Multivariate Analysis - Assignment 2

1) Assuming $g(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

$\exp(\cdot)$ is positive and increasing. $\therefore e^{-z^2/2} \geq 0 \ \forall \ z \in \mathbb{R}$.

$\Rightarrow \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) = g(z) \geq 0 \ \forall \ z \in \mathbb{R}$.

Now; to evaluate the Gaussian integral $\int_0^\infty e^{-x^2} dx$: we apply the substitution method as follows:

if $J = \int_0^\infty e^{-x^2} dx$, $\int_0^\infty e^{-y^2} dy = J$.

$J^2 = J \cdot J = \int_0^\infty e^{-x^2} dx \cdot \int_0^\infty e^{-y^2} dy = \int_0^\infty e^{-y^2} \left\{ \int_0^\infty e^{-x^2} dx \right\} dy$ [Since the first is a constant from the point of view of the second]

$= \int_0^\infty \left\{ \int_0^\infty e^{-y^2} \cdot e^{-x^2} dx \right\} dy$ [e^{-y^2} is invariant while integrating w.r.t x]

$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy$

let $t = x/y \Rightarrow x = yt \quad \therefore dx = y\, dt$

Now, $x = 0 \Rightarrow t = 0$

$x \to \infty \Rightarrow t \to \infty$

$\therefore J^2 = \int_0^\infty \int_0^\infty e^{-y^2t^2 - y^2} y\, dt\, dy = \underbrace{\int_0^\infty \int_0^\infty e^{-y^2(t^2+1)} y\, dt\, dy}$

$= \int_0^\infty \left[\frac{e^{-y^2(t^2+1)}}{(t^2+1) \cdot -2} \right]_{y=0}^{y=\infty} dt = \int_0^\infty \left[0 - \left(-\frac{1}{2(t^2+1)} \right) \right] dt = \int_0^\infty \frac{1}{2(t^2+1)} dt$

$= \frac{1}{2} \left[\tan^{-1} t \right]_{t=0}^{t=\infty} = \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}(0) = \frac{\pi}{4}$

$J^2 = \pi/4 \Rightarrow J = \sqrt{\pi}/2$ [J is positive as we are integrating a non-negative function]

Now, $\int_{-\infty}^\infty g(z) dz = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \exp\left(-\frac{z^2}{2}\right) dz$

$= \frac{2}{\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{z^2}{2}\right) dz = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-z^2/2} dz = \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{\pi}{2}} = 1.$

As, if $I = \int_0^\infty e^{-x^2/2} dx$, $I^2 = \int_0^\infty \left[\frac{e^{-y^2(t^2+1)}}{(t^2+1) \cdot -2} \cdot \frac{1}{1/2} \right]_{y=0}^{y=\infty} dt = \int_0^\infty \left[\frac{e^{-y^2(t^2+1)}}{-(t^2+1)} \right]_{y=0}^{y=\infty} dt$

$= \int_0^\infty \frac{dt}{1+t^2} = \left[\tan^{-1} \right]_{t=0}^{t=\infty} = \pi/2 - 0 = \pi/2$ & $I = \sqrt{I^2} = \sqrt{\frac{\pi}{2}} \Rightarrow I = \sqrt{\frac{\pi}{2}}$

(1) $g(z) \geq 0 \ \forall \ z \in \mathbb{R}$ & $\int_{-\infty}^\infty g(z) dz = 1 \Rightarrow g(z)$ is a density function.

2) Standardised Variables : $z_1 = \frac{y_1 - \mu_1}{\sigma_1}$, $z_2 = \frac{y_2 - \mu_2}{\sigma_2}$

Mean Squared Difference $= E[(z_1 - z_2)^2]$

$$= E\left[\left\{\frac{y_1 - \mu_1}{\sigma_1} - \frac{y_2 - \mu_2}{\sigma_2}\right\}^2\right] = E\left[\frac{\{\sigma_2(y_1-\mu_1) - \sigma_1(y_2-\mu_2)\}^2}{(\sigma_1\sigma_2)^2}\right]$$

$$= \frac{1}{(\sigma_1\sigma_2)^2} E\left[\{\sigma_2(y_1-\mu_1)\}^2 + \{\sigma_1(y_2-\mu_2)\}^2 - 2\sigma_1\sigma_2(y_1-\mu_1)(y_2-\mu_2)\right]$$

