Real Analysis qualifying review big list

Students at the University of Michigan

Thursday 1st July, 2021

1 Categorized (without solutions)

Problem 1 (Date: September 2011, tags: `integral convergence`). Let $f \in L_1([0,1], dx)$. Find

$$\lim_{n\to\infty} \frac{1}{n} \int_0^1 \log\left(1 + e^{nf(x)}\right)\, dx.$$

Problem 2 (Date: September 2011, tags: `bounded variation`). Suppose $f : [0,1] \to \mathbb{R}$ satisfies $f(x) - f(y) < x - y$ for all $x, y \in [0,1], x > y$. Show that f' exists almost everywhere on $[0,1]$ or give a counterexample.

Problem 3 (Date: September 2011, tags: `hölder's inequality`). Let (X, Ω, μ) be a finite measure space.

(i) Prove that for any $p < q,\ L_q(\mu) \subset L_p(\mu)$.

(ii) Assume that for any $t > 0$ there exists $E \in \Omega$ satisfying

$$0 < \mu(E) < t.$$

Prove that for any $1 < p < \infty$ there exists a function $f \in L_p(\mu)$ such that $f \notin L_q(\mu)$ for any $q > p$.

Problem 4 (Date: September 2011, tags: L^p `spaces`). For a real valued function $f(x, y)$ on $\mathbb{R}^2$ which is in L^2, show that $f(x+\epsilon, y+\epsilon) \to f(x, y)$ in L^2 when $\epsilon \to 0$.

Problem 5 (Date: September 2011, tags: `measurable sets?`). Let $f \in L_1([0,1])$ be a function such that $\int_E f(x)\, dx = 0$ for any measurable set $E \subset [0,1]$ of Lebesgue measure $1/2$. Prove that $f = 0$ a.e.

Problem 6 (Date: January 2011, tags: `measurable functions`).

Let A be a sequence of measurable subsets of $[0,1]$ such that $\inf m(A_n) > 0$, where m stands for the Lebesgue measure.

(i) Prove that there exists $x \in [0,1]$ which belongs to infinitely many of the sets A_n.

(ii) Does there necessarily exist a point which (does not?) belong to any of the sets A_n, except finitely many?

Problem 7 (Date: January 2011, tags: `integral convergence`). Let $\{f_n\} \subset L_1(\mu)$ be a decreasing sequence of functions such that $f_n \to f$ a.e. Prove that

$$\lim_{n\to\infty} \int f_n\, d\mu = \int f\, d\mu.$$

Problem 8 (Date: January 2011, tags: `simple function approximation`). Let $f : X \to [0, +\infty)$ be an integrable function on a measure space $(X, \mathcal{A}, \mu)$. Define the measure ν by $\nu(A) = \int_A f \, d\mu$.

(i) Prove that the measure ν is σ-additive.

(ii) Prove that if $g \in L_1(\nu)$, then $\int_X g \, d\nu = \int_X fg \, d\mu$.

(Hint: first, assume that g is a simple positive function. Then extend the result to non-negative integrable functions using limit theorems).

Problem 9 (Date: January 2011, tags: `integral inequalities`). Let $f_n : \mathbb{R} \to [0, 1]$ be functions such that $\sup_{x\in\mathbb{R}} f_n(x) = 1/n$ and $\int_{\mathbb{R}} f(x) \, dx = 1$. Set

$$F(x) = \sup_{n\in\mathbb{N}} f_n(x).$$

Find all possible values of $\int_{\mathbb{R}} F(x) \, dx$.

Problem 10 (Date: January 2011, tags: `integral convergence?`). Let $f \in L_\infty([0, 1])$. Prove that

$$\lim_{n\to\infty} \frac{\int_{[0,1]} |f(x)|^{n+1} \, dx}{\int_{[0,1]} |f(x)|^n \, dx} = \|f\|_\infty.$$

Problem 11 (Date: January 2011, tags: `egoroff's theorem`). Let E be the exceptional set in Egoroff's theorem. Is it possible to prove Egoroff's theorem with $l(E) = 0$ instead of $l(E) < e$?

Problem 12 (Date: January 2011, tags: `simple functions`). Let $f : X \to [0, \infty]$ be a measurable function. Assume that $\mu(X) < \infty$. Prove that $\int f \, d\mu < +\infty$ if and only if

$$\sum_{n=1}^{\infty} 2^n \mu(x \in X \mid f(x) \geq 2^n) < +\infty.$$

Problem 13 (Date: January 2011, tags: `dominated convergence`). Let $f_n, g_n, f, g \in L_1(\mu)$ be functions such that $f_n \to f$ a.e., $g_n \to g$ a.e. and $|f_n| \leq g_n$. Prove that if $\int g_n \, d\mu \to \int g \, d\mu$, then $\int f_n \, d\mu \to \int f \, d\mu$.

