Multivariate Analysis

Assignment 6

1) The probability density functions for G_1 & G_2 are given by $f(y|G_1)$, $f(y|G_2)$ where G_1, G_2 are multivariate normals of dimension $= p$

$$\cdot f(y|G_1) = \frac{1}{(\sqrt{2\pi})^p |\Sigma|^{1/2}} \exp\left\{-\frac{(y-\mu_1)^T \Sigma^{-1} (y-\mu_1)}{2}\right\}$$

$$\cdot f(y|G_2) = \frac{1}{(\sqrt{2\pi})^p |\Sigma|^{1/2}} \exp\left\{-\frac{(y-\mu_2)^T \Sigma^{-1} (y-\mu_2)}{2}\right\}$$

Now, $$\frac{f(y|G_1)}{f(y|G_2)} = \exp\left\{\frac{(y-\mu_2)^T \Sigma^{-1}(y-\mu_2) - (y-\mu_1)^T \Sigma^{-1}(y-\mu_1)}{2}\right\}$$

$$= \exp\left\{[(\mu_1-\mu_2)^T \Sigma^{-1} y + y^T \Sigma^{-1}(\mu_1-\mu_2) + \mu_2^T \Sigma^{-1}\mu_2 - \mu_1^T \Sigma^{-1}\mu_1]/2\right\}$$

We know that $(\mu_1-\mu_2)^T \Sigma^{-1} y$ is a scalar quantity.

$$\therefore \{(\mu_1-\mu_2)^T \Sigma^{-1} y\}^T = (\mu_1-\mu_2)^T \Sigma^{-1} y$$

$$\Rightarrow (\mu_1-\mu_2)\Sigma^{-1} y = y^T \Sigma^{-1}(\mu_1-\mu_2) \quad \{\text{As } \Sigma^{-1}, \Sigma \text{ are symmetric}\}$$

$$\therefore \frac{f(y|G_1)}{f(y|G_2)} = \exp\left\{(\mu_1-\mu_2)^T \Sigma^{-1} y + \frac{(\mu_2^T \Sigma^{-1}\mu_2 - \mu_1^T \Sigma^{-1}\mu_1)}{2}\right\}$$

Now, $$\mu_2^T \Sigma^{-1}\mu_2 - \mu_1^T \Sigma^{-1}\mu_1 = \mu_2^T \Sigma^{-1}\mu_2 - \mu_2^T \Sigma^{-1}\mu_1 + \mu_2^T \Sigma^{-1}\mu_1 - \mu_1^T \Sigma^{-1}\mu_1$$

$$= \mu_2^T \Sigma^{-1}(\mu_1+\mu_2) - (\mu_1+\mu_2)^T \Sigma^{-1}\mu_1$$

[but $(\mu_1+\mu_2)^T \Sigma^{-1}\mu_1$ is a scalar $\rightarrow (\mu_1+\mu_2)^T \Sigma^{-1}\mu_1 = \mu_1^T \Sigma^{-1}(\mu_1+\mu_2)$]

$$= \mu_2^T \Sigma^{-1}(\mu_1+\mu_2) - \mu_1^T \Sigma^{-1}(\mu_1+\mu_2)$$

$$= (\mu_2-\mu_1)^T \Sigma^{-1}(\mu_1+\mu_2)$$

$$\therefore \frac{f(y|G_1)}{f(y|G_2)} = \exp\left\{(\mu_1-\mu_2)^T \Sigma^{-1} y + (\mu_2-\mu_1)^T \Sigma^{-1}(\mu_1+\mu_2)/2\right\}$$

$$= \exp\left\{(\mu_1-\mu_2)^T \Sigma^{-1} y - \frac{(\mu_1-\mu_2)^T \Sigma^{-1}(\mu_1+\mu_2)}{2}\right\}$$

[Proved]

2) We classify y as G_2 iff:

$$(\mu_1-\mu_2)^T\Sigma^{-1}y \le \frac{1}{2}(\mu_1-\mu_2)^T\Sigma^{-1}(\mu_1+\mu_2)+\ln\left(\frac{P_2}{P_1}\right)$$

Also, if $y \in G_1$, then

$$a^T y \sim N(a^T\mu_1, a^T\Sigma a)$$

if $a = \{(\mu_1-\mu_2)^T\Sigma^{-1}\}^T \longrightarrow a^T\Sigma a = (\mu_1-\mu_2)^T\Sigma^{-1}\Sigma\Sigma^{-1}(\mu_1-\mu_2)$

$$= (\mu_1-\mu_2)^T\Sigma^{-1}(\mu_1-\mu_2) = \Delta^2$$

Now, $a^T y \sim N(a^T\mu_1, \Delta^2)$ with $a = \Sigma^{-1}(\mu_1-\mu_2)$

$$P[\text{classify as } G_2 \mid G_1] = P\left[(\mu_1-\mu_2)^T\Sigma^{-1}y \le \frac{1}{2}(\mu_1-\mu_2)^T\Sigma^{-1}(\mu_1+\mu_2) + \ln(P_2/P_1)\right]$$

$$= P\left[\frac{(\mu_1-\mu_2)^T\Sigma^{-1}y-(\mu_1-\mu_2)^T\Sigma^{-1}\mu_1}{\Delta} \le \frac{\frac{(\mu_1-\mu_2)^T\Sigma^{-1}(\mu_1+\mu_2)}{2}+\ln(P_2/P_1)-(\mu_1-\mu_2)^T\Sigma^{-1}\mu_1}{\Delta}\right]$$

Now, $\dfrac{(\mu_1-\mu_2)^T\Sigma^{-1}y-(\mu_1-\mu_2)^T\Sigma^{-1}\mu_1}{\Delta} = \omega \sim N(0, I)$

$\therefore P[\text{classify as } G_2 \mid G_1]$

$$= P\left[w \le \frac{-\frac{(\mu_1-\mu_2)^T\Sigma^{-1}(\mu_1-\mu_2)}{2}+\ln(P_2/P_1)}{\Delta}\right]$$

$$= P\left[\omega \le \frac{-\frac{\Delta^2}{2}+\ln(P_2/P_1)}{\Delta}\right]$$

$$= \phi\left[\frac{-\frac{1}{2}\Delta^2+\ln(P_2/P_1)}{\Delta}\right]$$

{Proved}