Real Analysis qualifying review big list

Students at the University of Michigan

Tuesday 13th July, 2021

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How to add to this document

This document contains a subset of the old real analysis qual problems, and solutions to some of them. To add a problem to this list, use the `problem` environment: this environment takes two arguments, the `date` the problem appeared on a qual, and `tags` describing the problem.

```
\begin{problem}{<Month> <Year>}{tag1, tag2, ...}
  <Problem statement>
\end{problem}
```

To add a solution, or a solution sketch, use the `solution` and `sketch` environments. These environments do not take any arguments.

Notation

Unless otherwise specified, m refers to the Lebesgue measure on $\mathbb{R}^n$ and subsets of $\mathbb{R}^n$, and m^* refers to the Lebesgue outer measure.

1 Categorized (without solutions)

1.1 Basic measure theory and integrability

Problem 1 (Date: 2013 draft, tags: `basic measure theory`)**.** Let $(X, \mathcal{A}, \mu)$ be a finite measure space. For a set $A \subset X$ define $\mu_*(A) = \mu(X) - \mu^*(X \setminus A)$, where μ^* is the outer measure. Prove that $\mu_*(A) \leq \mu^*(A)$ for any $A \subset X$.

Problem 2 (Date: 2013 draft, tags: `basic measure theory`)**.** Let $A \subset [0,1] \times [0,1]$ be the set of points (x, y) with decimal representations $x = 0.x_1x_2\ldots,\ \ y = 0.y_1y_2\ldots$ such that $x_ny_n = 5$ for all $n \in \mathbb{N}$. Prove that the set A is measurable and find its Lebesgue measure.

Problem 3 (Date: 2013 draft, tags: `convergence in measure`). Let $f_1, f_2, \ldots, f, g$ be measurable functions on a measure space $(X, \mathcal{A}, \mu)$. Assume that $f_n \to f$ in measure and $f_n \le g$ a.e. Prove that $f \le g$ a.e.

Problem 4 (Date: 2013 draft, tags: `basic measure theory`). Let $\{x_n\}_{n=1}^{\infty} \subset [0,1]$ be any sequence. For $n \in \mathbb{N}$ define the set $A_n \subset \mathbb{R}$ by

$$A_k = \bigcup_{n=k}^{\infty} \left(x_n - \frac{k}{n^3}, x_n + \frac{k}{n^3} \right).$$

Prove that $m(\bigcap_{k=1}^{\infty} A_k) = 0$, where m denotes the Lebesgue measure.

Problem 5 (Date: January 2014, tags: `outer regularity`). Prove or disprove: If E is an open subset of $\mathbb{R}$ with $m(E) = 1$ then there is a finite union of intervals F containing E with $m(F) < 1.1$.

Problem 6 (Date: September 2014, tags: `integrability?`). Let $\{f_k(x)\}$ be a sequence of nonnegative measurable functions on E and $m(E) < \infty$. Show that $\{f_k(x)\}$ converges in measure to 0 if and only if

$$\lim_{k\to\infty} \int_E \frac{f_k(x)}{1 + f_k(x)}\, dx = 0.$$

Problem 7 (Date: September 2014, tags: `integrability?`). Let $1 \le p < \infty$, $f \in L^p(\mathbb{R}^n)$. Let

$$f_*(\lambda) = m(\{x : |f(x)| > \lambda\}), \quad \lambda > 0$$

Show that

(i) $p \int_0^\infty \lambda^{p-1} f_*(\lambda)\, d\lambda = \int |f(x)|^p\, dx$

(ii) $\lim_{\lambda\to\infty} \lambda^p f_*(\lambda) = 0$

(iii) $\lim_{\lambda\to 0} \lambda^p f_*(\lambda) = 0$

Problem 8 (Date: September 2014, tags: `fat Cantor set`). Construct a measurable subset A of $(0,1)$ such that $m(A) < 1$ and $m\big(A \cap (a,b)\big) > 0$ for any $(a,b) \subset (0,1)$.

