

# Real Analysis qualifying review big list

Students at the University of Michigan

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## How to add to this document

This document contains a subset of the old real analysis qual problems, and solutions to some of them. To add a problem to this list, use the `problem` environment: this environment takes two arguments, the date the problem appeared on a qual, and tags describing the problem.

```
\begin{problem}{<Month> <Year>}{tag 1, tag 2, ...}  
  <Problem statement>  
\end{problem}
```

To add a solution, or a solution sketch, use the `solution` and `sketch` environments. These environments do not take any arguments.

## Notation

Unless otherwise specified,  $m$  refers to the Lebesgue measure on  $\mathbb{R}^n$  and subsets of  $\mathbb{R}^n$ , and  $m^*$  refers to the Lebesgue outer measure.

## 1 Categorized (without solutions)

### 1.1 Basic measure theory and integrability

**Problem 1** (Date: 2013 draft, tags: ). Let  $(X, \mathcal{A}, \mu)$  be a finite measure space. For a set  $A \subset X$  define  $\mu_*(A) = \mu(X) - \mu^*(X \setminus A)$ , where  $\mu^*$  is the outer measure. Prove that  $\mu_*(A) \leq \mu^*(A)$  for any  $A \subset X$ .

**Problem 2** (Date: 2013 draft, tags: ). Let  $f_1, f_2, \dots, f, g$  be measurable functions on a measure space  $(X, \mathcal{A}, \mu)$ . Assume that  $f_n \rightarrow f$  in measure and  $f_n \leq g$  a.e. Prove that  $f \leq g$  a.e.

**Problem 3** (Date: 2013 draft, tags: For  $n \in \mathbb{N}$  define the set  $A_n \subset \mathbb{R}$  by

$$A_k = \bigcup_{n=k}^{\infty} \left( x_n - \frac{k}{n^3}, x_n + \frac{k}{n^3} \right).$$

Prove that  $m(\bigcap_{k=1}^{\infty} A_k) = 0$ , where  $m$  denotes the Lebesgue measure.

**Problem 4** (Date: September 2014, tags: ). Let  $\{f_k(x)\}$  be a sequence of nonnegative measurable functions on  $E$  and  $m(E) < \infty$ . Show that  $\{f_k(x)\}$  converges in measure to 0 if and only if

$$\lim_{k \rightarrow \infty} \int_E \frac{f_k(x)}{1 + f_k(x)} dx = 0.$$

**Problem 5** (Date: September 2014, tags: ). Construct a measurable subset  $A$  of  $(0, 1)$  such that  $m(A) < 1$  and  $m(A \cap (a, b)) > 0$  for any  $(a, b) \subset (0, 1)$ .

**Problem 6** (Date: January 2015, tags: ). Let  $A, B \subset \mathbb{R}^d$ . Assume  $A \cup B$  is measurable, and  $m(A \cup B) < \infty$ . If

$$m(A \cup B) = m^*(A) + m^*(B)$$

Show that  $A$  and  $B$  are measurable.

(Hint: prove first that for any set  $A$ , there a measurable set  $U$ , with  $A \subset U$ , such that  $m^*(A) = m(U)$ .)

**Problem 7** (Date: January 2015, tags: ). Let  $f$  be a nonnegative measurable function on  $(0, 1)$ . Assume that there is a constant  $c$ , such that

$$\int_0^1 (f(x))^n dx = c, \quad n = 1, 2, \dots$$

Show that there is a measurable set  $E \subset (0, 1)$ , such that

$$f(x) = \chi_E(x), \quad \text{for a.e. } x \in (0, 1).$$

**Problem 8** (Date: May 2020, tags: ). Suppose  $f$  is a  $C^1$  function on  $\mathbb{R}$  satisfying  $f(0) = 0$ ,  $|f(x)| \leq |x|^{-1/2}$ ,  $x \neq 0$ . Let  $g$  be in  $L^1(\mathbb{R})$ .

(a) Show there is a constant  $C$  such that  $m\{|g| > \alpha\} \leq C/\alpha$  for all  $\alpha > 0$ .

