Multivariate Analysis

Assignment 5

Problem 1

Assume: 2 populations having the same covariance matrix Σ but different mean vectors u_1 & u_2.

Population 1 samples: $y_{11}, y_{12}, \dots, y_{1n_1}$ | each $y_{ij} \in \mathbb{R}_{1\times p}$

Population 2 samples: $y_{21}, y_{22}, \dots, y_{2n_2}$ | linear combination: $z = a'y$

$\therefore z_{1i} = a'y_{1i}$, $z_{2i} = a'y_{2i}$ and $\bar{z}_1 = \frac{1}{n_1}\sum_{i=1}^{n_1} z_{1i} = a'\bar{y}_1$, $\bar{z}_2 = \frac{1}{n_2}\sum_{i=1}^{n_2} z_{2i} = a'\bar{y}_2$

We wish to find a s.t $\frac{(\bar{z}_1 - \bar{z}_2)^2}{s_z^2}$ is maximized.

Now $\left\{\frac{\bar{z}_1 - \bar{z}_2}{s_z}\right\}^2 = \frac{[a'(\bar{y}_1 - \bar{y}_2)]^2}{a' S_{pl}\, a}$

$\frac{\partial}{\partial a} \frac{[a'(\bar{y}_1 - \bar{y}_2)]^2}{a' S_{pl}\, a} = 0 \Rightarrow (a' S_{pl}\, a)\, 2\,[a'(\bar{y}_1 - \bar{y}_2)](\bar{y}_1 - \bar{y}_2) - [a'(\bar{y}_1 - \bar{y}_2)]^2 (2 S_{pl}\, a) = 0$

$\Rightarrow (a' S_{pl}\, a)(\bar{y}_1 - \bar{y}_2) - [a'(\bar{y}_1 - \bar{y}_2)](S_{pl}\, a) = 0$

$\Rightarrow \frac{(a' S_{pl}\, a)}{[a'(\bar{y}_1 - \bar{y}_2)]} (\bar{y}_1 - \bar{y}_2) = S_{pl}\, a \Rightarrow a = c\, S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2)$

where $c = \frac{a' S_{pl}\, a}{a'(\bar{y}_1 - \bar{y}_2)}$

$\max\, [(\bar{z}_1 - \bar{z}_2)/s_z]^2 = \frac{\{[c\, S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2)]'(\bar{y}_1 - \bar{y}_2)\}^2}{[c\, S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2)]' S_{pl} [c\, S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2)]}$

$= \frac{\{(\bar{y}_1 - \bar{y}_2)'(S_{pl}^{-1})' c\, (\bar{y}_1 - \bar{y}_2)\}^2}{(\bar{y}_1 - \bar{y}_2)'(S_{pl}^{-1})' c^2 S_{pl} S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2)} = \frac{[(\bar{y}_1 - \bar{y}_2)' S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2)]^2 \cdot \cancel{c^2}}{(\bar{y}_1 - \bar{y}_2)' \cancel{S_{pl}^{-1}} \cdot \cancel{S_{pl}} \cdot S_{pl}^{-1}(y_1 - y_2)\, \cancel{c^2}}$

$= (\bar{y}_1 - \bar{y}_2)' S_{pl}^{-1} (\bar{y}_1 - \bar{y}_2)$.

$\therefore$ As c^2 (s) get cancelled out, we conclude that c can take any non-zero value.

if c is set to one, $\left(\frac{\bar{z}_1 - \bar{z}_2}{s_z}\right)^2$ is maximized with $a = S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2)$

- the maximizing vector a is not unique.