Problem 9 (Date: January 2015, tags: `Carathéodory's criterion`). Let $A, B \subset \mathbb{R}^d$. Assume $A \cup B$ is measurable, and $m(A \cup B) < \infty$. If

$$m(A \cup B) = m^*(A) + m^*(B)$$

Show that A and B are measurable.

(Hint: prove first that for any set A, there a measurable set U, with $A \subset U$, such that $m^*(A) = m(U)$.)

Problem 10 (Date: January 2015, tags: `simple function approximation`). Let f be a nonnegative measurable function on $(0,1)$. Assume that there is a constant c, such that

$$\int_0^1 (f(x))^n\, dx = c, \qquad n = 1, 2, \ldots$$

Show that there is a measurable set $E \subset (0,1)$, such that

$$f(x) = \chi_E(x), \qquad \text{for a.e. } x \in (0,1).$$

Problem 11 (Date: May 2020, tags: L^p `spaces`). Suppose f is a C^1 function on $\mathbb{R}$ satisfying $f(0) = 0$, $|f(x)| \le |x|^{-1/2}$, $x \neq 0$. Let g be in $L^1(\mathbb{R})$.

(a) Show there is a constant C such that $m\{|g| > \alpha\} \le C/\alpha$ for all $\alpha > 0$.

(b) Show that the function $h(x) = f(g(x))$ is in $L^1(\mathbb{R})$.

Problem 12 (Date: May 2020, tags: `triangle inequality?`). Let r_n, $n = 1, 2, \ldots$, be an enumeration of the rationals in the interval $[0,1]$ and consider the function $f : [0,1] \to \mathbb{R} \cup \infty$ defined by

$$f(x) \ = \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1}{|x - r_n|^{1/3}} \,, \quad 0 \le x \le 1 \,.$$

Show that $f \in L^2(0,1)$.

Problem 13 (Date: May 2020, tags: `outer regularity`). Let $E \subset (0,1)$ be a measurable set such that for any interval $(a,b) \subset (0,1)$, there exists an interval $(c,d) \subset (a,b) \setminus E$ with

$$d - c \ge \frac{a}{10}(b - a).$$

Prove that $m(E) = 0$.

Problem 14 (Date: September 2019, tags: `countable subadditivity?`). Let E be the set of all $x \in (0,1)$ such that there exists a sequence of irreducible fractions $\{p_n/q_n\}_{n \in \mathbb{N}}$ with $p_n, q_n \in \mathbb{N}$, $q_1 < q_2 < \cdots$ such that

$$\left| x - \frac{p_n}{q_n} \right| \ \le \ \frac{1}{q_n^3} \,, \quad n = 1, 2, \ldots$$

Prove that the Lebesgue measure of E is zero.

Problem 15 (Date: January 2012, tags: L^p `spaces`). Construct a function $f \in L_1(\mathbb{R})$ such that $f \notin L_2((a,b))$ for any interval $(a,b) \subseteq \mathbb{R}$.

Problem 16 (Date: September 2011, tags: `measurable sets?`). Let $f \in L_1([0,1])$ be a function such that $\int_E f(x)\,dx = 0$ for any measurable set $E \subset [0,1]$ of Lebesgue measure $1/2$. Prove that $f = 0$ a.e.

Problem 17 (Date: January 2011, tags: `measurable functions`).

Let A be a sequence of measurable subsets of $[0,1]$ such that $\inf m(A_n) > 0$, where m stands for the Lebesgue measure.

(i) Prove that there exists $x \in [0,1]$ which belongs to infinitely many of the sets A_n.

(ii) Does there necessarily exist a point which (does not?) belong to any of the sets A_n, except finitely many?

Problem 18 (Date: January 2011, tags: `simple function approximation`). Let $f : X \to [0, +\infty)$ be an integrable function on a measure space $(X, \mathcal{A}, \mu)$. Define the measure ν by $\nu(A) = \int_A f\,d\mu$.

