

Real Analysis qualifying review big list

Students at the University of Michigan

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How to add to this document

This document contains a subset of the old real analysis qual problems, and solutions to some of them. To add a problem to this list, use the `problem` environment: this environment takes two arguments, the date the problem appeared on a qual, and tags describing the problem.

```
\begin{problem}{<Month> <Year>}{tag 1, tag 2, ...}  
  <Problem statement>  
\end{problem}
```

To add a solution, or a solution sketch, use the `solution` and `sketch` environments. These environments do not take any arguments.

1 Categorized (without solutions)

Problem 1 (Date: September 2011, tags: bounded variation). Suppose $f : [0, 1] \rightarrow \mathbb{R}$ satisfies $f(x) - f(y) < x - y$ for all $x, y \in [0, 1], x > y$. Show that f' exists almost everywhere on $[0, 1]$ or give a counterexample.

Problem 2 (Date: September 2011, tags: hölder's inequality). Let (X, Ω, μ) be a finite measure space.

- (i) Prove that for any $p < q$, $L_q(\mu) \subset L_p(\mu)$.
- (ii) Assume that for any $t > 0$ there exists $E \in \Omega$ satisfying

$$0 < \mu(E) < t.$$

Prove that for any $1 < p < \infty$ there exists a function $f \in L_p(\mu)$ such that $f \notin L_q(\mu)$ for any $q > p$.

Problem 3 (Date: September 2011, tags: L^p spaces). For a real valued function $f(x, y)$ on \mathbb{R}^2 which is in L^2 , show that $f(x + \epsilon, y + \epsilon) \rightarrow f(x, y)$ in L^2 when $\epsilon \rightarrow 0$.

Problem 4 (Date: September 2011, tags: measurable sets?). Let $f \in L_1([0, 1])$ be a function such that $\int_E f(x) dx = 0$ for any measurable set $E \subset [0, 1]$ of Lebesgue measure $1/2$. Prove that $f = 0$ a.e.

Problem 5 (Date: January 2011, tags: measurable functions).

Let A be a sequence of measurable subsets of $[0, 1]$ such that $\inf m(A_n) > 0$, where m stands for the Lebesgue measure.

- (i) Prove that there exists $x \in [0, 1]$ which belongs to infinitely many of the sets A_n .

- (ii) Does there necessarily exist a point which (does not?) belong to any of the sets A_n , except finitely many?

Problem 6 (Date: January 2011, tags: integral convergence). Let $\{f_n\} \subset L_1(\mu)$ be a decreasing sequence of functions such that $f_n \rightarrow f$ a.e. Prove that

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu.$$

Problem 7 (Date: January 2011, tags: simple function approximation). Let $f : X \rightarrow [0, +\infty)$ be an integrable function on a measure space (X, \mathcal{A}, μ) . Define the measure ν by $\nu(A) = \int_A f d\mu$.

- (i) Prove that the measure ν is σ -additive.
- (ii) Prove that if $g \in L_1(\nu)$, then $\int_X g d\nu = \int_X fg d\mu$.
(Hint: first, assume that g is a simple positive function. Then extend the result to non-negative integrable functions using limit theorems).

Problem 8 (Date: January 2011, tags: integral inequalities). Let $f_n : \mathbb{R} \rightarrow [0, 1]$ be functions such that $\sup_{x \in \mathbb{R}} f_n(x) = 1/n$ and $\int_{\mathbb{R}} f(x) dx = 1$. Set

$$F(x) = \sup_{n \in \mathbb{N}} f_n(x).$$

Find all possible values of $\int_{\mathbb{R}} F(x) dx$.

Problem 9 (Date: January 2011, tags: integral convergence?). Let $f \in L_\infty([0, 1])$. Prove that

$$\lim_{n \rightarrow \infty} \frac{\int_{[0,1]} |f(x)|^{n+1} dx}{\int_{[0,1]} |f(x)|^n dx} = \|f\|_\infty.$$

Problem 10 (Date: January 2011, tags: egoroff's theorem). Let E be the exceptional set in Egoroff's theorem. Is it possible to prove Egoroff's theorem with $l(E) = 0$ instead of $l(E) < \epsilon$?

Problem 11 (Date: January 2011, tags: simple functions). Let $f : X \rightarrow [0, \infty]$ be a measurable function. Assume that $\mu(X) < \infty$. Prove that $\int f d\mu < +\infty$ if and only if

$$\sum_{n=1}^{\infty} 2^n \mu(x \in X \mid f(x) \geq 2^n) < +\infty.$$

Problem 12 (Date: January 2011, tags: dominated convergence). Let $f_n, g_n, f, g \in L_1(\mu)$ be functions such that $f_n \rightarrow f$ a.e., $g_n \rightarrow g$ a.e. and $|f_n| \leq g_n$. Prove that if $\int g_n d\mu \rightarrow \int g d\mu$, then $\int f_n d\mu \rightarrow \int f d\mu$.

