Important Points from *Lecture 19* by Prof. J D Cowan...

...from the course: 'Modeling and Signal Analysis for Neuroscientists'[1]

1. History of early efforts in Neural Networks and how it is intertwined with the history of Digital Computers

Jack Cowan was a graduate student at MIT from 1958 to 1962 where he worked closely with the founders of Neural-Network (NN) studies, namely McCulloch and Pitts, and other stalwarts of the field such as Norbert Wiener and Claude Shannon. He also collaborated with Dennis Gabor from England, the inventor of Holography who was also interested in Machine Learning and the working of the human brain. In 1962, Cowan asked three of his mentors, McCulloch, Pitts, and Wiener; what kind of mathematics would be needed to model biological and not artificial NNs (of which people in MIT were considered pioneers)? McCulloch suggested the use of the kind of mathematics used by Pitts and him in their 1943 paper, 'A logical calculus of the ideas immanent in nervous activity',[2] which showed that linear threshold elements in very simplified models of neurons could be used to calculate logical functions. Something capable of computing 'and', 'or', and 'not', having the ability to iterate with a virtually limitless memory would essentially be a universal Turing machine. Turing had interacted with John Von Neumann in Princeton and roughly in mid 1930s, Turing built upon the work of Gödel and the famous paper on the decision problem (how to decide whether a formula evaluates to true or false) was released. In the turn of the 20th century, German mathematician Hilbert claimed that it might be possible to completely reduce math to logic. In that, one would be able to reduce any problem to determining truth or falsity of specific sub-units. The idea then was to use this universal framework to solve all mathematical problems and in this direction, Alfred North Whitehead and Bertrand Russell went on to write the famous Principia Mathematica. However, they encountered logical paradoxes: propositions which are true and false at the same time leading to inconsistencies. This problem didn't seem to be resolvable.

In the mid 1920s, Gödel had identified the following three desirable properties of a logical system:

1. Can adequately represent whatever is required
2. Is consistent i.e one cannot prove something to be both true and false at the same time
3. Is complete i.e all expressible propositions fit into the domain of formulas

Gödel showed further that in any logical system, we can have upto two of the above three properties. This produced a big surprise for the entire community. Turing formalized Gödel's

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[1] https://www.youtube.com/watch?v=67HdtyJrPkA&t=1747s

[2] https://link.springer.com/article/10.1007/BF02478259

theorem into a machine involving binary codes, a tape containing a message, and a little program (control system) working on the tape with a few basic operations. All problems that could be solved to arrive at a boolean answer could be calculated by this machine. The operation would stop when a value was reached. But for certain problems, the computation would in theory, go on indefinitely. This corresponds to the undecidable propositions and was demonstrated by the halting problem. Overall, a Turing machine could simulate any other program and in his mid 20s, Turing developed his machine to decode German codes which involved many different rotors, spinning to produce various deciphering combinations. This was the first digital computer.

McCulloch and Pitts tried to imitate a universal Turing machine using basic nerve cells or neurons in early 1943 at the University of Chicago which led to modern developments in theory of artificial neural networks. But even till the early 1960s, these formulas could not be adapted for biological neural networks as the human brain contains no clocks or switches. Pitts advised Cowan to formulate the problem in a way which would allow him to use calculus. Norbert Wiener advised Cowan to formulate the whole problem as a random process and to use his technologies involving stochastic modelling from the 1930s which included the path integral method. This would leverage the statistics involved with large scale brain activity data. Although Cowan didn't fully understand this at the time, he chose the path of working out the differential equations approach plus the stochastic theory of neural networks.

Cowan joined the University of Chicago in 1967 and took over the committee of mathematical biology. Working with Hugh Wilson, Cowan published two papers after polishing his equations further. These Cowan-Wilson equations are still relevant to this day.

2. On dynamical attractors

A neural cortex has over fifty billion neurons. On flattening it out, it can be approximated to a two-dimensional sheet. Every tissue block contains roughly 250 thousand to 750 thousand neurons, where each neuron is connected to several thousand others. It is impossible to calculate the firing states of every single neuron simultaneously in a block of tissue as that would require building or modelling a replica of the brain. The adopted statistical approach and the Wilson-Cowan equations ultimately led to the notion of dynamical attractors which are systems associating 'stability' with 'behavior'. This was the first step in mathematical biology which helped understand attractor dynamics.