(i) Prove that the measure ν is σ-additive.

(ii) Prove that if $g \in L_1(\nu)$, then $\int_X g\,d\nu = \int_X fg\,d\mu$.

(Hint: first, assume that g is a simple positive function. Then extend the result to non-negative integrable functions using limit theorems).

Problem 19 (Date: January 2011, tags: `simple functions`). Let $f : X \to [0, \infty]$ be a measurable function. Assume that $\mu(X) < \infty$. Prove that $\int f \, d\mu < +\infty$ if and only if

$$\sum_{n=1}^{\infty} 2^n \mu(x \in X \mid f(x) \geq 2^n) < +\infty.$$

Problem 20 (Date: January 2012, tags: `measurable functions`). Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. Prove that the function f' is measurable.

Problem 21 (Date: January 2012, tags: L^p `spaces`). Let $1 \leq p < \infty$, and let $f \in L_p(\mu)$. Prove that

$$\lim_{t \to 0} t^p \mu\{x \mid |f(x)| > t\} = 0.$$

1.2 Integral convergence

Problem 22 (Date: 2013 draft, tags: `integral convergence`). Prove that

$$\lim_{n \to \infty} \int_{\mathbb{R}} \frac{\cos^n(\pi x)}{(x-n)^2 + 1} \, dx$$

exists and find it.

Problem 23 (Date: May 2011, tags: `integral convergence`). Let $f_n, g_n, f, g \in L_1(\mu)$ be functions such that $f_n \to f$ a.e., $g_n \to g$ a.e. and $|f_n| \leq g_n$. Prove that if $\int g_n \, d\mu \to \int g \, d\mu$, then $\int f_n \, d\mu \to \int f \, d\mu$. (Hint: use Fatou's Lemma.)

Problem 24 (Date: September 2019, tags: `integral convergence`). Let f be a measurable function on $(0, \infty)$, and for $n = 1, 2, \ldots$ let f_n be defined by

$$f_n(x) = f(x) e^{-x} \left[1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \right].$$

Suppose $f \in L^2[(0, \infty)]$. Prove that $\lim_{n \to \infty} \|f_n - f\|_{L^2[(0,\infty)]} = 0$

Problem 25 (Date: January 2012, tags: `integral convergence`). Let $(X, \mathcal{A}, \mu)$ be a finite measure space ($\mu(X) < \infty$). Assume that a sequence $\{f_n\}_{n=1}^{\infty} \subseteq L_1(\mu)$ satisfies the condition

$$\frac{1}{\sqrt{\mu(E)}} \int_E |f_n| \, d\mu \leq 1$$

for all $n \in \mathbb{N}$ and all sets E of positive measure. Prove that if $f_n \to f$ a.e., then $f \in L_1(\mu)$ and

$$\int_X f_n \, d\mu \to \int_X f \, d\mu.$$

Problem 26 (Date: January 2011, tags: `integral convergence`). Let $\{f_n\} \subset L_1(\mu)$ be a decreasing sequence of functions such that $f_n \to f$ a.e. Prove that

$$\lim_{n \to \infty} \int f_n \, d\mu = \int f \, d\mu.$$

Problem 27 (Date: September 2011, tags: L^p `spaces`). For a real valued function $f(x, y)$ on $\mathbb{R}^2$ which is in L^2, show that $f(x + \epsilon, y + \epsilon) \to f(x, y)$ in L^2 when $\epsilon \to 0$.

Problem 28 (Date: January 2011, tags: `dominated convergence`). Let $f_n, g_n, f, g \in L_1(\mu)$ be functions such that $f_n \to f$ a.e., $g_n \to g$ a.e. and $|f_n| \leq g_n$. Prove that if $\int g_n \, d\mu \to \int g \, d\mu$, then $\int f_n \, d\mu \to \int f \, d\mu$.