$\{\sigma_1, \sigma_2$ are constants$\}$

$$= \frac{1}{(\sigma_1\sigma_2)^2}\left\{\sigma_2^2 E[(y_1-\mu_1)^2] + \sigma_1^2 E[(y_1-\mu_1)^2] - 2\sigma_1\sigma_2 E[(y_1-\mu_1)(y_2-\mu_2)]\right\}$$

$\{E[(y_1-\mu_1)^2] = \sigma_1^2,\ E[(y_2-\mu_2)^2] = \sigma_2^2,\ E[(y_1-\mu_1)(y_2-\mu_2)] = cov(y_1, y_2) = \sigma_{12}\}$

$$= \frac{1}{(\sigma_1\sigma_2)^2}\{\sigma_2^2\sigma_1^2 + \sigma_1^2\sigma_2^2 - 2\sigma_1\sigma_2 \cdot \sigma_{12}\}$$

$$= \frac{1}{(\sigma_1\sigma_2)^2}\{2(\sigma_1\sigma_2)^2 - 2\sigma_1\sigma_2 \cdot \rho\sigma_1\sigma_2\} \quad [\sigma_{12} = \rho\sigma_1\sigma_2 \text{ (Given)}]$$

$$= \frac{1}{(\sigma_1\sigma_2)^2}\{2(\sigma_1\sigma_2)^2 - 2\rho(\sigma_1\sigma_2)^2\} = 2(1-\rho) \quad \{\text{Proved}\}$$

3) By definition of $\bar{y}$, $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$

$E[\bar{y}] = E[1/n \sum_{i=1}^{n} y_i] = 1/n\ E[\sum_{i=1}^{n} y_i] = 1/n \sum_{i=1}^{n} E[y_i]$

— by linearity of expectations.

Now, $y_i \sim N(\mu, \sigma^2)\ \forall\ i = 1(1)n \Rightarrow E[y_i] = \mu\ \forall\ i = 1(1)n$

$\therefore E[\bar{y}] = 1/n \sum_{i=1}^{n} \mu = 1/n \cdot n\mu = \mu$ {Proved}

4) $y_i \sim N(\mu, \sigma^2)\ \forall\ i = 1(1)n$

$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$

Consider the i^{th} observation : Now : $(y_i - \bar{y})(y_i - \bar{y})' = [(y_i - y')^2]$

$\therefore s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})(y_i - \bar{y})' = \frac{1}{n-1}\left\{\sum_{i=1}^{n} y_i y_i' + \sum_{i=1}^{n}\bar{y}\bar{y}' - \sum_{i=1}^{n} y_i\bar{y}' - \sum_{i=1}^{n}\bar{y}y_i'\right\}$

$= \frac{1}{n-1}\left[\sum_{i=1}^{n} y_i y_i' + n\bar{y}\bar{y}' - \sum_{i=1}^{n} y_i\bar{y}' - \sum_{i=1}^{n}\bar{y}\bar{y}_i'\right] = \frac{1}{n-1}\left[\sum_{i=1}^{n} y_i y_i' - n\bar{y}\bar{y}'\right]$

let $j = \begin{bmatrix}1\\1\\ \vdots \\1\end{bmatrix}$ $\therefore J = jj' = \begin{bmatrix}1\\ \vdots \\1\end{bmatrix}[1\,1 \cdots 1] = \begin{bmatrix}1 & 1 & \cdots & 1\\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1\end{bmatrix}_{n \times n}$

$\bar{y} = y'j/n$, y' being the data vector

$\Rightarrow n\bar{y} = y'j \Rightarrow n\bar{y}\bar{y}' = y'j \cdot \left(\frac{y'j}{n}\right)' = \frac{y'jj'y}{n} = 1/n\ y'jj'y$

Also, $\sum_{i=1}^{n} y_i y_i' = y'y$

$$\therefore s^2 = \frac{1}{n-1}\left(y'y - \frac{1}{n} y'jj'y\right)$$

$$= \frac{1}{n-1} y'\left\{I - \frac{1}{n}jj'\right\}y = \frac{1}{n-1} y'\left\{I - \frac{1}{n}J\right\}y$$

$$= \frac{1}{n-1} y'Hy \text{ is the required form.}$$

$$E[s^2] = E\left[\frac{1}{n-1} y'Hy\right] = \frac{1}{n-1} E[y'Hy]$$

We know, $E[X'AX] = tr\{A\Sigma\} + \mu' A \mu$

- where X: $n \times 1$ vector of random variables $E[X] = \mu$, $cov(X) = \Sigma$
  A: $n \times n$ matrix

$$\therefore E[s^2] = \frac{1}{n-1} E[y'Hy] = \frac{1}{n-1}\left[tr\{H\Sigma\} + \bar{\mu}' H \bar{\mu}\right]$$

where Σ: population covariance matrix of y.

