounded away from 0, and $\tau(p_0)$ bounded away from $\pm\infty$, and repeating the calculations of the previous case, we get the following two inequalities.

$$\frac{\ell_{\text{hyp}}(p_1)}{\ell_{\text{hyp}}(p_3)} \geq \frac{\ell_{\text{hyp}}(p_1)}{\ell_{\text{hyp}}(p_0) + 2\sinh^{-1}\left(\frac{\cosh\left(\frac{\ell_{\text{hyp}}(p_1)}{2}\right)}{f_0\varepsilon}\right)} \tag{35}$$

$$\frac{\ell_{\text{hyp}}(p_0)}{\ell_{\text{hyp}}(p_3)} \geq \frac{\ell_{\text{hyp}}(p_0)}{\ell_{\text{hyp}}(p_0) + 2\sinh^{-1}\left(\frac{\cosh\left(\frac{\ell_{\text{hyp}}(p_1)}{2}\right)}{f_0\varepsilon}\right)} \tag{36}$$

The right hand side of (36) must approach 0, and that forces either $\ell_{\text{hyp}}(p_1)$ or $\ell_{\text{hyp}}(p_2)$ to approach ∞. But that means the right hand term of (35) must approach 1, which cannot happen, by the hypothesis of case II. This means $\ell_{\text{hyp}}(p_0)$ goes to 0, proving the result in case II.

Reducing to case II. Suppose now that both p_1 and p_2 have an intersection number larger than 1 with λ. We can modify one of them to have a small intersection number with λ. First, we assume that λ_{Leb} is supported on a single minimal component, i.e. every leaf of λ_{Leb} is dense in the support. We now perform a local surgery on p_1: starting at a point on p_1 not contained in the support of λ_{Leb}, we follow along until we intersect λ_{Leb} for the first time. We denote this point by α. We now go along p_1 in the opposite direction, until we hit the support of λ_{Leb} again, but rather than stopping, we keep going until the arc has intersection number $0 < \delta < 1$ with λ_{Leb}. We then go back to α, and follow along a leaf of λ_{Leb} rather than p_1, until we hit the arc. This is guaranteed to happen by the minimality of λ_{Leb}. Once we hit the arc, we continue along the arc, and close up the curve. This gives a new simple closed curve which intersection number with λ is at most δ. This curve is our replacement for p_1 for a schematic of this local surgery). If λ_{Leb} is not minimal, we repeat this process for each minimal component. We pick p_2 in a manner such that p_0, p_1, and p_2 bound a pair of pants. Since $\delta < 1$, we have reduced to case II. This concludes the proof of the theorem. ∎

Remark (On the orientable version of Proposition 4.3). The same idea also works in the orientable setting, although the analysis of the various cases gets a little more delicate. The first change one needs to make is in the statement of the proposition: we no longer need to require p to be a one-sided atom, and correspondingly, either the length coordinate $\ell_{\text{hyp}}(p)$ can go to 0, or the twist coordinate $\tau(p_0)$ can go to $\pm\infty$. To see how the twist coordinate enters the picture, observe that (26), which was the main inequality of the proof, turns into the following in the orientable version.

$$\ell_{\text{hyp}}(p_4) \leq \ell_{\text{hyp}}(p_3) \leq \tau(p_0) + \ell_{\text{hyp}}(p_4) \tag{37}$$

Here, $\tau(p_0)$ is the twist parameter about p_0, and p_4 is the orthogeodesic multi-arc (there may be one or two orthogeodesics, depending on the two cases described below).

The proof splits up into two cases, depending on whether both sides of p are the same pair of pants, or distinct pairs of pants. This was not an issue in the non-orientable setting, since p was one-sided. If both sides of p are the same pair of pants, then the analysis is similar to what we just did, since the curve p_3 stays within a single pair of pants. In the other, p_3 goes through two pair of pants, and its length is a function of the twist parameters, as well the cuff lengths of four curves, rather than two curves, the four curves being the two remaining cuffs of each pair of pants. The analysis again splits up into two cases, depending on the intersection number of the cuffs with λ, but reducing all the other cases to case II becomes tricky because we need to simultaneously reduce the intersection number of two curves, rather than one, as in the non-orientable setting. This added complication obscures the main idea of the proof, which is why we chose to only prove the non-orientable version.

