3) y_p is uncorrelated with y_i $\forall$ $i \in \{1, 2, \dots, p-1\}$ [Given]

$\to \forall i \in \{1, 2, \dots, p-1\}$; $cov(y_p, y_i) = 0$

$\to \forall i \in \{1, 2, \dots, p-1\}$; $S_{ip} = S_{pi} = 0$ — (i)

let $a = [0\ 0\ \dots\ 0\ 1]^T_{1 \times p}$

$(S - \lambda I)\omega = 0$
- eigenvalue problem with solution: λ(eigenvalue), ω(eigenvector)

Now, $Sa = [S_{1p}\ \ S_{2p}\ \dots\ S_{pp}]^T$

$= [0\ \ 0\ \dots\ 0\ S_{pp}]^T$ (by (i))

$= [0\ \ 0\ \dots\ 0\ S_p^2]^T = S_p^2\,[0\ 0\ \dots\ 0\ 1]^T = S_p^2\, a$

$\therefore$ $Sa - S_p^2 a = 0 \Rightarrow (S - S_p^2 I)\,a = 0$

$\therefore$ S_p^2 is an eigenvalue of S with corresponding eigenvector as a.

4) A rotational transform preserves all mutual distances.

Let x_i, y_i be any two random non-zero vectors in the coordinate space.

$z_i = Ay_i$ and $\omega_i = Ax_i$. Now;

- $z_i^T z_i = y_i^T A^T A y_i = y_i^T I y_i = y_i^T y_i$ [Preserves distance from origin]
- $z_i^T \omega_i = y_i^T A^T A x_i = y_i^T I x_i = y_i^T x_i$ [Preserves dot products]

— As $A^T A = I$ (A is orthogonal)

$\therefore$ the transformation preserves mutual distances and dot products.

$\to$ A is a rotation matrix.

- $\hat{u}_k$: unit vector with the k^{th} component as 1.

$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_p^T \end{bmatrix}$ (let) $\quad \hat{u}_k = Ag_k$ for some g_k

$\Rightarrow A^T \hat{u}_k = A^T A g_k \Rightarrow A^T \hat{u}_k = g_k$

$\Rightarrow g_k = A^T \hat{u}_k = [a_1\ a_2\ \dots\ a_p]\,\hat{u}_k = a_k$

$\hat{u}_k$ (a unit vector) in the z-space is equivalent to a_k in the y-space. Thus, A rotates the axes to align with those of the principal components. [Proved]

5>

1. λ: eigenvalue of A with corresponding eigenvector x.

$(A-bI)x = Ax - bx = \lambda x - bx = (\lambda-b)x$ [Proved]

As $Ax = \lambda x$ by definition of eigenvalue & eigenvector.

2. $\underline{\lambda_i}$: i^{th} eigenvalue of cA with corresponding eigenvector $\underline{x_i}$.

Also, let λ_i : i^{th} eigenvalue of A with corresponding eigenvector x_i.

$\therefore (cA - \underline{\lambda_i} I)\underline{x_i} = 0$

& $(A - \lambda_i I)x_i = 0 \Rightarrow (cA - c\lambda_i I)x_i = 0$

$\Rightarrow (cA - (c\lambda_i)I)x_i = 0$ —— (i)

$\therefore$ An eigenvalue of A is also an eigenvector of cA with a different eigenvalue.

Since (i) holds WLG $\forall i$, A & cA have the exact same eigenvectors. And if $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ is the eigenvalue set of A, then $\{c\lambda_1, c\lambda_2, \dots, c\lambda_n\}$ is the eigenvalue set of cA. [As λ_i is an eigenvalue of $A \Rightarrow c\lambda_i$ is an eigenvalue of cA $\forall\ i = 1(1)n$]

3. Assign $R_k \leftarrow kR - (k-1)I$

4. Let λ_k be an eigenvalue of R_k with corresponding eigenvector x_k.

$\therefore (R_k - \lambda_k I)x_k = 0 \Rightarrow (kR - (k-1)I - \lambda_k I)x_k = 0$

$\Rightarrow (kR - (k-1+\lambda_k)I)x_k = 0$ —— (i)

if λ, x are an eigenvalue, eigenvector pair of R:

$(R - \lambda I)x = 0 \Rightarrow (kR - k\lambda I)x = 0$ —— (ii)

(i) & (ii) represent solutions to the same eigenvalue problem for kR.

$[(A - bI)x = (\lambda - b)x]$ $\therefore (kR - (k-1+\lambda_k)I)x = (k\lambda - (k-1+\lambda_k))x$

Now, $(k\lambda - (k-1+\lambda_k))x = 0$

$\Rightarrow k\lambda - (k-1+\lambda_k) = 0$

$\Rightarrow \lambda = \dfrac{k-1+\lambda_k}{k} = 1 + \dfrac{\lambda_k - 1}{k}$ {Proved}