- As $y = [y_1, y_2, \dots, y_n]$ where $y_i \sim N(\mu, \sigma^2)$; each y_i is independent of each other. [As they are independent random normal variables sharing the same mean & variance]

So, $\Sigma = \begin{bmatrix} \sigma^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma^2 \end{bmatrix}_{n\times n} = \sigma^2 I_n$, $H_{n\times n} = \begin{bmatrix} 1-1/n & -1/n & -1/n & \cdots & -1/n \\ -1/n & 1-1/n & -1/n & \cdots & -1/n \\ \vdots & \vdots & & \ddots & \vdots \\ -1/n & -1/n & -1/n & \cdots & 1-1/n \end{bmatrix}$

$$\bar{\mu}_{n\times 1} = \begin{bmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{bmatrix}$$

- $tr\{H\Sigma\} = tr\{H\sigma^2 I_n\} = tr\{\sigma^2 H I_n\} = \sigma^2 tr\{H\}$
  [As σ^2 is a scalar constant] $= \sigma^2 \sum_{i=1}^{n}(1-1/n) = \sigma^2 . n . (1 - 1/n)$
  $= \sigma^2 (n-1)$

- $\bar{\mu}' H \bar{\mu} = \bar{\mu}' H^2 \bar{\mu}$ {H is idempotent} $= \bar{\mu}' H H \bar{\mu} = \bar{\mu}' H' H \bar{\mu}$
  {H is symmetric} $= (H\bar{\mu})'(H\bar{\mu})$

$$H\bar{\mu} = \begin{bmatrix} 1-1/n & -1/n & -1/n & \cdots & -1/n \\ -1/n & 1-1/n & -1/n & \cdots & -1/n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1/n & -1/n & -1/n & \cdots & 1-1/n \end{bmatrix}_{n\times n} \begin{bmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{bmatrix}_{n\times 1} = \begin{bmatrix} 1-1/n - 1/n \cdots - 1/n \\ -1/n + 1 - 1/n \cdots - 1/n \\ \vdots \\ -1/n - 1/n \cdots + 1 - 1/n \end{bmatrix}_{n\times 1} . \mu$$

$$= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mu = \bar{0}_{n\times 1} \qquad \therefore (H\bar{\mu})'(H\bar{\mu}) = \bar{0}_{1\times n}\, \bar{0}_{n\times 1} = 0$$

$$\text{Thus, } E[s^2] = \frac{1}{n-1}\left[tr\{H\Sigma\} + \bar{\mu}' H \bar{\mu}\right] = \frac{1}{n-1}\left[\sigma^2(n-1) + 0\right] = \sigma^2$$

{Proved}

5) By definition, $\bar{y}_{p\times 1} = \frac{1}{n}\sum_{i=1}^{n} y_{i\,p\times 1}$

Now, $E[\bar{y}] = E[1/n \sum_{i=1}^{n} y_i] = \frac{1}{n} E[\sum_{i=1}^{n} y_i]$ {1/n is constant}

$= \frac{1}{n}\sum_{i=1}^{n} E[y_i]$ {Linearity of expectations}

Now, $E[y_i] = \mu_{p\times 1} \ \forall\ i = 1(1)n$ as $y_i \sim N_p(\mu, \Sigma) \ \forall\ i = 1(1)n$

$\therefore E[\bar{y}] = \frac{1}{n}\sum_{i=1}^{n} \mu_{p\times 1} = \frac{1}{n} \cdot n\mu_{p\times 1} = \mu_{p\times 1}$ {Proved}

6) $y_i \sim N_p(\mu, \Sigma) \ \forall\ i = 1(1)n$.

$S = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})(y_i - \bar{y})' = \frac{1}{n-1}\sum_{i=1}^{n}(y_i y_i' - \bar{y} y_i' - y_i \bar{y}' + \bar{y}\bar{y}')$

$= \frac{1}{n-1}\left\{\sum_{i=1}^{n} y_i y_i' - \sum_{i=1}^{n} \bar{y} y_i' - \sum_{i=1}^{n} y_i \bar{y}' + \sum_{i=1}^{n} \bar{y}\bar{y}'\right\}$

$= \frac{1}{n-1}\left\{\sum_{i=1}^{n} y_i y_i' - \bar{y}\sum_{i=1}^{n} y_i' - \left(\sum_{i=1}^{n} y_i\right)\bar{y}' + \sum_{i=1}^{n} \bar{y}\bar{y}'\right\}$