Problem 2

$$Z = a'y \ , \quad a = S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2)$$

$$\frac{(\bar{z}_1 - \bar{z}_2)^2}{s_z^2} = \frac{[a'(\bar{y}_1 - \bar{y}_2)]^2}{a' S_{pl}\, a} = \frac{\left[\{S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2)\}'(\bar{y}_1 - \bar{y}_2)\right]^2}{(S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2))' S_{pl}\,(S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2))}$$

$$= \frac{\{(\bar{y}_1 - \bar{y}_2)'(S_{pl}^{-1})'(\bar{y}_1 - \bar{y}_2)\}^2}{(\bar{y}_1 - \bar{y}_2)'(S_{pl}^{-1})' S_{pl} S_{pl}^{-1}(y_1 - y_2)} = \frac{\{(y_1 - y_2)' S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2)\}^2}{(y_1 - y_2)' \cancel{S_{pl}^{-1}}\, \cancel{S_{pl}}\, S_{pl}^{-1}(y_1 - y_2)}$$

- As S_{pl}^{-1} is symmetric.

$$= \frac{\{(\bar{y}_1 - \bar{y}_2)' S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2)\}^{\cancel{2}}}{\cancel{(\bar{y}_1 - \bar{y}_2)' S_{pl}^{-1}(y_1 - \bar{y}_2)}} = (\bar{y}_1 - \bar{y}_2)' S_{pl}^{-1}(\bar{y}_1 - \bar{y}_2) \qquad \text{[Proved]}$$

Problem 3

$Z_1 = a_1'y$ maximizes $a'Ha / a'Ea$. Thus, $\lambda_1 = a_1'Ha_1 / a_1'Ea_1$ (symbols carry their usual meanings). To show that z_1 & z_2 are uncorrelated: $r_{z_1 z_2} = \dfrac{s_{z_1 z_2}}{s_{z_1} s_{z_2}} = \dfrac{a_1' S a_2}{\sqrt{(a_1' S a_1)(a_2' S a_2)}}$

The pooled estimator of $\Sigma = S = \dfrac{E}{N-k}$, with $N = \sum_{i=1}^{k} n_i$.

where k = number of groups,
n_i = number of samples in group i.

$r_{z_1 z_2} = \dfrac{a_1' E a_2}{\sqrt{(a_1' E a_1)(a_2' E a_2)}}$. To show $a_1' E a_2 = 0$, consider:
$Ha_1 = \lambda_1 E a_1$, $Ha_2 = \lambda_2 E a_2$

$\therefore a_2' H a_1 = \lambda_1 a_2' E a_1$, $a_1' H a_2 = \lambda_2 a_1' E a_2$

Subtracting $\rightarrow$ $(\lambda_1 - \lambda_2) a_2' E a_1 = 0$ As $a_2' H a_1$ is symmetric.

Since, eigenvalues of $E^{-1}H$ are distinct, $\lambda_1 - \lambda_2 \neq 0$.

$\therefore a_2' E a_1 = 0$, $r_{z_1 z_2} = 0$.

Now, to show that $z_2 = a_2' y$ has the largest discriminant criterion.

$\lambda_2 = a_2' H a_2 / a_2' E a_2$ subject to the constraint $r_{z_1 z_2} = 0$.

$\rightarrow$ Using lagrangian γ:

$$\frac{\partial}{\partial a_2}\left(\frac{a_2' H a_2}{a_2' E a_2} + \gamma a_1' E a_2\right) = 0 \Rightarrow \frac{a_2' E a_2 \, 2 H a_2 - a_2' H a_2 \, 2 E a_2}{(a_2' E a_2)} + \gamma E a_1 = 0$$

$$\Rightarrow \frac{2 H a_2 - 2 \lambda_2 E a_2}{a_2' E a_2} + \gamma E a_1 = 0 \xrightarrow[\text{with } a_1]{\text{multipying}}$$

$$\frac{2 a_1' H a_2 - 2 \lambda_2 a_1' E a_2}{a_2' E a_2} + \gamma a_1' E a_1 = 0 \Rightarrow a_1' E a_2 = 0 \Rightarrow a_1' H a_2 = 0$$

As $\gamma = 0$, $H a_2 - \lambda_2 E a_2 = 0$. So, 2nd eigenvector of $E^{-1}H$ maximizes $\lambda_2 = a_2' H a_2 / a_2' E a_2$ subject to $r_{z_1 z_2} = 0$.