(Hint: follow the proof of Lebesgue dominated convergence theorem.)

Problem 13 (Date: January 2011, tags:). Let $f_n, f \in L_1(\mu)$ be such that $f_n \rightarrow f$ a.e. Prove that if $\|f_n\|_1 \rightarrow \|f\|_1$, then $f_n \rightarrow f$ in $L_1(\mu)$.

(Hint: use the previous problem)

Problem 14 (Date: January 2011, tags: change of variables). Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be L_1 -functions.

- (a) Prove that

$$\int_{\mathbb{R}} |f(x-y)g(y)| dm(y) < +\infty.$$

(b) Let

$$h(x) = \int_{\mathbb{R}} f(x-y)g(y)dm(y).$$

Prove that $h \in L_1(\mathbb{R})$ and $\|h\|_1 \leq \|f\|_1 \cdot \|g\|_1$.

Problem 15 (Date: January 2012, tags: measurable functions). Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function. Prove that the function f' is measurable.

Problem 16 (Date: January 2012, tags: L^p spaces). Let $1 \leq p < \infty$, and let $f \in L_p(\mu)$. Prove that

$$\lim_{t \rightarrow 0} t^p \mu\{x \mid |f(x)| > t\} = 0.$$

Problem 17 (Date: January 2012, tags: Lebesgue differentiation theorem, Hardy-Littlewood maximal estimate?). Let $f \in L_1(\mathbb{R})$. For $n \in \mathbb{N}$ define the function $g_n : \mathbb{R} \rightarrow \mathbb{R}$ as follows. For $k \in \mathbb{Z}$ and for $x \in [k/n, (k+1)/n)$ set

$$g_n(x) = n \int_{k/n}^{(k+1)/n} f(x) dx.$$

Prove that g_n converges to f a.e. and in $L_1(\mathbb{R})$.

Problem 18 (Date: January 2012, tags: integral convergence). Let (X, \mathcal{A}, μ) be a finite measure space ($\mu(X) < \infty$). Assume that a sequence $\{f_n\}_{n=1}^{\infty} \subseteq L_1(\mu)$ satisfies the condition

$$\frac{1}{\sqrt{\mu(E)}} \int_E |f_n| d\mu \leq 1$$

for all $n \in \mathbb{N}$ and all sets E of positive measure. Prove that if $f_n \rightarrow f$ a.e., then $f \in L_1(\mu)$ and

$$\int_X f_n d\mu \rightarrow \int_X f d\mu.$$

Problem 19 (Date: January 2012, tags: L^p spaces). Construct a function $f \in L_1(\mathbb{R})$ such that $f \notin L_2((a, b))$ for any interval $(a, b) \subseteq \mathbb{R}$.

Problem 20 (Date: September 2019, tags: countable subadditivity?). Let E be the set of all $x \in (0, 1)$ such that there exists a sequence of irreducible fractions $\{p_n/q_n\}_{n \in \mathbb{N}}$ with $p_n, q_n \in \mathbb{N}$, $q_1 < q_2 < \dots$ such that

$$\left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n^3}, \quad n = 1, 2, \dots$$

Prove that the Lebesgue measure of E is zero.

Problem 21 (Date: September 2019, tags: integral convergence). Let f be a measurable function on $(0, \infty)$, and for $n = 1, 2, \dots$ let f_n be defined by

$$f_n(x) = f(x)e^{-x} \left[1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} \right].$$

Suppose $f \in L^2[(0, \infty)]$. Prove that $\lim_{n \rightarrow \infty} \|f_n - f\|_{L^2[(0, \infty)]} = 0$

Problem 22 (Date: September 2019, tags: Fubini, Hölder). Let $f : \mathbb{R} \times (0, 1) \rightarrow \mathbb{R}$ be a measurable function such that for any $y \in (0, 1)$,

$$\int_{\mathbb{R}} f^2(x, y) dx \leq 1.$$

Prove there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$, with $\lim_{n \rightarrow \infty} x_n = +\infty$, such that

$$\lim_{n \rightarrow \infty} \int_0^1 |f(x_n, y)| dy = 0.$$

Problem 23 (Date: September 2019, tags: Hölder). Let (X, Ω, μ) be a measure space with $\mu(X) = 1$, and let $f \in L^2(\mu)$ be a non-negative function satisfying $\int_X f d\mu \geq 1$. Prove that

$$\mu(\{x \in X \mid f(x) > 1\}) \geq \frac{(\int_X f d\mu - 1)^2}{\int_X f^2 d\mu}.$$

Problem 24 (Date: September 2019, tags: absolute continuity). A function $f : (0, 1) \rightarrow \mathbb{R}$ is locally Lipschitz if for any $x \in (0, 1)$ there is an open interval I_x with $x \in I_x \subset (0, 1)$ and a constant C_x such that $|f(y) - f(y')| \leq C_x|y - y'|$ for $y, y' \in I_x$.