$= \frac{1}{n-1}\left\{\sum_{i=1}^{n} y_i y_i' - \bar{y}.n\bar{y}' - \cancel{n\bar{y}\bar{y}'} + \cancel{n\bar{y}\bar{y}'}\right\} = \frac{1}{n-1}\left\{\sum_{i=1}^{n} y_i y_i' - n\bar{y}\bar{y}'\right\}$

Now, $\text{cov}(y) = E[(y-\mu)(y-\mu)^T] = E(yy^T) - \mu\mu' = \Sigma$

And $\text{cov}(\bar{y}) = E[\{\bar{y} - E(\bar{y})\}\{\bar{y} - E(\bar{y})\}^T] = E(\bar{y}\bar{y}^T) - E[\bar{y}]E[\bar{y}^T]$

$= E(\bar{y}\bar{y}^T) - \mu\mu^T$ —— (i)

Also, $\text{cov}(\bar{y}) = \text{cov}\left(\frac{y_1 + y_2 + \dots y_n}{n}\right) = \left(\frac{1}{n}\right)^2 \text{cov}(y_1 + y_2 + \dots + y_n)$

$= \left(\frac{1}{n}\right)^2\{\text{cov}(y_1) + \text{cov}(y_2) + \dots + \text{cov}(y_n)\}$ [$y_i'(s)$ are independent]

$= \left(\frac{1}{n}\right)^2\{\Sigma + \Sigma + \dots + \Sigma\} = \frac{1}{n^2}.n\Sigma = \frac{\Sigma}{n}$ —— (ii)

$\therefore$ From (i) and (ii) $\rightarrow \frac{\Sigma}{n} = E(\bar{y}\bar{y}^T) - \mu\mu^T \Rightarrow E(\bar{y}\bar{y}^T) = \frac{\Sigma}{n} + \mu\mu^T$ —— (iii)

Finally, $E(S) = E\left[1/n-1\left\{\sum_{i=1}^{n} y_i y_i' - n\bar{y}\bar{y}'\right\}\right]$

$= \frac{1}{n-1}\left\{E\left[\sum_{i=1}^{n} y_i y_i'\right] + E[-n\bar{y}\bar{y}']\right\} = \frac{1}{n-1}\left\{\sum_{i=1}^{n} E[y_i y_i'] - nE[\bar{y}\bar{y}']\right\}$

$= \frac{1}{n-1}\left\{n(\Sigma + \mu\mu') - n\left(\frac{\Sigma}{n} + \mu\mu'\right)\right\}$ [From (iii), $E(yy^T) = \Sigma + \mu\mu'$]

$= \frac{1}{n-1}\{n\Sigma + \cancel{n\mu\mu'} - \Sigma - \cancel{n\mu\mu'}\}$

$= \frac{1}{n-1}.(n-1)\Sigma = \Sigma$ [Proved]

7) $\begin{bmatrix} x \\ y \end{bmatrix}$: bivariate vector with population mean $\begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}$ and population covariance matrix Σ

$$\Sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix}$$

Samples: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \bar{Y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

$$S_{xy} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \frac{1}{n-1}\left\{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}\right\}$$

$$= \frac{1}{n-1}\left\{X'Y - n\bar{X}\bar{Y}\right\}$$

Now; $\bar{X} = \frac{1}{n}X'j$ and $\bar{Y} = \frac{1}{n}Y'j$ $\qquad j = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{n\times 1}$

$\bar{X}, \bar{Y}$ are both 1×1 and hence symmetric.

$\therefore \bar{X} = \bar{X}', \bar{Y} = \bar{Y}' = \frac{1}{n} j'Y$

$$\therefore S_{xy} = \frac{1}{n-1}\left\{X'Y - n . X'jj'Y . \frac{1}{n^2}\right\} = \frac{1}{n-1}\left\{X'Y - \frac{1}{n}X'JY\right\}$$

$$= \frac{1}{n-1}X'\left(I_n - \frac{1}{n}J\right)Y = \frac{1}{n-1}X'H_nY$$

where $H_n = \left(I_n - \frac{1}{n}J\right)$, $J = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{n\times n}$

H_n is idempotent.