Similary, subject to $r_{z_1 z_3} = r_{z_2 z_3} = 0$: $H a_3 - \lambda_3 E a_3 = 0$

$\Rightarrow E^{-1} H a_3 = \lambda_3 a_3$. $\therefore a_3$ is the 3rd eigenvector of $E^{-1}H$ corresponding to its 3rd largest distinct eigenvalue λ_3.

$\therefore \lambda_3 = \dfrac{a_3' H a_3}{a_3' E a_3}$ is maximized and we have the corresponding 3rd discriminant function $z_3 = a_3' y$.

Also, the set of vectors a_1, a_2, a_3 is linearly independent.

notebook

March 21, 2021

```
In [1]: import numpy as np
        import scipy as sc
        from scipy.stats import f
        from scipy.stats import t
        import matplotlib.pyplot as plt
```

1 *Problem* 4

```
In [2]: data_males = [[15, 17, 24, 14],
                      [17, 15, 32, 26],
                      [15, 14, 29, 23],
                      [13, 12, 10, 16],
                      [20, 17, 26, 28],
                      [15, 21, 26, 21],
                      [15, 13, 26, 22],
                      [13, 5, 22, 22],
                      [14, 7, 30, 17],
                      [17, 15, 30, 27],
                      [17, 17, 26, 20],
                      [17, 20, 28, 24],
                      [15, 15, 29, 24],
                      [18, 19, 32, 28],
                      [18, 18, 31, 27],
                      [15, 14, 26, 21],
                      [18, 17, 33, 26],
                      [10, 14, 19, 17],
                      [18, 21, 30, 29],
                      [18, 21, 34, 26],
                      [13, 17, 30, 24],
                      [16, 16, 16, 16],
                      [11, 15, 25, 23],
                      [16, 13, 26, 16],
                      [16, 13, 23, 21],
                      [18, 18, 34, 24],
                      [16, 15, 28, 27],
                      [15, 16, 29, 24],
                      [18, 19, 32, 23],
```

```
              [18, 16, 33, 23],
              [17, 20, 21, 21],
              [19, 19, 30, 28]]

data_females = [[13, 14, 12, 21],
                [14, 12, 14, 26],
                [12, 19, 21, 21],
                [12, 13, 10, 16],
                [11, 20, 16, 16],
                [12, 9, 14, 18],
                [10, 13, 18, 24],
                [10, 8, 13, 23],
                [12, 20, 19, 23],
                [11, 10, 11, 27],
                [12, 18, 25, 25],
                [14, 18, 13, 26],
                [14, 10, 25, 28],
                [13, 16, 23, 28],
                [16, 21, 26, 26],
                [14, 17, 14, 14],
                [16, 16, 15, 23],
                [13, 16, 23, 24],
                [2, 6, 16, 21],
                [14, 16, 22, 26],
                [14, 17, 22, 28],
                [16, 13, 16, 14],
                [15, 14, 20, 26],
                [12, 10, 12, 9],
                [14, 17, 24, 23],
                [13, 15, 18, 20],
                [11, 16, 18, 28],
                [7, 7, 19, 18],
                [12, 15, 7, 28],
                [6, 5, 6, 13]]

data_females = np.asarray(data_females)
data_males = np.asarray(data_males)
# As there are two classes
k = 2
p = data_females.shape[1]

count_females = np.shape(data_females)[0]
count_males = np.shape(data_females)[0]
total_count = count_females + count_males

mean_females = np.mean(data_females, axis = 0)
mean_males = np.mean(data_males, axis = 0)
```

```
        covariance_females = np.cov(data_females.T)
        covariance_males = np.cov(data_males.T)

        pooled_covariance = ((count_females - 1) * covariance_females +
                             (count_males - 1) * covariance_males) / (total_count - k)
```