- (a) Prove that a locally Lipschitz function $f(\cdot)$ is absolutely continuous on any compact subinterval $[a, b] \subset (0, 1)$.
- (b) Give an example of a locally Lipschitz function $f : (0, 1) \rightarrow \mathbb{R}$ which extends to a continuous function on the closed interval $[0, 1]$, but is not absolutely continuous on $[0, 1]$.

Problem 25 (Date: May 2020, tags: outer regularity). Let $E \subset (0, 1)$ be a measurable set such that for any interval $(a, b) \subset (0, 1)$, there exists an interval $(c, d) \subset (a, b) \setminus E$ with

$$d - c \geq \frac{a}{10}(b - a).$$

Prove that $m(E) = 0$.

Problem 26 (Date: May 2020, tags: bounded variation). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a Lebesgue measurable function such that

$$f(y) \leq f(x) + (x^2 + y^2)(x - y) \quad \text{for } -\infty < y < x < \infty.$$

Show that the derivative function $x \rightarrow f'(x)$ exists a.e. on \mathbb{R} .

Problem 27 (Date: May 2020, tags: triangle inequality?). Let $r_n, n = 1, 2, \dots$, be an enumeration of the rationals in the interval $[0, 1]$ and consider the function $f : [0, 1] \rightarrow \mathbb{R} \cup \infty$ defined by

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1}{|x - r_n|^{1/3}}, \quad 0 \leq x \leq 1.$$

Show that $f \in L^2(0, 1)$.

Problem 28 (Date: May 2020, tags: L^p spaces). Suppose f is a C^1 function on \mathbb{R} satisfying $f(0) = 0$, $|f(x)| \leq |x|^{-1/2}$, $x \neq 0$. Let g be in $L^1(\mathbb{R})$.

- (a) Show there is a constant C such that $m\{|g| > \alpha\} \leq C/\alpha$ for all $\alpha > 0$.
- (b) Show that the function $h(x) = f(g(x))$ is in $L^1(\mathbb{R})$.

Problem 29 (Date: May 2011, tags: integral convergence). Let $f_n, g_n, f, g \in L_1(\mu)$ be functions such that $f_n \rightarrow f$ a.e., $g_n \rightarrow g$ a.e. and $|f_n| \leq g_n$. Prove that if $\int g_n d\mu \rightarrow \int g d\mu$, then $\int f_n d\mu \rightarrow \int f d\mu$. (Hint: use Fatou's Lemma.)

Problem 30 (Date: May 2011, tags: Egorov's theorem). Let $\{f_n : [0, 1] \rightarrow \mathbb{R}\}_{n=1}^{\infty}$ be a sequence of continuous functions such that $f_n(x) \rightarrow f(x)$ for any $x \in [0, 1]$. Does there exist a set $E \subset [0, 1]$ of Lebesgue measure 0 such that $f_n \rightarrow f$ uniformly on $[0, 1] \setminus E$?

2 Categorized (with solutions)

Problem 31 (Date: September 2011, tags: integral convergence). Let $f \in L_1([0, 1], dx)$. Find

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^1 \log \left(1 + e^{nf(x)} \right) dx.$$

Solution sketch. Step 1: Consider pointwise convergence in two different sets: $f(x) \leq 0$ and $f(x) > 0$. Step 2: Use convergence theorems in these two domains to get convergence of integral. \square

Problem 32 (Date: May 2020, tags: change of variables, integral convergence). Let f_n , $n = 1, 2, \dots$, be the sequence of functions on $(0, \infty)$ defined by

$$f_n(x) = \frac{1}{n} \left(1 - \frac{x}{n} \right)^n e^x, \quad 0 < x < n, \quad f_n(x) = 0, \quad x \geq n.$$

Prove that the sequence a_n , $n = 1, 2, \dots$, given by

$$a_n = \int_0^\infty f_n(x) dx \quad \text{converges and identify } a_\infty = \lim_{n \rightarrow \infty} a_n.$$

Solution sketch. Step 1: Change variables so that all integrals are over $[0, 1]$. Step 2: Observe that the new integrand is converging pointwise to 0 a.e. Step 3: Use monotone/dominated convergence theorem. \square

3 Uncategorized

Problem 33 (Date: January 2011, tags:). Let $f \in L_1([0, 1])$ be a function such that $f(x) > 0$ a.e.

(i) Prove that for any $0 < a < 1$

$$\inf_{m(A)=a} \int_A f dm > 0.$$

(ii) Does the previous statement hold for a function $f \in L_1(\mathbb{R})$ such that $f(x) > 0$ a.e.