(Hint: follow the proof of Lebesgue dominated convergence theorem.)

Problem 29 (Date: January 2011, tags:). Let $f_n, f \in L_1(\mu)$ be such that $f_n \to f$ a.e. Prove that if $\|f_n\|_1 \to \|f\|_1$, then $f_n \to f$ in $L_1(\mu)$.

(Hint: use the previous problem)

1.3 Integral inequalities

Problem 30 (Date: January 2014, tags: `Hardy-Littlewood maximal inequality`). Let $E \subset [0,1]$ be a measurable set, $m(E) \geq \frac{99}{100}$. Prove that there exists $x \in [0,1]$ such that for any $r \in (0,1)$,

$$m\big(E \cap (x-r, x+r)\big) \geq \frac{r}{4}.$$

Hint: One approach to this problem involves the Hardy-Littlewood maximal inequality.

Problem 31 (Date: January 2014, tags: `Hölder's inequality`). Find all $q \geq 1$, such that $f(x^2) \in L_q((0,1), m)$ for any $f(x) \in L_4((0,1), m)$, where m denotes the Lebesgue measure.

Problem 32 (Date: September 2014, tags: `Hölder's inequality`). Let $K = \{f : (0, +\infty) \to \mathbb{R} \mid \int_0^\infty f^4(x)\,dx \leq 1\}$. Evaluate

$$\sup_{f \in K} \int_0^\infty f^3(x) e^{-x}\,dx.$$

Problem 33 (Date: September 2019, tags: `Fubini, Hölder`). Let $f : \mathbb{R} \times (0,1) \to \mathbb{R}$ be a measurable function such that for any $y \in (0,1)$,

$$\int_{\mathbb{R}} f^2(x,y)\,dx \;\leq\; 1\,.$$

Prove there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$, with $\lim_{n\to\infty} x_n = +\infty$, such that

$$\lim_{n\to\infty} \int_0^1 |f(x_n, y)|\,dy \;=\; 0\,.$$

Problem 34 (Date: January 2011, tags: `integral inequalities?`). Let $f \in L_\infty([0,1])$. Prove that

$$\lim_{n\to\infty} \frac{\int_{[0,1]} |f(x)|^{n+1}\,dx}{\int_{[0,1]} |f(x)|^n\,dx} = \|f\|_\infty.$$

Problem 35 (Date: January 2011, tags: `integral inequalities`). Let $f_n : \mathbb{R} \to [0,1]$ be functions such that $\sup_{x\in\mathbb{R}} f_n(x) = 1/n$ and $\int_{\mathbb{R}} f(x)\,dx = 1$. Set

$$F(x) = \sup_{n\in\mathbb{N}} f_n(x).$$

Find all possible values of $\int_{\mathbb{R}} F(x)\,dx$.

Problem 36 (Date: September 2011, tags: `hölder's inequality`). Let (X, Ω, μ) be a finite measure space.

(i) Prove that for any $p < q,\ L_q(\mu) \subset L_p(\mu)$.

(ii) Assume that for any $t > 0$ there exists $E \in \Omega$ satisfying

$$0 < \mu(E) < t.$$

Prove that for any $1 < p < \infty$ there exists a function $f \in L_p(\mu)$ such that $f \notin L_q(\mu)$ for any $q > p$.

1.4 Miscellaneous

Problem 37 (Date: September 2011, tags: `bounded variation`). Suppose $f : [0,1] \to \mathbb{R}$ satisfies $f(x) - f(y) < x - y$ for all $x, y \in [0,1], x > y$. Show that f' exists almost everywhere on $[0,1]$ or give a counterexample.

Problem 38 (Date: January 2011, tags: `egoroff's theorem`). Let E be the exceptional set in Egoroff's theorem. Is it possible to prove Egoroff's theorem with $l(E) = 0$ instead of $l(E) < e$?

Problem 39 (Date: January 2011, tags: `change of variables`). Let $f, g : \mathbb{R} \to \mathbb{R}$ be L_1-functions.