This quantitative estimate of Proposition 4.3 gives us an easy proof of Theorem 4.2.

Proof of Theorem 4.2. Suppose that the theorem were false, and there was a foliation $[\lambda] \in \mathbb{P}\mathcal{MF}^-(\mathcal{N}_g)$ in the closure of $\mathcal{T}_\varepsilon^-(\mathcal{N}_g)$. Suppose p is a one-sided atom in λ. Then Proposition 4.3 tells us that the hyperbolic length of p goes to 0, but the length of p must be greater than ε in $\mathcal{T}_\varepsilon^-(\mathcal{N}_g)$. This contradicts our initial assumption, and the closure of $\mathcal{T}_\varepsilon^-(\mathcal{N}_g)$ can only intersect $\mathbb{P}\mathcal{MF}(\mathcal{N}_g)$ in the complement of $\mathbb{P}\mathcal{MF}^-(\mathcal{N}_g)$. ☐

Corollary 4.4. *The geometric limit set $\Lambda_{\text{geo}}(\text{MCG}(\mathcal{N}_g))$ is contained in $\mathbb{P}\mathcal{MF}^+(\mathcal{N}_g)$.*

Proof. Every point $p \in \mathcal{T}(\mathcal{N}_g)$ is contained in $\mathcal{T}_\varepsilon^-(\mathcal{N}_g)$ for some small enough ε. This means $\Lambda_{\text{geo},p}(\text{MCG}(\mathcal{N}_g))$ is contained in $\mathbb{P}\mathcal{MF}^+(\mathcal{N}_g)$ by Theorem 4.2. Taking the union over all p proves the result. ☐

5. Wandering set

In this section, we prove that the wandering set for the action of $\text{MCG}(\mathcal{N}_g)$ on $\mathbb{P}\mathcal{MF}(\mathcal{N}_g)$ is empty. This follows fairly easily from Conjecture 3.1, but because we only have a slightly weaker version of the result, i.e. Theorem 3.3, we need to work a little harder. However, the techniques used in the proof of Theorem 3.3 are robust enough to prove the result in this section.

Theorem 5.1. *The wandering set for the $\text{MCG}(\mathcal{N}_g)$ action on $\mathbb{P}\mathcal{MF}(\mathcal{N}_g)$ is empty.*

We prove this result by showing that for any measured foliation λ, there exists a mapping class φ such that φ moves λ by an arbitrarily small amount.

Proposition 5.2. *Let λ be a measured foliation. Then for any simple closed curve κ and any $\delta > 0$, there exists a $\varphi \in \text{MCG}(\mathcal{N}_g)$ such that the following inequalities hold.*

$$i(\lambda, \kappa) - \delta \leq i(\varphi(\lambda), \kappa) \leq i(\lambda, \kappa) + \delta$$

To prove this proposition, we will need to estimate how Dehn twists change intersection numbers.

Lemma 5.3. *Let λ be any measured foliation, γ any two-sided simple curve, and κ any simple curve (not necessarily two-sided). Denote the Dehn twist about γ by T_γ; the intersection number of $T_\gamma(\lambda)$ and κ is bounded by the following intersection numbers.*

$$i(\lambda,\kappa) - i(\lambda,\gamma)\cdot i(\gamma,\kappa) \le i(T_\gamma,\kappa) \le i(\lambda,\kappa) + i(\lambda,\gamma)\cdot i(\gamma,\kappa)$$

Proof. We approximate λ by simple curves, possibly one-sided, and use Proposition 3.4 of [FM11], which is the same inequality, but with λ replaced with a simple curve. □