(b) Show that the function  $h(x) = f(g(x))$  is in  $L^1(\mathbb{R})$ .

**Problem 9** (Date: May 2020, tags: ). Let  $r_n$ ,  $n = 1, 2, \dots$ , be an enumeration of the rationals in the interval  $[0, 1]$  and consider the function  $f : [0, 1] \rightarrow \mathbb{R} \cup \infty$  defined by

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1}{|x - r_n|^{1/3}}, \quad 0 \leq x \leq 1.$$

Show that  $f \in L^2(0, 1)$ .

**Problem 10** (Date: May 2020, tags: ). Let  $E \subset (0, 1)$  be a measurable set such that for any interval  $(a, b) \subset (0, 1)$ , there exists an interval  $(c, d) \subset (a, b) \setminus E$  with

$$d - c \geq \frac{a}{10}(b - a).$$

Prove that  $m(E) = 0$ .

**Problem 11** (Date: September 2019, tags: ). Let  $E$  be the set of all  $x \in (0, 1)$  such that there exists a sequence of irreducible fractions  $\{p_n/q_n\}_{n \in \mathbb{N}}$  with  $p_n, q_n \in \mathbb{N}$ ,  $q_1 < q_2 < \dots$  such that

$$\left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n^3}, \quad n = 1, 2, \dots$$

Prove that the Lebesgue measure of  $E$  is zero.

**Problem 12** (Date: January 2012, tags: ). Construct a function  $f \in L_1(\mathbb{R})$  such that  $f \notin L_2((a, b))$  for any interval  $(a, b) \subseteq \mathbb{R}$ .

**Problem 13** (Date: September 2011, tags: ). Let  $f \in L_1([0, 1])$  be a function such that  $\int_E f(x) dx = 0$  for any measurable set  $E \subset [0, 1]$  of Lebesgue measure  $1/2$ . Prove that  $f = 0$  a.e.

**Problem 14** (Date: January 2011, tags: ).

Let  $A$  be a sequence of measurable subsets of  $[0, 1]$  such that  $\inf m(A_n) > 0$ , where  $m$  stands for the Lebesgue measure.

- (i) Prove that there exists  $x \in [0, 1]$  which belongs to infinitely many of the sets  $A_n$ .
- (ii) Does there necessarily exist a point which (does not?) belong to any of the sets  $A_n$ , except finitely many?

**Problem 15** (Date: January 2011, tags: ). Let  $f : X \rightarrow [0, +\infty)$  be an integrable function on a measure space  $(X, \mathcal{A}, \mu)$ . Define the measure  $\nu$  by  $\nu(A) = \int_A f d\mu$ .

- (i) Prove that the measure  $\nu$  is  $\sigma$ -additive.
- (ii) Prove that if  $g \in L_1(\nu)$ , then  $\int_X g d\nu = \int_X fg d\mu$ .  
(Hint: first, assume that  $g$  is a simple positive function. Then extend the result to non-negative integrable functions using limit theorems).

**Problem 16** (Date: January 2011, tags: ). Let  $f : X \rightarrow [0, \infty]$  be a measurable function. Assume that  $\mu(X) < \infty$ . Prove that  $\int f d\mu < +\infty$  if and only if

$$\sum_{n=1}^{\infty} 2^n \mu(x \in X \mid f(x) \geq 2^n) < +\infty.$$

**Problem 17** (Date: January 2012, tags: ). Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable function. Prove that the function  $f'$  is measurable.

**Problem 18** (Date: January 2012, tags: ). Let  $1 \leq p < \infty$ , and let  $f \in L_p(\mu)$ . Prove that

$$\lim_{t \rightarrow 0} t^p \mu\{x \mid |f(x)| > t\} = 0.$$

## 1.2 Integral convergence

**Problem 19** (Date: 2013 draft, tags: ). Prove that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \frac{\cos^n(\pi x)}{(x - n)^2 + 1} dx$$

exists and find it.