(a) Prove that

$$\int_{\mathbb{R}} |f(x-y)g(y)| dm(y) < +\infty.$$

(b) Let

$$h(x) = \int_{\mathbb{R}} f(x-y)g(y) dm(y).$$

Prove that $h \in L_1(\mathbb{R})$ and $\|h\|_1 \leq \|f\|_1 \cdot \|g\|_1$.

Problem 40 (Date: January 2012, tags: `Lebesgue differentiation theorem, Hardy-Littlewood maximal estimate?`). Let $f \in L_1(\mathbb{R})$. For $n \in \mathbb{N}$ define the function $g_n : \mathbb{R} \to \mathbb{R}$ as follows. For $k \in \mathbb{Z}$ and for $x \in [k/n, (k+1)/n)$ set

$$g_n(x) = n \int_{k/n}^{(k+1)/n} f(x)\, dx.$$

Prove that g_n converges to f a.e. and in $L_1(\mathbb{R})$.

Problem 41 (Date: September 2019, tags: `absolute continuity`). A function $f : (0,1) \to \mathbb{R}$ is locally Lipschitz if for any $x \in (0,1)$ there is an open interval I_x with $x \in I_x \subset (0,1)$ and a constant C_x such that $|f(y) - f(y')| \leq C_x |y - y'|$ for $y, y' \in I_x$.

(a) Prove that a locally Lipschitz function $f(\cdot)$ is absolutely continuous on any compact subinterval $[a,b] \subset (0,1)$.

(b) Give an example of a locally Lipschitz function $f : (0,1) \to \mathbb{R}$ which extends to a continuous function on the closed interval $[0,1]$, but is not absolutely continuous on $[0,1]$.

Problem 42 (Date: May 2020, tags: `bounded variation`). Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function such that

$$f(y) \;\leq\; f(x) + (x^2 + y^2)(x - y) \quad \text{for } -\infty < y < x < \infty\,.$$

Show that the derivative function $x \to f'(x)$ exists a.e. on $\mathbb{R}$.

Problem 43 (Date: May 2011, tags: `Egorov's theorem`). Let $\{f_n : [0,1] \to \mathbb{R}\}_{n=1}^{\infty}$ be a sequence of continuous functions such that $f_n(x) \to f(x)$ for any $x \in [0,1]$. Does there exist a set $E \subset [0,1]$ of Lebesgue measure 0 such that $f_n \to f$ uniformly on $[0,1] \setminus E$?

Problem 44 (Date: January 2011, tags:). Let $f \in L_1([0,1])$ be a function such that $f(x) > 0$ a.e.

(i) Prove that for any $0 < a < 1$

$$\inf_{m(A)=a} \int_A f\, dm > 0.$$

(ii) Does the previous statement hold for a function $f \in L_1(\mathbb{R})$ such that $f(x) > 0$ a.e.

Problem 45 (Date: January 2015, tags: `Egorov's theorem`)**.** Let $E_k \subset [a,b]$, $k \in \mathbb{N}$ be measurable sets, and there exists $\delta > 0$ such that $m(E_k) \geq \delta$ for all k. Assume that $a_k \in \mathbb{R}$ satisfies

$$\sum_{k=1}^{\infty} |a_k| \chi_{E_k}(x) < \infty \qquad \text{for a.e. } x \in [a,b].$$

Show that

$$\sum_{k=1}^{\infty} |a_k| < \infty.$$

(For extra challenge, find a proof that does not use Egorov's theorem).

Problem 46 (Date: September 2014, tags: `Lebesgue differentiation`)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $\int_{\mathbb{R}} |f(x)|\, dx < \infty$. Show that the sequence

$$h_n(x) = \frac{1}{n} \sum_{k=1}^{n} f\left(x + \frac{k}{n}\right)$$

converges in $L_1(\mathbb{R})$.