Proof of Proposition 5.2. Consider a projective measured lamination $[\lambda] \in \mathbb{P}\mathcal{MF}(\mathcal{N}_g)$. If $[\lambda]$ contains one-sided leaves, we delete those one-sided curves and work with the resulting subsurface. If the restriction of λ to the resulting subsurface is empty, then any mapping class on the subsurface does not change λ, and the result holds. If the restriction of λ to the subsurface is non-empty, and λ can be approximated by simple two-sided curves and tells us that it can also be approximated by stable foliations of pseudo-Anosov maps on the subsurface. For any small neighbourhood of $[\lambda]$, pick a pseudo-Anosov φ whose stable foliation lies in that neighbourhood. Then the north-south dynamics of pseudo-Anosov maps tells us that $\varphi([\lambda])$ will be in the same neighbourhood, proving the result in this case.

Suppose now that the restriction of the foliation is not approximable by simple two-sided curves. Then there exists some minimal component that is not approximable by two-sided curves. Proposition 3.6 tells us that there is some ergodic measure supported on the underlying topological foliation that is not approximable by two-sided curves either. Let p be a generic point for this measure, and let η be a transverse arc whose left endpoint is p. Pick a flat structure on this surface such that λ is the vertical foliation, and η is a horizontal arc of length 1. The only way the ergodic measure is not approximable by a two-sided curve is the situation described in the proof of Theorem 3.3 where a leaf leaving p from the top returns infinitely often from the top again, with the orientation remaining unflipped. Recall that in this situation, we have an arc a a leaving p from the top, following the foliation, terminating at a point q that is ε distance to the right of p, and it comes back from the top with the orientation unflipped. We concatenate this arc with the segment along η' joining p and q. This is a simple two-sided curve, which we will call c (this curve c depends on ε, but we will suppress the dependence in the notation). Let c' be the tightening of this curve described in the paragraph before Lemma 3.13. If we do not have that $\ell_{\text{flat}}(c')$ is $o(\ell_{\text{flat}}(c))$, then Lemma 3.14 tells us that the ergodic measure can be approximated by simple two-sided curves. That means $\varepsilon \cdot \ell_{\text{flat}}(c')$ goes to 0 as ε goes to 0.

We now relate $\ell_{\text{flat}}(c')$ with $i(c',\kappa)$ via the following inequality, where c' can be any closed curve, and j_κ is a constant depending on κ and the flat geometry of the surface.

$$i(c',\kappa) \le j_\kappa \cdot \ell_{\text{flat}}(c') \tag{38}$$

The constant j_κ can be computed by cutting the surface along κ, and computing the length of the shortest essential arc starting and ending at the boundary component κ.

We pick an ε and curve c such that $\varepsilon \cdot \ell_{\text{flat}}(c') < \frac{\delta}{j_\kappa}$. We Dehn twist about the curve c: Lemma 5.3 tells us that the error term is $i(\lambda,c)\cdot i(c,\kappa) = \varepsilon \cdot i(c,\kappa)$. We bound the intersection number by $j_\kappa \cdot \ell_{\text{flat}}(c')$, which bounds the error term by δ, proving the result in the case when λ is not approximable by two-sided curves.

This handles all the cases, and proves the proposition. □

Proof of Theorem 5.1. Pick any small open set U in $\mathbb{P}\mathcal{MF}(\mathcal{N}_g)$, and any arbitrary local metric such that U has diameter at least 2. Pick a sequence of mapping classes $\{\varphi_i\}$ that move a point by distance at most $\frac{1}{2^i}$, using Proposition 5.2. Then for all i, $\varphi_i(U) \cap U$ is non-empty. This proves the result. $\square$

6. Failure of quasi-convexity for $\mathcal{T}_\varepsilon^-$

In the setting of Teichmüller geometry, convexity is usually too strong of a requirement. For instance, metric balls in Teichmüller space are not convex, but merely quasi-convex (see [LR11]).