1.1 (*a*)

```
In [3]: disc_coeff_vector = np.matmul(np.linalg.inv(pooled_covariance),
                                      (mean_males - mean_females))

        print('Required Discriminant Function Coefficient Vector:')
        print(np.ndarray.tolist(np.round(disc_coeff_vector, 4)))
```

```
Required Discriminant Function Coefficient Vector:
[0.5621, -0.2294, 0.4412, -0.2863]
```

1.2 (*b*)

```
In [4]: std_coefficients = np.multiply(np.sqrt(np.diag(pooled_covariance)),
                                       disc_coeff_vector)
        print('Standardised Coefficients:')
        print(np.ndarray.tolist(np.round(std_coefficients, 4)))
        print()

        print('Absolute Values of Standardised Coefficients:')
        print(np.ndarray.tolist(np.abs(np.round(std_coefficients, 4))))
```

```
Standardised Coefficients:
[1.4958, -0.9133, 2.3669, -1.3436]

Absolute Values of Standardised Coefficients:
[1.4958, 0.9133, 2.3669, 1.3436]
```

1.3 (*c*)

```
In [ ]: # Computing Student's Two Sample t-test statistics
        t_test_stats = []

        for variable in range(p):
          numerator = mean_males[variable] - mean_females[variable]
          pooled_sd = np.sqrt((covariance_males[variable, variable] +
                               covariance_females[variable, variable]) / 2)
          denominator = pooled_sd * np.sqrt(2 / count_males)
          t_test_stats.append(numerator / denominator)

        t_test_stats = np.asarray(t_test_stats)
```

```
        print('Two Sample t-test Statistics for the ' + str(p) + ' variables:')
        print(np.ndarray.tolist(np.round(t_test_stats, 4)))
```

1.4 (*d*)

```
In [6]: print('Ranking variables in order of their contribution to separating of the groups.')
        print('  Based on:')
        print('  Two Sample t-test Statistics = '
              + str(1 + np.argsort(-1 * np.abs(t_test_stats))))
        print('  Standardized Coefficient = '
              + str(1 + np.argsort(-1 * np.abs(std_coefficients))))
```

```
Ranking variables in order of their contribution to separating of the groups.
  Based on:
  Two Sample t-test Statistics = [3 1 2 4]
  Standardized Coefficient = [3 1 4 2]
```

$The\ same\ rankings\ are\ obtained\ based\ on\ Two\ Sample\ t-test\ Statistics\ and\ standardized\ coefficient\ values.$

1.5 (*e*)

```
In [7]: def partial_hotelling_two_sample_T_square(neglect = -1):
          # neglect can take values 0, 1, ..., (p - 1)
          # if neglect = -1, consider complete two sample T square
          global pooled_covariance
          global mean_females, mean_males
          global count_females, count_males
          global p

          indices = np.asarray([index for index in range(p) if index != neglect])
          cov = []

          for row in range(p):
            if row == neglect:
              continue
            cov.append([])
            for col in range(p):
              if col == neglect:
                continue
              cov[-1].append(pooled_covariance[row, col])

          cov = np.asmatrix(cov)
          mean_1 = mean_males[indices]
          mean_2 = mean_females[indices]
          difference = mean_1 - mean_2
```

```
    factor = (mean_males * mean_females) / (mean_males + mean_females)
    T_square = np.matmul(np.matmul(difference.T, np.linalg.inv(cov)), difference)

    return np.ravel(T_square)[0]

  complete_T_square = partial_hotelling_two_sample_T_square()
  nu = count_females + count_males - k
  factor = nu - p + 1
  alpha = 0.05

  critical_partial_F = f.isf(alpha, 1, factor)
  print('Critical Partial F value = ' + str(np.round(critical_partial_F, 4)))
  print()

  partial_F_s = []
  for drop in range(p):
    T_square_partial = partial_hotelling_two_sample_T_square(drop)
    F = factor * (complete_T_square - T_square_partial) / (nu + T_square_partial)
    partial_F_s.append(F)
    print('p = ' + str(drop + 1) + ': ')
    print('  Computed Partial F value = '  + str(np.round(F, 4)))
    print('  Standardized Coefficient = '  + str(np.round(std_coefficients[drop], 4)))
    print()

  partial_F_s = np.asarray(partial_F_s)

  print('Ranking variables in order of their contribution to separating the groups.')
  print('  Based on:')
  print('  Computed Partial F value = '
        + str(1 + np.argsort(-1 * np.abs(partial_F_s))))
  print('  Standardized Coefficient = '
        + str(1 + np.argsort(-1 * np.abs(std_coefficients))))
```

```
Critical Partial F value = 4.0162

p = 1:
  Computed Partial F value = 1.189
  Standardized Coefficient = 1.4958

p = 2:
  Computed Partial F value = 0.4386
  Standardized Coefficient = -0.9133

p = 3:
  Computed Partial F value = 3.2927
  Standardized Coefficient = 2.3669

p = 4:
```

```
  Computed Partial F value = 1.0822
  Standardized Coefficient = -1.3436

Ranking variables in order of their contribution to separating the groups.
  Based on:
  Computed Partial F value = [3 1 4 2]
  Standardized Coefficient = [3 1 4 2]
```