Problem 47 (Date: January 2014, tags: `Density of smooth/simple functions`)**.** Let $f \in L_1 \cap L_4$ (on some measure space). Prove that the function defined on $[1,4]$, given by the following formula

$$p \mapsto \|f\|_p$$

is continuous.

Problem 48 (Date: January 2014, tags: `Fubini`)**.** Let

$$\begin{aligned} E &\subset \{(x,y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq x\} \\ E_x &= \{y \mid (x,y) \in E\} \\ E_y &= \{x \mid (x,y) \in E\} \end{aligned}$$

and assume that $m(E_x) \geq x^3$ for any $x \in [0,1]$.

(i) Prove that there exists $y \in [0,1]$ such that $m(E_y) \geq \dfrac{1}{4}$.

(ii) (Hard) Prove that there exists $y \in [0,1]$ such that $m(E_y) \geq c$, where $c > 1/4$ is a constant independent of E. Find the optimal such c.

2 Categorized (with solutions)

Problem 49 (Date: September 2011, tags: `integral convergence`)**.** Let $f \in L_1([0,1], dx)$. Find

$$\lim_{n\to\infty} \frac{1}{n} \int_0^1 \log\left(1 + e^{nf(x)}\right)\, dx.$$

Solution sketch. Step 1: Consider pointwise convergence in two different sets: $f(x) \leq 0$ and $f(x) > 0$. Step 2: Use convergence theorems in these two domains to get convergence of integral. $\square$

Problem 50 (Date: May 2020, tags: `change of variables, integral convergence`)**.** Let $f_n,\ n = 1, 2, ...,$ be the sequence of functions on $(0, \infty)$ defined by

$$f_n(x) \ = \ \frac{1}{n}\left(1 - \frac{x}{n}\right)^n e^x \ , \ \ 0 < x < n, \quad f_n(x) = 0, \ \ x \geq n.$$

Prove that the sequence $a_n,\ n = 1, 2, \ldots,$ given by

$$a_n \ = \ \int_0^\infty f_n(x)\, dx \quad \text{converges and identify } a_\infty \ = \ \lim_{n\to\infty} a_n.$$

Solution sketch. Step 1: Change variables so that all integrals are over $[0, 1]$. Step 2: Observe that the new integrand is converging pointwise to 0 a.e. Step 3: Use monotone/dominated convergence theorem. □

Problem 51 (Date: January 2015, tags: `hölder's inequality`)**.** Let f be locally integrable on $\mathbb{R}^n$, $1 < p < \infty$. Show that the following are equivalent:

(i) $f \in L^p(\mathbb{R}^n)$.

(ii) there exist $M > 0$, such that for any finite collection of mutually disjoint measurable sets $E_1, E_2, \ldots, E_k$, with $0 < m(E_i) < \infty$ for $1 \leq i \leq k$,

$$\sum_{i=1}^k \left(\frac{1}{m(E_i)}\right)^{p-1} \left|\int_{E_i} f(x)\, dx\right|^p \leq M.$$

Solution sketch. For $(i) \implies (ii)$, use Hölder's inequality with f and the indicator functions of E_i. For $(ii) \implies (i)$, observe that the inequality in (ii) implies that $\int_E |fg| \leq M\|g\|_q$ for all $g \in L^q(E)$. This follows from approximating g by simple functions. The inequality shows that integration against f is a bounded linear functional on L^q, and therefore, f must be in L^p by the Riesz representation theorem. □

Problem 52 (Date: September 2019, tags: `Hölder`)**.** Let (X, Ω, μ) be a measure space with $\mu(X) = 1$, and let $f \in L^2(\mu)$ be a non-negative function satisfying $\int_X f\, d\mu \geq 1$. Prove that

$$\mu\left(\{x \in X \mid f(x) > 1\}\right) \ \geq \ \frac{\left(\int_X f\, d\mu - 1\right)^2}{\int_X f^2\, d\mu} \ .$$

