

# Real Analysis qualifying review big list

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## How to add to this document

This document contains a subset of the old real analysis qual problems, and solutions to some of them. To add a problem to this list, use the `problem` environment: this environment takes two arguments, the date the problem appeared on a qual, and tags describing the problem.

```
\begin{problem}{<Month> <Year>}{tag 1, tag 2, ...}
  <Problem statement>
\end{problem}
```

To add a solution, or a solution sketch, use the `solution` and `sketch` environments. These environments do not take any arguments.

## 1 Categorized (without solutions)

**Problem 1** (Date: September 2011, tags: `integral` `convergence`). Let  $f \in L_1([0, 1], dx)$ . Find

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^1 \log(1 + e^{nf(x)}) dx.$$

**Problem 2** (Date: September 2011, tags: `bounded variation`). Suppose  $f : [0, 1] \rightarrow \mathbb{R}$  satisfies  $f(x) - f(y) < x - y$  for all  $x, y \in [0, 1], x > y$ . Show that  $f'$  exists almost everywhere on  $[0, 1]$  or give a counterexample.

**Problem 3** (Date: September 2011, tags: `hölder's inequality`). Let  $(X, \Omega, \mu)$  be a finite measure space.

- (i) Prove that for any  $p < q$ ,  $L_q(\mu) \subset L_p(\mu)$ .
- (ii) Assume that for any  $t > 0$  there exists  $E \in \Omega$  satisfying

$$0 < \mu(E) < t.$$

Prove that for any  $1 < p < \infty$  there exists a function  $f \in L_p(\mu)$  such that  $f \notin L_q(\mu)$  for any  $q > p$ .

**Problem 4** (Date: September 2011, tags:  `$L^p$  spaces`). For a real valued function  $f(x, y)$  on  $\mathbb{R}^2$  which is in  $L^2$ , show that  $f(x + \epsilon, y + \epsilon) \rightarrow f(x, y)$  in  $L^2$  when  $\epsilon \rightarrow 0$ .

**Problem 5** (Date: September 2011, tags: `measurable sets?`). Let  $f \in L_1([0, 1])$  be a function such that  $\int_E f(x) dx = 0$  for any measurable set  $E \subset [0, 1]$  of Lebesgue measure  $1/2$ . Prove that  $f = 0$  a.e.

**Problem 6** (Date: January 2011, tags: measurable functions).

Let  $A$  be a sequence of measurable subsets of  $[0, 1]$  such that  $\inf m(A_n) > 0$ , where  $m$  stands for the Lebesgue measure.

- (i) Prove that there exists  $x \in [0, 1]$  which belongs to infinitely many of the sets  $A_n$ .
- (ii) Does there necessarily exist a point which (does not?) belong to any of the sets  $A_n$ , except finitely many?

**Problem 7** (Date: January 2011, tags: integral convergence). Let  $\{f_n\} \subset L_1(\mu)$  be a decreasing sequence of functions such that  $f_n \rightarrow f$  a.e. Prove that

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu.$$

**Problem 8** (Date: January 2011, tags: simple function approximation). Let  $f : X \rightarrow [0, +\infty)$  be an integrable function on a measure space  $(X, \mathcal{A}, \mu)$ . Define the measure  $\nu$  by  $\nu(A) = \int_A f d\mu$ .

- (i) Prove that the measure  $\nu$  is  $\sigma$ -additive.
- (ii) Prove that if  $g \in L_1(\nu)$ , then  $\int_X g d\nu = \int_X fg d\mu$ .  
(Hint: first, assume that  $g$  is a simple positive function. Then extend the result to non-negative integrable functions using limit theorems).

**Problem 9** (Date: January 2011, tags: integral inequalities). Let  $f_n : \mathbb{R} \rightarrow [0, 1]$  be functions such that  $\sup_{x \in \mathbb{R}} f_n(x) = 1/n$  and  $\int_{\mathbb{R}} f_n(x) dx = 1$ . Set

$$F(x) = \sup_{n \in \mathbb{N}} f_n(x).$$

Find all possible values of  $\int_{\mathbb{R}} F(x) dx$ .

**Problem 10** (Date: January 2011, tags: integral convergence?). Let  $f \in L_\infty([0, 1])$ . Prove that

$$\lim_{n \rightarrow \infty} \frac{\int_{[0,1]} |f(x)|^{n+1} dx}{\int_{[0,1]} |f(x)|^n dx} = \|f\|_\infty.$$

**Problem 11** (Date: January 2011, tags: egoroff's theorem). Let  $E$  be the exceptional set in Egoroff's theorem. Is it possible to prove Egoroff's theorem with  $l(E) = 0$  instead of  $l(E) < \epsilon$ ?

