The Laplacian on Riemannian manifolds and some geometric estimates of its eigenvalues

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1.1 Motivating the Laplacian

In complex analysis, holomorphic functions are the key object of study, not only because they're simple to define, but also because they have remarkably nice properties: although their definition requires them to be differentiable once, they turn out to be analytic (a regularity property), holomorphic functions defined on a disc achieve maximum and minimum on the boundary (a maximum principle), bounded holomorphic functions on $\mathbb{C}$ are constant (a Liouville property), etc. On flat $\mathbb{R}^n$, the closest analog to holomorphic functions are harmonic functions, i.e. functions f such that $\sum_i \frac{\partial^2}{\partial x_i^2} f = 0$. They are also nice in a similar manner, and satisfy all the mentioned properties.

One can generalize this notion even further, and define harmonic functions on Riemannian manifolds (henceforth whenever the word *manifold* is mentioned without any adjective, it will be implicitly assumed we're talking about a compact Riemannian manifold with no boundary). Having defined harmonic functions on manifolds, one would expect them to satisfy similar results. On a compact manifold, since all continuous functions are bounded, the Liouville type result would force harmonic functions to be constant: this might indicate the study of harmonic functions on compact manifolds is rather boring, and that is indeed the case. To make things interesting, one defines the notion of harmonicity for not just functions, but for all differential forms. One might hope that harmonic forms are not trivial, and at the same time, are a sufficiently small subset of the set of all differential forms such that it's easier to deal with them (after all, the harmonic functions were a one dimensional subspace of the infinite dimensional space of smooth functions on the manifold). For harmonic forms however, we won't really be using any of the nice properties that hold for harmonic functions. What makes harmonic forms so interesting is that they're solutions of some elliptic PDE on manifolds, and it's a general principle that solutions to elliptic PDE provide insight into the global topology of the space.

To define the notion for harmonicity for functions and forms on manifolds, we'll generalize the operator $\sum_i \frac{\partial^2}{\partial x_i^2}$ (while still calling it the Laplacian), and define harmonic forms to be the elements in the kernel of the operator.

1.2 Defining the Laplacian for functions and forms

To define the Laplacian for functions on a manifold, we recall an equivalent definition for Laplacian on $\mathbb{R}^n$, which is the following.

$$\Delta f = \operatorname{div}(\operatorname{grad}(f))$$

Luckily, both div and grad can be defined on manifolds. Writing out the definitions for the two operators on manifolds, the expression the Laplacian can be written in terms of the exterior derivative and the Hodge star operation.

$$\Delta f = (\star d \star)(df)$$

The above definition of the Laplacian extends verbatim to differential forms of all degrees. Is this the "right" definition though? As it turns out, it isn't. There are two problems with it: the first problem is that when applied to top dimensional forms, this operator sends everything to 0. This would mean that every top dimensional form is harmonic, which is something we don't want, since we would like the space of harmonic forms to be a finite-dimensional subspace of the space of forms. The second and more fundamental problem is that this operator is not self adjoint with respect to the L^2 inner product on the space of forms. Luckily, that's an easy problem to solve: we just add the adjoint of the operator to the operator. We'll call this symmetrized operator the Laplacian. If we denote by δ the operator that takes p-forms to $(p-1)$-forms, given by the expression $(-1)^{n(p+1)+1} \star d \star$, then the Laplacian has a particularly nice expression.

$$\Delta f = \delta df + d\delta f$$

As it turns out, this operator is not only self adjoint, but also has a finite-dimensional kernel (the latter fact is a consequence of the Hodge decomposition theorem).

1.3 Statement of the Hodge decomposition theorem

Now that we have defined the Laplacian on manifolds, we can state the Hodge decomposition theorem. We shall denote the inner product space of p-forms on the manifold by $E^p(M)$, and the subspace of harmonic p-forms by H^p. There is a particularly nice orthogonal direct sum decomposition of the space $E^p(M)$, which is given by the following theorem.

Theorem 1 (Hodge decomposition theorem). *The space of harmonic p-forms H^p is a finite dimensional subspace of the space of p-forms $E^p(M)$, and $E^p(M)$ has the following orthogonal direct sum decomposition.*

$$E^p(M) = \Delta(E^p) \oplus H^p$$

Here, $\Delta(E^p)$ denotes the image of $E^p(M)$ under the operator Δ.