The same individual variable rankings are obtained based on Partial F values and the standardized coefficient values.

2 *Problem* 5

In [8]:
```
method_1 = [[5.4, 6.0, 6.3, 6.7],
            [5.2, 6.2, 6.0, 5.8],
            [6.1, 5.9, 6.0, 7.0],
            [4.8, 5.0, 4.9, 5.0],
            [5.0, 5.7, 5.0, 6.5],
            [5.7, 6.1, 6.0, 6.6],
            [6.0, 6.0, 5.8, 6.0],
            [4.0, 5.0, 4.0, 5.0],
            [5.7, 5.4, 4.9, 5.0],
            [5.6, 5.2, 5.4, 5.8],
            [5.8, 6.1, 5.2, 6.4],
            [5.3, 5.9, 5.8, 6.0]]

method_2 = [[5.0, 5.3, 5.3, 6.5],
            [4.8, 4.9, 4.2, 5.6],
            [3.9, 4.0, 4.4, 5.0],
            [4.0, 5.1, 4.8, 5.8],
            [5.6, 5.4, 5.1, 6.2],
            [6.0, 5.5, 5.7, 6.0],
            [5.2, 4.8, 5.4, 6.0],
            [5.3, 5.1, 5.8, 6.4],
            [5.9, 6.1, 5.7, 6.0],
            [6.1, 6.0, 6.1, 6.2],
            [6.2, 5.7, 5.9, 6.0],
            [5.1, 4.9, 5.3, 4.8]]

method_3 = [[4.8, 5.0, 6.5, 7.0],
            [5.4, 5.0, 6.0, 6.4],
            [4.9, 5.1, 5.9, 6.5],
            [5.7, 5.2, 6.4, 6.4],
            [4.2, 4.6, 5.3, 6.3],
            [6.0, 5.3, 5.8, 5.4],
            [5.1, 5.2, 6.2, 6.5],
```

```
                [4.8, 4.6, 5.7, 5.7],
                [5.3, 5.4, 6.8, 6.6],
                [4.6, 4.4, 5.7, 5.6],
                [4.5, 4.0, 5.0, 5.9],
                [4.4, 4.2, 5.6, 5.5]]

method_1 = np.asarray(method_1)
method_2 = np.asarray(method_2)
method_3 = np.asarray(method_3)
methods = [method_1, method_2, method_3]

all_data = np.vstack(methods)

# As there are three methods
k = 3
n = methods[0].shape[0]
p = methods[0].shape[1]
N = all_data.shape[0]

total_mean = np.mean(all_data, 0)
means = []

for method in methods:
  means.append(np.mean(method, 0))
```

```
In [9]: # Computing between Sum of Squares
        H = np.zeros((p, p))
        for mean in means:
          unit = np.asmatrix(mean - total_mean)
          H += np.matmul(unit.T, unit)

        H = n * H
```

```
In [10]: # Computing within Sum of Squares
         E = np.zeros((p, p))

         for i in range(k):
           for j in range(n):
             unit = np.asmatrix(methods[i][j] - means[i])
             E += np.matmul(unit.T, unit)

         consider = np.matmul(np.linalg.inv(E), H)
```

