Multivariate Analysis

Assignment 8

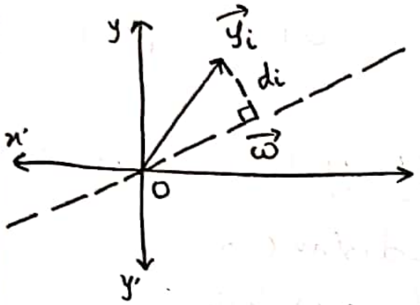
1) Y : centered data matrix of dimension $n \times p$ (let)

Y_i : $p \times 1$ vector representing the i^{th} datapoint

if ω is a unit vector :

Objective : $\underset{\omega}{\text{minimize}} \sum_{i=1}^{n}$ (squared orthogonal distance of Y_i from ω)

such that $\|\omega\| = 1$ — (I)



: Based on the figure

$$d_i^2 = \|y_i\|^2 - \|y_i \cdot \vec{w}\|^2$$
$$= y_i \cdot y_i^T - (y_i^T \cdot \omega)^2$$
$$= y_i^T \cdot y_i - \omega^T \cdot y_i \cdot y_i^T \omega$$

Now, $\underset{\|\omega\|=1}{\text{argmin}_\omega} \sum_{i=1}^{n} d_i^2 = \underset{\|\omega\|=1}{\text{argmin}_\omega} \sum_{i=1}^{n} (y_i^T \cdot y_i - \omega^T \cdot y_i \cdot y_i^T \omega)$

$= \underset{\|\omega\|=1}{\text{argmax}_\omega} \sum_{i=1}^{n} (\omega^T y_i y_i^T \omega) = \underset{\|\omega\|=1}{\text{argmax}_\omega} \omega^T \left(\sum_{i=1}^{n} y_i^T y_i\right) \omega$

[Y is centered $\rightarrow \bar{Y} = 0$] $= \underset{\|\omega\|=1}{\text{argmax}_\omega} \omega^T S \omega$ {S being the data covariance matrix}

$= \text{argmax}_\omega \; \omega^T S \omega + \lambda(1 - \omega^T \omega)$ [Use of lagrangian]

let $F(\omega) = \omega^T S \omega + \lambda(1 - \omega^T \omega)$

$\frac{\partial F}{\partial \omega} = 0 \Rightarrow 2S\omega - 2\lambda\omega = 0 \Rightarrow (S - I\lambda)\,\omega = 0$

$\Rightarrow (S - \lambda I)\omega = 0$ [Eigenvalue problem]

And $S\omega = \lambda\omega$

$\therefore \omega^T S \omega = \omega^T \lambda \omega = \lambda \omega^T \omega = \lambda$.

To maximize $\omega^T S \omega$ or to achieve (I), we consider the largest eigenvalue of S as λ and choose the corresponding eigenvector as ω. Hence, the direction of the first principal component is given by the eigenvector corresponding to the largest eigenvalue of S. [Proved]

2) Proposition: Principal Components are scale invariant.

We construct a counterexample to disprove the above Proposition. Consider the case presented below.

- After scaling the data; we see that the directions of the principal components change significantly.
  - this is illustrated both by
    - computing the eigenvectors
    - plotting the data along with the principal components.

A scale change in a variable leads to a change in the shape of the swarm of points in the dataset.

Based on the illustrations within the presented counterexample, the proposition is disproved. Therefore, principal components are not scale invariant. [Proved].