In fact, more can be said about the operator Δ when we restrict it to the space of smooth functions on the manifold.

Theorem 2 (Hodge theorem part II). *The operator $\Delta : C^\infty(M) \to C^\infty(M)$ has all real eigenvalues, with all eigenvalues greater than or equal to 0. The set of eigenvalues is a discrete subset of $[0, \infty)$, and the eigenvalues can be arbitrarily large. Furthermore, each eigenspace is finite dimensional, and the span of all the eigenfunctions is dense in $C^\infty(M)$ in both the L^2-norm as well as the $\|\cdot\|_\infty$ norm.*

An important corollary of Theorem 1 connects the space of harmonic forms to the De Rham cohomology spaces of the manifold.

Corollary 3. *The linear map π that projects closed p-forms to harmonic p-forms is surjective and $\pi(f) = \pi(g)$ if and only if f and g differ by an exact form. This means that the p^{th} De Rham cohomology space is isomorphic to the space of harmonic p-forms (and the isomorphism is an easily computable map).*

The isomorphism between the space of harmonic p-forms and the p^{th} cohomology space is particularly useful, as it lets us prove stuff about the dimension of the cohomology spaces by proving the same thing for the space of harmonic forms instead. We shall use this trick very often when we are proving stuff using the Bochner technique.

1.4 Idea of proof

The sketch of the proof we give is based on the proof in Warner's book ([War10] Ch. 6).

Before outlining the idea of the proof, let's reiterate what we want to prove. We want to prove that $E^p(M)$ has an orthogonal direct sum decomposition as $\Delta(E^p) \oplus H^p$. This is equivalent to proving that the space $(H^p)^\perp$ is equal to the space $\Delta(E^p)$. As in most statements asserting equality of two sets, showing the inclusion one way is significantly easier than showing the inclusion the other way. In this case, the following inclusion is fairly easy to show: it simply follows from the fact that Δ is self-adjoint.

$$\Delta(E^p) \subseteq (H^p)^\perp$$

The hard part is showing the reverse inclusion, i.e. the following inclusion.

$$(H^p)^\perp \subseteq \Delta(E^p)$$

Our goal is now the following: given an element $\alpha \in (H^p)^\perp$, we want to find an element $\omega \in E^p(M)$ such that the following equation is satisfied.

$$\Delta\omega = \alpha \tag{1}$$

Here comes the ingenious part: for every $\gamma \in E^p(M)$, one can consider the linear functional $l_\gamma : E^p(M) \to \mathbb{R}$ defined in the following manner.

$$l_\gamma(\psi) = \langle \gamma, \psi \rangle$$

Now suppose we had an ω such that equation (1) was satisfied. Then we can say the something about the functional l_ω as well. The functional will satisfy the following identity for all $\phi \in E^p(M)$ (as a consequence of the fact that Δ is self-adjoint).

$$l_\omega(\Delta\phi) = \langle \alpha, \phi \rangle \tag{2}$$

In fact, if some other functional l_γ satisfies equation (2), then γ satisfies equation (1). But equation (2) makes sense for any functional, and not just the functionals corresponding to elements of $E^p(M)$. We shall call functional that satisfies equation (2) a weak solution to equation (1).

As it turns out, finding weak solutions isn't too hard. In our case, the fact that a weak solution to equation (1) exists is simply a consequence of the following lemma, and the Hahn-Banach theorem.

Lemma 4. *If $\{\alpha_n\}$ is a sequence of smooth forms in $E^p(M)$ such that $\|\alpha_n\| \leqslant c$ and $\|\Delta\alpha_n\| \leqslant c$ for some fixed constant c and for all n. Then the sequence has a Cauchy subsequence.*

Remark. This lemma also proves the first part of theorem 1, which states that the space of harmonic forms is finite dimensional. That follows from the above lemma and the fact that bounded sequences have Cauchy subsequences iff the space is finite dimensional.

Finding the weak solution is the first step to solving equation (1). The next step is showing that the functional l we obtained that satisfies equation (2) actually comes from some smooth p-form γ. Then γ is an honest solution of equation 2. As it turns out, all weak solutions are actually honest solutions, as the following theorem demonstrates.

Theorem 5 (Regularity Theorem)**.** *If the functional l is a weak solution to $\Delta\omega = \alpha$, then there exists $\gamma \in E^p(M)$ such that $l = l_\gamma$, i.e. l is given by the following expression.*

$$l(\beta) = \langle \gamma, \beta \rangle$$

That means that $\Delta\gamma = \alpha$, and hence γ is a solution to the required equation.

