Assignment 3 (T Square Tests)

1. Consider the univariate test $H_0: \mu = \mu_0$ vs $H_1: \mu \neq \mu_0$, σ^2 known. A random sample of n observations $y_1, \cdots, y_n$ is available from $N(\mu, \sigma^2)$. Show that the test using the test statistic $z = (\bar{y} - \mu_0) / \left(\sigma / \sqrt{n}\right)$ is equivalent to the likelihood ratio test.

2. Consider the multivariate test $H_0: \mu = \mu_0$ vs $H_1: \mu \neq \mu_0$, Σ known. The $\mathbb{y}_1, \cdots, \mathbb{y}_n$ constitute a random sample from $N_p(\mu, \Sigma)$. Show that the test using the test statistic $z^2 = n(\bar{\mathbb{y}} - \mu_0)' \Sigma^{-1} (\bar{\mathbb{y}} - \mu_0)$ is equivalent to the likelihood ratio test.

3. In Table below, height and weight are given for a sample of 20 college students. Assume that this sample originated from the bivariate normal $N_2(\mu, \Sigma)$, where

$$\Sigma = \begin{bmatrix} 20 & 100 \\ 100 & 1000 \end{bmatrix}$$

Test the $H_0 : \mu = \begin{bmatrix} 70 \\ 170 \end{bmatrix}$ using $\alpha = 0.05$.

69	153;
74	175;
68	155;
70	135;
72	172;
67	150;
66	115;
70	137;
76	200;
68	130;
72	140;
79	265;
74	185;
67	112;
66	140;
71	150;
74	165;
75	185;
75	210;
71	149

4. Show that the characteristic form of the t-statistic, $t = \dfrac{\bar{y}_1 - \bar{y}_2}{\sqrt{\text{var}\left(\bar{y}_1 - \bar{y}_2\right)}}$, satisfies the formal definition of t variable.

5. Show that when $\mu = \mu_0$, the value of Σ that maximises the likelihood function

$$L(\mu, \Sigma) = \frac{1}{(2\pi)^{\frac{np}{2}} \left|\Sigma\right|^{\frac{n}{2}}} e^{-\sum_{i=1}^{n} (\mathrm{y}_i - \mu)' \Sigma^{-1} (\mathrm{y}_i - \mu)/2}$$

is

$$\Sigma = \sum_{i=1}^{n} \left(\mathrm{y}_i - \mu_0\right)\left(\mathrm{y}_i - \mu_0\right)' / n$$

and

$$\max_{H_0} L = \frac{n^{np/2}}{(2\pi)^{np/2} \left| \sum_i \left(\mathbf{y}_i - \mu_0\right) \left(\mathbf{y}_i - \mu_0\right)' \right|^{n/2}} e^{-np/2}$$

6. Using the probe word data given below, do the following:
   a. Test $H_0 : \mu_1 = (30{,}25{,}40{,}25{,}30)'$
   b. Obtain 95% simultaneous confidence intervals for $\mu_1, \ldots, \mu_5$.
   c. Obtain 95% Bonferroni confidence intervals for $\mu_1, \ldots, \mu_5$.
   d. Test the hypotheses $H_{0j} : \mu_j = \mu_{0j}$ for $j = 1,\ldots,5$ using t_j with a Bonferroni critical value.

51	36	50	35	42;
27	20	26	17	27;
37	22	41	37	30;
42	36	32	34	27;
27	18	33	14	29;
43	32	43	35	40;
41	22	36	25	38;
38	21	31	20	16;
36	23	27	25	28;
26	31	31	32	36;
29	20	25	26	25;

7. Show that $t^2(\mathbf{a}) = T^2$, where $t(\mathbf{a}) = \dfrac{\mathbf{a}'\bar{\mathbf{y}}_1 - \mathbf{a}'\bar{\mathbf{y}}_2}{\sqrt{\left[(n_1 + n_2)/n_1 n_2\right] \mathbf{a}'\mathbf{S}_{\mathbf{pl}}\mathbf{a}}}$,

$T^2 = \dfrac{n_1 n_2}{n_1 + n_2}\left(\bar{\mathbf{y}}_1 - \bar{\mathbf{y}}_2\right)' \mathbf{S}_{\mathbf{pl}}^{-1} \left(\bar{\mathbf{y}}_1 - \bar{\mathbf{y}}_2\right)$ and $\mathbf{a} = \mathbf{S}_{\mathbf{pl}}^{-1}\left(\bar{\mathbf{y}}_1 - \bar{\mathbf{y}}_2\right)$

8. The following four variables are measured on two species of Flea Beetles:

$y_1 =$ the distance of the transverse groove from the posterior border of the prothorax (µm).

$y_2 =$ the length of the elytra (in 0.01 mm).

$y_3 =$ the length of the second antennal joint (µm).

$y_4 =$ the length of the third antennal joint (µm).

The data are given in the table below.

(a) Test $H_0\colon\ \mu_1 = \mu_2$ using T^2

(b) Calculate the discriminant function coefficient vector $\mathbb{a} = \mathbb{S}_{pl}^{-1}\left(\bar{\mathbb{y}}_1 - \bar{\mathbb{y}}_2\right)$

(c) Substitute the vector $\mathbb{a}$ found in part (b) into $t^2(\mathbb{a}) = \dfrac{\left[\mathbb{a}'\left(\bar{\mathbb{y}}_1 - \bar{\mathbb{y}}_2\right)\right]^2}{\left[\left(n_1 + n_2\right)/n_1 n_2\right]\mathbb{a}'\mathbb{S}_{pl}\mathbb{a}}$ and verify whether it is equal to T^2 found in part (a).

Flea Beetles 1 (19 observations)

189 245 137 163;

192 260 132 217;

217 276 141 192;

221 299 142 213;

171 239 128 158;

192 262 147 173;

213 278 136 201;

192 255 128 185;

170 244 128 192;

201 276 146 186;

195 242 128 192;

205 263 147 192;

180 252 121 167;

192 283 138 183;

200 294 138 188;

192 277 150 177;

200 287 136 173;

181 255 146 183;

192 287 141 198;

Flea Beetles 2 (20 observations)

181 305 184 209;

158 237 133 188;

184 300 166 231;

171 273 162 213;

181 297 163 224;

181 308 160 223;

177 301 166 221;

198 308 141 197;

180 286 146 214;

177 299 171 192;

176 317 166 213;

192 312 166 209;

176 285 141 200;

169 287 162 214;

164 265 147 192;

181 308 157 204;

192 276 154 209;

181 278 149 235;

175 271 140 192;

197 303 170 205;

9. For the dataset in problem 7, do the following:

   (a) Find 95% simultaneous confidence intervals for $\mu_{1j} - \mu_{2j}, j = 1, \cdots, 4$.

   (b) Find 95% Bonferroni confidence intervals for $\mu_{1j} - \mu_{2j}, j = 1, \cdots, 4$.

10. For the dataset in problem 7, do the following:

   (a) Test $H_0 : \mathbf{C}\mu_1 = \mathbf{C}\mu_2$ with

$$\mathbf{C} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

(b) Test $H_0 : \mathbf{C}\mu_{\mathbf{1}} = \mathbf{C}\mu_{\mathbf{2}}$ with

$$\mathbf{C} = \begin{bmatrix} 1 & -3 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$