**Problem 12** (Date: January 2011, tags: simple functions). Let  $f : X \rightarrow [0, \infty]$  be a measurable function. Assume that  $\mu(X) < \infty$ . Prove that  $\int f d\mu < +\infty$  if and only if

$$\sum_{n=1}^{\infty} 2^n \mu(x \in X \mid f(x) \geq 2^n) < +\infty.$$

**Problem 13** (Date: January 2011, tags: dominated convergence). Let  $f_n, g_n, f, g \in L_1(\mu)$  be functions such that  $f_n \rightarrow f$  a.e.,  $g_n \rightarrow g$  a.e. and  $|f_n| \leq g_n$ . Prove that if  $\int g_n d\mu \rightarrow \int g d\mu$ , then  $\int f_n d\mu \rightarrow \int f d\mu$ .

(Hint: follow the proof of Lebesgue dominated convergence theorem.)

**Problem 14** (Date: January 2011, tags: ). Let  $f_n, f \in L_1(\mu)$  be such that  $f_n \rightarrow f$  a.e. Prove that if  $\|f_n\|_1 \rightarrow \|f\|_1$ , then  $f_n \rightarrow f$  in  $L_1(\mu)$ .

(Hint: use the previous problem)

**Problem 15** (Date: January 2011, tags: change of variables). Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  be  $L_1$ -functions.

(a) Prove that

$$\int_{\mathbb{R}} |f(x-y)g(y)| dm(y) < +\infty.$$

(b) Let

$$h(x) = \int_{\mathbb{R}} f(x-y)g(y) dm(y).$$

Prove that  $h \in L_1(\mathbb{R})$  and  $\|h\|_1 \leq \|f\|_1 \cdot \|g\|_1$ .

**Problem 16** (Date: January 2012, tags: measurable functions). Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable function. Prove that the function  $f'$  is measurable.

**Problem 17** (Date: January 2012, tags:  $L^p$  spaces). Let  $1 \leq p < \infty$ , and let  $f \in L_p(\mu)$ . Prove that

$$\lim_{t \rightarrow 0} t^p \mu\{x \mid |f(x)| > t\} = 0.$$

**Problem 18** (Date: January 2012, tags: Lebesgue differentiation theorem, Hardy-Littlewood maximal estimate?). Let  $f \in L_1(\mathbb{R})$ . For  $n \in \mathbb{N}$  define the function  $g_n : \mathbb{R} \rightarrow \mathbb{R}$  as follows. For  $k \in \mathbb{Z}$  and for  $x \in [k/n, (k+1)/n)$  set

$$g_n(x) = n \int_{k/n}^{(k+1)/n} f(x) dx.$$

Prove that  $g_n$  converges to  $f$  a.e. and in  $L_1(\mathbb{R})$ .

**Problem 19** (Date: January 2012, tags: integral convergence). Let  $(X, \mathcal{A}, \mu)$  be a finite measure space ( $\mu(X) < \infty$ ). Assume that a sequence  $\{f_n\}_{n=1}^{\infty} \subseteq L_1(\mu)$  satisfies the condition

$$\frac{1}{\sqrt{\mu(E)}} \int_E |f_n| d\mu \leq 1$$

for all  $n \in \mathbb{N}$  and all sets  $E$  of positive measure. Prove that if  $f_n \rightarrow f$  a.e., then  $f \in L_1(\mu)$  and

$$\int_X f_n d\mu \rightarrow \int_X f d\mu.$$

**Problem 20** (Date: January 2012, tags:  $L^p$  spaces). Construct a function  $f \in L_1(\mathbb{R})$  such that  $f \notin L_2((a, b))$  for any interval  $(a, b) \subseteq \mathbb{R}$ .

**Problem 21** (Date: September 2019, tags: countable subadditivity?). Let  $E$  be the set of all  $x \in (0, 1)$  such that there exists a sequence of irreducible fractions  $\{p_n/q_n\}_{n \in \mathbb{N}}$  with  $p_n, q_n \in \mathbb{N}$ ,  $q_1 < q_2 < \dots$  such that

$$\left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n^3}, \quad n = 1, 2, \dots$$

Prove that the Lebesgue measure of  $E$  is zero.

**Problem 22** (Date: September 2019, tags: integral convergence). Let  $f$  be a measurable function on  $(0, \infty)$ , and for  $n = 1, 2, \dots$  let  $f_n$  be defined by

$$f_n(x) = f(x)e^{-x} \left[ 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} \right].$$

Suppose  $f \in L^2[(0, \infty)]$ . Prove that  $\lim_{n \rightarrow \infty} \|f_n - f\|_{L^2[(0, \infty)]} = 0$

**Problem 23** (Date: September 2019, tags: Fubini, Hölder). Let  $f : \mathbb{R} \times (0, 1) \rightarrow \mathbb{R}$  be a measurable function such that for any  $y \in (0, 1)$ ,

$$\int_{\mathbb{R}} f^2(x, y) dx \leq 1.$$

Prove there exists a sequence  $\{x_n\}_{n \in \mathbb{N}}$ , with  $\lim_{n \rightarrow \infty} x_n = +\infty$ , such that

$$\lim_{n \rightarrow \infty} \int_0^1 |f(x_n, y)| dy = 0.$$

**Problem 24** (Date: September 2019, tags: Hölder). Let  $(X, \Omega, \mu)$  be a measure space with  $\mu(X) = 1$ , and let  $f \in L^2(\mu)$  be a non-negative function satisfying  $\int_X f d\mu \geq 1$ . Prove that

$$\mu(\{x \in X \mid f(x) > 1\}) \geq \frac{(\int_X f d\mu - 1)^2}{\int_X f^2 d\mu}.$$