Remark. The regularity theorem has vast generalizations which are routinely used to prove smoothness of weak solutions to PDEs. These generalizations apply to a large class of partial differential operators called elliptic operators (the Laplacian is an example of such an operator), and all the generalizations say something to the following effect: if u is a "weak" solution to the equation $Lu = f$, and f is k-times differentiable, then u actually is better than a weak solution, it's $(k + l)$-times differentiable, where l is some positive constant depending on the partial differential operator L.

It's clear from Theorem 5 that the weak solution we got using the Hahn-Banach theorem is actually a smooth solution lying in $E^p(M)$. This proves the Hodge decomposition theorem, modulo the proofs of Lemma 4 and Theorem 5. We will skip those proofs in this exposition, mainly because it will be a long detour into the realm of PDEs on manifolds, and we want to get to the geometric consequences as quickly as possible.

Proving Theorem 2 is much simpler, and simply utilises the min-max method to compute the eigenvalues of Δ.

This is not the only way of proving the theorem. A different approach is to construct an "approximate inverse" (also called a parametrix) for the operator Δ. That will show Δ is what is known as a Fredholm operator, and from that point on, the proof is just standard functional analysis. The details of this approach can be found in Griffith's book ([GH14], Ch. 0, Sec. 6).

1.5 Applications of Hodge decomposition

1.5.1 The Bochner technique

The Bochner technique is a way of proving the vanishing of certain cohomology classes of a manifold using constraints on the curvature of the manifold. A prototypical example of such a result is the following theorem.

Theorem 6. *If the Ricci curvature tensor of a manifold M is positive definite, then the first De Rham cohomology of M is trivial.*

The idea behind the Bochner technique is quite simple. We write down the Laplacian of a form (or a function derived from the form) as a sum of two terms: the first one is a non-negative term, and the second term is some function of the curvature, and can be made positive or non-negative by imposing suitable constraints on the curvature. A concrete example of this is the following formula.

$$-\Delta |\phi|^2 = 2\left(\sum_i |\nabla_{V_i}\phi|^2 + \left\langle \phi, \sum_{i,j} \omega^i \wedge \iota(V_j) R_{V_i V_j}(\phi) - \Delta\phi \right\rangle\right) \tag{3}$$

Here ϕ is a k-form, $\{V_i\}$ a local frame field, $\{\omega^i\}$ its dual coframe field, and R is the curvature endomorphism of the induced Levi-Civita connection on the bundle of k-forms. By Corollary 3 of the Hodge decomposition theorem, the dimension of the space of harmonic k-forms and the dimension of $H^k(M, \mathbb{R})$ are the same. That means if we want to show the k^{th} cohomology space of M is trivial, it will suffice to show that the only harmonic k-form is identically zero. Restricting equation (3) to harmonic forms, we get the following simplified equation, with which we'll work.

$$-\Delta |\phi|^2 = 2 \left(\sum_i |\nabla_{V_i} \phi|^2 + \underbrace{\left\langle \phi, \sum_{i,j} \omega^i \wedge \iota(V_j) R_{V_i V_j}(\phi) \right\rangle}_{F(\phi)} \right) \tag{4}$$

We shall denote the second term on the right hand side as $F(\phi)$. We are interested in the cases when $F(\phi) > 0$ and $F(\phi) \geqslant 0$ (the distinction between the two is that in the former, $F(\phi) = 0$ iff $\phi = 0$). It's clear that if the $F(\phi)$ is non-negative, then $-\Delta |\phi|^2 \geqslant 0$. Since we're working over a closed manifold, the maximum principle tells us that $|\phi|^2$ must be a constant, and as a result, both $\nabla\phi$ and $F(\phi)$ are 0. If the $F(\phi)$ is positive, then this tells us that the space of harmonic k-forms is trivial (where ϕ was taken to be a k-form), and hence the k^{th} cohomology is also trivial. Even in the case when $f(\phi)$ is merely non-negative, we can still bound the dimension of the space of harmonic forms, by using the fact that $\nabla\phi \equiv 0$. This means that the value of the k-form ϕ at any point x depends only on the value of the form at a fixed base point, say x_0. The value of ϕ at x is given by parallel transporting the form from x_0 to x along any curve; $\nabla\phi \equiv 0$ ensures that what we get is independent of the path chosen. That means the space of harmonic k-forms is of dimension at most $\binom{n}{k}$, and so is the dimension of $H^k(M, \mathbb{R})$. We encapsulate this in a theorem.