Definition 6.1 (Quasi-convexity)**.** A subset S of $\mathcal{T}(S)$ is said to be quasi-convex if there is some uniform constant $D > 0$ such that the geodesic segment joining any pair of points in S stays within distance D of S.

Our goal for this section will be to prove the following theorem.

Theorem 6.2. *For all $\varepsilon > 0$, and all $D > 0$, there exists a Teichmüller geodesic segment whose endpoints lie in $\mathcal{T}_\varepsilon^-$ such that some point in the interior of the geodesic is more than distance D from $\mathcal{T}_\varepsilon^-$.*

We begin by finding Teichmüller geodesic segments whose endpoints lie in $\mathcal{T}_\varepsilon^-$ such that at a point in the interior, some one-sided curve gets very short. Once we have arbitrarily short one-sided curves in the interior of the geodesic segments, estimates relating Teichmüller distance and ratios of hyperbolic lengths of curves will give us the result.

Proposition 6.3. *For all $g \geq 8$ and any $\delta > 0$, there exists a Teichmüller geodesic segment l whose endpoints lie in $\mathcal{T}_\varepsilon^-$, and a point p in l such that some one-sided curve has length less than δ with respect to the hyperbolic metric on p.*

To prove this result, we will need two lemmas relate hyperbolic and flat lengths.

Lemma 6.4. *Let q be any area 1 DQD on $\mathcal{N}_g$, and let γ be a simple closed curve of q. Suppose that $\ell_{\mathrm{hyp}}(\gamma) \leq \delta$ (with respect to the unique hyperbolic metric coming from the flat structure q). Then $\ell_{\mathrm{flat}}(\gamma) \leq k\sqrt{\delta}$, where k is some absolute constant.*

Sketch of proof. If $\ell_{\mathrm{hyp}}(\gamma) \leq \delta$, then there exists an annulus around γ of modulus proportional to $\frac{1}{\delta}$. By the results in [Min92], this annulus can be homotoped to be a primitive annulus, i.e. an annulus that does not pass through a singularity of the flat metric. Such annuli are either expanding, i.e. concentric circles in the flat metric, or flat, and in either case, we have an upper bound on the flat length of the core curve in terms of the modulus. This proves the result. $\square$

Lemma 6.5. *Let q be an area 1 DQD on $\mathcal{N}_g$, and consider the unique hyperbolic metric with the same conformal structure. Let A be a primitive annulus in q, i.e. an annulus whose interior does not pass through a singularity of the flat metric. Let the modulus of A be m. Then the hyperbolic length of the isotopy class of the core curve of the annulus is at most $\frac{\pi}{m}$.*

Sketch of proof. Without loss of generality, we can pass to the orientable double cover. This changes the hyperbolic lengths by at most a factor of two. Consider the interior of the annulus as a Riemann surface, and put the unique hyperbolic metric on that surface. With respect to this hyperbolic metric, the length of the core curve is $\frac{\pi}{m}$. Since the interior doesn't contain any singularities, the inclusion map is holomorphic, and holomorphic maps are distance reducing with respect to the hyperbolic metric. This proves the result. $\square$

To find a geodesic segment whose endpoints lie in $\mathcal{T}_\varepsilon^-$, we will construct a DQD q, and use Lemma 6.4 to find large enough t such that both $g_t(q)$ and $g_{-t}(q)$ are in $\mathcal{T}_\varepsilon^-$. We will then show that some one sided curve on q is very short using Lemma 6.5, which will prove Proposition 6.3.

Proof of Proposition 6.3. We will prove the result by constructing explicit examples in genus 4 and 9, and then connect sum orientable surfaces of genus j to get examples in genus $4+2j$ and $9+2j$.

We first list the two properties we require from the DQD q we want to construct, and show that having those properties proves the result.