**Problem 20** (Date: May 2011, tags: ). Let  $f_n, g_n, f, g \in L_1(\mu)$  be functions such that  $f_n \rightarrow f$  a.e.,  $g_n \rightarrow g$  a.e. and  $|f_n| \leq g_n$ . Prove that if  $\int g_n d\mu \rightarrow \int g d\mu$ , then  $\int f_n d\mu \rightarrow \int f d\mu$ . (Hint: use Fatou's Lemma.)

**Problem 21** (Date: September 2019, tags: ). Let  $f$  be a measurable function on  $(0, \infty)$ , and for  $n = 1, 2, \dots$  let  $f_n$  be defined by

$$f_n(x) = f(x)e^{-x} \left[ 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} \right].$$

Suppose  $f \in L^2[(0, \infty)]$ . Prove that  $\lim_{n \rightarrow \infty} \|f_n - f\|_{L^2[(0, \infty)]} = 0$

**Problem 22** (Date: January 2012, tags: ). Let  $(X, \mathcal{A}, \mu)$  be a finite measure space ( $\mu(X) < \infty$ ). Assume that a sequence  $\{f_n\}_{n=1}^\infty \subseteq L_1(\mu)$  satisfies the condition

$$\frac{1}{\sqrt{\mu(E)}} \int_E |f_n| d\mu \leq 1$$

for all  $n \in \mathbb{N}$  and all sets  $E$  of positive measure. Prove that if  $f_n \rightarrow f$  a.e., then  $f \in L_1(\mu)$  and

$$\int_X f_n d\mu \rightarrow \int_X f d\mu.$$

**Problem 23** (Date: January 2011, tags: ). Let  $\{f_n\} \subset L_1(\mu)$  be a decreasing sequence of functions such that  $f_n \rightarrow f$  a.e. Prove that

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu.$$

**Problem 24** (Date: September 2011, tags: ). For a real valued function  $f(x, y)$  on  $\mathbb{R}^2$  which is in  $L^2$ , show that  $f(x + \epsilon, y + \epsilon) \rightarrow f(x, y)$  in  $L^2$  when  $\epsilon \rightarrow 0$ .

**Problem 25** (Date: January 2011, tags: ). Let  $f_n, f \in L_1(\mu)$  be such that  $f_n \rightarrow f$  a.e. Prove that if  $\|f_n\|_1 \rightarrow \|f\|_1$ , then  $f_n \rightarrow f$  in  $L_1(\mu)$ .  
(Hint: use the previous problem)

### 1.3 Integral inequalities

**Problem 26** (Date: January 2014, tags: ). Let  $E \subset [0, 1]$  be a measurable set,  $m(E) \geq \frac{99}{100}$ . Prove that there exists  $x \in [0, 1]$  such that for any  $r \in (0, 1)$ ,

$$m(E \cap (x - r, x + r)) \geq \frac{r}{4}.$$

*Hint:* One approach to this problem involves the Hardy-Littlewood maximal inequality.

**Problem 27** (Date: January 2014, tags: ). Find all  $q \geq 1$ , such that  $f(x^2) \in L_q((0, 1), m)$  for any  $f(x) \in L_4((0, 1), m)$ , where  $m$  denotes the Lebesgue measure.

**Problem 28** (Date: September 2014, tags: ). Let  $K = \{f : (0, +\infty) \rightarrow \mathbb{R} \mid \int_0^\infty f^4(x) dx \leq 1\}$ . Evaluate

$$\sup_{f \in K} \int_0^\infty f^3(x) e^{-x} dx.$$

**Problem 29** (Date: September 2019, tags: ). Let  $f : \mathbb{R} \times (0, 1) \rightarrow \mathbb{R}$  be a measurable function such that for any  $y \in (0, 1)$ ,

$$\int_{\mathbb{R}} f^2(x, y) dx \leq 1.$$

Prove there exists a sequence  $\{x_n\}_{n \in \mathbb{N}}$ , with  $\lim_{n \rightarrow \infty} x_n = +\infty$ , such that

$$\lim_{n \rightarrow \infty} \int_0^1 |f(x_n, y)| dy = 0.$$

**Problem 30** (Date: January 2011, tags: ). Let  $f \in L_\infty([0, 1])$ . Prove that

$$\lim_{n \rightarrow \infty} \frac{\int_{[0,1]} |f(x)|^{n+1} dx}{\int_{[0,1]} |f(x)|^n dx} = \|f\|_\infty.$$