Theorem 7. *If F is non-negative, the k^{th} Betti number of M for $0 < k < \dim(M)$ is less than or equal to $\binom{n}{k}$. If F is strictly positive, then the k^{th} Betti number is 0.*

We have so far neglected to address the question of what the geometric meaning of $F(\phi)$ being positive or non-negative is. The Bochner technique wouldn't be very interesting if $F(\phi) > 0$ or $F(\phi) \geqslant 0$ was an extremely restrictive condition. In the case of 1-forms, $F(\phi)$ has a particularly simple expression.

$$F(\phi) = \mathrm{Ric}\left(\phi^\sharp, \phi^\sharp\right)$$

It is clear from this expression that if the Ricci curvature is positive definite, then $F(\phi) > 0$, and as a result, the first cohomology will vanish (this is how Theorem 6 is proved). In the case of general k-forms, there's no simple formula for $F(\phi)$, but there is a bilinear form, called the *curvature operator*, whose positivity determines the positivity of $F(\phi)$. The positivity of the curvature operator is weakly correlated with the positivity of the sectional curvature, in a manner which we'll make precise later.

The curvature operator Q is a bilinear form defined on the space of 2-forms which takes two 2-forms and outputs a smooth function.

$$Q(\phi \wedge \psi, \zeta \wedge \eta) = \left\langle R_{\phi^\sharp \psi^\sharp} \zeta^\sharp, \eta^\sharp \right\rangle$$

From the definition, it's not too hard to conclude that if the curvature operator is positive definite, then the sectional curvature is positive. On the other hand, if the sectional curvature is positive, all we can conclude is that the curvature operator is positive semi-definite (there are examples where the sectional curvature is positive, but the curvature operator is only positive semi-definite). We have the following theorem due to Meyer ([Mey71]; also found in [Wu88]) linking the curvature operator to $F(\phi)$.

Theorem 8. *If Q is a positive definite (respectively positive semi-definite), then $F(\phi) > 0$ (respectively $F(\phi) \geqslant 0$).*

A consequence of Theorems 7 and 8 is that a space with positive curvature operator has the same homology as that of the sphere. Which raises the question of whether a manifold with positive curvature operator is *diffeomorphic* to a sphere. In the cases of dimension 3 and 4, Hamilton ([Ham82]) showed that positive curvature operator indeed does imply that the manifold is diffeomorphic to a sphere.

In whatever we have seen about the Bochner technique, it's easy to lose sight of the importance of the fact that all this would have been much harder if we did not have a way of picking out a harmonic representative from each cohomology class. The Hodge decomposition theorem played a key role, albeit hidden, in everything we did so far.

1.5.2 Representations of compact Lie groups

Let G be a compact Lie group with a left-invariant metric. We are interested in finding some of the finite-dimensional subrepresentations of the regular representation of G, i.e. G acting by left translation on the Hilbert space $L^2(G)$. Let's focus our attention on the subspace $C^\infty(G)$. We have the following group action on this space.

$$(g \circ f)(x) = f(g \cdot x)$$

On this subspace, we also have the operator Δ taking $C^\infty(M)$ to itself. Because the metric is left-invariant, left multiplication is an isometry. And since Δ commutes with isometries, it commutes with the action of g, giving us the following identity for all $g \in G$.

$$\Delta(g \circ f) = g \circ (\Delta f)$$

This means that each eigenspace of Δ is left invariant by the action of G on that subspace, which means the action of G on each eigenspace gives a finite-dimensional representation of G. This observation is actually fairly useful, and is a key element of the proof of the first part of Peter-Weyl Theorem.

Theorem 9 (Peter-Weyl Theorem part I)**.** *The space of matrix coefficients of all finite dimensional representations of a compact Lie group G is dense in the space $C^\infty(G)$ with the $\|\cdot\|_\infty$ norm.*

The theorem is proved by using the previous observations to show that eigenfunctions of Δ are matrix coefficients of G, and then using the fact that eigenfunctions are dense in $C^\infty(M)$.

