REFEREE REPORT FOR "THE LIMIT SET OF NON-ORIENTABLE MAPPING CLASS GROUPS"

1. Summary of the paper

The paper under review studies the dynamics of the action of $MCG(N_g)$ on $T(N_g)$ where N_g is the genus g unoriented surface. Norbury and Gendulphe have some results and conjectures about this problems.

1. The quotient $T(N_g)/MCG(N_g)$ has infinite volume.
2. The action of $MCG(N_g)$ on the Thurston boundary is not minimal
3. There is an $MCG(N_g)$-equivariant finite covolume deformation retract of $T(N_g)$ which is T_ϵ^-, the set of points in the Teichmuller space that have no one-sided curves shorter than ϵ.

This paper answers a question of Gendulphe in (Question 19.1 in [Gen17]):

Theorem. *T_ϵ^- is not quasi-convex w.r.t the Teichmuller metric.*

The author also studies Conjecture 9.1 in [Gen17] to compute the limit set of $MCG(N_g)$. The conjecture is that the limit set is equal to $PMF^+(N_g)$, the set of projective measured foliations that does not contain one-sided leaves. The result gives a criterian of a large subset of $PMF^+(N_g)$ is in the limit set and the limit set is contained in $PMF^+(N_g)$, or the following inclusion.

$$\{\text{orientable and ergodic, unique ergodic or periodic}\} \subset \Lambda_{dyn} \subset \Lambda_{geom} \subset PMF^+(N_g)$$

Recently Conjecture 9.1 is fully proved by Erlandsson, Gendulphe, Pasquinelli, and Suoto with a different method.

Even though the 2nd main theorem has been improved by the other group, this paper under review is still interesting. It needs some modification according to the following.

2. Some details

1. It would be helpful if the author can include corresponding results about $MCG(S_g)$ in the 2nd chapter as comparison.

2. You have two different notions of limit set, could you state more precisely what did Erlandsson, Gendulphe, Pasquinelli, and Suoto prove? According what you wrote, they prove $\Lambda_{dyn} = \Lambda_{geom} = PMF^+(N_g)$, right? Is it possible to improve your method to conclude the same?

3. What is the definition of p_3 in Chapter 4? Page 16, last paragraph, which means its length should be going to ∞, why? Your formula is a bit weird, the left side is p_4?

4. The whole chapter 3 has weird notation with $l_{hyp}(p_i)$. It would make more sense to use l_i to represent the hyperbolic length w.r.t the metric m_i. Also please reorganize your computation to make it more logical. The final contradiction is also weird, what do you mean by this can't happen if $l_{hyp}(p_i)$ approaches ∞? I thought it should be zero by your assumption. Your equation (18) should be interpreted algebraically in a better way.

5. I understand that you use hyperbolic geometry estimates to argue, but if you can, please have a more intuitive explanation. Is it similar to Margulis lemma type result?

6. Why do you not have construction for lower genera?

7. Could you state more precisely how does your result (or related result about limit set) help with the counting problem for N_g?

8. In the introduction, you said on Page 2, that the key idea involved in Theorem 5.2 is to determine the limit set, where do you use this in the proof? If not, you can find a different bridge between these two results.