**Problem 31** (Date: January 2011, tags: ). Let  $f_n : \mathbb{R} \rightarrow [0, 1]$  be functions such that  $\sup_{x \in \mathbb{R}} f_n(x) = 1/n$  and  $\int_{\mathbb{R}} f(x) dx = 1$ . Set

$$F(x) = \sup_{n \in \mathbb{N}} f_n(x).$$

Find all possible values of  $\int_{\mathbb{R}} F(x) dx$ .

**Problem 32** (Date: September 2011, tags: ). Let  $(X, \Omega, \mu)$  be a finite measure space.

- (i) Prove that for any  $p < q$ ,  $L_q(\mu) \subset L_p(\mu)$ .
- (ii) Assume that for any  $t > 0$  there exists  $E \in \Omega$  satisfying

$$0 < \mu(E) < t.$$

Prove that for any  $1 < p < \infty$  there exists a function  $f \in L_p(\mu)$  such that  $f \notin L_q(\mu)$  for any  $q > p$ .

## 1.4 Miscellaneous

**Problem 33** (Date: September 2011, tags: ). Suppose  $f : [0, 1] \rightarrow \mathbb{R}$  satisfies  $f(x) - f(y) < x - y$  for all  $x, y \in [0, 1], x > y$ . Show that  $f'$  exists almost everywhere on  $[0, 1]$  or give a counterexample.

**Problem 34** (Date: January 2011, tags: ). Let  $E$  be the exceptional set in Egoroff's theorem. Is it possible to prove Egoroff's theorem with  $l(E) = 0$  instead of  $l(E) < \epsilon$ ?

**Problem 35** (Date: January 2011, tags: ). Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  be  $L_1$ -functions.

(a) Prove that

$$\int_{\mathbb{R}} |f(x - y)g(y)| dm(y) < +\infty.$$

(b) Let

$$h(x) = \int_{\mathbb{R}} f(x - y)g(y) dm(y).$$

Prove that  $h \in L_1(\mathbb{R})$  and  $\|h\|_1 \leq \|f\|_1 \cdot \|g\|_1$ .

**Problem 36** (Date: January 2012, tags:

). Let  $f \in L_1(\mathbb{R})$ . For  $n \in \mathbb{N}$  define the function  $g_n : \mathbb{R} \rightarrow \mathbb{R}$  as follows. For  $k \in \mathbb{Z}$  and for  $x \in [k/n, (k+1)/n)$  set

$$g_n(x) = n \int_{k/n}^{(k+1)/n} f(x) dx.$$

Prove that  $g_n$  converges to  $f$  a.e. and in  $L_1(\mathbb{R})$ .

**Problem 37** (Date: September 2019, tags:

). A function  $f : (0, 1) \rightarrow \mathbb{R}$  is locally Lipschitz if for any  $x \in (0, 1)$  there is an open interval  $I_x$  with  $x \in I_x \subset (0, 1)$  and a constant  $C_x$  such that  $|f(y) - f(y')| \leq C_x |y - y'|$  for  $y, y' \in I_x$ .

- (a) Prove that a locally Lipschitz function  $f(\cdot)$  is absolutely continuous on any compact subinterval  $[a, b] \subset (0, 1)$ .
- (b) Give an example of a locally Lipschitz function  $f : (0, 1) \rightarrow \mathbb{R}$  which extends to a continuous function on the closed interval  $[0, 1]$ , but is not absolutely continuous on  $[0, 1]$ .

**Problem 38** (Date: May 2020, tags:

). Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a Lebesgue measurable function such that

$$f(y) \leq f(x) + (x^2 + y^2)(x - y) \quad \text{for } -\infty < y < x < \infty.$$

Show that the derivative function  $x \rightarrow f'(x)$  exists a.e. on  $\mathbb{R}$ .