2 Eigenvalues of the Laplacian

If we consider the full spectrum of the Laplacian, we have so far only considered the eigenfunctions (and eigenforms) with eigenvalue 0. But part II of the Hodge theorem

(Theorem 2) tells us that in the case of functions, there are more eigenvalues and eigenfunctions to consider. As a first step in investigating them, one could look at the sequence of eigenvalues, and try to glean information about the geometry of the manifold through that numerical sequence. To misquote Mark Kac ([Kac66]), "Can one hear the shape of a manifold if it's played like a drum". The answer turns out to be no, as showed by Milnor ([Mil64]).

However, even if one can't determine the isometry type of the manifold from its spectrum, it is possible to get an estimate of certain geometric quantities like volume, diameter and curvature from the spectrum. One can also do the reverse, and estimate some of the eigenvalues from bounds on the curvature.

Before we look at these estimates, it will be instructional to look at an explicit description of all the eigenvalues (and the corresponding eigenfunctions) in the case of the flat torus and the round sphere (this follows the exposition given in Gallot, Hulin, and Lafontaine's book [GHL90]).

2.1 Exact computations

Flat torus A flat torus is the quotient of $\mathbb{R}^n$ by a lattice Γ of rank n. Using Fourier analytic methods, we can write down all the eigenfunctions of the Laplacian acting on the space of *complex* valued smooth functions (the fact that we're looking at complex valued eigenfunctions is not really important; we could just take the collection of real and imaginary part of each complex eigenfunction and that would give us the collection of real eigenfunctions). The eigenfunctions are indexed by the elements of the lattice dual[1] to Γ, whose elements we'll denote by λ^*. The eigenfunctions are given by the following formula.

$$f_{\lambda^*}(x) = \exp(2\pi i \langle \lambda^*, x \rangle)$$

The corresponding eigenvalues are $4\pi^2 |\lambda^*|^2$.

Round sphere A round sphere S^n is just the sphere of radius 1 centred at origin in $\mathbb{R}^{n+1}$ with the induced Riemannian metric. For any function f defined in a neighbourhood of the sphere, we can describe the Laplacian Δ of f (considered as a function on the sphere) in terms the Laplacian $\widetilde{\Delta}$ on $\mathbb{R}^n$ and $\frac{\partial}{\partial r}$.

$$\Delta f = \widetilde{\Delta} f + \frac{\partial^2 f}{\partial r^2} + n \frac{\partial f}{\partial r}$$

This expression makes it easy to find some of the eigenfunctions for the sphere. If f is a homogeneous harmonic polynomial of degree k (harmonic in the sense of $\widetilde{\Delta} f = 0$, i.e. harmonic on $\mathbb{R}^{n+1}$), then f restricted to S^n is an eigenfunction of the Laplacian with eigenvalue $k(k+n-1)$. It turns out that eigenfunctions of this form are the only kinds of eigenfunctions on the sphere (the way to prove it is to show that the span of these eigenfunctions is dense in $C^\infty(S^n)$).

One thing to observe in both the examples is that the k^{th} eigenvalue grows as k^2. It turns out that it is a more general phenomenon. The k^{th} eigenvalue of a manifold

[1] A lattice dual to Γ is the collection of all elements λ^* in $\mathbb{R}^n$ such that $\langle \lambda, \lambda^* \rangle \in \mathbb{Z}$ for all elements $\lambda \in \Gamma$. Since Γ is a full rank lattice, the dual Γ^* is also a full rank lattice.

is smaller than ck^2, where c is some constant that depends on the manifold (and the metric).

2.2 Estimates on the eigenvalues

In the case of the torus and the sphere, the eigenvalues were easy to compute because these spaces either had nice coordinates (and a nice metric with respect to those coordinates), in the case of the torus, or a large amount of symmetry, in the case of the sphere. In the case of arbitrary manifolds, it's usually not so easy; we will settle for estimates on the eigenvalues in terms of geometric properties of the manifold like curvature and diameter.

One of the key techniques we'll use to estimate the eigenvalues of the Laplacian is the min-max theorem.