(Hint: follow the proof of Lebesgue dominated convergence theorem.)

Problem 14 (Date: January 2011, tags:). Let $f_n, f \in L_1(\mu)$ be such that $f_n \to f$ a.e. Prove that if $\|f_n\|_1 \to \|f\|_1$, then $f_n \to f$ in $L_1(\mu)$.

(Hint: use the previous problem)

Problem 15 (Date: January 2011, tags: `change of variables`). Let $f, g : \mathbb{R} \to \mathbb{R}$ be L_1-functions.

(a) Prove that

$$\int_{\mathbb{R}} |f(x - y)g(y)| dm(y) < +\infty.$$

(b) Let

$$h(x) = \int_{\mathbb{R}} f(x - y)g(y) dm(y).$$

Prove that $h \in L_1(\mathbb{R})$ and $\|h\|_1 \leq \|f\|_1 \cdot \|g\|_1$.

Problem 16 (Date: January 2012, tags: `measurable functions`). Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. Prove that the function f' is measurable.

Problem 17 (Date: January 2012, tags: L^p `spaces`). Let $1 \le p < \infty$, and let $f \in L_p(\mu)$. Prove that

$$\lim_{t \to 0} t^p \mu\{x \mid |f(x)| > t\} = 0.$$

Problem 18 (Date: January 2012, tags: `Lebesgue differentiation theorem, Hardy-Littlewood maximal estimate?`). Let $f \in L_1(\mathbb{R})$. For $n \in \mathbb{N}$ define the function $g_n : \mathbb{R} \to \mathbb{R}$ as follows. For $k \in \mathbb{Z}$ and for $x \in [k/n, (k+1)/n)$ set

$$g_n(x) = n \int_{k/n}^{(k+1)/n} f(x)\, dx.$$

Prove that g_n converges to f a.e. and in $L_1(\mathbb{R})$.

Problem 19 (Date: January 2012, tags: `integral convergence`). Let $(X, \mathcal{A}, \mu)$ be a finite measure space ($\mu(X) < \infty$). Assume that a sequence $\{f_n\}_{n=1}^{\infty} \subseteq L_1(\mu)$ satisfies the condition

$$\frac{1}{\sqrt{\mu(E)}} \int_E |f_n|\, d\mu \le 1$$

for all $n \in \mathbb{N}$ and all sets E of positive measure. Prove that if $f_n \to f$ a.e., then $f \in L_1(\mu)$ and

$$\int_X f_n\, d\mu \to \int_X f\, d\mu.$$

Problem 20 (Date: January 2012, tags: L^p `spaces`). Construct a function $f \in L_1(\mathbb{R})$ such that $f \notin L_2((a,b))$ for any interval $(a,b) \subseteq \mathbb{R}$.

Problem 21 (Date: September 2019, tags: `countable subadditivity?`). Let E be the set of all $x \in (0,1)$ such that there exists a sequence of irreducible fractions $\{p_n/q_n\}_{n \in \mathbb{N}}$ with $p_n, q_n \in \mathbb{N}$, $q_1 < q_2 < \cdots$ such that

$$\left| x - \frac{p_n}{q_n} \right| \le \frac{1}{q_n^3}, \quad n = 1, 2, \ldots$$

Prove that the Lebesgue measure of E is zero.

Problem 22 (Date: September 2019, tags: `integral convergence`). Let f be a measurable function on $(0, \infty)$, and for $n = 1, 2, \ldots$ let f_n be defined by

$$f_n(x) = f(x) e^{-x} \left[1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \right].$$

Suppose $f \in L^2[(0,\infty)]$. Prove that $\lim_{n \to \infty} \|f_n - f\|_{L^2[(0,\infty)]} = 0$

Problem 23 (Date: September 2019, tags: `Fubini, Hölder`). Let $f : \mathbb{R} \times (0,1) \to \mathbb{R}$ be a measurable function such that for any $y \in (0,1)$,

$$\int_{\mathbb{R}} f^2(x,y)\, dx \le 1.$$

Prove there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$, with $\lim_{n \to \infty} x_n = +\infty$, such that

$$\lim_{n \to \infty} \int_0^1 |f(x_n, y)|\, dy = 0.$$

Problem 24 (Date: September 2019, tags: `Hölder`). Let (X, Ω, μ) be a measure space with $\mu(X) = 1$, and let $f \in L^2(\mu)$ be a non-negative function satisfying $\int_X f\, d\mu \geq 1$. Prove that

$$\mu\left(\{x \in X \mid f(x) > 1\}\right) \geq \frac{\left(\int_X f\, d\mu - 1\right)^2}{\int_X f^2\, d\mu} .$$