(a) There exists an embedded annulus in q with a very large modulus whose core curve is the square of a one-sided curve in $\pi_1(\mathcal{N}_g)$.
(b) The vertical and horizontal foliations decompose as a union of cylinders, i.e. the vertical and horizontal flow is periodic, and no closed orbit is a one-sided curve. Furthermore, deleting the core curves of the cylinders in the horizontal or vertical direction result in a disjoint union of *orientable* subsurfaces.

We now show why having these two properties proves the result. Suppose we have a DQD q satisfying (a) and (b). Lemma 6.5 tells us that satisfying (a) means that the one-sided curve whose square is the core curve of the annulus will be very short. To find a large enough t such that $g_t(q)$ has no one-sided curves shorter than ε, pick a t enough such that each vertical cylinder in $g_t(q)$ is at least $2k\sqrt{\varepsilon}$ wide. Consider now any closed curve who flat length is less than $k\sqrt{\varepsilon}$. It must either be homotopic to one of the core curves of the vertical cylinders, or can be homotoped to be completely contained in one of the subsurfaces obtained by deleting all the core curves. That is because it was neither of these cases, it would cross at least one of these cylinders, and since the cylinders are at least $2k\sqrt{\varepsilon}$ wide, the flat length of the curve would exceed $k\sqrt{\varepsilon}$. If the curve is the core curve of a cylinder, or completely contained in one of the subsurfaces, it must be two-sided, by condition (b).

This proves that all one-sided curves have flat length exceeding $k\sqrt{\varepsilon}$, and therefore hyperbolic length exceeding ε. The same argument also works for $g_{-t}(q)$, proving the result.

We now construct explicitly the DQDs satisfying conditions (a) and (b) in genus 4, 9, and above.

The $g = 4$ case. Consider the area 1 DQD on $\mathcal{N}_4$ depicted in Figure 12. We impose the following constraint on the depicted DQD: the edges $\{c, c', d, d'\}$ are all oriented at an angle of $\pm\frac{\pi}{4}$, and have the same length.

Observe that by making the length of c (and correspondingly c', d, and d') go to 0, while keeping the area 1 lets us embed an annulus of high modulus (pictured in yellow in Figure 12) around any curve in $\{c, c', d, d'\}$. This shows that the DQD we constructed satisfies condition (a).

Checking condition (b) is easy, but tedious. For convenience, we have labelled the core curves of the vertical cylinders in red, blue, and green: the reader can check that they are all two-sided, and deleting them results in orientable subsurfaces. In fact, deleting the core curves results in 2 pairs of pants.

The $g = 9$ case. Consider the area 1 DQD on $\mathcal{N}_9$ depicted in Figure 13. To keep the picture from getting cluttered, we describe the edge gluing maps in words: the edges labelled c are glued via the map $z \mapsto -\overline{z} + k$, the edges labelled b and e are glued via $z \mapsto -z + k$, where

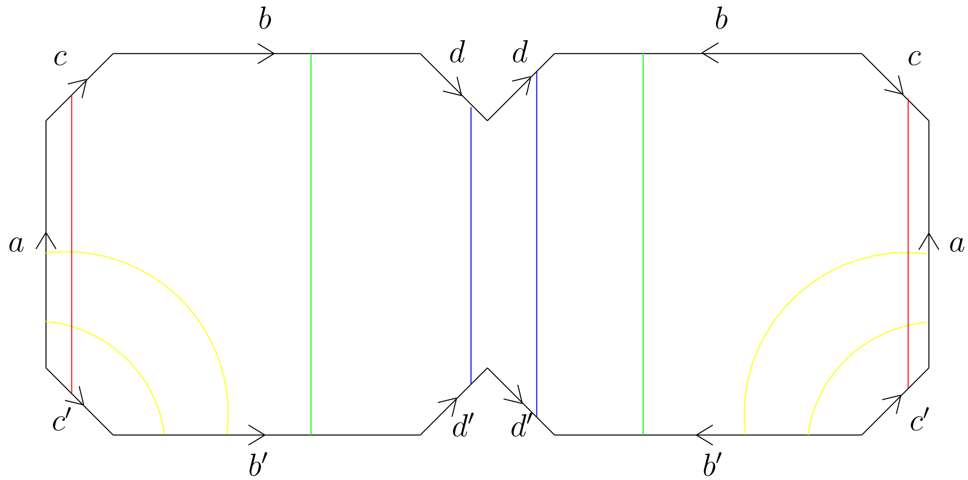


FIGURE 12. A DQD on $\mathcal{N}_4$.

