Real Analysis qualifying review big list

Students at the University of Michigan

Monday 28th June, 2021

1 Categorized (without solutions)

Problem 1 (Date: September 2011, tags: `integral convergence`)**.** Let $f \in L_1([0,1], dx)$. Find

$$\lim_{n\to\infty} \frac{1}{n} \int_0^1 \log\left(1 + e^{nf(x)}\right) dx.$$

Problem 2 (Date: September 2011, tags: `bounded variation`)**.** Suppose $f : [0,1] \to \mathbb{R}$ satisfies $f(x) - f(y) < x - y$ for all $x, y \in [0,1], x > y$. Show that f' exists almost everywhere on $[0,1]$ or give a counterexample.

Problem 3 (Date: September 2011, tags: `hölder's inequality`)**.** Let (X, Ω, μ) be a finite measure space.

(i) Prove that for any $p < q,\ L_q(\mu) \subset L_p(\mu)$.

(ii) Assume that for any $t > 0$ there exists $E \in \Omega$ satisfying

$$0 < \mu(E) < t.$$

Prove that for any $1 < p < \infty$ there exists a function $f \in L_p(\mu)$ such that $f \notin L_q(\mu)$ for any $q > p$.

Problem 4 (Date: September 2011, tags: L^p `spaces`)**.** For a real valued function $f(x, y)$ on $\mathbb{R}^2$ which is in L^2, show that $f(x + \epsilon, y + \epsilon) \to f(x, y)$ in L^2 when $\epsilon \to 0$.

Problem 5 (Date: September 2011, tags: `measurable sets?`)**.** Let $f \in L_1([0,1])$ be a function such that $\int_E f(x)\, dx = 0$ for any measurable set $E \subset [0,1]$ of Lebesgue measure $1/2$. Prove that $f = 0$ a.e.

Problem 6 (Date: January 2011, tags: `measurable functions`)**.**

Let A be a sequence of measurable subsets of $[0,1]$ such that $\inf m(A_n) > 0$, where m stands for the Lebesgue measure.

(i) Prove that there exists $x \in [0,1]$ which belongs to infinitely many of the sets A_n.

(ii) Does there necessarily exist a point which (does not?) belong to any of the sets A_n, except finitely many?

Problem 7 (Date: January 2011, tags: `integral convergence`)**.** Let $\{f_n\} \subset L_1(\mu)$ be a decreasing sequence of functions such that $f_n \to f$ a.e. Prove that

$$\lim_{n\to\infty} \int f_n\, d\mu = \int f\, d\mu.$$

Problem 8 (Date: January 2011, tags: `simple function approximation`). Let $f : X \to [0, +\infty)$ be an integrable function on a measure space $(X, \mathcal{A}, \mu)$. Define the measure ν by $\nu(A) = \int_A f \, d\mu$.

(i) Prove that the measure ν is σ-additive.

(ii) Prove that if $g \in L_1(\nu)$, then $\int_X g \, d\nu = \int_X fg \, d\mu$.

(Hint: first, assume that g is a simple positive function. Then extend the result to non-negative integrable functions using limit theorems).

Problem 9 (Date: January 2011, tags: `integral inequalities`). Let $f_n : \mathbb{R} \to [0, 1]$ be functions such that $\sup_{x\in\mathbb{R}} f_n(x) = 1/n$ and $\int_{\mathbb{R}} f(x) \, dx = 1$. Set

$$F(x) = \sup_{n\in\mathbb{N}} f_n(x).$$

Find all possible values of $\int_{\mathbb{R}} F(x) \, dx$.

Problem 10 (Date: January 2011, tags: `integral convergence?`). Let $f \in L_\infty([0, 1])$. Prove that

$$\lim_{n\to\infty} \frac{\int_{[0,1]} |f(x)|^{n+1} \, dx}{\int_{[0,1]} |f(x)|^n \, dx} = \|f\|_\infty.$$

Problem 11 (Date: January 2011, tags: `egoroff's theorem`). Let E be the exceptional set in Egoroff's theorem. Is it possible to prove Egoroff's theorem with $l(E) = 0$ instead of $l(E) < e$?

2 Categorized (with solutions)

3 Uncategorized

Problem 12 (Date: January 2011, tags:). Let $f : X \to [0, \infty]$ be a measurable function. Assume that $\mu(X) < \infty$. Prove that $\int f \, d\mu < +\infty$ if and only if

$$\sum_{n=1}^{\infty} 2^n \mu(x \in X \mid f(x) \geq 2^n) < +\infty.$$

Problem 13 (Date: January 2011, tags:). Let $f_n, g_n, f, g \in L_1(\mu)$ be functions such that $f_n \to f$ a.e., $g_n \to g$ a.e. and $|f_n| \leq g_n$. Prove that if $\int g_n \, d\mu \to \int g \, d\mu$, then $\int f_n \, d\mu \to \int f \, d\mu$.
(Hint: follow the proof of Lebesgue dominated convergence theorem.)

Problem 14 (Date: January 2011, tags:). Let $f_n, f \in L_1(\mu)$ be such that $f_n \to f$ a.e. Prove that if $\|f_n\|_1 \to \|f\|_1$, then $f_n \to f$ in $L_1(\mu)$.
(Hint: use the previous problem)

Problem 15 (Date: January 2011, tags:). Let $f, g : \mathbb{R} \to \mathbb{R}$ be L_1-functions.

(a) Prove that

$$\int_{\mathbb{R}} |f(x-y)g(y)| dm(y) < +\infty.$$

(b) Let

$$h(x) = \int_{\mathbb{R}} f(x-y)g(y) dm(y).$$

Prove that $h \in L_1(\mathbb{R})$ and $\|h\|_1 \leq \|f\|_1 \cdot \|g\|_1$.

Problem 16 (Date: January 2011, tags:)**.** Let Φ be the Cantor's function and let μ_Φ be the corresponding Lebesgue-Stiltjes measure. Calculate the integral

$$\int_{[0,1]} x d\mu_\Phi.$$

Problem 17 (Date: January 2011, tags:)**.** Let $f \in L_1([0,1])$ be a function such that $\int_E f\, dm = 0$ for any $E \subset [0,1]$ with $m(E) = 1/2$. Prove that $f = 0$.

Problem 18 (Date: January 2011, tags:)**.** Let $f \in L_1([0,1])$ be a function such that $f(x) > 0$ a.e.

(i) Prove that for any $0 < a < 1$

$$\inf_{m(A)=a} \int_A f\, dm > 0.$$

(ii) Does the previous statement hold for a function $f \in L_1(\mathbb{R})$ such that $f(x) > 0$ a.e.

Problem 19 (Date: January 2011, tags:)**.** Let (X, Ω, μ) be a finite measure space.

(i) Prove that for any $p < q,\ L_q(\mu) \subset L_p(\mu)$.

(ii) Assume that for any $t > 0$ there exists $E \in \Omega$ satisfying

$$0 < \mu(E) < t.$$

Prove that for any $1 < p < \infty$ there exists a function $f \in L_p(\mu)$ such that $f \notin L_q(\mu)$ for any $q > p$.