2.1 (*a*)

```
In [11]: # Computing eigenvectors of inverse(E).H

         eigen_values, eigen_vectors = np.linalg.eig(consider)
```

```
        positions = np.argsort(-1 * eigen_values)

        print('Eigen Vectors in Decreasing order of importance:')
        print()

        for vector_index, index in enumerate(positions):
          print('Eigen Vector ' + str(vector_index + 1) + ' = ', end = '')
          print(np.ndarray.tolist(np.round(eigen_vectors[:, index], 4)))
```

```
Eigen Vectors in Decreasing order of importance:

Eigen Vector 1 = [-0.013, 0.8274, -0.5496, -0.1149]
Eigen Vector 2 = [0.6057, -0.5857, -0.5088, 0.1767]
Eigen Vector 3 = [-0.2557, 0.0156, -0.4918, 0.8322]
Eigen Vector 4 = [0.8065, -0.3107, -0.0554, 0.4999]
```

2.2 (*b*)

```
In [12]: # Tests of significance

         sorted = eigen_values[positions]

         def test_statistic_V_m(m = 0):
           global N, p, k
           global sorted

           factor = N - 1 - (p + k) / 2
           sum = 0
           for index in range(m, p):
             sum += np.log(1 + sorted[index])

           return factor * sum

         # Computing relative importance of each eigen vector based on V_m test statistic
         print('Contribution to the separation based on V_m test statistic of:')
         for index in range(p):
           print('  Eigen Vector ' + str(index + 1) + ' = ', end = '')
           print(np.abs(np.round(test_statistic_V_m(index), 4)))
```

```
Contribution to the separation based on V_m test statistic of:
  Eigen Vector 1 = 45.4099
  Eigen Vector 2 = 3.9405
  Eigen Vector 3 = 0.0
  Eigen Vector 4 = 0.0
```

```
In [13]: # Computing relative importance of each eigen vector
         # based on weighted average of eigen values
```

```
        sum_eigen_values = np.sum(eigen_values)

        print('Contribution to the separation based on eigen value spectrum of:')
        for vector_index, index in enumerate(positions):
          print('  Eigen Vector ' + str(vector_index + 1) + ' = ', end = '')
          print(np.abs(np.round(eigen_values[index] / sum_eigen_values, 4)))
```

```
Contribution to the separation based on eigen value spectrum of:
  Eigen Vector 1 = 0.9535
  Eigen Vector 2 = 0.0465
  Eigen Vector 3 = 0.0
  Eigen Vector 4 = 0.0
```

- *The two procedures agree on the number of important discriminant functions.*
- *The relative importance of the functions as discerned based on the two methods are also similar.*

2.3 (*c*)

```
In [14]: principal_eigen_vector = eigen_vectors[:, positions[0]]
         print('Discriminant Function Coefficient Vector:')
         print(np.ndarray.tolist(np.round(principal_eigen_vector, 4)))
         print()

         S_pl = E / (N - k)
         std_coefficients = np.multiply(np.sqrt(np.diag(S_pl)), principal_eigen_vector)
         print('Standardised Coefficients:')
         print(np.ndarray.tolist(np.round(std_coefficients, 4)))
         print()

         print('Absolute Values of Standardised Coefficients:')
         print(np.ndarray.tolist(np.abs(np.round(std_coefficients, 4))))
```

```
Discriminant Function Coefficient Vector:
[-0.013, 0.8274, -0.5496, -0.1149]

Standardised Coefficients:
[-0.0083, 0.41, -0.326, -0.0668]

Absolute Values of Standardised Coefficients:
[0.0083, 0.41, 0.326, 0.0668]
```