**Problem 25** (Date: September 2019, tags: absolute continuity). A function  $f : (0, 1) \rightarrow \mathbb{R}$  is locally Lipschitz if for any  $x \in (0, 1)$  there is an open interval  $I_x$  with  $x \in I_x \subset (0, 1)$  and a constant  $C_x$  such that  $|f(y) - f(y')| \leq C_x |y - y'|$  for  $y, y' \in I_x$ .

- (a) Prove that a locally Lipschitz function  $f(\cdot)$  is absolutely continuous on any compact subinterval  $[a, b] \subset (0, 1)$ .
- (b) Give an example of a locally Lipschitz function  $f : (0, 1) \rightarrow \mathbb{R}$  which extends to a continuous function on the closed interval  $[0, 1]$ , but is not absolutely continuous on  $[0, 1]$ .

**Problem 26** (Date: May 2020, tags: outer regularity). Let  $E \subset (0, 1)$  be a measurable set such that for any interval  $(a, b) \subset (0, 1)$ , there exists an interval  $(c, d) \subset (a, b) \setminus E$  with

$$d - c \geq \frac{a}{10}(b - a).$$

Prove that  $m(E) = 0$ .

**Problem 27** (Date: May 2020, tags: bounded variation). Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a Lebesgue measurable function such that

$$f(y) \leq f(x) + (x^2 + y^2)(x - y) \quad \text{for } -\infty < y < x < \infty.$$

Show that the derivative function  $x \rightarrow f'(x)$  exists a.e. on  $\mathbb{R}$ .

**Problem 28** (Date: May 2020, tags: triangle inequality?). Let  $r_n$ ,  $n = 1, 2, \dots$ , be an enumeration of the rationals in the interval  $[0, 1]$  and consider the function  $f : [0, 1] \rightarrow \mathbb{R} \cup \infty$  defined by

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1}{|x - r_n|^{1/3}}, \quad 0 \leq x \leq 1.$$

Show that  $f \in L^2(0, 1)$ .

**Problem 29** (Date: May 2020, tags: change of variables, integral convergence). Let  $f_n$ ,  $n = 1, 2, \dots$ , be the sequence of functions on  $(0, \infty)$  defined by

$$f_n(x) = \frac{1}{n} \left(1 - \frac{x}{n}\right)^n e^x, \quad 0 < x < n, \quad f_n(x) = 0, \quad x \geq n.$$

Prove that the sequence  $a_n$ ,  $n = 1, 2, \dots$ , given by

$$a_n = \int_0^{\infty} f_n(x) dx \quad \text{converges and identify } a_{\infty} = \lim_{n \rightarrow \infty} a_n.$$

**Problem 30** (Date: May 2020, tags:  $L^p$  spaces). Suppose  $f$  is a  $C^1$  function on  $\mathbb{R}$  satisfying  $f(0) = 0$ ,  $|f(x)| \leq |x|^{-1/2}$ ,  $x \neq 0$ . Let  $g$  be in  $L^1(\mathbb{R})$ .

(a) Show there is a constant  $C$  such that  $m\{|g| > \alpha\} \leq C/\alpha$  for all  $\alpha > 0$ .

(b) Show that the function  $h(x) = f(g(x))$  is in  $L^1(\mathbb{R})$ .

**Problem 31** (Date: May 2011, tags: integral convergence). Let  $f_n, g_n, f, g \in L_1(\mu)$  be functions such that  $f_n \rightarrow f$  a.e.,  $g_n \rightarrow g$  a.e. and  $|f_n| \leq g_n$ . Prove that if  $\int g_n d\mu \rightarrow \int g d\mu$ , then  $\int f_n d\mu \rightarrow \int f d\mu$ .

(Hint: use Fatou's Lemma.)

**Problem 32** (Date: May 2011, tags: Egorov's theorem). Let  $\{f_n : [0, 1] \rightarrow \mathbb{R}\}_{n=1}^\infty$  be a sequence of continuous functions such that  $f_n(x) \rightarrow f(x)$  for any  $x \in [0, 1]$ . Does there exist a set  $E \subset [0, 1]$  of Lebesgue measure 0 such that  $f_n \rightarrow f$  uniformly on  $[0, 1] \setminus E$ ?

## 2 Categorized (with solutions)

### 3 Uncategorized

**Problem 33** (Date: January 2011, tags: ). Let  $f \in L_1([0, 1])$  be a function such that  $f(x) > 0$  a.e.

(i) Prove that for any  $0 < a < 1$

$$\inf_{m(A)=a} \int_A f dm > 0.$$

(ii) Does the previous statement hold for a function  $f \in L_1(\mathbb{R})$  such that  $f(x) > 0$  a.e.