A2 Multivariate Normal Distribution

1. Show that $g(z) = \frac{1}{\sqrt{2\pi}} e^{z^2/2}$ is a density.

2. For a bivariate normal distribution $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ with means μ_1 and μ_2 and with $\sigma_1 = \sqrt{\sigma_{11}},\ \sigma_2 = \sqrt{\sigma_{22}}$, and $\sigma_{12} = \rho\sigma_1\sigma_2$, where ρ is correlation, form the standardised variables

   $$z_1 = \frac{y_1 - \mu_1}{\sigma_1} \qquad\qquad z_2 = \frac{y_2 - \mu_2}{\sigma_2}$$

   show that the mean squared difference between the two standardised variables is $2(1 - \rho)$.

3. If $\bar{y}$ is the mean of a random sample $y_1, \ldots, y_n$ from $N(\mu, \sigma^2)$, then show that $E(\bar{y}) = \mu$

4. The sample variance of a random sample $y_1, \ldots, y_n$ from $N(\mu, \sigma^2)$ is $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2$.

   (a) Express s^2 in the quadratic form $\mathbf{y}'\mathbf{H}\mathbf{y}$ where $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ and $\mathbf{H}$ is the idempotent matrix $\mathbf{H} = \mathbf{I} - \mathbf{J}/n$.

   (b) Show that $E\left(s^2\right) = \sigma^2$ using the the formula for the expectation of quadratic form (see Assignment 1)

5. If $\bar{\mathbf{y}}$ is the mean of a random sample $\mathbf{y_1}, \ldots, \mathbf{y_n}$ from $N_p(\mu, \boldsymbol{\Sigma})$, then show that $E(\bar{\mathbf{y}}) = \mu$

6. The sample variance $\mathbf{S}$ of a random sample $\mathbf{y_1}, \ldots, \mathbf{y_n}$ from $N_p(\mu, \boldsymbol{\Sigma})$, is $\mathbf{S} = \dfrac{1}{n-1}\sum_{i=1}^{n}(\mathbf{y}_i - \bar{\mathbf{y}})(\mathbf{y}_i - \bar{\mathbf{y}})'$. Show that $E(\mathbf{S}) = \boldsymbol{\Sigma}$.

7. Consider a bivariate random vector $\begin{bmatrix} x \\ y \end{bmatrix}$ and let σ_{xy} denote the population covariance . For a sample $\left(x_1, y_1\right), \ldots, \left(x_n, y_n\right)$, let s_{xy} denote the sample covariance. Using the result
$E(\mathbf{x}'\mathbf{A}\mathbf{y}) = \text{tr}\left(\mathbf{A}\boldsymbol{\Sigma}_{\mathbf{yx}}\right) + \mu_{\mathbf{x}}'\mathbf{A}\mu_{\mathbf{y}}$, prove that

$$E\left(s_{xy}\right) = \sigma_{xy}$$

8. Prove that if $\bar{\mathbf{y}}$ and $\mathbf{S}$ are based on a random sample $\mathbf{y}_1, \ldots, \mathbf{y}_n$ from $N_p(\mu, \Sigma)$, then $\bar{\mathbf{y}}$ and $\mathbf{S}$ are independent.

9. A standardised vector $\mathbf{z}$ is obtained as

$$\mathbf{z} = (\mathbf{T}')^{-1}(\mathbf{y} - \mu)$$

where $\boldsymbol{\Sigma} = \mathbf{T}'\mathbf{T}$ is factored using the Cholesky procedure. Show that $E(\mathbf{z}) = \mathbf{0}$ and cov$(\mathbf{z}) = \mathbf{I}$.

10. Suppose $\mathbf{y}$ is $N_3(\mu, \Sigma)$, where

$$\mu = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 6 & 1 & -2 \\ 1 & 13 & 4 \\ -2 & 4 & 4 \end{bmatrix}$$

(a) Find the distribution of $z = 2y_1 - y_2 + 3y_3$.

(b) Find the joint distribution of $z_1 = y_1 + y_2 + y_3$ and $z_2 = y_1 - y_2 + 2y_3$.

(c) Find the distribution of y_2.

(d) Find the joint distribution of y_1 and y_3.

(e) Find the joint distribution of y_1, y_3, and $\frac{1}{2}\left(y_1 + y_2\right)$

11. Suppose $\mathbf{y}$ is $N_3(\mu, \Sigma)$, with

$$\mu = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 4 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Which of the following random variables are independent?

(a) y_1 and y_2

(b) y_1 and y_3

(c) y_2 and y_3

(d) $\left(y_1, y_2\right)$ and y_3

(e) $\left(y_1, y_3\right)$ and y_2

12. Suppose $\mathbf{y}$ and $\mathbf{x}$ are subvectors, such that $\mathbf{y}$ is 2×1 and $\mathbf{x}$ is 3×1, with μ and Σ partitioned accordingly:

$$\mu = \begin{bmatrix} 3 \\ -2 \\ 4 \\ -3 \\ 5 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 14 & -8 & 15 & 0 & 3 \\ -8 & 18 & 8 & 6 & -2 \\ 15 & 8 & 50 & 8 & 5 \\ 0 & 6 & 8 & 4 & 0 \\ 3 & -2 & 5 & 0 & 1 \end{bmatrix}$$

Assume that $\begin{bmatrix} \mathbf{y} \\ \mathbf{x} \end{bmatrix}$ is distributed as $N_5(\mu, \Sigma)$.

(a) Find $E\left(\mathbf{y} \,|\, \mathbf{x}\right)$

(b) Find $\operatorname{cov}(\mathbf{y} \mid \mathbf{x})$