k is some constant. All the other gluings are translation gluings. We impose the following

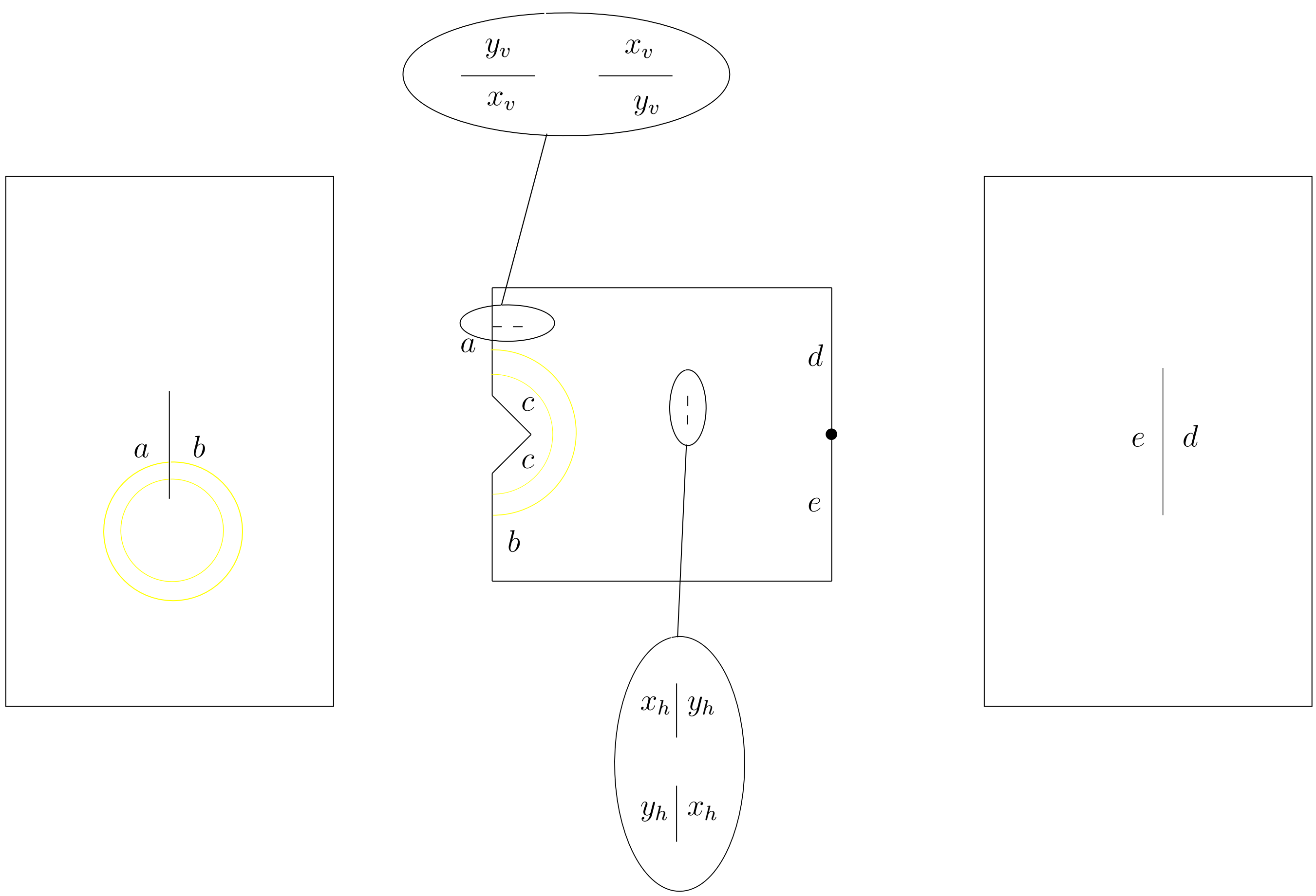


FIGURE 13. A DQD on $\mathcal{N}_9$. To display the gluing maps on the small slits, we have a zoomed in picture in the ellipses.