Based on the absolute values of the standardized coefficients, we conclude that y_2, and y_3 contribute significantly to the separation of groups but y_1 and y_4 do not.

```
In [15]: def partial_Wilk_s_two_sample_T_square(neglect = -1):
           # neglect can take values 0, 1, ..., (p - 1)
```

```
    # if neglect = -1, consider complete two sample T square
    global H, E

    if neglect == -1:
      return np.linalg.det(E) / np.linalg.det(H + E)

    indices = np.asarray([index for index in range(p) if index != neglect])
    H_consider = []
    E_consider = []

    for row in range(p):
      if row == neglect:
        continue
      H_consider.append([])
      E_consider.append([])

      for col in range(p):
        if col == neglect:
          continue
        H_consider[-1].append(H[row, col])
        E_consider[-1].append(E[row, col])

    H_consider = np.asmatrix(H_consider)
    E_consider = np.asmatrix(E_consider)

    wilk_s = np.linalg.det(E_consider) / np.linalg.det(H_consider + E_consider)

    return wilk_s
```

2.4 *(d)*

```
In [16]: nu_E = N - k
         nu_H = k - 1
         alpha = 0.05
         factor = (nu_E - p + 1) / nu_H

         critical_partial_F = f.isf(alpha, nu_H, nu_E - p + 1)
         print('Critical Partial F value = ' + str(np.round(critical_partial_F, 4)))
         print()

         wilk_s_total = partial_Wilk_s_two_sample_T_square()

         partial_F_s = []
         for drop in range(p):
           wilk_s = partial_Wilk_s_two_sample_T_square(drop) / wilk_s_total
           F = factor * (1 - wilk_s) / wilk_s
           partial_F_s.append(F)
```

```
        print('p = ' + str(drop + 1) + ': ')
        print('  Computed Partial F value = '  + str(np.round(F, 4)))
        print('  Standardized Coefficient = '  + str(np.round(std_coefficients[drop], 4)))
        print()

    partial_F_s = np.asarray(partial_F_s)

    print('Ranking variables in order of their contribution to separating the groups.')
    print('Based on:')
    print('  Computed Partial F value = '
          + str(1 + np.argsort(-1 * np.abs(partial_F_s))))
    print('  Standardized Coefficient = '
          + str(1 + np.argsort(-1 * np.abs(std_coefficients))))
```

```
Critical Partial F value = 3.3158

p = 1:
  Computed Partial F value = -1.0668
  Standardized Coefficient = -0.0083

p = 2:
  Computed Partial F value = -8.3064
  Standardized Coefficient = 0.41

p = 3:
  Computed Partial F value = -6.1863
  Standardized Coefficient = -0.326

p = 4:
  Computed Partial F value = -0.483
  Standardized Coefficient = -0.0668

Ranking variables in order of their contribution to separating the groups.
Based on:
  Computed Partial F value = [2 3 1 4]
  Standardized Coefficient = [2 3 4 1]
```

Partial F values rank the two worthy variables i.e. y_2*, and* y_3 *in the same order as standardized* $coefficitents$*.*

2.5 (*e*)