$$E(S_{xy}) = E\left[\frac{1}{n-1}X'H_nY\right] = \frac{1}{n-1}E[X'H_nY]$$

{Using $E(x'Ay) = tr(A\Sigma) + \mu_x' A \mu_y$ with $A = H_n$}

$= tr\{H_n\Sigma\} + \tilde{\mu}_x' H_n \tilde{\mu}_y$ where $\tilde{\mu}_x = \begin{bmatrix} \mu_x \\ \mu_x \\ \vdots \\ \mu_x \end{bmatrix}_{n\times 1}$ and $\tilde{\mu}_y = \begin{bmatrix} \mu_y \\ \mu_y \\ \vdots \\ \mu_y \end{bmatrix}_{n\times 1}$

- $H_n = \begin{bmatrix} 1-1/n & -1/n & \cdots & -1/n \\ -1/n & 1-1/n & \cdots & -1/n \\ \vdots & \vdots & \ddots & \vdots \\ -1/n & -1/n & \cdots & 1-1/n \end{bmatrix}$, $\Sigma = \sigma_{xy}$

$$\therefore tr\{H_n\Sigma\} = n\left(1 - \frac{1}{n}\right)\sigma_{xy}$$

- $H_n \cdot \tilde{\mu}_y = \underset{n\times n}{H_n}\begin{bmatrix} \mu_y \\ \mu_y \\ \vdots \\ \mu_y \end{bmatrix}_{n\times 1} = \begin{bmatrix} 1-1/n - 1/n \cdots - 1/n \\ -1/n + 1 - 1/n \cdots - 1/n \\ \vdots \\ -1/n - 1/n \cdots + 1 - 1/n \end{bmatrix}_{n\times 1} \mu_y = \tilde{0}_{n\times 1}$

Similarly, $H_n\tilde{\mu}_x = \tilde{0}_{n\times 1}$

Now, $\tilde{\mu}_x' H_n \tilde{\mu}_y = \tilde{\mu}_x' H_n' H_n \tilde{\mu}_y$ {H_n is idempotent, symmetric}

$$= (H_n\tilde{\mu}_x)'(H_n\tilde{\mu}_y) = \underset{1\times n}{\tilde{0}}\ \underset{n\times 1}{\tilde{0}} = 0$$

$$\therefore E(S_{xy}) = \frac{1}{n-1}E[X'H_nY] = \frac{1}{n-1}\left\{n\left(1 - \frac{1}{n}\right)\sigma_{xy} + 0\right\}$$

$$= \frac{1}{n-1} . n . \frac{n-1}{n}\sigma_{xy} = \sigma_{xy} \text{ (Proved)}$$

8) $y_i \sim N_p(\mu, \Sigma) \;\forall\; i = 1(1)n$

$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i, \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$

Assume: $y_1, y_2, \dots, y_n \overset{iid}{\sim} N_p(\mu, \Sigma)$

$E[\bar{y}] = \mu; \quad cov(\bar{y}) = \frac{\Sigma}{n}$

$\left\{ cov(\bar{y}) = cov\left(\frac{y_1+y_2+\dots+y_n}{n}\right) = \frac{1}{n^2}\{cov(y_1)+cov(y_2)+\dots+cov(y_n)\} \right.$ [As y_i's are iid]

$\left. = \frac{1}{n^2}(\Sigma+\Sigma+\dots+\Sigma) = \frac{1}{n^2}.n\Sigma = \frac{\Sigma}{n} \right\}$

$\therefore \bar{y} \sim N(\mu, \frac{\Sigma}{n})$

Also, $\bar{y} - y_j = \frac{1}{n}(y_1+y_2+\dots+y_{j-1}+y_{j+1}+\dots+y_j) - \frac{n-1}{n}y_j$

$\sim N\left(\frac{(n-1)}{n}\mu - \frac{(n-1)}{n}\mu, (n-1)\frac{\Sigma}{n^2} + (n-1)^2\frac{\Sigma}{n^2}\right)$

$\sim N(0, \frac{n-1}{n}\Sigma) \;\forall\; j = 1(1)n.$

Also, $\forall\; j=1(1)n;\; Cov(y_j - \bar{y}, \bar{y}) = cov(y_j, \bar{y}) - cov(\bar{y}, \bar{y})$

$= \frac{\Sigma}{n} - \frac{\Sigma}{n} = 0 \Rightarrow$ vector $[\bar{y}, y_1-\bar{y}, y_2-\bar{y}, \dots, y_n-\bar{y}]^T$ is a multivariate random normal distribution with covariance matrix of the form: $\begin{pmatrix} \Sigma/n & 0^T \\ 0 & \hat{\Sigma} \end{pmatrix}$ for some symmetric $\hat{\Sigma}$.