Theorem 10 (Min-max theorem)**.** *Suppose the eigenvalues of the Laplacian are written in an increasing order, where each eigenvalue is repeated according to their multiplicities.*

$$\lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots$$

In that case, λ_k is given by the following expression.

$$\lambda_k = \min_{U} \left\{ \max_{f} \left\{ \frac{\langle f, \Delta f, \rangle}{\|f\|^2} \middle| f \in U \textit{ and } f \neq 0 \right\} \middle| \dim(U) = k+1 \right\}$$

We will also need to look at eigenvalues of the Laplacian in geodesic balls on the manifold. Because geodesic balls have a boundary, the Laplacian on these spaces is *not* a self-adjoint operator. However, if we restrict the Laplacian to only those functions which are zero on the boundary, then the Laplacian becomes self adjoint again (this is why this boundary condition is of interest).

Consider a manifold M whose Ricci curvature is bounded below by some constant κ, i.e. the following inequality holds for all tangent vectors η.

$$\operatorname{Ric}(\eta, \eta) \geqslant \kappa(n-1)\,|\eta|^2 \tag{5}$$

In this case, it is possible to give an upper bound for the all the eigenvalues, and a lower bound for the lowest non-zero eigenvalue, and the bounds given are functions of κ. The upper bound is due to Cheng ([Che75]), and is encapsulated in the following theorem.

Theorem 11 (Cheng's theorem)**.** *Suppose M is a compact manifold whose Ricci curvature satisfies inequality* (5)*, and whose diameter is D. Suppose we list out the eigenvalues of Δ on M in the following manner,*

$$0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$$

where we repeat each eigenvalue according to its multiplicity, then λ_j satisfies the following inequality.

$$\lambda_j \leqslant {}_{\kappa}\lambda_0\left(\frac{D}{2(j+1)}\right) \tag{6}$$

Here, ${}_{\kappa}\lambda_0\left(\frac{D}{2(j+1)}\right)$ is the lowest eigenvalue on the open ball of radius $\frac{D}{2(j+1)}$ on the simply connected manifold with constant sectional curvature κ.

The idea of the proof is to find $j+1$ disjoint geodesic balls, and look at an eigenfunction corresponding to the lowest eigenvalue in each of the geodesic balls. This gives a $j+1$ dimensional subspace of $C^\infty(M)$, and then using a technical comparison theorem for the eigenvalues on the balls, and the min-max theorem, the inequality can be obtained.

Inequality (6) may seem a little disappointing because the dependence on κ is not quite clear. However, we have a fairly simple upper bound on ${}_\kappa\lambda_0\left(\frac{D}{2(j+1)}\right)$ in the case when $\kappa \leqslant 0$. This bound is due to Gage ([Gag80]).

Theorem 12. *If $\kappa \leqslant 0$, then the following inequality is satisfied.*

$$ {}_\kappa\lambda_0(\delta) \leqslant -a\kappa + \frac{b}{\delta^2} + \frac{c}{\sinh^2(\kappa\delta)} \tag{7} $$

Here a and b are positive constants, and c a real constant, all of which only depend on the dimension of the manifold.

Combining inequalities (6) and (7), we see the dependence on the curvature bound κ, as well as the phenomenon we saw in the case of the torus and the sphere, namely that the k^{th} eigenvalue is $O(k^2)$.

Now that we have obtained upper bounds for λ_1, we turn our attention towards getting a lower bound. The lower bound is given by an inequality due to Lichnerowicz ([Lic58]).

Theorem 13. *Suppose M is a compact manifold whose Ricci curvature satisfies inequality (5) for $\kappa > 0$. Then the lowest non-zero eigenvalue of M satisfies the following inequality.*

$$ \lambda_1 \geqslant n\kappa \tag{8} $$

Surprisingly, this inequality is proved by yet another application of a Bochner style identity, namely the following one.

$$ \frac{1}{2}\Delta\left(|\text{grad } f|^2\right) = |\text{Hess } f|^2 + \langle \text{grad } f, \text{grad } \Delta f\rangle + \text{Ric}(\text{grad } f, \text{grad } f) \tag{9} $$

Integrating both sides, and then using inequality (5), one gets the following inequality from which the result follows.

$$ 0 \geqslant \lambda\left(1 - \frac{1}{n}\right)(n\kappa - \lambda)\,\|f\|^2 $$

In fact, it turns out that if equality is achieved in inequality (8), then M is isometric to the sphere of curvature κ (this result is due to Obata [Oba62]). This would suggest that lowest non-zero eigenvalue λ_1 imposes a fair amount of rigidity on the structure of M.

The takeaway from all this is that even though we might not be able to hear the shape of a drum, we can certainly hear how large and how curved it is.

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