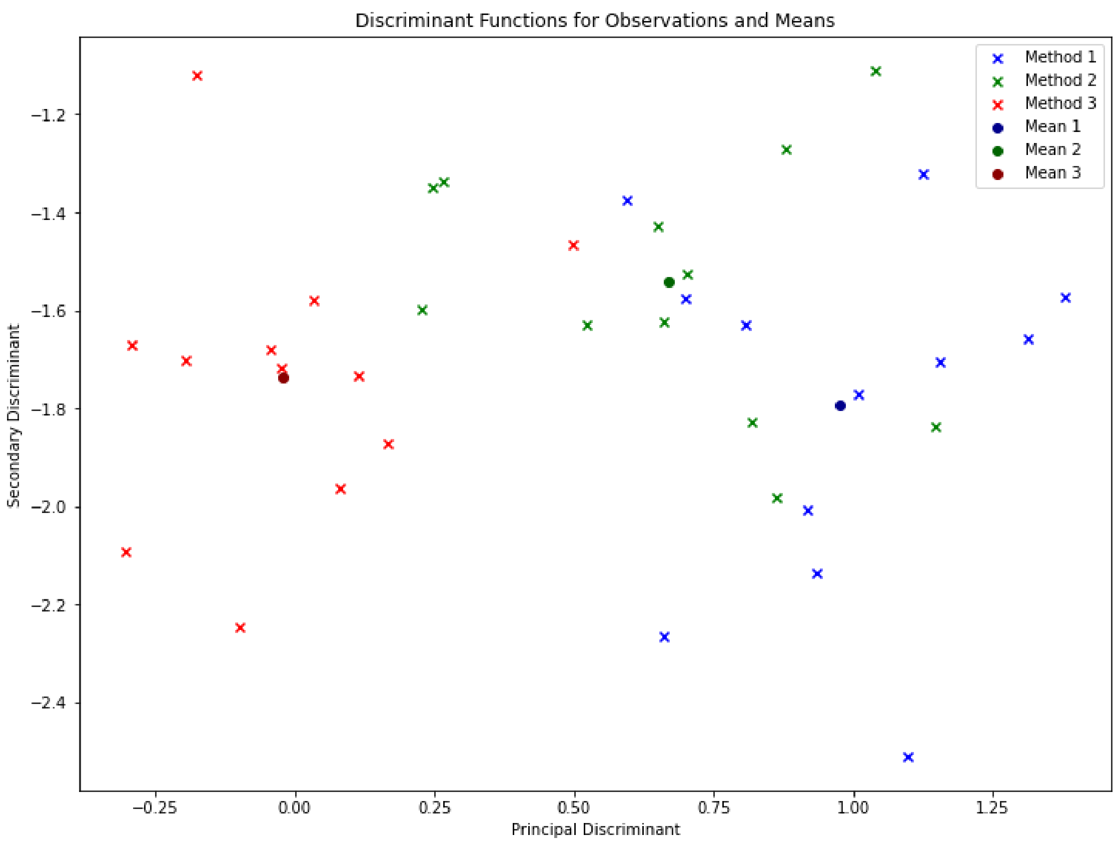
```
In [17]: # Constructing the Plots
         x = [[], [], [], []]
         y = [[], [], [], []]

         for index in range(N):
           unit = all_data[index]
```

```
    x[int(index / n)].append(np.dot(eigen_vectors[:, positions[0]], unit))
    y[int(index / n)].append(np.dot(eigen_vectors[:, positions[1]], unit))

for index in range(k):
    unit = means[index]
    x[k].append(np.dot(eigen_vectors[:, positions[0]], unit))
    y[k].append(np.dot(eigen_vectors[:, positions[1]], unit))

fig = plt.figure(figsize = (12, 9))
p = fig.add_subplot('111')
p.set_title('Discriminant Functions for Observations and Means')
p.set_xlabel('Principal Discriminant')
p.set_ylabel('Secondary Discriminant')
p.scatter(x[0], y[0], color = 'blue', label = 'Method 1', marker = 'x')
p.scatter(x[1], y[1], color = 'green', label = 'Method 2', marker = 'x')
p.scatter(x[2], y[2], color = 'red', label = 'Method 3', marker = 'x')
p.scatter(x[3][0], y[3][0], color = 'darkblue', label = 'Mean 1')
p.scatter(x[3][1], y[3][1], color = 'darkgreen', label = 'Mean 2')
p.scatter(x[3][2], y[3][2], color = 'darkred', label = 'Mean 3')
p.legend()
plt.show()
```



```
In [18]: # ^_^ Thank You
```