**Problem 39** (Date: May 2011, tags:

). Let  $\{f_n : [0, 1] \rightarrow \mathbb{R}\}_{n=1}^\infty$  be a sequence of continuous functions such that  $f_n(x) \rightarrow f(x)$  for any  $x \in [0, 1]$ . Does there exist a set  $E \subset [0, 1]$  of Lebesgue measure 0 such that  $f_n \rightarrow f$  uniformly on  $[0, 1] \setminus E$ ?

**Problem 40** (Date: January 2011, tags: ). Let  $f \in L_1([0, 1])$  be a function such that  $f(x) > 0$  a.e.

- (i) Prove that for any  $0 < a < 1$

$$\inf_{m(A)=a} \int_A f dm > 0.$$

- (ii) Does the previous statement hold for a function  $f \in L_1(\mathbb{R})$  such that  $f(x) > 0$  a.e.

**Problem 41** (Date: January 2015, tags:

). Let  $E_k \subset [a, b]$ ,  $k \in \mathbb{N}$  be measurable sets, and there exists  $\delta > 0$  such that  $m(E_k) \geq \delta$  for all  $k$ . Assume that  $a_k \in \mathbb{R}$  satisfies

$$\sum_{k=1}^\infty |a_k| \chi_{E_k}(x) < \infty \quad \text{for a.e. } x \in [a, b].$$

Show that

$$\sum_{k=1}^\infty |a_k| < \infty.$$

(For extra challenge, find a proof that does not use Egorov's theorem).

**Problem 42** (Date: September 2014, tags:

). Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function such that  $\int_{\mathbb{R}} |f(x)| dx < \infty$ . Show that the sequence

$$h_n(x) = \frac{1}{n} \sum_{k=1}^n f\left(x + \frac{k}{n}\right)$$

converges in  $L_1(\mathbb{R})$ .

**Problem 43** (Date: January 2014, tags: ). Let  $f \in L_1 \cap L_4$  (on some measure space). Prove that the function defined on  $[1, 4]$ , given by the following formula

$$p \mapsto \|f\|_p$$

is continuous.

**Problem 44** (Date: January 2014, tags: ). Let

$$\begin{aligned} E &\subset \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x\} \\ E_x &= \{y \mid (x, y) \in E\} \\ E_y &= \{x \mid (x, y) \in E\} \end{aligned}$$

and assume that  $m(E_x) \geq x^3$  for any  $x \in [0, 1]$ .

- (i) Prove that there exists  $y \in [0, 1]$  such that  $m(E_y) \geq \frac{1}{4}$ .
- (ii) (Hard) Prove that there exists  $y \in [0, 1]$  such that  $m(E_y) \geq c$ , where  $c > 1/4$  is a constant independent of  $E$ . Find the optimal such  $c$ .

## 2 Categorized (with solutions)

**Problem 45** (Date: September 2014, tags: ). Let  $1 \leq p < \infty, f \in L^p(\mathbb{R}^n)$ . Let

$$f_*(\lambda) = m(\{x : |f(x)| > \lambda\}), \quad \lambda > 0$$

Show that

- (i)  $p \int_0^\infty \lambda^{p-1} f_*(\lambda) d\lambda = \int |f(x)|^p dx$
- (ii)  $\lim_{\lambda \rightarrow \infty} \lambda^p f_*(\lambda) = 0$
- (iii)  $\lim_{\lambda \rightarrow 0} \lambda^p f_*(\lambda) = 0$

*Solution sketch.* For simplicity, prove the result for  $L^1$  functions, and use a change of variables argument to prove it for  $p > 1$ . For part (i), express  $f_*(\lambda)$  as the integral over  $\mathbb{R}$  of the indicator of  $\{x \mid |f(x)| > \lambda\}$ , and then use Fubini to swap integrals. For parts (ii) and (iii), note that  $f_*$  is decreasing, so it will suffice to prove it for a sequence of  $\lambda$  going to  $\infty$  and 0. We pick a sequence such that the sums of corresponding quantities are bounded above by an absolute constant times the integral of  $f$ : we can do this by interpreting the quantities in the sequence as areas of rectangles under the graph of the function  $f$ .  $\square$

**Problem 46** (Date: January 2014, tags: **Unclear what the tag should be**). Prove or disprove: If  $E$  is an open subset of  $\mathbb{R}$  with  $m(E) = 1$  then there is a finite union of intervals  $F$  containing  $E$  with  $m(F) < 1.1$ .