constraints on the DQD.

(i) The edges labelled c are oriented at an angle of $\pm\frac{\pi}{4}$, and the lengths of $\{x_h, y_h, x_v, y_v\}$ are $\frac{\ell_{\text{flat}}(c)}{4\sqrt{2}}$.

(ii) The left edge of x_v is aligned with the left edge of c, the left edge of y_v is aligned with the midpoint of c, the top edge of x_h is aligned with the top edge of c, and the top edge of y_h is aligned with the midpoint of c.

By making c smaller, while keeping the area equal to 1, one can embed an annulus of high modulus in the DQD, pictured in yellow in Figure 13. This shows that our construction satisfies condition (a).

To see that deleting the core curves of the horizontal cylinders results in orientable subsurfaces, note that deleting the core curves passing through c results in 2 pairs of pants, and a genus 3 orientable surface with one boundary component. This is again easy, but tedious to verify, so we leave the verification to the reader. This shows that the example satisfies condition (b).

The induction step. To get higher genus DQDs satisfying conditions (a) and (b), we start with the $g = 4$ and $g = 9$ examples and connect-sum an orientable surface using the slit construction. To ensure that the new surfaces still satisfy conditions (a) and (b), we need to ensure that the slit we construct if far away from the annulus of condition (a), as well as all the vertical and horizontal leaves passing through $\{c, c', d, d'\}$ in the $g = 4$ example, and the vertical and horizontal leaves passing through c in the $g = 9$ example. This will ensure that the resulting higher genus surface still satisfies conditions (a) and (b). □

To relate Teichmüller distance to hyperbolic lengths, we need Wolpert's lemma ([Wol79])

Lemma 6.6 (Wolpert's Lemma)**.** *Let M and M' be two points in $\mathcal{T}(\mathcal{S}_g)$, and let γ be a simple closed curve on $\mathcal{S}_g$. Let R be the Teichmüller distance between M and M'. Then the ratio of the hyperbolic length of γ and R are related by the following inequalities.*

$$\exp(-2R) \leq \frac{\ell_{\text{hyp}}(M, \gamma)}{\ell_{\text{hyp}}(M', \gamma)} \leq \exp(2R)$$

Using Proposition 6.3 and Wolpert's lemma, we can prove Theorem 6.2.

Proof of Theorem 6.2. Suppose that $\mathcal{T}_\varepsilon^-(\mathcal{N}_g)$ was indeed quasi-convex. That would mean that there exists some $R > 0$, depending on ε such that every point in the interior of any geodesic segment with endpoints in $\mathcal{T}_\varepsilon^-$ was within R distance of some point in $\mathcal{T}_\varepsilon^-(\mathcal{N}_g)$. Proposition 6.3 lets us construct a sequence of Teichmüller geodesic segments such that for on some interior point, the length of a given one-sided curve γ goes to 0. If those points were within distance R of $\mathcal{T}_\varepsilon^-$, there would be some point in $\mathcal{T}_\varepsilon^-$ where the length of γ was at most $\exp(2R)$ times the length of γ in the geodesic, by Wolpert's lemma. But since the length of γ in the geodesic goes to 0, the length in the corresponding closest point in $\mathcal{T}_\varepsilon^-$ must also go to 0. This violates the definition of $\mathcal{T}_\varepsilon^-$, giving us a contradiction, and proving the result. □

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