– Since we arrive at a multivariate normal distribution with 0's in non-diagonal elements of the covariance matrix; we conclude: $\bar{y}$ and $\underline{Y} = (y_1-\bar{y}, y_2-\bar{y}, \dots, y_n-\bar{y})^T$ are independent normal vectors.

Now, $\bar{y}$ is independent of Y^TY. But $Y^TY = (n-1)S^2$

$\Rightarrow \bar{y}$ is independent of S^2

$\Rightarrow \bar{y}$ and S are independent.

→ In terms of vector forms: $S = \frac{1}{n-1}y'Hy \Rightarrow (n-1)S = y'Hy$

where $H = I_n - \frac{J}{n} = I_n - \frac{jj'}{n}$, $j = [1]_{n\times 1}$

Also, $\bar{y} = \frac{y'j}{n} \Rightarrow \bar{y}\bar{y}' = \frac{1}{n^2}y'jj'y$

$\Rightarrow n\bar{y}\bar{y}' = y'\left(\frac{jj'}{n}\right)y$

Let $A = H$ and $B = \left(\frac{jj'}{n}\right)_{n\times n}$

$\therefore AB = \left(I_n - \frac{jj'}{n}\right)\left(\frac{jj'}{n}\right) = \frac{jj'}{n} - \frac{jj'jj'}{n^2}$

$= \frac{jj'}{n} - \frac{j(n)j'}{n^2} = \frac{jj'}{n} - \frac{jj'}{n} = 0$

$AB = 0 \Rightarrow Y'AY$ & $Y'BY$ are independent

$\Rightarrow Y'HY$ & $y'\frac{jj'}{n}y$ are independent

$\Rightarrow (n-1)S$ & $n\bar{y}\bar{y}'$ are indepent.

$\Rightarrow S$ & $\bar{y}\bar{y}'$ are independent (As n is constant)

$\Rightarrow S$ & $\bar{y}$ are independent. [Proved]

9) y is a random vector with population mean μ and population covariance matrix Σ (Assuming)

$\therefore E[y] = \mu$ and $E[(y-\mu)(y-\mu)^T] = \Sigma$

Given: $\Sigma = T'T$ and $z = (T')^{-1}(y-\mu)$

$E(z) = E[(T')^{-1}(y-\mu)] = (T')^{-1} E[(y-\mu)]$ {T is a constant}

$= (T')^{-1}\{E(y) - E(\mu)\} = (T')^{-1}[0-0] = 0$ [Proved]

$cov(z) = E[(z-E[z])(z-E[z])^T] = E[zz^T]$

$= E[(T')^{-1}(y-\mu)\{(T')^{-1}(y-\mu)\}^T]$

$= E[(T')^{-1}(y-\mu)(y-\mu)^T\{(T')^{-1}\}^T]$

$= (T')^{-1} E[(y-\mu)(y-\mu)^T]\{(T')^{-1}\}^T$

$= (T')^{-1}\Sigma\{(T')^{-1}\}^T = (T')^{-1}T'T\ \{(T')^{-1}\}^T$

$= I\ T\{(T')^{-1}\}^T = \{(T')^{-1}\cdot T'\}' = I' = I$ [Proved]

10) $y \sim N_3(\mu, \Sigma)$

$\mu = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} \mu_{y_1} \\ \mu_{y_2} \\ \mu_{y_3} \end{bmatrix}$ and $\Sigma = \begin{bmatrix} 6 & 1 & -2 \\ 1 & 13 & 4 \\ -2 & 4 & 4 \end{bmatrix} = \begin{bmatrix} \sigma_{y_1y_1} & \sigma_{y_1y_2} & \sigma_{y_1y_3} \\ \sigma_{y_2y_1} & \sigma_{y_2y_2} & \sigma_{y_2y_3} \\ \sigma_{y_3y_1} & \sigma_{y_3y_2} & \sigma_{y_3y_3} \end{bmatrix}$

a) $z = 2y_1 - y_2 + 3y_3$

$\mu_z = 2\mu_{y_1} - \mu_{y_2} + 3\mu_{y_3} = 2.3 - 1 + 3.4 = 17$

$var(z) = \sigma_z^2 = var(2y_1 - y_2 + 3y_3) = 4\,var(y_1) + var(y_2) + 9\,var(y_3)$
$+ 12\,cov(y_1, y_3) - 4\,cov(y_1, y_2) - 6\,cov(y_2, y_3)$

$= 4\sigma_{y_1y_1} + \sigma_{y_2y_2} + 9\sigma_{y_3y_3} + \{6\sigma_{y_1y_3} - 2\sigma_{y_1y_2} - 3\sigma_{y_2y_3}\}.2$

