Multivariate Analysis

Assignment 5

Problem 1

Assume: 2 populations having the same covariance matrix Σ but different mean vectors u_1 & u_2.

Population 1 samples: $y_{11}, y_{12}, \ldots, y_{1n_1}$ | each $y_{ij} \in \mathbb{R}_{1\times p}$

Population 2 samples: $y_{21}, y_{22}, \ldots, y_{2n_2}$ | linear combination: $z = a' y$

$\therefore z_{1i} = a' y_{1i}$, $z_{2i} = a' y_{2i}$ and $\bar{z}_1 = \frac{1}{n_1}\sum_{i=1}^{n_1} z_{1i} = a'\bar{y}_1$, $\bar{z}_2 = \frac{1}{n_2}\sum_{i=1}^{n_2} z_{2i} = a'\bar{y}_2$

We wish to find a s.t $\frac{(\bar{z}_1 - \bar{z}_2)^2}{s_z^2}$ is maximized.

Now $\left\{\frac{\bar{z}_1 - \bar{z}_2}{s_z}\right\}^2 = \frac{[a'(\bar{y}_1 - \bar{y}_2)]^2}{a' S_{pl} a}$

$$\frac{\partial}{\partial a} \frac{[a'(\bar{y}_1 - \bar{y}_2)]^2}{a' S_{pl} a} = 0 \Rightarrow (a' S_{pl} a)\, 2\, [a'(\bar{y}_1 - \bar{y}_2)](\bar{y}_1 - \bar{y}_2) - [a'(\bar{y}_1 - \bar{y}_2)]^2 (2 S_{pl} a) = 0$$

$$\Rightarrow (a' S_{pl} a)(\bar{y}_1 - \bar{y}_2) - [a'(\bar{y}_1 - \bar{y}_2)](S_{pl} a) = 0$$

$$\Rightarrow \frac{(a' S_{pl} a)}{[a'(\bar{y}_1 - \bar{y}_2)]}(\bar{y}_1 - \bar{y}_2) = S_{pl}\, a \Rightarrow a = c\, S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2)$$

where $c = \frac{a' S_{pl} a}{a'(\bar{y}_1 - \bar{y}_2)}$

$$\max [(\bar{z}_1 - \bar{z}_2)/s_z]^2 = \frac{\{[c\, S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2)]'(\bar{y}_1 - \bar{y}_2)\}^2}{[c\, S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2)]' S_{pl} [c\, S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2)]}$$

$$= \frac{\{(\bar{y}_1 - \bar{y}_2)'(S_{pl}^{-1})' c\, (\bar{y}_1 - \bar{y}_2)\}^2}{(\bar{y}_1 - \bar{y}_2)'(S_{pl}^{-1})' c^2 S_{pl} S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2)} = \frac{[(\bar{y}_1 - \bar{y}_2)' S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2)]^2 \cdot \cancel{c^2}}{(\bar{y}_1 - \bar{y}_2)' \cancel{S_{pl}^{-1}} \cdot \cancel{S_{pl}} \cdot S_{pl}^{-1}(y_1 - y_2)\, \cancel{c^2}}$$

$$= (\bar{y}_1 - \bar{y}_2)' S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2).$$

$\therefore$ As c^2 (s) get cancelled out, we conclude that c can take any non-zero value.

if c is set to one, $\left(\frac{\bar{z}_1 - \bar{z}_2}{s_z}\right)^2$ is maximized with $a = S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2)$

- the maximizing vector a is not unique.

Problem 2

$$Z = a'y \ , \quad a = S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2)$$

$$\frac{(\bar{Z}_1 - \bar{Z}_2)^2}{S_z^2} = \frac{[a'(\bar{y}_1 - \bar{y}_2)]^2}{a' S_{pl}\, a} = \frac{\left[\{S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2)\}'(\bar{y}_1 - \bar{y}_2)\right]^2}{(S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2))' S_{pl}\,(S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2))}$$

$$= \frac{\{(\bar{y}_1 - \bar{y}_2)'(S_{pl}^{-1})'(\bar{y}_1 - \bar{y}_2)\}^2}{(\bar{y}_1 - \bar{y}_2)'(S_{pl}^{-1})' S_{pl} S_{pl}^{-1}(y_1 - y_2)} = \frac{\{(y_1 - y_2)' S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2)\}^2}{(y_1 - y_2)' \cancel{S_{pl}^{-1}}\, \cancel{S_{pl}}\, S_{pl}^{-1}(y_1 - y_2)}$$

