A7 Canonical Correlation

Problems

1. Show that the squared canonical correlations and the coefficient vectors in the canonical variates are the eigenvalues and eigenvectors of the following eigenvalue problems.

$$\left(\mathbf{S}_{yy}^{-1}\mathbf{S}_{yx}\mathbf{S}_{xx}^{-1}\mathbf{S}_{xy} - r^2\mathbf{I}\right)\mathbf{a} = \mathbf{0} \qquad (1)$$

and

$$\left(\mathbf{S}_{xx}^{-1}\mathbf{S}_{xy}\mathbf{S}_{yy}^{-1}\mathbf{S}_{yx} - r^2\mathbf{I}\right)\mathbf{b} = \mathbf{0} \qquad (2)$$

Show that (1) and (2) yield the same eigenvalue r^2.

2. Using the scaling $s_u^2 = \mathbf{a}'\mathbf{S}_{yy}\mathbf{a} = 1$ and $s_v^2 = \mathbf{b}'\mathbf{S}_{xx}\mathbf{b} = 1$, show that

$$\mathbf{a} = \frac{1}{r}\mathbf{S}_{yy}^{-1}\mathbf{S}_{yx}\mathbf{b}$$

$$\mathbf{b} = \frac{1}{r}\mathbf{S}_{xx}^{-1}\mathbf{S}_{xy}\mathbf{a}$$

3. Define $\mathbf{C} = \mathbf{S}_{yy}^{-1/2}\mathbf{S}_{yx}\mathbf{S}_{xx}^{-1/2}$. Show that the eigenvalues of $\mathbf{C}\mathbf{C}'$ or $\mathbf{C}'\mathbf{C}$ are squared canonical correlations. If $\mathbf{p}_i$ and $\mathbf{q}_i$ are eigenvectors of $\mathbf{C}\mathbf{C}'$ and $\mathbf{C}'\mathbf{C}$, determine their relationship to the coefficient vectors $\mathbf{a}$ and $\mathbf{b}$ in the canonical variates.

4. Show that all canonical variates are uncorrelated with each other except for the pairs u_i and v_i, $i = 1,2,\ldots,s$.

5. The largest canonical correlation r_1 was defined as the maximum correlation between $u = \mathbf{a}'\mathbf{y}$ and $v = \mathbf{b}'\mathbf{x}$. It was shown that r_1^2 is the largest eigenvalue of $\mathbf{S}_{yy}^{-1}\mathbf{S}_{yx}\mathbf{S}_{xx}^{-1}\mathbf{S}_{xy}$. Show that the same result is obtained if r^2 is defined as the maximum value of $\left(\mathbf{a}'\mathbf{S}_{yx}\mathbf{b}\right)^2$ subject to the scaling $\mathbf{a}'\mathbf{S}_{yy}\mathbf{a} = 1$ and $\mathbf{b}'\mathbf{S}_{xx}\mathbf{b} = 1$.

6. Use the glucose data given below. (y_1, y_2, y_3) and (x_1, x_2, x_3) represent measurements of blood glucose levels on three occasions for 20 women.

   (a) Find the canonical correlations between (y_1, y_2, y_3) and (x_1, x_2, x_3).

   (b) Find the standardised coefficients for the canonical variates.

   (c) Test the significance of each canonical correlation.

Fasting			One Hour After Sugar Intake		
y_1	y_2	y_3	x_1	x_2	x_3
60	69	62	97	69	98
56	53	84	103	78	107
80	69	76	66	99	130
55	80	90	80	85	114
62	75	68	116	130	91
74	64	70	109	101	103
64	71	66	77	102	130
73	70	64	115	110	109
68	67	75	76	85	119
69	82	74	72	133	127
60	67	61	130	134	121
70	74	78	150	158	100
66	74	78	150	131	142
83	70	74	99	98	105
68	66	90	119	85	109
78	63	75	164	98	138
77	68	74	144	71	153
66	77	68	77	82	89
70	70	72	114	93	122
75	65	71	77	70	109