Problem 25 (Date: September 2019, tags: `absolute continuity`). A function $f : (0,1) \to \mathbb{R}$ is locally Lipschitz if for any $x \in (0,1)$ there is an open interval I_x with $x \in I_x \subset (0,1)$ and a constant C_x such that $|f(y) - f(y')| \leq C_x |y - y'|$ for $y, y' \in I_x$.

(a) Prove that a locally Lipschitz function $f(\cdot)$ is absolutely continuous on any compact subinterval $[a,b] \subset (0,1)$.

(b) Give an example of a locally Lipschitz function $f : (0,1) \to \mathbb{R}$ which extends to a continuous function on the closed interval $[0,1]$, but is not absolutely continuous on $[0,1]$.

Problem 26 (Date: May 2020, tags: `outer regularity`). Let $E \subset (0,1)$ be a measurable set such that for any interval $(a,b) \subset (0,1)$, there exists an interval $(c,d) \subset (a,b) \setminus E$ with

$$d - c \geq \frac{a}{10}(b-a).$$

Prove that $m(E) = 0$.

Problem 27 (Date: May 2020, tags: `bounded variation`). Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function such that

$$f(y) \leq f(x) + (x^2 + y^2)(x - y) \quad \text{for } -\infty < y < x < \infty .$$

Show that the derivative function $x \to f'(x)$ exists a.e. on $\mathbb{R}$.

Problem 28 (Date: May 2020, tags: `triangle inequality?`). Let $r_n,\ n = 1, 2, \ldots,$ be an enumeration of the rationals in the interval $[0,1]$ and consider the function $f : [0,1] \to \mathbb{R} \cup \infty$ defined by

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1}{|x - r_n|^{1/3}} , \quad 0 \leq x \leq 1 .$$

Show that $f \in L^2(0,1)$.

Problem 29 (Date: May 2020, tags: `change of variables, integral convergence`). Let $f_n,\ n = 1, 2, \ldots,$ be the sequence of functions on $(0, \infty)$ defined by

$$f_n(x) = \frac{1}{n}\left(1 - \frac{x}{n}\right)^n e^x , \ 0 < x < n, \quad f_n(x) = 0, \ x \geq n.$$

Prove that the sequence $a_n,\ n = 1, 2, \ldots,$ given by

$$a_n = \int_0^\infty f_n(x)\, dx \quad \text{converges and identify } a_\infty = \lim_{n \to \infty} a_n.$$

Problem 30 (Date: May 2020, tags: `$L^p$ spaces`). Suppose f is a C^1 function on $\mathbb{R}$ satisfying $f(0) = 0,\ |f(x)| \leq |x|^{-1/2},\ x \neq 0$. Let g be in $L^1(\mathbb{R})$.

(a) Show there is a constant C such that $m\{|g| > \alpha\} \leq C/\alpha$ for all $\alpha > 0$.

(b) Show that the function $h(x) = f(g(x))$ is in $L^1(\mathbb{R})$.

Problem 31 (Date: May 2011, tags: `integral convergence`). Let $f_n, g_n, f, g \in L_1(\mu)$ be functions such that $f_n \to f$ a.e., $g_n \to g$ a.e. and $|f_n| \leq g_n$. Prove that if $\int g_n \, d\mu \to \int g \, d\mu$, then $\int f_n \, d\mu \to \int f \, d\mu$.

(Hint: use Fatou's Lemma.)

Problem 32 (Date: May 2011, tags: `Egorov's theorem`). Let $\{f_n : [0,1] \to \mathbb{R}\}_{n=1}^{\infty}$ be a sequence of continuous functions such that $f_n(x) \to f(x)$ for any $x \in [0,1]$. Does there exist a set $E \subset [0,1]$ of Lebesgue measure 0 such that $f_n \to f$ uniformly on $[0,1] \setminus E$?

2 Categorized (with solutions)

3 Uncategorized

Problem 33 (Date: January 2011, tags:). Let $f \in L_1([0,1])$ be a function such that $f(x) > 0$ a.e.

(i) Prove that for any $0 < a < 1$

$$\inf_{m(A)=a} \int_A f \, dm > 0.$$

(ii) Does the previous statement hold for a function $f \in L_1(\mathbb{R})$ such that $f(x) > 0$ a.e.