• As S_{pl}^{-1} is symmetric.

$$= \frac{\{(\bar{y}_1 - \bar{y}_2)' S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2)\}^{\cancel{2}}}{\cancel{(\bar{y}_1 - \bar{y}_2)' S_{pl}^{-1}(y_1 - \bar{y}_2)}} = (\bar{y}_1 - \bar{y}_2)' S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2) \qquad [\text{Proved}]$$

Problem 3

$Z_1 = a_1'y$ maximizes $a'Ha / a'Ea$. Thus, $\lambda_1 = a_1' H a_1 / a_1' E a_1$ (symbols carry their usual meanings). To show that z_1 & z_2 are uncorrelated: $r_{z_1 z_2} = \dfrac{s_{z_1 z_2}}{s_{z_1} s_{z_2}} = \dfrac{a_1' S a_2}{\sqrt{(a_1' S a_1)(a_2' S a_2)}}$

The pooled estimator of $\Sigma = S = \dfrac{E}{N-k}$, with $N = \sum_{i=1}^{k} n_i$.

where k = number of groups,
n_i = number of samples in group i.

$r_{z_1 z_2} = \dfrac{a_1' E a_2}{\sqrt{(a_1' E a_1)(a_2' E a_2)}}$. To show $a_1' E a_2 = 0$, consider:

$H a_1 = \lambda_1 E a_1$, $H a_2 = \lambda_2 E a_2$

$\therefore a_2' H a_1 = \lambda_1 a_2' E a_1$, $a_1' H a_2 = \lambda_2 a_1' E a_2$

Subtracting $\rightarrow (\lambda_1 - \lambda_2) a_2' E a_1 = 0$ As $a_2' H a_1$ is symmetric.

Since, eigenvalues of $E^{-1}H$ are distinct, $\lambda_1 - \lambda_2 \neq 0$.

$\therefore a_2' E a_1 = 0$, $r_{z_1 z_2} = 0$.

Now, to show that $z_2 = a_2' y$ has the largest discriminant criterion. $\lambda_2 = a_2' H a_2 / a_2' E a_2$ subject to the constraint $r_{z_1 z_2} = 0$.

$\rightarrow$ Using lagrangian γ:

$$\frac{\partial}{\partial a_2}\left(\frac{a_2' H a_2}{a_2' E a_2} + \gamma a_1' E a_2\right) = 0 \Rightarrow \frac{a_2' E a_2 \, 2 H a_2 - a_2' H a_2 \, 2 E a_2}{(a_2' E a_2)} + \gamma E a_1 = 0$$

$$\Rightarrow \frac{2 H a_2 - 2 \lambda_2 E a_2}{a_2' E a_2} + \gamma E a_1 = 0 \xrightarrow[\text{with } a_1]{\text{multipying}}$$

$$\frac{2 a_1' H a_2 - 2 \lambda_2 a_1' E a_2}{a_2' E a_2} + \gamma a_1' E a_1 = 0 \Rightarrow a_1' E a_2 = 0 \Rightarrow a_1' H a_2 = 0$$

As $\gamma = 0$, $H a_2 - \lambda_2 E a_2 = 0$. So, 2nd eigenvector of $E^{-1}H$ maximizes $\lambda_2 = a_2' H a_2 / a_2' E a_2$ subject to $r_{z_1 z_2} = 0$.

Similary, subject to $r_{z_1 z_3} = r_{z_2 z_3} = 0$: $H a_3 - \lambda_3 E a_3 = 0$

$\Rightarrow E^{-1} H a_3 = \lambda_3 a_3$. $\therefore a_3$ is the 3rd eigenvector of $E^{-1}H$ corresponding to its 3rd largest distinct eigenvalue λ_3.

$\therefore \lambda_3 = \dfrac{a_3' H a_3}{a_3' E a_3}$ is maximized and we have the corresponding 3rd discriminant function $z_3 = a_3' y$.

Also, the set of vectors a_1, a_2, a_3 is linearly independent.