$= 4.6 + 13 + 9.4 + (6.-2 - 2.1 - 3.4).2 = 21$

$\therefore z \sim N(17, 21)$ {Sum of correlated normally distributed r.v(s) is also a normally distributed r.v}

b) $z_1 = y_1 + y_2 + y_3$

$\mu_{z_1} = \mu_{y_1} + \mu_{y_2} + \mu_{y_3} = 3+1+4 = 8$

$var(z_1) = \sigma_{z_1z_1} = var(y_1+y_2+y_3) = var(y_1) + var(y_2) + var(y_3) + \{cov(y_1, y_2)$
$+ cov(y_1, y_3) + cov(y_2, y_3)\}.2$

$= 6 + 13 + 4 + 2\{1 - 2 + 4\} = 29$

$z_2 = y_1 - y_2 + 2y_3$

$\mu_{z_2} = \mu_{y_1} - \mu_{y_2} + 2\mu_{y_3} = 3 - 1 + 2.4 = 10$

$var(z_2) = \sigma_{z_2z_2} = var(y_1 - y_2 + 2y_3) = var(y_1) + var(y_2) + 4\,var(y_3)$
$+ 2\{-cov(y_1, y_2) + 2\,cov(y_1, y_3) - 2\,cov(y_2, y_3)\}$

$= 6 + 13 + 4.4 + 2\{1 + 2.-2 - 2.4\} = 13$

$cov(z_1, z_2) = \sigma_{z_1 z_2} = cov(y_1+y_2+y_3, y_1-y_2+2y_3)$
$= cov(y_1, y_1) - cov(y_1, y_2) + 2\,cov(y_1, y_3)$
$+ cov(y_2, y_1) - cov(y_2, y_2) + 2\,cov(y_2, y_3)$
$+ cov(y_3, y_1) - cov(y_3, y_2) + 2\,cov(y_3, y_3)$

$= 6 - 1 - 2\cdot 2 + 1 - 13 + 2\cdot 4 + (-2) - 4 + 2\cdot 4 = -1$

$\therefore z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \sim N_2(\mu_z, \Sigma_z)$

with $\mu_z = \begin{bmatrix} 8 \\ 10 \end{bmatrix}$, $\Sigma_z = \begin{bmatrix} 29 & -1 \\ -1 & 13 \end{bmatrix}$

c) $y_2 \sim N(1, 13)$ as $\mu_{y_2} = 1$ and $\sigma_{y_2 y_2} = 13$

d) Joint Distribution of $y_1, y_3 \sim N_2(\mu_{\hat{y}}, \Sigma_{\hat{y}})$
with $\mu_{\hat{y}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $\Sigma_{\hat{y}} = \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}$ as $\sigma_{y_1 y_1} = 6$, $\sigma_{y_3 y_3} = 4$, $\sigma_{y_1 y_3} = -2$

e) Let $a \equiv y_1$, $b \equiv y_3$, $c \equiv \frac{1}{2}(y_1 + y_2)$
$\mu_a = \mu_{y_1} = 3$
$\mu_b = \mu_{y_3} = 4$
$\mu_c = \frac{1}{2}\mu_{y_1} + \frac{1}{2}\mu_{y_2} = \frac{1}{2}\cdot 3 + \frac{1}{2}\cdot 1 = 2$
$\sigma_a^2 = \sigma_{y_1 y_1} = 6$, $\sigma_b^2 = \sigma_{y_3 y_3} = 4$, $\sigma_c^2 = var\{\frac{1}{2}(y_1+y_2)\} = \frac{1}{4}\{var(y_1) + var(y_2) + 2\,cov(y_1, y_2)\} = 1/4\{6 + 13 + 1\} = 1/4 \cdot 20 = 5$
$\sigma_{ab} = \sigma_{y_1 y_3} = -2$
$\sigma_{ac} = cov(y_1, 1/2(y_1+y_2)) = cov(y_1, 1/2 y_1 + 1/2 y_2)$
$= cov(y_1, 1/2 y_1) + cov(y_1, 1/2 y_2) = 1/4 \cdot 6 + 1/4 = 7/4$
$\sigma_{bc} = cov(y_3, 1/2(y_1+y_2)) = cov(y_3, 1/2 y_1 + 1/2 y_2)$
$= 1/4\,cov(y_3, y_1) + 1/4\,cov(y_3, y_2) = 1/4(-2+4) = 1/2$
$\therefore$ Required Joint Distribution: $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \sim N_3(\tilde{\mu}, \tilde{\Sigma})$

where $\tilde{\mu} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ and $\tilde{\Sigma} = \begin{bmatrix} 6 & -2 & 7/4 \\ -2 & 4 & 1/2 \\ 7/4 & 1/2 & 5 \end{bmatrix}$