Solution sketch. Use Cauchy-Schwartz with f and the indicator of the set where $f(x) > 1$. □

3 Uncategorized

Problem 53 (Date: 2013 draft, tags:)**.** Let $\mu_1 \leq \mu_2 \leq \ldots$ be a sequence of positive absolutely continuous measures on a measure space $(X, \mathcal{A}, \rho)$. Assume that there exists a finite positive measure ν such that $\mu_n \leq \nu$ for all $n \in \mathbb{N}$. For $A \in \mathcal{A}$ set $\mu(A) = \lim_{n\to\infty} \mu_n(A)$. Prove that μ is an absolutely continuous measure.

(Hint: use Lebesgue–Radon–Nikodym Theorem.)

Problem 54 (Date: 2013 draft, tags:)**.** Let $f_1, f_2, \ldots, f : [0, 1] \to \mathbb{R}$ be non-decreasing functions such that $\sum_{n=1}^\infty f_n = f$. Prove that $\sum_{n=1}^\infty f_n' = f'$ a.e.

Problem 55 (Date: 2013 draft, tags:). (Hard) Prove that the sequence

$$f_n(x) = n^{1/2} \exp\left(-\frac{n^2x^2}{x+1}\right)$$

converges in $L_p([0,+\infty))$ for $1 \le p < 2$ and diverges for $p \ge 2$.

Problem 56 (Date: 2013 draft, tags:). Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space with $\mu(X) = \infty$. Construct a function $F : X \to \mathbb{R}$ such that $F \in L_p(\mu)$ for all $p > 1$, but $F \notin L_1(\mu)$.

Problem 57 (Date: 2013 draft, tags:). (Hard?) Let $K = \{f : (0,+\infty) \to \mathbb{R} \mid \int_0^\infty f^4(x)\,dx \le 1\}$. Evaluate

$$\sup_{f\in K} \int_0^\infty \frac{f^3(x)}{1+x}\,dx.$$

Problem 58 (Date: 2013 draft, tags:). Let $E \subset \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x$, and assume that $m(E_x) \ge x^3$ for any $x \in [0,1]$. Prove that there exists $y \in [0,1]$ such that

(i) $m(E_y) \ge \frac{1}{4}$;

(ii) $m(E_y) \ge \frac{3}{8}$;

Problem 59 (Date: 2013 draft, tags:).

(i) (Easy) Let $E \subset [0,1]$ be a measurable set, $m(E) \ge \frac{99}{100}$. Prove that there exists $x \in [0,1]$ such that for any $r \in (0,1)$,

$$m\big(E \cap (x-r, x+r)\big) \ge \frac{r}{4}.$$

(ii) (Hard) Let $E \subset [0,1]$ be a measurable set, $m(E) \ge \frac{1}{2}$. Prove that there exists $x \in [0,1]$ such that for any $r \in (0,1)$,

$$m\big(E \cap (x-r, x+r)\big) \ge \frac{r}{20}.$$

Problem 60 (Date: 2013 draft, tags:). Find all $q \ge 1$, such that $f(x^2) \in L_q((0,1))$ for any $f \in L_4((0,1))$.

Problem 61 (Date: 2013 draft, tags:). Let $E_n,\ n \in \mathbb{N}$ be measurable sets. Prove that the set of $x \in \mathbb{R}$ for which there exists at most 3 values of n such that $x \in E_k$, but $x \notin E_{k^n}$ for all $n \in \mathbb{N} \setminus \{1\}$ is measurable.

Problem 62 (Date: 2013 draft, tags:). (Hard) Let $g : \mathbb{R} \to (0,+\infty)$ be a 1-periodic function, and assume that $g \in L_1(0,1)$. Prove that if $f_n \to 0$ a.e. on $(0,1)$, and

$$|f_n(x)| \le g(nx) \quad \text{for all } x \in (0,1),$$

then $\int_0^1 f_n(x)\,dx \to 0$.