*Solution sketch.* False: consider the set  $E$  obtained by taking the unions of balls of radius  $\frac{1}{n^2}$  around  $n \in \mathbb{N}$ . This open set has finite measure, but any finite union of intervals containing it will have infinite measure.  $\square$

**Problem 47** (Date: 2013 draft, tags: ). Let  $A \subset [0, 1] \times [0, 1]$  be the set of points  $(x, y)$  with decimal representations  $x = 0.x_1x_2 \dots$ ,  $y = 0.y_1y_2 \dots$  such that  $x_n y_n = 5$  for all  $n \in \mathbb{N}$ . Prove that the set  $A$  is measurable and find its Lebesgue measure.

*Solution sketch.* Observe that for any  $(x, y) \in A$ ,  $x + y = 0.\bar{6}$ . This means  $A$  is contained in the graph of the continuous function  $y = 0.\bar{6} - x$ , which is a measure 0 set. Subsets of measure 0 sets are measurable, and have measure 0.  $\square$

**Problem 48** (Date: September 2011, tags: ). Let  $f \in L_1([0, 1], dx)$ . Find

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^1 \log(1 + e^{nf(x)}) dx.$$

*Solution sketch.* Step 1: Consider pointwise convergence in two different sets:  $f(x) \leq 0$  and  $f(x) > 0$ . Step 2: Use convergence theorems in these two domains to get convergence of integral.  $\square$

**Problem 49** (Date: May 2020, tags: ). Let  $f_n$ ,  $n = 1, 2, \dots$ , be the sequence of functions on  $(0, \infty)$  defined by

$$f_n(x) = \frac{1}{n} \left(1 - \frac{x}{n}\right)^n e^x, \quad 0 < x < n, \quad f_n(x) = 0, \quad x \geq n.$$

Prove that the sequence  $a_n$ ,  $n = 1, 2, \dots$ , given by

$$a_n = \int_0^\infty f_n(x) dx \quad \text{converges and identify } a_\infty = \lim_{n \rightarrow \infty} a_n.$$

*Solution sketch.* Step 1: Change variables so that all integrals are over  $[0, 1]$ . Step 2: Observe that the new integrand is converging pointwise to 0 a.e. Step 3: Use monotone/dominated convergence theorem.  $\square$

**Problem 50** (Date: January 2015, tags: ). Let  $f$  be locally integrable on  $\mathbb{R}^n$ ,  $1 < p < \infty$ . Show that the following are equivalent:

- (i)  $f \in L^p(\mathbb{R}^n)$ .
- (ii) there exist  $M > 0$ , such that for any finite collection of mutually disjoint measurable sets  $E_1, E_2, \dots, E_k$ , with  $0 < m(E_i) < \infty$  for  $1 \leq i \leq k$ ,

$$\sum_{i=1}^k \left( \frac{1}{m(E_i)} \right)^{p-1} \left| \int_{E_i} f(x) dx \right|^p \leq M.$$

*Solution sketch.* For (i)  $\implies$  (ii), use Hölder's inequality with  $f$  and the indicator functions of  $E_i$ . For (ii)  $\implies$  (i), observe that the inequality in (ii) implies that  $\int_E |fg| \leq M \|g\|_q$  for all  $g \in L^q(E)$ . This follows from approximating  $g$  by simple functions. The inequality shows that integration against  $f$  is a bounded linear functional on  $L^q$ , and therefore,  $f$  must be in  $L^p$  by the Riesz representation theorem.  $\square$

**Problem 51** (Date: September 2019, tags: ). Let  $(X, \Omega, \mu)$  be a measure space with  $\mu(X) = 1$ , and let  $f \in L^2(\mu)$  be a non-negative function satisfying  $\int_X f d\mu \geq 1$ . Prove that

$$\mu(\{x \in X \mid f(x) > 1\}) \geq \frac{(\int_X f d\mu - 1)^2}{\int_X f^2 d\mu}.$$