11) if y_i & y_j are normally distributed and $\sigma_{y_i y_j} = 0$, then y_i & y_j are independent. From Σ: $\sigma_{y_1 y_2} = -3 \neq 0$, $\sigma_{y_2 y_3} = \sigma_{y_1 y_3} = 0$
$\therefore$ (b) y_1 & y_3 and (c) y_2 & y_3 are independent.
(e) components of y_1 in (y_1, y_3) vinduces dependence with y_2
(d) (y_1, y_2) and y_3 are independent since: y_1 & y_3 are independent and y_2 & y_3 are independent.

12) $\mu_y = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $\mu_x = \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix}$ $\quad \Sigma = \begin{bmatrix} \Sigma_{yy}\ (2\times2) & \Sigma_{yx}\ (2\times3) \\ \Sigma_{xy}\ (3\times2) & \Sigma_{xx}\ (3\times3) \end{bmatrix}$

$\Sigma_{yy} = \begin{bmatrix} 14 & -8 \\ -8 & 18 \end{bmatrix}$, $\Sigma_{xx} = \begin{bmatrix} 50 & 8 & 5 \\ 8 & 4 & 0 \\ 5 & 0 & 1 \end{bmatrix}$ | Also, $\begin{bmatrix} y \\ x \end{bmatrix} \sim N_5(\mu, \Sigma)$

$\Sigma_{yx} = \begin{bmatrix} 15 & 0 & 3 \\ 8 & 6 & -2 \end{bmatrix}$, $\Sigma_{xy} = \begin{bmatrix} 15 & 8 \\ 0 & 6 \\ 3 & -2 \end{bmatrix}$

$$y|x \sim N_3\left(\mu_y + \Sigma_{yx}\Sigma_{xx}^{-1}(x - \mu_x),\ \Sigma_{yy} - \Sigma_{yx}\ \Sigma_{xx}^{-1}\Sigma_{xy}\right)$$

- $E[y|x] = \mu_y + \Sigma_{yx}\ \Sigma_{xx}^{-1}(x - \mu_x)$

$$= \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \begin{bmatrix} 15 & 0 & 3 \\ 8 & 6 & -2 \end{bmatrix} \cdot \frac{1}{|\Sigma_{xx}|} \cdot \operatorname{adj}(\Sigma_{xx}) \cdot \left[x - \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \begin{bmatrix} 15 & 0 & 3 \\ 8 & 6 & -2 \end{bmatrix} \cdot \frac{1}{36} \cdot \begin{bmatrix} 4 & -8 & -20 \\ -8 & 25 & 40 \\ -20 & 40 & 136 \end{bmatrix} \cdot \left(x - \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}\right)$$

$$= \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \frac{1}{36}\begin{bmatrix} 0 & 0 & 108 \\ 24 & 6 & -192 \end{bmatrix}\begin{bmatrix} x_1 - 4 \\ x_2 + 3 \\ x_3 - 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \frac{1}{36}\begin{bmatrix} 108(x_3 - 5) \\ 24(x_1 - 4) + 6(x_2 + 3) - 192(x_3 - 5) \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \frac{1}{36}\begin{bmatrix} 108x_3 - 540 \\ 24x_1 + 6x_2 - 192x_3 + 882 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 108x_3 - 432 \\ 24x_1 + 6x_2 + 192x_3 + 810 \end{bmatrix}$$

- $\operatorname{cov}(Y|X) = \Sigma_{yy} - \Sigma_{yx}(\Sigma_{xx})^{-1}\Sigma_{xy}$

$$= \begin{bmatrix} 14 & -8 \\ -8 & 18 \end{bmatrix} - \begin{bmatrix} 15 & 0 & 3 \\ 8 & 6 & -2 \end{bmatrix} \cdot \frac{1}{36}\begin{bmatrix} 4 & -8 & -20 \\ -8 & 25 & 40 \\ -20 & 40 & 136 \end{bmatrix}\begin{bmatrix} 15 & 8 \\ 0 & 6 \\ 3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -8 \\ -8 & 18 \end{bmatrix} - \frac{1}{36}\begin{bmatrix} 324 & -216 \\ -216 & 612 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -8 \\ -8 & 18 \end{bmatrix} - \begin{bmatrix} 9 & -6 \\ -6 & 17 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$