To perceive dynamical attractors, we need to plot the phase plane of points of local stability in the system; which are current (I) vs potential (E) graphs where we plot the locus of points where $I^{\cdot}$ and $E^{\cdot}$ are zero. Intersection of these lines known as nullclines represent attractors. A point of stability is a stable attractor which witnesses only small fluctuations. Now, if there exists two separate stable states, there has to be an unstable state in between (analogous to

Bolzano's Theorem of continuous functions in calculus). Based on this, we have three different types of dynamical attractors:

1. Node or Fixed Point: This type of attractor can be visualized as nullclines intersecting at a particular point. On a neural sheet, this manifests as constant firing activity throughout.
2. Limit Cycle: When the nullclines do not intersect at a particular point but execute a periodic decaying behavior, we have a periodic attractor. This manifests as burst-like firing activity.
3. Stable Focus: This is sort of an intermediate state between Node and Limit Cycle. The system's nullcline whines around pretending to reach a limit cycle but ultimately stops at a node. This manifests on a neural sheet as burst-like firing activity during initial periods which gradually decays into constant firing.

The first two types of attractors were discovered in 1972, however, the last one was missed out. Just by tuning the rate of recurrent excitation and inhibition, we can pass all the way from Node through Stable Focus to a Limit Cycle. Thus, using the given formulation, one can capture the global dynamics of brain tissue, pretty well.

3. Origin of Sigmoid Function

The concept of a heavy-side step function was introduced into neuronal dynamics when the idea of a critical threshold for neuron firing or spiking originated. However, this function was not continuous. When the stochastic model for pulses was proposed, phenomena such as exponential decay of potential in the post-spiking period could be explained and quantified. Considering the idea of a refractory period of neurons, the activation function flattened out resembling the integral of a probability density function. This would, in future, lead to sigmoid activation for NNs and other types of functions modelling firing rate such as $\frac{1}{2}.(1 + tanh(\frac{v}{2}))$. Cowan didn't identify a use-case for this to model general functions in feed-forward NN form at the time (and jokes about setting back the field by 25 years).

Wim van Drongelen and Jack Cowan were both interested in epileptic seizures. To try to fit the available data, Wim introduced a sigmoidal firing rate function. At very high levels of activity, the potential curve turned down resulting in something unimodal in nature. This seemingly violated the whole existing setup. Now, sigmoid can be thought of as an integral over a probability density of thresholds or rather the fraction of neurons that receive at least threshold excitation. Wim noted that this density function should behave similar to a sine curve and in that, it would be negative after a point. The interesting bit here was the realization that there were actually two kinds of thresholds. The first being from Hugh Wilson's modelling, while the second was due to the depolarization block which causes the firing to cease and the potential to normalize as explained by Hodgkin-Huxley. Thus, the absolute difference in the two sigmoids with differing initial points led to the unimodal curve. This is how two sigmoids get combined to produce a unimodal Gaussian:

- $S(x) = S_a(x).(1 - S_b(x))$

4. Incorporating Stochastic Processes into mean-field neural dynamics

The Wilson-Cowan equations captured the mean field dynamics of large scale cortical activity quite well. It stood the test of time, and was able to generalize to other phenomena and fit new observations and curated datasets. Mean field regime of neural dynamics involves study of neural activity based on existing data and modelling without building in, the effect of intrinsic noise. It has its origins in statistical physics. Cowan introduced the concept of population of neurons and its related statistical features. It is practically impossible to model or observe firing states of every single neuron simultaneously in a block of tissue as that would require building or modelling a replica of the brain. Thus a stochastic approach was adopted where the model parameters would suit the existing data regarding behavior of large neuron populations.

Cowan quotes Einstein saying, "The best model is the simplest model; but no simpler". He went on to use a two-state Markov Process, where the states represent quiescent neuronal state and active neuronal state respectively, with a sigmoid activation to go from quiescent to active and, in the original system, a probability of random occurence to go in the reverse direction. The original equations with the induced probability components due to the stochasticity could be solved using path integral techniques as was suggested to Cowan by Norbert Wiener. A stochastic version of the Wilson-Cowan equations factors-in all of these components. In future, for an observed neural activity producing random bursts, upon plotting the histogram of number of bursts per given size vs burst-size on a $log(log(.))$ scale, a straight line with slope close to -1.5 was obtained. This phenomenon is the signature of a random process called branching and annihilating random walk and fits the original theory proposed by Wilson-Cowan. Thus, we see how the stochastic modelling with Markov Chains was in fact quite successful. Although most brain models propagate activations and responses in the form of a feed-forward network, in reality, almost 90% of the connections in the brain are recurrent. This leaves scope for future research and modifications in the field of neural dynamics in terms of mathematical and statistical modelling.