*Solution sketch.* Use Cauchy-Schwartz with  $f$  and the indicator of the set where  $f(x) > 1$ .  $\square$



### 3 Uncategorized

**Problem 52** (Date: 2013 draft, tags: ). Let  $\mu_1 \leq \mu_2 \leq \dots$  be a sequence of positive absolutely continuous measures on a measure space  $(X, \mathcal{A}, \rho)$ . Assume that there exists a finite positive measure  $\nu$  such that  $\mu_n \leq \nu$  for all  $n \in \mathbb{N}$ . For  $A \in \mathcal{A}$  set  $\mu(A) = \lim_{n \rightarrow \infty} \mu_n(A)$ . Prove that  $\mu$  is an absolutely continuous measure.

(Hint: use Lebesgue–Radon–Nikodym Theorem.)

**Problem 53** (Date: 2013 draft, tags: ). Let  $f_1, f_2, \dots, f : [0, 1] \rightarrow \mathbb{R}$  be non-decreasing functions such that  $\sum_{n=1}^{\infty} f_n = f$ . Prove that  $\sum_{n=1}^{\infty} f'_n = f'$  a.e.

**Problem 54** (Date: 2013 draft, tags: ). (Hard) Prove that the sequence

$$f_n(x) = n^{1/2} \exp\left(-\frac{n^2 x^2}{x+1}\right)$$

converges in  $L_p([0, +\infty))$  for  $1 \leq p < 2$  and diverges for  $p \geq 2$ .

**Problem 55** (Date: 2013 draft, tags: ). Let  $(X, \mathcal{A}, \mu)$  be a  $\sigma$ -finite measure space with  $\mu(X) = \infty$ . Construct a function  $F : X \rightarrow \mathbb{R}$  such that  $F \in L_p(\mu)$  for all  $p > 1$ , but  $F \notin L_1(\mu)$ .

**Problem 56** (Date: 2013 draft, tags: ). (Hard?) Let  $K = \{f : (0, +\infty) \rightarrow \mathbb{R} \mid \int_0^\infty f^4(x) dx \leq 1\}$ . Evaluate

$$\sup_{f \in K} \int_0^\infty \frac{f^3(x)}{1+x} dx.$$

**Problem 57** (Date: 2013 draft, tags: ). (i) (Easy) Let  $E \subset [0, 1]$  be a measurable set,  $m(E) \geq \frac{99}{100}$ . Prove that there exists  $x \in [0, 1]$  such that for any  $r \in (0, 1)$ ,

$$m(E \cap (x-r, x+r)) \geq \frac{r}{4}.$$

(ii) (Hard) Let  $E \subset [0, 1]$  be a measurable set,  $m(E) \geq \frac{1}{2}$ . Prove that there exists  $x \in [0, 1]$  such that for any  $r \in (0, 1)$ ,

$$m(E \cap (x-r, x+r)) \geq \frac{r}{20}.$$

**Problem 58** (Date: 2013 draft, tags: ). Find all  $q \geq 1$ , such that  $f(x^2) \in L_q((0, 1))$  for any  $f \in L_4((0, 1))$ .

**Problem 59** (Date: 2013 draft, tags: ). Let  $E_n$ ,  $n \in \mathbb{N}$  be measurable sets. Prove that the set of  $x \in \mathbb{R}$  for which there exists at most 3 values of  $n$  such that  $x \in E_k$ , but  $x \notin E_{k^n}$  for all  $n \in \mathbb{N} \setminus \{1\}$  is measurable.

**Problem 60** (Date: 2013 draft, tags: ). (Hard) Let  $g : \mathbb{R} \rightarrow (0, +\infty)$  be a 1-periodic function, and assume that  $g \in L_1(0, 1)$ . Prove that if  $f_n \rightarrow 0$  a.e. on  $(0, 1)$ , and

$$|f_n(x)| \leq g(nx) \quad \text{for all } x \in (0, 1),$$

then  $\int_0^1 f_n(x) dx \rightarrow 0$ .